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Fibonacci Powers and a Fascinating Triangle

by Dale K. Hathaway and Stephen L. Brown

The Fibonacci sequence can be used as a starting point for an interesting project or research experience in a discrete mathematics class. The recurrence relation for the Fibonacci sequence is well known:

$$F_n = F_{n-1} + F_{n-2}, \quad (1)$$

for $n > 1$ with $F_0 = 1$ and $F_1 = 1$. Less well known is that there is a recurrence relation for F_n^2 , which can be found in at least two different ways.

Our first method involves writing the Fibonacci sequence in terms of its explicit formula

$$F_n = \frac{1}{\sqrt{5}}\alpha^{n+1} - \frac{1}{\sqrt{5}}\beta^{n+1},$$

for $n \geq 0$, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ [5], [6]. Squaring both sides results in an expression for F_n^2 :

$$F_n^2 = \frac{1}{5} [(\alpha^2)^{n+1} - 2(\alpha\beta)^{n+1} + (\beta^2)^{n+1}] = \frac{1}{5} [(\alpha^2)^{n+1} - 2(-1)^{n+1} + (\beta^2)^{n+1}].$$

Note that F_n^2 is a sum of three terms, whereas F_n was a sum of two terms. That is, F_n^2 is a linear combination of the $(n+1)$ st power of -1 , α^2 , and β^2 , the roots of $(t+1)(t-\alpha^2)(t-\beta^2) = 0$, or $t^3 - 2t^2 - 2t + 1 = 0$. This yields the recurrence relation

$$F_n^2 = 2F_{n-1}^2 + 2F_{n-2}^2 - F_{n-3}^2. \quad (2)$$

Another way to find this recurrence relation is by solving the system of equations

$$3^2 = 2^2a + 1^2b + 1^2c$$

$$5^2 = 3^2a + 2^2b + 1^2c$$

$$8^2 = 5^2a + 3^2b + 2^2c.$$

A drawback of this second approach is that the student needs to know in advance that the recurrence relation involves the previous three terms, rather than just the previous two.

A modified Fibonomial triangle. Either approach can be extended to find recurrence relations for F_n^3, F_n^4, \dots . To appreciate the resulting patterns, we have our students calculate all the recurrence relations through F_n^7 . Using a computer algebra system (*Matlab, Derive, Maple*, etc.) or graphing calculator, they can zip through some messy computations and focus on the resulting patterns.

The student now has a sequence of equations, beginning with

$$F_n = F_{n-1} + F_{n-2}$$

$$F_n^2 = 2F_{n-1}^2 + 2F_{n-2}^2 - F_{n-3}^2$$

$$F_n^3 = 3F_{n-1}^3 + 6F_{n-2}^3 - 3F_{n-3}^3 - F_{n-4}^3$$

$$F_n^4 = 5F_{n-1}^4 + 15F_{n-2}^4 - 15F_{n-3}^4 - 5F_{n-4}^4 + F_{n-5}^4$$

If we include the equation for power zero, $F_n^0 = 1$, at the top and move the F_n^m terms to the right-hand side of each equation, we obtain the coefficients set out in Figure 1 in a triangular scheme similar to Pascal's triangle. This triangle (a variation of the Fibonomial triangle, which has no negatives [1], [2]) has a wealth of patterns for students to uncover:

- Diagonal 1 gives the Fibonacci sequence.
- There is an alternating pattern of $+/-$ signs by double diagonals.
- Ignoring signs, there is a symmetry about the vertical axis through the middle of the triangle.

The adventurous student will soon discover many less obvious patterns as well.

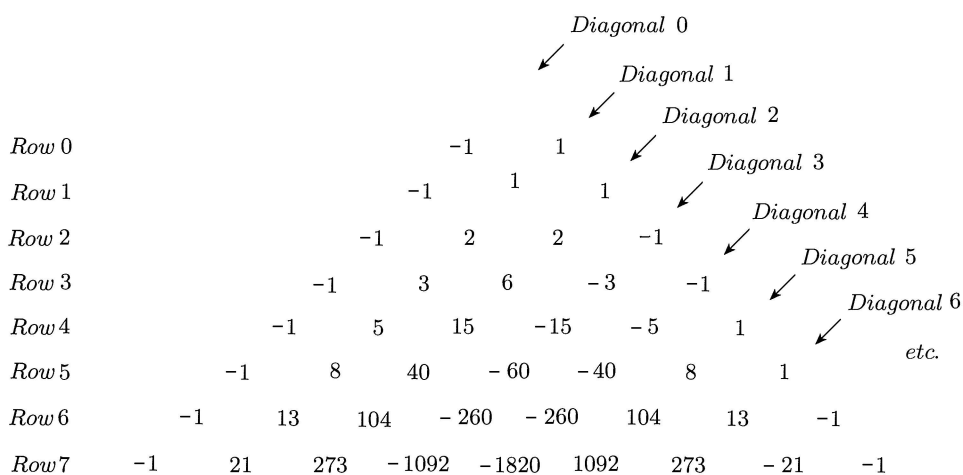


Figure 1. Modified Fibonomial triangle.

An important feature is uncovered if we examine how each entry is constructed from the previous entry in its diagonal by multiplying by a suitable fraction. Figure 2 shows the modified Fibonomial triangle with some multiplier values inserted. Do you see the pattern formed by the multipliers? The multipliers in diagonal m are the ratios of Fibonacci numbers, F_n/F_{n-m} , where n is the next row number!

Since each Fibonacci ratio is used as a multiplier to move down the diagonal one row at a time, these multipliers can be combined to yield an explicit formula for any entry by continually multiplying down a diagonal until the desired term is reached. For example, if we wish to find the value in the sixth row at the third diagonal, $F_{6,3}$, we multiply all the ratios on the third diagonal until we reach the sixth row:

$$\begin{aligned}
 F_{6,3} &= -1 \cdot \frac{F_3}{F_0} \cdot \frac{F_4}{F_1} \cdot \frac{F_5}{F_2} \cdot \frac{F_6}{F_3} \\
 &= -1 \cdot \frac{3}{1} \cdot \frac{5}{1} \cdot \frac{8}{2} \cdot \frac{13}{3} \\
 &= -260.
 \end{aligned}$$

The general relationship is expressed by the formula

$$F_{n,m} = (-1)^{\frac{m(m+1)-2}{2}} \cdot \prod_{k=1}^{n+1-m} \frac{F_{m+k-1}}{F_{k-1}}. \quad (3)$$

Thus, each entry in the triangle can be found as a product of Fibonacci ratios.

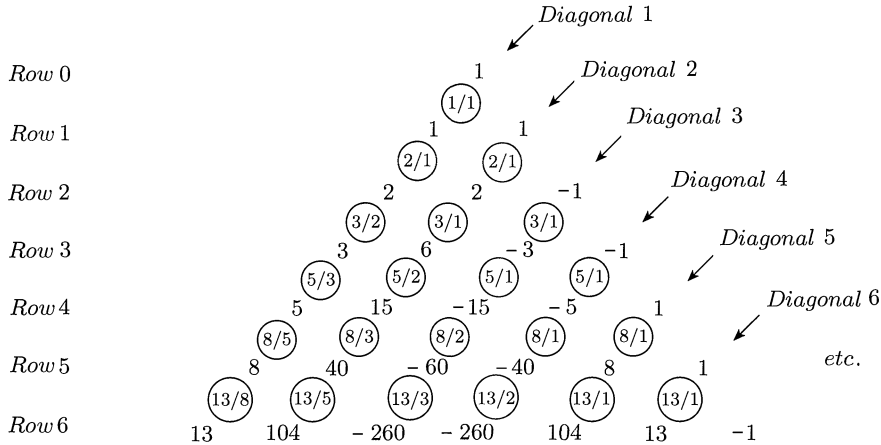


Figure 2. Modified Fibonacci triangle with multipliers circled (diagonal 0 is excluded).

It can be shown that the *Lucas numbers* also appear in the Fibonacci ratios F_n/F_{n-m} . The Lucas numbers, which follow the same recurrence formula as the Fibonacci numbers but with different initial conditions, are defined by

$$L_n = L_{n-1} + L_{n-2}$$

for $n > 1$ with $L_0 = 1$ and $L_1 = 3$. The Fibonacci ratio used to obtain the middle entry of row $2k + 1$ is a Lucas number. In fact, since the middle entry of row $2k + 1$ is in diagonal $k + 1$, the Fibonacci ratio used to obtain this entry is

$$\begin{aligned} \frac{F_n}{F_{n-m}} &= \frac{F_{2k+1}}{F_{2k+1-(k+1)}} \\ &= \frac{F_{2k+1}}{F_k}, \end{aligned}$$

which when reduced turns out to be L_k [4].

Divisibility patterns. Pascal's triangle provides some interesting patterns of divisibility by various numbers [3], [7]. What is amazing is that the Fibonacci triangle also has some nearly identical patterns of divisibility by the same numbers! These patterns are especially noticeable for small prime number divisors, though some other divisors also show similarities. See Figure 3, where the evenly divisible entries are shaded.

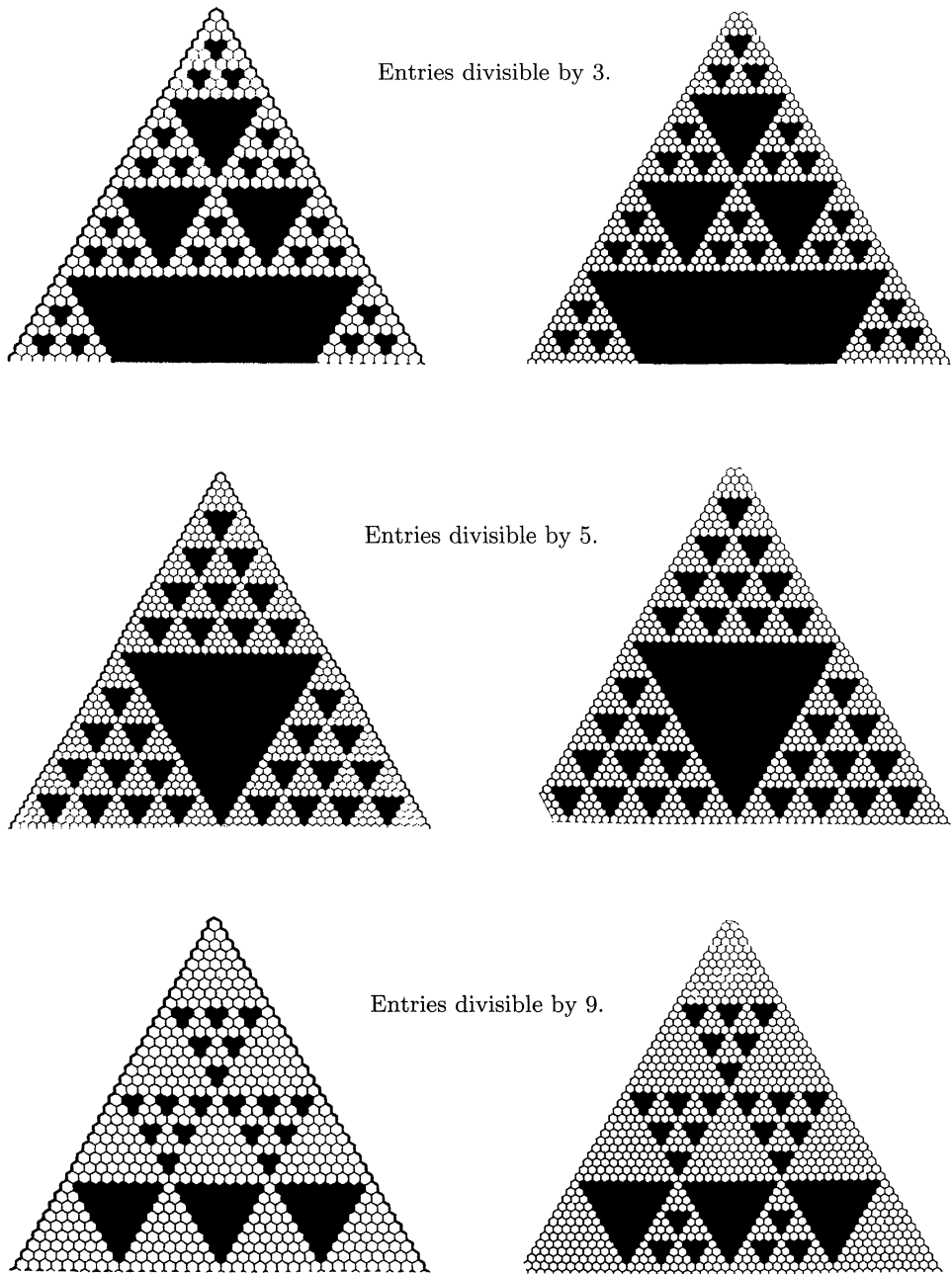


Figure 3. Divisibility of entries in Pascal's triangle (left) and the Fibonomial triangle (right).

The divisibility patterns are not always alike, however. Figure 4 shows the pattern of numbers divisible by 6, where the disparity can easily be seen.

By building on some of these ideas, a student could develop a significant senior project investigating the powers of Fibonacci numbers.

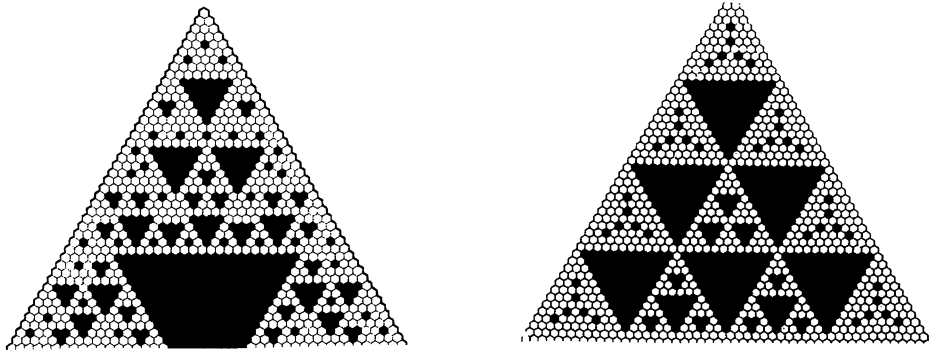


Figure 4. Entries divisible by 6 are shaded in Pascal's triangle (left) and the Fibonomial triangle (right).

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