# Kochen-Specker Theorem for a Single Qubit

[1]Toh Sing Poh & [1,2]Hishamuddin Zainuddin

[1]Laboratory of Computational Sciences and Informatics
Institute for Mathematical Research
Universiti Putra Malaysia,

[2]Department of Physics
Faculty of Science
Universiti Putra Malaysia

[1] singpoh@gmail.com

**Abstract**

The Kochen-Specker theorem (KS) states that noncontextual hidden variable theories are incompatible with quantum mechanics. Since the first proof of the theorem, von Neumann projection measurement has been used and the quantum system considered has dimensionality of at least three. However, recently generalized measurements represented by positive-operator-valued measures (POVM) have been applied to extend the proof to two dimensional quantum systems. This note gives numerical calculations on the first KS proof for a single qubit.

## Introduction

One of the main theorems on the impossibility of hidden variables in quantum mechanics is Kochen-Specker theorem (KS). This theorem says that any hidden variable theory that satisfies quantum mechanics must be contextual. More specifically, it asserts that, in Hilbert space of dimension ≥ 3, it is impossible to associate definite numerical values, 1 or 0, with every projection operator $P_m$, in such a way that, if a set of commuting $P_m$ satisfies $\sum P_m = 1$, the corresponding values $v(P_m)$ will also satisfy $\sum v(P_m) = 1$. Since the first proof of Kochen and Specker [1] using 117 vectors in $R^3$, there were many attempts to reduce the number of vector either via conceiving ingenious models or extending the system being considered to higher dimension. However, further advancements recently show that the proof can be extended to two dimensional quantum system through generalized measurement represented by positive operator-valued measured (POVM). In POVMs the number of available outcomes of a measurement may be higher than the dimensionality of the Hilbert space and $N$-outcome generalized measurement is represented by $N$-element POVM which consists of $N$ positive semidefinite operators $\{E_d\}$ that sum to identity. Each pair of elements are not mutually orthogonal. Here we show numerical calculations demonstrating idea of the first proof given by Cabello [2] that make use of five inscribed cubes in dodecahedron. For the single qubit system considered, restricted element of POVM is given by [3]

$$E = \frac{1}{N}\left(I + \vec{n} \bullet \vec{\sigma}\right) \qquad (1)$$

where $\vec{n}$ can be any unit vector and $\vec{\sigma}$ are Pauli matrices.

## POVM constructed from a single cube

Consider a cube centered at origin of a Cartesian coordinate system such that all the vectors joining origin to each vertex have unit length. Thus, the 8 unit vectors are $\vec{n}_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$, $\vec{n}_2 = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$, $\vec{n}_3 = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}})$, $\vec{n}_4 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}})$, $\vec{n}_{1'} = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}})$, $\vec{n}_{2'} = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}})$, $\vec{n}_{3'} = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}})$ and $\vec{n}_{4'} = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}})$. Vector sum of these vectors gives rise to zero. As every unit vector can be generally expressed in terms of polar and azimuthal angles as $(\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta)$, we can find $\vartheta$ and $\varphi$ for each vector above. By substitution of each pair of the angles into Eq. 1 with the explicit form as

$$E = \frac{1}{N}\begin{bmatrix} 1+\cos\vartheta & e^{-i\phi}\sin\vartheta \\ e^{i\phi}\sin\vartheta & 1-\cos\vartheta \end{bmatrix} \qquad (2)$$

where $N$ = 8 in this case, we can produce the following (Table 1) eight positive operators:

**Table 1 :** Eight Element POVM Constructed From a Cube

| | |
|---|---|
| $E_1 = \frac{1}{8}\begin{bmatrix} 1.7071 & 0.5-0.5i \\ 0.5+0.5i & 0.2929 \end{bmatrix}$ | $E_{1'} = \frac{1}{8}\begin{bmatrix} 0.2929 & 0.5-0.5i \\ 0.5+0.5i & 1.7071 \end{bmatrix}$ |
| $E_2 = \frac{1}{8}\begin{bmatrix} 1.7071 & 0.5-0.5i \\ -0.5+0.5i & 0.2929 \end{bmatrix}$ | $E_{2'} = \frac{1}{8}\begin{bmatrix} 0.2929 & -0.5-0.5i \\ -0.5+0.5i & 1.7071 \end{bmatrix}$ |
| $E_3 = \frac{1}{8}\begin{bmatrix} 1.7071 & -0.5+0.5i \\ -0.5-0.5i & 0.2929 \end{bmatrix}$ | $E_{3'} = \frac{1}{8}\begin{bmatrix} 0.2929 & -0.5+0.5i \\ -0.5-0.5i & 1.7071 \end{bmatrix}$ |
| $E_4 = \frac{1}{8}\begin{bmatrix} 1.7071 & 0.5+0.5i \\ 0.5-0.5i & 0.2929 \end{bmatrix}$ | $E_{4'} = \frac{1}{8}\begin{bmatrix} 0.2929 & 0.5+0.5i \\ 0.5-0.5i & 1.7071 \end{bmatrix}$ |

**Table 2 :** Vertices of Dodecahedron and Corresponding Polar and Azimuthal Angle

| Vertex | $x$ | $y$ | $z$ | $\vartheta$ | $\varphi$ |
|---|---|---|---|---|---|
| 1 | 0.0000 | 0.35682 | 0.93417 | 20.906 | 90 |
| 2 | 0.0000 | 0.35682 | -0.93417 | 159.094 | 90 |
| 3 | 0.0000 | -0.35682 | 0.93417 | 20.906 | 270 |
| 4 | 0.0000 | -0.35682 | -0.93417 | 159.094 | 270 |
| 5 | 0.93417 | 0.0000 | 0.35682 | 69.095 | 0 |
| 6 | 0.93417 | 0.0000 | -0.35682 | 110.905 | 0 |
| 7 | -0.93417 | 0.0000 | 0.35682 | 69.095 | 180 |
| 8 | -0.93417 | 0.0000 | -0.35682 | 110.905 | 180 |
| 9 | 0.35682 | 0.93417 | 0.0000 | 90 | 69.095 |
| 10 | 0.35682 | -0.93417 | 0.0000 | 90 | 290.905 |
| 11 | -0.35682 | 0.93417 | 0.0000 | 90 | 110.905 |
| 12 | -0.35682 | -0.93417 | 0.0000 | 90 | 249.095 |
| 13 | 0.57735 | 0.57735 | 0.57735 | 54.736 | 45 |
| 14 | 0.57735 | 0.57735 | -0.57735 | 125.264 | 45 |
| 15 | 0.57735 | -0.57735 | 0.57735 | 54.736 | 315 |
| 16 | 0.57735 | -0.57735 | -0.57735 | 125.264 | 315 |
| 17 | -0.57735 | 0.57735 | 0.57735 | 54.736 | 135 |
| 18 | -0.57735 | 0.57735 | -0.57735 | 125.264 | 135 |
| 19 | -0.57735 | -0.57735 | 0.57735 | 54.736 | 225 |
| 20 | -0.57735 | -0.57735 | -0.57735 | 125.264 | 225 |

Note that $\sum_i E_i = I$ for $i, i' = 1-4$, thus they form a POVM.

**POVMs constructed from five cubes inscribed in dodecahedron**

Dodecaheron is one of the platonic solids in 3 dimensions. It has 20 vertices and 20 pentagonal faces. As there are 5 inscribed cubes in dodecahedron, we could construct 5 POVMs via procedure outlined above. To this end, we need to know all the vertices of the 5 inscribed cubes. Consider a dodecahedron centered at the origin $(0,0,0)$. The coordinates of its 20 vertices [4] and corresponding $\vartheta$ and $\varphi$ are listed in Table 2.

All the above 20 vectors have been normalized. Each pentagonal face has side of length 0.71364 and each point in a pentagon forms two lines with length 1.15469 with two opposite points in the same pentagon (see Figure 1). Since each vertex of the dodecahedron share 3 pentagons (see Figure 2), there are 60 pairs of vertices that form a line of length 1.15469. These are the lines that constitute edges of the inscribed cubes which are shown in Figure 2.

Apart from checking the side lengths, we need to find groups of the above lines, with 3 elements each that are mutually orthogonal. After some tedious calculations, we found vertices that build up 5 cubes are as labeled in Figure 3.

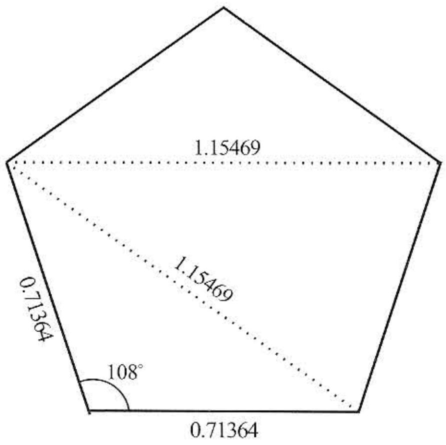


**Figure 1 :** Pentagonal Face of the Dodecahedron

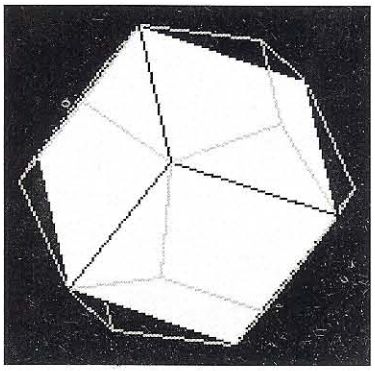

**Figure 2 :** Inscribed Cube Formed from Vertices of the Dodecahedron

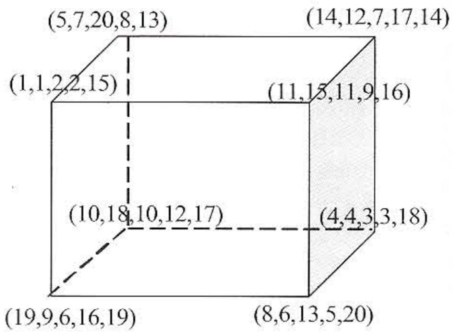


**Figure 3 :** Vertoces Labeled for the Five Inscribed Cubes respectively.

**Table 3 :** Positive Semidefinite Operator Corresponding to Vertices of Dodecahedron

| | |
|---|---|
| $E_1 = \begin{bmatrix} 0.2418 & -0.0446i \\ 0.0446i & 0.0082 \end{bmatrix}$ | $E_{11} = \begin{bmatrix} 0.1250 & -0.0446-0.1168i \\ -0.0446+0.1168i & 0.1250 \end{bmatrix}$ |
| $E_2 = \begin{bmatrix} 0.0082 & -0.0446i \\ 0.0446i & 0.2418 \end{bmatrix}$ | $E_{12} = \begin{bmatrix} 0.1250 & -0.0446+0.1168i \\ -0.0446-0.1168i & 0.1250 \end{bmatrix}$ |
| $E_3 = \begin{bmatrix} 0.2418 & 0.0446i \\ -0.0446i & 0.0082 \end{bmatrix}$ | $E_{13} = \begin{bmatrix} 0.1972 & 0.0722-0.0722i \\ 0.0722+0.0722i & 0.0528 \end{bmatrix}$ |
| $E_4 = \begin{bmatrix} 0.0082 & 0.0446i \\ -0.0446i & 0.2418 \end{bmatrix}$ | $E_{14} = \begin{bmatrix} 0.0528 & 0.0722-0.0722i \\ 0..0722+0.0722i & 0.1972 \end{bmatrix}$ |
| $E_5 = \begin{bmatrix} 0.1696 & 0.1168 \\ 0.1168 & 0.0804 \end{bmatrix}$ | $E_{15} = \begin{bmatrix} 0.1972 & 0.0722+0.0722i \\ 0.0722-0.0722i & 0.0528 \end{bmatrix}$ |
| $E_6 = \begin{bmatrix} 0.0804 & 0.1168 \\ 0.1168 & 0.1696 \end{bmatrix}$ | $E_{16} = \begin{bmatrix} 0.0528 & 0.0722+0.0722i \\ 0.0722-0.0722i & 0.1972 \end{bmatrix}$ |
| $E_7 = \begin{bmatrix} 0.1696 & -0.1168 \\ 0.1168 & 0.0804 \end{bmatrix}$ | $E_{17} = \begin{bmatrix} 0.1972 & -0.0722-0.0722i \\ -0.0722+0.0722i & 0.0528 \end{bmatrix}$ |
| $E_8 = \begin{bmatrix} 0.0804 & -0.1168 \\ -0.1168 & 0.1696 \end{bmatrix}$ | $E_{18} = \begin{bmatrix} 0.0528 & -0.0722-0.0722i \\ -0.0722+0.0722i & 0.1972 \end{bmatrix}$ |
| $E_9 = \begin{bmatrix} 0.1250 & 0.0446-01168i \\ 0.0446+0.1168i & 0.1250 \end{bmatrix}$ | $E_{19} = \begin{bmatrix} 0.1972 & -0.0722+0.0722i \\ -0.0722-0.0722i & 0.0528 \end{bmatrix}$ |
| $E_{10} = \begin{bmatrix} 0.1250 & 0.0446+0.1168i \\ 0.0446-0.1168i & 0.1250 \end{bmatrix}$ | $E_{20} = \begin{bmatrix} 0.0528 & -0.0722+0.0722i \\ -0.0722-0.0722i & 0.1972 \end{bmatrix}$ |

Once having all the vertices, it is easy to follow exactly the procedure outlined in the previous section to get 20 positive semi-definite operators as listed in Table 3.

Since there are 5 inscribed cubes, there are 5 corresponding POVMs. Pick any one of the cubes and about it's different diagonal once, there would be the other 4 cubes. This means that for any two of the 5 POVMs constructed, they share two elements and this becomes the crucial property used to prove Kochen Specker theorem as in Eq. 3.

$$\begin{aligned} &E_1+E_4+E_5+E_8+E_{10}+E_{11}+E_{14}+E_{19}=I \\ &E_1+E_4+E_6+E_7+E_9+E_{12}+E_{15}+E_{18}=I \\ &E_2+E_3+E_5+E_8+E_9+E_{12}+E_{16}+E_{17}=I \\ &E_2+E_3+E_6+E_7+E_{10}+E_{11}+E_{13}+E_{20}=I \\ &E_{13}+E_{20}+E_{15}+E_{18}+E_{16}+E_{17}+E_{14}+E_{19}=I \end{aligned} \quad (3)$$

As measurement gives every positive semi definite operator a '1' (yes) or '0' (no) result and since each of them appears twice at the left of the above set of equations in Eq. 3, so the total number of result 1 should be even. However, according to the measurement operator at the right of Eq. 3, there can only be five '1' results, which is odd. This means that the function of $\Sigma E_m = 1$ and $\Sigma v(E_m) = 1$ can't be satisfied simultaneously. Hence this gives the KS obstruction to noncontextual assignments.

**Acknowledgments**

We would like to thank Mr. Foo Kim Eng for his help in using Matlab to find vertices of five inscribed cubes. This work is jointly supported by Malaysian SAGA grant P55c, Akademi Sains Malaysia, Ministry of Science, Technology & Innovations and Fundamental grant 01-01-07-170FR, Ministry of Higher Education, Malaysia.

**References**

[1] Kochen, S. and Specker, E. 1967. The Problem of Hidden Variables in Quantum Mechanics. *J. Math. Mech.*, **17,** 59-87.

[2] Adan, C. 2003. Kochen-Specker Theorem for a Single Qubit Using Positive Operator-Valued Measures. *Phys. Rev. Lett.* **90**, 190401.

[3] Caves, C. M., Fuchs, C. A., Manne, K. K. and Renes, J. M. Gleason-Type Derivations of The Quantum Probability Rule for Generalized Measurements. *Foundation of Physics,* **34**, 193-209.

[4] http://www.mcs.surrey.ac.uk/Personal/R.Knott /Fibonacci/phi3DGeom.html#dodec