



UNIVERSITI PUTRA MALAYSIA

**RUNGE-KUTTA-NYSTROM METHODS FOR SOLVING
OSCILLATORY PROBLEMS**

**NORAZAK BIN SENU
FS 2010 23**

**RUNGE-KUTTA-NYSTROM METHODS FOR SOLVING
OSCILLATORY PROBLEMS**

NORAZAK BIN SENU

**DOCTOR OF PHILOSOPHY
UNIVERSITI PUTRA MALAYSIA**

2010



**RUNGE-KUTTA-NYSTRÖM METHODS FOR SOLVING
OSCILLATORY PROBLEMS**

By

NORAZAK BIN SENU

**Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia,
in Fulfilment of the Requirements for the Degree of Doctor of Philosophy**

February 2010



Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

**RUNGE-KUTTA-NYSTROM METHODS FOR
SOLVING OSCILLATORY PROBLEMS**

By

NORAZAK BIN SENU

February 2010

Chairman : Professor Dato' Mohamed bin Suleiman, PhD

Faculty : Science

New Runge-Kutta-Nyström (RKN) methods are derived for solving system of second-order Ordinary Differential Equations (ODEs) in which the solutions are in the oscillatory form. The dispersion and dissipation relations are imposed to get methods with the highest possible order of dispersion and dissipation. The derivation of Embedded Explicit RKN (ERKN) methods for variable step size codes are also given. The strategies in choosing the free parameters are also discussed. We analyze the numerical behavior of the RKN and ERKN methods both theoretically and experimentally and comparisons are made over the existing methods.

In the second part of this thesis, a Block Embedded Explicit RKN (BERKN) method are developed. The implementation of BERKN method is discussed. The numerical results are compared with non block method. We find that the new code on Block Embedded Explicit RKN (BERKN) method is more efficient for solving system of second-order ODEs directly.

Next, we discussed the derivation of Diagonally Implicit RKN (DIRKN) methods for solving stiff second order ODEs in which the solutions are oscillating functions. The dispersion and



dissipation relations are developed and again are imposed in the derivation of the methods. For solving oscillatory problems with high frequency, method with P-stability property is discussed. We also derive the Embedded Diagonally Implicit RKN (EDIRKN) methods for variable step size codes. To see the preciseness and effectiveness of the methods, the constant and variable step size codes are developed and numerical results are compared with current methods given in the literature.

Finally, the Parallel Embedded Explicit RKN (PERKN) method is developed. The parallel implementation of PERKN on the parallel machine is discussed. The performance of the PERKN algorithm for solving large system of ODEs are presented. We observe that the PERKN gives the better performance when solving large system of ODEs.

In conclusion, the new codes developed in this thesis are suitable for solving system of second-order ODEs in which the solutions are in the oscillatory form.



Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

**KAEDAH RUNGE-KUTTA-NYSTROM BAGI
MENYELESAIKAN MASALAH BERAYUNAN**

Oleh

NORAZAK BIN SENU

Februari 2010

Pengerusi : Professor Dato' Mohamed bin Suleiman, PhD

Fakulti : Sains

Kaedah baharu Runge-Kutta-Nyström (RKN) diterbitkan bagi menyelesaikan Persamaan Pembezaan Biasa (PPB) peringkat dua yang mana penyelesaiannya adalah dalam bentuk berayunan. Hubungan serakan dan lesapan dikenakan bagi mendapatkan kaedah dengan peringkat serakan dan lesapan setinggi yang mungkin. Penerbitan kaedah Benaman Tak Tersirat RKN (BTRKN) untuk kod panjang langkah berubah turut diberikan. Strategi pemilihan parameter bebas juga dibincangkan. Kami menganalisa kelakuan berangka bagi kaedah RKN dan BTRKN secara teori dan eksperimen serta perbandingan dibuat terhadap kaedah sedia ada.

Di dalam bahagian kedua tesis, kaedah Blok Benaman Tak Tersirat RKN (BBRKN) dibincangkan. Implimentasi ke atas kaedah BBRKN turut dibincangkan. Keputusan berangka dibandingkan dengan kaedah bukan blok. Kami perolehi bahawa kod baharu Blok Benaman Tak Tersirat RKN (BBRKN) adalah lebih efisien bagi menyelesaikan sistem PPB peringkat dua.



Seterusnya, kami membincangkan penerbitan kaedah Pepenjuru Tersirat (PTRKN) bagi menyelesaikan PPB kaku peringkat dua yang penyelesaiannya berbentuk berkala. Hubungan serakan dan lesapan dibangunkan dan sekali lagi diaplikasikan dalam penerbitan kaedah. Untuk menyelesaikan masalah berkala dengan frekuensi tinggi, kaedah dengan sifat P-kestabilan dibincangkan. Kami juga menerbitkan kaedah Benaman Pepenjuru Tersirat RKN (BPTRKN) bagi kod panjang langkah berubah. Untuk melihat kejituan dan keefisienan kaedah, kod panjang langkah tetap dan berubah dibangunkan serta keputusan berangka dibandingkan terhadap kaedah sedia ada.

Akhir sekali, kaedah Selari Benaman Tak Tersirat RKN (SBTRKN) dibangunkan. Implimentasi SBTRKN ke atas mesin selari dibincangkan. Prestasi algoritma SBTRKN bagi menyelesaikan sistem PPB berdimensi besar diberikan. Kami perolehi SBTRKN memberikan prestasi yang baik bila dilaksanakan terhadap sistem PPB berdimensi besar.

Kesimpulannya, kod baharu yang dibangunkan di dalam tesis ini sesuai untuk sistem PPB peringkat dua yang mana penyelesaian adalah dalam bentuk berayunan.

ACKNOWLEDGEMENTS

**In the Name of Allah the Most Compassionate,
the Most Merciful First and foremost**

First all, praise is for *Allah Subhanahu Wa Taala* for giving me the strength, guidance and patience to complete this thesis. May blessing and peace be upon Prophet Muhammad *Sallallahu Alaihi Wasallam*, who was sent for mercy to the world.

I wish to express my sincere and deepest gratitude to the chairman of the supervisory committee, YBhg. Professor Dato' Dr. Mohamed bin Suleiman for his invaluable advice, guidance, assistance and most of all, for his constructive criticisms. This work would not have been completed without his help that I received in various aspects of the research.

I am also grateful to the member of the supervisory committee, Associate Professor Dr. Fudziah bt Ismail and Professor Dr. Mohamed bin Othman. I also wish to express my thanks to all of my friends during my study in Universiti Putra Malaysia. I would like to thank all staffs of the Department of Mathematics. Their continuous help, encouragement and support are highly appreciated. I thank my employer, Universiti Putra Malaysia for providing me with the UPM scholarship which funded this research during most of my studies and also who granted me study leave.

Finally, I cannot put into words how much I appreciate the continuous support, understanding and patience of my wife, Norfifah, and my children, Nor Fatin Aqilah, Muhammad Farhan Aqil and Muhammad Fath Hadif and special thanks to my mother Hjh. Jamenah bt. Sirat and my father Hj. Senu bin Sabikan for their continuous encouragement. Thank you.



I certify that a Thesis Examination Committee has met on 22 February 2010 to conduct the final examination of Norazak bin Senu on his thesis entitled “Runge-Kutta-Nyström Methods for Solving Oscillatory Problems” in accordance with Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

Members of the Thesis Examination Committee were as follows:

Norihan Md. Arifin, PhD

Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Malik Hj Abu Hassan, PhD

Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Leong Wah Jun, PhD

Lecturer
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Bachok M. Taib, PhD

Professor
Faculty of Science
Universiti Sains Islam Malaysia
(External Examiner)

BUJANG BIN KIM HUAT, PhD

Professor and Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 15 April 2010



This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

Dato' Mohamed Suleiman, PhD

Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Fudziah Ismail, PhD

Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

Mohamed Othman, PhD

Professor
Faculty of Computer Science and Information Technology
Universiti Putra Malaysia
(Member)

HASANAH MOHD GHAZALI, PhD

Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 13 May 2010



DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

NORAZAK BIN SENU

Date: 22 February 2010

TABLE OF CONTENTS

		Page
ABSTRACT		ii
ABSTRAK		iv
ACKNOWLEDGEMENTS		vi
APPROVAL		vii
DECLARATION		ix
LIST OF TABLES		xiii
LIST OF FIGURES		xx
LIST OF ABBREVIATIONS		xxiv
CHAPTER		
1	INTRODUCTION	1
	1.1 Literature Review	1
	1.2 The Objectives of the Thesis	4
	1.3 Outline of the Thesis	5
	1.4 The Initial Value Problem	6
	1.5 Runge-Kutta-Nyström Method	7
	1.6 Algebraic Conditions for RKN Method	9
	1.7 Local Truncation Error	13
	1.8 Analysis of the Periodicity and Absolute Stability	15
	1.9 Analysis of Dispersion (Phase-lag) and Dissipation	23
	1.10 The Stiff Problem	28
2	AN EXPLICIT RUNGE-KUTTA-NYSTRÖM (RKN) METHODS FOR SOLVING OSCILLATORY PROBLEMS	30
	2.1 Introduction	30
	2.2 Derivation of Three-stage Third-order RKN Methods	33
	2.2.1 Problems Tested	41
	2.2.2 Numerical Results	43
	2.2.3 Discussion	55
	2.3 Derivation of Four-stage Fourth-order RKN Methods	56
	2.3.1 Numerical Results	63
	2.3.2 Discussion	75
	2.4 Derivation of Four-stage Fifth-order RKN Methods	76
	2.4.1 Numerical Results	83
	2.4.2 Discussion	94
3	AN EMBEDDED EXPLICIT RUNGE-KUTTA-NYSTRÖM METHODS (ERKN) FOR SOLVING OSCILLATORY PROBLEMS	95
	3.1 Introduction	95
	3.2 Derivation of Three-stage Embedded RKN Methods	97
	3.2.1 Derivation of 3(2) Pair RKN Methods	97
	3.2.2 Estimating the Error and Step Size Selection	102



	3.2.3	Estimating the Maximum Error, MAXE	103
	3.2.4	Numerical Results	104
3.3		Derivation of Four-stage Embedded RKN Methods	114
	3.3.1	Derivation of 4(3) Pairs RKN Methods	114
	3.3.2	Numerical Results	120
	3.3.3	Derivation of 5(4) Pairs RKN Methods	130
	3.3.4	Numerical Results	136
3.4		Discussion	146
4		BLOCK EMBEDDED EXPLICIT RUNGE-KUTTA-NYSTRÖM (BERKN) METHOD FOR SOLVING SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS	149
	4.1	Introduction	149
	4.2	Block Embedded Explicit RKN (BERKN) Method Type A	152
	4.2.1	Developing the Conditions for y_{n+2} and y'_{n+2}	152
	4.2.2	Stability Analysis of Block Explicit RKN Method	154
	4.2.3	Derivation of Third-order Block Embedded Explicit RKN (BERKN) Method	158
	4.3	Block Embedded Explicit RKN (BERKN) Method Type B	162
	4.3.1	Developing the Conditions for y_{n+2} and y'_{n+2}	162
	4.3.2	Stability Analysis of Block Explicit RKN Method	167
	4.3.3	Derivation of Third-order Block Embedded Explicit RKN (BERKN) Method	170
	4.4	Implementation of Block Method	174
	4.5	Numerical Results	175
	4.6	Discussion	183
5		DIAGONALLY IMPLICIT RUNGE-KUTTA-NYSTRÖM METHODS (DIRKN) FOR SOLVING OSCILLATORY PROBLEMS	185
	5.1	Introduction	185
	5.2	Development of the Consistent and Dispersion Relations	186
	5.3	Derivation of Diagonally Implicit Runge-Kutta-Nyström (DIRKN) Methods	195
	5.4	Derivation of Three-stage DIRKN Methods	197
	5.4.1	Problems Tested	203
	5.4.2	Numerical Results	206
	5.4.3	Discussion	213
	5.5	Derivation of Four-stage DIRKN Methods	214
	5.5.1	Numerical Results	227
	5.5.2	Discussion	238
6		AN EMBEDDED DIAGONALLY IMPLICIT RUNGE-KUTTA-NYSTRÖM METHODS (EDIRKN) FOR SOLVING OSCILLATORY PROBLEMS	240
	6.1	Introduction	240
	6.2	Derivation of 4(3) Pair EDIRKN Methods with Dispersion of High Order	242
	6.2.1	Numerical Results	246
	6.2.2	Discussion	253

6.3	Derivation of Embedded DIRKN Method with P-stability Property	254
6.3.1	Numerical Results	257
6.3.2	Discussion	263
7	PARALLEL EMBEDDED EXPLICIT RUNGE-KUTTA-NYSTRÖM (PERKN) METHOD FOR SOLVING SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS	264
7.1	Introduction	264
7.2	Parallel Programming	266
7.2.1	High Performance Computer Sunfire 1280 Architecture	266
7.2.2	Message Passing Interface (MPI)	268
7.2.3	Performance of Parallel Algorithm	268
7.3	Derivation of Parallel Embedded Explicit RKN (PERKN) Method	271
7.4	Implementation of PERKN5(4) Method on Parallel Machines	275
7.5	Problem Tested	277
7.6	Numerical Results	279
7.7	Discussion	284
8	CONCLUSION	286
8.1	Summary	286
8.2	Future Work	287
	BIBLIOGRAPHY	289
	BIODATA OF STUDENT	298
	LIST OF PUBLICATIONS	299



LIST OF TABLES

Table		Page
2.1	The RKN3(3,6,3)M method	35
2.2	The RKN3(3,6,3)S method	36
2.3	The RKN3(3,6,5) method	37
2.4	The RKN3(3,6,1) method	40
2.5	Summary of the characteristic of the third-order explicit RKN methods	40
2.6	Comparison results between RKN3(3,6, ∞), RKN3(3,6, ∞)HS, RKN3(3,6,5),RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS andRKN3(3,12,3)HS methods when solving Problem 2.1	45
2.7	Comparison results between RKN3(3,6, ∞), RKN3(3,6, ∞)HS, RKN3(3,6,5),RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS andRKN3(3,12,3)HS methods when solving Problem 2.2	46
2.8	Comparison results between RKN3(3,6, ∞), RKN3(3,6, ∞)HS, RKN3(3,6,5),RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS andRKN3(3,12,3)HS methods when solving Problem 2.3	47
2.9	Comparison results between RKN3(3,6, ∞), RKN3(3,6, ∞)HS, RKN3(3,6,5),RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS andRKN3(3,12,3)HS methods when solving Problem 2.4	48
2.10	Comparison results between RKN3(3,6, ∞), RKN3(3,6, ∞)HS, RKN3(3,6,5),RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS andRKN3(3,12,3)HS methods when solving Problem 2.5	49
2.11	Comparison results between RKN3(3,6, ∞), RKN3(3,6, ∞)HS, RKN3(3,6,5),RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS andRKN3(3,12,3)HS methods when solving Problem 2.6	50
2.12	Comparison results between RKN3(3,6, ∞), RKN3(3,6, ∞)HS, RKN3(3,6,5),RKN3(3,6,3)M, RKN3(3,6,3)S, RKN3(3,8,3)HS, RKN3(3,10,3)HS andRKN3(3,12,3)HS methods when solving Problem 2.7	51



2.13	The RKN4(4,8,5)M method	58
2.14	The RKN4(4,8,5)S method	59
2.15	The RKN4(4,8,7) method	61
2.16	Summary of the characteristic of the fourth-order RKN methods	63
2.17	Comparison results between RKN4(4,8,7), RKN4(4,8,5)M , RKN4(4,8,5)S,RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.1	65
2.18	Comparison results between RKN4(4,8,7), RKN4(4,8,5)M , RKN4(4,8,5)S,RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.2	66
2.19	Comparison results between RKN4(4,8,7), RKN4(4,8,5)M , RKN4(4,8,5)S,RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.3	67
2.20	Comparison results between RKN4(4,8,7), RKN4(4,8,5)M , RKN4(4,8,5)S,RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.4	68
2.21	Comparison results between RKN4(4,8,7), RKN4(4,8,5)M , RKN4(4,8,5)S,RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.5	69
2.22	Comparison results between RKN4(4,8,7), RKN4(4,8,5)M , RKN4(4,8,5)S,RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.6	70
2.23	Comparison results between RKN4(4,8,7), RKN4(4,8,5)M , RKN4(4,8,5)S,RKN4(4,4,5)D, RKN4(4,8,5)P, RKN4(4,8,5)Si and RKN4(4,10,5)HS methods when solving Problem 2.7	71
2.24	The RKN4(5,8,5)M method	77
2.25	The RKN4(5,8,5)S method	78
2.26	The RKN4(5,8,7) method	81
2.27	Summary of the characteristic of the fifth-order RKN methods	82
2.28	Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S,RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.1	84
2.29	Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S,RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.2	85



2.30	Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.3	86
2.31	Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.4	87
2.32	Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.5	88
2.33	Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.6	89
2.34	Comparison results between RKN4(5,8,7), RKN4(5,8,5)M, RKN4(5,8,5)S, RKN4(5,4,5)D and RKN4(5,4,5)B methods when solving Problem 2.7	90
3.1	The ERKN3(2)M method	98
3.2	The ERKN3(2)S method	100
3.3	The ERKN3(2)D5 method	101
3.4	The ERKN3(2)Z method	102
3.5	Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.1	105
3.6	Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.2	106
3.7	Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.3	107
3.8	Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.4	108
3.9	Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.5	109
3.10	Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.6	110
3.11	Comparison results between ERKN3(2)Z, ERKN3(2)D5, ERKN3(2)M, ERKN3(2)S and RK3(2)D for Problem 2.7	111
3.12	The ERKN4(3)M method	115



3.13	The ERKN4(3)S method	117
3.14	The ERKN4(3)D7 method	119
3.15	Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.1	121
3.16	Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.2	122
3.17	Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.3	123
3.18	Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.4	124
3.19	Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.5	125
3.20	Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.6	126
3.21	Comparison results between ERKN4(3)M, ERKN4(3)S, ERKN4(3)D7, ERKN4(3)D and ERKN4(3)P for Problem 2.7	127
3.22	The ERKN5(4)M method	131
3.23	The ERKN5(4)S method	133
3.24	The ERKN5(4)D7 method	135
3.25	Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.1	137
3.26	Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.2	138
3.27	Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.3	139
3.28	Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.4	140
3.29	Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.5	141
3.30	Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D, ERKN5(4)B and DOPRI5 for Problem 2.6	142
3.31	Comparison results between ERKN5(4)M, ERKN5(4)S, ERKN5(4)D,	



	ERKN5(4)B and DOPRI5 for Problem 2.7	143
4.1	The ERKN3(2) method for y_{n+2} and y'_{n+2} of Type A	160
4.2	The Block Embedded Explicit RKN (BERKN3(2)TA) of Type A method	161
4.3	The ERKN3(2) method for y_{n+2} and y'_{n+2} of Type B	173
4.4	The Block Embedded Explicit RKN (BERKN3(2)TB) of Type B method	173
4.5	Comparison results between BERKN3(2)TA, BERKN3(2)TB, RKN3(2)M,RK3(2)D and RK4(3)F for Problem 2.1	176
4.6	Comparison results between BERKN3(2)TA, BERKN3(2)TB, RKN3(2)M,RK3(2)D and RK4(3)F for Problem 2.3	177
4.7	Comparison results between BERKN3(2)TA, BERKN3(2)TB, RKN3(2)M,RK3(2)D and RK4(3)F for Problem 2.4	178
4.8	Comparison results between BERKN3(2)TA, BERKN3(2)TB, RKN3(2)M,RK3(2)D and RK4(3)F for Problem 2.7	179
4.9	Comparison results between BERKN3(2)TA, BERKN3(2)TB, RKN3(2)M,RK3(2)D and RK4(3)F for Problem 2.8	180
5.1	The DIRKN3(4,4)(a) method	198
5.2	The DIRKN3(4,4)(b) method	198
5.3	The DIRKN3(4,6)(a) method	200
5.4	The DIRKN3(4,6)(b) method	201
5.5	Summary of the characteristic of the three-stage fourth-order DIRKN methods	202
5.6	Comparing our results with the methods in the literature for Problem 5.1	207
5.7	Comparing our results with the methods in the literature for Problem 5.2	208
5.8	Comparing our results with the methods in the literature for Problem 5.3	209
5.9	Comparing our results with the methods in the literature for Problem 5.4	210
5.10	The DIRKN4(4,4)a method	217
5.11	The DIRKN4(4,4)b method	218
5.12	The DIRKN4(4,8) method	223
5.13	The DIRKN4(4,4)P method	227



5.14	Summary of the characteristic of the four-stage fourth-order DIRKN methods	228
5.15	Comparison numerical results between DIRKN4(4,4)(a), DIRKN4(4,4)(b), DIRKN4(4,6), DIRKN4(4,8) and DIRKN4(4,4)Raed methods for Problem 5.1	229
5.16	Comparison numerical results between DIRKN4(4,4)(a), DIRKN4(4,4)(b), DIRKN4(4,6), DIRKN4(4,8) and DIRKN4(4,4)Raed methods for Problem 5.2	230
5.17	Comparison numerical results between DIRKN4(4,4)(a), DIRKN4(4,4)(b), DIRKN4(4,6), DIRKN4(4,8) and DIRKN4(4,4)Raed methods for Problem 5.3	231
5.18	Comparison numerical results between DIRKN4(4,4)(a), DIRKN4(4,4)(b), DIRKN4(4,6), DIRKN4(4,8) and DIRKN4(4,4)Raed methods for Problem 5.4	232
5.19	Comparison results for DIRKN4(4,4)P with the DIRKN3(4,4)P-Fine, DIRKN4(4,6)P-Franco and DIRKN4(4,8)P-Franco methods for Problem 5.5	236
5.20	Comparison results for DIRKN4(4,4)P with the DIRKN3(4,4)P-Fine, DIRKN4(4,6)P-Franco and DIRKN4(4,8)P-Franco methods for Problem 5.6	236
6.1	The DIRKN34(3)6 method	243
6.2	The DIRKN44(3)8 method	245
6.3	Comparison results between DIRKN34(3)6, DIRKN44(3)8, DIRKN44(3)Raed and DIRKN34(3)Imoni for Problem 5.1	247
6.4	Comparison results between DIRKN34(3)6, DIRKN44(3)8, DIRKN44(3)Raed and DIRKN34(3)Imoni for Problem 5.2	248
6.5	Comparison results between DIRKN34(3)6, DIRKN44(3)8, DIRKN44(3)Raed and DIRKN34(3)Imoni for Problem 5.3	249
6.6	Comparison results between DIRKN34(3)6, DIRKN44(3)8, DIRKN44(3)Raed and DIRKN34(3)Imoni for Problem 5.4	250
6.7	The DIRKN4(3)P method	257
6.8	Comparison numerical results between EDIRKN34(3)P and DIRK4(3)B for Problem 5.5	258
6.9	Comparison numerical results between EDIRKN34(3)P and DIRK4(3)B for Problem 5.6	259



6.10	Comparison numerical results between EDIRKN4(3)P and DIRK4(3)B for Problem 5.7	259
6.11	Comparison numerical results between EDIRKN4(3)P and DIRK4(3)B for Problem 5.8	260
7.1	Hardware Configuration of Sunfire 1280	267
7.2	Runge-Kutta matrix suggested by Burrage (1990)	272
7.3	A new Runge-Kutta matrix adapted for PERKN method	272
7.4	The Parallel Embedded Explicit RKN (PERKN5(4)) method	274
7.5	Numerical results of sequential and parallel implementation of PERKN5(4) for Problem 7.1	280
7.6	Numerical results of sequential and parallel implementation of PERKN5(4) for Problem 7.2	282



LIST OF FIGURES

Figure		Page
2.1	Stability region for RKN3(3,6,3)M method with $m = 3$; $p = 3$; $q = 6$ and $r = 3$	35
2.2	Stability region for RKN3(3,6,3)S method with $m = 3$; $p = 3$; $q = 6$ and $r = 3$	36
2.3	Stability region for RKN3(3,6,5) method with $m = 3$; $p = 3$; $q = 6$ and $r = 5$	38
2.4	Histogram of accuracy for third-order RKN methods for Problem 2.1 with $h = 0.025$	52
2.5	Histogram of accuracy for third-order RKN methods for Problem 2.2 with $h = 0.025$	52
2.6	Histogram of accuracy for third-order RKN methods for Problem 2.3 with $h = 0.025$	53
2.7	Histogram of accuracy for third-order RKN methods for Problem 2.4 with $h = 0.025$	53
2.8	Histogram of accuracy for third-order RKN methods for Problem 2.5 with $h = 0.025$	54
2.9	Histogram of accuracy for third-order RKN methods for Problem 2.6 with $h = 0.025$	54
2.10	Stability region for RKN4(4,8,5)S method with $m = 4$; $p = 4$; and $q = 8$	60
2.11	Stability region for RKN4(4,8,5)S method for the range $-2 \leq \text{Re}(H) \leq 0$	60
2.12	Stability region for RKN4(4,8,7) method	62
2.13	Histogram of accuracy for fourth-order RKN methods for Problem 2.1 with $h = 0.025$	72
2.14	Histogram of accuracy for fourth-order RKN methods for Problem 2.2 with $h = 0.025$	72
2.15	Histogram of accuracy for fourth-order RKN methods for Problem 2.3 with $h = 0.025$	73
2.16	Histogram of accuracy for fourth-order RKN methods for Problem 2.4 with $h = 0.025$	73
2.17	Histogram of accuracy for fourth-order RKN methods for Problem 2.5 with $h = 0.05$	74



2.18	Histogram of accuracy for fourth-order RKN methods for Problem 2.6 with $h = 0.05$	74
2.19	Stability region for RKN4(5,8,5)S method	78
2.20	Stability region for RKN4(5,8,5)S method for the range $-2.5 \leq \text{Re}(H) \leq 0$	79
2.21	Stability region for RKN4(5,8,7) method	80
2.22	Stability region for RKN4(5,8,7) method for the range $-2.5 \leq \text{Re}(H) \leq 0$	82
2.23	Histogram of accuracy for fifth-order RKN methods for Problem 2.1 with $h = 0.025$	91
2.24	Histogram of accuracy for fifth-order RKN methods for Problem 2.2 with $h = 0.025$	91
2.25	Histogram of accuracy for fifth-order RKN methods for Problem 2.3 with $h = 0.05$	92
2.26	Histogram of accuracy for fifth-order RKN methods for Problem 2.4 with $h = 0.05$	92
2.27	Histogram of accuracy for fifth-order RKN methods for Problem 2.5 with $h = 0.1$	93
2.28	Histogram of accuracy for fifth-order RKN methods for Problem 2.6 with $h = 0.05$	93
3.1	Efficiency curve for 3(2) pair of RKN methods for Problem 2.1	112
3.2	Efficiency curve for 3(2) pair of RKN methods for Problem 2.2	112
3.3	Efficiency curve for 3(2) pair of RKN methods for Problem 2.3	113
3.4	Efficiency curve for 3(2) pair of RKN methods for Problem 2.6	113
3.5	Efficiency curve for 4(3) pair of RKN methods for Problem 2.1	128
3.6	Efficiency curve for 4(3) pair of RKN methods for Problem 2.2	128
3.7	Efficiency curve for 4(3) pair of RKN methods for Problem 2.5	129
3.8	Efficiency curve for 4(3) pair of RKN methods for Problem 2.7	129
3.9	Efficiency curve for 5(4) pair of RKN methods for Problem 2.1	144
3.10	Efficiency curve for 5(4) pair of RKN methods for Problem 2.2	144
3.11	Efficiency curve for 5(4) pair of RKN methods for Problem 2.3	145



3.12	Efficiency curve for 5(4) pair of RKN methods for Problem 2.6	145
4.1	Stability region for Block Explicit RKN method of Type A	161
4.2	Stability region for Block Explicit RKN method of Type B	172
4.3	Efficiency curve for block methods for Problem 2.1	181
4.4	Efficiency curve for block methods for Problem 2.3	181
4.5	Efficiency curve for block methods for Problem 2.4	182
4.6	Efficiency curve for block methods for Problem 2.7	182
4.7	Efficiency curve for block methods for Problem 2.8	183
5.1	Stability region for DIRKN3(4,6)(a) method	201
5.2	Stability region for DIRKN3(4,6)(b) method	202
5.3	Histogram of accuracy for three-stage forth-order DIRKN method for Problem 5.1 with $h = 0.01$	211
5.4	Histogram of accuracy for three-stage forth-order DIRKN method for Problem 5.2 method with $h = 0.25$	211
5.5	Histogram of accuracy for three-stage forth-order DIRKN method for Problem 5.3 method with $h = 0.0625$	212
5.6	Histogram of accuracy for three-stage forth-order DIRKN method for Problem 5.4 method with $h = 0.01$	212
5.7	Stability region for DIRKN4(4,8) method	224
5.8	Histogram of accuracy for four-stage forth-order DIRKN method for Problem 5.1 with $h = 0.01$	233
5.9	Histogram of accuracy for four-stage forth-order DIRKN method for Problem 5.2 with $h = 0.0625$	233
5.10	Histogram of accuracy for four-stage forth-order DIRKN method for Problem 5.3 with $h = 0.25$	234
5.11	Histogram of accuracy for four-stage forth-order DIRKN method for Problem 5.4 with $h = 0.01$	234
5.12	Histogram of accuracy for P-stable four-stage forth-order DIRKN method for Problem 5.5 with $h = 0.0025$	237
5.13	Histogram of accuracy for P-stable four-stage forth-order DIRKN method for Problem 5.6 with $h = 0.0625$	237



6.1	Efficiency curve for 4(3) pair DIRKN methods for Problem 5.1	251
6.2	Efficiency curve for 4(3) pair DIRKN methods for Problem 5.2	251
6.3	Efficiency curve for 4(3) pair DIRKN methods for Problem 5.3	252
6.4	Efficiency curve for 4(3) pair DIRKN methods for Problem 5.4	252
6.5	Stability region for embedded formula of DIRKN4(3)P	256
6.6	Efficiency curve for DIRKN4(3)P method for Problem 5.5	261
6.7	Efficiency curve for DIRKN4(3)P method for Problem 5.6	261
6.8	Efficiency curve for DIRKN4(3)P method for Problem 5.7	262
6.9	Efficiency curve for DIRKN4(3)P method for Problem 5.8	262
7.1	MPI program structure	269
7.2	Speedup for PERKN5(4) method on Parallel Machine when solving Problem 7.1	281
7.3	Efficiency for PERKN5(4) method on Parallel Machine when solving Problem 7.1	281
7.4	Speedup for PERKN5(4) method on Parallel Machine when solving Problem 7.2	283
7.5	Efficiency for PERKN5(4) method on Parallel Machine when solving Problem 7.2	283

