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HOMOTOPY ANALYSIS AND LEGENDRE MULTI-WAVELETS METHODS FOR SOLVING INTEGRAL EQUATIONS

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By

SAEED VAHDATI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

December 2009

DEDICATION

To

My Wife

For her countless sacrifices and endless patience and understanding

My Mother and My Father

For their support, encouragement and love

and

My Dear Teachers

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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December 2009

Chair: Dr. Zulkifly Abbas, PhD

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Due to the ability of function representation, hybrid functions and wavelets have a special position in research. In this thesis, we state elementary definitions, then we introduce hybrid functions and some wavelets such as Haar, Daubechies, Chebyshev, sine-cosine and linear Legendre multi wavelets. The construction of most wavelets are based on stepwise functions and the comparison between two categories of wavelets will become easier if we have a common construction of them.

The properties of the Floor function are used to find a function which is one on the interval $[0, 1)$ and zero elsewhere. The suitable dilation and translation parameters lead us to get similar function corresponding to the interval $[a, b)$. These functions and their combinations enable us to represent the stepwise functions as a function of floor function. We have applied this method on Haar wavelet, Sine-Cosine wavelet, Block - Pulse functions and Hybrid Fourier Block-Pulse functions to get the new representations of these functions.

The main advantage of the wavelet technique for solving a problem is its ability to transform complex problems into a system of algebraic equations. We use the

Legendre multi-wavelets on the interval $[0, 1)$ to solve the linear integro-differential and Fredholm integral equations of the second kind. We also use collocation points and linear legendre multi wavelets to solve an integro-differential equation which describes the charged particle motion for certain configurations of oscillating magnetic fields. Illustrative examples are included to reveal the sufficiency of the technique. In linear integro-differential equations and Fredholm integral equations of the second kind cases, comparisons are done with CAS wavelets and differential transformation methods and it shows that the accuracy of these results are higher than them.

Homotopy Analysis Method (HAM) is an analytic technique to solve the linear and nonlinear equations which can be used to obtain the numerical solution too. We extend the application of homotopy analysis method for solving Linear integro-differential equations and Fredholm and Volterra integral equations. We provide some numerical examples to demonstrate the validity and applicability of the technique. Numerical results showed the advantage of the HAM over the HPM, SCW, LLMW and CAS wavelets methods. For future studies, some problems are proposed at the end of this thesis.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

HOMOTOPY ANALYSIS AND LEGENDRE MULTI-WAVELETS METHODS FOR SOLVING INTEGRAL EQUATIONS

Oleh

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Fungsi hibrid dan wavelet adalah penting dalam penyelidikan perwakilan fungsi. Tesis ini dimulakan dengan kenyataan takrifan asas sebelum memperkenalkan fungsi hybrid dan wavelet seperti Haar, Daubechies, sin-cos dan wavelet berbilang linear Legendre. Pembentukan kebanyakan wavelet adalah berasaskan kepada fungsi bijaklangkah . Perbandingan antara dua kategori wavelet adalah lebih mudah dilaksanakan sekiranya berasaskan pembentukan yang seiras.

Sifat fungsi Floor telah digunakan untuk mendapatkan fungsi yang bernilai tunggal pada sela $[0, 1)$ dan sifar di luar sempadannya. Parameter kembangan dan ubahan yang bersesuaian membawa kepada perolehan fungsi sepadanan sela $[a, b)$. Fungsi ini serta gabungannya membenarkan perwakilan fungsi bijaklangkah sebagai fungsi kepada fungsi ”floor” . Kaedah ini telah digunakan ke atas wavelet Haar, sinkos, fungsi denyut blok dan fungsi hybrid Fourier denyut blok untuk mendapatkan perwakilan baru fungsi.

Faedah utama menggunakan kaedah wavelet untuk penyelesaian masalah ialah kebolehanannya untuk mengubah masalah kompleks kepada satu system persamaan algebra. Kaedah Legendre Wavelet Berbilang yang selanjar telah digunakan pada sela $[0, 1)$ untuk menyelesaikan persamaan pembza-kamiran dan persamaan kamiran Fredholm peringkat kedua. Titik kolokasi dan Legendre berbilang wavelet yang linear juga telah digunakan untuk penyelesaiaan persamaan pembeza-kamiran menerangkan pergerakan zarah bercas dibawah pengaruh medan magnet. Contoh kiraan juga dipamirkan untuk membuktikan keupayaan kaedah tersebut. Kaedah persamaan linear pembeza-kamiran dan persamaan kamiran Fredholm peringkat kedua menununjukkan kejituan tinggi bila dibandingkan dengan kaedah lain.

Kaedah Analisis Homotopi adalah teknik analitik untuk penyelesaian persamaan linear dan tidak linear secara berangka. Kaedah ini telah digunakan untuk penyelesaian persamaan pembeza-kamiran linear dan kamiran Fredholm dan Volterra. Beberapa contoh berangka dalam tesis ditunjukan untuk mementusahkan teknik ini dan kegunaannya. Akhir sekali, beberapa kerja lanjutan kepada tesis ini juga dicadangkan untuk penambahbaikan teknik ini.

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I certify that an Examination Committee has met on 31 December 2009 to conduct the final examination of **Saeed Vahdati** on his **PhD** thesis entitled "**Homotopy Analysis and Legendre Multi-Wavelets Methods for Solving Integral Equations**" in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The Committee recommends that the student be awarded the Doctor of Philosophy.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of Supervisory Committee were as follows:

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DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

Saeed Vahdati
Date:

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ABBREVIATIONS

PCBF	Piecewise Constant Basis Functions
BPF	Block-Pulse Functions
TF	Triangular Functions
RF	Rademacher Functions
WF	Walsh Vunctions
HF	Haar Functions
SCF	Sine-Cosine Functions
HFBPF	Hybrid Fourier and Block-Pulse Functions
OMI	Operational Matrix of Integration
POM	Product Operation Matrix
SCW	Sine-Cosine Wavelets
LLMW	Linear Legendre Multi Wavelets
HPM	Homotopy Perturbation Method
HAM	Homotopy Analysis Method
NLEE	Nonlinear evolution equation

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CHAPTER 1
INTRODUCTION

1.1 Background

In the classic integral equations of the second kind we can solve integral equations in some special cases, for example, when kernel is degenerate. But in most physical phenomena we are concerned with equations that are not a special case, so we can not obtain an exact solution for them. Therefore, it is necessary that we obtain an approximate solution for these integral equations. In recent years, schoolers have been considered functions such as: continuous orthogonal functions, piecewise continuous functions, orthogonal polynomials and Taylor polynomials. Functions and polynomials can be classified as follows: The first class contains piecewise continuous functions (for example Walsh, Block-Pulse and Haar functions). Orthogonal functions belong to the second class, for example: Legendre, Chebyshev, Hermit and Laguerre polynomials. The third class contains continuous orthogonal function approximation using piecewise continuous functions, orthogonal polynomials and Taylor polynomials has less accuracy so we use the orthogonal piecewise continuous functions, say hybrid functions, and wavelets have been considered by schoolers. Wavelet theory is a relatively new phenomena in applied mathematics. Its history returns to the recent two decades. Schooler's study such as Morlet, Arens, Fourgeau and Giard (1982) and Grossmann (1984) yield to wavelet theory. It is impressive that pay attention to this area has been increased, for a survey one can read [34].

Current success of the wavelet theory is related to two reasons, first we can consider wavelet theory as a combination of engineering science, physics and pure mathematics and on the other hand wavelets are relatively simple tools with various applications. So far wavelets have been used in areas such as: signal analysis [34], image processing [95], numerical solution of partial differential equations [16, 54], integral equations [22, 42, 55], integro-differential equations [11]. As an orthogonal system, wavelets have a special position in other systems of orthogonal functions.

The principle characterization of methods based on orthogonal functions is the approximation of differential and integral operators using concept of operational matrix. This is done as follows: first the solution of the system as an unknown function is expanded in terms of orthogonal functions with unknown coefficients then by operational matrix the equations that state behavior of the system are appeared in a linear or non-linear system, where its solution is the solution of the original system at various conditions.

For the first time, operational matrix of integration in Walsh domain was introduced [30] and a method was determined for solving some linear and non-linear differential systems and integral equations. In [20, 85], Walsh functions were used to estimate and identify linear systems that are independent of time. Operational matrix of integration for Block-Pulse functions was determined by the obtained linear transform between Walsh and Block-Pulse functions[21].

In this area some studies fared on the computational error in the methods based on operational matrix of integration in Walsh and Block-Pulse. The results showed that representation of non-smooth functions in Walsh and Block-Pulse domain have less accuracy [23].

1.2 Motivation and problem statement

This section is concerned primarily with the numerical solution of what are called Fredholm integral equations, but we begin by discussing the broader category of integral equations in general. In classifying integral equations, we say, very roughly, that those integral equations in which the integration domain varies with the independent variable in the equation are *Volterra integral equations*; and those in which the integration domain is fixed are Fredholm integral equations [10].

1.2.1 Volterra integral equations

The general form that is studied is

$$x(t) + \int_a^t k(t, s, x(s))ds = y(t), \qquad t \geq a. \tag{1.2.1}$$

The functions $k(t, s, u)$ and $y(t)$ are given, and $x(t)$ is unknown. This is a *nonlinear-volterra* integral equation, and it is in this form that the equation is most commonly applied and solved. Such equations can be thought of as generalization of:

$$x'(t) = f(t, x(t)), \qquad t \geq a, \qquad x(a) = x_0, \tag{1.2.2}$$

the *initial value problem for ordinary differential equations*. This equation is equivalent to the integral equation:

$$x(t) = x_0 + \int_a^t f(s, x(s))ds, \qquad t \geq a,$$

which is a special case of (1.2.1).

For an introduction to the theory of Volterra integral equations, see Miller [83]. These integral equations are not studied in this section, and the reader is referred to Brunner and Riele [19] and Linz [74]. Volterra integral equations are most commonly studied for functions x of one variable, as above, but there are examples of Volterra integral equations for functions of more than one variable.

1.2.2 Fredholm integral equations

The general form of such integral equations is

$$\lambda x(t) - \int_D k(t,s)x(s)ds = y(t), \qquad t \in D, \quad \lambda \neq 0, \tag{1.2.3}$$

with D a closed bounded set in $\mathbb{R}^m$, for some $m \geq 1$. The *kernel function* $k(t,s)$ is assumed to be absolutely integrable, and it is assumed to satisfy other properties that are sufficient to imply the Fredholm Alternative Theorem [10]. For $y \neq 0$, we have λ and y given, and we seek x; this is the *nonhomogeneous problem*. For $y = 0$, Equation (1.2.3) becomes an *eigenvalue problem*, and we seek both the *eigenvalue* λ and the *eigenfunction* x. The principal focus of the numerical methods presented in the following sections is the numerical solution of (1.2.3) with $y \neq 0$.

1.2.3 Integro-differential equations

An integro-differential equation is an equation involving one (or more) unknown functions $x(t)$, together with both differential and integral operations on x. Such a description covers a very broad class of functional relations and we restrict discussion here to the simplest types of one-dimensional integro-differential equation, which form a natural generalisation of Volterra and Fredholm integral equations. In particular we shall consider *nonlinear first order ordinary Volterra integro-differential equations* of the form

$$\begin{cases} x'(t) = g(t, x(t)) + \lambda \int_a^s k(t, s, x(s))ds, \\ x(a) = \alpha, \end{cases} \tag{1.2.4}$$

and *linear first and second order ordinary Fredholm integro-differential equations* of the form

$$\begin{cases} p(t)x''(t) + q(t)x'(t) + r(t)x(t) + \lambda \int_a^b k(t,s)x(s)ds = g(t), \\ \mathbf{C}x(\mathbf{r}) + \mathbf{D}x'(\mathbf{r}) = \mathbf{e}, \end{cases} \tag{1.2.5}$$

where

$$\mathbf{r} = (r_1, r_2, \ldots, r_m)^T, \qquad a \leq r_i \leq b,$$

$$x(\mathbf{r}) = (x(r_1), x(r_2), \ldots, x(r_m))^T,$$

$$x'(\mathbf{r}) = (x'(r_1), x'(r_2), \ldots, x'(r_m))^T,$$

and where for a pth order problem, $\mathbf{C}$, $\mathbf{D}$ are $p \times m$ matrices and $\mathbf{e}$ is a $p \times 1$ matrix.

In Equations (1.2.4) and (1.2.5) g, k, p, q are known functions and $\mathbf{r}$, $x(\mathbf{r})$ and $x'(\mathbf{r})$ are known vectors. Note the appearance in (1.2.4) and (1.2.5) of *boundary condition* equations. These are necessary to help to prove that a unique solution exists; the fact that such conditions are needed is evident by analogy with first order initial value and second order boundary value problems (set $\lambda = 0$ in (1.2.4) and (1.2.5)). It is the presence of these additional boundary conditions which makes the treatment of integro-differential equations significantly different from that of integral equations.

1.2.4 Degenerate kernel methods

Integral equation with a degenerate kernel function were introduced by Fredholm Alternative Theorem [10]. The degenerate kernel method is a well-known classical method for solving Fredholm integral equations of the second kind, and it is one of easiest numerical methods to define and analyze.

General theory:

Consider the integral equation (1.2.3). We assume throughout this and the following sections that D is a closed bounded set. Generally, it is an m-dimensional set with a piecewise smooth boundary; or it can be a piecewise smooth boundary itself. We usually work in the space $\mathcal{X} = C(D)$ with $\|\cdot\|_\infty$, and occasionally in $\mathcal{X} = L^2(D)$. The integral operator $\mathcal{K}$ of (1.2.3) is assumed to be a compact operator on $\mathcal{X}$ into $\mathcal{X}$.

The kernel function k is to be approximated by a sequence of kernel functions:

$$k_n(t,s) = \sum_{i=1}^{n} \alpha_{i,n}(t)\beta_{i,n}(s), \qquad n \geq 1, \tag{1.2.6}$$

in such a way that the associated integral operators $\mathcal{K}_n$ satisfy:

$$\lim_{n\to\infty} \|\mathcal{K} - \mathcal{K}_n\| = 0. \tag{1.2.7}$$

Generally, we want this convergence to be rapid to obtain rapid convergence of x_n to x, where x_n is the solution of the approximating equation:

$$\lambda x_n(t) - \int_D k_n(t,s)x_n(s)ds = y(t), \qquad t \in D. \tag{1.2.8}$$

Theorem 1.2.1 *Assume $\lambda - \mathcal{K} : \mathcal{X} \xrightarrow[onto]{1-1} \mathcal{X}$, with $\mathcal{X}$ a Banach space and $\mathcal{K}$ bounded. Further, assume $\{\mathcal{K}_n\}$ is a sequence of bounded linear operators with*

$$\lim_{n\to\infty} \|\mathcal{K} - \mathcal{K}_n\| = 0.$$

Then the operators $(\lambda - \mathcal{K}_n)^{-1}$ exist from $\mathcal{X}$ onto $\mathcal{X}$ for all sufficiently large n, say $n \geq N$, and

$$\left\|(\lambda - \mathcal{K}_n)^{-1}\right\| \leq \frac{\|(\lambda - \mathcal{K})^{-1}\|}{1 - \|(\lambda - \mathcal{K})^{-1}\| \, \|(\mathcal{K} - \mathcal{K}_n)\|}, \qquad n \geq N. \tag{1.2.9}$$

For the equations $(\lambda - \mathcal{K})x = y$ and $(\lambda - \mathcal{K}_n)x_n = y$, $n \geq N$, we have

$$\|x - x_n\| \leq \left\|(\lambda - \mathcal{K}_n)^{-1}\right\| \, \|(\mathcal{K}x - \mathcal{K}_n x)\|, \qquad n \geq N. \tag{1.2.10}$$