Solid Conductor in a Magnetic Field That Moves at a Uniform Velocity along a Channel

Mohd Noor Saad

Laboratory of Computational Sciences and Informatics
Institute for Mathematical Research
Universiti Putra Malaysia

Department of Mathematics
Faculty of Science
Univesiti Putra Malaysia

matnorsaad@yahoo.co.nz

Abstract

A two dimensional magnetic field is suddenly moved at a uniform velocity along a channel of infinite length containing an electrically conducting solid metal with magnetic permeability and magnetic diffusivity under the assumption of finite Reynolds number. An analytical solution is obtained for the case of one Fourier-component magnetic field. The numerical scheme is developed with the implementation of the boundary conditions. It is shown that in the steady state the numerical and the analytical results agree well for magnetic Reynolds number in the range of 1 to 1200. It is found that as magnetic Reynolds number increases the solid metal behaves more and more like a perfect conductor where the field lines are frozen in the moving solid metal. Pictures of evolution of field-lines with time are plotted for various times.

Introduction

The phenomenon of 'flux expulsion' in magnetohydrodynamics occurs at high magnetic Reynolds number whenever a flow with streamlines acts upon a magnetic field transverse to the flow. The purely kinematics aspects have been widely studied by Zel'dovich (1957), E. N. Parker (1963), R. L. Parker (1966) and Weiss (1966). It is also known (Galloway, Proctor & Weiss 1978; Proctor & Galloway 1979) that flux expulsion can persist even in situations where the magnetic field has a strong dynamic influence. H. A. Kamkar & H. K. Moffatt (1981) considered pressure-driven flow along a channel in the presence of an applied magnetic field which is periodic in the stream-wise direction and showed that flux expulsion due to reconnection of field lines occurs when the pressure gradient is sufficiently large which finally leads to a runaway effect.

In the flow where it is assumed that the magnetic Reynolds number, R_m is small the fluid velocity, u, does not affect the applied magnetic field. But in this study we want to consider the effects of finite, R_m on the fluid flow as well as the imposed magnetic field.

For simplicity we first consider a solid metal which at time $t = 0$ is moved at a constant velocity $U\hat{x}$. Because this problem can be solved analytically it provides a useful check on the numerical scheme we use later to investigate flow where R_m is finite.

In section 2 we formulate the solid conductor problem and derive the dimensionless governing equation. We investigate the analytical solution of this problem in section 3. In section 4 we derive the numerical scheme used and the implementation of the boundary conditions. The following section 5 gives numerical results and lastly the conclusions are summarised in section 6.

Solid Conductor Problem

Consider a channel $-a \leq z \leq a$ of infinite length and width containing a solid metal of magnetic permeability μ, electrical conductivity σ, and magnetic diffusivity η, where $\eta = 1/(\sigma\mu)$, being subjected to an applied two-dimensional periodic magnetostatic field $\boldsymbol{B}$ which travels parallel to the channel with a uniform velocity $-U\hat{x}$. The magnetic boundary conditions are

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=\pm a} = B_0 \sum_{n=1}^{N} b_n \cos[nk(x+Ut)]. \quad (1)$$

We make the assumption that the edge effects of the side walls $y = \pm a$, where $b > a$, may be neglected. Now this problem is equivalent to the situation where the applied magnetic field is stationary but at $t = 0$, the solid metal in the channel is suddenly moved at the velocity $U\hat{x}$.

Together with the irrotational and divergence conditions, viz.

$$\nabla' \times \boldsymbol{B}' = 0, \quad \nabla' . \boldsymbol{B}' = 0 \quad (2)$$

the governing induction equation is

$$\frac{\partial \boldsymbol{B}'}{\partial t'} = \nabla' \times (\boldsymbol{v}' \times \boldsymbol{B}') + \eta \nabla'^2 \boldsymbol{B}.$$

By putting $\boldsymbol{B}' = -\nabla'\psi' \times \hat{\boldsymbol{y}} = -\nabla' \times (\psi'\hat{\boldsymbol{y}})$ in the induction equation, where ψ' is the magnetic stream function, we obtain

$$\frac{\partial\psi'}{\partial t'}\,\hat{\boldsymbol{y}} + \boldsymbol{v}' \times \boldsymbol{B}' - \eta\nabla' \times \boldsymbol{B}' = \nabla'\varphi' \quad (3)$$

where φ is some scalar field and η is constant in space and time (which is likely for liquid metal). Taking the $y-$component of (3) we get a simpler governing equation

$$\frac{\partial\psi'}{\partial t'} + \boldsymbol{v}' \cdot \nabla'\psi' = \eta\nabla'^2\psi', \quad (4)$$

where $\partial\varphi'/\partial y' = 0$, because the y – boundaries are assumed to be perfectly conducting or short circuited, so that the y – component of $\acute{A}' = 0$.

Non-dimensionalisation

To non-dimensionalize (4) we write $x' = Lx$, $z' = Lz$, $\psi' = LB_0\,\psi$ where B_0 and L are the typical strength of the magnetic field and length respectively, (4) then becomes

$$\frac{\partial\psi}{\partial t} + \boldsymbol{v}\cdot\nabla\psi = \frac{1}{R_m}\nabla^2\psi \qquad \text{or} \qquad (5)$$

$$\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial x} = \frac{1}{R_m}\nabla^2\psi,$$

where $\boldsymbol{v} = \hat{x}$.

To derive an alternative formulation in terms of Poisson brackets we let $\boldsymbol{v} = -\nabla\times(\chi\hat{\boldsymbol{y}}) = \left(\frac{\partial\chi}{\partial z}, 0, -\frac{\partial\chi}{\partial x}\right)$,

where χ is the magnetic stream function, the term $\boldsymbol{v}\cdot\nabla\psi$ becomes

$$\boldsymbol{v}\cdot\nabla\psi = \frac{\partial\chi}{\partial z}\cdot\frac{\partial\psi}{\partial x} - \frac{\partial\chi}{\partial x}\cdot\frac{\partial\psi}{\partial z} = [\psi, \chi], \quad (6)$$

where the Poisson bracket notation is defined as $[\psi,\chi] = \psi_x\chi_z - \psi_z\chi_x$. Hence (5) can also be written as

$$\frac{\partial\psi}{\partial t} + [\psi,\chi] = \frac{1}{R_m}\nabla^2\psi. \qquad (7)$$

This is the form of equation used in the later numerical calculations.

Analytical Solution

We consider the simple case of one Fourier component (because it has an analytical solution), where a solid conductor is subjected to an applied magnetic field

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=\pm a} = B_0\cos k(x+Ut) = B_0\cos kX, \qquad (8)$$

where $X = x + Ut$, in a frame of reference stationary relative to the applied magnetic field. Now we need to obtain the magnetic stream function ψ as a result of the applied magnetic field (8). By writing $\boldsymbol{B} = \nabla\phi$ where ϕ is the magnetic potential function, the divergence condition (2) gives the Laplace's equation $\nabla^2\phi = 0$, whose solution can be shown to be

$$\phi(x,z,t) = \frac{B_0}{kc_1}\sinh(kz)\cos kX, \qquad (9)$$

where $c_1 = \cosh(ka)$. Since the magnetic field $\boldsymbol{B}$ has zero divergence it can be expressed as $\boldsymbol{B} = -\nabla\psi\times\hat{\boldsymbol{y}} = (\partial\psi/\partial z, 0, -\partial\psi/\partial X)$. Using (9) in $\boldsymbol{B} = \nabla\phi$ we have components of $\boldsymbol{B}$ as

$$\frac{\partial\psi}{\partial z} = -\frac{B_0}{c_1}\sinh(kz)\sin(kX)$$
$$-\frac{\partial\psi}{\partial X} = \frac{B_0}{c_1}\cosh(kz)\cos(kX),$$

whose solution is

$$\psi(X,z) = -\frac{B_0}{kc_1}\cosh(kz)\sin(kX). \qquad (10)$$

It is important to note here that we choose our frame of reference to be stationary with respect to the applied magnetic field, which is moving at a constant velocity $U\hat{x}$. From now onwards we use x to mean X just for convenience.

At steady state $\partial\boldsymbol{B}/\partial t = 0$, i.e. $\partial\psi/\partial t = 0$. So from (5) and (10) we obtain

$$\nabla^2\psi = R_m\frac{\partial\psi}{\partial x}, \quad \psi(\pm a) = -\frac{B_0}{k}\sin kx, \quad (11)$$

respectively.

In order to solve (11) we try a solution of the form
$$\psi = \mathrm{Im}[f(z)\,e^{ikx}], \qquad (12)$$

resulting in $f''(z) - \omega^2 f(z) = 0$, where $\omega = \pm\sqrt{k^2 + kR_m\mathrm{i}}$.

Writing $\sqrt{k^2 + kR_m\mathrm{i}} = \alpha + \beta$, where α, β are real, leads to

$$\alpha^2 - \beta^2 = k^2 \quad\text{and}\quad 2\alpha\beta = kR_m. \qquad (13)$$

The two solutions to these simultaneous equations (13) can be shown to be

$$\alpha = \sqrt{\frac{k}{2\Omega}}R_m, \beta = \sqrt{\frac{k}{2}\Omega}$$
$$\alpha = -\sqrt{\frac{k}{2\Omega}}R_m, \beta = -\sqrt{\frac{k}{2}\Omega},$$

Where $\Omega = \sqrt{R_m^2 + k^2} - k$. Now we can write

$$f(z) = Ae^{\omega z} + Be^{-\omega z}$$
where $\omega = \alpha + \beta\mathrm{i}$. $\qquad$ (14)

From the boundary conditions (11) we get $f(\pm a)$ =

$-B_0/k$, which when applied to (14) gives the values of A and B. So the solution of (11) after some algebra can be shown to be

$$\psi = -\frac{B_0}{k}.\frac{R(z)\sin kx + S(z)\cos kx}{P}, \quad (15)$$

where

$$R(z) = \cosh(\alpha z)\cosh(\alpha a)\cos(\beta z)\cos(\beta a) + \sinh(\alpha z)\sinh(\alpha a)\sin(\beta z)\sin(\beta a),$$
$$S(z) = \sinh(\alpha z)\cosh(\alpha a)\sin(\beta z)\cos(\beta a) - \cosh(\alpha z)\sinh(\alpha a)\cos(\beta z)\sin(\beta a),$$

and

$$P = \cosh^2(\alpha a)\cos^2(\beta a) + \sinh^2(\alpha a)\sin^2(\beta a).$$

Fieldlines are shown in Figure 1. It can be seen that as the magnetic Reynolds number increases from 1 to 120, more lines of force are swept in the direction of motion of the solid conductor. This is to be expected since as R_m increases, the field lines tend to become more "frozen-in" to the conductor.

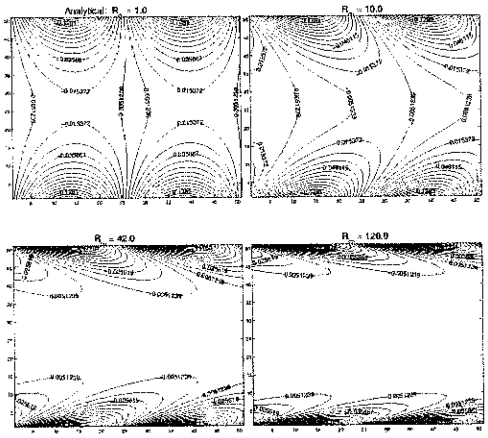

Figure 1: Contours of the analytical streamfunction for magnetic Reynolds number, R_m, which takes the values from 1 to 120. The number of contour lines is 30. The resolution is 50 by 50. The region is [0,1] by [-0.5,0.5].

Numerical scheme

The problem is to solve the governing equation

$$\frac{\partial\psi}{\partial t} + [\psi,\chi] = \frac{1}{R_m}\nabla^2\psi, \quad (16)$$
$$\chi(x,-a) = 0,$$
$$\chi(x,\ a) = 2Ua.$$

numerically on the domain [0,1] × [−a, a].
The numerical method used was the Russian scheme (Heerikhuisen [(2001) Appendix A]). Briefly the scheme works like this. Based on the induction equation (16) we calculate the predictor step, ψ^{n+1*}(17) at time step n + 1 *. Then we use the predicted values of ψ^{n+1*} to calculate the corrector value ψ^{n+1} as in (18) at time step n + 1, while using the same values of χ^n_{ij} and $\nabla^2\psi^n_{ij}$ as in (17).
We first calculate the predictor step,

$$\psi^{n+1*}_{ij} = \psi^n_{ij} - (\psi^n_x\chi^n_z - \psi^n_z\chi^n_x)\Delta t + \frac{1}{R_m}\nabla^2\psi^n_{ij}\Delta t, \quad (17)$$

where

$$\psi^n_x = \frac{\psi^n_{i+1,j} - \psi^n_{i-1,j}}{2\Delta}, \quad \psi^n_z = \frac{\psi^n_{i,j+1} - \psi^n_{i,j-1}}{2\Delta}$$
$$\chi^n_x = \frac{\chi^n_{i+1,j} - \chi^n_{i-1,j}}{2\Delta}, \quad \chi^n_z = \frac{\chi^n_{i,j+1} - \chi^n_{i,j-1}}{2\Delta}$$

and

$$\nabla^2\psi^n_{ij} = \frac{\psi^n_{i+1,j} - 2\psi^n_{ij} + \psi^n_{i-1,j}}{2h^2} + \frac{\psi^n_{i,j+1} - 2\psi^n_{ij} + \psi^n_{i,j-1}}{2h^2}.$$

The predictor values, ψ^{n+1*} are then used to calculate the corrector step ψ^{n+1}, for the next time step $t + \Delta t$,

$$\psi^{n+1}_{ij} = \psi^n_{ij} - (\psi^{n+1*}_x\chi^n_z - \psi^{n+1*}_z\chi^n_x)\Delta t + \frac{1}{R_m}\nabla^2\psi^n_{ij}\Delta t, \quad (18)$$

where

$$\psi^{n+1*}_x = \frac{\psi^{n+1*}_{i+1,j} - \psi^{n+1*}_{i-1,j}}{2\Delta},$$
$$\psi^{n+1*}_z = \frac{\psi^{n+1*}_{i,j+1} - \psi^{n+1*}_{i,j-1}}{2\Delta},$$

and $\nabla^2\psi^n_{ij}$ remain unchanged. To ensure stability the time step, Δt, for 2-D must satisfy the Courant-Friedrichs-Lewy condition

$$\Delta t \le \frac{\Delta}{\sqrt{2}\,|\boldsymbol{v}|_{\max}}, \quad (19)$$

where $|\boldsymbol{v}|_{\max}$ is the modulus of the maximum velocity of the fluid taken over all grid points. Since

$$|\boldsymbol{v}|_{\max} \le U = 1$$

we find,

$$\frac{1}{|\boldsymbol{v}|_{\max}} \ge 1.$$

Hence,

$$\Delta t \le \frac{\Delta}{\sqrt{2}}. \quad (20)$$

In addition, the time step must also satisfy the von Neumann criterion for stability (Richtmyer & Morton 1967) arising from the viscous term, which leads to

$$\Delta t \le \frac{\Delta^2}{\nu} = R_m\,\Delta^2. \quad (21)$$

Boundary Values

The domain is $[0,1] \times [-a,a]$. The calculations are carried out over all the points of the mesh including the boundaries. The point x_i, z_j corresponds to the point with co-ordinates

$$x_i = (i-1) * \Delta, \qquad i = 0, \ldots, m+1$$
$$z_j = -a + (j-1) * \Delta, \quad j = 1, \ldots, n.$$

In the finite difference expression for $\nabla^2 \psi$

$$\nabla^2 \psi = \frac{\psi_{i+1,j} - 2\psi_{ij} + \psi_{i-1,j}}{2\,h^2} + \frac{\psi_{i,j+1} - 2\psi_{ij} + \psi_{i,j-1}}{2\,h^2},$$

the calculations of the values of ψ on the boundaries i.e. $\psi(i,0)$ and $\psi(i,n+1)$, require the values of ψ on the ghostpoints. For example, in the calculation for $j = 1$ (boundary $z = -a$) we need to the values of $\psi(i,2)$ and the ghostpoint $\psi(i,0)$. Similarly for $j = n$ we need the values $\psi(i,n-1)$ and the ghost-point $\psi(i,n+1)$. We solve this by the "mirror image" technique where we set

$$\psi(i,0) = \psi(i,2),$$
$$\psi(i,n+1) = \psi(i,n-1), \quad i = 0, \ldots, m+1.$$

The justification is the x-component of the applied magnetic field on the boundaries $z = \pm a$ is zero i.e.

$$\frac{\partial \psi}{\partial z} \sim \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\,h} = 0$$

which implies that ψ does not change between the points on either side of the boundaries namely $(i,\ 0)$, $(i,2)$ and $(i,n-1)$, $(i,n+1)$.

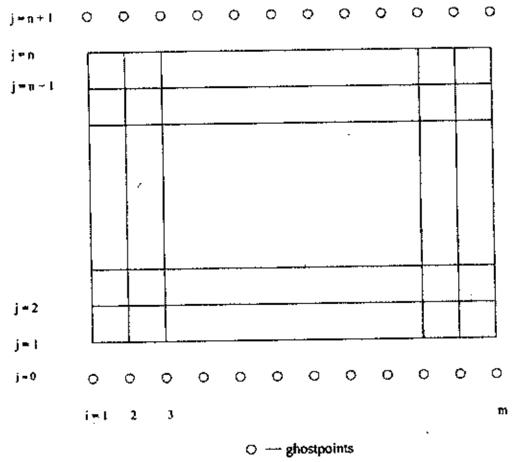

Figure 2: The mesh of the domain $[0,1] \times [-a,a]$ showing the ghost points next to the boundaries $z = \pm a$.

Numerical Results

The simulations were continued until a steady state was reached for a given value of magnetic Reynolds number in the range 1 to 120. The number of partitions is 50 by 50. When we compare the analytical streamlines (the first column in Figure 3) for the steady state with the numerically calculated streamlines (in the second column) for the same magnetic Reynolds number they look remarkably similar (only a few are reproduced here). As the magnetic Reynolds number increases the solid metal behaves more and more like a perfect conductor. The fieldlines in the shape of half-a-loop on the boundaries are swept in the direction of the motion and become flatter as R_m increases as shown in Figure 3. For values of R_m of about 800, the fieldlines are swept into a narrow layer on the boundaries. The collapsing boundary layers look stable as R_m increases. To see whether they are accurate for the given grid spacing we increase the resolution of the magnetic fieldlines just for the case of $R_m = 800$. It is shown that the narrow layer of fieldlines on the boundaries has a thickness of approximately within 8% of a, the channel width. When the resolutions are increased from 50 by 50 to 150 by 150, the thin boundary layers of magnetic field remain within 4 grid points for 50 by 50; within 8 grid points for 100 by 100; and within 12 grid points for 150 by 150.

To study the evolution of fieldlines with time, at high R_m, two simulations for $R_m = 42$ and $R_m = 800$ were carried out. For $R_m = 800$ it is clear from Figure 4 that both pairs of vortices travel the whole wavelength (of 1 unit length) in 1 time unit. During the travel "frozen-in" effect is at work with reconnections of fieldlines taking place on the boundaries. The initial fieldline shape remains unchanged.

Some half loops are severed from the boundaries but connect to form vortices where two new pairs of loops are formed. After $t = 0.7$ twelve new pairs of loops are formed but as time goes on loops of vortices gradually disappear through reconnection with the flux on the boundaries and through ohmic losses. By $t = 3$, only one pair of loops is left. In the end what remains is a thin layer of flux on the boundaries. The severing of fieldlines mostly involves those that join the two boundaries. Within just a period of $t = 2.0$, only closed loop fieldlines originating from the boundaries remain and they are inclined in the direction of motion of the solid metal. A similar flux expulsion effect was found by Kamkar and Moffatt (1982). They studied a pressure-driven flow along a channel in the presence of an applied periodic magnetic field.

Conclusion

The numerical and analytical results agree well for the case of one Fourier-component source, for R_m in the range of 1 to 1200. We found that as R_m increases the solid metal behaves more and more like a perfect conductor where the fieldlines are frozen in the moving solid metal. The fieldlines are swept in the direction of the motion. It would be interesting to extend the present problem to the case where we have a conducting fluid instead a solid metal.

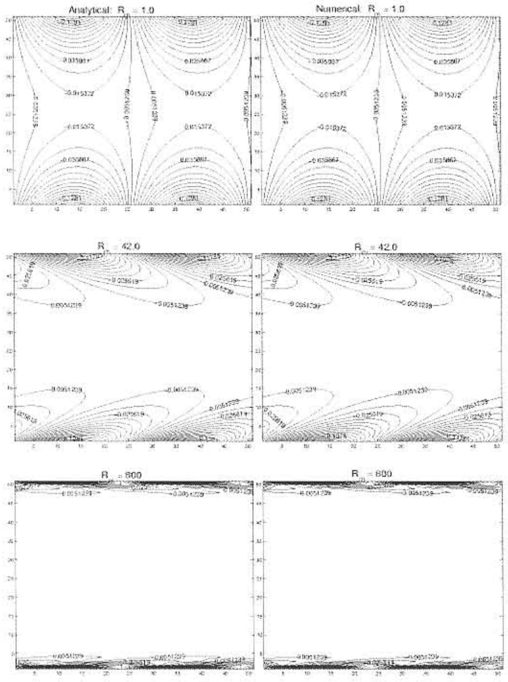

Figure 3: The left column shows the analytical field-lines and the second column shows steady state numerical fieldlines for increasing values of magnetic Reynolds number (viz 1-800). The number of stream-function contour lines is 30. The region is [0,1] × [−0.5, 0.5]. The number of partitions is 50 by 50.

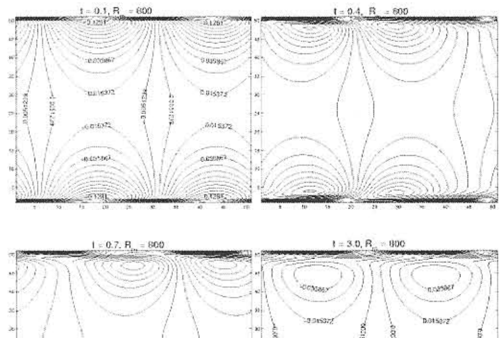

Figure 4: Evolution of fieldlines for t = 0.1 to 18.0 showing "frozen-in" effect of magnetic field at high R_m = 800.

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