A rich structure related to the construction of holomorphic matrix functions

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Abstract

The problem of designing controllers that are robust with respect to uncertainty leads to questions that are in the areas of operator theory and several complex variables. One direction is the engineering problem of μ-synthesis, which has led to the study of certain inhomogeneous domains such as the symmetrised polydisc and the tetrablock. The μ-synthesis problem involves the construction of holomorphic matrix valued functions on the disc, subject to interpolation conditions and a boundedness condition.

In more detail, let $\lambda_1, \dots, \lambda_n$ be distinct points in the disc, and let $W_1, \dots, W_n$ be 2×2 matrices. The μ-synthesis problem related to the symmetrised bidisc involves finding a holomorphic 2×2 matrix function F on the disc such that $F(\lambda_j) = W_j$ for all j, and the spectral radius of $F(\lambda)$ is less than or equal to 1 for all λ in the disc. The μ-synthesis problem related to the tetrablock involves finding a holomorphic 2×2 matrix function F on the disc such that $F(\lambda_j) = W_j$ for all j, and the structured singular value (for the diagonal matrices with entries in $\mathbb{C}$) of $F(\lambda)$ is less than or equal to 1 for all λ in the disc.

For the symmetrised bidisc and for the tetrablock, we study the structure of interconnections between the matricial Schur class, the Schur class of the bidisc, the set of pairs of positive kernels on the bidisc subject to a boundedness condition, and the set of holomorphic functions from the disc into the given inhomogeneous domain. We use the theory of reproducing kernels and Hilbert function spaces in these connections. We give a solvability criterion for the interpolation problem that arises from the μ-synthesis problem related to the tetrablock. Our strategy for this problem is the following: (i) reduce the μ-synthesis problem to an interpolation problem in the set of holomorphic functions from the disc into the tetrablock; (ii) induce a duality between this set and the Schur class of the bidisc; and then (iii) use Hilbert space models for this Schur class to obtain necessary and sufficient conditions for solvability.

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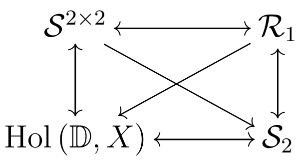
Contents

Chapter 1. Introduction and historical remarks

1.1 Introduction

Research by several mathematicians over the last few years indicates a rich structure of interconnections between four naturally arising objects of analysis. We introduce this structure, and construct correspondences to illustrate it. We use the theory of reproducing kernels and Hilbert function spaces to aid in these constructions. Better understanding of the structure is expected to contribute in solving extremal problems in the context of control engineering.

The rich structure can be summarised by the following diagram:



which we call the *rich saltire*. Whereas $\mathcal{S}^{2\times 2}$ and $\mathcal{S}_2$ are classical objects that have been studied greatly, $\operatorname{Hol}(\mathbb{D}, X)$ and $\mathcal{R}_1$ are relatively new and have been introduced over the last two decades in connection with the robust stabilisation problem (see, for example, [1, 3, 11]). The objects are defined as follows:

$$\operatorname{Hol}(\mathbb{D}, X) := \{\text{holomorphic functions from } \mathbb{D} \text{ to } X\},$$

where $\mathbb{D}$ is the open unit disc and X is either the symmetrised bidisc or the tetrablock; $\mathcal{S}^{2\times 2}$ is the 2×2 matricial Schur class; $\mathcal{S}_2$ is the Schur class of the bidisc $\mathbb{D}^2$; and $\mathcal{R}_1$ is the set of pairs (N, M) of holomorphic kernels on $\mathbb{D}^2$ such that the function defined by

$$(z, \lambda, w, \mu) \mapsto 1 - (1 - \overline{w}z)N(z, \lambda, w, \mu) - (1 - \overline{\mu}\lambda)M(z, \lambda, w, \mu),$$

is a rank 1 kernel on $\mathbb{D}^2$.

In the case of the tetrablock, we apply our results to obtain a solvability criterion for an interpolation problem from the disc to the set of 2×2 matrices with entries in $\mathbb{C}$, subject to a boundedness condition. Similar results were obtained for the symmetrised bidisc by J. Agler [UC San Diego, USA], Z. A. Lykova [Newcastle University, UK] and N. J. Young [Leeds and Newcastle Universities, UK] in [3], we formalise these results in the context of the rich saltire. Our strategy to obtain the criterion is the following: (i) reduce the problem to an interpolation problem in the set of holomorphic functions from the disc into the tetrablock; (ii) induce a duality between this set and $\mathcal{S}_2$; and then (iii) use Hilbert space models for $\mathcal{S}_2$ to obtain necessary and sufficient conditions for solvability. The criterion states that the interpolation problem is sovable if and only if there exists positive $3n$-square matrices N, of rank at most 1, and M that satisfy a matrix inequality obtained from the interpolation data (see Theorem 4.4.2).

This research is a step towards the use of several complex variables as a tool for representing and analysing the uncertainty of models used in engineering design, particularly in the design of robust automatic controllers.

In this thesis, any results that we use have a reference to the people who proved them or where we found them. All results without references are proved by D. C. Brown, Z. A. Lykova and N. J. Young.

1.2 H^∞ control and μ-synthesis

H^∞ control is a topic in control engineering and was heavily developed during the 1980's. Previously, control engineering theory tried to approximate desired frequency domain performance, in the sense of mean-square-error. The main methods for classical control often relied on trial and error, and so H^∞ engineering arose to provide a more precise method for optimising worst-case error in the frequency domain. This more precise approach is useful in converting an engineering problem into a problem that can be treated with a mathematical optimisation package. Numerous authors have covered the topic of H^∞ control, for example, B. A. Francis [University of Toronto, Canada] in [40], and J. W. Helton [UC San Diego, USA] and O. Merino [University of Rhode Island, USA] in [43].

D. Sarason [UC Berkeley, USA] gave an effective technique in [59] to deal with certain interpolation problems that arise from H^∞ control, in particular, the Carathéodory and Nevanlinna-Pick problems. His technique is operator theoretic and so demonstrates a connection between these interpolation problems and operator theory. There has been a lot of research to develop connections of this type, for example, the book [56] of J. R. Partington [Leeds University, UK] studies the problems of recovery and worst-case identification, and gives the application of these to H^∞ control.

An important aspect of modern control engineering is robustness. One approach to the design of stabilising controllers, for linear time-invariant systems, that are robust with respect to structured uncertainty, is that of H^∞ control, which leads to an optimisation problem over a class of holomorphic matrix functions on the disc. The book [57] of Partington focuses on the connections between linear operators and linear systems, and considers the stability of such systems. It includes a theorem of M. C. Smith [University of Cambridge, UK] from [60], which connects stability with transfer functions and coprime factorisations. In addition, it draws on a number of papers, for example, [42] by Smith with T. T. Georgiou [University of Minnesota, USA], to show that the gap topology is the correct topology to measure the distance between two linear systems in regards to robustness.

More recent papers of Partington include research on inner functions and Toeplitz kernels, for example, [28, 29] with I. Chalendar [Université du Lyon, France] and P. Gorkin [Bucknell University, USA], and [26, 27] with M. C. Câmara [Instituto Superior Técnico, Portugal], respectively.

The symbol μ denotes the structured singular value, of an operator or matrix, corresponding to a given uncertainty class. It is a type of cost function that generalises the operator and H^∞ norms, and was introduced by J. C. Doyle [Caltech, USA] and G. Stein [Honeywell Laboratories, USA] in [33], with further work by Doyle in [31] and [32]. The motivation for the structured singular value was the desire to achieve a less conservative stabilising controller by incorporating known structural information. The μ-synthesis problem involves the construction of holomorphic matrix valued functions on the disc which are subject to interpolation conditions and a boundedness condition. It can be shown, for certain cases of μ-synthesis, it is equivalent to construct holomorphic functions from the disc to a particular inhomogeneous domain, subject to interpolation conditions. Attempts to solve cases of the μ-synthesis problem have led to the study of a number of these domains.

A good description of robust stabilisation and μ-synthesis can be found in the book [36] of G. E. Dullerud [University of Illinois, USA] and F. G. Paganini [Universidad ORT, Uruguay]. For our purposes a *structure* is a linear subspace of

$$\mathcal{M}_{n\times m}(\mathbb{C}) := \{[m_{ij}]_{i,j=1}^{n,m} : m_{ij} \in \mathbb{C} \text{ for all } 1 \le i \le n, 1 \le j \le m\}.$$

For a structure $\mathcal{E}$, we define the *structured singular value* of $M \in \mathcal{M}_{m\times n}(\mathbb{C})$ by

$$\mu_{\mathcal{E}}(M) := \frac{1}{\inf\{||E|| : E \in \mathcal{E} \text{ and } I - ME \text{ is singular}\}},$$

where we set $\mu_{\mathcal{E}}(M) = 0$ if $I - ME$ is non-singular for all $E \in \mathcal{E}$. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc. Then the *μ-synthesis problem*, for a structure $\mathcal{E}$, is to construct a holomorphic matrix function $F : \mathbb{D} \to \mathcal{M}_{m\times n}(\mathbb{C})$, which satisfies a finite number of interpolation conditions, and is such that $\mu_{\mathcal{E}}(F(\lambda)) \le 1$ for all $\lambda \in \mathbb{D}$.

We highlight two special cases of the structured singular value. If we consider the structure $\mathcal{E} = \mathcal{M}_{n\times m}(\mathbb{C})$, then $\mu_{\mathcal{E}}$ is the operator norm $||\cdot||$. If we consider the structure which is the linear subspace of $\mathcal{M}_{n\times n}(\mathbb{C}) := \mathcal{M}_n(\mathbb{C})$ given by

$$\mathcal{E} = \{\lambda I : \lambda \in \mathbb{C} \text{ and } I \text{ is the identity matrix in } \mathcal{M}_n(\mathbb{C})\},$$

then $\mu_{\mathcal{E}}$ is the spectral radius ρ of a square matrix. One reason we highlight these two cases is that they can be considered extremal cases. Indeed, for any structure $\mathcal{E}$ and any $M \in \mathcal{M}_{m\times n}(\mathbb{C})$, we have $\mu_{\mathcal{E}}(M) \le ||M||$ and if, in addition, $m = n$ and $\mathcal{E}$ contains the identity matrix then $\rho(M) \le \mu_{\mathcal{E}}(M)$.

The other reason we highlight these cases is that the associated μ-synthesis problems become examples of more familiar problems. In the case that $\mu_\mathcal{E} = ||\cdot||$, the μ-synthesis problem becomes the classical Nevanlinna-Pick problem as discussed by many authors, including J. A. Ball [Virginia Tech, USA], I. Gohberg [Tel Aviv University, Israel] and L. Rodman [College of William and Mary, USA] in [15]. In the case that $\mu_\mathcal{E} = \rho$, the μ-synthesis problem becomes the spectral Nevanlinna-Pick problem as discussed by, for example, H. Bercovici [Indiana University, USA], C. Foiaş [Indiana University, USA] and A. Tannenbaum [Stony Brook University, USA] in [19]. In particular, if we take $\mu_\mathcal{E} = \rho$ in the case of 2×2 matrices then the μ-synthesis problem becomes the 2×2 spectral Nevanlinna-Pick problem, which is an interpolation problem that can be stated as follows.

Question 1.2.1. Let $\lambda_1, \dots, \lambda_k$ be distinct points in $\mathbb{D}$. Let $W_1, \dots, W_k \in \mathcal{M}_2(\mathbb{C})$ be such that $\rho(W_j) \leq 1$ for $j = 1, \dots, n$. Does there exist a holomorphic function $F : \mathbb{D} \to \mathcal{M}_2(\mathbb{C})$ such that $F(\lambda_j) = W_j$ for all $j = 1, \dots, k$, and $\rho(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$?

Agler and Young showed in [11] that this question is equivalent to an interpolation problem in the set of holomorphic functions from the disc to the symmetrised bidisc. Moreover, Agler, Lykova and Young showed in [3] that this interpolation problem can be used to find a criterion for which Question 1.2.1 is solvable.

We highlighted the extremal case in the setting of 2×2 matrices, that is, the case of $\mu = \rho$. The 'next' case we can consider is the structure that contains the diagonal matrices in $\mathcal{M}_2(\mathbb{C})$, that is, we consider the structure

$$\text{Diag} := \left\{ \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} : z, w \in \mathbb{C} \right\},$$

in this case, the μ-synthesis problem can be stated as follows.

Question 1.2.2. Let $\lambda_1, \dots, \lambda_k$ be distinct points in $\mathbb{D}$. Let $W_1, \dots, W_k \in \mathcal{M}_2(\mathbb{C})$ be such that $\mu_{\text{Diag}}(W_j) \leq 1$ for $j = 1, \dots, n$. Does there exist a holomorphic function $F : \mathbb{D} \to \mathcal{M}_2(\mathbb{C})$ such that $F(\lambda_j) = W_j$ for all $j = 1, \dots, k$, and $\mu_{\text{Diag}}(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$?

Young with A. A. Abouhajar [Newcastle University, UK] and M. C. White [Newcastle University, UK] showed in [1] that this question is equivalent to an interpolation problem in the set of holomorphic functions from the disc to the tetrablock. We show, in Chapter 4, that this interpolation problem can be used to find a criterion for which Question 1.2.2 is solvable. The strategy is: (i) to induce a duality between the set of holomorphic functions from the disc to the tetrablock, and a subset of $\mathcal{S}_2$; and then (ii) use Hilbert space models for $\mathcal{S}_2$ to obtain necessary and sufficient conditions for solvability.

1.3 Main results

Our first main result appears in Section 4.3.1, and gives the existence of a function in

$$\mathcal{S}^{2\times 2} := \{F : \mathbb{D} \to \mathcal{M}_2(\mathbb{C}) : F \text{ is holomorphic and } ||F(\lambda)|| \le 1 \text{ for all } \lambda \in \mathbb{D}\}$$

for each holomorphic function from the disc to the tetrablock. Let

$$\overline{\mathbb{E}} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 z w \ne 0 \text{ for all } z, w \in \overline{\mathbb{D}}\}$$

be the tetrablock, and let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Then the result is:

Theorem 4.3.1. *Let* $x = (x_1, x_2, x_3) \in \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. *Then there exists a unique*

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$$

such that $x = (F_{11}, F_{22}, \det F)$, $|F_{12}| = |F_{21}|$ *almost everywhere on* $\mathbb{T}$, F_{12} *is either outer or* 0, *and* $F_{12}(0) \ge 0$. *Moreover, we have*

$$1 - \overline{\Psi(w, x(\mu))}\Psi(z, x(\lambda)) = (1 - \overline{w}z)\overline{\gamma(\mu, w)}\gamma(\lambda, z) + \eta(\mu, w)^*(I - F(\mu)^* F(\lambda))\eta(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$, *where*

$$\Psi(z, x(\lambda)) = \frac{x_3(\lambda)z - x_1(\lambda)}{x_2(\lambda)z - 1}, \gamma(\lambda, z) = \frac{F_{21}(\lambda)}{1 - F_{22}(\lambda)z} \text{ and } \eta(\lambda, z) = \begin{bmatrix} 1 \\ z\gamma(\lambda, z) \end{bmatrix}$$

for all $z, \lambda \in \mathbb{D}$.

The proof of Theorem 4.3.1 is constructive, and is used to produce a map from $\operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ to $\mathcal{S}^{2\times 2}$. The proof uses the inner-outer factorisation of non-zero H^∞ functions to construct the appropriate function in $\mathcal{S}^{2\times 2}$. The identity is proved by using the realisation formula for functions on $\mathbb{D}$, and the fact that, for the constructed function F, we have

$$\Psi(z, x(\lambda)) = F_{11}(\lambda) + F_{12}(\lambda)z(1 - F_{22}(\lambda)z)^{-1}F_{21}(\lambda)$$

for all $z, \lambda \in \mathbb{D}$. The identity is a useful tool in the construction of a number of our other correspondences.

Our second main result appears in Section 4.4, and comes from our study of the rich structure of interconnections between $\operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$, $\mathcal{S}^{2\times 2}$,

$$\mathcal{S}_2 := \{\text{holomorphic functions from } \mathbb{D}^2 \text{ to } \overline{\mathbb{D}}\}$$

and $\mathcal{R}_1$, where $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$ is the closed unit disc. The result gives a criterion for the solvability of an interpolation problem in $\operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. The strategy is: (i) to induce a duality between $\text{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ and a subset of $\mathcal{S}_2$; and then (ii) use Hilbert space models for $\mathcal{S}_2$ to obtain necessary and sufficient conditions for solvability. The result is:

Theorem 4.4.1. *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$, and let $(x_{1j}, x_{2j}, x_{3j}) \in \overline{\mathbb{E}}$ be such that $x_{1j}x_{2j} \neq x_{3j}$ for $j = 1, \ldots, n$. Then the following are equivalent.*

(i) *There exists a holomorphic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ satisfying*

$$x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j}) \text{ for } j = 1, \ldots, n;$$

(ii) *there exists a rational $\overline{\mathbb{E}}$-inner function x satisfying*

$$x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j}) \text{ for } j = 1, \ldots, n;$$

(iii) *for every distinct points $z_1, z_2, z_3 \in \mathbb{D}$, there exist positive $3n$-square matrices $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ of rank at most 1, and $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ such that, for $1 \leq i, j \leq n$ and $1 \leq l, k \leq 3$,*

$$1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1} = (1 - \overline{z_l} z_k) N_{il,jk} + (1 - \overline{\lambda_i} \lambda_j) M_{il,jk};$$

(iv) *for some distinct points $z_1, z_2, z_3 \in \mathbb{D}$, there exist positive $3n$-square matrices $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ of rank at most 1, and $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ such that*

$$\left[1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1}\right] \geq \left[(1 - \overline{z_l} z_k) N_{il,jk}\right] + \left[(1 - \overline{\lambda_i} \lambda_j) M_{il,jk}\right].$$

Although Theorem 4.4.1 concerns the solvability of an $\overline{\mathbb{E}}$-interpolation problem, by use of the result of Abouhajar, White and Young in [1] that connects $\overline{\mathbb{E}}$-interpolation problems with the μ-synthesis problems described by Question 1.2.2, we obtain the following criterion for which the associated μ-synthesis problem is solvable.

Theorem 4.4.2. *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$, and let*

$$W_j = \begin{bmatrix} w_{11}^j & w_{12}^j \\ w_{21}^j & w_{22}^j \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$$

be such that $\mu_{\text{Diag}}(W_j) \leq 1$ and $w_{11}^j w_{22}^j \neq \det W_j$ for $j = 1, \ldots, n$. Set $(x_{1j}, x_{2j}, x_{3j}) = (w_{11}^j, w_{22}^j, \det W_j) \in \overline{\mathbb{E}}$ for each $j = 1, \ldots, n$. Then the following are equivalent.

(i) *There exists a holomorphic function $F : \mathbb{D} \to \mathcal{M}_2(\mathbb{C})$ such that $F(\lambda_j) = W_j$ for $j = 1, \ldots, n$, and $\mu_{\text{Diag}}(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$;*

(ii) *there exists a holomorphic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ satisfying*

$$x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j}) \text{ for } j = 1, \ldots, n;$$

(iii) *for some distinct points* $z_1, z_2, z_3 \in \mathbb{D}$*, there exist positive* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1*, and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that*

$$\left[1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1}\right] \geq [(1 - \overline{z_l} z_k) N_{il,jk}] + \left[(1 - \overline{\lambda_i}\lambda_j) M_{il,jk}\right].$$

1.4 Description of results by section

In Chapter 2, we consider the connections between $\mathcal{S}^{2\times 2}$, $\mathcal{S}_2$ and $\mathcal{R}_1$. In Section 2.1, we give a realisation formula for functions on $\mathbb{D}$, and, in Section 2.2, we use this to define a map from $\mathcal{S}^{2\times 2}$ to $\mathcal{S}_2$. In Section 2.3, we define a map from $\mathcal{S}^{2\times 2}$ to the set

$$\mathcal{R}_1 := \{(N, M) : N, M, K_{N,M} \text{ are holomorphic kernels on } \mathbb{D}^2 \text{ and } K_{N,M} \text{ has rank } 1\},$$

where $K_{N,M} : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ is the function given by

$$K_{N,M}(z, \lambda, w, \mu) := 1 - (1 - \overline{w} z) N(z, \lambda, w, \mu) - (1 - \overline{\mu}\lambda) M(z, \lambda, w, \mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. The map takes each $F \in \mathcal{S}^{2\times 2}$ to the pair of kernels $(N_F, M_F) \in \mathcal{R}_1$, which are defined by

$$N_F(z, \lambda, w, \mu) := \overline{\gamma(\mu, w)}\gamma(\lambda, z) \text{ and } M_F(z, \lambda, w, \mu) := \eta(\mu, w)^* \frac{I - F(\mu)^* F(\lambda)}{1 - \overline{\mu}\lambda} \eta(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. The functions γ and η are given by

$$\gamma(\lambda, z) = (1 - F_{22}(\lambda) z)^{-1} F_{21}(\lambda) \text{ and } \eta(\lambda, z) = \begin{bmatrix} 1 \\ z\gamma(\lambda, z) \end{bmatrix}$$

for all $z, \lambda \in \mathbb{D}$. In Section 2.4, we construct a function $\Xi \in \mathcal{S}^{2\times 2}$ for each pair of kernels in the subset of $\mathcal{R}_1$ defined by

$$\mathcal{R}_{11} := \{(N, M) \in \mathcal{R}_1 : N \text{ has rank } 1\},$$

and use this to define a set map from $\mathcal{R}_{11}$ to $\mathcal{S}^{2\times 2}$. We show that by taking (N_Ξ, M_Ξ) we get back the pair of kernels in $\mathcal{R}_{11}$. In Section 2.5, we give a set map from $\mathcal{R}_1$ to $\mathcal{S}_2$, and, in Section 2.6, we give a set map from $\mathcal{S}_2$ to $\mathcal{R}_1$. Where possible, we investigate how the maps of this chapter interact with one another.

In Chapter 3, we study the structure of interconnections between the sets $\operatorname{Hol}(\mathbb{D}, X)$, $\mathcal{S}^{2\times 2}$, $\mathcal{S}_2$ and $\mathcal{R}_1$ in the case that X is the *symmetrised bidisc*. In Section 3.1, we give some historic remarks on the symmetrised bidisc,

$$\Gamma := \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \overline{\mathbb{D}}\},$$

and describe the connection between Γ and μ-synthesis. In Section 3.2, we discuss the necessary background for Γ, including the distinguished boundary of Γ and Γ-inner functions. In Section 3.3, we construct the maps that illustrate the interconnections between $\mathrm{Hol}\,(\mathbb{D},\Gamma)$, $\mathcal{S}^{2\times 2}$, $\mathcal{S}_2$ and $\mathcal{R}_1$. Some of the results in this section are contained in [3, 4, 8, 12], and references are given for where each of these results originally appeared. We formalise the results and bring them together in order to better understand the rich structure; we fill the gaps and add any connections that do not appear in these papers.

In most of Section 3.3, we consider the connections between $\mathrm{Hol}\,(\mathbb{D},\Gamma)$, $\mathcal{S}^{2\times 2}$ and $\mathcal{S}_2$. We construct a unique function in $\mathcal{S}^{2\times 2}$ for each function in $\mathrm{Hol}\,(\mathbb{D},\Gamma)$, and provide a map from $\mathcal{S}^{2\times 2}$ that recovers the function in $\mathrm{Hol}\,(\mathbb{D},\Gamma)$. We produce a bijection between $\mathrm{Hol}\,(\mathbb{D},\Gamma)$ and the subset of $\mathcal{S}_2$ that contains the functions φ for which $\varphi(\cdot,\lambda)$ has the form

$$z \mapsto \frac{a(\lambda)z + b(\lambda)}{b(\lambda)z + c(\lambda)}$$

for all $\lambda \in \mathbb{D}$. In the remainder of Section 3.3, we consider the connections between $\mathrm{Hol}\,(\mathbb{D},\Gamma)$ and the kernel set $\mathcal{R}_1$. We use the results of Chapter 2 to define a set map from $\mathcal{R}_{11}$ to $\mathrm{Hol}\,(\mathbb{D},\Gamma)$. Where possible, we investigate how the maps of Chapter 2 interact with the maps involving $\mathrm{Hol}\,(\mathbb{D},\Gamma)$ that were obtained in the first half of the section.

In Section 3.4, we give a criterion for the solvability of a Γ-interpolation problem, and discuss the process of obtaining a solution to this problem. These results are proved in [3]. We give concluding remarks on how the criterion for solvability of the Γ-interpolation problem connects with the associated μ-synthesis problem.

In Chapter 4, we study the structure of interconnections between the sets $\mathrm{Hol}\,(\mathbb{D},X)$, $\mathcal{S}^{2\times 2}$, $\mathcal{S}_2$ and $\mathcal{R}_1$ in the case that X is the *tetrablock*. In Section 4.1, we give some historic remarks on the tetrablock,

$$\overline{\mathbb{E}} = \{(x_1,x_2,x_3) \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 z w \neq 0 \text{ for all } z,w \in \overline{\mathbb{D}}\},$$

and describe the connection between $\overline{\mathbb{E}}$ and μ-synthesis. In Section 4.2, we discuss the necessary background for $\overline{\mathbb{E}}$, including the distinguished boundary of $\overline{\mathbb{E}}$ and $\overline{\mathbb{E}}$-inner functions. In Section 4.3, we construct the maps that illustrate the interconnections between $\mathrm{Hol}\,(\mathbb{D},\overline{\mathbb{E}})$, $\mathcal{S}^{2\times 2}$, $\mathcal{S}_2$ and $\mathcal{R}_1$.

In the majority of Section 4.3, we consider the connections between $\mathrm{Hol}\,(\mathbb{D},\overline{\mathbb{E}})$, $\mathcal{S}^{2\times 2}$ and $\mathcal{S}_2$. We construct a unique function in $\mathcal{S}^{2\times 2}$ for each function in $\mathrm{Hol}\,(\mathbb{D},\overline{\mathbb{E}})$, and provide a map from $\mathcal{S}^{2\times 2}$ that recovers the function in $\mathrm{Hol}\,(\mathbb{D},\overline{\mathbb{E}})$. We produce a surjection from $\mathrm{Hol}\,(\mathbb{D},\overline{\mathbb{E}})$ to the subset of $\mathcal{S}_2$ that contains the functions φ for which $\varphi(\cdot,\lambda)$ has the form

$$z \mapsto \frac{a(\lambda)z + b(\lambda)}{c(\lambda)z + 1}$$

for all $\lambda \in \mathbb{D}$, where c is holomorphic and if $a(\lambda) = b(\lambda)c(\lambda)$ for some $\lambda \in \mathbb{D}$, then, in addition, $|c(\lambda)| \leq 1$. We finish Section 4.3 by considering the connections between

$\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ and the kernel set $\mathcal{R}_1$. Using the results of Chapter 2, we define a set map from $\mathcal{R}_{11}$ to $\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. Where possible, we investigate how the maps of Chapter 2 interact with the maps involving $\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ that were obtained in the rest of the section.

In Section 4.4, we prove a criterion for the solvability of an $\overline{\mathbb{E}}$-interpolation problem. The strategy is: (i) to induce a duality between $\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ and a subset of $\mathcal{S}_2$, and then (ii) use Hilbert space models for $\mathcal{S}_2$ to obtain necessary and sufficient conditions for solvability. We give concluding remarks on how the criterion for solvability of the $\overline{\mathbb{E}}$-interpolation problem connects with the associated μ-synthesis problem.

An appendix contains the necessary supplementary material. Chapter A contains examples from control engineering, and Chapter B contains the required background material. In Section B.1, we give definitions and results we wish to use but not discuss in depth. In Section B.2, we give a realisation formula and a number of related results. In Section B.3, the required definitions and results from the theory of reproducing kernels and Hilbert function spaces are given.

1.5 Historical remarks

Engineers often represent modelling error as a linear fractional transformation of an unknown element of a structured uncertainty class, in this case, the problem of designing controllers that are robust with respect to uncertainty leads to questions that are in the areas of operator theory and several complex variables. One direction is the engineering problem of μ-synthesis, which has led to the study of certain inhomogeneous domains that enjoy a rich function theory and operator theory.

The topics of this section represent a rich area of research with many active authors. As we cannot cover everything, our aim is to illustrate some of the areas and give an insight into the type of research that has been carried out. In particular, we want to highlight some of the alternative branches of research to ours, which exist in this area.

In [11], Agler and Young proved that there is an equivalence between the solvability of the 2×2 spectral Nevanlinna-Pick problem and the solvability of an interpolation problem in the set of holomorphic functions from the disc to the symmetrised bidisc. The spectral Nevanlinna-Pick problem is a special case of the μ-synthesis problem, and is a variant of the classical Nevanlinna-Pick problem, as studied by Pick in 1916 and Nevanlinna in 1919. The paper [11] includes a realisation formula for holomorphic functions from the disc to the symmetrised bidisc, which is a useful tool in the study of the domain, and answers, for dimension two, the problem posed by Ball and Young in [18]: to find a realisation formula for holomorphic functions from the disc to the symmetrised polydisc.

Since Agler and Young's first paper on the subject, the study has led to other domains related to cases of μ-synthesis. D. J. Ogle [Newcastle University, UK] studied the symmetrised n-disc in his thesis [53]. Ogle proves a necessary condition for the solvability of the spectral Nevanlinna-Pick problem that extends the necessary condition of Agler and Young in [9] for the 2×2 spectral Nevanlinna-Pick problem. He uses the approach of

Agler and Young to reduce the $n \times n$ spectral Nevanlinna-Pick problem to an interpolation problem in the set of holomorphic functions from the disc to the symmetrised n-disc, and then consider commuting n-tuples of operators that have this domain as a complete spectral set. Ogle uses this condition to find an upper bound on the Carathéodory distance between two points in the symmetrised n-disc.

Abouhajar, White and Young introduced the tetrablock in [1]. They showed that there is a relationship between a special case of μ-synthesis and interpolation in the set of holomorphic functions from the disc to the tetrablock. One of the main results of the paper is a Schwarz lemma for the tetrablock. They describe a large group of automorphisms of the tetrablock, and conjecture that this group of automorphisms is the automorphism group of the tetrablock. It was proved later, by Young in [63], that the conjecure is correct.

Agler, Lykova and Young introduced the pentablock in [5]. The paper studies the complex geometry of the pentablock, and derives a group of automorphisms of the pentablock. It was later shown, by L. Kosiński [Jagiellonian University, Poland] in [47], that this group of automorphisms is the automorphism group of the pentablock. Agler, Lykova and Young also show how the pentablock arises from a special case of μ-synthesis, and that this connection is more subtle than the similar connections that exist for the symmetrised bidisc and the tetrablock.

Aside from their use in the study of μ-synthesis, these domains turn out to have many properties of interest to specialists in several complex variables and operator theory. We give some examples. C. Costara [Ovidius University, Romania] showed in [30] that the symmetrised bidisc is not biholomorphic to a convex set. A. Edigarian [Jagiellonian University, Poland] improved on this result in [37] by showing that the symmetrised bidisc cannot be exhausted by domains biholomoprhic to convex ones. Combining this with an earlier result of Agler and Young in [12], that the Carathédory distance, Kobayashi distance and Lempert function coincide on the symmetrised bidisc, it follows that the symmetrised bidisc is a non-convex domain that satisfies the result of the Lempert Theorem (see [45, Theorem 11.2.1]).

In [51], N. Nikolov [Bulgarian Academy of Sciences, Bulgaria], P. Pflug [Oldenburg University, Germany] and W. Zwonek [Jagiellonian University, Poland] proved that, for n greater than two, the Lempert function of the symmetrised n-disc is not a distance. In particular, the Carathéodory distance and Lempert function of these domains do not coincide. As a result, these domains cannot be exhausted by domains biholomorphic to covex domains; this had previously been shown directly by Nikolov in [50]. In addition, the authors show that there exist, for any dimension greater than two, bounded pseudoconvex domains that cannot be exhausted by domains biholomorphic to covex domains, but for which the Carathéodory distance and Lempert function coincide.

Edigarian, Kosiński and Zwonek showed in [39] that the tetrablock is an example of a domain that cannot be exhausted by domains biholomorphic to convex domains, but for which the Carathéodory distance and Lempert function coincide. More information on these topics can be found in the book [45] of Pflug with M. Jarnicki [Jagiellonian University, Poland]. The book is about the study of invariant pseudodistances and pseudometrics in several complex variables, and is a useful collection of many results from this area. The authors include chapters on the symmetrised polydisc and on Lempert's theorem, and sections on complex geodesics in the symmetrised bidisc and in the tetrablock.

Edigarian and Zwonek studied the geometric properties of the symmetrised polydisc in [38]. They describe all proper holomorphic mappings of the symmetrised polydisc, and apply their results to the study of the spectral unit ball in $\mathcal{M}_n(\mathbb{C}^n)$. They show that, for a proper holomorphic self-map of the spectral unit ball, there exists a finite Blaschke product such that the spectrum of the map evaluated at a point in the ball is equal to the Blaschke product applied to elements of the spectrum of that point. This is a partial generalisation of a result of White with T. J. Ransford [Laval University, Canada] in [58].

In [48], Kosiński and Zwonek discuss three notions of m-extremal holomorphic maps, and the relations between them in the general case, and in special cases, including the symmetrised bidisc and the tetrablock. They showed that weak 3-extremal maps in the symmetrised bidisc are rational, which gives a partial positive answer to the question of Agler, Lykova and Young in [4] that asks if this is true for m-extremal maps. In [4], the authors introduced the class of m-extremal maps, and explored it in relation to the finite interpolation problem for holomorphic functions from the disc to the symmetrised bidisc. They give a sequence of necessary conditions for solvability that are of strictly increasing strength.

Agler and Young proved a Commutant Lifting Theorem for the symmetrised bidisc in [9], which led to the study of Γ-contractions. In [10], Agler and Young developed a model theory for Γ-contractions; they show that any Γ-contraction is unitarily equivalent to the restriction to a common invariant subspace of the orthogonal direct sum of a Γ-unitary and the adjoint of a pure Γ-isometry. This was taken further by T. Bhattacharya [IIT Kharagpur, India], S. Pal [IIT Bombay, India] and S. Shyam Roy [IISER Kolkata, India] in [21]. They construct an explicit Γ-isometric dilation for any Γ-contraction, the existence of which follows from the results of Agler and Young. Moreover, they show that a commuting pair of operators is a Γ-contraction if and only if the fundamental equation of the pair can be solved with a solution of numerical radius less than or equal to one.

In [20], Bhattacharya constructed a tetrablock-isometric dilation for a tetrablock-contraction whose fundamental operators satisfy certain commutativity conditions. Pal showed in [54] that there is a tetrablock-contraction which does not dilate to a tetrablock-isometry, and so demonstrated the failure of rational dilation on a domain in $\mathbb{C}^3$. In a different direction, M. A Dritschel [Newcastle University, UK] and S. McCullough [University of Florida, USA] showed in [34] the failure of rational dilation on a triply connected domain in $\mathbb{C}$.

In [55], Pal showed that every tetrablock-contraction can be uniquely written as a direct sum of a tetrablock-unitary and a completely non-unitary tetrablock-contraction. Moreover, for certain conditions on the fundamental operators of the tetrablock-contraction, he showed the tetrablock-contraction is the restriction to a common invariant subspace of the orthogonal direct sum of a tetrablock-unitary and a pure tetrablock-co-isometry.

Shyam Roy with G. Misra [IISc Bangalore, India] and G. Zheng [Chalmers and Gothenburg Universities, Sweden] studied in [49] a class of weighted Bergman spaces on the symmetrised polydisc that isometrically embed as a subspace in the corresponding weighted Bergman space on the polydisc. Using their embedding, the authors compute the kernel function for the weighted Bergman spaces on the symmetrised polydisc. In particular, they show that the collection of all these kernel functions contains the Szegő and Bergman kernels on the symmetrised polydisc.

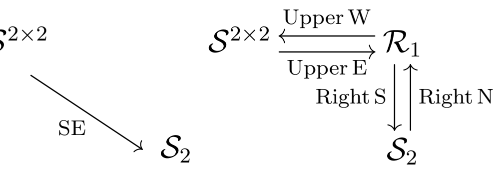
The theory of reproducing kernels and Hilbert function spaces is a useful tool in treating certain interpolation problems. One use comes from a result of Agler in [2], that functions in the Schur class of the bidisc have a realisation in terms of a pair of kernels. As the proof of this is non-constructive, these pairs have been studied by a number of authors in order to produce a canonical pair, for example, in [16] by Ball with C. Sadosky [Howard University, USA] and V. Vinnikov [Ben Gurion University of the Negev, Israel], in [22] by K. Bickel [Bucknell University, USA], and in [23] by Bickel with G. Knese [Washington University, USA].

In [6], Agler with J. E. McCarthy [Washington University, USA] developed an operator theoretic approach to interpolation problems of Pick type. The book is also a good introduction to the theory of reproducing kernels and Hilbert function spaces. Other authors have considered this topic too, for example, Ball with T. T. Trent [University of Alabama, USA] in [17], and Dritschel and McCullough in [35]. More recently, in [7], Agler and McCarthy obtained a criterion for solving a Pick interpolation problem on a basic open set, and its generalisation to extending bounded free holomorphic functions off free varieties. In addition, they give a description of all solutions of a solvable Pick problem.

The study of the inhomogeneous domains, and related topics, brings together a diverse range of researchers including pure and applied mathematicians, computer scientists and engineers. This is because, in addition to the interest from several complex variables and operator theory, there is interest in the applications of the area to problems in control theory. Evidence that the area has a thriving international community can be seen, for example, in the 2014 international workshop 'Function theory in several complex variables in relation to modelling uncertainty' at the ICMS in Edinburgh, which was ICMS/EPSRC/LMS/Newcastle University funded. The workshop was well attended and there were many interesting talks from specialists in mathematics and engineering. Details of the presentations given can be found on the workshop website: `http://www.icms.org.uk/workshops/functiontheory#presentations`.

Chapter 2. The realisation formula and kernels on $\mathbb{D}^2$

In this chapter, we construct maps between $\mathcal{S}^{2\times 2}$, $\mathcal{S}_2$ and $\mathcal{R}_1$. Where possible, we investigate how the maps interact with each other. We label the maps in accordance with the following diagrams:



The maps in this chapter are used in Chapter 3 to study the rich saltire in the case of the symmetrised bidisc, and in Chapter 4 to study the rich saltire in the case of the tetrablock, however they are independent of either set and so we have collected them in this chapter.

2.1 The realisation formula

Recall that $\mathcal{S}^{2\times 2}$ is the set of holomorphic functions $F : \mathbb{D} \to \mathcal{M}_2(\mathbb{C})$ such that $||F(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$. For $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$, we define a linear fractional transformation by

$$\mathcal{F}_{F(\lambda)}(z) := F_{11}(\lambda) + F_{12}(\lambda)z(1 - F_{22}(\lambda)z)^{-1}F_{21}(\lambda)$$

for all $z, \lambda \in \mathbb{D}$. Furthermore, we define two functions by

$$\gamma(\lambda, z) := (1 - F_{22}(\lambda)z)^{-1}F_{21}(\lambda) \text{ and } \eta(\lambda, z) := \begin{bmatrix} 1 \\ z\gamma(\lambda, z) \end{bmatrix}$$

for all $z, \lambda \in \mathbb{D}$. If $F_{21} = 0$ then γ is the zero map. Note that, for $z, \lambda \in \mathbb{D}$, since $|F_{22}(\lambda)| \leq 1$, we have $1 - F_{22}(\lambda)z \neq 0$, and so $\mathcal{F}_{F(\lambda)}(z)$, $\gamma(\lambda, z)$ and $\eta(\lambda, z)$ are all well defined.

Proposition 2.1.1. *Let* $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. *Then*

$$1 - \overline{\mathcal{F}_{F(\mu)}(w)}\mathcal{F}_{F(\lambda)}(z) = \overline{\gamma(\mu, w)}(1 - \overline{w}z)\gamma(\lambda, z) + \eta(\mu, w)^*(I - F(\mu)^*F(\lambda))\eta(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. *Moreover, for* $z, \lambda \in \mathbb{D}$, *we have* $\mathcal{F}_{F(\lambda)}(z)$ *is holomorphic and* $|\mathcal{F}_{F(\lambda)}(z)| \leq 1$.

Proof. The required equality follows immediately if we let $H = G = U = V = \mathbb{C}$, $P = F(\lambda)$, $Q = F(\mu)$, $X = z$ and $Y = w$ in Proposition B.2.1. Moreover, by Corollary B.2.2, since $||F(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$,

$$|\mathcal{F}_{F(\lambda)}(z)| \leq 1 \text{ for all } z, \lambda \in \mathbb{D}.$$

We note that, by Remark B.2.3, $1 - F_{22}(\lambda)z \neq 0$ for all $z, \lambda \in \mathbb{D}$. Now, by Remark B.2.4, since F is holomorphic on $\mathbb{D}$, we have $\mathcal{F}_{F(\lambda)}(z)$ is holomorphic. □

2.2 SE : $\mathcal{S}^{2\times 2} \to \mathcal{S}_2$

Recall that $\mathcal{S}_2$ is the set of holomorphic functions from $\mathbb{D}^2$ to $\overline{\mathbb{D}}$. Proposition 2.1.1 shows, for each $F \in \mathcal{S}^{2\times 2}$, there is such a function. This motivates the following definition.

Definition 2.2.1. *We define* SE : $\mathcal{S}^{2\times 2} \to \mathcal{S}_2$ *by*

$$F \mapsto \mathrm{SE}\,(F) : \mathbb{D}^2 \to \overline{\mathbb{D}}$$

for all $F \in \mathcal{S}^{2\times 2}$*, where*

$$\mathrm{SE}\,(F)(z, \lambda) := -\mathcal{F}_{F(\lambda)}(z) = -F_{11}(\lambda) - F_{12}(\lambda)z(1 - F_{22}(\lambda)z)^{-1}F_{21}(\lambda)$$

for all $z, \lambda \in \mathbb{D}$.

That SE is well defined follows immediately from Proposition 2.1.1.

Remark 2.2.2. Let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$. If either $F_{21} = 0$ or $F_{12} = 0$, then

$$\mathrm{SE}\,(F)(z, \lambda) = -\mathcal{F}_{F(\lambda)}(z) = -F_{11}(\lambda)$$

for all $z, \lambda \in \mathbb{D}$. In this case, SE (F) is independent of z and, in general, can lose information about F.

2.3 Upper E : $\mathcal{S}^{2\times 2} \to \mathcal{R}_1$

Recall that $\mathcal{R}_1$ is the set of pairs (N, M) of holomorphic kernels on $\mathbb{D}^2$ such that $K_{N,M}$ is a rank 1 kernel on $\mathbb{D}^2$, where $K_{N,M}$ is defined by

$$K_{N,M}(z, \lambda, w, \mu) = 1 - (1 - \overline{w}z)N(z, \lambda, w, \mu) - (1 - \overline{\mu}\lambda)M(z, \lambda, w, \mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. By Proposition 2.1.1,

$$1 - \overline{\mathcal{F}_{F(\mu)}(w)}\mathcal{F}_{F(\lambda)}(z) = \overline{\gamma(\mu, w)}(1 - \overline{w}z)\gamma(\lambda, z) + \eta(\mu, w)^*(I - F(\mu)^*F(\lambda))\eta(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$, where

$$\gamma(\lambda, z) = (1 - F_{22}(\lambda)z)^{-1}F_{21}(\lambda) \text{ and } \eta(\lambda, z) = \begin{bmatrix} 1 \\ z\gamma(\lambda, z) \end{bmatrix}$$

for all $z, \lambda \in \mathbb{D}$. Define $N_F : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ and $M_F : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ by

$$N_F(z, \lambda, w, \mu) := \overline{\gamma(\mu, w)}\gamma(\lambda, z) \text{ and } M_F(z, \lambda, w, \mu) := \eta(\mu, w)^* \frac{I - F(\mu)^* F(\lambda)}{1 - \overline{\mu}\lambda}\eta(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Clearly N_F and M_F are well defined. The following lemma motivates our definition of Upper E.

Proposition 2.3.1. *Let* $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$*. Then* N_F*,* M_F *and* K_{N_F,M_F} *are holomorphic kernels on* $\mathbb{D}^2$*. Moreover,*

$$K_{N_F,M_F}(z, \lambda, w, \mu) = \overline{\mathcal{F}_{F(\mu)}(w)}\mathcal{F}_{F(\lambda)}(z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$*, that is,* K_{N_F,M_F} *has rank* 1*.*

Proof. By Proposition B.3.22, since $\gamma : \mathbb{D}^2 \to \mathbb{C}$, we have N_F is a kernel on $\mathbb{D}^2$. By Corollary B.3.32, since $\eta : \mathbb{D}^2 \to \mathbb{C}^2$, we have M_F is a kernel on $\mathbb{D}^2$. By Proposition 2.1.1,

$$1 - \overline{\mathcal{F}_{F(\mu)}(w)}\mathcal{F}_{F(\lambda)}(z) = (1 - \overline{w}z)N_F(z, \lambda, w, \mu) + (1 - \overline{\mu}\lambda)M_F(z, \lambda, w, \mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Hence

$$K_{N_F,M_F}(z, \lambda, w, \mu) = \overline{\mathcal{F}_{F(\mu)}(w)}\mathcal{F}_{F(\lambda)}(z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. By Corollary B.3.23, K_{N_F,M_F} is a rank 1 kernel on $\mathbb{D}^2$. It is clear that N_F, M_F and K_{N_F,M_F} define holomorphic kernels. □

Remark 2.3.2. Let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. If $F_{21} = 0$, and so γ is the zero map, then $N_F = 0$ and is the trivial kernel on $\mathbb{D}^2$. If $F_{21} \neq 0$ then, by Corollary B.3.23, the kernel N_F has rank 1, since

$$N_F(z, \lambda, w, \mu) = \overline{\gamma(\mu, w)}\gamma(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$.

Definition 2.3.3. *We define* Upper E : $\mathcal{S}^{2\times 2} \to \mathcal{R}_1$ *by*

$$\text{Upper E}\,(F) = (N_F, M_F)$$

for all $F \in \mathcal{S}^{2\times 2}$*.*

That Upper E is well defined follows immediately from Proposition 2.3.1.

Remark 2.3.4. Let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. If $F_{21} = 0$ then $N_F = 0$ and

$$\begin{aligned} M_F(z,\lambda,w,\mu) &= \frac{1}{1-\overline{\mu}\lambda} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^* \begin{bmatrix} \overline{F_{11}(\mu)}F_{11}(\lambda) & \overline{F_{11}(\mu)}F_{12}(\lambda) \\ \overline{F_{12}(\mu)}F_{11}(\lambda) & \overline{F_{12}(\mu)}F_{12}(\lambda) + \overline{F_{22}(\mu)}F_{22}(\lambda) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1 - \overline{F_{11}(\mu)}F_{11}(\lambda)}{1-\overline{\mu}\lambda} \end{aligned}$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. If either $F_{21} = 0$ or $F_{12} = 0$, then, by Proposition 2.3.1 and Remark 2.2.2,

$$K_{N_F, M_F} = \overline{\mathcal{F}_{F(\mu)}(w)}\mathcal{F}_{F(\lambda)}(z) = \overline{F_{11}(\mu)}F_{11}(\lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Clearly, if $F_{21} = 0$ then we lose information about F when we pass to (N_F, M_F), since we only retain F_{11}.

Remark 2.3.5. We could consider the following alternative realisation formula. Let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. Define

$$\nu(\lambda, z) := F_{12}(\lambda)(1 - F_{11}(\lambda)z)^{-1} \text{ and } \upsilon(\lambda, z) := \begin{bmatrix} z\nu(\lambda, z) \\ 1 \end{bmatrix}$$

for all $z, \lambda \in \mathbb{D}$. If $F_{12} = 0$ then ν is the zero map. Let

$${}_FN(z,\lambda,w,\mu) = \overline{\nu(\mu, w)}\nu(\lambda, z) \text{ and } {}_FM(z,\lambda,w,\mu) = \upsilon(\mu,w)^* \frac{I - F(\mu)^*F(\lambda)}{1-\overline{\mu}\lambda}\upsilon(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Then, similarly to the proof of Proposition 2.3.1, it can be shown that $({}_FN, {}_FM) \in \mathcal{R}_1$. Similarly to Remark 2.3.4, if $F_{12} = 0$ then ${}_FN = 0$ and

$${}_FM(z,\lambda,w,\mu) = \frac{1 - \overline{F_{22}(\mu)}F_{22}(\lambda)}{1-\overline{\mu}\lambda}$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Moreover, if either $F_{12} = 0$ or $F_{21} = 0$, then

$$K_{{}_FN, {}_FM}(z,\lambda,w,\mu) = \overline{F_{22}(\mu)}F_{22}(\lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Thus we lose information about F if $F_{12} = 0$.

In this thesis, we consider the realisation formula as in Definition 2.2.1.

2.4 Upper W : $\mathcal{R}_{11} \to \mathcal{S}^{2\times 2}$

Let $F \in \mathcal{S}^{2\times 2}$. By Remark 2.3.2, there are two possibilities, either the kernel N_F is 0 or it has rank 1. As N_F is used to map F into $\mathcal{R}_1$, the image of Upper E is contained in the proper subset of $\mathcal{R}_1$ containing (N, M) such that either N is 0 or has rank 1. Clearly, we want Upper W to map from this subset, and so, for convenience, we separate into the two possibilities. Recall that

$$K_{N,M}(z,\lambda,w,\mu) = 1 - (1-\overline{w}z)N(z,\lambda,w,\mu) - (1-\overline{\mu}\lambda)M(z,\lambda,w,\mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Then we define two subsets of $\mathcal{R}_1$.

Definition 2.4.1. *We define $\mathcal{R}_{10} \subseteq \mathcal{R}_1$ by*

$$\mathcal{R}_{10} := \{(N,M) : N = 0; M, K_{N,M} \text{ are holomorphic kernels on } \mathbb{D}^2 \text{ and } K_{N,M} \text{ has rank } 1\}$$

and $\mathcal{R}_{11} \subseteq \mathcal{R}_1$ by

$$\mathcal{R}_{11} := \{(N,M) : N, M, K_{N,M} \text{ are holomorphic kernels on } \mathbb{D}^2 \text{ and } N, K_{N,M} \text{ have rank } 1\}.$$

In the proof of the next theorem, we describe the procedure by which we construct a function in $\mathcal{S}^{2\times 2}$ from a pair of kernels in $\mathcal{R}_{11}$, we refer to this as Procedure UW. First, we give a lemma which provides the existence of functions required in the construction.

Lemma 2.4.2. *Let $(N,M) \in \mathcal{R}_{11}$. Then $\mathcal{H}_N$ and $\mathcal{H}_{K_{N,M}}$ are 1-dimensional, and*

$$N(z,\lambda,w,\mu) = \overline{e_N(w,\mu)}e_N(z,\lambda) \text{ and } K_{N,M}(z,\lambda,w,\mu) = \overline{e_{K_{N,M}}(w,\mu)}e_{K_{N,M}}(z,\lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$, where $\{e_N\}$ and $\{e_{K_{N,M}}\}$ are orthonormal bases of $\mathcal{H}_N$ and $\mathcal{H}_{K_{N,M}}$, respectively.

Proof. Since $(N,M) \in \mathcal{R}_{11}$, we have N and $K_{N,M}$ are rank 1 kernels, and so $\mathcal{H}_N$ and $\mathcal{H}_{K_{N,M}}$ are 1-dimensional. Let $\{e_N\}$ and $\{e_{K_{N,M}}\}$ be orthonormal bases of $\mathcal{H}_N$ and $\mathcal{H}_{K_{N,M}}$, respectively. By Proposition B.3.6,

$$N(z,\lambda,w,\mu) = \overline{e_N(w,\mu)}e_N(z,\lambda) \text{ and } K_{N,M}(z,\lambda,w,\mu) = \overline{e_{K_{N,M}}(w,\mu)}e_{K_{N,M}}(z,\lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. □

Theorem 2.4.3. *Let $(N,M) \in \mathcal{R}_{11}$. Then there is a function $\Xi \in \mathcal{S}^{2\times 2}$ such that*

$$\Xi(\lambda)\begin{pmatrix} 1 \\ zf(z,\lambda) \end{pmatrix} = \begin{pmatrix} g(z,\lambda) \\ f(z,\lambda) \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$, where $f \in \mathcal{H}_N$ and $g \in \mathcal{H}_M$ are such that

$$N(z,\lambda,w,\mu) = \overline{f(w,\mu)}f(z,\lambda) \text{ and } K_{N,M}(z,\lambda,w,\mu) = \overline{g(w,\mu)}g(z,\lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$.

Proof. (Procedure UW). Let $(N, M) \in \mathcal{R}_{11}$. Lemma 2.4.2 guarantees the existence of $f \in \mathcal{H}_N$ and $g \in \mathcal{H}_M$ such that

$$N(z, \lambda, w, \mu) = \overline{f(w, \mu)} f(z, \lambda) \text{ and } K_{N,M}(z, \lambda, w, \mu) = \overline{g(w, \mu)} g(z, \lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. By Corollary B.3.8, the functions $v_{z,\lambda} = CM(\cdot, \cdot, z, \lambda) \in \mathcal{H}_M$ satisfy

$$M(z, \lambda, w, \mu) = \langle v_{z,\lambda}, v_{w,\mu} \rangle_{\mathcal{H}_M}$$

for all $z, \lambda, w, \mu \in \mathbb{D}$, where C is the conjugate linear operator. By the definition of $K_{N,M}$, we obtain

$$\overline{g(w, \mu)} g(z, \lambda) = 1 - (1 - \overline{w} z) \overline{f(w, \mu)} f(z, \lambda) - (1 - \overline{\mu} \lambda) \langle v_{z,\lambda}, v_{w,\mu} \rangle_{\mathcal{H}_M}$$

and hence

$$\overline{g(w, \mu)} g(z, \lambda) + \overline{f(w, \mu)} f(z, \lambda) + \langle v_{z,\lambda}, v_{w,\mu} \rangle_{\mathcal{H}_M} = 1 + \overline{w} z \overline{f(w, \mu)} f(z, \lambda) + \overline{\mu} \lambda \langle v_{z,\lambda}, v_{w,\mu} \rangle_{\mathcal{H}_M}$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. It follows that

$$\left\langle \begin{pmatrix} \begin{pmatrix} g(z,\lambda) \\ f(z,\lambda) \end{pmatrix} \\ v_{z,\lambda} \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} g(w,\mu) \\ f(w,\mu) \end{pmatrix} \\ v_{w,\mu} \end{pmatrix} \right\rangle_{\mathbb{C}^2 \oplus \mathcal{H}_M} = \left\langle \begin{pmatrix} \begin{pmatrix} 1 \\ z f(z,\lambda) \end{pmatrix} \\ \lambda v_{z,\lambda} \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} 1 \\ w f(w,\mu) \end{pmatrix} \\ \mu v_{w,\mu} \end{pmatrix} \right\rangle_{\mathbb{C}^2 \oplus \mathcal{H}_M}$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Hence, by Proposition B.1.22, there is an isometry

$$L_0 : \overline{\text{span}} \left\{ \begin{pmatrix} \begin{pmatrix} 1 \\ z f(z,\lambda) \end{pmatrix} \\ \lambda v_{z,\lambda} \end{pmatrix} : z, \lambda \in \mathbb{D} \right\} \to \mathbb{C}^2 \oplus \mathcal{H}_M$$

such that

$$L_0 \begin{pmatrix} \begin{pmatrix} 1 \\ z f(z,\lambda) \end{pmatrix} \\ \lambda v_{z,\lambda} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} g(z,\lambda) \\ f(z,\lambda) \end{pmatrix} \\ v_{z,\lambda} \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$. As the proof of Proposition B.1.22 is constructive, this isometry is uniquely defined. We extend L_0 to a contraction L on $\mathbb{C}^2 \oplus \mathcal{H}_M$ by defining L to be 0 on $(\overline{\text{span}}\{\begin{pmatrix} \begin{pmatrix} 1 \\ z f(z,\lambda) \end{pmatrix} \\ \lambda v_{z,\lambda} \end{pmatrix} : z, \lambda \in \mathbb{D}\})^{\perp}$. If we write

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{C}^2 \oplus \mathcal{H}_M \to \mathbb{C}^2 \oplus \mathcal{H}_M,$$

then $A : \mathbb{C}^2 \to \mathbb{C}^2$, $B : \mathcal{H}_M \to \mathbb{C}^2$, $C : \mathbb{C}^2 \to \mathcal{H}_M$ and $D : \mathcal{H}_M \to \mathcal{H}_M$ satisfy

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \left(\begin{pmatrix} 1 \\ zf(z,\lambda) \end{pmatrix} \atop \lambda v_{z,\lambda} \right) = \left(\begin{pmatrix} g(z,\lambda) \\ f(z,\lambda) \end{pmatrix} \atop v_{z,\lambda} \right)$$

for all $z, \lambda \in \mathbb{D}$. By expanding the last equality, we obtain

$$A \begin{pmatrix} 1 \\ zf(z,\lambda) \end{pmatrix} + B\lambda v_{z,\lambda} = \begin{pmatrix} g(z,\lambda) \\ f(z,\lambda) \end{pmatrix}$$

and

$$C \begin{pmatrix} 1 \\ zf(z,\lambda) \end{pmatrix} + D\lambda v_{z,\lambda} = v_{z,\lambda}$$

for all $z, \lambda \in \mathbb{D}$. By Remark B.2.3, since L is a contraction, $I_{\mathcal{H}_M} - D\lambda$ is invertible for all $\lambda \in \mathbb{D}$. Thus, we can write

$$(I_{\mathcal{H}_M} - D\lambda)^{-1} C \begin{pmatrix} 1 \\ zf(z,\lambda) \end{pmatrix} = v_{z,\lambda}$$

for all $z, \lambda \in \mathbb{D}$. It follows that

$$\left(A + B\lambda(I_{\mathcal{H}_M} - D\lambda)^{-1} C\right) \begin{pmatrix} 1 \\ zf(z,\lambda) \end{pmatrix} = \begin{pmatrix} g(z,\lambda) \\ f(z,\lambda) \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$. Since L is a contraction, by Corollary B.2.2 and Remark B.2.4,

$$||\mathcal{F}_L(\lambda)|| = ||A + B\lambda(I_{\mathcal{H}_M} - D\lambda)^{-1} C|| \leq 1 \text{ for all } \lambda \in \mathbb{D},$$

and $\mathcal{F}_L(\lambda) = A + B\lambda(I_{\mathcal{H}_M} - D\lambda)^{-1} C$ is holomorphic on $\mathbb{D}$. Set

$$\Xi(\lambda) = A + B\lambda(I_{\mathcal{H}_M} - D\lambda)^{-1} C$$

for all $\lambda \in \mathbb{D}$. Since A and $B\lambda(I_{\mathcal{H}_M} - D\lambda)^{-1} C$ are operators from $\mathbb{C}^2$ to $\mathbb{C}^2$, we have $\Xi \in \mathcal{S}^{2\times 2}$. Moreover, $\Xi(\lambda)$ satisfies the required identity for all $\lambda \in \mathbb{D}$. □

Remark 2.4.4. We could apply Procedure UW to a pair $(N, M) \in \mathcal{R}_{10}$ by taking the representation f of N to be 0. From this we would obtain a function $\Xi \in \mathcal{S}^{2\times 2}$ such that

$$\Xi(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} g(z,\lambda) \\ 0 \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$. If we let $\Xi = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then it follows easily that $a = g(z, \cdot)$ and $c = 0$. There is not much we can say about b and d beyond that they must be such that $\Xi \in \mathcal{S}^{2\times 2}$.

Since the representations f and g used in Procedure UW are not unique, the function Ξ may not be unique. However, we can say something about the relationship between Ξ and another function obtained by Procedure UW using different representations.

Proposition 2.4.5. *Let $(N, M) \in \mathcal{R}_{11}$. Let $f_1, f_2 \in \mathcal{H}_N$; $g_1, g_2 \in \mathcal{H}_{K_{N,M}}$ and $v_1, v_2 : X \to \mathcal{H}_M$ be such that*

$$N(z, \lambda, w, \mu) = \overline{f_1(w, \mu)} f_1(z, \lambda) = \overline{f_2(w, \mu)} f_2(z, \lambda),$$

$$K_{N,M}(z, \lambda, w, \mu) = \overline{g_1(w, \mu)} g_1(z, \lambda) = \overline{g_2(w, \mu)} g_2(z, \lambda)$$

and

$$M(z, \lambda, w, \mu) = \langle v_1(z, \lambda), v_1(w, \mu)\rangle_{\mathcal{H}_M} = \langle v_2(z, \lambda), v_2(w, \mu)\rangle_{\mathcal{H}_M}$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Let Ξ_1 and Ξ_2 be constructed from (N, M) by Procedure UW using f_1, g_1, v_1 and f_2, g_2, v_2, respectively. Then

$$\Xi_2 = \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi_1 \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix}$$

for some $\zeta_1, \zeta_2 \in \mathbb{T}$.

Proof. By Theorem 2.4.3, we have

$$\Xi_1(\lambda) \begin{pmatrix} 1 \\ z f_1(z, \lambda) \end{pmatrix} = \begin{pmatrix} g_1(z, \lambda) \\ f_1(z, \lambda) \end{pmatrix} \text{ and } \Xi_2(\lambda) \begin{pmatrix} 1 \\ z f_2(z, \lambda) \end{pmatrix} = \begin{pmatrix} g_2(z, \lambda) \\ f_2(z, \lambda) \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$. By Proposition B.1.30, we have $f_2 = \zeta_f f_1$ and $g_2 = \zeta_g g_1$ for some $\zeta_f, \zeta_g \in \mathbb{T}$. Hence

$$\Xi_2(\lambda) \begin{pmatrix} 1 \\ z f_2(z, \lambda) \end{pmatrix} = \Xi_2(\lambda) \begin{bmatrix} 1 & 0 \\ 0 & \zeta_f \end{bmatrix} \begin{pmatrix} 1 \\ z f_1(z, \lambda) \end{pmatrix}$$

and

$$\begin{pmatrix} g_2(z, \lambda) \\ f_2(z, \lambda) \end{pmatrix} = \begin{bmatrix} \zeta_g & 0 \\ 0 & \zeta_f \end{bmatrix} \begin{pmatrix} g_1(z, \lambda) \\ f_1(z, \lambda) \end{pmatrix} = \begin{bmatrix} \zeta_g & 0 \\ 0 & \zeta_f \end{bmatrix} \Xi_1(\lambda) \begin{pmatrix} 1 \\ z f_1(z, \lambda) \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$. By subtracting the two equations, we obtain

$$0 = \left(\Xi_2(\lambda) \begin{bmatrix} 1 & 0 \\ 0 & \zeta_f \end{bmatrix} - \begin{bmatrix} \zeta_g & 0 \\ 0 & \zeta_f \end{bmatrix} \Xi_1(\lambda) \right) \begin{pmatrix} 1 \\ z f_1(z, \lambda) \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$. Set

$$\Xi_2(\lambda) \begin{bmatrix} 1 & 0 \\ 0 & \zeta_f \end{bmatrix} - \begin{bmatrix} \zeta_g & 0 \\ 0 & \zeta_f \end{bmatrix} \Xi_1(\lambda) := A(\lambda) = \begin{bmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{bmatrix}$$

for all $\lambda \in \mathbb{D}$. Then

$$a_{11}(\lambda) + a_{12}(\lambda)zf_1(z,\lambda) = 0 \text{ and } a_{21}(\lambda) + a_{22}(\lambda)zf_1(z,\lambda) = 0$$

for all $z, \lambda \in \mathbb{D}$. Letting $z = 0$, we obtain $a_{11}(\lambda) = 0 = a_{21}(\lambda)$ for all $\lambda \in \mathbb{D}$. Let $z, \lambda \in \mathbb{D}$ be such that $zf_1(z,\lambda) \neq 0$. Then $a_{12}(\lambda) = 0 = a_{22}(\lambda)$, and hence $A(\lambda) = 0$.

Conversely, let $z, \lambda \in \mathbb{D}$ be such that $zf_1(z,\lambda) = 0$. Since N is a rank 1 holomorphic kernel on $\mathbb{D}^2$, we have $f_1 \neq 0$ and, by Proposition B.3.12, f_1 is holomorphic on $\mathbb{D}^2$. Thus, by Corollary B.1.28, there is a sequence $(z_i, \lambda_i)_{i=1}^{\infty}$ in $\mathbb{D}^2$ such that

$$\lim_{i\to\infty} (z_i, \lambda_i) = (z, \lambda),$$

and $z_i f_1(z_i, \lambda_i) \neq 0$ for each $i \in \mathbb{N}$. It follows that $A(\lambda_i) = 0$ for each $i \in \mathbb{N}$, and, since A is holomorphic on $\mathbb{D}$, we have $A(\lambda) = \lim_{i\to\infty} A(\lambda_i) = 0$. Consequently, $A(\lambda) = 0$ for all $\lambda \in \mathbb{D}$, that is,

$$\Xi_2(\lambda) = \begin{bmatrix} \zeta_g & 0 \\ 0 & \zeta_f \end{bmatrix} \Xi_1(\lambda) \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_f} \end{bmatrix} \text{ for all } \lambda \in \mathbb{D},$$

where $\zeta_f, \zeta_g \in \mathbb{T}$. □

Remark 2.4.6. Let $(N, M) \in \mathcal{R}_{10}$. Let $f : (z,\lambda) \mapsto 0$ for all $z, \lambda \in \mathbb{D}$, so that $N(z,\lambda,w,\mu) = \overline{f(w,\mu)}f(z,\lambda) = 0$ for all $z, \lambda, w, \mu \in \mathbb{D}$. Moreover, let $v_1, v_2 : \mathbb{D}^2 \to \mathcal{H}_M$ and $g_1, g_2 \in \mathcal{H}_{K_{N,M}}$ be such that

$$M(z,\lambda,w,\mu) = \langle v_1(z,\lambda), v_1(w,\mu)\rangle_{\mathcal{H}_M} = \langle v_2(z,\lambda), v_2(w,\mu)\rangle_{\mathcal{H}_M}$$

and

$$K_{N,M}(z,\lambda,w,\mu) = \overline{g_1(w,\mu)}g_1(z,\lambda) = \overline{g_2(w,\mu)}g_2(z,\lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Then, by Proposition B.1.30, $g_2 = \zeta g_1$ for some $\zeta \in \mathbb{T}$. Let Ξ_1 and Ξ_2 be constructed from (N, M) by Procedure UW using f, g_1, v_1 and f, g_2, v_2, respectively. Then, by Remark 2.4.4,

$$\Xi_1 = \begin{bmatrix} g_1(z,\cdot) & b_1 \\ 0 & d_1 \end{bmatrix} \text{ and } \Xi_2 = \begin{bmatrix} \zeta g_1(z,\cdot) & b_2 \\ 0 & d_2 \end{bmatrix}$$

for all $z \in \mathbb{D}$. There is not much we can say about b_1, d_1 and b_2, d_2 as we do not have an effective way to compare them.

From Proposition 2.4.5 we obtain the following result, which motivates our definition of Upper W.

Proposition 2.4.7. *Let $(N, M) \in \mathcal{R}_{11}$. Let Ξ be constructed from (N, M) by Proce-*

dure UW. Then

$$\left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} : \zeta_1, \zeta_2 \in \mathbb{T} \right\} \text{ is a subset of } \mathcal{S}^{2\times 2},$$

and is the set of all functions that can be constructed from (N, M) by Procedure UW. Moreover, it is independent of which function Ξ is used to define it.

Proof. By Proposition 2.4.5, any function constructed from (N, M) by Procedure UW belongs to the set. To see that any function in the set can be constructed from (N, M) by Procedure UW, let $\zeta_1, \zeta_2 \in \mathbb{T}$ and consider the function

$$\begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix}.$$

Let f and g be the representations of N and $K_{N,M}$, respectively, used in Procedure UW to construct Ξ. By the proof of Proposition 2.4.5, applying Procedure UW to (N, M) using the representations $\zeta_2 f$ and $\zeta_1 g$, we obtain the function

$$\begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix}.$$

Hence any function in the set can be constructed from (N, M) by Procedure UW, and so, in addition, the set is contained in $\mathcal{S}^{2\times 2}$.

For the last statement, suppose Ξ_1 is any other function constructed from (N, M) by Procedure UW. Then, by Proposition 2.4.5,

$$\Xi_1 = \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\xi_2} \end{bmatrix}$$

for some $\xi_1, \xi_2 \in \mathbb{T}$. Let $\zeta_1, \zeta_2 \in \mathbb{T}$. Then

$$\begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi_1 \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \begin{bmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\xi_2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} = \begin{bmatrix} \zeta_1\xi_1 & 0 \\ 0 & \zeta_2\xi_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2\xi_2} \end{bmatrix},$$

where $\zeta_1\xi_1, \zeta_2\xi_2 \in \mathbb{T}$. Similarly,

$$\begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \begin{bmatrix} \overline{\xi_1} & 0 \\ 0 & \overline{\xi_2} \end{bmatrix} \Xi_1 \begin{bmatrix} 1 & 0 \\ 0 & \xi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} = \begin{bmatrix} \zeta_1\overline{\xi_1} & 0 \\ 0 & \zeta_2\overline{\xi_2} \end{bmatrix} \Xi_1 \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2\overline{\xi_2}} \end{bmatrix},$$

where $\zeta_1\overline{\xi_1}, \zeta_2\overline{\xi_2} \in \mathbb{T}$. It follows that

$$\left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} : \zeta_1, \zeta_2 \in \mathbb{T} \right\} = \left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi_1 \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} : \zeta_1, \zeta_2 \in \mathbb{T} \right\},$$

and so the set is independent of which function Ξ is used to define it. □

Remark 2.4.8. Although the result of Proposition 2.4.7 may not hold for $(N, M) \in \mathcal{R}_{10}$, we can say the following. Let $(N, M) \in \mathcal{R}_{10}$, and let $g \in \mathcal{H}_{K_{N,M}}$ be such that

$$K_{N,M}(z, \lambda, w, \mu) = \overline{g(w, \mu)}g(z, \lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Then the set of all functions that can be constructed from (N, M) by Procedure UW is contained in the set

$$\left\{ \Xi = \begin{bmatrix} \zeta g(z, \cdot) & b \\ 0 & d \end{bmatrix} : \zeta \in \mathbb{T};\ b, d \text{ are functions such that } \Xi \in \mathcal{S}^{2\times 2} \right\}.$$

Moreover, by Remark 2.4.6, any function constructed from (N, M) by Procedure UW has the form $\Xi = \begin{bmatrix} \zeta g(z, \cdot) & b \\ 0 & d \end{bmatrix}$ for some $\zeta \in \mathbb{T}$, where b, d are functions such that $\Xi \in \mathcal{S}^{2\times 2}$. Hence, this set is independent of which function g is used to define it.

Definition 2.4.9. *We define* Upper W *as the set map from* $\mathcal{R}_{10} \cup \mathcal{R}_{11}$ *to* $\mathcal{S}^{2\times 2}$ *given in following way. For* $(N, M) \in \mathcal{R}_{10}$*,*

$$\text{Upper W}\,((N, M)) = \left\{ \Xi = \begin{bmatrix} \zeta g(z, \cdot) & b \\ 0 & d \end{bmatrix} : \zeta \in \mathbb{T};\ b, d \textit{ are functions such that } \Xi \in \mathcal{S}^{2\times 2} \right\},$$

where $g \in \mathcal{H}_{K_{N,M}}$ *is such that* $K_{N,M}(z, \lambda, w, \mu) = \overline{g(w, \mu)}g(z, \lambda)$ *for all* $z, \lambda, w, \mu \in \mathbb{D}$*. For* $(N, M) \in \mathcal{R}_{11}$*,*

$$\text{Upper W}\,((N, M)) = \left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} : \zeta_1, \zeta_2 \in \mathbb{T} \right\},$$

where $\Xi \in \mathcal{S}^{2\times 2}$ *is constructed from* (N, M) *by Procedure UW.*

That Upper W is well defined follows immediately from Proposition 2.4.7 and Remark 2.4.8, since $\mathcal{R}_{10} \cap \mathcal{R}_{11} = \emptyset$. We now look at how this map interacts with Upper E.

Proposition 2.4.10. *Let* $(N, M) \in \mathcal{R}_{11}$*. Then, for all* $F \in \text{Upper W}\,((N, M))$*, we have*

$$\text{Upper E}\,(F) = (N, M).$$

Proof. Let $\Xi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{S}^{2\times 2}$ be constructed from (N, M) by Procedure UW, and let $F \in \text{Upper W}\,((N, M))$. Then

$$F = \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} = \begin{bmatrix} \zeta_1 a & \overline{\zeta_2}\zeta_1 b \\ \zeta_2 c & d \end{bmatrix}$$

for some $\zeta_1, \zeta_2 \in \mathbb{T}$. Hence

$$N_F(z,\lambda,w,\mu) = \overline{\frac{\zeta_2 c(\mu)}{1-d(\mu)w}} \cdot \frac{\zeta_2 c(\lambda)}{1-d(\lambda)z} = \overline{\frac{c(\mu)}{1-d(\mu)w}} \cdot \frac{c(\lambda)}{1-d(\lambda)z} = N_\Xi(z,\lambda,w,\mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Moreover,

$$\begin{aligned} M_F(z,\lambda,w,\mu) &= \begin{bmatrix} 1 & \frac{\overline{w}\,\overline{\zeta_2}\,\overline{c(\mu)}}{1-\overline{d(\mu)}\,\overline{w}} \end{bmatrix} \frac{I - F(\mu)^* F(\lambda)}{1-\overline{\mu}\lambda} \begin{bmatrix} 1 \\ \frac{z\zeta_2 c(\lambda)}{1-d(\lambda)z} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{\overline{w}\,\overline{\zeta_2}\,\overline{c(\mu)}}{1-\overline{d(\mu)}\,\overline{w}} \end{bmatrix} \frac{I - \begin{bmatrix} 1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi(\mu)^* \begin{bmatrix} \overline{\zeta_1} & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi(\lambda) \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix}}{1-\overline{\mu}\lambda} \begin{bmatrix} 1 \\ \frac{z\zeta_2 c(\lambda)}{1-d(\lambda)z} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{\overline{w}\,\overline{\zeta_2}\,\overline{c(\mu)}}{1-\overline{d(\mu)}\,\overline{w}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \frac{I - \Xi(\mu)^*\Xi(\lambda)}{1-\overline{\mu}\lambda} \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{z\zeta_2 c(\lambda)}{1-d(\lambda)z} \end{bmatrix} \end{aligned}$$

and so

$$M_F(z,\lambda,w,\mu) = \begin{bmatrix} 1 & \frac{\overline{w}\,\overline{c(\mu)}}{1-\overline{d(\mu)}\,\overline{w}} \end{bmatrix} \frac{I - \Xi(\mu)^*\Xi(\lambda)}{1-\overline{\mu}\lambda} \begin{bmatrix} 1 \\ \frac{z c(\lambda)}{1-d(\lambda)z} \end{bmatrix} = M_\Xi(z,\lambda,w,\mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. It follows that $\operatorname{Upper E}(F) = (N_F, M_F) = (N_\Xi, M_\Xi)$.

If $N_\Xi = N$ and $M_\Xi = M$, then we have $\operatorname{Upper E}(F) = (N, M)$. Let f and g be the representations of N and $K_{N,M}$, respectively, used in the construction of Ξ. Then

$$N(z,\lambda,w,\mu) = \overline{f(w,\mu)} f(z,\lambda) \text{ and } K_{N,M}(z,\lambda,w,\mu) = \overline{g(w,\mu)} g(z,\lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Thus, by Theorem 2.4.3,

$$\Xi(\lambda) \begin{pmatrix} 1 \\ z f(z,\lambda) \end{pmatrix} = \begin{pmatrix} g(z,\lambda) \\ f(z,\lambda) \end{pmatrix}$$

and so

$$a(\lambda) + b(\lambda) z f(z,\lambda) = g(z,\lambda) \text{ and } c(\lambda) + d(\lambda) z f(z,\lambda) = f(z,\lambda)$$

for all $z, \lambda \in \mathbb{D}$. Since $\Xi \in \mathcal{S}^{2\times 2}$, we have $|d(\lambda)| \le 1$ for all $\lambda \in \mathbb{D}$, and so $1 - d(\lambda)z \neq 0$ for all $z, \lambda \in \mathbb{D}$. Hence

$$\frac{c(\lambda)}{1-d(\lambda)z} = f(z,\lambda) \text{ and } \mathcal{F}_{\Xi(\lambda)}(z) = a(\lambda) + b(\lambda) z (1-d(\lambda)z)^{-1} c(\lambda) = g(z,\lambda)$$

for all $z, \lambda \in \mathbb{D}$. It follows that

$$N_\Xi(z,\lambda,w,\mu) = \overline{f(w,\mu)} f(z,\lambda) = N(z,\lambda,w,\mu)$$

and

$$\overline{\mathcal{F}_{\Xi(\mu)}(w)}\mathcal{F}_{\Xi(\lambda)}(z) = \overline{g(w,\mu)}g(z,\lambda) = K_{N,M}(z,\lambda,w,\mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. By Proposition 2.1.1,

$$1 - \overline{\mathcal{F}_{\Xi(\mu)}(w)}\mathcal{F}_{\Xi(\lambda)}(z) = (1-\overline{w}z)N_{\Xi}(z,\lambda,w,\mu) + (1-\overline{\mu}\lambda)M_{\Xi}(z,\lambda,w,\mu)$$

and so

$$K_{N,M}(z,\lambda,w,\mu) = 1 - (1-\overline{w}z)N(z,\lambda,w,\mu) - (1-\overline{\mu}\lambda)M_{\Xi}(z,\lambda,w,\mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. It follows that $M_{\Xi}(z,\lambda,w,\mu) = M(z,\lambda,w,\mu)$ for all $z, \lambda, w, \mu \in \mathbb{D}$. Thus Upper E $(F) = (N, M)$. □

Remark 2.4.11. Let $(N, M) \in \mathcal{R}_{10}$. Then, for all $F \in$ Upper W $((N, M))$, we have Upper E $(F) = (N, M)$. Indeed, let $g \in \mathcal{H}_{K_{N,M}}$ be such that

$$K_{N,M}(z,\lambda,w,\mu) = \overline{g(w,\mu)}g(z,\lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$, and let $F \in$ Upper W $((N, M))$. Then, for some $\zeta \in \mathbb{T}$, we can write

$$F = \begin{bmatrix} \zeta g(z,\cdot) & b \\ 0 & d \end{bmatrix},$$

where b and d are functions such that $F \in \mathcal{S}^{2\times 2}$. By Remark 2.3.4, $N_F = 0 = N$ and

$$M_F(z,\lambda,w,\mu) = \frac{1 - \overline{g(w,\mu)}g(z,\lambda)}{1-\overline{\mu}\lambda} = \frac{1 - K_{N,M}(z,\lambda,w,\mu)}{1-\overline{\mu}\lambda} = M(z,\lambda,w,\mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Hence Upper E $(F) = (N_F, M_F) = (N, M)$, as required.

Proposition 2.4.12. *Let* $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$ *be such that* $F_{21} \neq 0$*. Then*

$$\text{Upper W} \circ \text{Upper E}\,(F) = \left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} F \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} : \zeta_1, \zeta_2 \in \mathbb{T} \right\}.$$

Proof. We have Upper E $(F) = (N_F, M_F)$, where

$$N_F(z,\lambda,w,\mu) = \overline{\frac{F_{21}(\mu)}{1-F_{22}(\mu)w}} \cdot \frac{F_{21}(\lambda)}{1-F_{22}(\lambda)z}$$

and

$$M_F(z,\lambda,w,\mu) = \begin{bmatrix} 1 & \dfrac{\overline{wF_{21}(\mu)}}{1-\overline{F_{22}(\mu)w}} \end{bmatrix} \frac{I - F(\mu)^*F(\lambda)}{1-\overline{\mu}\lambda} \begin{bmatrix} 1 \\ \dfrac{zF_{21}(\lambda)}{1-F_{22}(\lambda)z} \end{bmatrix}$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. By Proposition 2.3.1, $K_{N_F,M_F}(z,\lambda,w,\mu) = \overline{\mathcal{F}_{F(\mu)}(w)}\mathcal{F}_{F(\lambda)}(z)$ for all

$z, \lambda, w, \mu \in \mathbb{D}$. Let $\Xi \in \mathcal{S}^{2\times 2}$ be constructed from (N_F, M_F) by Procedure UW using the representations

$$(z, \lambda) \mapsto \frac{F_{21}(\lambda)}{1 - F_{22}(\lambda) z}$$

of N_F, and $(z, \lambda) \mapsto \mathcal{F}_{F(\lambda)}(z)$ of K_{N_F, M_F}. Then, by Theorem 2.4.3,

$$\Xi(\lambda) \begin{pmatrix} 1 \\ \dfrac{z F_{21}(\lambda)}{1 - F_{22}(\lambda) z} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{F(\lambda)}(z) \\ \dfrac{F_{21}(\lambda)}{1 - F_{22}(\lambda) z} \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$. Moreover,

$$F(\lambda) \begin{pmatrix} 1 \\ \dfrac{z F_{21}(\lambda)}{1 - F_{22}(\lambda) z} \end{pmatrix} = \begin{pmatrix} F_{11}(\lambda) + \dfrac{F_{12}(\lambda) z F_{21}(\lambda)}{1 - F_{22}(\lambda) z} \\ F_{21}(\lambda) + \dfrac{F_{22}(\lambda) z F_{21}(\lambda)}{1 - F_{22}(\lambda) z} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{F(\lambda)}(z) \\ \dfrac{F_{21}(\lambda)}{1 - F_{22}(\lambda) z} \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$. Hence

$$(\Xi(\lambda) - F(\lambda)) \begin{pmatrix} 1 \\ \dfrac{z F_{21}(\lambda)}{1 - F_{22}(\lambda) z} \end{pmatrix} = 0$$

for all $z, \lambda \in \mathbb{D}$. Set $\Xi(\lambda) - F(\lambda) := A(\lambda) = \begin{bmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{bmatrix}$ for all $\lambda \in \mathbb{D}$. Then

$$a_{11}(\lambda) + a_{12}(\lambda) \frac{z F_{21}(\lambda)}{1 - F_{22}(\lambda) z} = 0 \text{ and } a_{21}(\lambda) + a_{22}(\lambda) \frac{z F_{21}(\lambda)}{1 - F_{22}(\lambda) z} = 0$$

for all $z, \lambda \in \mathbb{D}$. Letting $z = 0$, we obtain $a_{11}(\lambda) = 0 = a_{21}(\lambda)$ for all $\lambda \in \mathbb{D}$.

Fix $0 \neq z \in \mathbb{D}$, and suppose $\lambda \in \mathbb{D}$ is such that $F_{21}(\lambda) \neq 0$. Then $a_{12}(\lambda) = 0 = a_{22}(\lambda)$, and hence $A(\lambda) = 0$. Now suppose $\lambda \in \mathbb{D}$ is such that $F_{21}(\lambda) = 0$. By Theorem B.1.25, since F_{21} is a non-zero holomorphic function on $\mathbb{D}$, the zeros of F_{21} are isolated. Thus there is a sequence $(\lambda_i)_{i=1}^{\infty}$ in $\mathbb{D}$ such that

$$\lim_{i \to \infty} \lambda_i = \lambda,$$

and $F_{21}(\lambda_i) \neq 0$ for each $i \in \mathbb{N}$. Since $A(\lambda_i) = 0$ for each $i \in \mathbb{N}$, and A is holomorphic on $\mathbb{D}$, we have $A(\lambda) = \lim_{i \to \infty} A(\lambda_i) = 0$. It follows that $A(\lambda) = 0$ for all $\lambda \in \mathbb{D}$, and hence $\Xi(\lambda) = F(\lambda)$ for all $\lambda \in \mathbb{D}$. Consequently,

$$\text{Upper W} \circ \text{Upper E}\,(F) = \text{Upper W}\,((N_F, M_F)) = \left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} F \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} : \zeta_1, \zeta_2 \in \mathbb{T} \right\},$$

as required. □

In Proposition 2.4.12, the reason we require $F_{21} \neq 0$ is: we lose information about F when we pass to (N_F, M_F) in the case that $F_{21} = 0$ (see Remark 2.3.4).

2.5 Right S : $\mathcal{R}_1 \to \mathcal{S}_2$

Recall that

$$\mathcal{R}_1 := \{(N, M) : N, M, K_{N,M} \text{ are holomorphic kernels on } \mathbb{D}^2 \text{ and } K_{N,M} \text{ has rank } 1\},$$

where

$$K_{N,M}(z, \lambda, w, \mu) = 1 - (1 - \overline{w}z)N(z, \lambda, w, \mu) - (1 - \overline{\mu}\lambda)M(z, \lambda, w, \mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$.

Definition 2.5.1. *We define* Right S *to be the set-valued map from* $\mathcal{R}_1$ *to* $\mathcal{S}_2$ *given by*

$$\text{Right S}\,(N, M) = \{\zeta f_{N,M} : \zeta \in \mathbb{T}\}$$

for all $(N, M) \in \mathcal{R}_1$*, where* $f_{N,M} : \mathbb{D}^2 \to \mathbb{C}$ *is holomorphic and satisfies*

$$K_{N,M}(z, \lambda, w, \mu) = \overline{f_{N,M}(w, \mu)} f_{N,M}(z, \lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$.

Proposition 2.5.2. Right S *is well defined.*

Proof. Let $(N, M) \in \mathcal{R}_1$. Then, by Proposition B.3.10, there is a holomorphic $f : \mathbb{D}^2 \to \mathbb{C}$ such that

$$K_{N,M}(z, \lambda, w, \mu) = \overline{f(w, \mu)} f(z, \lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Suppose g is another such function. Then, by Proposition B.1.30, $g = \xi f$ for some $\xi \in \mathbb{T}$. It follows that

$$\{\zeta f : \zeta \in \mathbb{T}\} = \{\zeta g : \zeta \in \mathbb{T}\},$$

and so Right S (N, M) is independent of which function f is used to define it.

Now, let $\zeta \in \mathbb{T}$. Then $\zeta f : \mathbb{D}^2 \to \mathbb{C}$ is holomorphic and, by Corollary B.1.6, since

$$1 - \overline{f(w, \mu)} f(z, \lambda) = (1 - \overline{w}z)N(z, \lambda, w, \mu) + (1 - \overline{\mu}\lambda)M(z, \lambda, w, \mu) \geq 0$$

for all $z, \lambda, w, \mu \in \mathbb{D}$, we have

$$|\zeta f(z, \lambda)| = |f(z, \lambda)| \leq 1$$

for all $z, \lambda \in \mathbb{D}$. Hence $\zeta f \in \mathcal{S}_2$, and it follows that Right S is well defined. $\square$

We now consider how Right S interacts with some of the other maps we defined in this chapter.

Proposition 2.5.3. *Let $F \in \mathcal{S}^{2\times 2}$. Then*

$$\text{Right S} \circ \text{Upper E}(F) = \{\zeta \, \text{SE}(F) : \zeta \in \mathbb{T}\}.$$

Proof. We have $\text{SE}(F)(z,\lambda) = -\mathcal{F}_{F(\lambda)}(z)$ for all $z, \lambda \in \mathbb{D}$. Moreover, $\text{Upper E}(F) = (N_F, M_F)$, where, by Proposition 2.3.1,

$$K_{N_F,M_F}(z,\lambda,w,\mu) = \overline{\mathcal{F}_{F(\mu)}(w)}\mathcal{F}_{F(\lambda)}(z) = \overline{(-\mathcal{F}_{F(\mu)}(w))}(-\mathcal{F}_{F(\lambda)}(z))$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. It follows that

$$\text{Right S} \circ \text{Upper E}(F) = \text{Right S}((N_F, M_F)) = \{\zeta \, \text{SE}(F) : \zeta \in \mathbb{T}\},$$

as required. □

Proposition 2.5.4. *Let $(N, M) \in \mathcal{R}_{11}$. Then*

$$\text{Right S}((N, M)) = \{\text{SE}(F) : F \in \text{Upper W}((N, M))\}.$$

Proof. Let $\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix}$ be constructed from (N, M) by Procedure UW. Then

$$\text{Upper W}((N, M)) = \left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} : \zeta_1, \zeta_2 \in \mathbb{T} \right\}$$

and

$$\begin{aligned} \text{SE}\left(\begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} \right)(z,\lambda) =& \text{SE}\left(\begin{bmatrix} \zeta_1 \Xi_{11} & \zeta_1 \overline{\zeta_2} \Xi_{12} \\ \zeta_2 \Xi_{21} & \Xi_{22} \end{bmatrix} \right)(z,\lambda) \\ =& -\zeta_1 \Xi_{11}(\lambda) - \frac{\zeta_1 \overline{\zeta_2} \Xi_{12}(\lambda) \zeta_2 \Xi_{21}(\lambda) z}{1 - \Xi_{22}(\lambda) z} \\ =& \zeta_1 \left(-\Xi_{11}(\lambda) - \frac{\Xi_{12}(\lambda)\Xi_{21}(\lambda) z}{1 - \Xi_{22}(\lambda) z} \right) = \zeta_1 \, \text{SE}(\Xi)(z,\lambda) \end{aligned}$$

for all $z, \lambda \in \mathbb{D}$ and $\zeta_1, \zeta_2 \in \mathbb{T}$. It follows that

$$\{\text{SE}(F) : F \in \text{Upper W}((N, M))\} = \{\zeta \, \text{SE}(\Xi) : \zeta \in \mathbb{T}\}.$$

Hence, by Proposition 2.5.3,

$$\text{Right S} \circ \text{Upper E}(\Xi) = \{\text{SE}(F) : F \in \text{Upper W}((N, M))\}.$$

By Proposition 2.4.10, Upper E $(\Xi) = (N, M)$, and so

$$\text{Right S}\left((N,M)\right) = \left\{\text{SE}\left(F\right) : F \in \text{Upper W}\left((N,M)\right)\right\},$$

as required. □

Remark 2.5.5. Let $(N, M) \in \mathcal{R}_{10}$. Then

$$\text{Right S}\left((N,M)\right) = \left\{\text{SE}\left(F\right) : F \in \text{Upper W}\left((N,M)\right)\right\}.$$

Indeed, let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \text{Upper W}\left((N,M)\right)$. Then $F_{21} = 0$, and $F_{11} = \zeta g(z, \cdot)$ for some $\zeta \in \mathbb{T}$, where $g \in \mathcal{H}_{K_{N,M}}$ is such that

$$K_{N,M}(z,\lambda,w,\mu) = \overline{g(w,\mu)}g(z,\lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. It follows that $\text{SE}\,(F)(z,\lambda) = -\zeta g(z,\lambda)$ for all $z, \lambda \in \mathbb{D}$, and so

$$\text{Right S}\left((N,M)\right) = \{\zeta g : \zeta \in \mathbb{T}\} = \left\{\text{SE}\left(F\right) : F \in \text{Upper W}\left((N,M)\right)\right\}.$$

2.6 Right N : $\mathcal{S}_2 \to \mathcal{R}_1$

The following theorem gives the realisation formula for functions on $\mathbb{D}^2$. We omit the proof of this theorem.

Theorem 2.6.1. [2, Proof of Theorem 1.12] *Let $\varphi \in \mathcal{S}_2$. Then there are holomorphic kernels N and M on $\mathbb{D}^2$ such that*

$$1 - \overline{\varphi(\mu_1,\mu_2)}\varphi(\lambda_1,\lambda_2) = (1 - \overline{\mu_1}\lambda_1)N(\lambda_1,\lambda_2,\mu_1,\mu_2) + (1 - \overline{\mu_2}\lambda_2)M(\lambda_1,\lambda_2,\mu_1,\mu_2)$$

for all $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{D}$.

Remark 2.6.2. Theorem 2.6.1 gives the realisation formula in terms of kernels, but there is an alternative statement, which is a consequence of [2, Theorem 1.12]. Let $\varphi \in \mathcal{S}_2$. Then there is a Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and a contractive operator $\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}$ such that

$$\varphi(\lambda) = A + B\lambda_P(I_{\mathcal{H}} - D\lambda_P)^{-1}C,$$

where $\lambda_P = \lambda_1 I_{\mathcal{H}_1} \oplus \lambda_2 I_{\mathcal{H}_2}$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$, for all $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$.

The proof of Theorem 2.6.1 is non-constructive, and so it does not give a particular pair (N, M). Pairs of kernels that satisfy Theorem 2.6.1 are known as *Agler kernels*. There has been research by a number of authors to produce a constructive proof of Theorem 2.6.1, and thus a canonical pair of Agler kernels (see, for example, [16], [22] and [23]).

Recall that $\mathcal{R}_1$ is the set of pairs (N, M) of holomorphic kernels on $\mathbb{D}^2$ such that $K_{N,M}$ is a rank 1 holomorphic kernel on $\mathbb{D}^2$, where

$$K_{N,M}(z, \lambda, w, \mu) = 1 - (1 - \overline{w}z)N(z, \lambda, w, \mu) - (1 - \overline{\mu}\lambda)M(z, \lambda, w, \mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$.

Lemma 2.6.3. *Let $\varphi \in \mathcal{S}_2$. Then every pair of Agler kernels for φ belongs to $\mathcal{R}_1$.*

Proof. Let (N, M) be a pair of Agler kernels for $\varphi \in \mathcal{S}_2$. Then

$$K_{N,M}(z, \lambda, w, \mu) = \overline{\varphi(w, \mu)}\varphi(z, \lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Hence, by Corollary B.3.23, $K_{N,M}$ is a holomorphic kernel on $\mathbb{D}^2$ and has rank 1. Since N and M are holomorphic kernels on $\mathbb{D}^2$, we have $(N, M) \in \mathcal{R}_1$. □

Definition 2.6.4. *We define* Right N *to be the set-valued map from $\mathcal{S}_2$ to $\mathcal{R}_1$ given by*

$$\text{Right N}(\varphi) = \{A \in \mathcal{R}_1 : A \textit{ is a pair of Agler kernels for } \varphi\}$$

for all $\varphi \in \mathcal{S}_2$.

That Right N is well defined follows immediately from Lemma 2.6.3. We now consider how Right N interacts with Right S.

Proposition 2.6.5. *Let $\varphi \in \mathcal{S}_2$. Then, for all $A \in \text{Right N}(\varphi)$,*

$$\text{Right S}(A) = \{\zeta\varphi : \zeta \in \mathbb{T}\}.$$

Proof. We have

$$\text{Right N}(\varphi) = \{A \in \mathcal{R}_1 : A \text{ is a pair of Agler kernels for } \varphi\}.$$

Let $A = (N, M) \in \text{Right N}(\varphi)$. Then

$$K_{N,M}(z, \lambda, w, \mu) = \overline{\varphi(w, \mu)}\varphi(z, \lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Hence $\text{Right S}(A) = \{\zeta\varphi : \zeta \in \mathbb{T}\}$. □

Proposition 2.6.6. *Let $(N, M) \in \mathcal{R}_1$. Let $f : \mathbb{D}^2 \to \mathbb{C}$ be holomorphic and satisfy*

$$K_{N,M}(z, \lambda, w, \mu) = \overline{f(w, \mu)}f(z, \lambda)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. Then, for all $\varphi \in \text{Right S}((N, M))$,

$$\text{Right N}(\varphi) = \text{Right N}(f).$$

Proof. By the proof of Proposition 2.5.2, $f \in \mathcal{S}_2$. Clearly $(N, M) \in \operatorname{Right N}(f)$. Moreover,

$$\operatorname{Right S}((N, M)) = \{\zeta f : \zeta \in \mathbb{T}\}.$$

Let $\varphi \in \operatorname{Right S}((N, M))$. Then $\varphi = \zeta f$ for some $\zeta \in \mathbb{T}$. Now, for all $(P, Q) \in \operatorname{Right N}(\varphi)$ and all $(R, S) \in \operatorname{Right N}(f)$, we have

$$K_{P,Q}(z, \lambda, w, \mu) = \overline{\zeta f(w, \mu)} \zeta f(z, \lambda) = \overline{f(w, \mu)} f(z, \lambda) = K_{R,S}(z, \lambda, w, \mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. It follows that $\operatorname{Right N}(\varphi) = \operatorname{Right N}(f)$. □

Chapter 3. The symmetrised bidisc

3.1 Introduction

Agler and Young introduced the symmetrised bidisc in [9]. The main result of [9] is a commutant lifting theorem for the symmetrised bidisc. As an application of this theorem, they proved a necessary condition for the solvability of the 2×2 spectral Nevanlinna-Pick problem, which is a special case of the μ-synthesis problem. This connection motivated the study of the symmetrised bidisc.

We define the *open* and *closed symmetrised bidiscs* to be the sets

$$G := \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{D}\} \text{ and } \Gamma := \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \overline{\mathbb{D}}\},$$

respectively. The sets G and Γ are the images of the open and closed bidiscs under the symmetrisation map

$$(z_1, z_2) \mapsto (z_1 + z_2, z_1 z_2),$$

which inspires their names. We note that the geometry of G is different to that of the bidisc. (For the next proposition, see Definition B.1.23 and Definition B.1.24 for the notions of hypoconvexity and polynomial convexity.)

Proposition 3.1.1. [12, Theorem 2.3] *The set G is hypoconvex, polynomially convex and starlike about $(0, 0)$, however, it is not convex.*

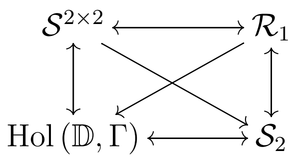
Let tr and det be the trace and determinant, respectively. In [11], using the fact that a 2×2 matrix M has both of its eigenvalues in $\overline{\mathbb{D}}$ if and only if $(\mathrm{tr}\ M, \det M) \in \Gamma$, Agler and Young showed that the solvability of the 2×2 spectral Nevanlinna-Pick problem is equivalent to the solvability of an interpolation problem from $\mathbb{D}$ into Γ. More precisely they proved the following theorem.

Theorem 3.1.2. [11, Theorem 1.1] *Let $\lambda_1, \dots, \lambda_n$ be distinct points in $\mathbb{D}$. Let $W_1, \dots, W_n \in \mathcal{M}_2(\mathbb{C})$ be such that $\rho(W_j) \leq 1$ for $j = 1, \dots, n$, and all or none of which are scalar matrices. Then the following are equivalent.*

(i) *There is a holomorphic function $F : \mathbb{D} \to \mathcal{M}_2(\mathbb{C})$ such that $F(\lambda_j) = W_j$ for $j = 1, \dots, n$, and $\rho(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$.*

(ii) *There is an $h \in \mathrm{Hol}(\mathbb{D}, \Gamma)$ such that $h(\lambda_j) = (\mathrm{tr}\ W_j, \det W_j)$ for $j = 1, \dots, n$.*

In this chapter, we discuss some background material for the symmetrised bidisc and define Γ-inner functions. Afterwards, we focus on the construction of the maps that illustrate the rich structure of interconnections between the sets $\mathcal{S}^{2\times 2}$, $\mathcal{S}_2$, $\mathrm{Hol}\,(\mathbb{D},\Gamma)$ and $\mathcal{R}_1$. The maps can be summarised by the rich saltire:



We use the maps produced to give conditions for the solvability of the μ-synthesis problem in Theorem 3.1.2.

To understand the rich structure between the sets, we need a number of results from [3, 4, 8, 12]. We give these results when they are needed and include the proofs when they provide insight into the rich structure.

3.2 Background

We give, as defined in [4], a useful function in the study of Γ. Let $\Phi : \mathbb{C}^3 \to \mathbb{C}$ be given by

$$\Phi(z,s,p) = \frac{2zp - s}{2 - zs}$$

for all $p \in \mathbb{C}$ and $z, s \in \mathbb{C}$ such that $zs \neq 2$. Note that, since Φ is a rational function and rational functions are holomorphic, Φ is holomorphic everywhere that $zs \neq 2$. Hence Φ is defined and holomorphic on $\mathbb{D} \times \Gamma$ since, for all $z \in \mathbb{D}$ and $(s,p) \in \Gamma$, we have $|zs| < 2$. We call attention to a special case. Let $(s,p) \in \Gamma$ be such that $s^2 = 4p$. Then

$$\Phi(z,s,p) = \frac{2z\frac{s^2}{4} - s}{2 - zs} = \frac{-\frac{1}{2}s(2 - zs)}{2 - zs} = -\frac{1}{2}s.$$

The following boundary is also useful in the study of Γ. First we give the general definition.

Definition 3.2.1. [1, pp. 739-740] *Let X be a domain in $\mathbb{C}^n$ and $\overline{X}$ be its closure. We denote by $\mathcal{A}(X)$ the algebra of continuous functions on $\overline{X}$ that are holomorphic on X. A* boundary *for $\overline{X}$ is a subset of $\overline{X}$ on which every function in $\mathcal{A}(X)$ attains its maximum modulus. By* [24, Corollary 2.2.10], *if $\overline{X}$ is polynomially convex then there is a smallest closed boundary of $\overline{X}$ that is contained in all closed boundaries of $\overline{X}$. We call this boundary the* distinguished boundary *of $\overline{X}$, and denote it by $b\overline{X}$. This boundary is also known as the* Šilov boundary *of $\mathcal{A}(X)$.*

By Proposition 3.1.1, Γ is polynomially convex. Hence the distinguished boundary of Γ exists and is the Šilov boundary of the algebra $\mathcal{A}(\Gamma)$ of continuous functions on Γ that are holomorphic on G. The following characterisation of $b\Gamma$ is more useful.

Proposition 3.2.2. [12, Theorem 2.4] *The distinguished boundary of* Γ *is the symmetrisation of the* 2*-torus, that is,*

$$b\Gamma = \{(z+w, zw) : z, w \in \mathbb{T}\}.$$

Moreover, $b\Gamma$ *is a Möbius band.*

In the following proposition we give alternative characterisations of G, Γ, $b\Gamma$ and the *topological boundary* of Γ, that is, the set $\partial\Gamma := \Gamma \setminus G$.

Proposition 3.2.3. [12, Theorem 2.1, Corollary 2.2],[4, Proposition 3.2] *Let* $(s,p) \in \mathbb{C}^2$. *Then*

(i) $(s,p) \in G \iff |s - \bar{s}p| < 1 - |p|^2$
$\iff |s| < 2$ *and, for all* $w \in \mathbb{T}$, $|\Phi(w,s,p)| < 1$;

(ii) $(s,p) \in \Gamma \iff |s| \leq 2$ *and* $|s - \bar{s}p| \leq 1 - |p|^2$
$\iff |s| \leq 2$ *and, for all* w *in a dense subset of* $\mathbb{T}$, $|\Phi(w,s,p)| \leq 1$;

(iii) $(s,p) \in b\Gamma \iff |s| \leq 2, |p| = 1$ *and* $s = \bar{s}p$;

(iv) $(s,p) \in \partial\Gamma \iff |s| \leq 2$ *and* $|s - \bar{s}p| = 1 - |p|^2$
$\iff$ *there exist* $z \in \mathbb{T}$ *and* $w \in \overline{\mathbb{D}}$ *such that* $s = z + w, p = zw$;

(v) *if* $(w,s,p) \in \mathbb{T} \times \Gamma$ *then* $|\Phi(w,s,p)| = 1 \iff w(s - \bar{s}p) = 1 - |p|^2$.

In Proposition 3.2.3 (ii), we have a dense subset of $\mathbb{T}$ because Φ is not defined for points of the form $(w, 2\overline{w}, \overline{w}^2) \in \mathbb{T} \times \Gamma$. In fact, these are the only points in $\mathbb{T} \times \Gamma$ for which Φ is not defined. Indeed, for $w \in \mathbb{T}$ and $(s,p) \in \Gamma$,

$$2 - ws = 0 \iff 2 = ws \iff 2\overline{w} = |w|^2 s = s \iff (s,p) = (2\overline{w}, \overline{w}^2).$$

The last equivalence holds since s is the sum of two elements in $\overline{\mathbb{D}}$, and $\overline{w} \in \mathbb{T}$ gives $|s| = 2$.

An important subset of $\operatorname{Hol}(\mathbb{D}, \Gamma)$ is the collection of Γ-inner functions. A Γ-inner function is the analogue for $\operatorname{Hol}(\mathbb{D}, \Gamma)$ of inner functions in $\operatorname{Hol}(\mathbb{D}, \overline{\mathbb{D}})$.

Definition 3.2.4. [4, Definition 6.1] *A* Γ-inner function *is a function* $h \in \operatorname{Hol}(\mathbb{D}, \Gamma)$ *such that, for almost all* $\lambda \in \mathbb{T}$, *the radial limit*

$$\lim_{r \to 1^-} h(r\lambda) \in b\Gamma.$$

We note that if $h \in \operatorname{Hol}(\mathbb{D}, \Gamma)$ then we may consider h as the function $(s,p) : \mathbb{D} \to \Gamma$ defined by

$$(s,p)(\lambda) = (s(\lambda), p(\lambda)) = h(\lambda) \text{ for all } \lambda \in \mathbb{D}.$$

It follows that if $h = (s,p)$ is a Γ-inner function then p is an inner function.

3.3 Relations between the sets

In this section, we construct maps between $\mathcal{S}^{2\times 2}$, $\mathcal{S}_2$, $\mathrm{Hol}(\mathbb{D},\Gamma)$ and $\mathcal{R}_1$, which illustrate the rich structure of interconnections summarised by the rich saltire. We label the maps in accordance with the following diagrams. The subscript $_{\mathrm{SB}}$ denotes that we have $\mathrm{Hol}(\mathbb{D},\Gamma)$, and so consider the *s*ymmetrised *b*idisc.

$$\begin{array}{ccc} & \mathcal{S}^{2\times 2} & \\ \scriptstyle\text{Left S}_{\mathrm{SB}} & \downarrow\uparrow & \scriptstyle\text{Left N}_{\mathrm{SB}} \\ & \mathrm{Hol}(\mathbb{D},\Gamma) \underset{\text{Lower E}_{\mathrm{SB}}}{\overset{\text{Lower W}_{\mathrm{SB}}}{\leftrightarrows}} \mathcal{S}_2 & \end{array} \qquad \begin{array}{ccc} \mathcal{S}^{2\times 2} & & \\ & \searrow^{\text{SE}} & \\ & & \mathcal{S}_2 \end{array} \qquad \begin{array}{ccc} \mathcal{S}^{2\times 2} \underset{\text{Upper E}}{\overset{\text{Upper W}}{\leftrightarrows}} \mathcal{R}_1 \\ \scriptstyle\text{Right S}\ \downarrow\uparrow\ \scriptstyle\text{Right N} \\ \mathcal{S}_2 \end{array} \qquad \begin{array}{ccc} & & \mathcal{R}_1 \\ & \swarrow^{\text{SW}_{\mathrm{SB}}} & \\ \mathrm{Hol}(\mathbb{D},\Gamma) & & \end{array}$$

3.3.1 *Schur class of the bidisc and* Left $\mathrm{N}_{\mathrm{SB}} : \mathrm{Hol}(\mathbb{D},\Gamma) \to \mathcal{S}^{2\times 2}$

We begin this section with the construction of a unique function $F \in \mathcal{S}^{2\times 2}$ for each $h \in \mathrm{Hol}(\mathbb{D},\Gamma)$. It is appropriate to include the realisation of $\Phi(z,h(\lambda))$ that is related to F. We show later that $\Phi(z,h(\lambda))$, as a function on the bidisc, belongs to $\mathcal{S}_2$, and that this realisation is a powerful tool in producing a number of the maps in the rich saltire.

Let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. Then the linear fractional transformation $\mathcal{F}_{F(\lambda)}(z)$ is given by

$$\mathcal{F}_{F(\lambda)}(z) := F_{11}(\lambda) + F_{12}(\lambda)z(1 - F_{22}(\lambda)z)^{-1}F_{21}(\lambda),$$

where $z, \lambda \in \mathbb{D}$.

Theorem 3.3.1. [3, Proposition 6.1] *Let $h = (s,p) \in \mathrm{Hol}(\mathbb{D},\Gamma)$. Then there exists a unique*

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$$

such that $h = (\mathrm{tr}\, F, \det F)$, $F_{11} = F_{22}$, $|F_{12}| = |F_{21}|$ almost everywhere on $\mathbb{T}$, F_{21} is either outer or 0, and $F_{21}(0) \geq 0$. Moreover, we have

$$1 - \overline{\Phi(w,h(\mu))}\Phi(z,h(\lambda)) = (1-\overline{w}z)\overline{\gamma(\mu,w)}\gamma(\lambda,z) + \eta(\mu,w)^*(I - F(\mu)^*F(\lambda))\eta(\lambda,z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$, where

$$\gamma(\lambda,z) = (1 - F_{22}(\lambda)z)^{-1}F_{21}(\lambda) \text{ and } \eta(\lambda,z) = \begin{bmatrix} 1 \\ z\gamma(\lambda,z) \end{bmatrix}$$

for all $z, \lambda \in \mathbb{D}$.

Proof. First, we show that such an F exists and is unique. Let $h = (s,p) \in \mathrm{Hol}(\mathbb{D},\Gamma)$.

Suppose that $\frac{1}{4}s^2 = p$. Then it is clear that

$$F = \begin{bmatrix} \frac{1}{2}s & 0 \\ 0 & \frac{1}{2}s \end{bmatrix}$$

is the only matrix to satisfy all of the required conditions. In particular, F is holomorphic and, since, by Proposition 3.2.3 (ii), $|s(\lambda)| \leq 2$ for all $\lambda \in \mathbb{D}$, we have

$$||F(\lambda)|| = \frac{1}{2}|s(\lambda)| \leq 1$$

for all $\lambda \in \mathbb{D}$. That it is the only matrix follows since we have $|F_{21}||F_{12}| = |\frac{1}{4}s^2 - p| = 0$ and $|F_{12}| = |F_{21}|$.

Now suppose that $\frac{1}{4}s^2 \neq p$. Then $\frac{1}{4}s^2 - p$ is a non-zero H^∞ function, and so, by Theorem B.1.21, it has a unique inner-outer factorisation of the form $\varphi e^C = \frac{1}{4}s^2 - p$, where φ is inner, e^C is outer and $e^C(0) \geq 0$. Set

$$F = \begin{bmatrix} \frac{1}{2}s & \varphi e^{\frac{1}{2}C} \\ e^{\frac{1}{2}C} & \frac{1}{2}s \end{bmatrix}.$$

Then, except for the condition that $F \in \mathcal{S}^{2\times 2}$, it is easy to check that F is the only matrix satisfying the required conditions. In particular,

$$\det F = \frac{1}{4}s^2 - \varphi e^C = \frac{1}{4}s^2 - (\frac{1}{4}s^2 - p) = p,$$

and, since $|\varphi| = 1$ almost everywhere on $\mathbb{T}$, we have $|F_{12}| = |F_{21}|$ almost everywhere on $\mathbb{T}$. That it is the only matrix follows from the uniqueness of the representation φe^C and the requirements that F_{21} be outer, and that $|F_{12}| = |F_{21}|$ almost everywhere on $\mathbb{T}$.

We still need to show that $F \in \mathcal{S}^{2\times 2}$. Clearly F is holomorphic, since inner and outer functions are holomorphic. To check that $||F(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$, it is equivalent, by Corollary B.1.6, to check that $I - F(\lambda)^*F(\lambda)$ is positive semidefinite for all $\lambda \in \mathbb{D}$. To do this, we show that the diagonal entries of $I - F(\lambda)^*F(\lambda)$ are non-negative and $\det(I - F(\lambda)^*F(\lambda)) \geq 0$ for all $\lambda \in \mathbb{D}$. Since $|F_{12}| = |F_{21}|$ almost everywhere on $\mathbb{T}$, and $F_{21}F_{12} = \frac{1}{4}s^2 - p$, we have

$$|F_{12}|^2 = |F_{21}|^2 = |F_{21}F_{12}| = \left|\frac{1}{4}s^2 - p\right|$$

almost everywhere on $\mathbb{T}$. By Proposition B.1.29, for almost every $\lambda \in \mathbb{T}$,

$$I - F(\lambda)^*F(\lambda) = \begin{bmatrix} 1 - \frac{1}{4}|s(\lambda)|^2 - |\frac{1}{4}s(\lambda)^2 - p(\lambda)| & -\frac{1}{2}\overline{s(\lambda)}F_{12}(\lambda) - \frac{1}{2}\overline{F_{21}(\lambda)}s(\lambda) \\ -\frac{1}{2}\overline{F_{12}(\lambda)}s(\lambda) - \frac{1}{2}\overline{s(\lambda)}F_{21}(\lambda) & 1 - |\frac{1}{4}s(\lambda)^2 - p(\lambda)| - \frac{1}{4}|s(\lambda)|^2 \end{bmatrix}$$

and

$$\det\left(I - F(\lambda)^* F(\lambda)\right) = 1 - 2\left(\frac{1}{4}|s(\lambda)|^2 - |\frac{1}{4}s(\lambda)^2 - p(\lambda)|\right) + |p(\lambda)|^2.$$

Let $D_1(\lambda)$ and $D_2(\lambda)$ be the diagonal entries of $I - F(\lambda)^* F(\lambda)$. For any $\lambda \in \mathbb{T}$, by continuity and since $(s,p) : \mathbb{D} \to \Gamma$, we can write $s(\lambda) = z_1 + z_2$ and $p(\lambda) = z_1 z_2$ for some $z_1, z_2 \in \overline{\mathbb{D}}$. Thus, for almost every $\lambda \in \mathbb{T}$,

$$\begin{aligned} D_1(\lambda) = D_2(\lambda) &= 1 - \frac{1}{4}|z_1+z_2|^2 - |\frac{1}{4}(z_1+z_2)^2 - z_1 z_2| \\ &= 1 - \frac{1}{4}|z_1+z_2|^2 - \frac{1}{4}|z_1 - z_2|^2 \\ &= 1 - \frac{1}{4}(|z_1|^2 + z_1\overline{z_2} + \overline{z_1}z_2 + |z_2|^2) - \frac{1}{4}(|z_1|^2 - z_1\overline{z_2} - \overline{z_1}z_2 + |z_2|^2) \\ &= 1 - \frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2 \geq 1 - \frac{1}{2} - \frac{1}{2} = 0 \end{aligned}$$

and

$$\begin{aligned} \det\left(I - F(\lambda)^* F(\lambda)\right) &= 1 - 2\left(\frac{1}{4}|z_1+z_2|^2 - |\frac{1}{4}(z_1+z_2)^2 - z_1 z_2|\right) + |z_1 z_2|^2 \\ &= 1 - |z_1|^2 - |z_2|^2 + |z_1 z_2|^2 \\ &= (1-|z_1|^2)(1-|z_2|^2) \geq (1-1)(1-1) = 0. \end{aligned}$$

Hence, by Corollary B.1.6, $||F(\lambda)|| \leq 1$ for almost every $\lambda \in \mathbb{T}$. It follows from the Maximum Modulus Principle that $||F(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$, as required.

It remains to show that

$$1 - \overline{\Phi(w, h(\mu))}\Phi(z, h(\lambda)) = (1 - \overline{w}z)\overline{\gamma(\mu, w)}\gamma(\lambda, z) + \eta(\mu, w)^*(I - F(\mu)^* F(\lambda))\eta(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. First we note that

$$\begin{aligned} \mathcal{F}_{F(\lambda)}(z) =& F_{11}(\lambda) + \frac{F_{12}(\lambda)F_{21}(\lambda)z}{1 - F_{22}(\lambda)z} = \frac{1}{2}s(\lambda) + \frac{(\frac{1}{4}s(\lambda)^2 - p(\lambda))z}{1 - \frac{1}{2}s(\lambda)z} \\ =& \frac{\frac{1}{2}s(\lambda) - \frac{1}{4}s(\lambda)^2 z + (\frac{1}{4}s(\lambda)^2 - p(\lambda))z}{(1 - \frac{1}{2}s(\lambda)z)} = \frac{\frac{1}{2}s(\lambda) - p(\lambda)z}{1 - \frac{1}{2}s(\lambda)z} \\ =& -\frac{2p(\lambda)z - s(\lambda)}{2 - s(\lambda)z} = -\Phi(z, h(\lambda)) \end{aligned}$$

for all $z, \lambda \in \mathbb{D}$. Hence, by Proposition 2.1.1,

$$\begin{aligned} 1 - \overline{\Phi(w, h(\mu))}\Phi(z, h(\lambda)) =& 1 - \mathcal{F}_{F(\mu)}(w)^* \mathcal{F}_{F(\lambda)}(z) \\ =& (1 - \overline{w}z)\overline{\gamma(\mu, w)}\gamma(\lambda, z) + \eta(\mu, w)^*(I - F(\mu)^* F(\lambda))\eta(\lambda, z) \end{aligned}$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. □

Definition 3.3.2. *We define* Left N_{SB} : $\mathrm{Hol}(\mathbb{D},\Gamma) \to \mathcal{S}^{2\times 2}$ *by*

$$\text{Left}\, N_{SB}(h) = F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

for each $h \in \mathrm{Hol}(\mathbb{D},\Gamma)$*, where* $F \in \mathcal{S}^{2\times 2}$ *and satisfies* $h = (\mathrm{tr}\, F, \det F)$*,* $F_{11} = F_{22}$*,* $|F_{12}| = |F_{21}|$ *almost everywhere on* $\mathbb{T}$*,* F_{21} *is either outer or* 0*, and* $F_{21}(0) \geq 0$*.*

Clearly the function F, as defined in Definition 3.3.2, must also satisfy $F_{21}F_{12} = \frac{1}{4}s^2 - p$. That Left N_{SB} is well defined follows immediately from Theorem 3.3.1. In the case that h is a Γ-inner function, there is an alternative proof of the realisation from Theorem 3.3.1 of $\Phi(z, h(\lambda))$, which we give now.

Proposition 3.3.3. [3, Proposition 7.1] *Let* $h = (s,p)$ *be a* Γ*-inner function such that* $s^2 \neq 4p$*. Then there is a Hilbert space* H*, a holomorphic function* $F : \mathbb{D} \to \mathcal{B}(\mathbb{C}^2, H)$*, and an outer function* $g \in H^\infty$ *such that* $|g(\xi)|^2 = 1 - \frac{1}{4}|s(\xi)|^2$ *for almost every* $\xi \in \mathbb{T}$*, which satisfy*

$$\begin{aligned} 1 - \overline{\Phi(w, h(\mu))}\Phi(z, h(\lambda)) =& (1 - \overline{w}z)\left\langle \frac{g(\lambda)}{1 - \frac{1}{2}zs(\lambda)}, \frac{g(\mu)}{1 - \frac{1}{2}ws(\mu)} \right\rangle_{\mathbb{C}} \\ &+ (1 - \overline{\mu}\lambda)\left\langle F(\lambda)\begin{pmatrix} 1 \\ \frac{zg(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix}, F(\mu)\begin{pmatrix} 1 \\ \frac{wg(\mu)}{1 - \frac{1}{2}ws(\mu)} \end{pmatrix} \right\rangle_H \end{aligned}$$

for all $\mu, \lambda \in \mathbb{D}$ *and* $w, z \in \mathbb{C}$ *such that* $1 - \frac{1}{2}s(\mu)w \neq 0$ *and* $1 - \frac{1}{2}s(\lambda)z \neq 0$*.*

Proof. By Proposition 3.2.3 (iii), $|p(\lambda)| = 1$ and $s(\lambda) = \overline{s(\lambda)}p(\lambda)$ for almost every $\lambda \in \mathbb{T}$. Hence

$$\Phi(z, h(\lambda)) = \frac{2zp(\lambda) - s(\lambda)}{2 - zs(\lambda)} = \frac{2zp(\lambda) - \overline{s(\lambda)}p(\lambda)}{2 - zs(\lambda)} = \frac{z - \frac{1}{2}\overline{s(\lambda)}}{1 - \frac{1}{2}zs(\lambda)}p(\lambda)$$

for almost every $\lambda \in \mathbb{T}$ and every $z \in \mathbb{C}$ such that $1 - \frac{1}{2}s(\lambda)z \neq 0$. It follows that

$$\begin{aligned} &1 - \overline{\Phi(w, h(\lambda))}\Phi(z, h(\lambda)) = \\ &= 1 - \overline{\frac{w - \frac{1}{2}\overline{s(\lambda)}}{1 - \frac{1}{2}ws(\lambda)}p(\lambda)}\frac{z - \frac{1}{2}\overline{s(\lambda)}}{1 - \frac{1}{2}zs(\lambda)}p(\lambda) = 1 - \frac{\overline{w} - \frac{1}{2}s(\lambda)}{1 - \frac{1}{2}\overline{ws(\lambda)}}\frac{z - \frac{1}{2}\overline{s(\lambda)}}{1 - \frac{1}{2}zs(\lambda)}|p(\lambda)| \\ &= \frac{(1 - \frac{1}{2}zs(\lambda) - \frac{1}{2}\overline{ws(\lambda)} + \frac{1}{4}\overline{w}z|s(\lambda)|^2) - (\overline{w}z - \frac{1}{2}\overline{ws(\lambda)} - \frac{1}{2}zs(\lambda) + \frac{1}{4}|s(\lambda)|^2)}{\overline{(1 - \frac{1}{2}ws(\lambda))}(1 - \frac{1}{2}zs(\lambda))} \\ &= \frac{1 - \frac{1}{4}|s(\lambda)|^2 + \overline{w}z(\frac{1}{4}|s(\lambda)|^2 - 1)}{\overline{(1 - \frac{1}{2}ws(\lambda))}(1 - \frac{1}{2}zs(\lambda))} = \frac{(1 - \frac{1}{4}|s(\lambda)|^2)(1 - \overline{w}z)}{\overline{(1 - \frac{1}{2}ws(\lambda))}(1 - \frac{1}{2}zs(\lambda))} \end{aligned}$$

for almost every $\lambda \in \mathbb{T}$ and every $w, z \in \mathbb{C}$ such that $1 - \frac{1}{2}s(\lambda)w \neq 0$ and $1 - \frac{1}{2}s(\lambda)z \neq 0$.

By Theorem B.1.21, the non-zero H^∞ function $\frac{1}{4}s^2 - p$ has an inner-outer factorisation of the form $\varphi g_0 = \frac{1}{4}s^2 - p$, where φ is inner and $g_0 \in H^\infty$ is outer. Since $|\varphi(\lambda)| = 1$,

$|p(\lambda)| = 1$ and $s(\lambda) = \overline{s(\lambda)}p(\lambda)$ for almost every $\lambda \in \mathbb{T}$, we have

$$
\begin{aligned}
|g_0(\lambda)|^2 = \varphi(\lambda) g_0(\lambda) \overline{\varphi(\lambda) g_0(\lambda)} &= \left(\frac{1}{4}s(\lambda)^2 - p(\lambda)\right)\overline{\left(\frac{1}{4}s(\lambda)^2 - p(\lambda)\right)} \\
&= \frac{1}{16}|s(\lambda)|^4 - \frac{1}{4}s(\lambda)^2\overline{p(\lambda)} - \frac{1}{4}\overline{s(\lambda)}^2 p(\lambda) + |p(\lambda)|^2 \\
&= \frac{1}{16}|s(\lambda)|^4 - \frac{1}{4}|s(\lambda)|^2 - \frac{1}{4}|s(\lambda)|^2 + 1 \\
&= \left(\frac{1}{4}|s(\lambda)|^2 - 1\right)^2
\end{aligned}
$$

for almost every $\lambda \in \mathbb{T}$. Since $|s(\lambda)| \leq 2$, we have $1 - \frac{1}{4}|s(\lambda)|^2 \geq 0$, and hence

$$
|g_0(\lambda)| = 1 - \frac{1}{4}|s(\lambda)|^2
$$

for almost every $\lambda \in \mathbb{T}$. Set $g = g_0^{\frac{1}{2}}$. Then $g \in H^\infty$ is an outer function such that $|g(\lambda)|^2 = 1 - \frac{1}{4}|s(\lambda)|^2$ for almost every $\lambda \in \mathbb{T}$. Thus we can write

$$
1 - \overline{\Phi(w, h(\lambda))}\Phi(z, h(\lambda)) = (1 - \overline{w}z)\left\langle \frac{g(\lambda)}{1 - \frac{1}{2}zs(\lambda)}, \frac{g(\lambda)}{1 - \frac{1}{2}ws(\lambda)} \right\rangle_{\mathbb{C}}
$$

and so, by expanding the right side,

$$
1 + \left\langle \frac{zg(\lambda)}{1 - \frac{1}{2}zs(\lambda)}, \frac{wg(\lambda)}{1 - \frac{1}{2}ws(\lambda)} \right\rangle_{\mathbb{C}} = \overline{\Phi(w, h(\lambda))}\Phi(z, h(\lambda)) + \left\langle \frac{g(\lambda)}{1 - \frac{1}{2}zs(\lambda)}, \frac{g(\lambda)}{1 - \frac{1}{2}ws(\lambda)} \right\rangle_{\mathbb{C}}
$$

for almost every $\lambda \in \mathbb{T}$ and every $w, z \in \mathbb{C}$ such that $1 - \frac{1}{2}s(\lambda)w \neq 0$ and $1 - \frac{1}{2}s(\lambda)z \neq 0$. It follows that

$$
\left\langle \begin{pmatrix} 1 \\ \dfrac{zg(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix}, \begin{pmatrix} 1 \\ \dfrac{wg(\lambda)}{1 - \frac{1}{2}ws(\lambda)} \end{pmatrix} \right\rangle_{\mathbb{C}^2} = \left\langle \begin{pmatrix} \Phi(z, h(\lambda)) \\ \dfrac{g(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix}, \begin{pmatrix} \Phi(w, h(\lambda)) \\ \dfrac{g(\lambda)}{1 - \frac{1}{2}ws(\lambda)} \end{pmatrix} \right\rangle_{\mathbb{C}^2}
$$

for almost every $\lambda \in \mathbb{T}$ and every $w, z \in \mathbb{C}$ such that $1 - \frac{1}{2}s(\lambda)w \neq 0$ and $1 - \frac{1}{2}s(\lambda)z \neq 0$. That is, for almost every $\lambda \in \mathbb{T}$, the Grammian of the vectors

$$
\left\{ \begin{pmatrix} 1 \\ \dfrac{zg(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix} : z \in \mathbb{C} \text{ and } 1 - \frac{1}{2}s(\lambda)z \neq 0 \right\} \subseteq \mathbb{C}^2
$$

is equal to the Grammian of the vectors

$$
\left\{ \begin{pmatrix} \Phi(z, h(\lambda)) \\ \dfrac{g(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix} \; z \in \mathbb{C} \text{ and } 1 - \frac{1}{2}s(\lambda)z \neq 0 \right\} \subseteq \mathbb{C}^2.
$$

Thus, by Proposition B.1.22, for almost every $\lambda \in \mathbb{T}$, there is an isometry L_λ such that

$$L_\lambda \left(\begin{pmatrix} 1 \\ \dfrac{zg(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix} \right) = \begin{pmatrix} \Phi(z, h(\lambda)) \\ \dfrac{g(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix}$$

for every $z \in \mathbb{C}$ such that $1 - \frac{1}{2}s(\lambda)z \neq 0$.

Now, we define a function Θ on $\mathbb{D}$ by

$$\Theta(\lambda) := \begin{bmatrix} -\frac{1}{2}s(\lambda) & \dfrac{p(\lambda) - \frac{1}{4}s(\lambda)^2}{g(\lambda)} \\ g(\lambda) & \frac{1}{2}s(\lambda) \end{bmatrix}$$

for every $\lambda \in \mathbb{D}$. Hence

$$\begin{aligned} \Theta(\lambda) \begin{pmatrix} 1 \\ \dfrac{zg(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2}s(\lambda) + \dfrac{p(\lambda) - \frac{1}{4}s(\lambda)^2}{g(\lambda)} \dfrac{zg(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \\ g(\lambda) + \frac{1}{2}s(\lambda)\dfrac{zg(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{-\frac{1}{2}s(\lambda) + \frac{1}{4}zs(\lambda)^2 + zp(\lambda) - \frac{1}{4}zs(\lambda)^2}{1 - \frac{1}{2}zs(\lambda)} \\ \dfrac{g(\lambda) - \frac{1}{2}zs(\lambda)g(\lambda) + \frac{1}{2}zs(\lambda)g(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{zp(\lambda) - \frac{1}{2}s(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \\ \dfrac{g(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix} = \begin{pmatrix} \Phi(z, h(\lambda)) \\ \dfrac{g(\lambda)}{1 - \frac{1}{2}zs(\lambda)} \end{pmatrix} \end{aligned}$$

for all $\lambda \in \mathbb{D}$ and $z \in \mathbb{C}$ such that $1 - \frac{1}{2}zs(\lambda) \neq 0$. Clearly Θ is holomorphic and, for almost every $\lambda \in \mathbb{T}$, we have $\Theta(\lambda)$ exists and is equal to the isometry L_λ. It follows from the Maximum Modulus Principle that $||\Theta(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$, and so $\Theta \in \mathcal{S}^{2\times 2}$.

By Proposition B.3.30, the function defined by

$$(\lambda, \mu) \mapsto \frac{I - \Theta(\mu)^*\Theta(\lambda)}{1 - \overline{\mu}\lambda}$$

is an $\mathcal{M}_2(\mathbb{C})$-valued kernel on $\mathbb{D}$, and so, by Corollary B.3.20, there is a Hilbert space $\mathcal{H}$ and a holomorphic function $F : \mathbb{D} \to \mathcal{B}(\mathbb{C}^2, \mathcal{H})$ such that

$$I - \Theta(\mu)^*\Theta(\lambda) = (1 - \overline{\mu}\lambda)F(\mu)^*F(\lambda)$$

for all $\mu, \lambda \in \mathbb{D}$. It follows that

$$
\begin{aligned}
&(1-\overline{\mu}\lambda)\left\langle F(\lambda)\begin{pmatrix}1\\ \dfrac{zg(\lambda)}{1-\frac{1}{2}zs(\lambda)}\end{pmatrix}, F(\mu)\begin{pmatrix}1\\ \dfrac{wg(\mu)}{1-\frac{1}{2}ws(\mu)}\end{pmatrix}\right\rangle_{\mathcal{H}} = \\
&= \left\langle (I-\Theta(\mu)^*\Theta(\lambda))\begin{pmatrix}1\\ \dfrac{zg(\lambda)}{1-\frac{1}{2}zs(\lambda)}\end{pmatrix}, \begin{pmatrix}1\\ \dfrac{wg(\mu)}{1-\frac{1}{2}ws(\mu)}\end{pmatrix}\right\rangle_{\mathbb{C}^2} \\
&= \left\langle \begin{pmatrix}1\\ \dfrac{zg(\lambda)}{1-\frac{1}{2}zs(\lambda)}\end{pmatrix}, \begin{pmatrix}1\\ \dfrac{wg(\mu)}{1-\frac{1}{2}ws(\mu)}\end{pmatrix}\right\rangle_{\mathbb{C}^2} - \left\langle \begin{pmatrix}\Phi(z,h(\lambda))\\ \dfrac{g(\lambda)}{1-\frac{1}{2}zs(\lambda)}\end{pmatrix}, \begin{pmatrix}\Phi(w,h(\mu))\\ \dfrac{g(\mu)}{1-\frac{1}{2}ws(\mu)}\end{pmatrix}\right\rangle_{\mathbb{C}^2} \\
&= 1+\overline{w}z\left\langle \frac{g(\lambda)}{1-\frac{1}{2}zs(\lambda)}, \frac{g(\mu)}{1-\frac{1}{2}ws(\mu)}\right\rangle_{\mathbb{C}} \\
&\qquad - \overline{\Phi(w,h(\mu))}\Phi(z,h(\lambda)) - \left\langle \frac{g(\lambda)}{1-\frac{1}{2}zs(\lambda)}, \frac{g(\mu)}{1-\frac{1}{2}ws(\mu)}\right\rangle_{\mathbb{C}} \\
&= 1-\overline{\Phi(w,h(\mu))}\Phi(z,h(\lambda)) - (1-\overline{w}z)\left\langle \frac{g(\lambda)}{1-\frac{1}{2}zs(\lambda)}, \frac{g(\mu)}{1-\frac{1}{2}ws(\mu)}\right\rangle_{\mathbb{C}}
\end{aligned}
$$

and so

$$
\begin{aligned}
1-\overline{\Phi(w,h(\mu))}\Phi(z,h(\lambda)) =& (1-\overline{w}z)\left\langle \frac{g(\lambda)}{1-\frac{1}{2}zs(\lambda)}, \frac{g(\mu)}{1-\frac{1}{2}ws(\mu)}\right\rangle_{\mathbb{C}} \\
&+ (1-\overline{\mu}\lambda)\left\langle F(\lambda)\begin{pmatrix}1\\ \dfrac{zg(\lambda)}{1-\frac{1}{2}zs(\lambda)}\end{pmatrix}, F(\mu)\begin{pmatrix}1\\ \dfrac{wg(\mu)}{1-\frac{1}{2}ws(\mu)}\end{pmatrix}\right\rangle_{\mathcal{H}}
\end{aligned}
$$

for all $\mu, \lambda \in \mathbb{D}$ and $w, z \in \mathbb{C}$ such that $1-\frac{1}{2}s(\mu)w \neq 0$ and $1-\frac{1}{2}s(\lambda)z \neq 0$, which is the required identity. □

Remark 3.3.4. [3, Remark 7.2] It is natural to ask what the relationship is between the function F from Theorem 3.3.1 and the function Θ from Proposition 3.3.3. Let $h = (s, p)$ be a Γ-inner function such that $s^2 \neq 4p$. Recall that

$$
F = \begin{bmatrix} \frac{1}{2}s & \varphi e^{\frac{1}{2}C} \\ e^{\frac{1}{2}C} & \frac{1}{2}s \end{bmatrix},
$$

where φ is inner, e^C is outer and $\varphi e^C = \frac{1}{4}s^2 - p$, and

$$
\Theta = \begin{bmatrix} -\frac{1}{2}s & \frac{p-\frac{1}{4}s^2}{g} \\ g & \frac{1}{2}s \end{bmatrix},
$$

where $g \in H^\infty$ is an outer function such that $|g(\lambda)|^2 = 1-\frac{1}{4}|s(\lambda)|^2$ for almost every $\lambda \in \mathbb{T}$. Then

$$
F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\Theta,
$$

since, by the uniqueness of the inner-outer factorisation of $\frac{1}{4}s^2 - p$ in the proofs of Theorem 3.3.1 and Proposition 3.3.3, we have $g = e^{\frac{1}{2}C}$ and

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Theta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}s & \frac{p - \frac{1}{4}s^2}{g} \\ g & \frac{1}{2}s \end{bmatrix} = \begin{bmatrix} \frac{1}{2}s & \frac{\frac{1}{4}s^2 - p}{g} \\ g & \frac{1}{2}s \end{bmatrix}.$$

3.3.2 Left $\mathrm{S_{SB}} : \mathcal{S}^{2\times 2} \to \mathrm{Hol}\,(\mathbb{D}, \Gamma)$

Definition 3.3.5. *We define* Left $\mathrm{S_{SB}} : \mathcal{S}^{2\times 2} \to \mathrm{Hol}\,(\mathbb{D}, \Gamma)$ *by*

$$F \mapsto (\mathrm{tr}\ F, \det F)$$

for all $F \in \mathcal{S}^{2\times 2}$.

Proposition 3.3.6. *The map* Left $\mathrm{S_{SB}}$ *is well defined.*

Proof. Let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. Clearly Left $\mathrm{S_{SB}}\,(F)$ is holomorphic since

$$\mathrm{tr}\ F = F_{11} + F_{22} \text{ and } \det F = F_{11}F_{22} - F_{21}F_{12}.$$

Let $\lambda \in \mathbb{D}$. Denote the eigenvalues of $F(\lambda)$ by $F_1(\lambda)$ and $F_2(\lambda)$. Then

$$\mathrm{tr}\ F(\lambda) = F_1(\lambda) + F_2(\lambda) \text{ and } \det F(\lambda) = F_1(\lambda)F_2(\lambda).$$

Moreover, since $||F(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$, we have $\rho(F(\lambda)) \leq 1$, and so $|F_1(\lambda)| \leq 1$ and $|F_2(\lambda)| \leq 1$. Hence

$$(\mathrm{tr}\ F(\lambda), \det F(\lambda)) = (F_1(\lambda) + F_2(\lambda), F_1(\lambda)F_2(\lambda)) \in \Gamma$$

for all $\lambda \in \mathbb{D}$. It follows that Left $\mathrm{S_{SB}}\,(F) = (\mathrm{tr}\ F, \det F) \in \mathrm{Hol}\,(\mathbb{D}, \Gamma)$. ☐

We now have a map $\mathcal{S}^{2\times 2} \to \mathrm{Hol}\,(\mathbb{D}, \Gamma)$ and a map $\mathrm{Hol}\,(\mathbb{D}, \Gamma) \to \mathcal{S}^{2\times 2}$, and so we can investigate how these maps interact.

Proposition 3.3.7. Left $\mathrm{S_{SB}} \circ$ Left $\mathrm{N_{SB}} = \mathrm{id}_{\mathrm{Hol}\,(\mathbb{D},\Gamma)}$.

Proof. Let $h \in \mathrm{Hol}\,(\mathbb{D}, \Gamma)$. Then Left $\mathrm{N_{SB}}\,(h) = F \in \mathcal{S}^{2\times 2}$ as defined in Definition 3.3.2. In particular, $(\mathrm{tr}\ F, \det F) = h$. It follows that Left $\mathrm{S_{SB}}\,(F) = h$, and hence

$$\text{Left } \mathrm{S_{SB}} \circ \text{Left } \mathrm{N_{SB}}\,(h) = h.$$

Consequently, Left $\mathrm{S_{SB}} \circ$ Left $\mathrm{N_{SB}} = \mathrm{id}_{\mathrm{Hol}\,(\mathbb{D},\Gamma)}$. ☐

Example 3.3.8. Let $F(\lambda) = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{bmatrix}$ for all $\lambda \in \mathbb{D}$. Then $F \in \mathcal{S}^{2\times 2}$ and Left $\mathrm{N_{SB}} \circ$ Left $\mathrm{S_{SB}}\,(F) \neq F$.

Proof. Clearly F is holomorphic on $\mathbb{D}$, and

$$||F(\lambda)|| = \max\{|\lambda^2|, |\lambda|\} = |\lambda| \leq 1$$

for all $\lambda \in \mathbb{D}$. Hence $F \in \mathcal{S}^{2\times 2}$. Thus we can apply Left $\mathrm{S_{SB}}$ to obtain

$$\text{Left}\,\mathrm{S_{SB}}\,(F)(\lambda) = (\text{tr } F(\lambda), \det F(\lambda)) = (\lambda^2 + \lambda, \lambda^3)$$

for all $\lambda \in \mathbb{D}$. Define $h = (s, p) \in \text{Hol}\,(\mathbb{D}, \Gamma)$ by $h(\lambda) := (\lambda^2 + \lambda, \lambda^3)$ for all $\lambda \in \mathbb{D}$, and let

$$\text{Left}\,\mathrm{N_{SB}}\,(h) := G \in \mathcal{S}^{2\times 2}.$$

Then the function G is defined as in Definition 3.3.2. In particular,

$$G = \begin{bmatrix} \frac{1}{2}s & G_{12} \\ G_{21} & \frac{1}{2}s \end{bmatrix},$$

where $G_{21}G_{12} = \frac{1}{4}s^2 - p$. Thus

$$G_{21}(\lambda)G_{12}(\lambda) = \frac{1}{2}(\lambda^2 + \lambda)^2 - \lambda^3$$

for all $\lambda \in \mathbb{D}$, and so $G_{21}G_{12} \neq 0$. Since $F_{21}F_{12} = 0$, we have $G \neq F$. It follows that Left $\mathrm{N_{SB}} \circ$ Left $\mathrm{S_{SB}}\,(F) \neq F$. □

3.3.3 $\mathrm{SW_{SB}} : \mathcal{R}_{11} \to \text{Hol}\,(\mathbb{D}, \Gamma)$

First we give a proposition which motivates our definition of $\mathrm{SW_{SB}}$. The idea is to follow Procedure UW with the map Left $\mathrm{S_{SB}}$.

Proposition 3.3.9. *Let $(N, M) \in \mathcal{R}_{11}$. Let Ξ be constructed from (N, M) by Procedure UW. Then*

$$\{\text{Left}\,\mathrm{S_{SB}}\,(F) : F \in \text{Upper W}\,((N, M))\} = \left\{ \left(\text{tr} \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} \Xi, \zeta \det \Xi \right) : \zeta \in \mathbb{T} \right\} \subseteq \text{Hol}\,(\mathbb{D}, \Gamma).$$

Proof. Let $F \in \text{Upper W}\,((N, M))$. Then $F = \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} \Xi \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix}$ for some $\zeta_1, \zeta_2 \in \mathbb{T}$, and so

$$\text{Left}\,\mathrm{S_{SB}}\,(F) = (\text{tr } F, \det F) = \left(\text{tr} \begin{bmatrix} \zeta_1 & 0 \\ 0 & 1 \end{bmatrix} \Xi, \zeta_1 \det \Xi \right).$$

It follows that

$$\{\operatorname{Left} \mathrm{S}_{\mathrm{SB}}(F) : F \in \operatorname{Upper} \mathrm{W}((N,M))\} = \left\{ \left(\operatorname{tr} \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} \Xi, \zeta \det \Xi \right) : \zeta \in \mathbb{T} \right\}.$$

Moreover, by Proposition 3.3.6, since $F \in \mathcal{S}^{2\times 2}$, we have $\operatorname{Left} \mathrm{S}_{\mathrm{SB}}(F) \in \operatorname{Hol}(\mathbb{D}, \Gamma)$. □

Definition 3.3.10. *We define* $\mathrm{SW}_{\mathrm{SB}}$ *to be the set-valued map from* $\mathcal{R}_{11}$ *to* $\operatorname{Hol}(\mathbb{D}, \Gamma)$ *given by*

$$\mathrm{SW}_{\mathrm{SB}}((N,M)) = \left\{ \left(\operatorname{tr} \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} \Xi, \zeta \det \Xi \right) : \zeta \in \mathbb{T} \right\}$$

for all $(N,M) \in \mathcal{R}_{11}$*, where* $\Xi \in \mathcal{S}^{2\times 2}$ *is constructed from* (N,M) *by Procedure UW.*

That $\mathrm{SW}_{\mathrm{SB}}$ is well defined follows from Proposition 3.3.9 and the observation that, as Upper W is independent of which function Ξ is used to define it, the set

$$\left\{ \left(\operatorname{tr} \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} \Xi, \zeta \det \Xi \right) : \zeta \in \mathbb{T} \right\}$$

is independent of the choice of Ξ.

By Proposition 3.3.9,

$$\{\operatorname{Left} \mathrm{S}_{\mathrm{SB}}(F) : F \in \operatorname{Upper} \mathrm{W}((N,M))\} = \mathrm{SW}_{\mathrm{SB}}((N,M))$$

for all $(N,M) \in \mathcal{R}_{11}$. We have the following other interactions with $\mathrm{SW}_{\mathrm{SB}}$.

Proposition 3.3.11. *Let* $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$ *be such that* $F_{21} \neq 0$*. Then*

$$\mathrm{SW}_{\mathrm{SB}} \circ \operatorname{Upper} \mathrm{E}(F) = \left\{ \operatorname{Left} \mathrm{S}_{\mathrm{SB}} \left(\begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} F \right) : \zeta \in \mathbb{T} \right\}.$$

Proof. By Proposition 2.4.12,

$$\operatorname{Upper} \mathrm{W} \circ \operatorname{Upper} \mathrm{E}(F) = \left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} F \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} : \zeta_1, \zeta_2 \in \mathbb{T} \right\}.$$

Hence

$$\begin{aligned} \mathrm{SW}_{\mathrm{SB}} \circ \operatorname{Upper} \mathrm{E}(F) &= \left\{ \operatorname{Left} \mathrm{S}_{\mathrm{SB}} \left(\begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} F \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} \right) : \zeta_1, \zeta_2 \in \mathbb{T} \right\} \\ &= \left\{ \left(\operatorname{tr} \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} F \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix}, \det \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} F \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} \right) : \zeta_1, \zeta_2 \in \mathbb{T} \right\} \\ &= \left\{ \operatorname{Left} \mathrm{S}_{\mathrm{SB}} \left(\begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} F \right) : \zeta \in \mathbb{T} \right\}, \end{aligned}$$

as required. □

Corollary 3.3.12. *Let $h = (s, p) \in \mathrm{Hol}\,(\mathbb{D}, \Gamma)$ be such that $s^2 \neq p$. Then*

$$\mathrm{SW_{SB}} \circ \mathrm{Upper\,E} \circ \mathrm{Left\,N_{SB}}\,(h) = \left\{ \left(\frac{1}{2}(\zeta + 1)s, \zeta p \right) : \zeta \in \mathbb{T} \right\}.$$

Proof. Let $F = \mathrm{Left\,N_{SB}}\,(h)$. Then, in particular,

$$F = \begin{bmatrix} \frac{1}{2}s & F_{12} \\ F_{21} & \frac{1}{2}s \end{bmatrix},$$

$F_{21} \neq 0$ and $\det F = p$. By Proposition 3.3.11,

$$\begin{aligned} \mathrm{SW_{SB}} \circ \mathrm{Upper\,E}\,(F) &= \left\{ \mathrm{Left\,S_{SB}} \left(\begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} F \right) : \zeta \in \mathbb{T} \right\} \\ &= \left\{ \mathrm{Left\,S_{SB}} \left(\begin{bmatrix} \zeta \frac{1}{2}s & \zeta F_{12} \\ F_{21} & \frac{1}{2}s \end{bmatrix} \right) : \zeta \in \mathbb{T} \right\} \\ &= \left\{ \left(\frac{1}{2}(\zeta + 1)s, \zeta \det F \right) : \zeta \in \mathbb{T} \right\}. \end{aligned}$$

Hence $\mathrm{SW_{SB}} \circ \mathrm{Upper\,E} \circ \mathrm{Left\,N_{SB}}\,(h) = \left\{ \left(\frac{1}{2}(\zeta + 1)s, \zeta p \right) : \zeta \in \mathbb{T} \right\}$. □

We note that if $\zeta = 1$ then $\left(\frac{1}{2}(\zeta + 1)s, \zeta p \right) = (s, p)$. Hence, by Corollary 3.3.12, for all $h \in \mathrm{Hol}\,(\mathbb{D}, \Gamma)$, we have $h \in \mathrm{SW_{SB}} \circ \mathrm{Upper\,E} \circ \mathrm{Left\,N_{SB}}\,(h)$.

3.3.4 $\mathrm{Lower\,E_{SB}} : \mathrm{Hol}\,(\mathbb{D}, \Gamma) \to \mathcal{S}_2$

The definition of $\mathrm{Lower\,E_{SB}}$ comes from the relationship between $\mathrm{Hol}\,(\mathbb{D}, \Gamma)$ and a particular subset of $\mathcal{S}_2$. The relationship uses the function Φ. Recall that, for $h = (s, p) \in \mathrm{Hol}\,(\mathbb{D}, \Gamma)$ and $z, \lambda \in \mathbb{D}$, we have

$$\Phi(z, h(\lambda)) = \frac{2zp(\lambda) - s(\lambda)}{2 - zs(\lambda)}.$$

Thus, for all $\lambda \in \mathbb{D}$, we have $\Phi(\cdot, h(\lambda))$ is a linear fractional map with the property '$b = c$'. By the property '$b = c$' we refer to the general form of a linear fractional map,

$$z \mapsto \frac{az + b}{cz + d}.$$

In our example, $a = 2p(\lambda)$, $b = -s(\lambda)$, $c = -s(\lambda)$ and $d = 2$, and hence $b = c$. Moreover, for all $\lambda \in \mathbb{D}$, by Proposition 3.2.3 (i), $h(\lambda) \in \Gamma$ if and only if $|s(\lambda)| \leq 2$ and, for all w in a dense subset of $\mathbb{T}$, we have

$$|\Phi(w, h(\lambda))| \leq 1.$$

This motivates the following definition of the subset of $\mathcal{S}_2$.

Definition 3.3.13. *We define $\mathcal{S}_2^{b=c}$ to be the subset of $\mathcal{S}_2$ which contains those φ such that, for all $\lambda \in \mathbb{D}$, we have $\varphi(\cdot, \lambda)$ is a linear fractional map with the property '$b = c$'.*

We now give the relationship between $\operatorname{Hol}(\mathbb{D}, \Gamma)$ and $\mathcal{S}_2^{b=c}$.

Proposition 3.3.14. [3, Proposition 5.2] *Let φ be a function on $\mathbb{D}^2$. Then $\varphi \in \mathcal{S}_2^{b=c}$ if and only if there exists a function $h \in \operatorname{Hol}(\mathbb{D}, \Gamma)$ such that*

$$\varphi(z, \lambda) = \Phi(z, h(\lambda)) \text{ for all } z, \lambda \in \mathbb{D}.$$

Moreover, if $\varphi \in \mathcal{S}_2^{b=c}$ then its corresponding function h is unique.

Proof. First, suppose $h = (s, p) \in \operatorname{Hol}(\mathbb{D}, \Gamma)$. Define $\varphi(z, \lambda) := \Phi(z, h(\lambda))$ for all $z, \lambda \in \mathbb{D}$. Since h is holomorphic and maps into Γ, and since Φ is holomorphic on $\mathbb{D} \times \Gamma$, we infer that φ is holomorphic on $\mathbb{D}^2$. For any $\lambda \in \mathbb{D}$, by Proposition 3.2.3 (i), for all w in a dense subset of $\mathbb{T}$, we have

$$|\Phi(w, h(\lambda))| \leq 1.$$

Hence, for any $z, \lambda \in \mathbb{D}$, by the Maximum Modulus Principle,

$$|\Phi(z, h(\lambda))| \leq 1,$$

and so $\varphi(z, \lambda) \in \overline{\mathbb{D}}$. It follows that $\varphi \in \mathcal{S}_2$. Moreover,

$$\varphi(z, \lambda) = \frac{2zp(\lambda) - s(\lambda)}{2 - zs(\lambda)} = \frac{2p(\lambda)z + (-s(\lambda))}{(-s(\lambda))z + 2}$$

for all $z, \lambda \in \mathbb{D}$. It follows that $\varphi \in \mathcal{S}_2^{b=c}$.

Conversely, suppose that $\varphi \in \mathcal{S}_2^{b=c}$. Then, for all $\lambda \in \mathbb{D}$, we have $\varphi(\cdot, \lambda)$ is a linear fractional map with the property '$b = c$'. Thus we can write

$$\varphi(z, \lambda) = \frac{a(\lambda)z + b(\lambda)}{b(\lambda)z + d(\lambda)}$$

for all $z, \lambda \in \mathbb{D}$, where a, b, d are functions from $\mathbb{D}$ to $\mathbb{C}$. Since $\varphi \in \mathcal{S}_2$, for any $\lambda \in \mathbb{D}$, up to cancellation, $\varphi(\cdot, \lambda)$ does not have a pole at 0, and so $d(\lambda) \neq 0$. Hence, without loss of generality, we can write

$$\varphi(z, \lambda) = \frac{a(\lambda)z + b(\lambda)}{b(\lambda)z + 1}$$

for all $z, \lambda \in \mathbb{D}$. Set $h(\lambda) = (-2b(\lambda), a(\lambda))$ for all $\lambda \in \mathbb{D}$. Then, since $b(\lambda) = \varphi(0, \lambda)$ and

$$a(\lambda)z = \varphi(z, \lambda)\,(b(\lambda)z + 1) - b(\lambda) = \varphi(z, \lambda)\,(\varphi(0, \lambda)z + 1) - \varphi(0, \lambda)$$

for all $z, \lambda \in \mathbb{D}$, we have h is holomorphic on $\mathbb{D}$. Now,

$$\Phi(z, h(\lambda)) = \frac{2za(\lambda) + 2b(\lambda)}{2 + z2b(\lambda)} = \frac{a(\lambda)z + b(\lambda)}{b(\lambda)z + 1} = \varphi(z, \lambda)$$

for all $z, \lambda \in \mathbb{D}$. Thus, by the continuity of Φ and since

$$|\Phi(z, h(\lambda))| = |\varphi(z, \lambda)| \leq 1 \text{ for all } z, \lambda \in \mathbb{D},$$

we have $|\Phi(w, h(\lambda))| \leq 1$ for every $\lambda \in \mathbb{D}$ and every w in a dense subset of $\mathbb{T}$. Moreover,

$$|2b(\lambda)| = 2|\varphi(0, \lambda)| \leq 2$$

for all $\lambda \in \mathbb{D}$. Hence, by Proposition 3.2.3 (ii), $h(\lambda) \in \Gamma$ for all $\lambda \in \mathbb{D}$. It follows that there exists an $h \in \mathrm{Hol}\,(\mathbb{D}, \Gamma)$ such that $\varphi(z, \lambda) = \Phi(z, h(\lambda))$ for all $z, \lambda \in \mathbb{D}$.

For uniqueness, suppose that $\varphi \in \mathcal{S}_2^{b=c}$. Let $h, g \in \mathrm{Hol}\,(\mathbb{D}, \Gamma)$ be such that

$$\Phi(z, h(\lambda)) = \varphi(z, \lambda) = \Phi(z, g(\lambda))$$

for all $z, \lambda \in \mathbb{D}$. Then, if $h = (s, p)$ and $g = (q, r)$, we have

$$\frac{2zp(\lambda) - s(\lambda)}{2 - zs(\lambda)} = \frac{2zr(\lambda) - q(\lambda)}{2 - zq(\lambda)}$$

and so

$$-2p(\lambda)q(\lambda)z^2 + (4p(\lambda) + q(\lambda)s(\lambda))z - 2s(\lambda) = -2r(\lambda)s(\lambda)z^2 + (4r(\lambda) + q(\lambda)s(\lambda))z - 2q(\lambda)$$

for all $z, \lambda \in \mathbb{D}$. By equating coefficients, we obtain $s(\lambda) = q(\lambda)$, $p(\lambda)q(\lambda) = r(\lambda)s(\lambda)$ and

$$4p(\lambda) + q(\lambda)s(\lambda) = 4r(\lambda) + q(\lambda)s(\lambda)$$

for all $\lambda \in \mathbb{D}$. It follows that $s = q$, $p = r$ and hence $h = g$. □

Definition 3.3.15. *We define* $\mathrm{Lower\,E_{SB}} : \mathrm{Hol}\,(\mathbb{D}, \Gamma) \to \mathcal{S}_2^{b=c}$ *by*

$$\mathrm{Lower\,E_{SB}}\,(h)(z, \lambda) := \Phi(z, h(\lambda)) = \frac{2zp(\lambda) - s(\lambda)}{2 - zs(\lambda)}$$

for all $h = (s, p) \in \mathrm{Hol}\,(\mathbb{D}, \Gamma)$ *and all* $z, \lambda \in \mathbb{D}$.

That $\mathrm{Lower\,E_{SB}}$ is well defined follows immediately from Proposition 3.3.14

3.3.5 $\mathrm{Lower\,W_{SB}} : \mathcal{S}_2^{b=c} \to \mathrm{Hol}\,(\mathbb{D}, \Gamma)$

The proof of Proposition 3.3.14 provides the construction of a unique function in $\mathrm{Hol}\,(\mathbb{D}, \Gamma)$ for each function in $\mathcal{S}_2^{b=c}$. We use this construction to define the map $\mathrm{Lower\,W_{SB}}$.

Definition 3.3.16. *We define* $\mathrm{Lower\,W_{SB}} : \mathcal{S}_2^{b=c} \to \mathrm{Hol}\,(\mathbb{D}, \Gamma)$ *by*

$$\mathrm{Lower\,W_{SB}}\,(\varphi) = (-2b, a)$$

for $\varphi \in \mathcal{S}_2^{b=c}$*, where* φ *can be written* $\varphi(z,\lambda) = \frac{a(\lambda)z+b(\lambda)}{b(\lambda)z+1}$ *for all* $z, \lambda \in \mathbb{D}$*.*

That $\mathrm{Lower\,W_{SB}}$ is well defined follows from Proposition 3.3.14 and the observation that, if $\varphi \in \mathcal{S}_2^{b=c}$ is such that $\varphi(z,\lambda) = b(\lambda)$ for all $z, \lambda \in \mathbb{D}$, then

$$\varphi(z,\lambda) = \frac{b(\lambda)^2 z + b(\lambda)}{b(\lambda)+1}$$

for all $z, \lambda \in \mathbb{D}$, and so $\mathrm{Lower\,W_{SB}}(\varphi) = (-2b, b^2)$. The maps $\mathrm{Lower\,W_{SB}}$ and $\mathrm{Lower\,E_{SB}}$ are mutually inverse.

Proposition 3.3.17. *The following relations hold.*

(i) $\mathrm{Lower\,W_{SB}} \circ \mathrm{Lower\,E_{SB}} = \mathrm{id}_{\mathrm{Hol}\,(\mathbb{D},\Gamma)}$.

(ii) $\mathrm{Lower\,E_{SB}} \circ \mathrm{Lower\,W_{SB}} = \mathrm{id}_{\mathcal{S}_2^{b=c}}$.

Proof. (i) Let $h = (s,p) \in \mathrm{Hol}\,(\mathbb{D},\Gamma)$. Then $\mathrm{Lower\,E_{SB}}(h) \in \mathcal{S}_2^{b=c}$, and

$$\mathrm{Lower\,E_{SB}}(h)(z,\lambda) = \Phi(z, h(\lambda)) = \frac{2zp(\lambda) - s(\lambda)}{2 - zs(\lambda)} = \frac{p(\lambda)z - \frac{1}{2}s(\lambda)}{-\frac{1}{2}s(\lambda)z + 1}$$

for all $z, \lambda \in \mathbb{D}$. Hence

$$\mathrm{Lower\,W_{SB}} \circ \mathrm{Lower\,E_{SB}}(h) = \left(-2\left(-\frac{1}{2}s\right), p\right) = h.$$

It follows that $\mathrm{Lower\,W_{SB}} \circ \mathrm{Lower\,E_{SB}} = \mathrm{id}_{\mathrm{Hol}\,(\mathbb{D},\Gamma)}$.

(ii) Let $\varphi \in \mathcal{S}_2^{b=c}$, where φ can be written $\varphi(z,\lambda) = \frac{a(\lambda)z+b(\lambda)}{b(\lambda(z)+1}$ for all $z, \lambda \in \mathbb{D}$. Then $\mathrm{Lower\,W_{SB}}(\varphi) = (-2b, a)$. Hence

$$\mathrm{Lower\,E_{SB}} \circ \mathrm{Lower\,W_{SB}}(\varphi)(z,\lambda) = \Phi(z, -2b(\lambda), a(\lambda)) = \frac{2a(\lambda)z + 2b(\lambda)}{2 + 2b(\lambda)z} = \varphi(z,\lambda)$$

for all $z, \lambda \in \mathbb{D}$. It follows that $\mathrm{Lower\,E_{SB}} \circ \mathrm{Lower\,W_{SB}} = \mathrm{id}_{\mathcal{S}_2^{b=c}}$. □

3.3.6 Relations between the remaining maps

We now consider how some of the maps we defined in this section interact with some of the maps in Chapter 2.

Proposition 3.3.18. $\mathrm{SE} \circ \mathrm{Left\,N_{SB}} = \mathrm{Lower\,E_{SB}}$.

Proof. Let $h \in \mathrm{Hol}\,(\mathbb{D},\Gamma)$. Then $\mathrm{Left\,N_{SB}}(h) = F \in \mathcal{S}^{2\times 2}$ as defined in Theorem 3.3.1. By the proof of Theorem 3.3.1,

$$\mathrm{SE}\,(F)(z,\lambda) = -\mathcal{F}_{F(\lambda)}(z) = \Phi(z, h(\lambda))$$

for all $z, \lambda \in \mathbb{D}$. Hence

$$\mathrm{SE} \circ \mathrm{Left\,N_{SB}}\,(h)(z,\lambda) = \Phi(z, h(\lambda)) = \mathrm{Lower\,E_{SB}}\,(h)(z,\lambda)$$

for all $z, \lambda \in \mathbb{D}$. It follows that $\mathrm{SE} \circ \mathrm{Left\,N_{SB}} = \mathrm{Lower\,E_{SB}}$. □

Corollary 3.3.19. *The following relations hold.*

(i) $\mathrm{Lower\,W_{SB}} \circ \mathrm{SE} \circ \mathrm{Left\,N_{SB}} = \mathrm{id}_{\mathrm{Hol}\,(\mathbb{D},\Gamma)}$.

(ii) $\mathrm{SE} \circ \mathrm{Left\,N_{SB}} \circ \mathrm{Lower\,W_{SB}} = \mathrm{id}_{\mathcal{S}_2^{b=c}}$.

Proof. The results follow immediately from Proposition 3.3.18 and Proposition 3.3.17. □

Proposition 3.3.20. *Let* $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. *If* $F_{11} = F_{22}$ *then*

$$\mathrm{Lower\,E_{SB}} \circ \mathrm{Left\,S_{SB}}\,(F) = \mathrm{SE}\,(F).$$

Proof. We have

$$\mathrm{SE}\,(F)(z,\lambda) = -F_{11}(\lambda) - \frac{F_{12}(\lambda)F_{21}(\lambda)z}{1 - F_{11}(\lambda)z} = \frac{-F_{11}(\lambda) + (F_{11}(\lambda)^2 - F_{12}(\lambda)F_{21}(\lambda))z}{1 - F_{11}(\lambda)z}$$

for all $z, \lambda \in \mathbb{D}$. Moreover, $\mathrm{Left\,S_{SB}}\,(F) = (\mathrm{tr}\ F, \det F) = (2F_{11}, F_{11}^2 - F_{21}F_{12})$ and so

$$\begin{aligned} \mathrm{Lower\,E_{SB}} \circ \mathrm{Left\,S_{SB}}\,(F)(z,\lambda) &= \Phi(z, 2F_{11}(\lambda), F_{11}(\lambda)^2 - F_{21}(\lambda)F_{12}(\lambda)) \\ &= \frac{2z(F_{11}^2(\lambda) - F_{21}(\lambda)F_{12}(\lambda)) - 2F_{11}(\lambda)}{2 - 2zF_{11}(\lambda)} \\ &= \frac{-F_{11}(\lambda) + (F_{11}(\lambda)^2 - F_{12}(\lambda)F_{21}(\lambda))z}{1 - F_{11}(\lambda)z} \end{aligned}$$

for all $z, \lambda \in \mathbb{D}$. It follows that $\mathrm{Lower\,E_{SB}} \circ \mathrm{Left\,S_{SB}}\,(F) = \mathrm{SE}\,(F)$. □

However, for an arbitrary $F \in \mathcal{S}^{2\times 2}$, we may have $\mathrm{Lower\,E_{SB}} \circ \mathrm{Left\,S_{SB}}\,(F) \neq \mathrm{SE}\,(F)$, as illustrated by the following example.

Example 3.3.21. Let $f(z) = \dfrac{1-2z}{2-z}$ and $g(z) = \dfrac{1+2z}{2+z}$ for all $z \in \mathbb{D}$. Set $F = \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}$. Then $F \in \mathcal{S}^{2\times 2}$ and $\mathrm{Lower\,E_{SB}} \circ \mathrm{Left\,S_{SB}}\,(F) \neq \mathrm{SE}\,(F)$.

Proof. A Blaschke factor has the form

$$B_a(z) = \frac{\overline{a}}{|a|}\frac{a - z}{1 - \overline{a}z}$$

for all $z \in \mathbb{D}$ and some $a \in \mathbb{D}$. Hence $f = B_{\frac{1}{2}}$ and $g = B_{-\frac{1}{2}}$. By Remark B.1.20, f and g are holomorphic functions on $\mathbb{D}$ such that $|f(z)| \leq 1$ and $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. It follows that F is holomorphic on $\mathbb{D}$, and

$$||F(\lambda)|| = \max\{|f(\lambda)|, |g(\lambda)|\} \leq 1$$

for all $\lambda \in \mathbb{D}$. Hence $F \in \mathcal{S}^{2\times 2}$.

Clearly $f \neq g$. Moreover,

$$\mathrm{SE}\,(F)(0,\lambda) = -F_{11}(\lambda) - \frac{F_{12}(\lambda)F_{21}(\lambda)\cdot 0}{1 - F_{22}(\lambda)\cdot 0} = -f(\lambda)$$

and

$$\mathrm{Lower}\,\mathrm{E_{SB}} \circ \mathrm{Left}\,\mathrm{S_{SB}}\ (F)\,(0,\lambda) = \frac{2\cdot 0\cdot \det F(\lambda) - \mathrm{tr}\ F(\lambda)}{2 - 0\cdot \mathrm{tr}\ F(\lambda)} = \frac{-(f(\lambda)+g(\lambda))}{2}$$

for all $\lambda \in \mathbb{D}$. If $\mathrm{Lower}\,\mathrm{E_{SB}} \circ \mathrm{Left}\,\mathrm{S_{SB}}\,(F) = \mathrm{SE}\,(F)$, then $f(\lambda) = g(\lambda)$ for all $\lambda \in \mathbb{D}$, which is a contradiction. It follows that $\mathrm{Lower}\,\mathrm{E_{SB}} \circ \mathrm{Left}\,\mathrm{S_{SB}}\,(F) \neq \mathrm{SE}\,(F)$. □

Proposition 3.3.22. *Let $\varphi \in \mathcal{S}_2^{b=c}$. Then*

$$\mathrm{Right}\,\mathrm{S} \circ \mathrm{Upper}\,\mathrm{E} \circ \mathrm{Left}\,\mathrm{N_{SB}} \circ \mathrm{Lower}\,\mathrm{W_{SB}}\,(\varphi) = \{\zeta\varphi : \zeta \in \mathbb{T}\}\,.$$

Proof. By Corollary 3.3.19 (ii),

$$\mathrm{SE} \circ \mathrm{Left}\,\mathrm{N_{SB}} \circ \mathrm{Lower}\,\mathrm{W_{SB}}\,(\varphi) = \varphi.$$

Moreover, by Proposition 2.5.3, since $\mathrm{Left}\,\mathrm{N_{SB}} \circ \mathrm{Lower}\,\mathrm{W_{SB}}\,(\varphi) \in \mathcal{S}^{2\times 2}$, we have

$$\mathrm{Right}\,\mathrm{S} \circ \mathrm{Upper}\,\mathrm{E}\,(\mathrm{Left}\,\mathrm{N_{SB}} \circ \mathrm{Lower}\,\mathrm{W_{SB}}\,(\varphi)) = \{\zeta\,\mathrm{SE}\,(\mathrm{Left}\,\mathrm{N_{SB}} \circ \mathrm{Lower}\,\mathrm{W_{SB}}\,(\varphi)) : \zeta \in \mathbb{T}\}\,.$$

It follows that $\mathrm{Right}\,\mathrm{S} \circ \mathrm{Upper}\,\mathrm{E} \circ \mathrm{Left}\,\mathrm{N_{SB}} \circ \mathrm{Lower}\,\mathrm{W_{SB}}\,(\varphi) = \{\zeta\varphi : \zeta \in \mathbb{T}\}\,.$ □

3.4 Criterion for solvability

In this section, we present a criterion for the solvability of the μ-synthesis problem given by Question 1.2.1. In addition, we give a number of related results, which can be seen to arise from the rich structure we have been studying. The proofs of the results in this section are contained in [3].

Theorem 3.4.1. *Let $\lambda_1, \dots, \lambda_n$ be distinct points in $\mathbb{D}$, and let $W_j \in \mathcal{M}_2(\mathbb{C})$ be such that $\rho(W_j) \leq 1$ for $j = 1, \dots, n$, and none of which are scalar multiples of the identity. Set $(s_j, p_j) = (\mathrm{tr}\ W_j, \det W_j) \in \Gamma$ for each $j = 1, \dots, n$. Then the following are equivalent.*

(i) *There exists a holomorphic function $F : \mathbb{D} \to \mathcal{M}_2(\mathbb{C})$ such that $F(\lambda_j) = W_j$ for $j = 1, \dots, n$, and $\rho(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$;*

(ii) *there exists a holomorphic function* $h : \mathbb{D} \to \Gamma$ *satisfying*

$$h(\lambda_j) = (s_j, p_j) \text{ for } j = 1, \ldots, n;$$

(iii) *for some distinct points* $z_1, z_2, z_3 \in \mathbb{D}$, *there exist positive* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1, *and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that*

$$\left[1 - \overline{\frac{2z_l p_i - s_i}{2 - z_l s_i}} \frac{2z_k p_j - s_{1j}}{2 - z_k s_j}\right] \geq [(1 - \overline{z_l} z_k) N_{il,jk}] + \left[(1 - \overline{\lambda_i}\lambda_j) M_{il,jk}\right].$$

Theorem 3.4.1 follows easily from a combination of the following theorem with [3, Theorem 8.4].

Theorem 3.4.2. [3, Theorem 8.1] *Let* $\lambda_1, \ldots, \lambda_n$ *be distinct points in* $\mathbb{D}$, *and let* $(s_j, p_j) \in \Gamma$ *be such that* $s_j^2 \neq 4p_j$ *for* $j = 1, \ldots, n$. *Then the following are equivalent.*

(i) *There exists a holomorphic function* $h : \mathbb{D} \to \Gamma$ *satisfying*

$$h(\lambda_j) = (s_j, p_j) \text{ for } j = 1, \ldots, n;$$

(ii) *there exists a rational* Γ*-inner function* h *satisfying*

$$h(\lambda_j) = (s_j, p_j) \text{ for } j = 1, \ldots, n;$$

(iii) *for every distinct points* $z_1, z_2, z_3 \in \mathbb{D}$, *there exist positive* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1, *and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that, for* $1 \leq i, j \leq n$ *and* $1 \leq l, k \leq 3$,

$$1 - \overline{\frac{2z_l p_i - s_i}{2 - z_l s_i}} \frac{2z_k p_j - s_j}{2 - z_k s_j} = (1 - \overline{z_l} z_k) N_{il,jk} + (1 - \overline{\lambda_i}\lambda_j) M_{il,jk};$$

(iv) *for some distinct points* $z_1, z_2, z_3 \in \mathbb{D}$, *there exist positive* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1, *and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that*

$$\left[1 - \overline{\frac{2z_l p_i - s_i}{2 - z_l s_i}} \frac{2z_k p_j - s_j}{2 - z_k s_j}\right] \geq [(1 - \overline{z_l} z_k) N_{il,jk}] + \left[(1 - \overline{\lambda_i}\lambda_j) M_{il,jk}\right].$$

The proof of Theorem 3.4.2 shows constructively that (iv) implies (i), which provides a procedure by which a solution h can be obtained once a pair (N, M) is known. The authors of [3] call this Procedure SW, and it is essentially Procedure UW followed by the Left $\mathrm{S_{SB}}$ map. More specifically we have:

Procedure SW_{SB}. [3, p. 2503] Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$, and let $(s_j, p_j) \in \Gamma$ be such that $s_j^2 \neq 4p_j$ for $j = 1, \ldots, n$. For some distinct points $z_1, z_2, z_3 \in \mathbb{D}$, suppose N

and M are positive $3n$-square matrices such that N has rank at most 1, and the matrix inequality as in Theorem 3.4.2 (iv) holds. Then:

1. Choose scalars $\gamma_{jk} \in \mathbb{C}$ such that $N = [\overline{\gamma_{il}}\gamma_{jk}]_{i,j=1,l,k=1}^{n,3}$.

2. Choose a Hilbert space $\mathcal{H}$ and vectors $v_{jk} \in \mathcal{H}$ such that $M = [\langle v_{jk}, v_{il}\rangle_{\mathcal{H}}]_{i,j=1,l,k=1}^{n,3}$.

3. Choose a contraction $\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{C}^2 \oplus \mathcal{H} \to \mathbb{C}^2 \oplus \mathcal{H}$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \left(\begin{pmatrix} 1 \\ z_k\gamma_{jk} \end{pmatrix} \atop \lambda_j v_{jk} \right) = \left(\begin{pmatrix} -\Phi(z_k, s_j, p_j) \\ \gamma_{jk} \end{pmatrix} \atop v_{jk} \right),$$

   for all $j = 1, \ldots, n$ and $k = 1, 2, 3$.

4. Let $h(\lambda) = (\mathrm{tr}\,, \det)(A + B\lambda(I - D\lambda)^{-1}C)$ for all $\lambda \in \mathbb{D}$.

Now, we have $h \in \mathrm{Hol}\,(\mathbb{D}, \Gamma)$ and $h(\lambda_j) = (s_j, p_j)$ for $j = 1, \ldots, n$. $\square$

The following proposition shows that every interpolating function can be obtained by applying Procedure SW_{SB} to a general solution (N, M) of the matrix inequality such that the rank of N is less than or equal to 1.

Proposition 3.4.3. [3, Proposition 10.1] *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$, and let $(s_j, p_j) \in \Gamma$ be such that $s_j^2 \neq 4p_j$ for $j = 1, \ldots, n$. Then every holomorphic function $h : \mathbb{D} \to \Gamma$ satisfying*

$$h(\lambda_j) = (s_j, p_j), \text{ for } j = 1, \ldots, n,$$

arises by Procedure SW_{SB} from a pair of positive $3n$-square matrices $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ of rank at most 1, and $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ satisfying

$$\left[1 - \overline{\frac{2z_l p_i - s_i}{2 - z_l s_i}} \frac{2z_k p_j - s_j}{2 - z_k s_j}\right] \geq [(1 - \overline{z_l} z_k) N_{il,jk}] + \left[(1 - \overline{\lambda_i}\lambda_j) M_{il,jk}\right],$$

where z_1, z_2, z_3 are distinct points in $\mathbb{D}$.

The following proposition shows that, in order to use Theorem 3.4.2 to determine if there is an interpolating function, it is sufficient to search over a compact set for a pair (N, M) that satisfies the matrix inequality and such that the rank of N is 1.

Proposition 3.4.4. [3, Proposition 11.1] *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$, and let $(s_j, p_j) \in \Gamma$ be such that $s_j^2 \neq 4p_j$ for $j = 1, \ldots, n$. Then the interpolation problem*

$$\lambda_j \in \mathbb{D} \mapsto (s_j, p_j) \in \Gamma, \text{ for all } j = 1, \ldots, n,$$

is solvable if and only if, for some distinct points $z_1, z_2, z_3 \in \mathbb{D}$*, there exist positive* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank* 1*, and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *that satisfy*

$$\left[1 - \overline{\frac{2z_l p_i - s_i}{2 - z_l s_i}} \frac{2z_k p_j - s_j}{2 - z_k s_j}\right] \geq [(1 - \overline{z_l} z_k) N_{il,jk}] + \left[(1 - \overline{\lambda_i}\lambda_j) M_{il,jk}\right],$$

and

$$|N_{il,jk}| \leq \frac{1}{(1 - \frac{1}{2}|s_i|)(1 - \frac{1}{2}|s_j|)} \text{ and}$$

$$|M_{il,jk}| \leq \frac{2}{|1 - \overline{\lambda_i}\lambda_j|} \sqrt{1 + \frac{1}{(1 - \frac{1}{2}|s_j|)^2}} \sqrt{1 + \frac{1}{(1 - \frac{1}{2}|s_j|)^2}}$$

for all $1 \leq i, j \leq n$ *and* $1 \leq l, k \leq 3$*.*

Chapter 4. The tetrablock

4.1 Introduction

The tetrablock was introduced by Abouhajar, White and Young in [1]. The authors studied the complex geometry of the tetrablock. One of the main results of the paper is a Schwarz lemma for the tetrablock. Motivation to study the tetrablock came from a special case of the μ-synthesis problem. The authors showed that the solvability of this special case is equivalent to the solvability of an interpolation problem into the tetrablock.

We define the *open tetrablock* to be the set

$$\mathbb{E} := \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \text{ for all } z, w \in \overline{\mathbb{D}}\},$$

and denote its closure by $\overline{\mathbb{E}}$. More explicitly we have the following.

Proposition 4.1.1. [1, Theorem 2.4] *The closed tetrablock satisfies*

$$\overline{\mathbb{E}} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \textit{ for all } z, w \in \mathbb{D}\}.$$

It is shown in [1] that the tetrablock intersects $\mathbb{R}^3$ in a regular tetrahedron, which inspires its name. The following result about the geometry of the tetrablock is also given. (For the next proposition, see Definition B.1.23 and Definition B.1.24 for the notions of hypoconvexity and polynomial convexity.)

Proposition 4.1.2. [64, Lemma 2.2][1, Theorem 2.7, Theorem 2.9] *The tetrablock is hypoconvex, polynomially convex, starlike about* $(0,0,0)$*, and not convex.*

Recall that

$$\text{Diag} = \left\{ \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix} : z, w \in \mathbb{C} \right\},$$

and $\mu_{\text{Diag}}(M) = (\inf\{||E|| : E \in \text{Diag and } I - ME \text{ is singular}\})^{-1}$ for all $M \in \mathcal{M}_2(\mathbb{C})$, where $\mu_{\text{Diag}}(M) = 0$ if $I - ME$ is non-singular for all $E \in \text{Diag}$. The μ-synthesis problem for μ_{Diag} is:

Question 4.1.3. Let $\lambda_1, \dots, \lambda_n$ be distinct points in $\mathbb{D}$. Let $W_1, \dots, W_n \in \mathcal{M}_2(\mathbb{C})$ be such that $\mu_{\text{Diag}}(W_j) \leq 1$ for $j = 1, \dots, n$. Does there exist a holomorphic function $F : \mathbb{D} \to \mathcal{M}_2(\mathbb{C})$ such that $F(\lambda_j) = W_j$ for all $j = 1, \dots, n$, and $\mu_{\text{Diag}}(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$?

It can be shown that $x \in \overline{\mathbb{E}}$ if and only if there exists an

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$$

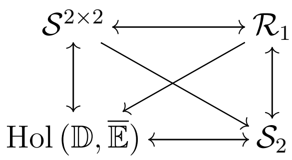
such that $\mu_{\mathrm{Diag}}(M) \leq 1$ and $x = (m_{11}, m_{22}, \det M)$. In [1], Abouhajar, White and Young used this fact to show that the solvability of Question 4.1.3 is equivalent to the solvability of an interpolation problem from $\mathbb{D}$ to $\overline{\mathbb{E}}$. More precisely they proved the following theorem.

Theorem 4.1.4. [1, Theorem 9.2] *Let $\lambda_1, \dots, \lambda_n$ be distinct points in $\mathbb{D}$. Let $W_1, \dots, W_n \in \mathcal{M}_2(\mathbb{C})$ be such that $\mu_{\mathrm{Diag}}(W_j) \leq 1$ and $w_{11}^j w_{22}^j \neq \det W_j$ for $j = 1, \dots, n$. Then the following are equivalent.*

(i) *There is a holomorphic function $F : \mathbb{D} \to \mathcal{M}_2(\mathbb{C})$ such that $F(\lambda_j) = W_j$ for $j = 1, \dots, n$, and $\mu_{\mathrm{Diag}}(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$.*

(ii) *There is an $x \in \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ such that $x(\lambda_j) = (w_{11}^j, w_{22}^j, \det W_j)$ for $j = 1, \dots, n$.*

In this chapter, we discuss some background material for the tetrablock and define $\overline{\mathbb{E}}$-inner functions. Afterwards, we focus on the construction of the maps that illustrate the rich structure of interconnections between the sets $\mathcal{S}^{2\times 2}, \mathcal{S}_2, \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ and $\mathcal{R}_1$. The maps can be summarised by the rich saltire:



We use the maps we produce to obtain conditions for the solvability of the μ-synthesis problem in Theorem 4.1.4.

4.2 Background

The following two functions are useful in the study of the tetrablock. As defined in [1], let $\Psi, \Upsilon : \mathbb{C}^4 \to \mathbb{C}$ be given by

$$\Psi(z, x_1, x_2, x_3) = \frac{x_3 z - x_1}{x_2 z - 1}, \text{ for } x_2 z \neq 1, \text{ and } \Upsilon(z, x_1, x_2, x_3) = \frac{x_3 z - x_2}{x_1 z - 1}, \text{ for } x_1 z \neq 1,$$

where $z, x_1, x_2, x_3 \in \mathbb{C}$. Note that, since Ψ is a rational function and rational functions are holomorphic, Ψ is holomorphic everywhere that $x_2 z \neq 1$. Similarly, Υ is holomorphic everywhere that $x_1 z \neq 1$. We prove the following proposition which says that $x_2 z \neq 1$ and $x_1 z \neq 1$ whenever $z \in \mathbb{D}$ and $(x_1, x_2, x_3) \in \overline{\mathbb{E}}$.

Proposition 4.2.1. *The functions Ψ and Υ are defined and holomorphic on $\mathbb{D} \times \overline{\mathbb{E}}$. Moreover, if $(x_1, x_2, x_3) \in \overline{\mathbb{E}}$ then $|x_1| \leq 1$ and $|x_2| \leq 1$.*

Proof. Let $x = (x_1, x_2, x_3) \in \overline{\mathbb{E}}$. Then $(x_2 - x_3 z)w \neq 1 - x_1 z$ for all $z, w \in \mathbb{D}$. In particular, for $z = 0$ we have $x_2 w \neq 1$ for all $w \in \mathbb{D}$, and for $w = 0$ we have $0 \neq x_1 z - 1$ for all $z \in \mathbb{D}$. It follows that Ψ and Υ are defined and holomorphic on $\mathbb{D} \times \overline{\mathbb{E}}$.

Moreover, suppose $|x_2| > 1$ and let

$$w = \frac{1}{x_2}.$$

Then $w \in \mathbb{D}$, and hence $1 = x_2 w \neq 1$, which is a contradiction. It follows that $|x_2| \leq 1$. Similarly, we can show that $|x_1| \leq 1$. □

Remark 4.2.2. The proof of Proposition 4.2.1 can be easily modified to show that Ψ and Υ are defined and holomorphic on $\overline{\mathbb{D}} \times \mathbb{E}$, and if $(x_1, x_2, x_3) \in \mathbb{E}$ then $|x_1| < 1$ and $|x_2| < 1$.

The tetrablock is related to Γ in the following way. Let $(x_1, x_2, x_3) \in \overline{\mathbb{E}}$ be such that $x_1 x_2 = x_3$. Then, by Proposition 4.2.1, $|x_1| \leq 1$ and $|x_2| \leq 1$. It follows that

$$(x_1 + x_2, x_3) = (x_1 + x_2, x_1 x_2) \in \Gamma.$$

We call attention to the special case that $(x_1, x_2, x_3) \in \overline{\mathbb{E}}$ is such that $x_1 x_2 = x_3$, in this case,

$$\Psi(z, x_1, x_2, x_3) = \frac{x_1(x_2 z - 1)}{x_2 z - 1} = x_1 \text{ and } \Upsilon(z, x_1, x_2, x_3) = \frac{x_2(x_1 z - 1)}{x_1 z - 1} = x_2.$$

The functions Ψ and Υ are related by the following equation. For $(x_1, x_2, x_3) \in \mathbb{C}^3$ and $z \in \mathbb{C}$ such that $x_1 z \neq 1$, we have $\Psi(z, x_1, x_2, x_3) = \Upsilon(z, x_2, x_1, x_3)$. Through this relationship, when working on $\mathbb{D} \times \overline{\mathbb{E}}$, it usually suffices to consider only one of these functions, since, clearly, if $(x_1, x_2, x_3) \in \overline{\mathbb{E}}$ then $(x_2, x_1, x_3) \in \overline{\mathbb{E}}$.

We have the following alternative characterisations of $\mathbb{E}$ and $\overline{\mathbb{E}}$.

Theorem 4.2.3. [1, Theorem 2.2] *Let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. The following are equivalent.*

(i) $x \in \mathbb{E}$;

(iia) $|\Upsilon(z, x)| < 1$ *for all* $z \in \overline{\mathbb{D}}$, *and if* $x_1 x_2 = x_3$ *then, in addition,* $|x_1| < 1$;

(iib) $|\Psi(z, x)| < 1$ *for all* $z \in \overline{\mathbb{D}}$, *and if* $x_1 x_2 = x_3$ *then, in addition,* $|x_2| < 1$;

(iiia) $|x_2 - \overline{x_1} x_3| + |x_1 x_2 - x_3| < 1 - |x_1|^2$;

(iiib) $|x_1 - \overline{x_2} x_3| + |x_1 x_2 - x_3| < 1 - |x_2|^2$;

(iva) $-|x_1|^2 + |x_2|^2 + |x_3|^2 + 2|x_1 - \overline{x_2} x_3| < 1$ *and* $|x_1| < 1$;

(ivb) $|x_1|^2 - |x_2|^2 + |x_3|^2 + 2|x_2 - \overline{x_1} x_3| < 1$ *and* $|x_2| < 1$;

(v) $|x_1|^2 + |x_2|^2 - |x_3|^2 + 2|x_1 x_2 - x_3| < 1$ *and* $|x_3| < 1$;

(vi) *there is a* 2×2 *matrix* $A = [a_{ij}]_{i,j=1}^2$ *such that* $||A|| < 1$ *and* $x = (a_{11}, a_{22}, \det A)$;

(vii) *there is a symmetric* 2×2 *matrix* $A=[a_{ij}]_{i,j=1}^2$ *such that* $||A||<1$ *and* $x=(a_{11},a_{22},\det A)$;

(viii) $|x_1-\overline{x_2}x_3|+|x_2-\overline{x_1}x_3|<1-|x_3|^2$;

(ix) $|x_3|<1$ *and there exist* $\beta_1,\beta_2\in\mathbb{C}$ *such that* $|\beta_1|+|\beta_2|<1, x_1=\beta_1+\overline{\beta_2}x_3$ *and* $x_2=\beta_2+\overline{\beta_1}x_3$.

Theorem 4.2.4. [1, Theorem 2.4] *Let* $x=(x_1,x_2,x_3)\in\mathbb{C}^3$. *The following are equivalent.*

(i) $x\in\overline{\mathbb{E}}$;

(iia) $|\Upsilon(z,x)|\le 1$ *for all* $z\in\mathbb{D}$, *and if* $x_1x_2=x_3$ *then, in addition,* $|x_1|\le 1$;

(iib) $|\Psi(z,x)|\le 1$ *for all* $z\in\mathbb{D}$, *and if* $x_1x_2=x_3$ *then, in addition,* $|x_2|\le 1$;

(iiia) $|x_2-\overline{x_1}x_3|+|x_1x_2-x_3|\le 1-|x_1|^2$ *and if* $x_1x_2=x_3$ *then, in addition,* $|x_2|\le 1$;

(iiib) $|x_1-\overline{x_2}x_3|+|x_1x_2-x_3|\le 1-|x_2|^2$ *and if* $x_1x_2=x_3$ *then, in addition,* $|x_1|\le 1$;

(iva) $-|x_1|^2+|x_2|^2+|x_3|^2+2|x_1-\overline{x_2}x_3|\le 1$ *and* $|x_1|\le 1$;

(ivb) $|x_1|^2-|x_2|^2+|x_3|^2+2|x_2-\overline{x_2}x_3|\le 1$ *and* $|x_2|\le 1$;

(v) $|x_1|^2+|x_2|^2-|x_3|^2+2|x_1x_2-x_3|\le 1$ *and* $|x_3|\le 1$;

(vi) *there is a* 2×2 *matrix* $A=[a_{ij}]_{i,j=1}^2$ *such that* $||A||\le 1$ *and* $x=(a_{11},a_{22},\det A)$;

(vii) *there is a symmetric* 2×2 *matrix* $A=[a_{ij}]_{i,j=1}^2$ *such that* $||A||\le 1$ *and* $x=(a_{11},a_{22},\det A)$;

(viii) $|x_1-\overline{x_2}x_3|+|x_2-\overline{x_1}x_3|\le 1-|x_3|^2$ *and if* $|x_3|=1$ *then, in addition,* $|x_1|\le 1$;

(ix) $|x_3|\le 1$ *and there exist* $\beta_1,\beta_2\in\mathbb{C}$ *such that* $|\beta_1|+|\beta_2|\le 1, x_1=\beta_1+\overline{\beta_2}x_3$ *and* $x_2=\beta_2+\overline{\beta_1}x_3$.

We recall Definition 3.2.1 of the distinguished boundary of a domain in $\mathbb{C}^n$. By Proposition 4.1.2, $\overline{\mathbb{E}}$ is polynomially convex, and hence the distinguished boundary $b\overline{\mathbb{E}}$ of $\overline{\mathbb{E}}$ exists and is the Šilov boundary of $\mathcal{A}(\mathbb{E})$, where $\mathcal{A}(\mathbb{E})$ is the algebra of continuous functions on $\overline{\mathbb{E}}$ that are holomorphic on $\mathbb{E}$. We have the following alternative descriptions of $b\overline{\mathbb{E}}$.

Theorem 4.2.5. [1, Theorem 7.1] *Let* $x=(x_1,x_2,x_3)\in\mathbb{C}^3$. *The following are equivalent.*

(i) $x\in b\overline{\mathbb{E}}$;

(ii) $x\in\overline{\mathbb{E}}$ *and* $|x_3|=1$;

(iii) $x_1=\overline{x_2}x_3, |x_3|=1$ *and* $|x_2|\le 1$;

(iv) *either* $x_1x_2\ne x_3$ *and* $\Psi(\cdot,x)$ *is an automorphism of* $\mathbb{D}$, *or* $x_1x_2=x_3$ *and* $|x_1|=|x_2|=|x_3|=1$;

(v) x *is a peak point of* $\overline{\mathbb{E}}$;

(vi) *there is a* 2×2 *unitary matrix* $U=[u_{ij}]_{i,j=1}^2$ *such that* $x=(u_{11},u_{22},\det U)$;

(vii) *there is a symmetric* 2×2 *unitary matrix* $U = [u_{ij}]_{i,j=1}^2$ *such that* $x = (u_{11}, u_{22}, \det U)$.

By a *peak point* of $\overline{\mathbb{E}}$, we mean a point p for which there is a function $f \in \mathcal{A}(\mathbb{E})$ such that $f(p) = 1$ and $|f(x)| < 1$ for all $x \in \overline{\mathbb{E}} \setminus \{p\}$. Clearly, any peak point belongs to $b\overline{\mathbb{E}}$.

In regards to the topological structure of $b\overline{\mathbb{E}}$, we have the following result.

Corollary 4.2.6. [1, Corollary 7.2] *The distinguished boundary* $b\overline{\mathbb{E}}$ *is homeomorphic to* $\overline{\mathbb{D}} \times \mathbb{T}$.

Proof. The map $f : \overline{\mathbb{D}} \times \mathbb{T} \to b\overline{\mathbb{E}}$ defined by

$$f((x_2, x_3)) = (\overline{x_2}x_3, x_2, x_3) \text{ for all } (x_2, x_3) \in \overline{\mathbb{D}} \times \mathbb{T}$$

is a homeomorphism. □

An important subset of $\operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ is the collection of $\overline{\mathbb{E}}$-inner functions. An $\overline{\mathbb{E}}$-inner function is the analogue for $\operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ of inner functions in $\operatorname{Hol}(\mathbb{D}, \overline{\mathbb{D}})$.

Definition 4.2.7. *An* $\overline{\mathbb{E}}$-inner function *is a function* $x \in \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ *such that, for almost all* $\lambda \in \mathbb{T}$*, the radial limit*

$$\lim_{r \to 1^-} x(r\lambda) \in b\overline{\mathbb{E}}.$$

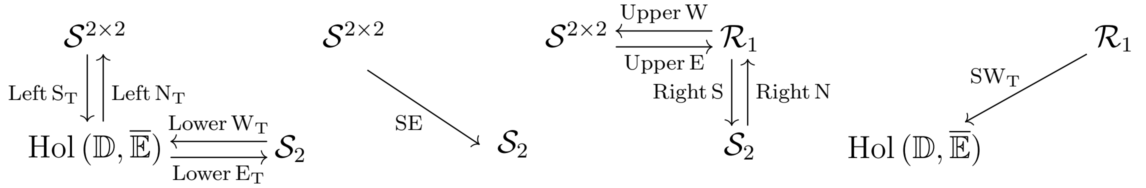
We note that if $x \in \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ then we may consider x as the function $(x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathbb{E}}$ defined by

$$(x_1, x_2, x_3)(\lambda) = (x_1(\lambda), x_2(\lambda), x_3(\lambda)) = x(\lambda) \in \overline{\mathbb{E}} \text{ for all } \lambda \in \mathbb{D}.$$

It follows that if $x = (x_1, x_2, x_3)$ is an $\overline{\mathbb{E}}$-inner function then x_3 is an inner function.

4.3 Relations between the sets

In this section, we construct maps between $\mathcal{S}^{2\times 2}, \mathcal{S}_2, \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ and $\mathcal{R}_1$, which illustrate the rich structure of interconnections summarised by the rich saltire. We label the maps in accordance with the following diagrams. The subscript $_\mathrm{T}$ denotes that we have $\operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$, and so consider the *t*etrablock.



4.3.1 Schur class of the bidisc and Left $\mathrm{N_T} : \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}}) \to \mathcal{S}^{2\times 2}$

We begin this section with the construction of a unique function $F \in \mathcal{S}^{2\times 2}$ for each $x \in \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. It is appropriate to include the realisation of $\Psi(z, x(\lambda))$ that is related to

F. We show later that $\Psi(z, x(\lambda))$, as a function on the bidisc, belongs to $\mathcal{S}_2$, and that this realisation is a powerful tool in producing a number of the maps in the rich saltire.

Let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. Then the linear fractional transformation $\mathcal{F}_{F(\lambda)}(z)$ is given by

$$\mathcal{F}_{F(\lambda)}(z) := F_{11}(\lambda) + F_{12}(\lambda)z(1 - F_{22}(\lambda)z)^{-1}F_{21}(\lambda),$$

where $z, \lambda \in \mathbb{D}$.

Theorem 4.3.1. *Let $x = (x_1, x_2, x_3) \in \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. Then there exists a unique*

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$$

such that $x = (F_{11}, F_{22}, \det F)$, $|F_{12}| = |F_{21}|$ almost everywhere on $\mathbb{T}$, F_{21} is either outer or 0, and $F_{21}(0) \geq 0$. Moreover, we have

$$1 - \overline{\Psi(w, x(\mu))}\Psi(z, x(\lambda)) = (1 - \overline{w}z)\overline{\gamma(\mu, w)}\gamma(\lambda, z) + \eta(\mu, w)^*(I - F(\mu)^*F(\lambda))\eta(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$, where

$$\gamma(\lambda, z) = (1 - F_{22}(\lambda)z)^{-1}F_{21}(\lambda) \text{ and } \eta(\lambda, z) = \begin{bmatrix} 1 \\ z\gamma(\lambda, z) \end{bmatrix}$$

for all $z, \lambda \in \mathbb{D}$.

Proof. First, we show that such an F exists and is unique. Let $x = (x_1, x_2, x_3) \in \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. Suppose that $x_1x_2 = x_3$. Then it is clear that

$$F = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}$$

is the only matrix satisfying all of the required conditions. In particular, F is holomorphic and, since, by Proposition 4.2.1, $|x_1(\lambda)| \leq 1$ and $|x_2(\lambda)| \leq 1$ for all $\lambda \in \mathbb{D}$, we have

$$||F(\lambda)|| = \left|\left| \begin{bmatrix} x_1(\lambda) & 0 \\ 0 & x_2(\lambda) \end{bmatrix} \right|\right| = \max\{|x_1(\lambda)|, |x_2(\lambda)|\} \leq 1$$

for all $\lambda \in \mathbb{D}$. That it is the only matrix follows since we have $|F_{21}||F_{12}| = |x_1x_2 - x_3| = 0$ and $|F_{12}| = |F_{21}|$.

Now suppose that $x_1x_2 \neq x_3$. Then $x_1x_2 - x_3$ is a non-zero H^∞ function, and so, by Theorem B.1.21, it has a unique inner-outer factorisation of the form $\varphi e^C = x_1x_2 - x_3$, where φ is inner, e^C is outer and $e^C(0) \geq 0$. Set

$$F = \begin{bmatrix} x_1 & \varphi e^{\frac{1}{2}C} \\ e^{\frac{1}{2}C} & x_2 \end{bmatrix}.$$

Then, except for the condition that $F \in \mathcal{S}^{2\times 2}$, it is easy to check that F is the only matrix satisfying the required conditions. In particular,

$$\det F = x_1x_2 - \varphi e^C = x_1x_2 - x_1x_2 + x_3 = x_3,$$

and, since $|\varphi| = 1$ almost everywhere on $\mathbb{T}$, we have $|F_{12}| = |F_{21}|$ almost everywhere on $\mathbb{T}$. That it is the only matrix follows from the uniqueness of the representation φe^C and the requirements that F_{21} be outer, and that $|F_{12}| = |F_{21}|$ almost everywhere on $\mathbb{T}$.

We still need to check that $F \in \mathcal{S}^{2\times 2}$. Clearly F is holomorphic, since inner and outer functions are holomorphic. To show that $||F(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$, it is equivalent, by Corollary B.1.6, to show that $I - F(\lambda)^*F(\lambda)$ is positive semidefinite for all $\lambda \in \mathbb{D}$. To do this, we show that the diagonal entries of $I - F(\lambda)^*F(\lambda)$ are non-negative and $\det(I - F(\lambda)^*F(\lambda)) \geq 0$ for all $\lambda \in \mathbb{D}$. Since $|F_{12}| = |F_{21}|$ almost everywhere on $\mathbb{T}$, and $F_{21}F_{12} = x_1x_2 - x_3$, we have

$$|F_{12}|^2 = |F_{21}|^2 = |F_{21}F_{12}| = |x_1x_2 - x_3|$$

almost everywhere on $\mathbb{T}$. By Proposition B.1.29, for almost every $\lambda \in \mathbb{T}$,

$$I - F(\lambda)^*F(\lambda) = \begin{bmatrix} 1 - |x_1(\lambda)|^2 - |x_1(\lambda)x_2(\lambda) - x_3(\lambda)| & -\overline{x_1(\lambda)}F_{12}(\lambda) - \overline{F_{21}(\lambda)}x_2(\lambda) \\ -\overline{F_{12}(\lambda)}x_1(\lambda) - \overline{x_2(\lambda)}F_{21}(\lambda) & 1 - |x_1(\lambda)x_2(\lambda) - x_3(\lambda)| - |x_2(\lambda)|^2 \end{bmatrix}$$

and

$$\det(I - F(\lambda)^*F(\lambda)) = 1 - |x_1(\lambda)|^2 - 2|x_1(\lambda)x_2(\lambda) - x_3(\lambda)| - |x_2(\lambda)|^2 + |x_3(\lambda)|^2.$$

Let $D_1(\lambda)$ and $D_2(\lambda)$ be the diagonal entries of $I - F(\lambda)^*F(\lambda)$. By Theorem 4.2.4 (iiia) and (iiib), since $x : \mathbb{D} \to \overline{\mathbb{E}}$, we have

$$|x_2(\lambda) - \overline{x_1(\lambda)}x_3(\lambda)| + |x_1(\lambda)x_2(\lambda) - x_3(\lambda)| \leq 1 - |x_1(\lambda)|^2$$

and

$$|x_1(\lambda) - \overline{x_2(\lambda)}x_3(\lambda)| + |x_1(\lambda)x_2(\lambda) - x_3(\lambda)| \leq 1 - |x_2(\lambda)|^2$$

for all $\lambda \in \mathbb{D}$. Since these two inequalities continue to hold for almost every $\lambda \in \mathbb{T}$, it follows that

$$D_1(\lambda) \geq |x_2(\lambda) - \overline{x_1(\lambda)}x_3(\lambda)| \geq 0 \text{ and } D_2(\lambda) \geq |x_1(\lambda) - \overline{x_2(\lambda)}x_3(\lambda)| \geq 0$$

for almost every $\lambda \in \mathbb{T}$. Moreover, by Theorem 4.2.4 (v), we have

$$|x_1(\lambda)|^2 + |x_2(\lambda)|^2 - |x_3(\lambda)|^2 + 2|x_1(\lambda)x_2(\lambda) - x_3(\lambda)| \leq 1$$

for all $\lambda \in \mathbb{D}$. Since this inequality also holds for almost every $\lambda \in \mathbb{T}$, it follows that

$$\det\left(I - F(\lambda)^* F(\lambda)\right) \geq 0$$

for almost every $\lambda \in \mathbb{T}$. Hence, by Corollary B.1.6, $||F(\lambda)|| \leq 1$ for almost every $\lambda \in \mathbb{T}$. By the Maximum Modulus Principle we obtain $||F(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$, as required.

It remains to show that

$$1 - \overline{\Psi(w, x(\mu))}\Psi(z, x(\lambda)) = (1 - \overline{w}z)\overline{\gamma(\mu, w)}\gamma(\lambda, z) + \eta(\mu, w)^*(I - F(\mu)^* F(\lambda))\eta(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. First we note that

$$\begin{aligned}\mathcal{F}_{F(\lambda)}(z) =& F_{11}(\lambda) + \frac{F_{12}(\lambda)F_{21}(\lambda)z}{1 - F_{22}(\lambda)z} = x_1(\lambda) + \frac{(x_1(\lambda)x_2(\lambda) - x_3(\lambda))z}{1 - x_2(\lambda)z}\\ =& \frac{x_1(\lambda) - x_1(\lambda)x_2(\lambda)z + x_1(\lambda)x_2(\lambda)z - x_3(\lambda)z}{1 - x_2(\lambda)z} = \frac{x_1(\lambda) - x_3(\lambda)z}{1 - x_2(\lambda)z}\\ =& \frac{x_3(\lambda)z - x_1(\lambda)}{x_2(\lambda)z - 1} = \Psi(z, x(\lambda))\end{aligned}$$

for all $z, \lambda \in \mathbb{D}$. Hence, by Proposition 2.1.1,

$$\begin{aligned}1 - \overline{\Psi(w, x(\mu))}\Psi(z, x(\lambda)) =& 1 - \mathcal{F}_{F(\mu)}(w)^* \mathcal{F}_{F(\lambda)}(z)\\ =& (1 - \overline{w}z)\overline{\gamma(\mu, w)}\gamma(\lambda, z) + \eta(\mu, w)^*(I - F(\mu)^* F(\lambda))\eta(\lambda, z)\end{aligned}$$

for all $z, \lambda, w, \mu \in \mathbb{D}$. □

Definition 4.3.2. *We define* $\text{Left}\,\text{N}_\text{T} : \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}}) \to \mathcal{S}^{2\times 2}$ *by*

$$\text{Left}\,\text{N}_\text{T}\,(x) = F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

for each $x = (x_1, x_2, x_3) \in \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$*, where* $F \in \mathcal{S}^{2\times 2}$ *and satisfies* $x = (F_{11}, F_{22}, \det F)$*,* $|F_{12}| = |F_{21}|$ *almost everywhere on* $\mathbb{T}$*,* F_{21} *is either outer or* 0*, and* $F_{21}(0) \geq 0$*.*

Clearly the function F, as defined in Definition 4.3.2, must also satisfy $F_{21}F_{12} = x_1x_2 - x_3$. That $\text{Left}\,\text{N}_\text{T}$ is well defined follows immediately from Theorem 4.3.1. In the case that x is an $\overline{\mathbb{E}}$-inner function, there is an alternative proof of the realisation of $\Psi(z, x(\lambda))$ from Theorem 4.3.1, which we give now.

Proposition 4.3.3. *Let* $x = (x_1, x_2, x_3)$ *be an* $\overline{\mathbb{E}}$*-inner function such that* $x_1x_2 \neq x_3$*. Then there is a Hilbert space* H*, a holomorphic function* $F : \mathbb{D} \to \mathcal{B}(\mathbb{C}^2, H)$*, and an outer* *function $g \in H^\infty$ such that $|g(\xi)|^2 = 1 - |x_2(\xi)|^2$ for almost every $\xi \in \mathbb{T}$, which satisfy*

$$\begin{aligned} 1 - \overline{\Psi(w, x(\mu))}\Psi(z, x(\lambda)) =& (1 - \overline{w}z) \left\langle \frac{g(\lambda)}{1 - x_2(\lambda)z}, \frac{g(\mu)}{1 - x_2(\mu)w} \right\rangle_{\mathbb{C}} \\ &+ (1 - \overline{\mu}\lambda) \left\langle F(\lambda) \begin{pmatrix} 1 \\ \frac{g(\lambda)z}{1 - x_2(\lambda)z} \end{pmatrix}, F(\mu) \begin{pmatrix} 1 \\ \frac{g(\mu)w}{1 - x_2(\mu)w} \end{pmatrix} \right\rangle_H \end{aligned}$$

for all $\mu, \lambda \in \mathbb{D}$ and $w, z \in \mathbb{C}$ such that $1 - x_2(\lambda)z \neq 0$ and $1 - x_2(\mu)w \neq 0$.

Proof. By Theorem 4.2.5 (iii), $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$, $|x_3(\lambda)| = 1$ and $|x_2(\lambda)| \leq 1$ for almost every $\lambda \in \mathbb{T}$. Thus

$$\Psi(z, x(\lambda)) = \frac{x_3(\lambda)z - x_1(\lambda)}{x_2(\lambda)z - 1} = \frac{x_3(\lambda)z - \overline{x_2(\lambda)}x_3(\lambda)}{x_2(\lambda)z - 1} = x_3(\lambda)\frac{z - \overline{x_2(\lambda)}}{x_2(\lambda)z - 1}$$

for almost every $\lambda \in \mathbb{T}$ and every $z \in \mathbb{C}$ such that $1 - x_2(\lambda)z \neq 0$. It follows that

$$\begin{aligned} 1 - &\overline{\Psi(w, x(\lambda))}\Psi(z, x(\lambda)) = \\ &= 1 - \overline{x_3(\lambda)\frac{w - \overline{x_2(\lambda)}}{x_2(\lambda)w - 1}}x_3(\lambda)\frac{z - \overline{x_2(\lambda)}}{x_2(\lambda)z - 1} = 1 - |x_3(\lambda)|^2 \frac{\overline{w} - x_2(\lambda)}{\overline{x_2(\lambda)w - 1}} \frac{z - \overline{x_2(\lambda)}}{x_2(\lambda)z - 1} \\ &= \frac{(|x_2(\lambda)|^2\overline{w}z - \overline{x_2(\lambda)w} - x_2(\lambda)z + 1) - (\overline{w}z - zx_2(\lambda) - \overline{wx_2(\lambda)} + |x_2(\lambda)|^2)}{\overline{(x_2(\lambda)w - 1)}(x_2(\lambda)z - 1)} \\ &= \frac{|x_2(\lambda)|^2(\overline{w}z - 1) + 1 - \overline{w}z}{\overline{(x_2(\lambda)w - 1)}(x_2(\lambda)z - 1)} = \frac{(|x_2(\lambda)|^2 - 1)(\overline{w}z - 1)}{\overline{(x_2(\lambda)w - 1)}(x_2(\lambda)z - 1)} \end{aligned}$$

for almost every $\lambda \in \mathbb{T}$ and every $w, z \in \mathbb{C}$ such that $1 - x_2(\lambda)z \neq 0$ and $1 - x_2(\mu)w \neq 0$.

By Theorem B.1.21, the non-zero H^∞ function $x_1x_2 - x_3$ has an inner-outer factorisation of the form $\varphi g_0 = x_1x_2 - x_3$, where φ is inner and $g_0 \in H^\infty$ is outer. Since $|x_3(\lambda)| = 1$, $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$ and $|x_2(\lambda)| \leq 1$ for almost every $\lambda \in \mathbb{T}$, we have

$$\begin{aligned} |g_0(\lambda)|^2 =& \varphi(\lambda)g_0(\lambda)\overline{\varphi(\lambda)g_0(\lambda)} = (x_1(\lambda)x_2(\lambda) - x_3(\lambda))\overline{(x_1(\lambda)x_2(\lambda) - x_3(\lambda))} \\ =& |x_1(\lambda)x_2(\lambda)|^2 - x_1(\lambda)x_2(\lambda)\overline{x_3(\lambda)} - \overline{x_1(\lambda)}\,\overline{x_2(\lambda)}x_3(\lambda) + |x_3(\lambda)|^2 \\ =& |x_1(\lambda)|^2|x_2(\lambda)|^2 - |x_2(\lambda)|^2 - |x_1(\lambda)|^2 + 1 \\ =& |x_2(\lambda)|^4 - |x_2(\lambda)|^2 - |x_2(\lambda)|^2 + 1 \\ =& \left(|x_2(\lambda)|^2 - 1\right)^2 \end{aligned}$$

for almost every $\lambda \in \mathbb{T}$. Since $|x_2(\lambda)| \leq 1$, we have $1 - |x_2(\lambda)|^2 \geq 0$, and hence $|g_0(\lambda)| = 1 - |x_2(\lambda)|^2$ for almost every $\lambda \in \mathbb{T}$. Set $g = g_0^{\frac{1}{2}}$. Then $g \in H^\infty$ is an outer function such that $|g(\lambda)|^2 = 1 - |x_2(\lambda)|^2$ for almost every $\lambda \in \mathbb{T}$. Thus we can write

$$1 - \overline{\Psi(w, x(\lambda))}\Psi(z, x(\lambda)) = (1 - \overline{w}z) \left\langle \frac{g(\lambda)}{1 - x_2(\lambda)z}, \frac{g(\lambda)}{1 - x_2(\lambda)w} \right\rangle_{\mathbb{C}}$$

and so, by expanding the right side,

$$1+\left\langle \frac{g(\lambda)z}{1-x_2(\lambda)z}, \frac{g(\lambda)w}{1-x_2(\lambda)w}\right\rangle_{\mathbb{C}} = \overline{\Psi(w,x(\lambda))}\Psi(z,x(\lambda))+\left\langle \frac{g(\lambda)}{1-x_2(\lambda)z}, \frac{g(\lambda)}{1-x_2(\lambda)w}\right\rangle_{\mathbb{C}}$$

for almost every $\lambda \in \mathbb{T}$ and every $w, z \in \mathbb{C}$ such that $1-x_2(\lambda)w \neq 0$ and $1-x_2(\lambda)z \neq 0$. It follows that

$$\left\langle \begin{pmatrix} 1 \\ \dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix}, \begin{pmatrix} 1 \\ \dfrac{g(\lambda)w}{1-x_2(\lambda)w}\end{pmatrix}\right\rangle_{\mathbb{C}^2} = \left\langle \begin{pmatrix} \Psi(z,x(\lambda)) \\ \dfrac{g(\lambda)}{1-x_2(\lambda)z}\end{pmatrix}, \begin{pmatrix} \Psi(w,x(\lambda)) \\ \dfrac{g(\lambda)}{1-x_2(\lambda)w}\end{pmatrix}\right\rangle_{\mathbb{C}^2}$$

for almost every $\lambda \in \mathbb{T}$ and every $w, z \in \mathbb{C}$ such that $1-x_2(\lambda)w \neq 0$ and $1-x_2(\lambda)z \neq 0$. That is, for almost every $\lambda \in \mathbb{T}$, the Grammian of the vectors

$$\left\{ \begin{pmatrix} 1 \\ \dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix} : z \in \mathbb{C} \text{ and } 1-x_2(\lambda)z \neq 0 \right\} \subseteq \mathbb{C}^2$$

is equal to the Grammian of the vectors

$$\left\{ \begin{pmatrix} \Psi(z,x(\lambda)) \\ \dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix} : z \in \mathbb{C} \text{ and } 1-x_2(\lambda)z \neq 0 \right\} \subseteq \mathbb{C}^2.$$

Thus, by Proposition B.1.22, for almost every $\lambda \in \mathbb{T}$, there exists an isomety L_λ such that

$$L_\lambda\left(\begin{pmatrix} 1 \\ \dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix}\right) = \begin{pmatrix} \Psi(z,x(\lambda)) \\ \dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix}$$

for every $z \in \mathbb{C}$ such that $1-x_2(\lambda)z \neq 0$.

Now, we define Θ on $\mathbb{D}$ by $\Theta(\lambda) := \begin{bmatrix} x_1(\lambda) & \dfrac{x_1(\lambda)x_2(\lambda)-x_3(\lambda)}{g(\lambda)} \\ g(\lambda) & x_2(\lambda)\end{bmatrix}$ for all $\lambda \in \mathbb{D}$. Hence

$$\begin{aligned}\Theta(\lambda)\begin{pmatrix} 1 \\ \dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix} &= \begin{pmatrix} x_1(\lambda)+\dfrac{x_1(\lambda)x_2(\lambda)-x_3(\lambda)}{g(\lambda)}\dfrac{g(\lambda)z}{1-x_2(\lambda)z} \\ g(\lambda)+x_2(\lambda)\dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix} \\ &= \begin{pmatrix} \dfrac{x_1(\lambda)-x_1(\lambda)x_2(\lambda)z+x_1(\lambda)x_2(\lambda)z-x_3(\lambda)z}{1-x_2(\lambda)z} \\ \dfrac{g(\lambda)-g(\lambda)x_2(\lambda)z+x_2(\lambda)g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix} \\ &= \begin{pmatrix} \dfrac{x_3(\lambda)z-x_1(\lambda)}{x_2(\lambda)z-1} \\ \dfrac{g(\lambda)}{1-x_2(\lambda)z}\end{pmatrix} = \begin{pmatrix} \Psi(z,x(\lambda)) \\ \dfrac{g(\lambda)}{1-x_2(\lambda)z}\end{pmatrix}\end{aligned}$$

for all $\lambda \in \mathbb{D}$ and $z \in \mathbb{C}$ such that $1 - x_2(\lambda)z \neq 0$. Clearly Θ is holomorphic and, for almost every $\lambda \in \mathbb{T}$, we have $\Theta(\lambda)$ exists and is equal to the isometry L_λ. It follows from the Maximum Modulus Principle that $||\Theta(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$, and so $\Theta \in \mathcal{S}^{2\times 2}$.

By Proposition B.3.30, the function defined by

$$(\lambda, \mu) \mapsto \frac{I - \Theta(\mu)^*\Theta(\lambda)}{1 - \overline{\mu}\lambda}$$

is an $\mathcal{M}_2(\mathbb{C})$-valued kernel on $\mathbb{D}$, and so, by Corollary B.3.20, there is a Hilbert space $\mathcal{H}$ and a holomorphic $F : \mathbb{D} \to \mathcal{B}(\mathbb{C}^2, \mathcal{H})$ such that

$$I - \Theta(\mu)^*\Theta(\lambda) = (1 - \overline{\mu}\lambda)F(\mu)^*F(\lambda)$$

for all $\mu, \lambda \in \mathbb{D}$. It follows that

$$\begin{aligned}
&(1-\overline{\mu}\lambda)\left\langle F(\lambda)\begin{pmatrix} 1 \\ \dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix}, F(\mu)\begin{pmatrix} 1 \\ \dfrac{g(\mu)w}{1-x_2(\mu)w}\end{pmatrix}\right\rangle_{\mathcal{H}} = \\
&= \left\langle (I - \Theta(\mu)^*\Theta(\lambda))\begin{pmatrix} 1 \\ \dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix}, \begin{pmatrix} 1 \\ \dfrac{g(\mu)w}{1-x_2(\mu)w}\end{pmatrix}\right\rangle_{\mathbb{C}^2} \\
&= \left\langle \begin{pmatrix} 1 \\ \dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix}, \begin{pmatrix} 1 \\ \dfrac{g(\mu)w}{1-x_2(\mu)w}\end{pmatrix}\right\rangle_{\mathbb{C}^2} - \left\langle \begin{pmatrix} \Psi(z, x(\lambda)) \\ \dfrac{g(\lambda)}{1-x_2(\lambda)z}\end{pmatrix}, \begin{pmatrix} \Psi(w, x(\mu)) \\ \dfrac{g(\mu)}{1-x_2(\mu)w}\end{pmatrix}\right\rangle_{\mathbb{C}^2} \\
&= 1 + \overline{w}z\left\langle \frac{g(\lambda)}{1-x_2(\lambda)z}, \frac{g(\mu)}{1-x_2(\mu)w}\right\rangle_{\mathbb{C}} \\
&\quad - \Psi(z, x(\lambda))\overline{\Psi(w, x(\mu))} - \left\langle \frac{g(\lambda)}{1-x_2(\lambda)z}, \frac{g(\mu)}{1-x_2(\mu)w}\right\rangle_{\mathbb{C}} \\
&= 1 - \overline{\Psi(w, x(\mu))}\Psi(z, x(\lambda)) - (1 - \overline{w}z)\left\langle \frac{g(\lambda)}{1-x_2(\lambda)z}, \frac{g(\mu)}{1-x_2(\mu)w}\right\rangle_{\mathbb{C}}
\end{aligned}$$

and so

$$\begin{aligned}
1 - \overline{\Psi(w, x(\mu))}\Psi(z, x(\lambda)) =& (1 - \overline{w}z)\left\langle \frac{g(\lambda)}{1-x_2(\lambda)z}, \frac{g(\mu)}{1-x_2(\mu)w}\right\rangle_{\mathbb{C}} \\
&+ (1-\overline{\mu}\lambda)\left\langle F(\lambda)\begin{pmatrix} 1 \\ \dfrac{g(\lambda)z}{1-x_2(\lambda)z}\end{pmatrix}, F(\mu)\begin{pmatrix} 1 \\ \dfrac{g(\mu)w}{1-x_2(\mu)w}\end{pmatrix}\right\rangle_{\mathcal{H}}
\end{aligned}$$

for all $\mu, \lambda \in \mathbb{D}$ and $w, z \in \mathbb{C}$ such that $1 - x_2(\mu)w \neq 0$ and $1 - x_2(\lambda)z \neq 0$, which is the required identity. □

Remark 4.3.4. It is natural to ask what the relationship is between the function F from Theorem 4.3.1 and the function Θ from Proposition 4.3.3. Let $x = (x_1, x_2, x_3)$ be an

$\overline{\mathbb{E}}$-inner function such that $x_1x_2 \neq x_3$. Recall that

$$F = \begin{bmatrix} x_1 & \varphi e^{\frac{1}{2}C} \\ e^{\frac{1}{2}C} & x_2 \end{bmatrix},$$

where φ is inner, e^C is outer and $\varphi e^C = x_1x_2 - x_3$, and

$$\Theta = \begin{bmatrix} x_1 & \frac{x_1x_2 - x_3}{g} \\ g & x_2 \end{bmatrix},$$

where $g \in H^\infty$ is an outer function such that $|g(\lambda)|^2 = 1 - |x_2(\lambda)|^2$ for almost every $\lambda \in \mathbb{T}$. Then

$$F = \Theta,$$

since, by the uniqueness of the inner-outer factorisation of $x_1x_2 - x_3$ in the proofs of Theorem 4.3.1 and Proposition 4.3.3, we have $g = e^{\frac{1}{2}C}$.

4.3.2 Left $\mathrm{S_T} : \mathcal{S}^{2\times 2} \to \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$

Definition 4.3.5. *We define* Left $\mathrm{S_T} : \mathcal{S}^{2\times 2} \to \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ *by*

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \mapsto (F_{11}, F_{22}, \det F)$$

for all $F \in \mathcal{S}^{2\times 2}$.

Proposition 4.3.6. *The map* Left $\mathrm{S_T}$ *is well defined.*

Proof. Let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. Clearly Left $\mathrm{S_T}\,(F)$ is holomorphic since

$$\det F = F_{11}F_{22} - F_{21}F_{12}.$$

By Theorem 4.2.4 (vi), since $||F(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$,

$$\mathrm{Left\,S_T}\,(F)(\lambda) = (F_{11}(\lambda), F_{22}(\lambda), \det F(\lambda)) \in \overline{\mathbb{E}}$$

for all $\lambda \in \mathbb{D}$. It follows that Left $\mathrm{S_T}\,(F) \in \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$. □

We now have a map $\mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}}) \to \mathcal{S}^{2\times 2}$ and a map $\mathcal{S}^{2\times 2} \to \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$, and so we can investigate how they interact.

Proposition 4.3.7. Left $\mathrm{S_T} \circ$ Left $\mathrm{N_T} = \mathrm{id}_{\mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})}$.

Proof. Let $x = (x_1, x_2, x_3) \in \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$. Then Left $\mathrm{N_T}\,(x) = F \in \mathcal{S}^{2\times 2}$ as defined in Definition 4.3.2. In particular, $F_{11} = x_1$, $F_{22} = x_2$ and $\det F = x_3$. It follows that Left $\mathrm{S_T}\,(F) = x$, and hence

$$\mathrm{Left\,S_T} \circ \mathrm{Left\,N_T}\,(x) = x.$$

Therefore $\text{Left}\,\text{S}_\text{T} \circ \text{Left}\,\text{N}_\text{T} = \text{id}_{\text{Hol}\,(\mathbb{D},\overline{\mathbb{E}})}$. □

Example 4.3.8. Let $F(\lambda) = \dfrac{\lambda}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ for all $\lambda \in \mathbb{D}$. Then $F \in \mathcal{S}^{2\times 2}$ and $\text{Left}\,\text{N}_\text{T} \circ \text{Left}\,\text{S}_\text{T}\,(F) \neq F$.

Proof. Clearly F is holomorphic on $\mathbb{D}$, and

$$||F(\lambda)|| = \frac{|\lambda|}{\sqrt{2}}\left|\left|\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\right|\right| \leq \frac{|\lambda|}{\sqrt{2}}\sqrt{2} \leq 1$$

for all $\lambda \in \mathbb{D}$. Hence $F \in \mathcal{S}^{2\times 2}$. Thus we can apply $\text{Left}\,\text{S}_\text{T}$ to obtain

$$\text{Left}\,\text{S}_\text{T}\,(F)(\lambda) = \left(\frac{\lambda}{\sqrt{2}}, 0, 0\right)$$

for all $\lambda \in \mathbb{D}$. Define $x = (x_1, x_2, x_3) \in \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ by $x(\lambda) = \left(\frac{\lambda}{\sqrt{2}}, \frac{\lambda}{\sqrt{2}}, \frac{\lambda^2}{2}\right)$ for all $\lambda \in \mathbb{D}$, and let

$$\text{Left}\,\text{N}_\text{T}\,(x) := G \in \mathcal{S}^{2\times 2}.$$

Then the function G is defined as in Definition 4.3.2. In particular, since $x_1 \cdot 0 = 0$, we have

$$G = \begin{bmatrix}x_1 & 0\\ 0 & 0\end{bmatrix} \neq F.$$

Hence $\text{Left}\,\text{N}_\text{T} \circ \text{Left}\,\text{S}_\text{T}\,(F) \neq F$. □

4.3.3 $\text{SW}_\text{T} : \mathcal{R}_{11} \to \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$

The idea for SW_T is to follow Procedure UW with the map $\text{Left}\,\text{S}_\text{T}$. The following proposition facilitates this.

Proposition 4.3.9. *Let $(N, M) \in \mathcal{R}_{11}$. Let $\Xi = \begin{bmatrix}\Xi_{11} & \Xi_{12}\\ \Xi_{21} & \Xi_{22}\end{bmatrix}$ be constructed from (N, M) by Procedure UW. Then*

$$\{\text{Left}\,\text{S}_\text{T}\,(F) : F \in \text{Upper}\,\text{W}\,((N, M))\} = \{(\zeta\Xi_{11}, \Xi_{22}, \zeta \det \Xi) : \zeta \in \mathbb{T}\} \subseteq \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}}).$$

Proof. Let $F \in \text{Upper}\,\text{W}\,((N, M))$. Then $F = \begin{bmatrix}\zeta_1 & 0\\ 0 & \zeta_2\end{bmatrix}\Xi\begin{bmatrix}1 & 0\\ 0 & \overline{\zeta_2}\end{bmatrix}$ for some $\zeta_1, \zeta_2 \in \mathbb{T}$, and so

$$\text{Left}\,\text{S}_\text{T}\,(F) = \left(\zeta_1\Xi_{11}, \Xi_{22}, \det\begin{bmatrix}\zeta_1 & 0\\ 0 & \zeta_2\end{bmatrix}\Xi\begin{bmatrix}1 & 0\\ 0 & \overline{\zeta_2}\end{bmatrix}\right) = (\zeta_1\Xi_{11}, \Xi_{22}, \zeta_1 \det \Xi)\,.$$

It follows that

$$\{\text{Left}\,\text{S}_\text{T}\,(F) : F \in \text{Upper}\,\text{W}\,((N, M))\} = \{(\zeta\Xi_{11}, \Xi_{22}, \zeta \det \Xi) : \zeta \in \mathbb{T}\}\,.$$

Moreover, by Proposition 4.3.6, since $F \in \mathcal{S}^{2\times 2}$, we have $\operatorname{Left} \mathrm{S_T}(F) \in \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. □

Definition 4.3.10. *We define* $\mathrm{SW_T}$ *to be the set-valued map from* $\mathcal{R}_{11}$ *to* $\operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ *given by*

$$\mathrm{SW_T}((N, M)) = \{(\zeta \Xi_{11}, \Xi_{22}, \zeta \det \Xi) : \zeta \in \mathbb{T}\}$$

for all $(N, M) \in \mathcal{R}_{11}$*, where* $\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$ *is constructed from* (N, M) *by Procedure UW.*

That $\mathrm{SW_T}$ is well defined follows from Proposition 4.3.9 and the observation that, as Upper W is independent of which function Ξ is used to define it, the set

$$\{(\zeta \Xi_{11}, \Xi_{22}, \zeta \det \Xi) : \zeta \in \mathbb{T}\}$$

is independent of the choice of Ξ.

By Proposition 4.3.9,

$$\{\operatorname{Left} \mathrm{S_T}(F) : F \in \operatorname{Upper W}((N, M))\} = \mathrm{SW_T}((N, M))$$

for all $(N, M) \in \mathcal{R}_{11}$. We have the following other interactions with $\mathrm{SW_T}$.

Proposition 4.3.11. *Let* $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$ *be such that* $F_{21} \neq 0$*. Then*

$$\mathrm{SW_T} \circ \operatorname{Upper E}(F) = \left\{ \operatorname{Left} \mathrm{S_T}\left(\begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} F \right) : \zeta \in \mathbb{T} \right\}.$$

Proof. By Proposition 2.4.12,

$$\operatorname{Upper W} \circ \operatorname{Upper E}(F) = \left\{ \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} F \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} : \zeta_1, \zeta_2 \in \mathbb{T} \right\}.$$

Hence

$$\begin{aligned} \mathrm{SW_T} \circ \operatorname{Upper E}(F) &= \left\{ \operatorname{Left} \mathrm{S_T}\left(\begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} F \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} \right) : \zeta_1, \zeta_2 \in \mathbb{T} \right\} \\ &= \left\{ \left(\zeta_1 F_{11}, F_{22}, \det \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix} F \begin{bmatrix} 1 & 0 \\ 0 & \overline{\zeta_2} \end{bmatrix} \right) : \zeta_1, \zeta_2 \in \mathbb{T} \right\} \\ &= \left\{ \operatorname{Left} \mathrm{S_T}\left(\begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} F \right) : \zeta \in \mathbb{T} \right\}, \end{aligned}$$

as required. □

Corollary 4.3.12. *Let* $x = (x_1, x_2, x_3) \in \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ *be such that* $x_1 x_2 \neq x_3$*. Then*

$$\mathrm{SW_T} \circ \operatorname{Upper E} \circ \operatorname{Left} \mathrm{N_T}(x) = \{(\zeta x_1, x_2, \zeta x_3) : \zeta \in \mathbb{T}\}.$$

Proof. Let $F = \text{Left}\,\text{N}_\text{T}\,(x)$. Then, in particular,

$$F = \begin{bmatrix} x_1 & F_{12} \\ F_{21} & x_2 \end{bmatrix},$$

$F_{21} \neq 0$ and $\det F = x_3$. By Proposition 4.3.11,

$$\begin{aligned} \text{SW}_\text{T} \circ \text{Upper}\,\text{E}\,(F) &= \left\{ \text{Left}\,\text{S}_\text{T} \left(\begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} F \right) : \zeta \in \mathbb{T} \right\} \\ &= \left\{ \text{Left}\,\text{S}_\text{T} \left(\begin{bmatrix} \zeta x_1 & \zeta F_{12} \\ F_{21} & x_2 \end{bmatrix} \right) : \zeta \in \mathbb{T} \right\} \\ &= \{(\zeta x_1, x_2, \zeta \det F) : \zeta \in \mathbb{T}\}. \end{aligned}$$

Hence $\text{SW}_\text{T} \circ \text{Upper}\,\text{E} \circ \text{Left}\,\text{N}_\text{T}\,(x) = \{(\zeta x_1, x_2, \zeta x_3) : \zeta \in \mathbb{T}\}$. □

We note that if $\zeta = 1$ then $(\zeta x_1, x_2, \zeta x_3) = (x_1, x_2, x_3)$. Hence, by Corollary 4.3.12, for all $x \in \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$, we have $x \in \text{SW}_\text{T} \circ \text{Upper}\,\text{E} \circ \text{Left}\,\text{N}_\text{T}\,(x)$.

4.3.4 Lower $\text{E}_\text{T} : \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}}) \to \mathcal{S}_2$

The definition of Lower E_T comes from the relationship between $\text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ and a particular subset of $\mathcal{S}_2$. The relationship uses the function Ψ. Recall that, for $x = (x_1, x_2, x_3) \in \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ and $z, \lambda \in \mathbb{D}$, we have

$$\Psi(z, x(\lambda)) = \frac{x_3(\lambda)z - x_1(\lambda)}{x_2(\lambda)z - 1}.$$

Hence, for all $\lambda \in \mathbb{D}$, we have $\Psi(\cdot, x(\lambda))$ is a linear fractional map. Moreover, for all $\lambda \in \mathbb{D}$, by Theorem 4.2.4 (iib), $x(\lambda) \in \overline{\mathbb{E}}$ if and only if $|\Psi(z, x(\lambda))| \leq 1$ for all $z \in \mathbb{D}$, and if $x_1(\lambda)x_2(\lambda) = x_3(\lambda)$ then, in addition, $|x_2(\lambda)| \leq 1$. This fact and the following lemma motivate our definition of the subset of $\mathcal{S}_2$.

Lemma 4.3.13. *Let $\varphi \in \mathcal{S}_2$ be such that, for all $\lambda \in \mathbb{D}$, we have $\varphi(\cdot, \lambda)$ is a linear fractional map. Then φ can be written as*

$$\varphi(z, \lambda) = \frac{a(\lambda)z + b(\lambda)}{c(\lambda)z + 1}$$

for all $z, \lambda \in \mathbb{D}$, where a, b, c are functions from $\mathbb{D}$ to $\mathbb{C}$. Moreover, b is holomorphic, and if c is holomorphic then so is a.

Proof. Since, for all $\lambda \in \mathbb{D}$, we have $\varphi(\cdot, \lambda)$ is a linear fractional map, we can write

$$\varphi(z, \lambda) = \frac{a(\lambda)z + b(\lambda)}{c(\lambda)z + d(\lambda)}$$

for all $z, \lambda \in \mathbb{D}$, where a, b, c, d are functions from $\mathbb{D}$ to $\mathbb{C}$. Since $\varphi \in \mathcal{S}_2$, for any $\lambda \in \mathbb{D}$, up to cancellation, $\varphi(\cdot, \lambda)$ does not have a pole at 0, and so $d(\lambda) \neq 0$. Thus, without loss of generality, we can write

$$\varphi(z, \lambda) = \frac{a(\lambda)z + b(\lambda)}{c(\lambda)z + 1}$$

for all $z, \lambda \in \mathbb{D}$. Moreover, since $b(\lambda) = \varphi(0, \lambda)$ for all $\lambda \in \mathbb{D}$, we have b is holomorphic. Now suppose c is holomorphic. Then

$$a(\lambda)z = \varphi(z, \lambda)(c(\lambda)z + 1) - b(\lambda)$$

for all $z, \lambda \in \mathbb{D}$, and so a is holomorphic. □

Definition 4.3.14. *We define $\mathcal{S}_2^{\mathrm{lf}}$ to be the subset of $\mathcal{S}_2$ which contains those φ such that, for all $\lambda \in \mathbb{D}$, we have $\varphi(\cdot, \lambda)$ is a linear fractional map of the form*

$$\varphi(z, \lambda) = \frac{a(\lambda)z + b(\lambda)}{c(\lambda)z + 1}$$

for all $z, \lambda \in \mathbb{D}$, where c is holomorphic, and if $a(\lambda) = b(\lambda)c(\lambda)$ for some $\lambda \in \mathbb{D}$, then, in addition, $|c(\lambda)| \leq 1$.

We now give the relationship between $\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ and $\mathcal{S}_2^{\mathrm{lf}}$.

Proposition 4.3.15. *Let φ be a function on $\mathbb{D}^2$. Then $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$ if and only if there exists a function $x \in \mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ such that*

$$\varphi(z, \lambda) = \Psi(z, x(\lambda)) \text{ for all } z, \lambda \in \mathbb{D}.$$

Proof. First, suppose $x = (x_1, x_2, x_3) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. Define $\varphi(z, \lambda) := \Psi(z, x(\lambda))$ for all $z, \lambda \in \mathbb{D}$. Since x is holomorphic and maps into $\overline{\mathbb{E}}$, and since Ψ is holomorphic on $\mathbb{D} \times \overline{\mathbb{E}}$, we infer that φ is holomorphic on $\mathbb{D}^2$. For all $z, \lambda \in \mathbb{D}$, by Theorem 4.2.4 (iib), $|\Psi(z, x(\lambda))| \leq 1$, and so $\varphi(z, \lambda) \in \overline{\mathbb{D}}$. It follows that $\varphi \in \mathcal{S}_2$. Moreover,

$$\varphi(z, \lambda) = \frac{x_3(\lambda)z - x_1(\lambda)}{x_2(\lambda)z - 1}$$

for all $z, \lambda \in \mathbb{D}$, where x_2 is holomorphic. By Theorem 4.2.4 (iib), if $x_1(\lambda)x_2(\lambda) = x_3(\lambda)$ for some $\lambda \in \mathbb{D}$, then, in addition, $|x_2(\lambda)| \leq 1$. It follows that $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$.

Conversely, suppose that $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$. Then

$$\varphi(z, \lambda) = \frac{a(\lambda)z + b(\lambda)}{c(\lambda)z + 1}$$

for all $z, \lambda \in \mathbb{D}$, where c is holomorphic, and if $a(\lambda) = b(\lambda)c(\lambda)$ for some $\lambda \in \mathbb{D}$, then, in addition, $|c(\lambda)| \leq 1$. Moreover, by Lemma 4.3.13, both a and b are holomorphic. Set

$$x(\lambda) = (b(\lambda), -c(\lambda), -a(\lambda))$$

for all $\lambda \in \mathbb{D}$. Then x is holomorphic on $\mathbb{D}$, and

$$|\Psi(z, x(\lambda)| = |\varphi(z, \lambda)| \leq 1 \text{ for all } z, \lambda \in \mathbb{D}.$$

Hence, by Theorem 4.2.4 (iib), $x(\lambda) \in \overline{\mathbb{E}}$ for all $\lambda \in \mathbb{D}$. It follows that there is an $x \in \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ such that $\varphi(z, \lambda) = \Psi(z, x(\lambda))$ for all $z, \lambda \in \mathbb{D}$. □

Definition 4.3.16. *We define* $\mathrm{Lower}\,\mathrm{E_T} : \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}}) \to \mathcal{S}_2^{\mathrm{lf}}$ *by*

$$\mathrm{Lower}\,\mathrm{E_T}\,(x)(z, \lambda) := \Psi(z, x(\lambda)) = \frac{x_3(\lambda)z - x_1(\lambda)}{x_2(\lambda) - 1}$$

for all $x = (x_1, x_2, x_3) \in \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ *and all* $z, \lambda \in \mathbb{D}$.

That $\mathrm{Lower}\,\mathrm{E_T}$ is well defined follows immediately from Proposition 4.3.15.

4.3.5 $\mathrm{Lower}\,\mathrm{W_T} : \mathcal{S}_2^{\mathrm{lf}} \to \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$

In terms of the uniqueness of a function $x \in \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ as in Proposition 4.3.15, we obtain the following result.

Proposition 4.3.17. *Let* $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$. *Suppose* $x = (x_1, x_2, x_3)$ *and* $y = (y_1, y_2, y_3)$ *are functions in* $\mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ *such that*

$$\Psi(z, x(\lambda)) = \varphi(z, \lambda) = \Psi(z, y(\lambda)) \text{ for all } z, \lambda \in \mathbb{D}.$$

Then,

(i) *if* $x_1x_2 \neq x_3$, *we have* $x = y$;

(ii) *if* $x_1x_2 = x_3$, *we have* $y = (x_1, y_2, x_1y_2)$.

Proof. Since $\Psi(z, x(\lambda)) = \varphi(z, \lambda) = \Psi(z, y(\lambda))$ for all $z \in \mathbb{D}$, we have

$$\frac{x_3z - x_1}{x_2z - 1} = \frac{y_3z - y_1}{y_2z - 1}$$

and so $x_3y_2z^2 - (x_1y_2 + x_3)z + x_1 = y_3x_2z^2 - (y_1x_2 + y_3)z + y_1$ for all $z \in \mathbb{D}$. By equating coefficients, we obtain

$$x_1 = y_1, x_3y_2 = y_3x_2 \text{ and } x_1y_2 + x_3 = y_1x_2 + y_3.$$

For (ii), suppose $x_1x_2 = x_3$. Then $y_1 = x_1$ and

$$y_3 = x_1y_2 + x_3 - y_1x_2 = x_1y_2 + x_1x_2 - x_1x_2 = x_1y_2.$$

That is, $y = (x_1, y_2, x_1y_2)$.

For (i), suppose instead that $x_1x_2 \neq x_3$. Then $f := x_3 - x_1x_2 \neq 0$ and is holomorphic on $\mathbb{D}$. Hence, by Theorem B.1.25, the zeros of f are isolated. Let $\lambda \in \mathbb{D}$. Then, since $x_3y_2 = (x_1y_2 + x_3 - x_1x_2)x_2$, we have

$$(x_3(\lambda) - x_1(\lambda)x_2(\lambda))y_2(\lambda) = (x_3(\lambda) - x_1(\lambda)x_2(\lambda))x_2(\lambda).$$

If $f(\lambda) \neq 0$ then clearly $y_2(\lambda) = x_2(\lambda)$. If $f(\lambda) = 0$ then there is a sequence $(\lambda_n)_{n=1}^{\infty}$ in $\mathbb{D}$ such that $\lim_{n\to\infty} \lambda_n = \lambda$, and $f(\lambda_n) \neq 0$ for each $n \in \mathbb{N}$. Hence $y_2(\lambda_n) = x_2(\lambda_n)$ for each $n \in \mathbb{N}$, and so $y_2(\lambda) = x_2(\lambda)$. Either way, we obtain $y_2(\lambda) = x_2(\lambda)$. It follows that

$$y_1 = x_1, y_2 = x_2 \text{ and } y_3 = x_1x_2 + x_3 - x_1x_2 = x_3,$$

that is, $y = x$. $\square$

Consequently, in some cases, the function $x \in \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ as in Proposition 4.3.15 may not be unique. However, the proof of Proposition 4.3.15 provides the construction of a function in $\mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ for each function in $\mathcal{S}_2^{\mathrm{lf}}$. We use this construction to define $\mathrm{Lower\,W_T}$.

Definition 4.3.18. *We define* $\mathrm{Lower\,W_T} : \mathcal{S}_2^{\mathrm{lf}} \to \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ *by:*

(i) *for* $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$ *such that* $\varphi(z, \lambda) = \dfrac{a(\lambda)z + b(\lambda)}{c(\lambda)z + 1}$ *for all* $z, \lambda \in \mathbb{D}$, *where* $a \neq bc$, *the map*

$$\mathrm{Lower\,W_T}\,(\varphi) = (b, -c, -a);$$

(ii) *for* $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$ *such that* $\varphi(z, \lambda) = b(\lambda)$ *for all* $z, \lambda \in \mathbb{D}$, *that is,* $a = bc$, *the set map*

$$\mathrm{Lower\,W_T}\,(\varphi) = \{(b, -d, -bd) : d \text{ is holomorphic and } |d(\lambda)| \leq 1 \text{ for all } \lambda \in \mathbb{D}\}.$$

By the proof of Proposition 4.3.15 and by Proposition 4.3.17, $\mathrm{Lower\,W_T}$ is well defined.

Proposition 4.3.19. *The following relations hold.*

(i) *For* $x = (x_1, x_2, x_3) \in \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ *such that* $x_1x_2 \neq x_3$, *we have*

$$\mathrm{Lower\,W_T} \circ \mathrm{Lower\,E_T}\,(x) = x.$$

(ii) *For* $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$ *such that* $\varphi(z, \lambda) = \dfrac{a(\lambda)z + b(\lambda)}{c(\lambda)z + 1}$ *for all* $z, \lambda \in \mathbb{D}$, *where* $a \neq bc$, *we have*

$$\mathrm{Lower\,E_T} \circ \mathrm{Lower\,W_T}\,(\varphi) = \varphi.$$

(iii) *For* $x = (x_1, x_2, x_3) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ *such that* $x_1 x_2 = x_3$*, we have*

$$\begin{aligned}&\mathrm{Lower\,W_T} \circ \mathrm{Lower\,E_T}(x) = \\ &\qquad =\{(x_1, -d, -x_1 d) : d \textit{ is holomorphic and } |d(\lambda)| \leq 1 \textit{ for all } \lambda \in \mathbb{D}\}.\end{aligned}$$

(iv) *For* $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$ *such that* $\varphi(z, \lambda) = b(\lambda)$ *for all* $z, \lambda \in \mathbb{D}$*, we have*

$$\mathrm{Lower\,E_T}(x) = \varphi \textit{ for all } x \in \mathrm{Lower\,W_T}(\varphi).$$

Proof. (i) Let $x = (x_1, x_2, x_3) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ be such that $x_1 x_2 \neq x_3$. Then $\mathrm{Lower\,E_T}(x) = \varphi \in \mathcal{S}_2^{\mathrm{lf}}$, where

$$\varphi(z, \lambda) = \Psi(z, x(\lambda)) = \frac{x_3(\lambda) z - x_1(\lambda)}{x_2(\lambda) z - 1} = \frac{-x_3(\lambda) z + x_1(\lambda)}{-x_2(\lambda) z + 1}$$

for all $z, \lambda \in \mathbb{D}$. Since $x_1 x_2 \neq x_3$, we have

$$\mathrm{Lower\,W_T}(\varphi) = (x_1, x_2, x_3) = x.$$

It follows that $\mathrm{Lower\,W_T} \circ \mathrm{Lower\,E_T}(x) = x$.

(ii) Let $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$ be such that $\varphi(z, \lambda) = \frac{a(\lambda) z + b(\lambda)}{c(\lambda) z + 1}$ for all $z, \lambda \in \mathbb{D}$, where $a \neq bc$. Then $\mathrm{Lower\,W_T}(\varphi) = (b, -c, -a) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. Moreover,

$$\mathrm{Lower\,E_T}((b, -c, -a))(z, \lambda) = \Psi(z, b(\lambda), -c(\lambda), -a(\lambda)) = \frac{-a(\lambda) z - b(\lambda)}{-c(\lambda) z - 1} = \varphi(z, \lambda)$$

for all $z, \lambda \in \mathbb{D}$. It follows that $\mathrm{Lower\,E_T} \circ \mathrm{Lower\,W_T}(\varphi) = \varphi$.

(iii) Let $x = (x_1, x_2, x_3) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ be such that $x_1 x_2 = x_3$. Then $\mathrm{Lower\,E_T}(x) = \varphi \in \mathcal{S}_2^{\mathrm{lf}}$, where

$$\varphi(z, \lambda) = \Psi(z, x(\lambda)) = \frac{x_1(\lambda) x_2(\lambda) z - x_1(\lambda)}{x_2(\lambda) z - 1} = x_1(\lambda)$$

for all $z, \lambda \in \mathbb{D}$. Hence

$$\begin{aligned}&\mathrm{Lower\,W_T} \circ \mathrm{Lower\,E_T}(x) = \\ &\qquad =\{(x_1, -d, -x_1 d) : d \text{ is holomorphic and } |d(\lambda)| \leq 1 \text{ for all } \lambda \in \mathbb{D}\}.\end{aligned}$$

(iv) Let $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$ be such that $\varphi(z, \lambda) = b(\lambda)$ for all $z, \lambda \in \mathbb{D}$. Then

$$\mathrm{Lower\,W_T}(\varphi) =\{(b, -d, -bd) : d \text{ is holomorphic and } |d(\lambda)| \leq 1 \text{ for all } \lambda \in \mathbb{D}\},$$

and is a subset of $\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$. Moreover, for any $x = (b, -d, -bd) \in \mathrm{Lower\,W_T}(\varphi)$, we have

$$\text{Lower E}_\text{T}\,(x)(z,\lambda) = \Psi(z, b(\lambda), -d(\lambda), -b(\lambda)d(\lambda)) = \frac{-b(\lambda)d(\lambda) - b(\lambda)}{-d(\lambda) - 1} = b(\lambda) = \varphi(z,\lambda)$$

for all $z, \lambda \in \mathbb{D}$. Hence $\text{Lower E}_\text{T}\,(x) = \varphi$ for all $x \in \text{Lower W}_\text{T}\,(\varphi)$. $\square$

By Proposition 4.3.19 (iii), it is clear that, for $x = (x_1, x_2, x_3) \in \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ such that $x_1 x_2 = x_3$, we have

$$x \in \text{Lower W}_\text{T} \circ \text{Lower E}_\text{T}\,(x).$$

4.3.6 Relations between the remaining maps

We now consider how some of the maps we defined in this section interact with some of the maps in Chapter 2.

Proposition 4.3.20. $\text{SE} \circ \text{Left N}_\text{T} = \text{Lower E}_\text{T}$.

Proof. Let $x \in \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$. Then $\text{Left N}_\text{T}\,(x) = F \in \mathcal{S}^{2\times 2}$ as defined in Theroem 4.3.1. By the proof of Theorem 4.3.1,

$$\text{SE}\,(F)(z,\lambda) = \mathcal{F}_{F(\lambda)}(z) = \Psi(z, x(\lambda))$$

for all $z, \lambda \in \mathbb{D}$. Hence

$$\text{SE} \circ \text{Left N}_\text{T}\,(x)(z,\lambda) = \Psi(z, x(\lambda)) = \text{Lower E}_\text{T}\,(x)(z,\lambda)$$

for all $z, \lambda \in \mathbb{D}$. It follows that $\text{SE} \circ \text{Left N}_\text{T} = \text{Lower E}_\text{T}$. $\square$

Corollary 4.3.21. *The following relations hold.*

(i) *For $x = (x_1, x_2, x_3) \in \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ such that $x_1 x_2 \neq x_3$, we have*

$$\text{Lower W}_\text{T} \circ \text{SE} \circ \text{Left N}_\text{T}\,(x) = x.$$

(ii) *For $\varphi \in \mathcal{S}_2^{\text{lf}}$ such that $\varphi(z,\lambda) = \dfrac{a(\lambda)z + b(\lambda)}{c(\lambda)z + 1}$ for all $z, \lambda \in \mathbb{D}$, where $a \neq bc$, we have*

$$\text{SE} \circ \text{Left N}_\text{T} \circ \text{Lower W}_\text{T}\,(\varphi) = \varphi.$$

(iii) *For $x = (x_1, x_2, x_3) \in \text{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ such that $x_1 x_2 = x_3$, we have*

$$\begin{aligned}&\text{Lower W}_\text{T} \circ \text{SE} \circ \text{Left N}_\text{T}\,(x) = \\ &\qquad = \{(x_1, -d, -x_1 d) : d \textit{ is holomorphic and } |d(\lambda)| \leq 1 \textit{ for all } \lambda \in \mathbb{D}\}.\end{aligned}$$

(iv) For $\varphi \in \mathcal{S}_2^{\mathrm{lf}}$ such that $\varphi(z, \lambda) = b(\lambda)$ for all $z, \lambda \in \mathbb{D}$, we have

$$\mathrm{SE} \circ \mathrm{Left\,N_T}\,(x) = \varphi \text{ for all } x \in \mathrm{Lower\,W_T}\,(\varphi).$$

Proof. The results follow immediately from Proposition 4.3.20 and Proposition 4.3.19. □

Proposition 4.3.22. $\mathrm{Lower\,E_T} \circ \mathrm{Left\,S_T} = \mathrm{SE}$.

Proof. Let $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathcal{S}^{2\times 2}$. Then

$$\mathrm{SE}\,(F)(z, \lambda) = F_{11}(\lambda) + \frac{F_{12}(\lambda)F_{21}(\lambda)z}{1 - F_{22}(\lambda)z} = \frac{F_{11}(\lambda) - (F_{11}(\lambda)F_{22}(\lambda) - F_{12}(\lambda)F_{21}(\lambda))z}{1 - F_{22}(\lambda)z}$$

for all $z, \lambda \in \mathbb{D}$. Moreover, $\mathrm{Left\,S_T}\,(F) = (F_{11}, F_{22}, \det F)$ and so

$$\begin{aligned} \mathrm{Lower\,E_T} \circ \mathrm{Left\,S_T}\,(F)(z, \lambda) &= \Psi(z, F_{11}(\lambda), F_{22}(\lambda), \det F(\lambda)) \\ &= \frac{\det F(\lambda)z - F_{11}(\lambda)}{F_{22}(\lambda)z - 1} \\ &= \frac{F_{11}(\lambda) - (F_{11}(\lambda)F_{22}(\lambda) - F_{21}(\lambda)F_{12}(\lambda))z}{1 - F_{22}(\lambda)z} \\ &= \mathrm{SE}\,(F)(z, \lambda) \end{aligned}$$

for all $z, \lambda \in \mathbb{D}$. It follows that $\mathrm{Lower\,E_T} \circ \mathrm{Left\,S_T} = \mathrm{SE}$. □

4.4 Criterion for solvability

In this section, we present a criterion for the solvability of the μ-synthesis problem given by Question 1.2.2. In addition, we give a number of related results, which can be seen to arise from the rich structure we have been studying.

Theorem 4.4.1. *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$, and let $(x_{1j}, x_{2j}, x_{3j}) \in \overline{\mathbb{E}}$ be such that $x_{1j}x_{2j} \neq x_{3j}$ for $j = 1, \ldots, n$. Then the following are equivalent.*

(i) *There exists a holomorphic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ satisfying*

$$x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j}) \text{ for } j = 1, \ldots, n;$$

(ii) *there exists a rational $\overline{\mathbb{E}}$-inner function x satisfying*

$$x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j}) \text{ for } j = 1, \ldots, n;$$

(iii) *for every distinct points $z_1, z_2, z_3 \in \mathbb{D}$, there exist positive $3n$-square matrices $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ of rank at most 1, and $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ such that, for*

$1 \le i, j \le n$ *and* $1 \le l, k \le 3$,

$$1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1} = (1 - \overline{z_l} z_k) N_{il,jk} + (1 - \overline{\lambda_i} \lambda_j) M_{il,jk};$$

(iv) *for some distinct points* $z_1, z_2, z_3 \in \mathbb{D}$, *there exist positive* $3n$*-square matrices* $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1, *and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that*

$$\left[1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1}\right] \ge [(1 - \overline{z_l} z_k) N_{il,jk}] + \left[(1 - \overline{\lambda_i} \lambda_j) M_{il,jk}\right].$$

Proof. Clearly (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (iv). To complete the proof, we need to show that (iii) $\Longrightarrow$ (ii), (iv) $\Longrightarrow$ (i) and (i) $\Longrightarrow$ (iii).

(iii) $\Longrightarrow$ (ii): Suppose (iii) holds. Then, since N is positive and has rank 1, there are $\gamma_{jk} \in \mathbb{C}$ such that, for all $j = 1, \ldots, n$ and $k = 1, 2, 3$, we have

$$N_{il,jk} = \overline{\gamma_{il}} \gamma_{jk}.$$

Similarly, since M is positive, there is a Hilbert space H of dimension at most $3n$ and vectors $v_{jk} \in H$ such that, for all $j = 1, \ldots, n$ and $k = 1, 2, 3$, we have

$$M_{il,jk} = \langle v_{jk}, v_{il} \rangle_H.$$

Recall that $\Psi(z_k, x_{1j}, x_{2j}, x_{3j}) = \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1}$. Then, as in Procedure UW, we can show that, for $j = 1, \ldots, n$ and $k = 1, 2, 3$, the Grammian of the vectors

$$\left(\begin{pmatrix} \Psi(z_k, x_{1j}, x_{2j}, x_{3j}) \\ \gamma_{jk} \end{pmatrix} \atop v_{jk} \right) \in \mathbb{C}^2 \oplus H$$

is equal to the Grammian of the vectors

$$\left(\begin{pmatrix} 1 \\ z_k \gamma_{jk} \end{pmatrix} \atop \lambda_j v_{jk} \right) \in \mathbb{C}^2 \oplus H.$$

Hence, by Proposition B.1.22, there is a unitary L on $\mathbb{C}^2 \oplus H$ such that

$$L \left(\begin{pmatrix} 1 \\ z_k \gamma_{jk} \end{pmatrix} \atop \lambda_j v_{jk} \right) = \left(\begin{pmatrix} \Psi(z_k, x_{1j}, x_{2j}, x_{3j}) \\ \gamma_{jk} \end{pmatrix} \atop v_{jk} \right)$$

for all $j = 1, \ldots, n$ and $k = 1, 2, 3$. If we write $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then

$$\begin{pmatrix} \Psi(z_k, x_{1j}, x_{2j}, x_{3j}) \\ \gamma_{jk} \end{pmatrix} = A \begin{pmatrix} 1 \\ z_k\gamma_{jk} \end{pmatrix} + B\lambda_j v_{jk} \text{ and } v_{jk} = C \begin{pmatrix} 1 \\ z_k\gamma_{jk} \end{pmatrix} + D\lambda_j v_{jk}$$

for all $j = 1, \ldots, n$ and $k = 1, 2, 3$. Thus

$$v_{jk} = (I - D\lambda_j)^{-1} C \begin{pmatrix} 1 \\ z_k\gamma_{jk} \end{pmatrix},$$

and so

$$\begin{pmatrix} \Psi(z_k, x_{1j}, x_{2j}, x_{3j}) \\ \gamma_{jk} \end{pmatrix} = (A + B\lambda_j(I - D\lambda_j)^{-1}C) \begin{pmatrix} 1 \\ z_k\gamma_{jk} \end{pmatrix}$$

for all $j = 1, \ldots, n$ and $k = 1, 2, 3$. Now, let

$$\Xi(\lambda) = A + B\lambda(I - D\lambda)^{-1}C = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}$$

for all $\lambda \in \mathbb{D}$. Since L is unitary and H is finite dimensional, Ξ is a rational 2×2 inner function. Hence the function defined by $x := (a, d, \det \Xi)$ is a rational $\overline{\mathbb{E}}$-inner function.

If we show that x satisfies $x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j})$ for all $j = 1, \ldots, n$, then we are done. We have shown that

$$\begin{pmatrix} \Psi(z_k, x_{1j}, x_{2j}, x_{3j}) \\ \gamma_{jk} \end{pmatrix} = \Xi(\lambda_j) \begin{pmatrix} 1 \\ z_k\gamma_{jk} \end{pmatrix}$$

for all $j = 1, \ldots, n$ and $k = 1, 2, 3$. Hence

$$\Psi(z_k, x_{1j}, x_{2j}, x_{3j}) = a(\lambda_j) + b(\lambda_j)z_k\gamma_{jk} \text{ and } \gamma_{jk} = c(\lambda_j) + d(\lambda_j)z_k\gamma_{jk},$$

and so

$$\Psi(z_k, x_{1j}, x_{2j}, x_{3j}) = a(\lambda_j) + b(\lambda_j)z_k(1 - d(\lambda_j)z_k)^{-1}c(\lambda_j)$$

for all $j = 1, \ldots, n$ and $k = 1, 2, 3$. Thus, for each $j = 1, \ldots, n$, the linear fractional maps

$$\frac{x_{1j} - x_{3j}z}{1 - x_{2j}z} \text{ and } \frac{a(\lambda_j) - (a(\lambda_j)d(\lambda_j) - b(\lambda_j)c(\lambda_j))z}{1 - d(\lambda_j)z}$$

agree at three distinct points in $\mathbb{D}$, and it follows that they are the same map. By Proposition 4.3.17, since $x_{1j}x_{2j} \neq x_{3j}$ for $j = 1, \ldots, n$, we have

$$a(\lambda_j) = x_{1j}, d(\lambda_j) = x_{2j} \text{ and } \det \Xi(\lambda_j) = a(\lambda_j)d(\lambda_j) - b(\lambda_j)c(\lambda_j) = x_{3j},$$

and so $x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j})$ for all $j = 1, \ldots, n$.

(iv) $\implies$ (i): This proof is similar to (iii) $\implies$ (ii), the difference is that, for $j = 1, \ldots, n$ and $k = 1, 2, 3$, the Grammian of the vectors

$$\left(\begin{pmatrix} \Psi(z_k, x_{1j}, x_{2j}, x_{3j}) \\ \gamma_{jk} \end{pmatrix} \atop v_{jk}\right) \in \mathbb{C}^2 \oplus H$$

is less than or equal to the Grammian of the vectors

$$\left(\begin{pmatrix} 1 \\ z_k \gamma_{jk} \end{pmatrix} \atop \lambda_j v_{jk}\right) \in \mathbb{C}^2 \oplus H.$$

Hence, there is a contraction $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ on $\mathbb{C}^2 \oplus H$ such that

$$L\left(\begin{pmatrix} 1 \\ \gamma_{jk} \end{pmatrix} \atop v_{jk}\right) = \left(\begin{pmatrix} \Psi(z_k, x_{1j}, x_{2j}, x_{3j}) \\ z_k\gamma_{jk} \end{pmatrix} \atop \lambda_j v_{jk}\right)$$

for all $j = 1, \ldots, n$ and $k = 1, 2, 3$. Now, let

$$\Xi(\lambda) = A + B\lambda(I - D\lambda)^{-1}C = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}$$

for all $\lambda \in \mathbb{D}$. Since L is a contraction, $\Xi \in \mathcal{S}^{2\times 2}$, and so $x := (a, d, \det \Xi) \in \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$. That $x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j})$ for all $j = 1, \ldots, n$, follows as in (iii) $\implies$ (ii).

(i) $\implies$ (iii): Suppose there is a holomorphic function $x = (x_1, x_2, x_3) : \mathbb{D} \to \overline{\mathbb{E}}$ such that $x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j})$ for all $j = 1, \ldots, n$. By Theorem 4.3.1, since $x_{1j}x_{2j} \neq x_{3j}$ for $j = 1, \ldots, n$, there is a holomorphic function

$$F = \begin{bmatrix} x_1 & f_1 \\ f_2 & x_2 \end{bmatrix} : \mathbb{D} \to \mathcal{M}_2(\mathbb{C})$$

such that $f_2 \neq 0$, $||F(\lambda)|| \leq 1$ for all $\lambda \in \mathbb{D}$, and

$$1 - \overline{\Psi(w, x(\mu))}\Psi(z, x(\lambda)) = (1 - \overline{w}z)\overline{\gamma(\mu, w)}\gamma(\lambda, z) + (1 - \overline{\mu}\lambda)\eta(\mu, w)^* \frac{I - F(\mu)^*F(\lambda)}{1 - \overline{\mu}\lambda}\eta(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$, where

$$\gamma(\lambda, z) = (1 - x_2(\lambda)z)^{-1}f_2(\lambda) \text{ and } \eta(\lambda, z) = \begin{bmatrix} 1 \\ \gamma(\lambda, z)z \end{bmatrix}.$$

Let z_1, z_2, z_3 be any distinct points in $\mathbb{D}$. Then, in particular, for $1 \leq i, j \leq n$ and

$1 \leq l, k \leq 3$, we have

$$\begin{aligned}
1-&\overline{\Psi(z_l, x_{1i}, x_{2i}, x_{3i})}\Psi(z_k, x_{1j}, x_{2j}, x_{3j}) = \\
&= (1-\overline{z_l}z_k)\overline{\gamma(\lambda_i, z_l)}\gamma(\lambda_j, z_k) + (1-\overline{\lambda_i}\lambda_j)\eta(\lambda_i, z_l)^*\frac{I - F(\lambda_i)^*F(\lambda_j)}{1-\overline{\lambda_i}\lambda_j}\eta(\lambda_j, z_k).
\end{aligned}$$

By Corollary B.3.23, since $F \in \mathcal{S}^{2\times 2}$ and $f_2 \neq 0$, the map $(z, \lambda, w, \mu) \mapsto \overline{\gamma(\mu, w)}\gamma(\lambda, z)$ is a rank 1 kernel on $\mathbb{D}^2$. By Corollary B.3.32, since $F \in \mathcal{S}^{2\times 2}$, the map

$$(z, \lambda, w, \mu) \mapsto \eta(\mu, w)^*\frac{I - F(\mu)^*F(\lambda)}{1-\overline{\mu}\lambda}\eta(\lambda, z)$$

is a kernel on $\mathbb{D}^2$. Hence the $3n$-square matrices

$$N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3} := \left[\overline{\gamma(\lambda_i, z_l)}\gamma(\lambda_j, z_k)\right]_{i,j=1,l,k=1}^{n,3}$$

and

$$M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3} := \left[\eta(\lambda_i, z_l)^*\frac{I - F(\lambda_i)^*F(\lambda_j)}{1-\overline{\lambda_i}\lambda_j}\eta(\lambda_j, z_k)\right]_{i,j=1,l,k=1}^{n,3}$$

are positive semidefinite. Moreover, N has rank 1 and, for $1 \leq i, j \leq n$ and $1 \leq l, k \leq 3$,

$$1 - \overline{\Psi(z_l, x_{1i}, x_{2i}, x_{3i})}\Psi(z_k, x_{1j}, x_{2j}, x_{3j}) = (1-\overline{z_l}z_k)N_{il,jk} + (1-\overline{\lambda_i}\lambda_j)M_{il,jk}.$$

It follows that (i) $\Longrightarrow$ (iii). ☐

As a corollary of Theorem 4.4.1, we obtain the following criterion for the solvability of the associated μ-synthesis problem.

Theorem 4.4.2. *Let $\lambda_1, \dots, \lambda_n$ be distinct points in $\mathbb{D}$, and let*

$$W_j = \begin{bmatrix} w_{11}^j & w_{12}^j \\ w_{21}^j & w_{22}^j \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$$

be such that $\mu_{\mathrm{Diag}}(W_j) \leq 1$ and $w_{11}^j w_{22}^j \neq \det W_j$ for $j = 1, \dots, n$. Set $(x_{1j}, x_{2j}, x_{3j}) = (w_{11}^j, w_{22}^j, \det W_j) \in \overline{\mathbb{E}}$ for each $j = 1, \dots, n$. Then the following are equivalent.

(i) *There exists a holomorphic function $F : \mathbb{D} \to \mathcal{M}_2(\mathbb{C})$ such that $F(\lambda_j) = W_j$ for $j = 1, \dots, n$, and $\mu_{\mathrm{Diag}}(F(\lambda)) \leq 1$ for all $\lambda \in \mathbb{D}$;*

(ii) *there exists a holomorphic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ satisfying*

$$x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j}) \text{ for } j = 1, \dots, n;$$

(iii) *for some distinct points $z_1, z_2, z_3 \in \mathbb{D}$, there exist positive $3n$-square matrices $N =$*

$[N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *of rank at most* 1*, and* $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ *such that*

$$\left[1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1}\right] \geq [(1 - \overline{z_l} z_k) N_{il,jk}] + \left[(1 - \overline{\lambda_i}\lambda_j) M_{il,jk}\right].$$

Proof. Since $w_{11}^j w_{22}^j \neq \det W_j$ for $j = 1, \ldots, n$, we have $x_{1j} x_{2j} \neq x_{3j}$ for $j = 1, \ldots, n$. Hence the theorem follows from a combination of Theorem 4.1.4 and Theorem 4.4.1. □

The proof of Theorem 4.4.1 provides a procedure by which a solution x to an $\overline{\mathbb{E}}$-interpolation problem can be obtained from a pair (N, M) satisfying the conditions of Theorem 4.4.1 (iv). We call this Procedure SW_T, and it is essentially Procedure UW followed by the Left $\mathrm{S_T}$ map. More specifically we have:

Procedure SW_T. Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$, and let $(x_{1j}, x_{2j}, x_{3j}) \in \overline{\mathbb{E}}$ be such that $x_{1j} x_{2j} \neq x_{3j}$ for $j = 1, \ldots, n$. For some distinct points $z_1, z_2, z_3 \in \mathbb{D}$, suppose N and M are positive $3n$-square matrices such that N has rank at most 1, and the matrix inequality as in Theorem 4.4.1 (iv) holds. Then:

1. Choose scalars $\gamma_{jk} \in \mathbb{C}$ such that $N = [\overline{\gamma_{il}} \gamma_{jk}]_{i,j=1,l,k=1}^{n,3}$.

2. Choose a Hilbert space $\mathcal{H}$ and vectors $v_{jk} \in \mathcal{H}$ such that $M = [\langle v_{jk}, v_{il} \rangle_{\mathcal{H}}]_{i,j=1,l,k=1}^{n,3}$.

3. Choose a contraction $\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathbb{C}^2 \oplus \mathcal{H} \to \mathbb{C}^2 \oplus \mathcal{H}$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \left(\begin{pmatrix} 1 \\ z_k \gamma_{jk} \end{pmatrix} \atop \lambda_j v_{jk} \right) = \left(\begin{pmatrix} \Psi(z_k, x_{1j}, x_{2j}, x_{3j}) \\ \gamma_{jk} \end{pmatrix} \atop v_{jk} \right)$$

   for all $j = 1, \ldots, n$ and $k = 1, 2, 3$.

4. Let $x = (a, d, \det \Xi)$, where $\Xi(\lambda) = A + B\lambda(I - D\lambda)^{-1} C = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}$ for all $\lambda \in \mathbb{D}$.

Now, we have $x \in \mathrm{Hol}\,(\mathbb{D}, \overline{\mathbb{E}})$ and $x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j})$ for $j = 1, \ldots, n$. □

The following proposition shows that every interpolating function can be obtained by applying Procedure SW_T to a general solution (N, M) of the matrix inequality such that the rank of N is less than or equal to 1.

Proposition 4.4.3. *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$, and let $(x_{1j}, x_{2j}, x_{3j}) \in \overline{\mathbb{E}}$ be such that $x_{1j} x_{2j} \neq x_{3j}$ for $j = 1, \ldots, n$. Then every holomorphic function $x : \mathbb{D} \to \overline{\mathbb{E}}$ satisfying*

$$x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j}), \text{ for } j = 1, \ldots, n,$$

arises by Procedure SW_T from a pair of positive $3n$-square matrices $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ of rank at most 1, and $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ satisfying

$$\left[1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i} z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j} z_k - 1}\right] \geq [(1 - \overline{z_l} z_k) N_{il,jk}] + \left[(1 - \overline{\lambda_i} \lambda_j) M_{il,jk}\right],$$

where z_1, z_2, z_3 are distinct points in $\mathbb{D}$.

Proof. Suppose $x = (x_1, x_2, x_3) \in \mathrm{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ is such that $x(\lambda_j) = (x_{1j}, x_{2j}, x_{3j})$ for all $j = 1, \ldots, n$. By Theorem 4.3.1, since $x_{1j} x_{2j} \neq x_{3j}$, there is a function

$$F = \begin{bmatrix} x_1 & f_1 \\ f_2 & x_2 \end{bmatrix} \in \mathcal{S}^{2\times 2}$$

such that $f_2 \neq 0$, and

$$1 - \overline{\Psi(w, x(\mu))} \Psi(z, x(\lambda)) = (1 - \overline{w} z) \overline{\gamma(\mu, w)} \gamma(\lambda, z) + (1 - \overline{\mu}\lambda) \eta(\mu, w)^* \frac{I - F(\mu)^* F(\lambda)}{1 - \overline{\mu}\lambda} \eta(\lambda, z)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$, where

$$\gamma(\lambda, z) = (1 - x_2(\lambda) z)^{-1} f_2(\lambda) \text{ and } \eta(\lambda, z) = \begin{bmatrix} 1 \\ z\gamma(z, \lambda) \end{bmatrix}.$$

By Proposition B.3.30, since $F \in \mathcal{S}^{2\times 2}$, we have

$$K : (\lambda, \mu) \mapsto \frac{I - F(\mu)^* F(\lambda)}{1 - \overline{\mu}\lambda}$$

is an $\mathcal{M}_2(\mathbb{C})$-valued kernel on $\mathbb{D}$. Hence, by Corollary B.3.20, there is a conjugate analytic map $U : \mathbb{D} \to \mathcal{B}(\mathbb{C}^2, \mathcal{H}_K)$ such that

$$\frac{I - F(\mu)^* F(\lambda)}{1 - \overline{\mu}\lambda} = U(\mu)^* U(\lambda)$$

for all $\lambda, \mu \in \mathbb{D}$. Let z_1, z_2, z_3 be any distinct points in $\mathbb{D}$. Then, in particular, for all $1 \leq i, j \leq n$ and $1 \leq l, k \leq 3$, we have

$$\begin{aligned} 1 - &\overline{\Psi(z_l, x_{1i}, x_{2i}, x_{3i})} \Psi(z_k, x_{1j}, x_{2j}, x_{3j}) = \\ &= (1 - \overline{z_l} z_k) \overline{\gamma(\lambda_i, z_l)} \gamma(\lambda_j, z_k) + (1 - \overline{\lambda_i} \lambda_j) \langle U(\lambda_i)^* U(\lambda_j) \eta(z_k, \lambda_j), \eta(z_l, \lambda_i) \rangle_{\mathbb{C}^2} \\ &= (1 - \overline{z_l} z_k) \overline{\gamma(\lambda_i, z_l)} \gamma(\lambda_j, z_k) + (1 - \overline{\lambda_i} \lambda_j) \langle U(\lambda_j) \eta(z_k, \lambda_j), U(\lambda_i) \eta(z_l, \lambda_i) \rangle_{\mathcal{H}_K} \end{aligned}$$

It follows that the positive $3n$-square matrices

$$N = \left[\overline{\gamma(z_l, \lambda_i)} \gamma(z_k, \lambda_j)\right]_{i,j=1,l,k=1}^{n,3} \quad \text{and } M = [\langle U(\lambda_j) \eta(z_k, \lambda_j), U(\lambda_i) \eta(z_l, \lambda_i) \rangle_{\mathcal{H}_K}]_{i,j=1,l,k=1}^{n,3}$$

satisfy the matrix inequality and the rank of N is less than or equal to 1. We now apply

Procedure SW_T to N and M. Choose $\gamma_{jk} = \gamma(\lambda_j, z_k)$ for $1 \le j \le n$ and $1 \le k \le 3$, $\mathcal{H} = \mathcal{H}_K$, and $v_{jk} = U(\lambda_j)\eta(\lambda_j, z_k)$ for $1 \le j \le n$ and $1 \le k \le 3$. As in the proof of Theorem 4.4.1, for $z, \lambda \in \mathbb{D}$, the Grammian of the vectors

$$\begin{pmatrix} \begin{pmatrix} 1 \\ z\gamma(\lambda, z) \end{pmatrix} \\ \lambda U(\lambda)\eta(\lambda, z) \end{pmatrix} \in \mathbb{C}^2 \oplus \mathcal{H}_K$$

is equal to the Grammian of the vectors

$$\begin{pmatrix} \begin{pmatrix} \Psi(z, x(\lambda)) \\ \gamma(\lambda, z) \end{pmatrix} \\ U(\lambda)\eta(\lambda, z) \end{pmatrix} \in \mathbb{C}^2 \oplus \mathcal{H}_K.$$

Hence, by Proposition B.1.22, there is a contraction L on $\mathbb{C}^2 \oplus \mathcal{H}_K$ such that

$$L \begin{pmatrix} \begin{pmatrix} 1 \\ z\gamma(\lambda, z) \end{pmatrix} \\ \lambda U(\lambda)\eta(\lambda, z) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \Psi(z, x(\lambda) \\ \gamma(\lambda, z) \end{pmatrix} \\ U(\lambda)\eta(\lambda, z) \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$. Choose $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = L$ in step (iii) of Procedure SW_T. Then we obtain a function $y \in \operatorname{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ such that $y(\lambda_j) = (x_{1j}, x_{2j}, x_{3j})$ for $j = 1, \ldots, n$.

If $y = x$ then we are done. We have shown that, for $z, \lambda \in \mathbb{D}$, we have

$$\begin{pmatrix} \begin{pmatrix} \Psi(z, x(\lambda) \\ \gamma(\lambda, z) \end{pmatrix} \\ U(\lambda)\eta(\lambda, z) \end{pmatrix} = L \begin{pmatrix} \begin{pmatrix} 1 \\ z\gamma(\lambda, z) \end{pmatrix} \\ \lambda U(\lambda)\eta(\lambda, z) \end{pmatrix} = \begin{pmatrix} A \begin{pmatrix} 1 \\ z\gamma(\lambda, z) \end{pmatrix} + B\lambda U(\lambda)\eta(\lambda, z) \\ C \begin{pmatrix} 1 \\ z\gamma(\lambda, z) \end{pmatrix} + D\lambda U(\lambda)\eta(\lambda, z) \end{pmatrix}.$$

Hence

$$\begin{pmatrix} \Psi(z, x(\lambda)) \\ \gamma(\lambda, z) \end{pmatrix} = A \begin{pmatrix} 1 \\ z\gamma(\lambda, z) \end{pmatrix} + B\lambda U(\lambda)\eta(\lambda, z) \text{ and } (1 - D\lambda)U(\lambda)\eta(\lambda, z) = C \begin{pmatrix} 1 \\ z\gamma(\lambda, z) \end{pmatrix},$$

and so

$$\begin{pmatrix} \Psi(z, x(\lambda)) \\ \gamma(\lambda, z) \end{pmatrix} = (A + B\lambda(I - D\lambda)^{-1}C) \begin{pmatrix} 1 \\ z\gamma(\lambda, z) \end{pmatrix} = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix} \begin{pmatrix} 1 \\ z\gamma(\lambda, z) \end{pmatrix}$$

for all $z, \lambda \in \mathbb{D}$. It follows that

$$\Psi(z, x(\lambda)) = a(\lambda) + b(\lambda)z\gamma(\lambda, z) \text{ and } \gamma(\lambda, z) = c(\lambda) + d(\lambda)z\gamma(\lambda, z),$$

and so, letting $\Xi = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\Psi(z, x(\lambda)) = a(\lambda) + \frac{b(\lambda)c(\lambda)z}{1 - d(\lambda)z} = \frac{\det \Xi(\lambda)z - a(\lambda)}{d(\lambda)z - 1}$$

for all $z, \lambda \in \mathbb{D}$. By Proposition 4.3.17 (i), $y = (a, d, \det \Xi) = (x_1, x_2, x_3) = x$. $\square$

The following proposition shows that, in order to use Theorem 4.4.1 to determine if there is an interpolating function, it is sufficient to search over a compact set for a pair (N, M) that satisfies the matrix inequality and such that the rank of N is 1.

Proposition 4.4.4. *Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$, and let $(x_{1j}, x_{2j}, x_{3j}) \in \overline{\mathbb{E}}$ be such that $x_{1j}x_{2j} \neq x_{3j}$ for $j = 1, \ldots, n$. Then the interpolation problem*

$$\lambda_j \in \mathbb{D} \mapsto (x_{1j}, x_{2j}, x_{3j}) \in \overline{\mathbb{E}}, \text{ for all } j = 1, \ldots, n,$$

is solvable if and only if, for some distinct points $z_1, z_2, z_3 \in \mathbb{D}$, there exist positive $3n$-square matrices $N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ of rank 1, and $M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3}$ that satisfy

$$\left[1 - \overline{\frac{z_l x_{3i} - x_{1i}}{x_{2i}z_l - 1}} \frac{z_k x_{3j} - x_{1j}}{x_{2j}z_k - 1}\right] \geq [(1 - \overline{z_l}z_k)N_{il,jk}] + \left[(1 - \overline{\lambda_i}\lambda_j)M_{il,jk}\right],$$

and

$$|N_{il,jk}| \leq \frac{1}{(1 - |x_{2i}|)(1 - |x_{2j}|)} \text{ and}$$

$$|M_{il,jk}| \leq \frac{2}{|1 - \overline{\lambda_i}\lambda_j|} \sqrt{1 + \frac{1}{(1 - |x_{2i}|)^2}} \sqrt{1 + \frac{1}{(1 - |x_{2j}|)^2}}$$

for all $1 \leq i, j \leq n$ and $1 \leq l, k \leq 3$.

Proof. Sufficiency follows from Theorem 4.4.1 (iv) $\Longrightarrow$ (i). For necessity, recall the proof of Theorem 4.4.1 (i) $\Longrightarrow$ (iii), from which it follows that the matrix inequality is satisfied for

$$N = [N_{il,jk}]_{i,j=1,l,k=1}^{n,3} = \left[\overline{\gamma(\lambda_i, z_l)}\gamma(\lambda_j, z_k)\right]_{i,j=1,l,k=1}^{n,3}$$

of rank 1, and

$$M = [M_{il,jk}]_{i,j=1,l,k=1}^{n,3} = \left[\eta(\lambda_i, z_l)^* \frac{I - F(\lambda_i)^* F(\lambda_j)}{1 - \overline{\lambda_i}\lambda_j} \eta(\lambda_j, z_k)\right]_{i,j=1,l,k=1}^{n,3},$$

where $||F(\lambda_j)|| \leq 1$, $\gamma(\lambda_j, z_k) = (1 - x_{2j}z_k)^{-1} f_2(\lambda_j)$,

$$\eta(\lambda_j, z_k) = \begin{bmatrix} 1 \\ \gamma(\lambda_j, z_k)z_k \end{bmatrix}$$

and $|f_2(\lambda_j)| \leq 1$ for all $j = 1, \ldots, n$. Hence

$$|\gamma(\lambda_j, z_k)| \leq \frac{1}{|1 - x_{2j} z_k|} \leq \frac{1}{1 - |x_{2j}|},$$

and so $|N_{il,jk}| \leq \dfrac{1}{(1-|x_{2i}|)(1-|x_{2j}|)}$ for all $j = 1, \ldots, n$ and $k = 1, 2, 3$. Moreover,

$$||\eta(\lambda_j, z_k)||^2_{\mathbb{C}^2} = \left|\left|\begin{bmatrix}\gamma(\lambda_j, z_k) z_k \\ 1\end{bmatrix}\right|\right|^2_{\mathbb{C}^2} = 1 + |\gamma(\lambda_j, z_k) z_k|^2 \leq 1 + \frac{1}{(1-|x_{2j}|)^2},$$

and so

$$\begin{aligned}|M_{il,jk}| &\leq ||\eta(\lambda_i, z_l)||_{\mathbb{C}^2} \frac{||I - F(\lambda_i)^* F(\lambda_j)||}{|1 - \overline{\lambda_i}\lambda_j|} ||\eta(\lambda_j, z_k)||_{\mathbb{C}^2} \\ &\leq \frac{2}{|1 - \overline{\lambda_i}\lambda_j|} \sqrt{1 + \frac{1}{(1-|x_{2i}|)^2}} \sqrt{1 + \frac{1}{(1-|x_{2j}|)^2}}\end{aligned}$$

for all $j = 1, \ldots, n$ and $k = 1, 2, 3$. Thus, if the given $\overline{\mathbb{E}}$-interpolation problem is solvable, then there exist positive $3n$-square matrices satisfying the required conditions. □

Chapter A. Examples from control engineering

In this chapter, we sketch the reduction of a robust stabilisation problem to a spectral Nevanlinna-Pick problem, and give an example to illustrate the connections with control engineering.

A.1 Reduction of a robust stabilisation problem to a spectral Nevanlinna-Pick problem

The content in this section is taken from [3, Section 2].

Figure 1 depicts a feedback system with uncertainty, where Δ, G and K are finite-dimensional linear time-invariant systems. We identify Δ, G and K with their transfer functions, which are real rational matrix-valued functions of the frequency domain. The *nominal plant* is the plant (model of a physical system) which the designer adopts as a representation of the system. For the system in Figure 1, we assume that the nominal plant

$$G = [G_{ij}]_{i,j=1}^{3}$$

is given and it is *proper*, that is, its entries are rational functions that have a finite limit at infinity. We model uncertainty with an uncertainty set $\mathbf{\Delta}$, and the assumption that the true plant is given by Figure 2 for some unknown $\Delta \in \mathbf{\Delta}$.

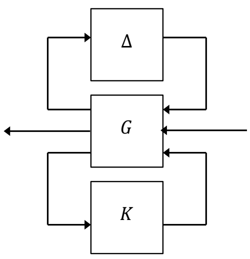


Figure 1. [3, p. 2475].

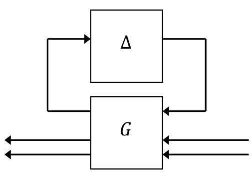


Figure 2. [3, p. 2475].

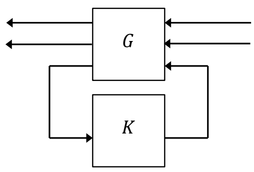


Figure 3. [3, p. 2476].

Let RH^∞ be the space of real rational matrix-valued functions that are bounded and analytic on $\mathbb{H} := \{s : \mathrm{Re}\,(s) > 0\}$. A system as in Figure 1 is *well posed* if the transfer functions between different branches of the interconnection are well defined, and it is *stable* if these transfer functions belong to RH^∞. A system as in Figure 3 is *internally* *stable* if the transfer function

$$\begin{bmatrix} I \\ G \end{bmatrix} (I + KG)^{-1} \begin{bmatrix} I & K \end{bmatrix}$$

belongs to RH^∞. If there exists a controller $K \in RH^\infty$ such that the lower loop of Figure 1 is well posed and internally stable, then we say G is *stabilisable.* If there exists a controller $K \in RH^\infty$ such that the system in Figure 1 is well posed and stable for all $\Delta \in \boldsymbol{\Delta}$, then we say G is *robustly stabilisable with respect to* $\boldsymbol{\Delta}$.

Suppose G is stabilisable. By [36, Lemma 5.4], K stabilises G if and only if K stabilises G_{33}. The set of all stabilising controllers of G_{33} can be parameterised by [36, Theorem 5.13 and Theorem 5.14]. First, we need to say how to define a structure from an uncertainty set. For a given property $\mathcal{P}$, we define the uncertainty set

$$\boldsymbol{\Delta} = \{\Delta \in \mathcal{M}_{n\times m}(\mathbb{C}) : ||\Delta|| \leq 1 \text{ and } \Delta \text{ satisfies property } \mathcal{P}\}.$$

As in [36, p. 256], we assume that $\mathcal{P}$ does not impose any norm restrictions. Moreover, we assume that if Δ satisfies $\mathcal{P}$ then so does $\alpha\Delta$ for every $\alpha > 0$. This means that the structure

$$\mathcal{E}_{\boldsymbol{\Delta}} = \{\Delta \in \mathcal{M}_{n\times m}(\mathbb{C}) : \Delta \text{ satisfies property } \mathcal{P}\}$$

is a cone. Recall that, for $M \in \mathcal{M}_{m\times n}(\mathbb{C})$, the structured singular value of M is

$$\mu_{\mathcal{E}_{\boldsymbol{\Delta}}}(M) = \frac{1}{\inf\{||\Delta|| : \Delta \in \mathcal{E}_{\boldsymbol{\Delta}} \text{ and } I - M\Delta \text{ is singular}\}}.$$

To illustrate, we give an example. Consider the uncertainty set

$$\boldsymbol{\Delta} = \{\delta I_n : |\delta| \leq 1\},$$

where I_n is the identity matrix in $\mathcal{M}_n(\mathbb{C})$. Then $\mathcal{E}_{\boldsymbol{\Delta}} = \{\delta I_n : \delta \in \mathbb{C}\}$, and, as shown in [36, p. 257], we have $\mu_{\mathcal{E}_{\boldsymbol{\Delta}}} = \rho$, the spectral radius.

Proposition A.1.1. [3, Proposition 2.2] *Let* $G = [G_{ij}]_{i,j=1}^3$ *be a stabilisable plant. Let* G_{33} *have the doubly coprime factorisation*

$$G_{33} = \hat{N}\hat{M}^{-1} = \tilde{M}^{-1}\tilde{N}$$

over RH^∞*, where* $\tilde{N}, \tilde{M}, \tilde{X}, \tilde{Y}, \hat{N}, \hat{M}, \hat{X}, \hat{Y} \in RH^\infty$ *and satisfy*

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} \hat{M} & \hat{Y} \\ \hat{N} & \hat{X} \end{bmatrix} = I$$

(by [36, Proposition 5.10]*, every proper real rational plant admits such a factorisation). Let the zero matrix belong to* $\boldsymbol{\Delta} \subset RH^\infty$*. Then there exists a controller* $K \in RH^\infty$ *such*

that the system in Figure 1 is internally stable for all $\Delta \in \mathbf{\Delta}$ *if and only if there exists a* $Q \in RH^\infty$ *such that* $\hat{X}(\infty) - \hat{N}(\infty)Q(\infty)$ *is nonsingular and*

$$\sup_{s \in \mathbb{H}} \mu_{\mathcal{E}_{\mathbf{\Delta}}}((T_1 - T_2QT_3)(s)) < 1,$$

where $T_1 = G_{11} + G_{13}\hat{Y}\tilde{M}G_{31}$*,* $T_2 = G_{13}\hat{M}$ *and* $T_3 = \tilde{M}G_{31}$*. Moreover, the general robustly stabilising controller of the system in Figure 1 for the uncertainty set* $\mathbf{\Delta}$ *is given by*

$$K = (\hat{Y} - \hat{M}Q)(\hat{X} - \hat{N}Q)^{-1} = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M})$$

for some $Q \in RH^\infty$ *such that* $\hat{X}(\infty) - \hat{N}(\infty)Q(\infty)$ *is nonsingular and*

$$\sup_{s \in \mathbb{H}} \mu_{\mathcal{E}_{\mathbf{\Delta}}}((T_1 - T_2QT_3)(s)) < 1.$$

Example A.1.2. [3, p. 2478] Using Proposition A.1.1, we reduce the robust stabilisation problem for the nominal plant G with the uncertainty set

$$\mathbf{\Delta} = \{\delta I_n : |\delta| \leq 1\}$$

to: *Find* $Q \in RH^\infty$ *such that* $\hat{X}(\infty) - \hat{N}(\infty)Q(\infty)$ *is nonsingular and*

$$\sup_{s \in \mathbb{H}} \rho((T_1 - T_2QT_3)(s)) < 1.$$

Suppose T_2 and T_3 are scalar matrix functions, and let $s_1, \ldots, s_n$ be the zeros of T_2T_3 in $\mathbb{H}$. If $s_1, \ldots, s_n$ are simple then

$$\{T_1 - T_2QT_3 : Q \in RH^\infty\} = \{F \in RH^\infty : F(s_j) = T_1(s_j) \text{ for } j = 1, \ldots, n\}.$$

In this case, the problem is: *Find* $F \in RH^\infty$ *such that* $F(s_j) = T_1(s_j)$ *for* $j = 1, \ldots, n$*, and*

$$\sup_{s \in \mathbb{H}} \rho(F(s)) < 1.$$

Now, by application of a Cayley transform, this becomes an instance of the spectral Nevanlinna-Pick problem; as shown in [3, Section 3], when G_{11} and G_{33} are 2×2 matrix functions, the theory of the symmetrised bidisc can be used to analyse this problem. In [3, Section 4], there is a worked numerical example in which it is shown that there exists a robustly stabilising controller for a certain plant with the uncertainty set

$$\mathbf{\Delta} = \{\delta I_n : |\delta| \leq 1\}$$

if and only if a certain parameter c satisfies $|c| < \dfrac{1}{4 - 2\sqrt{3}}$.

A.2 Example from robust control

The content in this section is taken from [36, Section 0.2].

In this section, we consider a concrete example to illustrate the effect of feedback when modelling a physical system with uncertainty. Feedback can be used to achieve system stability. A physical system is unlikely to be at an equilibrium point, however, a stable system is insensitive to uncertainty about its initial conditions. This means that the state trajectory of a stable system does not diverge when the initial state is slightly perturbed from an equilibrium point. An exponentially stable system returns to the equilibrium point at a fast rate after it has been slightly perturbed. To stabilise a system at an equilibrium point we use the control input to make the equilibrium point exponentially stable; this can be achieved by a state feedback control law (see [36, p. 10]).

Another reason we may apply feedback to a system is to improve aspects of the dynamic behaviour. There may be environmental factors which affect the behaviour of the system, or external commands which act on the system. The problem is that these influences may be unknown when we design the control system. Thus we would like the system to be insensitive to these influences, and again we can appeal to feedback control to achieve this.

When we apply feedback to a system, there is an important issue to consider. We may not have access to a complete description of the physical system, or a complete description may be more complicated than we would like. Thus, when modelling the physical system it can be useful to make approximations or simplifications of certain aspects of the system. We must now ask what effect this has when feedback is applied.

Example A.2.1 (Position control of an electric motor)**.** [36, pp. 13-14]. Figure 4 depicts an electric motor which receives unknown inputs.

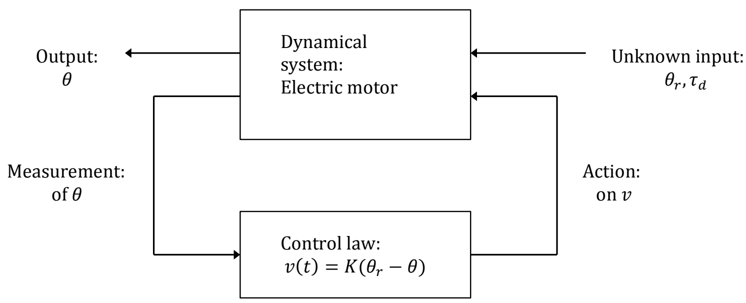


Figure 4. [36, p. 12].

A voltage v is applied to the motor windings, which results in a torque τ applied to the motor shaft. We characterise the behaviour of the system by its output θ, the angular position of the shaft. We employ a feedback control law, which receives a measurement of θ and acts on v, to ensure θ follows a reference command θ_r despite the effect of an unknown resisting torque τ_d. The control system acts according to the law

$$v(t) = K(\theta_r - \theta),$$

where $K > 0$ is a constant to be designed. As shown in [36, pp. 13-14], the dynamics of the system are given by

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -K & -1\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0 & 0 \\ K & -1\end{bmatrix}\begin{bmatrix}\theta_r \\ \tau_d\end{bmatrix},$$

and the system is exponentially stable in the absence of external inputs. Now it is important to note that, although the inputs θ_r and τ_d are unknown when K is designed, we do need some information about them. For example, we may say that the disturbances should lie in a prescribed set $\mathcal{D}$, in which case, we would consider the worst possible behaviour the system could have for any element in $\mathcal{D}$. As in [36, pp. 14], we suppose θ_r and τ_d are constant over time. In this case, the states $(\theta, \dot{\theta})$ converge asymptotically to

$$\theta(\infty) = \theta_r - \frac{\tau_d}{K} \text{ and } \dot{\theta}(\infty) = 0.$$

It follows that θ will track θ_r despite the effect of τ_d whenever K is sufficiently large.

We now return to the issue of approximations and simplifications made in modelling the system. In Example A.2.1, the effect of inductance in the electric circuit was considered negligible and so it was neglected in the model (see [36, p. 13]). However, if inductance is accounted for in the model, it can be shown that the system becomes unstable for sufficiently large K. This means that feedback has caused the modelling error we considered negligible to make the system unusable.

It is clear that feedback can make a system both insensitive to uncertainty and more sensitive to uncertainty. This tradeoff is a fundamental issue in feedback design. Of course, we could have worked with a model that did account for inductance, but even then we would be neglecting other aspects of the system, for example, bending dynamics of the motor shaft. There are always neglected effects, and how reliable our analysis is depends on whether these effects can truly be neglected.

Detailed models can be infinite dimensional and more than what a computer can accurately simulate. This means that stabilising the system can be very difficult, if not impossible. However, it may be that it is enough to stabilise a low dimensional model of the system. That is, a feedback design which makes the low dimensional model insensitive to uncertainty may also make the real system insensitive to uncertainty. In this way, there is no correct model of a system. Instead, we seek a model that can compensate for any remaining uncertainty by means of feedback control.

Chapter B. Background material

B.1 General background

In this section, we give some definitions and results that we use. Let $\mathcal{A}$ be a unital C^* algebra with unit $1_{\mathcal{A}}$ and let $a \in \mathcal{A}$. Then we denote by $\sigma(a)$ the *spectrum* of a, that is, the set

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a \text{ is not invertible}\}.$$

We denote by $\rho(a)$ the *spectral radius* of a, that is,

$$\rho(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Definition B.1.1. [46, p. 244] *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$. Then we say that a is* positive semidefinite *and write $a \geq 0$ if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$.*

Proposition B.1.2. [46, Theorem 4.2.2 (iii)] *Let $\mathcal{A}$ be a unital C^*-algebra. Let $a, b \in \mathcal{A}$ be such that $a, b \geq 0$. Then $a + b \geq 0$.*

The following theorem gives an alternative description of a positive element of a C^*-algebra. We primarily use two corollaries of this theorem.

Theorem B.1.3. [46, Theorem 4.2.6] *Let $\mathcal{A}$ be a unital C^*-algebra and let $a \in \mathcal{A}$. Then $a \geq 0$ if and only if $a = b^*b$ for some $b \in \mathcal{A}$.*

Remark B.1.4. As a consequence of B.1.3, $a^*a \geq 0$ for all a in a unital C^*-algebra.

Corollary B.1.5. [46, Corollary 4.2.7] *Let $\mathcal{A}$ be a unital C^*-algebra. Let $a \in \mathcal{A}$ be such that $a \geq 0$. Then $b^*ab \geq 0$ for all $b \in \mathcal{A}$.*

Corollary B.1.6. *Let $\mathcal{A}$ be a C^*-algebra with identity $1_{\mathcal{A}}$ and let $a \in \mathcal{A}$. Then*

$$1_{\mathcal{A}} - a^*a \geq 0 \text{ if and only if } ||a|| \leq 1.$$

Proof. By Remark B.1.4, we have $\sigma(a^*a) \subseteq [0, \infty)$. Hence

$$1_{\mathcal{A}} - a^*a \geq 0 \iff 1 - \sigma(a^*a) \subseteq [0, \infty) \iff \sigma(a^*a) \subseteq [-1, 1] \iff ||a||^2 \leq 1,$$

since $||a||^2 = \rho(a^*a)$ for elements in a C^*-algebra. □

The following theorem provides a useful characterisation of continuous linear functionals on a Hilbert space.

Theorem B.1.7 (Riesz-Fréchet Theorem)**.** [62, Theorem 6.8] *Let H be a Hilbert space. Let f be a continuous linear functional on H. Then there exists a unique $y \in H$ such that $f(x) = \langle x, y\rangle$ for all $x \in H$. Moreover, $||y|| = ||f||$.*

Let H be a Hilbert space. We define $\mathcal{B}(H)$ to be the C^*-algebra of all bounded linear operators $T : H \to H$ with norm given by

$$||T|| = \sup\{||Tx||_H : ||x||_H \leq 1\}$$

and involution given by $* : T \mapsto T^*$, where the bounded linear operator $T^* : H \to H$ is called the *adjoint* of T and defined by

$$\langle T^*x, y\rangle = \langle x, Ty\rangle \text{ for all } x, y \in H.$$

For any $T \in \mathcal{B}(H)$,

$$T \geq 0 \text{ if and only if } \langle Tx, x\rangle \geq 0 \text{ for all } x \in H.$$

We define the identity operator I in $\mathcal{B}(H)$ by $Ix = x$ for all $x \in H$.

Remark B.1.8. Let H and G be Hilbert spaces. Then we can similarly define $\mathcal{B}(H, G)$ to be the Banach space of all bounded linear operators $T : H \to G$ with norm

$$||T|| = \sup\{||Tx||_G : ||x||_H \leq 1\}$$

and involution $* : T \mapsto T^*$, where the bounded linear operator $T^* : G \to H$ is the adjoint of T and defined by

$$\langle T^*x, y\rangle_H = \langle x, Ty\rangle_G \text{ for all } x \in G \text{ and } y \in H.$$

Although $\mathcal{B}(H, G)$ is not a C^*-algebra, we obtain analogous results to Remark B.1.4, Corollary B.1.5 and Corollary B.1.6. Indeed, let $T \in \mathcal{B}(H, G)$. Then

$$\langle T^*Th, h\rangle_H = \langle Th, Th\rangle_G = ||Th||_G^2 \geq 0 \text{ for all } h \in H,$$

and so $T^*T \geq 0$. Moreover,

$$||T|| \leq 1 \iff \langle (I - T^*T)h, h\rangle = ||h|| - ||Th|| \geq 0 \text{ for all } h \in H \quad \iff I - T^*T \geq 0.$$

Lastly, suppose $H = G$ and $T \geq 0$. Then, if $B \in \mathcal{B}(U, H)$ for some Hilbert space U,

$$\langle B^*TBu, u\rangle_U = \langle TBu, Bu\rangle_H \geq 0 \text{ for all } u \in U,$$

and so $B^*TB \geq 0$.

The Banach algebra $\mathcal{M}_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$ is a finite dimensional C^*-algebra with the involution that takes each element to its conjugate transpose. The *conjugate transpose* of a matrix $M = [m_{ij}]_{i=1,j=1}^{n,m} \in \mathcal{M}_{n\times m}(\mathbb{C})$ is the matrix

$$M^* = ([\overline{m_{ij}}]_{i=1,j=1}^{n,m})^T = [\overline{m_{ji}}]_{i=1,j=1}^{m,n} \in \mathcal{M}_{m\times n}(\mathbb{C}).$$

Let $\mathcal{V}$ be a complex vector space. Recall that a *semi-inner product* $\langle\cdot,\cdot\rangle$ on $\mathcal{V}$ relaxes the definition of an inner product to allow $\langle v, v\rangle = 0$ for $0 \neq v \in \mathcal{V}$. In the next proposition, we note that a semi-inner product still satisfies the Cauchy-Schwarz inequality.

Proposition B.1.9. [46, Proposition 2.1.1 (i)] *Let $\mathcal{V}$ be a complex vector space. Let $\langle\cdot,\cdot\rangle$ be a semi-inner product on $\mathcal{V}$. Then $\langle\cdot,\cdot\rangle$ satisfies*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

for all $x, y \in \mathcal{V}$.

We now consider the class of spaces known as Hardy spaces. The Hilbert space structure of the Hardy space H^2_d is particularly useful to us.

Definition B.1.10. *Let Ω be an open set in $\mathbb{C}$ and X a Banach space. Then we say a map $f : \Omega \to X$ is* holomorphic *if for every $z_0 \in \Omega$ there exists an $f'(z_0) \in X$ such that*

$$\lim_{z\to z_0} \left|\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right|\right|_X = 0.$$

Definition B.1.11. [57, Definition 1.2.1, Definition 1.4.1] *Let $1 \leq p \leq \infty$. We define the* Hardy space H^p_d *to be the set of holomorphic functions $f : \mathbb{D} \to \mathbb{C}^d$ for which*

$$||f||_{p,d} := \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} ||f(re^{i\theta})||^p_{\mathbb{C}^d} d\theta\right)^{\frac{1}{p}} < \infty, \text{ when } 1 \leq p < \infty,$$

and

$$||f||_{p,d} := \sup_{|z|<1} ||f(z)||_{\mathbb{C}^d} < \infty, \text{ when } p = \infty.$$

We denote by H^p the Hardy space H^p_1.

For $1 \leq p < q \leq \infty$, it follows from Hölder's inequality that $H^q_d \subseteq H^p_d$. We note that H^p_d is a Banach space for $1 \leq p \leq \infty$, and H^2_d is a Hilbert space with inner product given by

$$\langle f, g\rangle_{H^2_d} = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} \langle f(re^{i\theta}), g(re^{i\theta})\rangle_{\mathbb{C}^d} d\theta$$

for all $f, g \in H^2_d$.

Remark B.1.12. [57, p. 13] Let $f : \mathbb{D} \to \mathbb{C}^d$ and $1 \leq p \leq \infty$. Then it is easy to check that $f \in H^p_d$ if and only if $f_j \in H^p$ for all $1 \leq j \leq d$, where $f_j : \mathbb{D} \to \mathbb{C}$ maps $z \in \mathbb{D}$ to the j-th entry of $f(z)$ and

$$[f_j]_{j=1}^d(z) := [f_j(z)]_{j=1}^d = f(z)$$

for all $z \in \mathbb{D}$.

We can say more about the connection between H^2_d and H^2. Define the *Hilbert direct sum* of Hilbert spaces $H_1, \ldots, H_n$ to be the Hilbert space $H_1 \oplus \cdots \oplus H_n$ given by the vector space direct sum with inner product

$$\langle (h_1, \cdots, h_n), (f_1, \cdots, f_n) \rangle = \langle h_1, f_1 \rangle_{H_1} + \cdots + \langle h_n, f_n \rangle_{H_n}$$

for all $h_1, f_1 \in H_1, \ldots, h_n, f_n \in H_n$.

Proposition B.1.13. [57, p. 45] *There is a unitary between H^2_d and the Hilbert direct sum of d copies of H^2.*

Proof. Define $U : H^2_d \to \bigoplus_{i=1}^d H^2$ by $Uf = (f_1, \ldots, f_d)$ for all $f = [f_j]_{j=1}^d \in H^2_d$. Clearly U is linear and injective. By Remark B.1.12, $f \in H^2_d$ if and only if $(f_1, \ldots, f_d) \in \bigoplus_{i=1}^d H^2$. Hence U is surjective. Let $f = [f_j]_{j=1}^d, g = [g_j]_{j=1}^d \in H^2_d$. Then

$$\begin{aligned}\langle f, g \rangle_{H^2_d} &= \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \langle f(re^{i\theta}), g(re^{i\theta}) \rangle_{\mathbb{C}^d} d\theta = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^d \langle f_j(re^{i\theta}), g_j(re^{i\theta}) \rangle_{\mathbb{C}} d\theta \\ &= \sum_{j=1}^d \langle f_j, g_j \rangle_{H^2} = \langle (f_1, \ldots, f_d), (g_1, \ldots, g_d) \rangle_{\bigoplus_{i=1}^d H^2} = \langle Uf, Ug \rangle_{\bigoplus_{i=1}^d H^2}.\end{aligned}$$

It follows that U is a unitary between H^2_d and $\bigoplus_{i=1}^d H^2$. □

We give another characterisation of H^2_d. Define the *Hilbert tensor product* of Hilbert spaces $H_1, \ldots, H_n$ as the Hilbert space $H_1 \otimes_H \cdots \otimes_H H_n$ given by the completion of the algebraic tensor product with respect to the inner product

$$\langle h_1 \otimes \cdots \otimes h_n, f_1 \otimes \cdots \otimes f_n \rangle = \langle h_1, f_1 \rangle_{H_1} \cdots \langle h_n, f_n \rangle_{H_n}$$

for all $h_1, f_1 \in H_1, \ldots, h_n, f_n \in H_n$.

Remark B.1.14. [46, Remark 2.6.8] Let H and K be Hilbert spaces and let the dimension of K be n. If $\{e_i\}_{i=1}^n$ is an orthonormal basis for K then

$$h_1 \oplus \cdots \oplus h_n \mapsto \sum_{i=1}^n h_i \otimes_H e_i, \text{ where } h_i \in H \text{ for } 1 \leq i \leq n,$$

is a unitary between $\bigoplus_{i=1}^n H$ and $H \otimes_H K$. Consequently, elements of $H \otimes_H K$ can be written (uniquely, if $\{e_i\}$ is specified) as

$$\sum_{i=1}^n h_i \otimes_H e_i, \text{ where } h_i \in H \text{ for } 1 \leq i \leq n,$$

and so are finite sums of the *simple tensors* $h \otimes_H k$, where $h \in H$ and $k \in K$.

Corollary B.1.15. *There is a unitary between H_d^2 and $H^2 \otimes_H \mathbb{C}^d$.*

Proof. By Proposition B.1.13, there is a unitary between H_d^2 and $\bigoplus_{i=1}^d H^2$. By Remark B.1.14, there is a unitary between $\bigoplus_{i=1}^d H^2$ and $H^2 \otimes_H \mathbb{C}^d$. Hence there is a unitary between H_d^2 and $H^2 \otimes_H \mathbb{C}^d$. More concretely, define $U : H_d^2 \to H^2 \otimes_H \mathbb{C}^d$ by

$$U(f) = \sum_{i=1}^d f_i \otimes_H e_i$$

for all $f = [f_i]_{i=1}^d \in H_d^2$, where $\{e_i\}_{i=1}^d$ is an orthonormal basis for $\mathbb{C}^d$. Then U is a unitary between H_d^2 and $H^2 \otimes_H \mathbb{C}^d$. □

Remark B.1.16. We identify elements of H_d^2 as elements of $H^2 \otimes_H \mathbb{C}^d$ via the unitary in Corollary B.1.15. More specifically, we consider $[f_i]_{i=1}^d \in H_d^2$ to be $\sum_{i=1}^d f_i \otimes_H e_i$, where the $e_i = [e_{ij}]_{j=1}^d$ satisfy $e_{ii} = 1$ and $e_{ij} = 0$ when $j \neq i$. This representation is unique, by Remark B.1.14, since we have specified the basis.

It is possible to factorise functions in H^p. First, we define the functions that are the factors.

Definition B.1.17. [44, p. 62] *An* inner function *is a holomorphic function $h : \mathbb{D} \to \mathbb{D}$ such that*

$$\lim_{r \to 1^-} h(r\lambda) \in \mathbb{T}$$

for almost every $\lambda \in \mathbb{T}$. A non-constant inner function without zeros which is positive at the origin is called a singular *inner function.*

Definition B.1.18. [44, p. 62] *An* outer function *is a holomorphic function g on $\mathbb{D}$ that has the form*

$$g(z) = c \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} k(\theta) d\theta\right), \text{ for } z \in \mathbb{D},$$

where $c \in \mathbb{T}$ and k is a real-valued integrable function on $\mathbb{T}$.

We call an automorphism of $\mathbb{D}$ a *Möbius transformation*. A Möbius transformation has the form

$$f(z) = e^{i\theta} \frac{a - z}{1 - \bar{a}z}$$

for some $a \in \mathbb{D}$ and $\theta \in [0, 2\pi)$.

Definition B.1.19. [6, p. 2] *A* Blaschke factor *is a Möbius transformation that is positive at zero (or has positive derivative at zero if it vanishes there):*

$$B_0(z) := z \text{ or } B_a(z) := \frac{\overline{a}}{|a|}\frac{a-z}{1-\overline{a}z}, \text{ where } a \in \mathbb{D}.$$

A finite Blaschke product *is a finite product of Blaschke factors times a unimodular constant:*

$$B(z) = e^{i\theta} z^M \prod_{j=1}^{N} B_{a_j}(z).$$

An infinite Blaschke product *is a non-zero infinite product of Blaschke factors times a unimodular constant:*

$$B(z) = e^{i\theta} z^M \prod_{j=1}^{\infty} B_{a_j}(z), \text{ where } \sum_{j=1}^{\infty}(1-|a_j|) < \infty.$$

The requirement that Blaschke factors are positive at zero guarantees that an infinite Blaschke product converges to a holomorphic function on $\mathbb{D}$*.*

Remark B.1.20. [57, Example 1.3.2] Blaschke factors and products are examples of inner functions.

Theorem B.1.21 (Factorisation Theorem)**.** [44, p. 67, p. 69] *Let* $1 \leq p \leq \infty$*. Let* f *be a non-zero function in* H^p*. Then* f *is uniquely expressible in the form* $f = Bsg$*, where* B *is a Blaschke product,* s *is a singular inner function and* g *is an outer function in* H^p*.*

We give the following useful construction of an isometry between two collections, of elements in Hilbert space, which have the same Grammian. Let $\{x_i\}_{i\in I}$ be a collection of elements in an inner product space. Then the *Grammian* of $\{x_i\}_{i\in I}$ is the matrix $G = [G_{ij}]_{i,j\in I}$ defined by

$$G_{ij} = \langle x_j, x_i \rangle$$

for all $i, j \in I$.

Proposition B.1.22. *Let* H *and* K *be Hilbert spaces and let* I *be a set. Let* $\{x_i\}_{i\in I}$ *be a collection of elements in* H *and* $\{y_i\}_{i\in I}$ *be a collection of elements in* K*. If* $\{x_i\}_{i\in I}$ *and* $\{y_i\}_{i\in I}$ *have the same Grammian, then there exists an isometry* $L : \overline{\text{span}}\{x_i : i \in I\} \to K$ *such that* $Lx_i = y_i$ *for all* $i \in I$*.*

Proof. Define a map $L_0 : \text{span}\{x_i : i \in I\} \to K$ by setting

$$L_0\left(\sum_{i\in I}\lambda_i x_i\right) = \sum_{i\in I}\lambda_i y_i,$$

where only finitely many λ_i are non-zero and $\lambda_i \in \mathbb{C}$ for all $i \in I$. Clearly L_0 is linear and $L_0 x_i = y_i$ for all $i \in I$. Let $n \in \mathbb{N}$. Then, for any $\lambda_{i_1}, \ldots, \lambda_{i_n} \in \mathbb{C}$, since $\{x_i\}$ and $\{y_i\}$

have the same Grammian,

$$\left\|\sum_{k=1}^{n}\lambda_{i_k}x_{i_k}\right\|^2=\sum_{k,l=1}^{n}\lambda_{i_k}\overline{\lambda_{i_l}}\langle x_{i_k},x_{i_l}\rangle=\sum_{k,l=1}^{n}\lambda_{i_k}\overline{\lambda_{i_l}}\langle y_{i_k},y_{i_l}\rangle=\left\|\sum_{k=1}^{n}\lambda_{i_k}y_{i_k}\right\|^2.$$

Hence L_0 is a well defined isometry. It follows that L_0 extends to an isometry L on $\overline{\text{span}}\{x_i : i \in I\}$ which satisfies the required conditions. □

We could extend the isometry L from Proposition B.1.22 to H, but the extension may not be an isometry. For example, define L to be 0 on $(\overline{\text{span}}\{x_i : i \in I\})^{\perp}$. We require the following notions of convexity.

Definition B.1.23. [61, p. 1] *Let Y be a compact set in $\mathbb{C}^n$. Then the* polynomial convex hull $\widehat{Y}$ *of Y is defined by*

$$\widehat{Y}=\{z\in\mathbb{C}^n : |P(z)|\le\sup_{w\in Y}|P(w)| \text{ for all polynomials } P \text{ on } \mathbb{C}^n\}.$$

We say that Y is polynomially convex *if $\widehat{Y}=Y$.*

Definition B.1.24. [61, p. 2] *Let X be a set in $\mathbb{C}^n$. Then X is called* hypoconvex *if its complement is the union of complex affine hyperplanes, that is, complex $(n-1)$-dimensional affine planes.*

We note that other authors use different terminology, for example, in [52] and [64], *linearly convex* is used instead of hypoconvex. We use the following result from complex analysis.

Theorem B.1.25. [13, p. 127] *Let Ω be a domain in $\mathbb{C}$. Let f be a non-zero holomorphic function on Ω. Then the zeros of f are isolated.*

The following results are from the theory of several complex variables. The statement of the corollary is given so that it is immediately applicable.

Theorem B.1.26. [41, Theorem 1] *Let Ω be a domain in $\mathbb{C}^n$. Let f be a holomorphic function on Ω. If $f=0$ on a non-empty open subset $U\subseteq\Omega$, then $f=0$ on Ω.*

Proposition B.1.27. [41, Lemma 24] *Let Ω be a domain in $\mathbb{C}^n$. Let f be a non-zero holomorphic function on Ω. Then the zero set of f is a closed nowhere dense set in Ω.*

Corollary B.1.28. *Let $f:\mathbb{D}^2\to\mathbb{C}$ be a non-zero holomorphic function. Let $z,\lambda\in\mathbb{D}$ be such that $zf(z,\lambda)=0$. Then there is a sequence $(z_i,\lambda_i)_{i=1}^{\infty}$ in $\mathbb{D}^2$ such that*

$$\lim_{i\to\infty}(z_i,\lambda_i)=(z,\lambda)$$

and $z_i f(z_i,\lambda_i)\neq 0$ for each $i\in\mathbb{N}$.

Proof. Let $g(w,\mu) = wf(w,\mu)$ for all $w, \mu \in \mathbb{D}$. Then g is the product of two holomorphic functions, and hence holomorphic. Suppose $g = 0$ on $\mathbb{D}^2$. Then

$$f(w,\mu) = 0$$

for all $0 \neq w \in \mathbb{D}$ and all $\mu \in \mathbb{D}$. That is, $f = 0$ on the non-empty open subset $(\mathbb{D}\setminus\{0\})\times\mathbb{D}$ of $\mathbb{D}^2$. Hence, by Theorem B.1.26, $f = 0$ on $\mathbb{D}^2$, which is a contradiction. It follows that g is a non-zero holomorphic function on $\mathbb{D}^2$. Let

$$\mathcal{Z} = \{(w,\mu) \in \mathbb{D}^2 : g(w,\mu) = 0\}.$$

Then $(z,\lambda) \in \mathcal{Z}$ and, by Proposition B.1.27, $\mathbb{D}^2 \setminus \mathcal{Z}$ is dense in $\mathbb{D}^2$. Let $(w,\mu) \in \mathcal{Z}$. It follows that there is a sequence $(w_i,\mu_i)_{i=1}^{\infty}$ in $\mathbb{D}^2$ such that

$$\lim_{i\to\infty}(w_i,\mu_i) = (w,\mu)$$

and $g(w_i,\mu_i) \neq 0$ for each $i \in \mathbb{N}$, as required. □

The following two results are used frequently, and so we give them here for convenience.

Proposition B.1.29. *Let* $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$. *Then*

$$I - M^*M = \begin{bmatrix} 1 - |m_{11}|^2 - |m_{21}|^2 & -\overline{m_{11}}m_{12} - \overline{m_{21}}m_{22} \\ -\overline{m_{12}}m_{11} - \overline{m_{22}}m_{21} & 1 - |m_{12}|^2 - |m_{22}|^2 \end{bmatrix}$$

and $\det(I - M^*M) = 1 - |m_{11}|^2 - |m_{21}|^2 - |m_{12}|^2 - |m_{22}|^2 + |\det M|^2$.

Proof. The first equality is easy to check, and using it we obtain the second. □

Proposition B.1.30. *Let X be a set and let $f, g : X \to \mathbb{C}$ be functions. Then*

$$\overline{f(y)}f(x) = \overline{g(y)}g(x) \text{ for all } x, y \in X$$

if and only if $f = \zeta g$ for some $\zeta \in \mathbb{T}$.

Proof. Sufficiency is clear. For necessity, suppose

$$\overline{f(y)}f(x) = \overline{g(y)}g(x) \text{ for all } x, y \in X.$$

Then $|g(y)|^2 = |f(y)|^2$ for all $y \in X$. If $f(y) = 0$ for all $y \in X$, then $f = g = 0$. Otherwise, there is a $y \in X$ such that $f(y) \neq 0$, and so $f = \zeta g$, where

$$\zeta = \frac{\overline{g(y)}}{\overline{f(y)}}$$

and $\zeta \in \mathbb{T}$, since $|g(y)|^2 = |f(y)|^2$. □

B.2 A realisation formula

The results in this section are used to prove the realisation formula in Section 2.1. Since these results hold in a more general context than is required in that section, we give them here. Some of the results are needed elsewhere too.

Let H, G, U and V be Hilbert spaces. Let P be an operator such that

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} : H \oplus U \to G \oplus V$$

and let $X : V \to U$ be an operator for which $I - P_{22}X$ is invertible in $\mathcal{B}(V)$. Then we denote by $\mathcal{F}_P(X)$ the linear fractional transformation

$$\mathcal{F}_P(X) := P_{11} + P_{12}X(I - P_{22}X)^{-1}P_{21} : H \to G.$$

Proposition B.2.1. [8, Lemma 1.7] *Let H, G, U and V be Hilbert spaces. Let*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \text{ and } Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

be operators from $H \oplus U$ to $G \oplus V$, and let X and Y be operators from V to U for which $I - P_{22}X$ and $I - Q_{22}Y$ are invertible in $\mathcal{B}(V)$. Then

$$\begin{aligned} I_H - \mathcal{F}_Q(Y)^*\mathcal{F}_P(X) = \\ &= Q_{21}^*(I_V - Y^*Q_{22}^*)^{-1}(I_V - Y^*X)(I_V - P_{22}X)^{-1}P_{21} \\ &\quad + \begin{bmatrix} I_H & Q_{21}^*(I_V - Y^*Q_{22}^*)^{-1}Y^* \end{bmatrix} (I_{H\oplus U} - Q^*P) \begin{bmatrix} I_H \\ X(I_V - P_{22}X)^{-1}P_{21} \end{bmatrix}. \end{aligned}$$

Proof. We prove the equality by expanding both sides. Let $S_1 = Q_{21}^*(I_V - Y^*Q_{22}^*)^{-1} \in \mathcal{B}(V, H)$ and $S_2 = (I_V - P_{22}X)^{-1}P_{21} \in \mathcal{B}(H, V)$. On the left side we have

$$\begin{aligned} I_H - \mathcal{F}_Q(Y)^*\mathcal{F}_P(X) = \\ &= I_H - (Q_{11}^* + Q_{21}^*(I_V - Y^*Q_{22}^*)^{-1}Y^*Q_{12}^*)(P_{11} + P_{12}X(I_V - P_{22}X)^{-1}P_{21}) \\ &= I_H - Q_{11}^*P_{11} - Q_{11}^*P_{12}XS_2 - S_1Y^*Q_{12}^*P_{11} - S_1Y^*Q_{12}^*P_{12}XS_2. \end{aligned}$$

Let $R_1 = S_1(I_V - Y^*X)S_2 \in \mathcal{B}(H)$ and

$$R_2 = \begin{bmatrix} I_H & S_1Y^* \end{bmatrix} (I_{H\oplus U} - Q^*P) \begin{bmatrix} I_H \\ XS_2 \end{bmatrix} \in \mathcal{B}(H).$$

Then the right side equals $R_1 + R_2$. Moreover, $R_1 = S_1S_2 - S_1Y^*XS_2$ and

$$\begin{aligned} R_2 &= \begin{bmatrix} I_H & S_1Y^* \end{bmatrix} \begin{bmatrix} I_H - Q_{11}^*P_{11} - Q_{21}^*P_{21} & -Q_{11}^*P_{12} - Q_{21}^*P_{22} \\ -Q_{12}^*P_{11} - Q_{22}^*P_{21} & I_U - Q_{12}^*P_{12} - Q_{22}^*P_{22} \end{bmatrix} \begin{bmatrix} I_H \\ XS_2 \end{bmatrix} \\ &= \begin{bmatrix} I_H - Q_{11}^*P_{11} - Q_{21}^*P_{21} + S_1Y^*(-Q_{12}^*P_{11} - Q_{22}^*P_{21}) \\ -Q_{11}^*P_{12} - Q_{21}^*P_{22} + S_1Y^*(I_U - Q_{12}^*P_{12} - Q_{22}^*P_{22}) \end{bmatrix}^T \begin{bmatrix} I_H \\ XS_2 \end{bmatrix} \\ &= I_H - Q_{11}^*P_{11} - Q_{21}^*P_{21} + S_1Y^*(-Q_{12}^*P_{11} - Q_{22}^*P_{21}) \\ &\quad - Q_{11}^*P_{12}XS_2 - Q_{21}^*P_{22}XS_2 + S_1Y^*(I_U - Q_{12}^*P_{12} - Q_{22}^*P_{22})XS_2 \\ &= I_H - Q_{11}^*P_{11} - Q_{21}^*P_{21} - S_1Y^*Q_{12}^*P_{11} - S_1Y^*Q_{22}^*P_{21} \\ &\quad - Q_{11}^*P_{12}XS_2 - Q_{21}^*P_{22}XS_2 + S_1Y^*XS_2 - S_1Y^*Q_{12}^*P_{12}XS_2 - S_1Y^*Q_{22}^*P_{22}XS_2. \end{aligned}$$

We note that all of the terms in the expansion of $I_H - \mathcal{F}_Q(Y)^*\mathcal{F}_P(X)$ appear in the expansion of R_2, and so we examine the remaining terms of $R_1 + R_2$. That is,

$$\begin{aligned} R_1 + R_2 - (I_H - \mathcal{F}_Q(Y)^*\mathcal{F}_P(X)) &= \\ &= (S_1S_2 - S_1Y^*XS_2) - Q_{21}^*P_{21} - S_1Y^*Q_{22}^*P_{21} \\ &\quad - Q_{21}^*P_{22}XS_2 + S_1Y^*XS_2 - S_1Y^*Q_{22}^*P_{22}XS_2 \\ &= S_1S_2 - Q_{21}^*P_{21} - S_1Y^*Q_{22}^*P_{21} - Q_{21}^*P_{22}XS_2 - S_1Y^*Q_{22}^*P_{22}XS_2 \\ &= S_1\big(I_V - (I_V - Y^*Q_{22}^*)(I_V - P_{22}X) - Y^*Q_{22}^*(I_V - P_{22}X) \\ &\quad - (I_V - Y^*Q_{22}^*)P_{22}X - Y^*Q_{22}^*P_{22}X\big)S_2 \\ &= S_1\big(I_V - I_V + Y^*Q_{22}^* + P_{22}X - Y^*Q_{22}^*P_{22}X - Y^*Q_{22}^* + Y^*Q_{22}^*P_{22}X \\ &\quad - P_{22}X + Y^*Q_{22}^*P_{22}X - Y^*Q_{22}^*P_{22}X\big)S_2 \\ &= S_1 \cdot 0 \cdot S_2 = 0. \end{aligned}$$

It follows that $I_H - \mathcal{F}_Q(Y)^*\mathcal{F}_P(X) = R_1 + R_2$, as required. □

Corollary B.2.2. *Let H, G, U and V be Hilbert spaces. Let P be an operator such that*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} : H \oplus U \to G \oplus V$$

and $||P|| \leq 1$. Let $X : V \to U$ be an operator such that $||X|| \leq 1$ and $I - P_{22}X$ is invertible. Then $||\mathcal{F}_P(X)|| \leq 1$.

Proof. By Proposition B.2.1,

$$\begin{aligned} I_H - \mathcal{F}_P(X)^*\mathcal{F}_P(X) &= \\ &= P_{21}^*(I_V - X^*P_{22}^*)^{-1}(I_V - X^*X)(I_V - P_{22}X)^{-1}P_{21} \\ &\quad + \begin{bmatrix} I_H & P_{21}^*(I_V - X^*P_{22}^*)^{-1}X^* \end{bmatrix} (I_{H\oplus U} - P^*P) \begin{bmatrix} I_H \\ X(I_V - P_{22}X)^{-1}P_{21} \end{bmatrix}. \end{aligned}$$

Let $A = (I_V - P_{22}X)^{-1}P_{21} : H \to V$ and

$$B = \begin{bmatrix} I_H \\ X(I_V - P_{22}X)^{-1}P_{21} \end{bmatrix} = \begin{bmatrix} I_H \\ XA \end{bmatrix} : H \to H \oplus U.$$

Then

$$I_H - \mathcal{F}_P(X)^*\mathcal{F}_P(X) = A^*(I_V - X^*X)A + B^*(I_{H\oplus U} - P^*P)B.$$

By Remark B.1.8, since $||X|| \leq 1$ and $||P|| \leq 1$, we have

$$A^*(I_V - X^*X)A \geq 0 \text{ and } B^*(I_{H\oplus U} - P^*P)B \geq 0.$$

Thus, by Proposition B.1.2,

$$I_H - \mathcal{F}_P(X)^*\mathcal{F}_P(X) \geq 0.$$

It follows from Remark B.1.8 that $||\mathcal{F}_P(X)|| \leq 1$. □

Remark B.2.3. Suppose, in addition, $||X|| < 1$ in Corollary B.2.2. Then

$$||P_{22}X|| \leq ||P_{22}||\,||X|| \leq ||X|| < 1$$

and so $I_V - P_{22}X$ is automatically invertible.

Remark B.2.4. Suppose, in addition, $H = G = \mathbb{C}^n$, $U = V$ and $X = z \cdot I_V$ in Corollary B.2.2. Then, by Remark B.2.3, $I_V - zP_{22}$ is invertible for all $z \in \mathbb{D}$. Moreover, the linear fractional transformation $\mathcal{F}_P$, given by

$$\mathcal{F}_P(z) = P_{11} + zP_{12}(I_V - zP_{22})^{-1}P_{21} \text{ for all } z \in \mathbb{D},$$

is holomorphic on $\mathbb{D}$.

B.3 Reproducing kernels and Hilbert function spaces

In this section we give the required definitions and results from the theory of reproducing kernels and Hilbert function spaces. We include a number of additional results that are used frequently in this thesis.

B.3.1 Kernels

Definition B.3.1. [14, p. 344] *Let X be a set and $k : X \times X \to \mathbb{C}$ be a function. Then k is a* positive semidefinite function *if, for all $x_1, \dots, x_n \in X$ and $c_1, \dots, c_n \in \mathbb{C}$,*

$$\sum_{i,j=1}^{n} \overline{c_j}c_i k(x_j, x_i) \geq 0.$$

Definition B.3.2. [6, Definition 2.22] *A* kernel *on a set X is a hermitian symmetric positive semidefinite function $k : X \times X \to \mathbb{C}$, where by hermitian symmetric we mean $k(x, y) = \overline{k(y, x)}$ for all $x, y \in X$.*

We note that Definition B.3.2 is what Agler and McCarthy call a *weak kernel* in [6]. Their definition requires that, in addition, kernels be non-zero on the diagonal. We use Definition B.3.2 for convenience, since we do not need to make this distinction. This is similar to Aronszajn's approach in [14]. We now have an abstract definition of a kernel, but it is also possible to construct a kernel from the following type of Hilbert space.

Definition B.3.3. [6, Definition 2.1] *A* Hilbert function space *on a set X is a Hilbert space $\mathcal{H}$ of functions on X such that evaluation at each point of X is a continuous linear functional on $\mathcal{H}$.*

We note that if we defined kernels to be non-zero on the diagonal then in Definition B.3.3 we would require that, in addition, evaluation at each point of X be non-zero, which is the definition of Hilbert function space given in [6]. We now show how to construct a kernel from a given Hilbert function space.

Definition B.3.4. [6, p. 17] *Let $\mathcal{H}$ be a Hilbert function space on a set X. Let ε_x denote the function on $\mathcal{H}$ given by evaluation at $x \in X$. Then ε_x is a continuous linear functional on $\mathcal{H}$, and so, by Theorem B.1.7, there is a unique element $k_x \in \mathcal{H}$ such that*

$$f(x) = \varepsilon_x(f) = \langle f, k_x \rangle$$

for all $f \in \mathcal{H}$. We call k_x the reproducing kernel *at x since it reproduces the value of each function at x. Moreover, for all $x, y \in X$, since $k_y \in \mathcal{H}$,*

$$k_y(x) = \varepsilon_x(k_y) = \langle k_y, k_x \rangle.$$

We define the kernel function *of $\mathcal{H}$ to be the function $k : X \times X \to \mathbb{C}$ given by*

$$k(x, y) := k_y(x)$$

for all $x, y \in X$.

Proposition B.3.5. *Let $\mathcal{H}$ be a Hilbert function space on a set X with kernel function k. Then k is a kernel on X.*

Proof. Clearly k is hermitian symmetric. Let $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{C}$. Then

$$\sum_{i,j=1}^{n} \overline{c_i} c_j k(x_i, x_j) = \left\langle \sum_{j=1}^{n} c_j k_{x_j}, \sum_{i=1}^{n} c_i k_{x_i} \right\rangle = \left\| \sum_{j=1}^{n} c_j k_{x_j} \right\|^2 \geq 0$$

and hence k is positive semidefinite. It follows that k is a kernel on X. □

As a consequence of Proposition B.3.5, we refer to the kernel function of a Hilbert function space as the kernel of that space. The following proposition characterises the kernel of a Hilbert function space in terms of an orthonormal basis for the space.

Proposition B.3.6. [6, Proposition 2.18] *Let $\mathcal{H}$ be a Hilbert function space on X with kernel k. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Then, for all $x, y \in X$,*

$$k(x,y) = \sum_{i\in I} \overline{e_i(y)} e_i(x).$$

Proof. By Parseval's equation, since $k_x \in \mathcal{H}$ for all $x \in X$, we have

$$k(x,y) = \langle k_y, k_x \rangle = \sum_{i\in I} \langle k_y, e_i \rangle \langle e_i, k_x \rangle = \sum_{i\in I} \overline{e_i(y)} e_i(x)$$

for all $x, y \in X$. □

We note that an orthonormal basis for a Hilbert space may be uncountably infinite. In this case, we need to be more precise about the sum in Proposition B.3.6. As in [46, pp. 25-26], a family of elements in a linear topological space V is called *summable* if the ordered net of all finite partial sums, of elements of this family, converges in V.

We have shown how to construct a kernel from a given Hilbert function space. The following theorem shows how to construct a Hilbert function space from a given kernel.

Theorem B.3.7 (Moore-Aronszajn)**.** [6, Theorem 2.23] *Let k be a kernel on a set X. Then there is a unique Hilbert function space on X with kernel k.*

Proof. Let $\mathcal{V}_k := \operatorname{span}\{k(\cdot, x) : x \in X\}$. It can be easily verified that $\mathcal{V}_k$ is a complex vector space. Define a map $\langle \cdot, \cdot \rangle_{\mathcal{V}_k} : \mathcal{V}_k \times \mathcal{V}_k \to \mathbb{C}$ by

$$\left\langle \sum_{i=1}^{n} \lambda_i k(\cdot, y_i), \sum_{j=1}^{m} \mu_j k(\cdot, x_j) \right\rangle_{\mathcal{V}_k} := \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \overline{\mu_j} k(x_j, y_i)$$

for all $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$ and $\mu_1, \ldots, \mu_m, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$. It is easy to check that $\langle \cdot, \cdot \rangle_{\mathcal{V}_k}$ is a semi-inner product, and so, by Proposition B.1.9,

$$|\langle u, v \rangle_{\mathcal{V}_k}|^2 \leq \langle u, u \rangle_{\mathcal{V}_k} \langle v, v \rangle_{\mathcal{V}_k}$$

for all $u, v \in \mathcal{V}_k$. Consider the subset $\mathcal{N} = \left\{ v \in \mathcal{V}_k : \langle v, v \rangle_{\mathcal{V}_k} = 0 \right\} \subseteq \mathcal{V}_k$. Since

$$0 \leq |\langle \eta, v \rangle_{\mathcal{V}_k}|^2 \leq \langle \eta, \eta \rangle_{\mathcal{V}_k} \langle v, v \rangle_{\mathcal{V}_k} = 0$$

for all $\eta \in \mathcal{N}$ and $v \in \mathcal{V}_k$, we obtain $\langle \eta, v \rangle_{\mathcal{V}_k} = 0$ for all $\eta \in \mathcal{N}$ and $v \in \mathcal{V}_k$. With this fact, it is easy to verify that $\mathcal{N}$ is a linear subspace of $\mathcal{V}_k$. Hence $H_k := \mathcal{V}_k / \mathcal{N}$ is a vector space. Define a map $\langle \cdot, \cdot \rangle_{H_k} : H_k \times H_k \to \mathbb{C}$ by

$$\langle u + \mathcal{N}, v + \mathcal{N} \rangle_{H_k} := \langle u, v \rangle_{\mathcal{V}_k}$$

for all $u+\mathcal{N}, v+\mathcal{N} \in H_k$. Then it is easy to check that $\langle \cdot,\cdot \rangle_{H_k}$ is a well defined inner product on H_k. Let $\mathcal{H}_k$ be the completion of H_k with respect to the norm given by

$$||v+\mathcal{N}||^2_{\mathcal{H}_k} = \langle v+\mathcal{N}, v+\mathcal{N}\rangle_{H_k}$$

for all $v+\mathcal{N} \in H_k$. Then $\mathcal{H}_k$ is a Hilbert space.

We want $\mathcal{H}_k$ to be a Hilbert function space on X with kernel k. Define

$$f(x) := \langle f, k(\cdot,x)+\mathcal{N}\rangle_{\mathcal{H}_k}$$

for all $f \in \mathcal{H}_k$ and all $x \in X$. Then $\mathcal{H}_k$ is a space of functions on X. Let ε_x denote the function on $\mathcal{H}_k$ given by evaluation at $x \in X$. Clearly ε_x is linear and $\varepsilon_x(f) = \langle f, k(\cdot,x)+\mathcal{N}\rangle_{\mathcal{H}_k}$ for all $f \in \mathcal{H}_k$. Moreover, by Proposition B.1.9,

$$|\varepsilon_x(f)|^2 = |\langle f, k(\cdot,x)+\mathcal{N}\rangle_{\mathcal{H}_k}|^2 \leq \langle f,f\rangle_{\mathcal{H}_k}\langle k(\cdot,x)+\mathcal{N}, k(\cdot,x)+\mathcal{N}\rangle_{\mathcal{H}_k} = k(x,x)||f||^2_{\mathcal{H}_k}$$

for all $f \in \mathcal{H}_k$. It follows that ε_x is continuous and so $\mathcal{H}_k$ is a Hilbert function space on X. Since $k(\cdot,x)+\mathcal{N}$ is the unique reproducing kernel of $\mathcal{H}_k$ at $x \in X$, the kernel of $\mathcal{H}_k$ is given by

$$\langle k(\cdot,y)+\mathcal{N}, k(\cdot,x)+\mathcal{N}\rangle_{\mathcal{H}_k} = \langle k(\cdot,y), k(\cdot,x)\rangle_{\mathcal{V}_k} = k(x,y)$$

for all $x,y \in X$. Hence k is the kernel of $\mathcal{H}_k$.

It remains to show that $\mathcal{H}_k$ is unique for k. Suppose that $\mathcal{H}$ is another Hilbert function space on X with kernel k. Let k_x denote the reproducing kernel of $\mathcal{H}_k$ at $x \in X$. Clearly

$$\operatorname{span}\{k_x : x \in X\} \subseteq \mathcal{H}.$$

Since $\mathcal{H}$ is complete, $\mathcal{H}_k \subseteq \mathcal{H}$. Since $\mathcal{H}_k$ is closed, $\mathcal{H} = \mathcal{H}_k \oplus (\mathcal{H}_k)^\perp$. Let $f \in (\mathcal{H}_k)^\perp$. Then, since

$$f(x) = \langle f, k_x\rangle = 0$$

for all $x \in X$, we have $f = 0$. It follows that $\mathcal{H} = \mathcal{H}_k$. Now let $f \in \mathcal{H}$. Then $f = \lim_{n\in\mathbb{N}} f_n$, where each $f_n \in \operatorname{span}\{k_x : x \in X\}$, that is, $f_n = \sum_{x\in X} \alpha_x k_x$, where only finitely many of the α_x are non-zero. Hence

$$||f_n||^2_{\mathcal{H}} = \sum_{x,y\in X} \alpha_x\overline{\alpha_y}\langle k_x, k_y\rangle_{\mathcal{H}} = \sum_{x,y\in X} \alpha_x\overline{\alpha_y} k(y,x) = \sum_{x,y\in X} \alpha_x\overline{\alpha_y}\langle k_x,k_y\rangle_{\mathcal{H}_k} = ||f_n||^2_{\mathcal{H}_k}$$

and so

$$||f||_{\mathcal{H}} = \lim_{n\in\mathbb{N}} ||f_n||_{\mathcal{H}} = \lim_{n\in\mathbb{N}} ||f_n||_{\mathcal{H}_k} = ||f||_{\mathcal{H}_k}.$$

It follows that $\mathcal{H}$ and $\mathcal{H}_k$ are the same Hilbert function space. □

As a consequence of Theorem B.3.7, for a kernel k on a set X, we denote by $\mathcal{H}_k$ the Hilbert function space on X with kernel k. The following corollary shows that a kernel k

can be represented as the Grammian of a collection of elements in $\mathcal{H}_k$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}_k$. Then we define the *conjugate linear operator* by

$$\sum_{i\in I} c_i e_i \mapsto \sum_{i\in I} \overline{c_i} e_i, \text{ where } c_i \in \mathbb{C} \text{ for all } i \in I.$$

Corollary B.3.8. [6, Theorem 2.53] *Let k be a kernel on a set X. Define the maps $f, g : X \to \mathcal{H}_k$ by $f(x) = k(\cdot, x)$ and $g(x) = Ck(\cdot, x)$ for all $x \in X$, respectively, where C is the conjugate linear operator. Then*

$$k(x, y) = \langle f(y), f(x)\rangle_{\mathcal{H}_k} = \langle g(x), g(y)\rangle_{\mathcal{H}_k}$$

for all $x, y \in X$.

Proof. Let $f : X \to \mathcal{H}_k$ be defined by $f(x) = k_x$ for all $x \in X$, where k_x is the unique reproducing kernel of $\mathcal{H}_k$ at $x \in X$. Then, for all $x, y \in X$,

$$k(x, y) = \langle k_y, k_x\rangle = \langle f(y), f(x)\rangle.$$

Now, let $g : X \to \mathcal{H}_k$ be defined by $g(x) = Cf(x)$ for all $x \in X$, where C is the conjugate linear operator. Then, by Parseval's equation,

$$k(x, y) = \sum_{i\in I} \langle k_y, e_i\rangle\langle e_i, k_x\rangle = \sum_{i\in I} \langle e_i, Ck_y\rangle\langle Ck_x, e_i\rangle = \langle Ck_x, Ck_y\rangle = \langle g(x), g(y)\rangle$$

for all $x, y \in X$. □

It follows from the construction of $\mathcal{H}_k$ that it is the space of minimal dimension for which the representation in Corollary B.3.8 can be realised. This leads to the following definition.

Definition B.3.9. *The* rank of a kernel k *on a set X is the dimension of $\mathcal{H}_k$.*

Proposition B.3.10. *Let k be a kernel on a set X and let $n < \infty$. Then k has rank n if and only if there exist linearly independent functions $f_1, \dots, f_n \in \mathcal{H}_k$ such that*

$$k(x, y) = \overline{f_1(y)} f_1(x) + \cdots + \overline{f_n(y)} f_n(x)$$

for all $x, y \in X$.

Proof. For necessity, suppose k has rank n. Then the dimension of $\mathcal{H}_k$ is n, and so there is an orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathcal{H}_k$. Hence, by Proposition B.3.6,

$$k(x, y) = \overline{e_1(y)} e_1(x) + \cdots + \overline{e_n(y)} e_n(x)$$

for all $x, y \in X$.

For sufficiency, suppose there are linearly independent functions $f_1, \ldots, f_n \in \mathcal{H}_k$ such that

$$k(x, y) = \overline{f_1(y)} f_1(x) + \cdots + \overline{f_n(y)} f_n(x)$$

for all $x, y \in X$. Let $f \in \mathcal{H}_k$. Then

$$f(y) = \langle f, k(\cdot, y) \rangle = \left\langle f, \sum_{i=1}^{n} \overline{f_i(y)} f_i \right\rangle = \sum_{i=1}^{n} f_i(y) \langle f, f_i \rangle$$

for all $y \in X$. Hence $f = \sum_{i=1}^{n} \langle f, f_i \rangle f_i$, and so the functions $f_1, \ldots, f_n$ span $\mathcal{H}_k$. Since $f_1, \ldots, f_n$ are linearly independent, $\{f_1, \ldots, f_n\}$ is a basis for $\mathcal{H}_k$. Moreover, since $n < \infty$, this basis generates an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathcal{H}_k$. Hence $\mathcal{H}_k$ has dimension n. It follows that k has rank n. □

We are particularly interested in kernels which produce Hilbert function spaces of holomorphic functions. This motivates the following definition.

Definition B.3.11. [6, p. 15] *Let X be a domain in $\mathbb{C}^d$. Then a kernel on X is a* holomorphic kernel *on X if it is holomorphic in the first variable and conjugate holomorphic in the second. A Hilbert function space on X is a* holomorphic space *on X if the functions belonging to it are holomorphic.*

Proposition B.3.12. [14, pp. 344 - 345] *Let k be a holomorphic kernel on a domain X in $\mathbb{C}^d$. Then $\mathcal{H}_k$ is a space of holomorphic functions on X.*

Proof. If k is a holomorphic kernel then every function in H_k, as constructed in the proof of Theorem B.3.7, is holomorphic. Let $f \in \mathcal{H}_k$. Then

$$f = \lim_{n \in \mathbb{N}} f_n, \text{ where each } f_n \in H_k.$$

Let K be a compact subset of X, let $u = \sup_{n \in \mathbb{N}} ||f_n||_{\mathcal{H}_k}$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that, for every $x, y \in K$,

$$||y - x||_{\mathbb{C}^d} < \delta \implies ||k(\cdot, y) - k(\cdot, x)||_{\mathcal{H}_k} < \frac{\varepsilon}{4u}.$$

Moreover, we can cover K by open discs of radius δ. This cover has a finite subcover, and every $y \in K$ belongs to an element of this subcover. Hence there is a finite subset $\{y_1, \ldots, y_m\} \subseteq K$ such that, for every $y \in K$, we have $||y - y_j||_{\mathbb{C}^d} < \delta$ for some $1 \leq j \leq m$. Let $y \in K$. Then there is a $j \in \{1, \ldots, m\}$ such that

$$||k(\cdot, y) - k(\cdot, y_j)||_{\mathcal{H}_k} < \frac{\varepsilon}{4u}.$$

Choose N to be such that

$$|f(y_j) - f_n(y_j)| = |\langle f - f_n, k(\cdot, y_j) \rangle| \leq ||f - f_n||_{\mathcal{H}_k} ||k(\cdot, y_j)||_{\mathcal{H}_k} \leq \frac{\varepsilon}{2}$$

for all $n \geq N$. Then, for every $n \geq N$,

$$\begin{aligned} |f(y) - f_n(y)| &\leq |\langle f - f_n, k(\cdot, y) - k(\cdot, y_j)\rangle| + |f(y_j) - f_n(y_j)| \\ &\leq ||f - f_n||_{\mathcal{H}_k} ||k(\cdot, y) - k(\cdot, y_j)||_{\mathcal{H}_k} + \frac{\varepsilon}{2} \\ &< 2u\frac{\varepsilon}{4u} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

It follows that f_n is a sequence of holomorphic functions converging uniformly to f on every compact subset of X, and so f is holomorphic on X. □

Proposition B.3.13. [6, p. 17] *Let $\mathcal{H}$ be a holomorphic space on a domain X in $\mathbb{C}^d$. Then the kernel of $\mathcal{H}$ is a holomorphic kernel.*

Proof. Let k be the kernel of $\mathcal{H}$ and let $\lambda \in X$. Since $k(\cdot, \lambda) = k_\lambda \in \mathcal{H}$, we have $k(\cdot, \lambda)$ is holomorphic. Since $\overline{k(\lambda, \cdot)} = k(\cdot, \lambda) \in \mathcal{H}$, we have $k(\lambda, \cdot)$ is conjugate holomorphic. □

B.3.2 Operator-valued kernels

Similarly to how scalar-valued kernels are defined, we can define operator-valued kernels. Some of the results for scalar-valued kernels hold for operator-valued kernels as well.

Definition B.3.14. [6, Definition 2.57] *Let H be a Hilbert space. A* $\mathcal{B}(H)$-valued kernel *on a set X is a map $K : X \times X \to \mathcal{B}(H)$ such that, for all $x_1, \ldots, x_n \in X$ and $u_1, \ldots, u_n \in H$,*

$$\sum_{i,j=1}^{n} \langle K(x_i, x_j)u_j, u_i\rangle_H \geq 0.$$

We note that Definition B.3.14 is what Agler and McCarthy call a *weak $\mathcal{B}(H)$-valued kernel* in [6]. Their definition requires that, in addition, $\mathcal{B}(H)$-valued kernels be non-zero on the diagonal. We do not need to make this distinction, and so use Definition B.3.14.

Definition B.3.15. [6, Definition 2.59] *Let H be a Hilbert space. An* H-valued Hilbert function space *on a set X is a Hilbert space $\mathcal{H}$ of maps from X to H such that evaluation at each point of X is a continuous linear map from $\mathcal{H}$ to H.*

We note that if we defined $\mathcal{B}(H)$-valued kernels to be non-zero on the diagonal then in Definition B.3.15 we would require that, in addition, evaluation at each point of X be non-zero, which is the definition given in [6]. We now show how to construct a $\mathcal{B}(H)$-valued kernel from a given H-valued Hilbert function space.

Definition B.3.16. *Let H be a Hilbert space. Let $\mathcal{H}$ be an H-valued Hilbert function space on a set X, and let $\varepsilon_x : \mathcal{H} \to H$ be given by evaluation at $x \in X$. For $x \in X$ and $u \in H$, define $\varepsilon_{x,u} : \mathcal{H} \to \mathbb{C}$ by*

$$\varepsilon_{x,u}(f) := \langle \varepsilon_x(f), u\rangle_H$$

for all $f \in \mathcal{H}$*. Then it is easy to check that* $\varepsilon_{x,u}$ *is a continuous linear functional. Hence, by Theorem B.1.7, there is a unique element* $K_{x,u} \in \mathcal{H}$ *such that*

$$\langle f(x), u\rangle_H = \langle \varepsilon_x(f), u\rangle_H = \varepsilon_{x,u}(f) = \langle f, K_{x,u}\rangle_{\mathcal{H}}$$

for all $f \in \mathcal{H}$*. We call* $K_{x,u}$ *the* inner-product-reproducing kernel *for* u *at* x *since it reproduces the inner product of* $f(x)$ *and* u *for each* $f \in \mathcal{H}$*. Moreover, for all* $x, y \in X$ *and* $u, v \in H$*, since* $K_{x,u} \in \mathcal{H}$*,*

$$\langle K_{y,v}(x), u\rangle_H = \langle K_{y,v}, K_{x,u}\rangle_{\mathcal{H}}.$$

We define the $\mathcal{B}(H)$-valued kernel function *of* $\mathcal{H}$ *to be* $K : X \times X \to \mathcal{B}(H)$ *given by*

$$K(x, y)v := K_{y,v}(x)$$

for all $x, y \in X$ *and* $v \in H$*.*

Proposition B.3.17. *Let* H *be a Hilbert space. Let* $\mathcal{H}$ *be an* H*-valued Hilbert function space on a set* X *with* $\mathcal{B}(H)$*-valued kernel function* K*. Then* K *is a* $\mathcal{B}(H)$*-valued kernel on* X*.*

Proof. Let $x_1, \dots, x_n \in X$ and $u_1, \dots, u_n \in H$. Then

$$\sum_{i,j=1}^{n} \langle K(x_i, x_j)u_j, u_i\rangle_H = \left\langle \sum_{j=1}^{n} K_{x_j,u_j}, \sum_{i=1}^{n} K_{x_i,u_i} \right\rangle_{\mathcal{H}} = \left\| \sum_{j=1}^{n} K_{x_j,u_j} \right\|_{\mathcal{H}}^2 \geq 0,$$

as required. □

As a consequence of Proposition B.3.17, we refer to the $\mathcal{B}(H)$-valued kernel function of an H-valued Hilbert function space as the $\mathcal{B}(H)$-valued kernel of that space. In the following theorem, we describe how to construct an H-valued Hilbert function space from a given $\mathcal{B}(H)$-valued kernel.

Theorem B.3.18. [6, Theorem 2.60] *Let* H *be a Hilbert space. Let* K *be a* $\mathcal{B}(H)$*-valued kernel on a set* X*. Then there is a unique* H*-valued Hilbert function space on* X *that has* K *as its* $\mathcal{B}(H)$*-valued kernel.*

Proof. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for H. Define a function $k : X \times I \to \mathbb{C}$ by

$$k((y, j), (x, i)) := \langle K(y, x)e_i, e_j\rangle_H$$

for all $x, y \in X$ and $i, j \in I$. It is easy to check that k is a kernel on X. Hence, by Theorem B.3.7, there is a unique Hilbert function space $\mathcal{H}_k$ on $X \times I$ that has k as its kernel. Let $\mathcal{H}$ be the set of all maps $F : X \to H$ given by

$$F(x) = \sum_{i\in I} f(x, i)e_i$$

for all $x \in X$ and some $f \in \mathcal{H}_k$. Then $\mathcal{H}$ is an H-valued Hilbert function space on X for the inner product given by

$$\langle F, G \rangle_{\mathcal{H}} = \langle f, g \rangle_{\mathcal{H}_k}$$

for all $F, G \in \mathcal{H}$, where $F(x) = \sum_{i \in I} f(x,i)e_i$ and $G(x) = \sum_{i \in I} g(x,i)e_i$ for all $x \in X$ and some $f, g \in \mathcal{H}_k$. Now, for each $x \in X$ and $i \in I$, define

$$K_{xi} := \sum_{i \in I} k(\cdot, (x,i))e_i \in \mathcal{H}.$$

Then, for all $F \in \mathcal{H}$, where $F(x) = \sum_{i \in I} f(x,i)e_i$ for all $x \in X$ and some $f \in \mathcal{H}_k$, we have

$$\langle F, K_{xi} \rangle_{\mathcal{H}} = \langle f, k(\cdot, (x,i)) \rangle_{\mathcal{H}_k} = f(x,i) = \langle F(x), e_i \rangle_H$$

for all $x \in X$ and $i \in I$. In particular, for all $x, y \in X$ and $i, j \in I$,

$$\langle K_{xi}(y), e_j \rangle_H = \langle K_{xi}, K_{yj} \rangle_{\mathcal{H}} = k((y,j),(x,i)) = \langle K(y,x)e_i, e_j \rangle_H.$$

It follows that $K_{xi}(y) = K(y,x)e_i$ for all $x \in X$ and $i \in I$. Hence K is the $\mathcal{B}(H)$-valued kernel for $\mathcal{H}$. That $\mathcal{H}$ is unique follows by the fact that $\mathcal{H}_k$ is unique. □

As a consequence of Theorem B.3.18, for a $\mathcal{B}(H)$-valued kernel K on a set X, we denote by $\mathcal{H}_K$ the H-valued Hilbert function space on X with $\mathcal{B}(H)$-valued kernel K.

Corollary B.3.19. [6, Theorem 2.62] *Let H be a Hilbert space. Let K be a $\mathcal{B}(H)$-valued kernel on a set X. Then the maps $F : X \to \mathcal{B}(H, \mathcal{H}_K)$ and $G : X \to \mathcal{B}(\mathcal{H}_K, H)$ defined by $F(x) = K(\cdot, x)$ and $G(x) = K(x, \cdot)$ for all $x \in X$, respectively, satisfy*

$$K(x,y) = F(x)^* F(y) = G(x)G(y)^*$$

for all $x, y \in X$.

Proof. Let $F : X \to \mathcal{B}(H, \mathcal{H}_K)$ be defined by $F(x) = K(\cdot, x)$ for all $x \in X$. Then

$$F(x)h = K(\cdot, x)h = K_{x,h} \in \mathcal{H}_K$$

for all $x \in X$ and $h \in H$, where $K_{x,h}$ is the unique inner-product-reproducing kernel for h at x. Hence, for all $x, y \in X$ and $h, g \in H$,

$$\langle F(x)^* F(y)h, g \rangle_H = \langle K_{y,h}, K_{x,g} \rangle_{\mathcal{H}_K} = \langle K(x,y)h, g \rangle_H.$$

It follows that $F(x)^* F(y) = K(x,y)$ for all $x, y \in X$. That $F(x)$ is bounded, for $x \in X$, follows since

$$||F(x)||^2 = \sup_{||h||_H \leq 1} \langle F(x)^* F(x)h, h \rangle = \sup_{||h||_H \leq 1} \langle K(x,x)h, h \rangle \leq ||K(x,x)||^2.$$

Let $G : X \to \mathcal{B}(\mathcal{H}_K, H)$ be defined by $G(x) = F(x)^*$ for all $x \in X$. Then

$$K(x, y) = F(x)^*F(y) = G(x)G(y)^*$$

for all $x, y \in X$. □

Let H be a Hilbert space, and K be a $\mathcal{B}(H)$-valued kernel on a domain X in $\mathbb{C}^d$. Then we call K an *analytic* $\mathcal{B}(H)$*-valued kernel* on X if K is analytic in the first variable and conjugate analytic in the second.

Corollary B.3.20. [6, Theorem 2.67] *Let H be a Hilbert space. Let K be an analytic $\mathcal{B}(H)$-valued kernel on a domain X in $\mathbb{C}^d$. Then the conjugate analytic map $F : X \to \mathcal{B}(H, \mathcal{H}_K)$ and the analytic map $G : X \to \mathcal{B}(\mathcal{H}_K, H)$ defined by $F(x) = K(\cdot, x)$ and $G(x) = K(x, \cdot)$ for all $x \in X$, respectively, satisfy*

$$K(x, y) = F(x)^*F(y) = G(x)G(y)^*$$

for all $x, y \in X$.

Proof. Let $F : X \to \mathcal{B}(H, \mathcal{H}_K)$ and $G : X \to \mathcal{B}(\mathcal{H}_K, H)$ be defined by $F(x) = K(\cdot, x)$ and $G(x) = F(x)^*$ for all $x \in X$. Then, by the proof Corollary B.3.19,

$$K(x, y) = F(x)^*F(y) = G(x)G(y)^* \text{ for all } x, y \in X.$$

For each $x \in X$, since K is an analytic $\mathcal{B}(H)$-valued kernel, $K(x, y)$ is conjugate analytic in y. It follows that F is conjugate analytic, and hence G is analytic. □

Remark B.3.21. Let H be a Hilbert space. Let K be a $\mathcal{B}(H)$-valued kernel on a set X. Suppose $\dim H = 1$. Then K is a kernel on X and $\mathcal{H}_K$ is a Hilbert function space on X with kernel K. Suppose, in addition, X is a domain in $\mathbb{C}^d$ and K is an analytic kernel. Then, by Corollary B.3.20, for all $x, y \in X$, we have

$$K(x, y) = G(x)G(y)^*,$$

where $G : X \to \mathcal{B}(\mathcal{H}_K, H)$ is the analytic map defined by $G(x) = K(x, \cdot)$ for all $x \in X$.

B.3.3 Additional results

In this section, we give some additional results related to the theory of reproducing kernels. These results are used frequently in this thesis, and so we have collected them here.

Proposition B.3.22. *Let $f : X \to \mathbb{C}^d$ be a function on a set X. Define $k : X \times X \to \mathbb{C}$ by $k(x, y) = f(y)^*f(x)$ for all $x, y \in X$. Then k is a kernel on X.*

Proof. It is easy to check that k is hermitian symmetric. Now, let $x_1, \dots, x_n \in X$ and $c_1, \dots, c_n \in \mathbb{C}$. Then

$$\sum_{i,j=1}^{n} \overline{c_j} c_i k(x_j, x_i) = \left\langle \sum_{j=1}^{n} \overline{c_j} f(x_j), \sum_{i=1}^{n} \overline{c_i} f(x_i) \right\rangle_{\mathbb{C}^d} = \left\| \sum_{j=1}^{n} \overline{c_j} f(x_j) \right\|_{\mathbb{C}^d}^2 \geq 0.$$

It follows that k is a positive semidefinite function, and hence k is a kernel on X. □

Corollary B.3.23. *Let $f = [f_i]_{i=1}^d : X \to \mathbb{C}^d$ be a function on a set X. Define $k : X \times X \to \mathbb{C}$ by $k(x, y) = f(y)^* f(x)$ for all $x, y \in X$. Then k is a kernel on X, and the rank of k is the number of $f_1, \dots, f_d$ that are linearly independent.*

Proof. By Proposition B.3.22, k is a kernel on X. For all $x, y \in X$,

$$k(x, y) = f(y)^* f(x) = \sum_{i=1}^{d} \overline{f_i(y)} f_i(x).$$

If $f_1, \dots, f_d \in \mathcal{H}_k$ are linearly independent then, by Proposition B.3.10, k has rank d. Otherwise, we use elimination to obtain linearly independent functions $g_1, \dots, g_n \in \mathcal{H}_k$, where $1 \leq n \leq d - 1$, such that

$$k(x, y) = f(y)^* f(x) = \sum_{i=1}^{n} \overline{g_i(y)} g_i(x).$$

In this case, by Proposition B.3.10, the rank of k is n. However, n is also the number of $f_1, \dots, f_d$ that are linearly independent. □

Using a similar proof to that of Proposition B.3.22, we can prove the following more general result.

Proposition B.3.24. *Let $f : X \to \mathbb{C}^d$ be a function on a set X, and let Y be a set. Let $M : Y \times Y \to \mathcal{M}_d(\mathbb{C})$ be a function such that the matrix $[M(y_i, y_j)]_{i,j=1}^n \in \mathcal{M}_{dn}(\mathbb{C})$ is positive semidefinite for all $y_1, \dots, y_n \in Y$. Define $k : (X \times Y) \times (X \times Y) \to \mathbb{C}$ by*

$$k(x_1, y_1, x_2, y_2) = f(x_2)^* M(y_2, y_1) f(x_1)$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then k is a kernel on $X \times Y$.

Proof. To see that k is hermitian symmetric, let $y_1, y_2 \in Y$ and $x_1, x_2 \in X$. Then the matrix

$$\mathcal{M} = \begin{bmatrix} M(y_1, y_1) & M(y_1, y_2) \\ M(y_2, y_1) & M(y_2, y_2) \end{bmatrix} \in \mathcal{M}_{2d}(\mathbb{C})$$

is positive semidefinite. In particular, $\mathcal{M} = \mathcal{M}^*$ and so $M(y_2, y_1) = M(y_1, y_2)^*$. Hence

$$k(x_1, y_1, x_2, y_2) = f(x_2)^* M(y_2, y_1) f(x_1) = (f(x_1)^* M(y_1, y_2) f(x_2))^* = \overline{k(x_2, y_2, x_1, y_1)}.$$

To see that k is a positive semidefinite function, let $x_1, \dots, x_n \in X$ and $y_1, \dots, y_n \in Y$. Then the matrix

$$\mathcal{M} = [M(y_i, y_j)]_{i,j=1}^n \in \mathcal{M}_{dn}(\mathbb{C})$$

is positive semidefinite. By Theorem B.1.3, $\mathcal{M} = \mathcal{N}^*\mathcal{N}$ for some $\mathcal{N} = [N_{ij}]_{i,j=1}^n \in \mathcal{M}_{dn}(\mathbb{C})$, where $N_{ij} \in \mathcal{M}_d(\mathbb{C})$ for $1 \le i, j \le n$. It follows that

$$M(y_i, y_j) = \sum_{k=1}^n N_{ki}^* N_{kj}$$

for all $1 \le i, j \le n$. Thus, for all $c_1, \dots, c_n \in \mathbb{C}$,

$$\begin{aligned}\sum_{i,j=1}^n \overline{c_j} c_i k(x_j, y_j, x_i, y_i) &= \sum_{i,j=1}^n \overline{c_j} c_i f(x_i)^* \left(\sum_{k=1}^n N_{ki}^* N_{kj}\right) f(x_j) \\ &= \sum_{k=1}^n \left\langle \sum_{j=1}^n \overline{c_j} N_{kj} f(x_j), \sum_{i=1}^n \overline{c_i} N_{ki} f(x_i) \right\rangle_{\mathbb{C}^d} = \sum_{k=1}^n \left\| \sum_{j=1}^n \overline{c_j} N_{kj} f(x_j) \right\|_{\mathbb{C}^d}^2 \ge 0.\end{aligned}$$

It follows that k is a kernel on $X \times Y$. □

In Section B.1, we defined the Hardy space H_d^2, in fact, this is a $\mathbb{C}^d$-valued Hilbert function space. It is natural to ask what its $\mathcal{M}_d(\mathbb{C})$-valued kernel is.

Example B.3.25. [57, p. 6] It is well known that the kernel for the Hardy space H^2 is the function $k_S : \mathbb{D}^2 \to \mathbb{C}$ given by

$$k_S(z, w) = \frac{1}{1 - \overline{w} z}$$

for all $z, w \in \mathbb{D}$. This kernel is called the *Szegő kernel*.

By Corollary B.1.15, H_d^2 and $H^2 \otimes_H \mathbb{C}^d$ are isomorphic as Hilbert spaces. By Remark B.1.16, we identify $k_S(\cdot, \lambda) \otimes_H v$ as a function in H_d^2. Hence, we may ask how this function interacts with functions in H_d^2. The following proposition says that $k_S(\cdot, \lambda) \otimes_H v$ is the inner-product-reproducing kernel for v at λ, and so the $\mathcal{M}_d(\mathbb{C})$-valued kernel of H_d^2 is the map $K_S : \mathbb{D}^2 \to \mathcal{M}_d(\mathbb{C})$ given by

$$K_S(\lambda, \mu) v = k_S(\lambda, \mu) \otimes_H v$$

for all $\lambda, \mu \in \mathbb{D}$ and $v \in \mathbb{C}^d$.

Proposition B.3.26. *Let $\lambda \in \mathbb{D}$ and $v \in \mathbb{C}^d$. Then*

$$\langle h, k_S(\cdot, \lambda) \otimes_H v \rangle_{H_d^2} = \langle h(\lambda), v \rangle_{\mathbb{C}^d}$$

for all $h \in H_d^2$.

Proof. Let $h = [h_i]_{i=1}^d \in H^2_d$, where $h_1, \dots, h_n \in H^2$. For $1 \le i \le d$, let $e_i = [e_{ij}]_{j=1}^d \in \mathbb{C}^d$ be such that $e_{ii} = 1$ and $e_{ij} = 0$ if $j \neq i$. Then, for $\lambda \in \mathbb{D}$ and $v \in \mathbb{C}^d$,

$$\begin{aligned}\langle h, k_S(\cdot,\lambda) \otimes_H v\rangle_{H^2_d} &= \left\langle \sum_{i=1}^d h_i \otimes_H e_i, k_S(\cdot,\lambda) \otimes_H v \right\rangle_{H^2 \otimes_H \mathbb{C}^d} \\ &= \sum_{i=1}^d \langle h_i, k_S(\cdot,\lambda)\rangle_{H^2} \langle e_i, v\rangle_{\mathbb{C}^d} \\ &= \sum_{i=1}^d h_i(\lambda) \langle e_i, v\rangle_{\mathbb{C}^d} = \langle h(\lambda), v\rangle_{\mathbb{C}^d},\end{aligned}$$

since $h(\lambda) = \sum_{i=1}^d h_i(\lambda) e_i$. $\square$

In this thesis, we require that certain functions defined using an element of $\mathcal{S}^{2\times 2}$ are kernels. We can, more generally, define these functions using an element of $\mathcal{S}^{d\times d}$, where

$$\mathcal{S}^{d\times d} := \{F : \mathbb{D} \to \mathcal{M}_d(\mathbb{C}) : F \text{ is holomorphic and } ||F(\lambda)|| \le 1 \text{ for all } \lambda \in \mathbb{D}\}.$$

In order to show that the functions are kernels, we use the fact that a certain matrix is positive semidefinite. Thus, we begin by proving this fact.

Lemma B.3.27. *Let $F \in \mathcal{S}^{d\times d}$. Define an operator $T_F : H^2_d \to H^2_d$ by*

$$(T_F h)(\lambda) = F(\lambda) h(\lambda)$$

for all $h \in H^2_d$ and $\lambda \in \mathbb{D}$. Then $||T_F|| \le 1$ and $T_F \in \mathcal{B}(H^2_d)$. Moreover,

$$T_F^*(k_S(\cdot,\lambda) \otimes_H v) = k_S(\cdot,\lambda) \otimes_H F(\lambda)^* v$$

for all $\lambda \in \mathbb{D}$ and $v \in \mathbb{C}^d$.

Proof. Clearly T_F is linear. Since F is holomorphic, $T_F h$ is holomorphic for all $h \in H^2_d$. Since $||F(\lambda)|| \le 1$ for all $\lambda \in \mathbb{D}$,

$$||T_F h||^2_{H^2_d} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} ||F(re^{i\theta}) h(re^{i\theta})||^2_{\mathbb{C}^d} d\theta \le \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} ||h(re^{i\theta})||^2_{\mathbb{C}^d} d\theta = ||h||^2_{H^2_d}$$

for all $h \in H^2_d$. Hence $T_F h \in H^2_d$ and $||T_F|| \le 1$. It follows that $T_F \in \mathcal{B}(H^2_d)$.

Let $\lambda \in \mathbb{D}$ and $v \in \mathbb{C}^d$. By Proposition B.3.26, $\langle h, k_S(\cdot,\lambda) \otimes_H v\rangle_{H^2_d} = \langle h(\lambda), v\rangle_{\mathbb{C}^d}$ for all $h \in H^2_d$. Hence

$$\langle h, T_F^*(k_S(\cdot,\lambda) \otimes_H v)\rangle_{H^2_d} = \langle T_F h, k_S(\cdot,\lambda) \otimes_H v\rangle_{H^2_d} = \langle (T_F h)(\lambda), v\rangle_{\mathbb{C}^d} = \langle F(\lambda) h(\lambda), v\rangle_{\mathbb{C}^d}$$

and

$$\langle h, k_S(\cdot,\lambda) \otimes_H F(\lambda)^* v\rangle_{H^2_d} = \langle h(\lambda), F(\lambda)^* v\rangle_{\mathbb{C}^d} = \langle F(\lambda) h(\lambda), v\rangle_{\mathbb{C}^d}$$

for all $h \in H^2_d$. It follows that $T_F^*(k_S(\cdot,\lambda) \otimes_H v) = k_S(\cdot,\lambda) \otimes_H F(\lambda)^* v$. □

Proposition B.3.28. *Let $F \in \mathcal{S}^{d\times d}$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{D}$. Then the matrix*

$$\mathcal{M} = \left[\frac{I - F(\lambda_i)F(\lambda_j)^*}{1 - \overline{\lambda_j}\lambda_i}\right]_{i,j=1}^n \in \mathcal{M}_{dn}(\mathbb{C})$$

is positive semidefinite.

Proof. Let $v = [v_i]_{i=1}^n \in \mathbb{C}^{dn}$, where $v_i \in \mathbb{C}^d$ for $1 \le i \le n$. Then

$$\begin{aligned}
\langle \mathcal{M}v, v\rangle_{\mathbb{C}^{dn}} &= \left\langle \left[\sum_{j=1}^n \frac{I - F(\lambda_i)F(\lambda_j)^*}{1 - \overline{\lambda_j}\lambda_i} v_j\right]_{i=1}^n, [v_i]_{i=1}^n \right\rangle_{\mathbb{C}^{dn}} \\
&= \sum_{i=1}^n \left\langle \sum_{j=1}^n \frac{I - F(\lambda_i)F(\lambda_j)^*}{1 - \overline{\lambda_j}\lambda_i} v_j, v_i \right\rangle_{\mathbb{C}^d} \\
&= \sum_{i,j=1}^n k_S(\lambda_i, \lambda_j)\big(\langle v_j, v_i\rangle_{\mathbb{C}^d} - \langle F(\lambda_i)F(\lambda_j)^* v_j, v_i\rangle_{\mathbb{C}^d}\big) \\
&= \sum_{i,j=1}^n \langle k_S(\cdot,\lambda_j), k_S(\cdot,\lambda_i)\rangle_{H^2}\big(\langle v_j, v_i\rangle_{\mathbb{C}^d} - \langle F(\lambda_j)^* v_j, F(\lambda_i)^* v_i\rangle_{\mathbb{C}^d}\big) \\
&= \sum_{i,j=1}^n \langle k_S(\cdot,\lambda_j) \otimes_H v_j, k_S(\cdot,\lambda_i) \otimes_H v_i\rangle_{H^2_d} \\
&\qquad - \langle k_S(\cdot,\lambda_j) \otimes_H F(\lambda_j)^* v_j, k_S(\cdot,\lambda_i) \otimes_H F(\lambda_i)^* v_i\rangle_{H^2_d},
\end{aligned}$$

where k_S is the Szegő kernel. Let T_F be the operator defined in Lemma B.3.27. By Corollary B.1.6, since $||T_F|| \le 1$ and $T_F \in \mathcal{B}(H^2_d)$, we have $I - T_F T_F^* \ge 0$. Moreover,

$$T_F^*(k_S(\cdot,\lambda) \otimes_H v) = k_S(\cdot,\lambda) \otimes_H F(\lambda)^* v$$

for all $\lambda \in \mathbb{D}$ and $v \in \mathbb{C}^d$. Hence

$$\begin{aligned}
\langle \mathcal{M}v, v\rangle_{\mathbb{C}^{dn}} &= \sum_{i,j=1}^n \langle k_S(\cdot,\lambda_j) \otimes_H v_j, k_S(\cdot,\lambda_i) \otimes_H v_i\rangle_{H^2_d} \\
&\qquad - \langle T_F^*(k_S(\cdot,\lambda_j) \otimes_H v_j), T_F^*(k_S(\cdot,\lambda_i) \otimes_H v_i)\rangle_{H^2_d} \\
&= \sum_{i,j=1}^n \langle k_S(\cdot,\lambda_j) \otimes_H v_j - T_F T_F^*(k_S(\cdot,\lambda_j) \otimes_H v_j), k_S(\cdot,\lambda_i) \otimes_H v_i\rangle_{H^2_d} \\
&= \left\langle (I - T_F T_F^*)\left(\sum_{j=1}^n k_S(\cdot,\lambda_j) \otimes_H v_j\right), \sum_{i=1}^n k_S(\cdot,\lambda_i) \otimes_H v_i \right\rangle_{H^2_d} \ge 0.
\end{aligned}$$

It follows that $\mathcal{M}$ is positive semidefinite. □

Corollary B.3.29. *Let $F \in \mathcal{S}^{d\times d}$ and let $\lambda_1, \dots, \lambda_n \in \mathbb{D}$. Then the matrix*

$$\mathcal{M} = \left[\frac{I - F(\lambda_j)^* F(\lambda_i)}{1 - \overline{\lambda_j}\lambda_i}\right]_{i,j=1}^n \in \mathcal{M}_{dn}(\mathbb{C})$$

is positive semidefinite.

Proof. Let $G(\lambda) = F(\overline{\lambda})^*$ for all $\lambda \in \mathbb{D}$, and set $\mu_i = \overline{\lambda_i}$ for $1 \leq i \leq n$. By Proposition B.3.28, we have

$$\left[\frac{I - G(\mu_i)G(\mu_j)^*}{1 - \overline{\mu_j}\mu_i}\right]_{i,j=1}^n = \left[\frac{I - F(\overline{\mu_i})^* F(\overline{\mu_j})}{1 - \overline{\mu_j}\mu_i}\right]_{i,j=1}^n = \left[\frac{I - F(\lambda_i)^* F(\lambda_j)}{1 - \overline{\lambda_i}\lambda_j}\right]_{i,j=1}^n = \mathcal{M}^T$$

is positive semidefinite. It follows that $\mathcal{M}$ is positive semidefinite. □

Now, using Corollary B.3.29, we can show that the functions we require are indeed kernels. The first is an $\mathcal{M}_d(\mathbb{C})$-valued kernel on $\mathbb{D}$, and the second is a kernel on $X \times \mathbb{D}$.

Proposition B.3.30. *Let $F \in \mathcal{S}^{d\times d}$. Define $K : \mathbb{D} \times \mathbb{D} \to \mathcal{M}_d(\mathbb{C})$ by*

$$K(\lambda_1, \lambda_2) = \frac{I - F(\lambda_2)^* F(\lambda_1)}{1 - \overline{\lambda_2}\lambda_1}$$

for all $\lambda_1, \lambda_2 \in \mathbb{D}$. Then K is an $\mathcal{M}_d(\mathbb{C})$-valued kernel on $\mathbb{D}$.

Proof. Let $\lambda_1, \dots, \lambda_n \in \mathbb{D}$. By Corollary B.3.29, the matrix $\mathcal{K} = [K(\lambda_i, \lambda_j)]_{i,j=1}^n$ is positive semidefinite. Let $v = [v_i]_{i=1}^n \in \mathbb{C}^{dn}$, where $v_i \in \mathbb{C}^d$ for $1 \leq i \leq n$. Then

$$\sum_{i,j=1}^n \langle K(\lambda_i, \lambda_j)v_j, v_i\rangle_{\mathbb{C}^d} = \left\langle \left[\sum_{j=1}^n K(\lambda_i, \lambda_j)v_j\right]_{i=1}^n, [v_i]_{i=1}^n \right\rangle_{\mathbb{C}^{dn}} = \langle \mathcal{K}v, v\rangle_{\mathbb{C}^{dn}} \geq 0.$$

It follows that K is an $\mathcal{M}_d(\mathbb{C})$-valued kernel on $\mathbb{D}$. □

Proposition B.3.31. *Let $f : X \to \mathbb{C}^d$ be a function on a set X, and let $F \in \mathcal{S}^{d\times d}$. Define a function $k : (X \times \mathbb{D}) \times (X \times \mathbb{D}) \to \mathbb{C}$ by*

$$k(x_1, \lambda_1, x_2, \lambda_2) = f(x_2)^* \frac{I - F(\lambda_2)^* F(\lambda_1)}{1 - \overline{\lambda_2}\lambda_1} f(x_1)$$

for all $x_1, x_2 \in X$ and $\lambda_1, \lambda_2 \in \mathbb{D}$. Then k is a kernel on $X \times \mathbb{D}$.

Proof. Let $\lambda_1, \dots, \lambda_n \in \mathbb{D}$. By Corollary B.3.29, the matrix

$$\left[\frac{I - F(\lambda_j)^* F(\lambda_i)}{1 - \overline{\lambda_j}\lambda_i}\right]_{i,j=1}^n \in \mathcal{M}_{dn}(\mathbb{C})$$

is positive semidefinite. Hence, by Proposition B.3.24, k is a kernel on $X \times \mathbb{D}$. □

Corollary B.3.32. *Let $f : X \times \mathbb{D} \to \mathbb{C}^d$ be a function on a set X, and let $F \in \mathcal{S}^{d\times d}$. Define a function $k : (X \times \mathbb{D}) \times (X \times \mathbb{D}) \to \mathbb{C}$ by*

$$k(x_1, \lambda_1, x_2, \lambda_2) = f(x_2, \lambda_2)^* \frac{I - F(\lambda_2)^* F(\lambda_1)}{1 - \overline{\lambda_2}\lambda_1} f(x_1, \lambda_1)$$

for all $x_1, x_2 \in X$ and $\lambda_1, \lambda_2 \in \mathbb{D}$. Then k is a kernel on $X \times \mathbb{D}$.

Proof. By Proposition B.3.31, the function given by

$$((x_1, \lambda_1), \lambda_1, (x_2, \lambda_2), \lambda_2) \mapsto f(x_2, \lambda_2)^* \frac{I - F(\lambda_2)^* F(\lambda_1)}{1 - \overline{\lambda_2}\lambda_1} f(x_1, \lambda_1)$$

is a kernel on $(X \times \mathbb{D}) \times \mathbb{D}$. Hence, it is a hermitian symmetric positive semidefinite function. Using this fact, it is easy to check that k is a hermitian symmetric positive semidefinite function, that is, k is a kernel on $X \times \mathbb{D}$. □

The following result is used frequently in this thesis, and so we give it here for convenience. For kernels N and M on $\mathbb{D}^2$, we define a function $K_{N,M} : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ by

$$K_{N,M}(z, \lambda, w, \mu) = 1 - (1 - \overline{w}z)N(z, \lambda, w, \mu) - (1 - \overline{\mu}\lambda)M(z, \lambda, w, \mu)$$

for all $z, \lambda, w, \mu \in \mathbb{D}$.

Proposition B.3.33. *Let N and M be kernels on $\mathbb{D}^2$. Suppose $K_{N,M}$ is a positive semidefinite function. Then $K_{N,M}$ is a kernel on $\mathbb{D}^2$. Suppose, in addition, N and M are holomorphic kernels. Then $K_{N,M}$ is a holomorphic kernel on $\mathbb{D}^2$.*

Proof. It is easy to check that $K_{N,M}$ is hermitian symmetric. If $K_{N,M}$ is positive semidefinite, then $K_{N,M}$ is a kernel on $\mathbb{D}^2$. Suppose, in addition, N and M are holomorphic kernels. Then they are holomorphic in the first variable and conjugate holomorphic in the second. Let

$$f(z, w) = 1 - \overline{w}z$$

for all $z, w \in \mathbb{D}$. Then clearly f is holomorphic in the first variable and conjugate holomorphic in the second. It follows that $K_{N,M}$ is a holomorphic kernel. □

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