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An Investigation into Three Dimensional Probabilistic Polyforms

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AN INVESTIGATION INTO THREE DIMENSIONAL PROBABILISTIC POLYFORMS

By

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Honors Capstone Project

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Olivet Nazarene University

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Bachelor of Science

in

Mathematics (Teaching) and Elementary Education

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An Investigation into Three Dimensional Probabilistic Polyforms

Danielle Vander Schaaf

Olivet Nazarene University

Abstract

Polyforms are created by taking squares, equilateral triangles, and regular hexagons and placing them side by side to generate larger shapes. This project addressed three-dimensional polyforms and focused on cubes. I investigated the probabilities of certain shape outcomes to discover what these probabilities could tell us about the polyforms' characteristics and vice versa. From my findings, I was able to derive a formula for the probability of two different polyform patterns which add to a third formula found prior to my research. In addition, I found the probability that 8 cubes randomly attached together one by one would form a 2x2x2 cube. Finally, I discovered a strong correlation between the probability of a polyform and its number of exposed edges, and I noticed a possible relationship between a polyform's probability and its graph representation.

Polyforms are created by taking squares, equilateral triangles, and hexagons and placing them side by side to generate larger shapes. In this paper, I will address three dimensional polyforms specifically cubes. These polycubes will be created by randomly choosing one face of the previous polycube to attach another cube and so on. While these polycubes are no strangers to the mathematics community, I intend to approach them with a new angle and look at them probabilistically. Are some shapes more probable than others? What is the probability of ending up with a larger cube? What can these probabilities say about the polycubes? I plan to answer all these questions and more.

To demonstrate the approach used, consider an initial cube. There are six places to add the next cube. Adding a cube in any one of the six places results in a rectangular prism or straight polyform as in Figure 1. When there are two cubes in the polyform, we will refer to this as degree two. From this 2nd degree polyform, there are ten places where a third cube can be added. Of these ten locations, eight of them produce an L-shaped polyform and two of them produce a larger straight polyform (Figure 2). The shapes in Figure 2 are degree three. The probability of the L-shaped polyform is $\frac{4}{5}$, and the probability of the straight polyform is $\frac{1}{5}$ (See Table 2, Appendix).

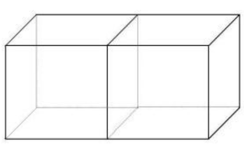


Figure 1

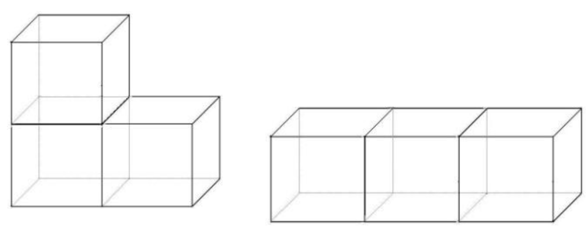


Figure 2

As these polycubes are constructed, it is possible to add the next cube in multiple ways. One variation is to count each location rather than each face. For example, in the L-shaped polyform in Figure 2, there are two different faces that can be filled to create the 2x2 square polyform. This could be thought of as a single location rather than two faces; however, for the duration of this paper, each face will be considered separately as I believe it leads to more interesting results.

Each polyform's probability is found by examining the polyform/polyforms of the previous degree (lesser polyforms) which lead to the polyform in question (greater polyform). For each lesser polyform, you count the number of faces which, if a cube was added there, would lead to the greater polyform. You then divide that number by the total number of exposed faces (surface area) on the lesser polyform to get a fraction. Finally, you multiply that fraction by the probability of the lesser polyform to get a new fraction. If the greater polyform could only be created by one lesser polyform, this new fraction is the probability of the greater polyform. If the greater polyform could be created by several lesser polyforms, you must add the fractions obtained by each of the lesser polyforms together to get the probability of the greater polyform.

Mark Lockwood and Dale Hathaway, in their article in the Journal of Recreational Mathematics, derived a formula to find the probability of a straight cubic polyform for any degree n of cubes.

$$\text{P (Straight polycube of degree n} \ \geq \ 3) \ = \prod_{i=3}^{n} \frac{1}{2i-1}$$

They were able to find this formula because each straight cubic polyform can only be created from the straight cubic polyform of the previous degree.

In my work with three-dimensional polyforms, I found two more shapes which can only be produced by one shape of the previous degree. I will refer to the first shape as the flat W-shape with Figure 3 acting as our base shape.

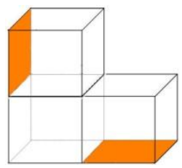


Figure 3

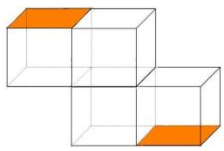


Figure 4

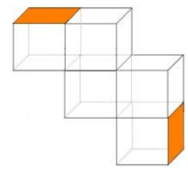


Figure 5

By adding a cube to either orange face in Figure 3, Figure 4 will result. Similarly, adding a cube to either orange face in Figure 4 will result in Figure 5, and adding a cube to either orange face in Figure 5 will result in the flat W-shape of degree 6.

As described earlier, the process for finding the probability of a polyform mainly involves the polyforms of the previous degree which can lead to the given polyform. Thus, to find a formula for probability, I needed to find a formula which would give the surface area of the polyforms of the previous degree by using the degree of the polyform for which the probability is desired. I began by observing that as the degree of the polyforms of this shape increases by one, the total number of exposed faces or the surface area increases by four. We know this will be true for any polyform of the flat W-shape with $n \geq 2$ because for each increase of one degree, one cube is added. One cube has six faces; however, one face on the added cube covers one face of the previous polyform. Covering the face on the existing polyform decreases its surface area by one, and the covered face on the added cube means that only five of its faces will be adding to the new surface area. Taking one face away and adding five faces results in an increase of four faces to the surface area.

I found this formula to be SA= 4(n-1) + 2 where n is the degree of the polyform for which the probability is desired. I first took (n-1) because we needed to find the surface area of the polyform whose degree is one less than the degree of the given polyform. I then multiplied the (n-1) by 4 to represent the constant increase of four faces for each added cube. Finally, I added two because the two cubes, one at each end of the W shape, are only attached to one other cube. Thus, they have five exposed faces instead of four so an extra face must be added for each of those cubes. The formula reduces to SA= 4n – 2. For example, n = 2 will give the surface area for a 1^{st} degree polyform, n = 3 will give the surface area for a 2^{nd} degree polyform, and so on.

As shown above, from any polyform of this shape, there are two face options for creating the polyform of this shape of the next degree. In following the process for finding probabilities, we divide 2 by 4n-2, thus obtaining $\frac{1}{2n-1}$. Now, we must multiply this fraction by the probability of the polyform of previous degree. Because the probability of our base shape is $\frac{4}{5}$, and each successive fraction will be in the form $\frac{1}{2n-1}$, the probability of any polyform of the flat W shape will have a 4 in the numerator. Therefore, the formula to find the probability of a W-shaped polyform of any degree n is:

$$\mathrm{P\ (W-shaped\ polyform\ of\ degreen\ n\ } \geq 3) = 4 * \prod_{i=3}^{n} \frac{1}{2i-1}$$

Our second shape will be referred to as the curly W-shape with Figure 6 acting as our base shape. Figures 7 and 8 demonstrate successive degrees of the pattern. Before going any further, I should explain that for n ≥ 4, there will be two polyforms at each degree which follow the curly W pattern. I will refer to the polyforms in Figures 7a and 8a as the Right Hand Result (RHR) and the polyforms in Figures 7b and 8b as the Left Hand Result (LHR).

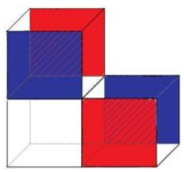


Figure 6

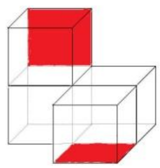


Figure 7a

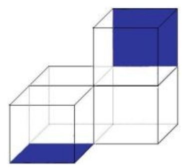


Figure 7b

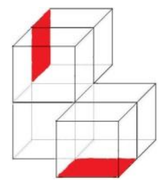


Figure 8a

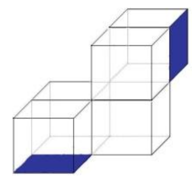


Figure 8b

This pattern begins with the L shape as seen in Figure 6. For the RHR of degree 4, the next cube must be added to either of the red faces shown in Figure 6. For the LHR of degree 4, the next cube must be added to either of the blue faces shown in Figure 6. From Figure 7a, adding a cube to either red face will result in the RHR of degree 5, shown in Figure 8a. From Figure 7b, adding a cube to either blue face will result in the LHR of degree 5, shown in Figure 8b. Adding a cube to either red face in Figure 8a would result in the RHR of degree 6 and so on. Likewise, adding a cube to either blue face in Figure 8b would result in the LHR of degree 6.

To obtain the curly W-shape, the next cube must be added to one of the tip faces of an L shape within the polyform and must not be in the same “plane” as any of the L shapes. Also, the next cube may not be added to a face whose opposite face already has an attached cube. The probability of obtaining the RHR of the curly W-shape for any degree n is as follows:

$$\mathrm{P}\ (\mathrm{RHR\ of\ the\ curly\ W-shape\ polyform\ od\ degree\ n}\ \geq 3) = 4 * \prod_{i=3}^{n} \frac{1}{2i-1}$$

(The formula for the probability of the LHR is the same as for the RHR). This formula is the same as the formula for the flat W-shaped polyforms because both start with the same base shape, both shapes have the same surface area for polyforms of the same degree, and both have two face options on each polyform for creating the polyform of the next degree.

One of the questions I pondered as I began this research was how probable it was that at a cubic degree, a new, larger cube would result. Because of the large number of shapes possible as n increases, I was only able to find the probability for a 2x2x2 cube of degree 8. This probability is ((26,178,075)/(23,478,840,000)) or 0.001115. There are several routes to obtain the 2x2x2 cube (see Figure 9). While I have only analyzed the complete set of polyforms for degree $n \leq 5$, I did investigate the polyforms of degree 6 and 7 which lead to a 2x2x2 cube. I found it interesting that ½ of the 4^{th} degree polyforms lead to the 2x2x2 cube, but only 2 of the 29 5^{th} degree polyforms lead to the 2x2x2 cube.

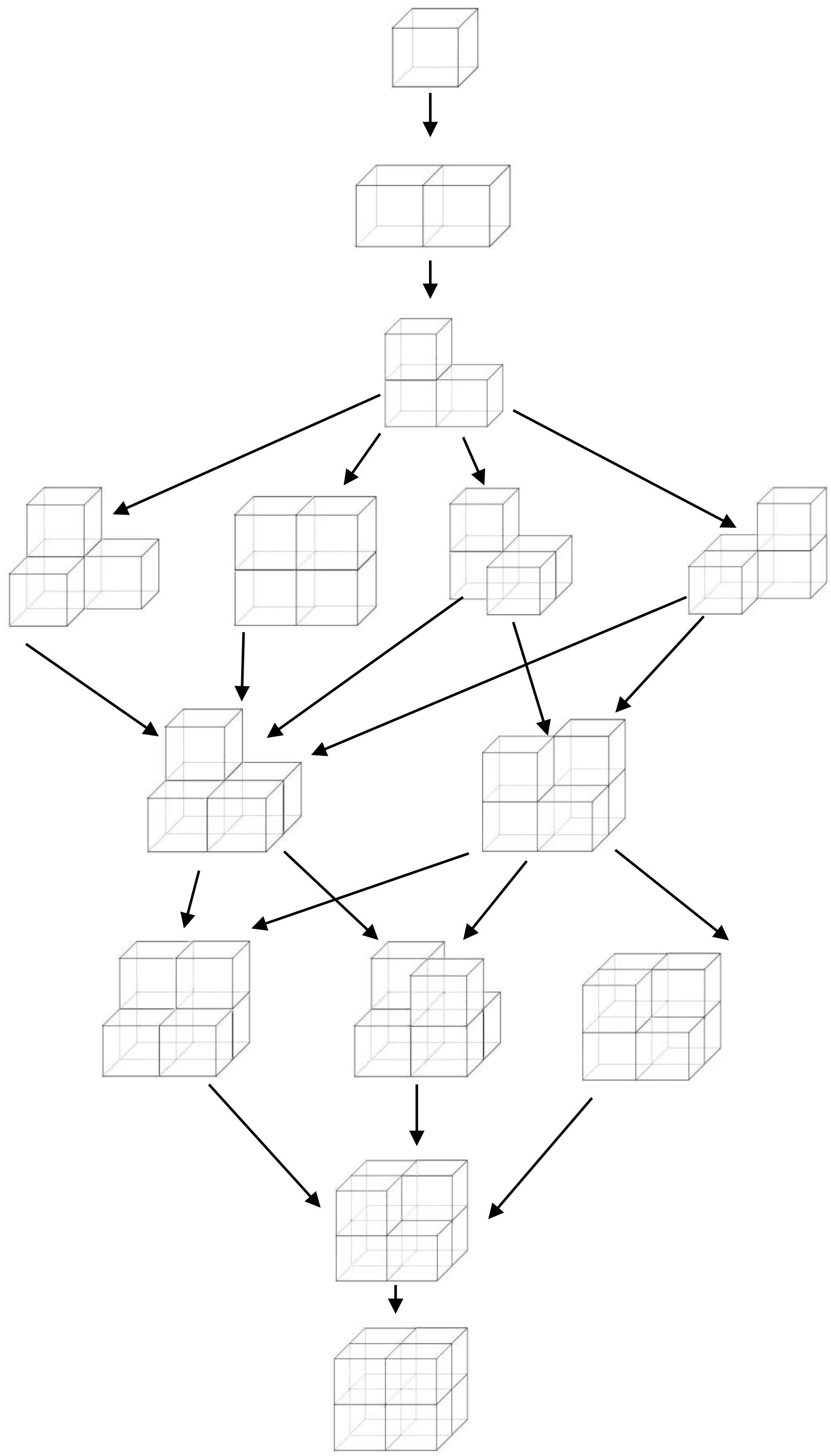
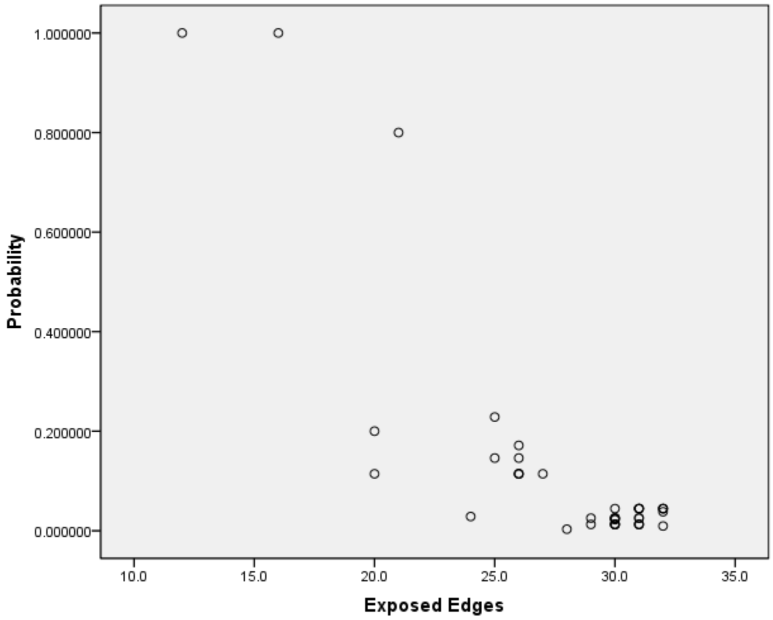


Figure 9



Figure

Another question I investigated was whether there were any relationships between properties of the polyforms and their probabilities. Figure 10 shows the scatter plot between the number of exposed edges and the probabilities for all the polyforms. Because the graph appeared to show a correlation, I completed a basic regression. The equation for this regression is $y = 1.34 - 0.044x$ where x is the number of exposed edges and y is the probability. The p-value was 0.000 and the correlation was 0.834 which indicates a strong relationship between the probabilities and the number of exposed edges. Since there were only three polyforms whose probabilities were higher than 0.5, I decided to check the correlation between probability and exposed edges if those probabilities were treated as outliers. This new regression had an equation of y = 0.463 – 0.014x. The p-value was still 0.000, but the correlation was only 0.747. Thus, even though the three influential polyforms were removed, there is still a strong relationship between probability and exposed edges.

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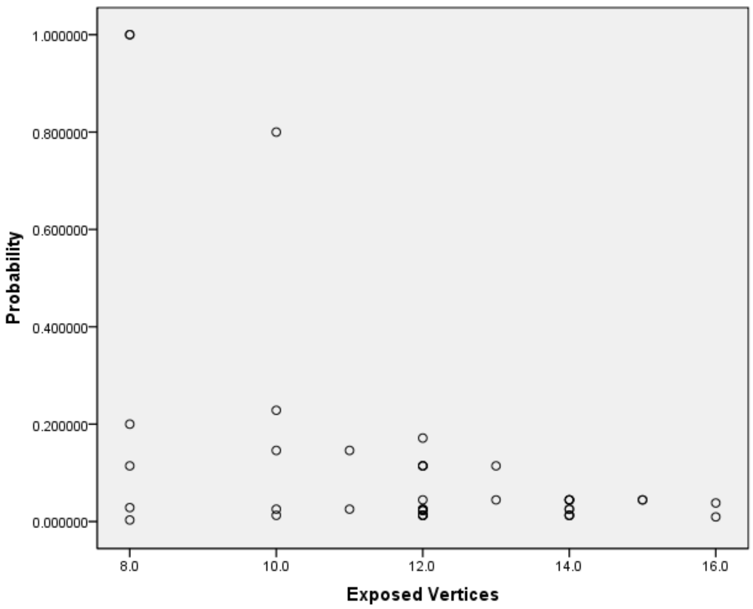
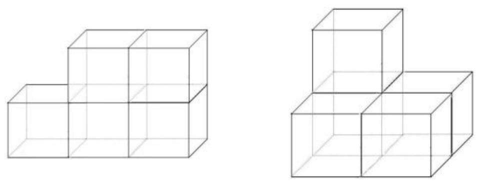


Figure 11

Figure 11 shows the scatter plot between the number of exposed vertices and the probabilities for all the polyforms. There did not appear to be any significant correlation between these two variables, but I still ran a basic linear regression. The equation for the relationship between probabilities and exposed vertices was

Y = 0.736 – 0.051x where x is the number of exposed vertices and y is the probability. The p-value was 0.000, and the correlation was 0.488. Again, I decided to run a second regression where the three polyforms with very large probabilities were treated as outliers. The new regression had an equation of y = 0.181 – 0.010x with a p-value of 0.000 and a correlation of 0.367. These outcomes confirmed my hypothesis that while the results were significant, the correlation between the probabilities of these polyforms and their number of exposed vertices was not as strong as the correlation between the probabilities and the number of exposed edges.

Finally, I compared the graph representations of the polyforms to their probabilities. There were several results that I found to be significant. First, two 5^{th} degree polyforms, 5D and 5V shown in Figure 12, shared the largest probability, $\frac{46}{315}$ (See Table 5, Appendix). Both polyforms also shared the same graph representation as shown in Figure 13.



5D **5V**

Figure 12

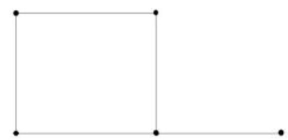


Figure 13

They were the only two 5^{th} degree polyforms with this representation and the only two 5^{th} degree polyforms whose graph had a closed circuit. In addition, of the nine 5^{th} degree polyforms whose stretched-out graph representations resemble Figure 14, eight of those have a probability of $^{14}/_{315}$ and the other has a probability of $^{7}/_{315}$ (See Table 5, Appendix).

Figure 14

There are sixteen 5^{th} degree polyforms who share a graph representation as shown in Figure 15, and their probabilities are either $^{1}/_{315}$, $^{4}/_{315}$, or $^{8}/_{315}$. Finally, there are two 5^{th} degree polyforms whose probabilities are $^{3}/_{315}$ and $^{12}/_{315}$, and they both have the same graph representation as shown in Figure 16 (See Table 5, Appendix).

Figure 15

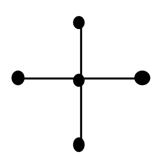


Figure 16

I found it very interesting that 5th degree polyforms sharing the same graph representation had probabilities which were multiples of each other.

This project is only skimming the surface of the topic of three dimensional polyforms and their probabilities. Some areas which I think deserve further research include the possibility of two-tiered probability formulas and the relationship between graph representations and probabilities. I was able to find formulas which will give the probabilities of certain shape outcomes for any degree. I was able to derive these formulas because each successive polyform in those shape patterns could only be created from one polyform of the previous degree. Each new shape in the pattern was found by adding one cube to the previous polyform. I think it may be possible to find two-tiered formulas for shape patterns which require two cubes to be added to create the next polyform in the shape pattern, and further research may be lead to discovering such a formula. In addition, it would be very interesting to see if polyforms of degree 6, 7, 8, etc, which share the same graph representation, still have probabilities which are multiples of each other. My hope is that I or another person will research this topic further in the future.

Appendix

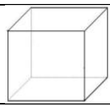
Shape	Probability	Surface Area	Exposed Edges	Exposed Vertices
	1.0	6	12	8

Table 1. 1st Degree Polyforms

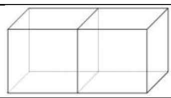
Shape	Probability	Surface Area	Exposed Edges	Exposed Vertices
	1.0	10	16	8

Table 2. 2nd Degree Polyforms

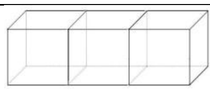
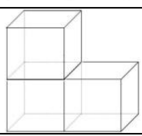
Shape	Probability	Surface Area	Exposed Edges	Exposed Vertices
 **3A**	$^{1}/_{5}$	14	20	8
3B 	$^{4}/_{5}$	14	21	10

Table 3. 3rd Degree Polyforms

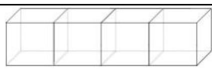
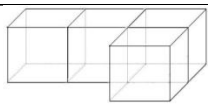
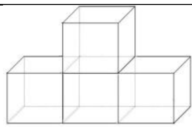
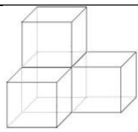
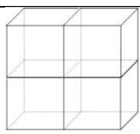
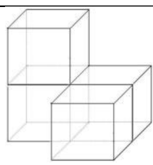
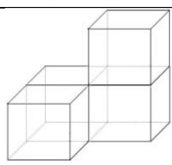
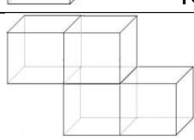
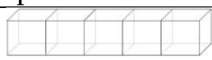
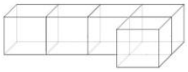
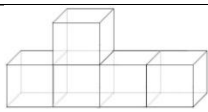
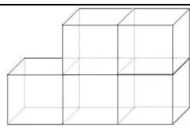
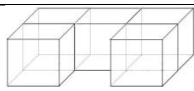
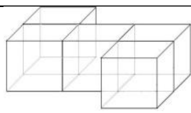
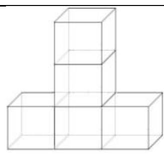
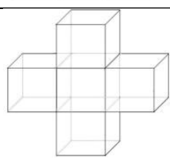
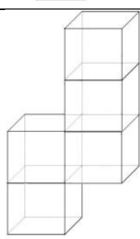
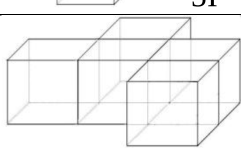
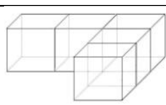
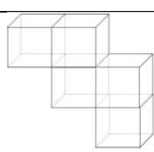
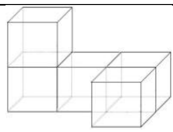
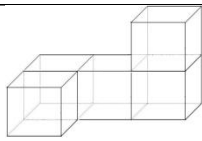
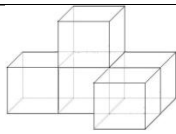
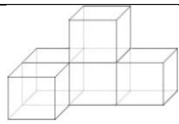
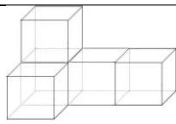
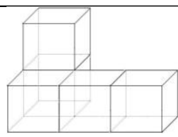
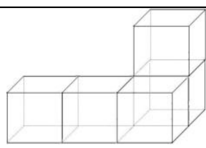
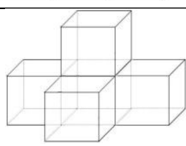
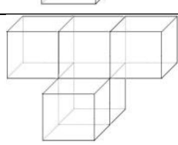
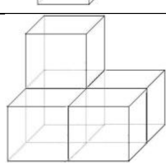
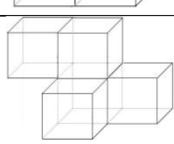
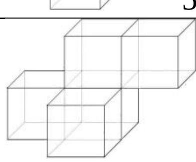
Shape	Probability	Surface Area	Exposed Edges	Exposed Vertices
 4A	$^{1}/_{35}$	18	24	8
 4B	$^{8}/_{35}$	18	25	10
 4C	$^{6}/_{35}$	18	26	12
 4D	$^{4}/_{35}$	18	27	13
 4E	$^{4}/_{35}$	16	20	8
 4F	$^{4}/_{35}$	18	26	12
 4G	$^{4}/_{35}$	18	26	12
 4H	$^{4}/_{35}$	18	26	12

Table 4. 4th Degree Polyforms

Shape	Probability	Surface Area	Exposed Edges	Exposed Vertices
 5A	$^{1}/_{315}$	22	28	8
 5B	$^{8}/_{315}$	22	29	10
 5C	$^{14}/_{315}$	22	30	12
 5D	$^{46}/_{315}$	20	25	10
 5E	$^{4}/_{315}$	22	30	12
 5F	$^{4}/_{315}$	22	30	12
 5G	$^{7}/_{315}$	22	30	12
 5H	$^{3}/_{315}$	22	32	16
 5I	$^{8}/_{315}$	22	30	12
 5J	$^{14}/_{315}$	22	31	14
 5K	$^{4}/_{315}$	22	29	10
 5L	$^{4}/_{315}$	22	31	14

 5M	$^{4}/_{315}$	22	30	12
 5N	$^{4}/_{315}$	22	30	12
 5O	$^{14}/_{315}$	22	31	14
 5P	$^{14}/_{315}$	22	31	14
 5Q	$^{14}/_{315}$	22	31	13
 5R	$^{8}/_{315}$	22	30	12
 5S	$^{8}/_{315}$	22	30	12
 5T	$^{12}/_{315}$	22	32	16
 5U	$^{14}/_{315}$	22	32	13
 5V	$^{46}/_{315}$	20	26	11
 5W	$^{14}/_{315}$	22	32	15
 5X	$^{14}/_{315}$	22	32	15

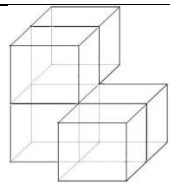
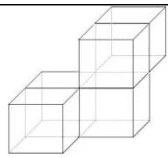
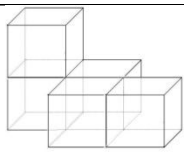
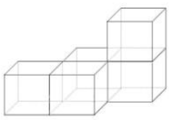
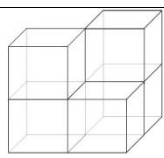
 5Y	$^{4}/_{315}$	22	31	14
 5Z	$^{4}/_{315}$	22	31	14
 5AA	$^{8}/_{315}$	22	31	14
 5BB	$^{8}/_{315}$	22	31	14
 5CC	$^{8}/_{315}$	22	30	11

Table 5. 5th Degree Polyforms

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