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A novel transversely isotropic strength criterion for soils

based on a mobilized plane approach

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28   ABSTRACT

29   The peak shear strength rules of transversely isotropic soils are stress state dependent and dependent on relative

30   orientation between bedding plane and principal stress. Accordingly, the shear strength of transversely isotropic

31   soils exhibits two primary characteristics: *(i)* the strength curve on the deviatoric plane is asymmetrical with

32   respect to three principal stress axes; *(ii)* the shear strength changes with the direction angle of the bedding plane

33   when the intermediate principal stress coefficient is a constant. In this paper, the mobilized plane is introduced and

34   used to reveal the failure mechanism of soils. By projecting the microstructure tensor of transversely isotropic

35   soils onto the normal of the mobilized plane, the directionality of the transversely isotropic soils is introduced into

36   the friction rules on the mobilized plane, and a transversely isotropic strength parameter is proposed. The

37   proposed strength parameter can extend isotropic strength criteria into transversely isotropic strength criteria. This

38   mobilized plane approach is used to establish a novel transversely isotropic nonlinear unified strength criterion

39   (TI-NUSC). The difficulty to establish a unified description of the asymmetrical strength curve and its evolution

40   with direction angle is overcome by the established criterion. Comparisons between available test results and the

41   TI-NUSC shows that the TI-NUSC can successfully describe these two primary peak strength characteristics.

42   KEYWORDS: Shear strength; Anisotropy; Fabric/structure of soils; Friction; Failure; Sands;

1 INTRODUCTION

Anisotropy is a significant property of soils and is intrinsically related to the microstructure of soils. Anisotropic soils exhibit inherent and induced anisotropy (Casagrande & Carillo, 1944). Induced anisotropy is attributed to plastic deformation associated with loading, while inherent anisotropy is typically treated as a fabric property in the virgin state before any loadings occur. The simplest form of anisotropy is transverse isotropy, which is a ubiquitous property of naturally deposited soils. The transverse isotropy has a remarkable influence on the peak shear strength, which varies with the relative orientation between the loading and bedding plane. The maximum variation of the bearing capacity for transversely isotropic soils in different loading directions is approximately 35% (Oda *et al.*, 1978). Such a variation cannot be represented by the isotropic strength criteria. The neglect of the transverse isotropy in engineering design is potentially hazardous. A proper description of the strength variation rules for transversely isotropic soils has important implications for the analysis of slope stability (Su & Liao, 1999) and the bearing capacity of shallow foundations (Fu & Dafalias, 2011) and embankments (Zdravkovic *et al.*, 2002; Sun *et al.*, 2004), amongst other applications.

Laboratory tests have been conducted to study the strength characteristics of transversely isotropic soils. These tests have included plane strain tests (Oda *et al.*, 1978; Tatsuoka *et al.*, 1990), true triaxial tests (Lam & Tatsuoka, 1988; Kirkgard & Lade, 1993) and hollow cylinder tests (Nishimura *et al.*, 2007; Lade *et al.*, 2014; Yang *et al.*, 2016). The experimental findings show that transversely isotropic soils exhibit two primary peak shear strength chatacteristics: *(i)* The strength curve on the deviatoric plane is not symmetrical with respect to the three principal stress axes when compared with the isotropic strength curve, meanwhile, the effects of the intermediate principal stress coefficient b (=$(\sigma_2-\sigma_3)/(\sigma_1-\sigma_3)$) on the strength parameter φ is no longer independent of stress direction; *(ii)* the direction angle δ between the normal direction of the bedding plane and the vertical direction greatly affects the peak shear strength and the strength parameter. Actually, the effects of δ and b are coupled and the strength curve on the deviatoric plane evolves with δ.

66   Compared with the experimental study of peak strength characteristics, research into the corresponding

67   strength theory is poorly developed. In general, there are four methods to establish transversely isotropic strength

68   criteria. Firstly, mathematical method, such as the coordinate rotation method (Abelev & Lade, 2004) and the

69   method of modified Lode angle-based shape function (Mortara, 2010; Lü *et al.*, 2011), was proposed to describe

70   the asymmetry of the strength curve with respect to the three stress axes. However, these mathematical methods

71   are only applied to the coaxial condition, i.e., δ=0°. Secondly, the fabric tensor and the stress tensor can be

72   combined to establish a transversely isotropic strength criterion. The combined tensor of the fabric tensor and the

73   stress tensor was proposed (Tobita, 1988) and introduced into an isotropic strength criterion to describe the

74   strength characteristics of transversely isotropic soils (Yao *et al.*, 2017). Additionally, the joint invariant of the

75   stress tensor and fabric tensor has been defined (Li & Dafalias, 2002; Dafalias *et al.*, 2004) and used to develop

76   the strength criteria for transversely isotropic soils (Gao *et al.*, 2010; Gao & Zhao, 2012). Thirdly, the method of

77   projecting the microstructure tensor onto the generalized loading direction was proposed to establish transversely

78   isotropic strength criterion by defining an anisotropic parameter (Pietruszczak & Mroz, 2000; Pietruszczak &

79   Mroz, 2001). The anisotropic parameter could be 'married' to an isotropic criterion (Lade, 2007; Xiao *et al.*, 2012;

80   Kong *et al.*, 2013; Lü *et al.*, 2016). Nevertheless, the combination of the fabric/microstructure tensor and the

81   stress tensor can only characterize the monotonic decrease of the shear strength as δ increases but cannot describe

82   the non-monotonic variation rules of the shear strength. Thus, a high-order equation was suggested by

83   Pietruszczak & Guo (2013) and used to modify the Lade criterion (Rodriguez & Lade, 2013). It is a mathematical

84   approach, in which a least squares method based on polynomial regression was used to determine the parameters.

85   Consequently, not all parameters in the high-order equation have physical meanings. The fourth method is based

86   on a new view that the relative orientation between the mobilized plane and the fabric direction could be used to

87   reflect the non-monotonic variation rules of the shear strength (Liu & Indraratna, 2011; Yao & Kong, 2012;

88   Oboudi *et al.*, 2016; Chang & Bennett, 2017). However, the non-monotonic variation applies only to a constant *b*.

89   These methods can describe either of the two primary characteristics revealed by experiments, but they are not

90   good enough to describe both of them in a unified way.

91   This paper presents a novel strength criterion for transversely isotropic soils based on a mobilized plane

92   approach which can couple the effects of b and δ in a more physically meaningful way. The concept of the

93   mobilized plane is introduced to reveal the shear failure mechanism and the strength rules of transversely isotropic

94   soils. Firstly, a two dimensional (2D) strength parameter is proposed by projecting the 2D microstructure tensor

95   onto the direction of the mobilized plane to reveal effects of δ on strength rules. Then, a 3D transversely isotropic

96   strength parameter is proposed by a similar projection under the 3D stress condition. The TI-NUSC is established

97   by combining the 3D strength parameter and the NUSC. Comparisons between the established TI-NUSC and the

98   experimental data available indicates the reasonable predictive capability of the TI-NUSC on accounting the

99   effects of δ and b on peak shear strength rules of transversely isotropic soils.

100   2   SHEAR FAILURE MECHANISM OF TRANSVERSELY ISOTROPIC SOILS

101   Shear failure is assumed to occur when the ratio of shear stress (τ) to normal stress (σ) acting on a specific

102   plane reaches a critical value (Matsuoka & Nakai, 1974; Wood, 1990; Pietruszczak & Mroz, 2001; Liu &

103   Indraratna, 2011; Lu *et al.*, 2017; Ma *et al.*, 2017). The specific plane can be called the mobilized plane. For

104   isotropic materials, the mobilized plane depends only on the stress values when failure occurs, and it is

105   independent of the loading direction. The failure condition can be written as follows:

106 $$\frac{\tau}{\sigma} = f\left(\Phi\right)$$ (1)

107   where Φ is a generalized material parameter. It can be the cohesive strength (c) and the internal friction angle (φ)

108   for soils, and only be φ for cohesionless soils.

109   The essences of the failure mechanism for soils are the direction of the mobilized plane and the critical value

110   that the shear-normal stress ratio can reach. For transversely isotropic materials, the determined method of the

111   mobilized plane is similar to that for isotropic materials. However, the direction of the mobilized plane and the

112 critical value that the shear-normal stress ratio can reach are all related not only to the stress values but also to the

113 relative orientation between loading direction and bedding plane. The failure condition can be expressed as

114 follows:

$$\frac{\tau}{\sigma} = f(\Phi, \Theta) \tag{2}$$

116 where Θ is a generalized direction angle. As shown in Fig. 1, three components of Θ, i.e., δ, ω and θ, are direction

117 angles of **D** in the *Oxyz* principal stress space, and two of them are independent due to the identical equation

118 $\cos^2\delta + \cos^2\omega + \cos^2\theta = 1$.

119 Experimental works under the condition that δ or θ changes with $\omega = 90°$ have been conducted to study peak

120 shear strength rules of transversely isotropic soils (Lam & Tatsuoka, 1988; Lade *et al.*, 2014; Yang *et al.*, 2016).

121 But failure criteria to fully describe the experimental findings have not been developed. How to reveal and

122 describe these experimental strength rules by the friction rule on the mobilized plane is a difficult and important

123 task.

124 *2.1 Microstructure tensor for transversely isotropic soils*

125 The direction of the mobilized plane and the critical value that the shear-normal stress ratio can reach are

126 closely related to the fabric of soils, which can be measured by a microstructure or fabric tensor **A** (Tobita, 1988;

127 Pietruszczak & Mroz, 2000) and quantified by the orientations of contacts, particles or voids (Yang *et al.*, 2008;

128 Shire *et al.*, 2013). The eigenvalues of **A** are expressed as a_1, a_2 and a_3. For transversely isotropic soils, $a_2=a_3$.

129 Thus, tensor **A** is represented as follows:

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_3 \end{bmatrix} = \eta_0 \left(\begin{bmatrix} \Omega_1 & 0 & 0 \\ 0 & \Omega_3 & 0 \\ 0 & 0 & \Omega_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \tag{3}$$

131 where $\eta_0 = (a_1 + 2a_3)/3$ is the average value of the eigenvalues of **A** and reflects the average level of material

132 properties in different directions. Ω_1 and Ω_3 are the eigenvalues of the deviatoric tensor, and $\Omega_1 + 2\Omega_3 = 0$. Ω_1 or Ω_3

133 reflects the degree of anisotropy.

134   In the plane perpendicular to the bedding plane, the transverse isotropy can be described by the 2D

135   microstructure tensor $\mathbf{A^{2D}}$, which is

136
$$\mathbf{A^{2D}} = \begin{bmatrix} a_1 & 0 \\ 0 & a_3 \end{bmatrix} = \eta_0^{2D}\left(\begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{4}$$

137   where $\eta_0^{2D} = (a_1 + a_3)/2$ is the average of the eigenvalues of $\mathbf{A^{2D}}$, and $\Omega_1 + \Omega_3 = 0$.

138   *2.2 Effect of the directionality of transversely isotropic soils on strength rules*

139   For a 2D stress state, the relative orientation between the bedding plane and mobilized plane is shown in Fig.

140   2. Fig. 2(a) and Fig. 2(b) are schematic diagrams of the physics and geometry, respectively. As shown in Fig. 2(b),

141   the z- and y-axes are the stress direction axes in the Oyz stress space, and the normal vector of the bedding plane is

142    $\mathbf{D}$, which is coaxial with the x-axis in the Oyx physical space. The vector $\mathbf{D}$ in Oyz can be represented by a

143   trigonometric function of the direction angle δ, i.e., $\mathbf{D}=(\cos\delta, \sin\delta)$. ζ is the angle between $\mathbf{D}$ and the normal vector

144   of the critical mobilized plane, and $\zeta=\min(\zeta_1, \zeta_2)$, where ζ_1 is the angle between $\mathbf{D}$ and the normal vector of

145   mobilized plane 1, and ζ_2 is the angle between $\mathbf{D}$ and the normal vector of mobilized plane 2. The normal vector of

146   the critical mobilized plane is denoted as $\mathbf{N^{2D}}$ in Oyz and as $\mathbf{\hat{N}^{2D}}$ in Oyx. The failure condition on the mobilized

147   plane for cohesionless soils can be expressed as follows:

148
$$\frac{\tau_n}{\sigma_n} = \tan\varphi \tag{5}$$

149   where $\tau_n = (\sigma_1 - \sigma_3)\sqrt{\sigma_1\sigma_3}/(\sigma_1 + \sigma_3)$ and $\sigma_n = 2\sigma_1\sigma_3/(\sigma_1 + \sigma_3)$ are the shear and normal stress acting on AC,

150   respectively. The equivalent form of Eq. (5), i.e., $\sqrt{\sigma_1/\sigma_3} = \tan(45° + \varphi/2)$, is used to determine the position of

151   AC, as shown in Fig. 2(b), and $OA = \sqrt{\sigma_1}$, $OC = \sqrt{\sigma_3}$. Thus, the normal vector of AC

152   is $\mathbf{N^{2D}} = \left(\sqrt{\sigma_3/(\sigma_1 + \sigma_3)}, \sqrt{\sigma_1/(\sigma_1 + \sigma_3)}\right)$ in Oyz and is $\mathbf{\hat{N}^{2D}} = (\hat{n}_1^{2D}, \hat{n}_3^{2D}) = (\cos\zeta, \sin\zeta)$ in Oyx.

153   The 2D transversely isotropic strength parameter is proposed by projecting the 2D microstructure tensor onto

154   the normal vector ($\mathbf{\hat{N}^{2D}}$) of the critical mobilized plane, which is similar to the definition of the anisotropic

155   parameter by Pietruszczak and Mroz (2000).

$$\eta^{2D} = \mathbf{N}^{2D} \mathbf{A}^{2D} \left(\mathbf{N}^{2D} \right)^{T} = \eta_0^{2D} \left[1 + \Omega_1 \left(\hat{n}_1^{2D} \right)^2 + \Omega_3 \left(\hat{n}_3^{2D} \right)^2 \right] \tag{6}$$

This projection of the 2D microstructure tensor onto the normal vector of the critical mobilized plane is the reflection of the effect of the 2D microstructure on frictional characteristics. That is, η^{2D} can reflect the changing rules of internal friction angle by combining Eq. (5) and Eq. (6), i.e., $\eta^{2D} = \tan\varphi|_\delta$. Where $\varphi|_\delta$ is an extended version of φ which varies with δ.

$$\frac{\tau_n}{\sigma_n} = \tan\varphi|_\delta = \eta_0^{2D} \left[1 + \Omega_1 \left(\hat{n}_1^{2D} \right)^2 + \Omega_3 \left(\hat{n}_3^{2D} \right)^2 \right] \tag{7}$$

Further, substituting $\Omega_1 + \Omega_3 = 0$ and $\left(\hat{n}_1^{2D} \right)^2 + \left(\hat{n}_2^{2D} \right)^2 = 1$ into Eq. (7) yields

$$\tan\varphi|_\delta = \eta_0^{2D} \left\{ 1 + \Omega_3 \left[1 - 2 \left(\hat{n}_1^{2D} \right)^2 \right] \right\} \tag{8}$$

where $\hat{n}_1^{2D}$ is the cosine of ζ, which can be calculated in terms of $\mathbf{N}^{2D}$ and $\mathbf{D}$.

$$\hat{n}_1^{2D} = \frac{\mathbf{N}^{2D} \cdot \mathbf{D}}{\left| \mathbf{N}^{2D} \right| \left| \mathbf{D} \right|} = \cos\delta \sqrt{\frac{\sigma_3}{\sigma_1 + \sigma_3}} + \sin\delta \sqrt{\frac{\sigma_1}{\sigma_1 + \sigma_3}} \tag{9}$$

$\hat{n}_1^{2D} = \cos\left(45^\circ + \varphi/2 - \delta \right)$ is derived by combining $\sigma_1/\sigma_3 = (1+\sin\varphi|_\delta)/(1-\sin\varphi|_\delta)$ with Eq. (9), thus, $\zeta = 45^\circ + \varphi|_\delta/2 - \delta$ is obtained. By substituting Eq. (9) into Eq. (8), we obtain an expression which describes how $\varphi|_\delta$ changes with δ.

$$\tan\varphi|_\delta = \eta_0^{2D} \left\{ 1 + \Omega_3 \left[1 - 2 \left(\cos\delta \sqrt{\frac{\sigma_3}{\sigma_1 + \sigma_3}} + \sin\delta \sqrt{\frac{\sigma_1}{\sigma_1 + \sigma_3}} \right)^2 \right] \right\} \tag{10}$$

Two material parameters η_0^{2D} and Ω_3 can be collectively solved by two linear equations which correspond $\varphi|_\delta$ at two different δ values. Taking the case of transversely isotropic Toyoura sand (Oda *et al.*, 1978) as a demonstrative example here. The friction angles $\varphi|_{\delta=0^\circ} = 49.44^\circ$ and $\varphi|_{\delta=90^\circ} = 44.22^\circ$ were obtained from plane strain tests on vertical ($\delta=0^\circ$) and horizontal ($\delta=90^\circ$) samples as shown in Fig. 3 for $\sigma_3=196$ kPa. $\left(\sigma_1/\sigma_3\right)\big|_{\delta=0^\circ} = 7.324$ and $\left(\sigma_1/\sigma_3\right)\big|_{\delta=90^\circ} = 5.610$ Therefore, $\eta_0^{2D}=1.067$ and $\Omega_3=0.126$ can be solved from Eq. (10).

The variation of η^{2D} associated with angle ζ_1 can be obtained from Eq. (8). As shown in Fig. 4, η^{2D} for mobilized plane 1 increases monotonously as ζ_1 increases from 0° to 90°. But for mobilized plane 2, the η^{2D} increases first and then decreases. Soil will fail along the critical mobilized plane which is close to the bedding

178   plane (i.e. the plane of weakness). The critical mobilized plane is mobilized plane 1 when $0°≤\zeta_1<69.72°$ and is

179   shown as the red solid line. The critical mobilized plane can be either of the two mobilized plane and $\zeta_2=\zeta_1$, when

180   $\zeta_1=69.72°$. After this, $\zeta_2<\zeta_1$, and the critical mobilized plane becomes mobilized plane 2 when $69.72°<\zeta_1≤90°$ and

181   is shown as the green solid line in Fig. 4. The closer the mobilized plane is to the bedding plane, the more easily

182   the soil fails.

183   A further understanding of the failure mechanism of transversely isotropic soils can be obtained by the

184   correspondence between Fig. 4 and Fig. 5 in terms of the relationship between ζ_1 and δ ($\zeta_1=45°+\varphi|_\delta/2-\delta$).

185   Correspondingly, the relationship between η^{2D} and δ can be calculated using Eq. (10), as shown in Fig. 5. When

186   $0°<\delta<90°$, the critical mobilized plane is mobilized plane 1 and η^{2D} initially decreases and then increases slightly

187   with δ. A_δ-B_δ-C_δ in Fig. 5 corresponds to A_ζ-B_ζ-C_ζ in Fig. 4. When $-90°<\delta<0°$, the critical mobilized plane will be

188   mobilized plane 2, and the corresponding relationship illustrated by the green solid line in Fig. 5 is opposite to that

189   of $0°<\delta<90°$. The theoretical predictions ($\varphi|_\delta=\arctan\eta^{2D}$) can capture the experimental observed non-monotonic

190   variation of φ with the increase of δ for the transversely isotropic sand (Oda *et al.*, 1978; Matsuoka *et al.*, 1984;

191   Tatsuoka *et al.*, 1990) as shown in Fig. 6.

192   3   3D TRANSVERSELY ISOTROPIC STRENGTH PARAMETER

193   Based on the understanding of the failure mechanism, a comprehensive and unified 3D transversely isotropic

194   strength parameter can be proposed by introducing the concept of the mobilized plane into the anisotropic

195   parameter. For isotropic soils under a 3D stress state, the position of the mobilized plane in the stress space is

196   closely related to the stress values only. The loading direction does not affect the shear strength characteristics,

197   and the φ-b relationship is constant. For transversely isotropic soils, soils themselves possess directionality. The

198   direction of the mobilized plane is affected by stress values and loading direction. The relative orientation between

199   the bedding plane and the mobilized plane will affect the peak shear strength characteristics (Liu & Indraratna,

200   2011; Yao & Kong, 2012) and needs to be analysed.

201   *3.1   Relative orientation between the bedding plane and mobilized plane*

202   The arbitrary relative orientation between the bedding plane and the mobilized plane can be described using

203   two angles (Lam & Tatsuoka, 1988; Mroz & Maciejewski, 2002; Lü *et al.*, 2016). However, for the condition that

204   $\omega=90°$, that is x and $\tilde{x}$ in coincide as shown in Fig. 7, one angle δ or θ ($\delta+\theta=90°$) is sufficient to describe the

205   relative orientation. The physical coordinate system $O\tilde{x}\tilde{y}\tilde{z}$ is introduced for describing the direction of the

206   mobilized plane in physical space, where the $\tilde{z}$-axis is coaxial with the normal vector **D** of the bedding plane.

207   The direction angle of **D** in $Oxyz$ is δ; thus, **D**$=(\cos\delta, 0, \sin\delta)$. The normal vector of the mobilized plane ABC is **N**

208   in $Oxyz$, and **N**$=(n_x, n_y, n_z)$. The direction cosine n_i ($i=x, y, z$) with respect to the σ_i axis can be expressed as

209   follows:

$$n_i = \sqrt{\frac{\sigma_x \sigma_y \sigma_z}{\sigma_i \left(\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x\right)}} \tag{11}$$

211   The normal vector of the mobilized plane in $O\tilde{x}\tilde{y}\tilde{z}$ is expressed as $\tilde{\mathbf{N}}=\left(\tilde{n}_{\tilde{x}}, \tilde{n}_{\tilde{y}}, \tilde{n}_{\tilde{z}}\right)$. As shown in Fig. 7,

212   $\tilde{n}_{\tilde{z}} = \cos\zeta$ and ζ is the angle between **D** and **N**. Thus, $\tilde{n}_{\tilde{z}} = n_z \cos\delta + n_y \sin\delta$ and can be used to describe the

213   relative orientation of **D** and **N**.

214   The relative orientation between the normal of the bedding plane and the normal of the mobilized plane is

215   closely related to the stress distribution that acts on a transversely isotropic soil element. Due to the directionality

216   of transversely isotropic soils, effects of stress distribution is not symmetrical about the three principal stress axes.

217   We can partition the deviatoric plane into six stress distribution sectors as shown in Fig. 8. The stresses σ_1, σ_2 and

218   σ_3 acted on a transversely isotropic soil element are distributed in different directions in each sector. The

219   symmetry of stress distributions is actually associated with δ. The symmetry axis will be the z-axis when $\delta=0°$, the

220   x-axis when $\delta=45°$ and the y-axis when $\delta=90°$. Accordingly, strength characteristics possess the same symmetry

221   with the stress distribution.

222   *3.2   A new 3D transversely isotropic strength parameter*

223   The method of projection (Pietruszczak & Mroz, 2000; Pietruszczak & Mroz, 2001) is extended to 3D to

224   propose a 3D transversely isotropic strength parameter. The effect of the 3D microstructure on the frictional

225   strength is taken into account by projecting the 3D microstructure tensor $\mathbf{A}$ onto the normal vector $\overset{\circ}{N}$ of the

226   mobilized plane. When combined with the transversely isotropic condition, a 3D strength parameter η can be

227   obtained:

228
$$\eta = \eta_0 \left[1 + \Omega_3 \left(1 - 3m_\rho^2 \right) \right] \tag{12}$$

229   where η_0 reflects the average value of η, and Ω_3 reflects the degree of anisotropy. m_ρ is a function that used to

230   reflect the change rules of η with the increase of δ. To reflect the strength characteristics of different axes related

231   to δ when $\omega=90°$, a more comprehensive function with the following two features is proposed: *(i)* it is

232   independent of δ along the *x*-axis ($\sigma_z=\sigma_y$); *(ii)* the change rules under the condition that $\sigma_x=\sigma_y$ or $\sigma_x=\sigma_z$ are

233   adjustable. The function that satisfies these two features is the key to develop a 3D transversely isotropic strength

234   parameter.

235   The following interpolation function m_ρ is used in this paper for describing these two features

236
$$m_\rho = \rho\psi m_{\mathrm{I}} + \left(1 - \rho\psi \right) m_{\mathrm{II}} \tag{13}$$

237   where ρ is the distribution coefficient that controls the variability of η as δ increases. ψ is a stress state dependent

238   function and its expression is

239
$$\psi = \frac{\left(\sigma_z - \sigma_y \right)^2}{\left(\sigma_z - \sigma_x \right)^2 + \left(\sigma_x - \sigma_y \right)^2 + \left(\sigma_y - \sigma_z \right)^2} \tag{14}$$

240   where $\psi=0$ when $\sigma_z=\sigma_y$, and Eq. (13) becomes $m_\rho = m_{\mathrm{II}}$; $\psi=1/2$ when $\sigma_x=\sigma_y$ or $\sigma_x=\sigma_z$, and Eq. (13) becomes

241    $m_\rho = 1/2\,\rho m_{\mathrm{I}} + \left(1 - 1/2\,\rho \right) m_{\mathrm{II}}$.

242   In Eq. (13), $m_{\mathrm{I}} = n_{\overset{\circ}{z}}$ is the direction cosine of $\overset{\circ}{N}$, and its expression is $m_{\mathrm{I}} = n_z \cos\delta + n_y \sin\delta$. m_{II} ensures

243   that the strength parameter η along the *x*-axis ($\sigma_z=\sigma_y$) is independent of δ, specified by the relationship:

244
$$m_{\mathrm{II}} = n_z \cos^2\delta + n_y \sin^2\delta \tag{15}$$

245   By substituting Eqs. (14)-(15) together with $m_{\mathrm{I}} = n_z \cos\delta + n_y \sin\delta$ into Eq. (13), a comprehensive expression

246   of m_ρ that satisfies the two features mentioned above can be derived. Then, the expression of η is obtained from

247   Eq. (12) as follows:

248
$$\eta = \eta_0 + \eta_0\Omega_3 \left\{ 1 - 3\left[\rho\psi\left(n_z \cos\delta + n_y \sin\delta\right) + \left(1 - \rho\psi\right)\left(n_z \cos^2\delta + n_y \sin^2\delta\right)\right]\right\} \tag{16}$$

249   It can be used to reflect strength variation rules for different stress distribution. For the stress state of the

250   x-axis ($\sigma_z=\sigma_y$), $\psi=0$ and $n_z=n_y=\sigma_x/(2\sigma_x+\sigma_y)$, η describes the strength parameter variation rules of the x-axis, and is

251   expressed as follows:

252
$$\eta = \eta_0\left[1 + \Omega_3\left(1 - m_{\mathrm{II}}\right)\right] = \eta_0\left[1 + \Omega_3\left(1 - 3n_z\right)\right] \tag{17}$$

253   Eq. (17) indicates that η is independent of δ when $\sigma_z=\sigma_y$. For the stress state of the z-axis ($\sigma_x=\sigma_y$) or y-axis ($\sigma_x=\sigma_z$),

254     $\psi=1/2$ is obtained from Eq. (14) and Eq. (12) becomes

255
$$\eta = \eta_0 + \eta_0\Omega_3 \left\{ 1 - 3\left[\frac{1}{2}\rho\left(n_z \cos\delta + n_y \sin\delta\right) + \left(1 - \frac{1}{2}\rho\right)\left(n_z \cos^2\delta + n_y \sin^2\delta\right)\right]^2\right\} \tag{18}$$

256   As shown in Fig. 9, the trend of η changes from monotonic for $\rho=0$ (the dashed line) to non-monotonic for $\rho=2$

257   (the solid line) when $\sigma_x=\sigma_y$.

258   The 3D transversely isotropic strength parameter η possesses the two features mentioned above and makes it

259   possible to describe the coupled effects of b and δ in a unified way. In the following section η is combined with an

260   isotropic strength criterion to develop a transversely isotropic strength criterion.

261   4   TRANSVERSELY ISOTROPIC NONLINEAR UNIFIED STRENGTH CRITERION

262   The NUSC (Yao et al., 2004; Lu, 2006; Du et al., 2010; Wang et al., 2018) is applicable to various isotropic

263   geomaterials, including soil, concrete, and rock. The failure surface of the NUSC in principal stress space is

264   continuous, smooth and convex as shown in Fig. 10. Strength curve on the deviatoric plane, which reflects the

265   effect of b on the shear strength, can continuously change from the Drucker-Prager (D-P) strength circle (upper

266   bound) to the Matsuoka-Nakai (M-N) strength curved triangle (lower bound). By using a power function, the

267   cohesion effect, friction effect and hydrostatic pressure effect can be characterized on the meridian plane.

268   Expression of the NUSC can be written as

$$\frac{\overline{q}_\alpha}{\overline{p}} = \alpha \frac{3\sqrt{\overline{I}_1^2 - 3\overline{I}_2}}{\overline{I}_1} + (1-\alpha)\frac{6}{3\sqrt{(\overline{I}_1\overline{I}_2 - \overline{I}_3)/(\overline{I}_1\overline{I}_2 - 9\overline{I}_3)} - 1} = M_f \qquad (19)$$

270   where $\overline{q}_\alpha$ is the equivalent shear strength, $\overline{p} = (\overline{\sigma}_1 + \overline{\sigma}_2 + \overline{\sigma}_3)/3$ is mean stress in the transitional stress space. α

271   is a material parameter that reflects the triaxial extension-compression strength ratio at the reference stress, and M_f

272   is the failure stress ratio at the reference stress. $\overline{I}_1$, $\overline{I}_2$ and $\overline{I}_3$ are the first, second and third stress invariants in

273   the transitional stress space respectively, defined as

$$\begin{cases} \overline{I}_1 = \overline{\sigma}_1 + \overline{\sigma}_2 + \overline{\sigma}_3 \\ \overline{I}_2 = \overline{\sigma}_1\overline{\sigma}_2 + \overline{\sigma}_2\overline{\sigma}_3 + \overline{\sigma}_3\overline{\sigma}_1 \\ \overline{I}_3 = \overline{\sigma}_1\overline{\sigma}_2\overline{\sigma}_3 \end{cases} \qquad (20)$$

275   The stress tensor in the transitional stress space is expressed as

$$\overline{\sigma}_{ij} = \sigma_{ij} + \left[p_r \left(\frac{p + \sigma_0}{p_r} \right)^n - p \right] \delta_{ij} \qquad (21)$$

277   where $p = (\sigma_1 + \sigma_2 + \sigma_3)/3$ is the mean stress, p_r is the reference stress, σ_0 is the three-dimensional tensile

278   strength, n is the hydrostatic pressure effect index and δ_{ij} is the Kronecker symbol.

279   *4.1 Nonlinear unified strength criterion for transversely isotropic soils*

280   Due to the parameter independence and extensibility of the NUSC, the 3D strength parameter η can be easily

281   introduced into the NUSC. For cohesionless soils, $\sigma_0=0$ and $n=1$, the simplified form of the NUSC adopted in this

282   study can be expressed as

$$\frac{q_\alpha}{p} = \alpha \frac{3\sqrt{I_1^2 - 3I_2}}{I_1} + (1-\alpha)\frac{6}{3\sqrt{(I_1I_2 - I_3)/(I_1I_2 - 9I_3)} - 1} = M_f \qquad (22)$$

284   where M_f degrades into the stress ratio in normal stress space when $\sigma_0=0$ and $n=1$. It is the reflection of the

285   frictional characteristics, which can be calculated by the internal friction angle φ_c under triaxial compressive

286   conditions, i.e., $M_f=6\sin\varphi_c/(3-\sin\varphi_c)$. I_1, I_2 and I_3 are the first, second and third stress invariants in the normal

287   stress space respectively, defined as

$$\begin{cases} I_1 = \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 = \sigma_1\sigma_2\sigma_3 \end{cases} \tag{23}$$

289   The frictional characteristics of transversely isotropic soils are strongly dependent on the loading direction

290   (Imam *et al.*, 2002; Lade, 2008; Gao *et al.*, 2010). Thus, taking η as an extension of the failure stress ratio M_f, the

291   TI-NUSC can be expressed as follows:

$$\frac{q_\alpha}{p} = \frac{1}{p}\left[\alpha\sqrt{I_1^2 - 3I_2} + (1-\alpha)\frac{2I_1}{3\sqrt{(I_1 I_2 - I_3)/(I_1 I_2 - 9I_3)} - 1} \right] = \eta_0\left[1 + \Omega_3\left(1 - 3m_\rho^2\right)\right] \tag{24}$$

293   Fig. 11 provides a graphical comparison of the NUSC and TI-NUSC when $\delta=0°$ at the same internal friction

294   angle $\left.\varphi_c\right|_{\delta=0°}$. It clearly shows that the TI-NUSC strength surface is asymmetrical with respect to σ_x- and σ_y-axes.

295   *4.2 Coupled effects of b and δ on strength rules*

296   The effects of *b* on strength rules within each of the six stress distribution sectors introduced in Fig. 8 are the

297   same for isotropic soils and can be well described by the NUSC. But for transversely isotropic soils, effects of *b*

298   are no longer symmetrical about three principal stress axes at the same time. Strength rules are closely related to

299   the stress distributions in the six sectors when δ is constant. The coupled effects of *b* and δ make transversely

300   isotropic soils exhibit the two primary peak shear strength characteristics described in the introduction.

301   4.2.1   Effects of *b* on strength rules when $\delta=0°$

302   As shown in Fig. 12, three stress distribution sectors are sufficient due to the symmetry with respect to the

303   σ_z-axis when $\delta=0°$. In this case, Eq. (13) becomes

$$m_\rho = n_z = \sqrt{\frac{\sigma_x\sigma_y}{\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x}} \tag{25}$$

305   The stresses (σ_1, σ_2 and σ_3) are distributed in these three sectors as shown in Fig. 12. Thus, the specific

306   expression of m_ρ can be obtained from Eq. (25) in terms of $R=\sigma_1/\sigma_3$ and *b* in these sectors as follows:

307   In sector I ($\sigma_x=\sigma_2$, $\sigma_y=\sigma_3$ and $\sigma_z=\sigma_1$):

308
$$m_\rho = \sqrt{\frac{(bR-b+1)}{R(bR-b+1)+(bR-b+1)+R}}$$
(26)

309 In sector II ($\sigma_x=\sigma_1$, $\sigma_y=\sigma_3$ and $\sigma_z=\sigma_2$):

310
$$m_\rho = \sqrt{\frac{R}{R(bR-b+1)+(bR-b+1)+R}}$$
(27)

311 In sector III ($\sigma_x=\sigma_1$, $\sigma_y=\sigma_2$ and $\sigma_z=\sigma_3$):

312
$$m_\rho = \sqrt{\frac{R(bR-b+1)}{R(bR-b+1)+(bR-b+1)+R}}$$
(28)

313   η_0 and Ω_3 in Eq. (24) can be obtained from internal friction angles measured under triaxial compression

314 perpendicular and along the bedding plane ($\varphi_c|_{\delta=0^\circ}$ and $\varphi_c|_{\delta=90^\circ}$). For conventional triaxial compression condition

315 (i.e., $b=0$), Eq. (24) can be simplified in terms of the principal stress ratio R_c (= $(1+\sin\varphi_c)/(1-\sin\varphi_c)$)

316
$$\frac{q_\alpha}{p} = \eta_c = \frac{3(R_c-1)}{R_c+2} = \eta_0\left[1+\Omega_3\left(1-3m_\rho^{\ 2}\right)\right]$$
(29)

317 where $\eta_c|_{\delta=0^\circ}$ and $\eta_c|_{\delta=90^\circ}$ can be obtained from the left side of Eq. (29), and $m_\rho|_{\delta=0^\circ}$ and $m_\rho|_{\delta=90^\circ}$ can be

318 obtained from Eq. (26) and Eq. (28), respectively. These are substituted into Eq. (29) and a binary system of linear

319 equations is thus obtained:

320
$$\begin{cases} \eta_c|_{\delta=0^\circ} = \eta_0\left[1+\Omega_3\left(1-3m_\rho^2|_{\delta=0^\circ}\right)\right] \\ \eta_c|_{\delta=90^\circ} = \eta_0\left[1+\Omega_3\left(1-3m_\rho^2|_{\delta=90^\circ}\right)\right] \end{cases}$$
(30)

321 η_0 and Ω_3 are then collectively solved by Eq. (30).

322   On the same deviatoric plane, $p=200$ kPa and $\alpha=1/3$ is taken as a special case for demonstration. For a fixed

323 value of $\varphi_c|_{\delta=0^\circ} = 35^\circ$ and values of $\varphi_c|_{\delta=90^\circ}$ between 30° and 40°, the solved values of η_0 and Ω_3 are listed in

324 Table 1.

325   As shown in Fig. 13, the fixed value of $\varphi_c|_{\delta=0^\circ} = 35^\circ$ makes each strength curve on the deviatoric plane pass

326 through the same point, and the TI-NUSC curves are symmetrical with respect to σ_z only. The NUSC curve is

327 shown by a black solid line when $\varphi_c|_{\delta=90^\circ} = 35^\circ$. As $\varphi_c|_{\delta=90^\circ}$ increases from 35° to 40°, η_0 increases, and Ω_3

328   decreases from 0. The size of the corresponding TI-NUSC curves increase, which means that the shear strength

329   along the bedding plane is higher than that along the normal of the bedding plane. As $\varphi_c\big|_{\delta=90°}$ decreases from 35°

330   to 30°, η_0 decreases, and Ω_3 increases from 0. The size of the corresponding TI-NUSC curves decrease, as the

331   bedding plane is the plane of weakness.

332   As indicated by the solid dots in Fig. 13, the maximum shear strength dots of the TI-NUSC are close to but

333   not directly on the σ_x- and σ_y-axes. The maximum shear strength dots are in sector II when $\Omega_3>0$, and the

334   maximum shear strength dots are in sector III when $\Omega_3<0$. The maximum shear strength move further away from

335   the axes as the absolute value of Ω_3 increases.

336   Corresponding to Fig. 13, Fig. 14 shows the φ-b curves for different values of $\varphi_c\big|_{\delta=90°}$. In sector I, σ_1 is

337   perpendicular to the bedding plane. The φ-b curves for different $\varphi_c\big|_{\delta=90°}$ are almost the same. Thus, the effect of

338   anisotropy on φ can be ignored in this sector. In sector II, σ_2 is perpendicular to the bedding plane, and the effect

339   of anisotropy increases significantly as b decreases from 1 to 0. In sector III, σ_3 is perpendicular to the bedding

340   plane. There are a series of approximately parallel φ-b curves for different $\varphi_c\big|_{\delta=90°}$. The φ-b curves and the

341   corresponding stress distribution in sectors IV, V and VI are symmetrical to these in sectors III, II and I,

342   respectively. The NUSC can be used to approximately describe the strength behaviour when the stress distribution

343   is consistent with that in sector I. In sector III, the approximately parallel φ-b curves could also be estimated by

344   multiplying φ_c in the NUSC by a scaling factor, particularly when the degree of anisotropy is small. However, the

345   transverse isotropy must be considered by η when the stress distribution is consistent with that in sector II, that is,

346   the isotropic strength criterion is incapable of describing behaviour under these conditions.

347   The effects of η_0 or Ω_3 can also be obtained, in a similar way to previous analysis. For a given value of

348    $\Omega_3=0.186$, the effect of η_0 on the strength curve on the deviatoric plane is shown in Fig. 15. The strength curves

349   vary from a quasi-circle to a curved triangle on the same deviatoric plane as η_0 increases. For a given value of

350    $\eta_0=1.267$, the effect of Ω_3 on the strength curve is shown in Fig. 16. The strength curves seem to be pulled up as

351   Ω_3 increases.

        352   The value of α, which reflects the triaxial extension-compression strength ratio, is constant for a given

353   geomaterial. As α changes from 0 to 1, the corresponding TI-NUSC curves are shown in Fig. 17 for

354     $\varphi_c\big|_{\delta=0^o} = 35^o$ and $\varphi_c\big|_{\delta=90^o} = 30^o$. There are a series of continuous, smooth and convex strength curves, in which

355   the lower bound (α=0) and upper bound (α=1) are derived from the M-N criterion and D-P criterion, respectively.

356   The maximum shear strength dots for α=0 are in sector II rather than on the σ_x- and σ_y-axes. As α increases, the

357   dots in sector II move further away from the σ_x- and σ_y-axes. For α=1, the maximum shear strength dots return to

358   the -σ_x- and -σ_y-axes.

359   4.2.2   Effects of δ on strength rules

360   Four parameters η_0=1.267, Ω_3=0.186, α=1/3 and ρ=1.200 are taken as a case study to analyse the effect of δ

361   on the strength rules of transversely isotropic soils. The strength curves on the deviatoric plane of p=200 kPa with

362   the different δ values are predicted in Fig. 18. The strength curves of δ (=0°, 22.5°, 45°) shown in Fig. 18(a) and

363   those of 90°-δ (=90°, 67.5°, 45°) shown in Fig. 18(b) are symmetrical about the σ_x-axis. The maximum shear

364   strength dots of the strength curves close to the principal stress axes also change with δ, as shown by solid dots in

365   Fig. 18. The corresponding φ-b curves are shown in Fig. 19. The φ-b curves in sectors I-VI of Fig. 19(a) are the

366   same with that in sectors IV-III-II-I-VI-V of Fig. 19(b), which also corresponds to the symmetry of strength

367   curves. As δ increases from 0° to 90°, the variations of φ in sectors II, I and VI are predicted as shown in Fig. 20.

368   The variations of φ as δ increases in sectors III, IV and V are inversely symmetric with those in sectors II, I and VI,

369   respectively.

370   4.2.3   Coupled effects of b and δ on strength rules

371   The strength curves shown in Fig. 18 can be extended into a strength surface shown in Fig. 21, which shows

372   that the strength curve on the deviatoric plane evolves with δ. The non-monotonic variation of shear strength

373   under the triaxial compression as δ increases is illustrated by this strength surface. Correspondingly, the coupled

374 effect of b and δ on the strength parameter φ can be described by the φ-δ-b surface as shown in Fig. 22, which is

375 the combination of Fig. 19 and Fig. 20. It is shown that the φ-b curve evolves with δ and the φ-δ curve changes

376 with b in each of six sectors. These two variation rules shown in Fig. 22 are associated with the material direction

377 and stress condition, respectively.

378 *4.3 Determined methods of material parameters*

379 There are four material parameters, i.e., η_0, Ω_3, α and ρ, with clear physical meaning in the established

380 TI-NUSC. These material parameters can be determined by at least four specific types of test results, as shown in

381 Fig. 23.

382 4.3.1 Determination of η_0 and Ω_3

383 $\varphi_c\big|_{\delta=0^\circ}$ and $\varphi_c\big|_{\delta=90^\circ}$ are used to solve η_0 and Ω_3 from Eq. (30). $\varphi_c\big|_{\delta=0^\circ}$ and $\varphi_c\big|_{\delta=90^\circ}$ are the internal friction

384 angles obtained under conventional triaxial compression from samples with $\delta=0^\circ$ and $\delta=90^\circ$, respectively, as

385 shown in Fig. 23(a) and (b).

386 4.3.2 Determination of α

387 The method for calculating α for isotropic soils (Yao *et al.*, 2004; Lu, 2006; Du *et al.*, 2010) can be extended

388 for transversely isotropic soils with a constant direction angle δ as follows:

389
$$\alpha = \left| \frac{\left(\dfrac{q_e}{q_c}\right) - \left(\dfrac{q_e}{q_c}\right)_{MN}}{\left(\dfrac{q_e}{q_c}\right)_{DP} - \left(\dfrac{q_e}{q_c}\right)_{MN}} \right|_\delta \tag{31}$$

390 Under the condition that $\delta=0^\circ$ and $q_c = \left(q_c\right)_{MN} = \left(q_c\right)_{DP}$, Eq. (31) can be transformed into Eq. (32).

391
$$\alpha = \frac{\eta_e\big|_{\delta=0^\circ} - \eta_{MN}\big|_{\delta=0^\circ}}{\eta_{DP}\big|_{\delta=0^\circ} - \eta_{MN}\big|_{\delta=0^\circ}} \tag{32}$$

392 In Eq. (32), $\eta_{DP}|_{\delta=0^\circ}=3(R_e|_{\delta=0^\circ}-1)/(2R_e|_{\delta=0^\circ}+1)$, $\eta_{MN}|_{\delta=0^\circ}=3(R_e|_{\delta=0^\circ}-1)/(R_e|_{\delta=0^\circ}+2)$ and

393 $\eta_e|_{\delta=0^\circ}=\eta_0[1+\Omega_3(R_e|_{\delta=0^\circ}-1)/(R_e|_{\delta=0^\circ}+2)]$, where $R_e\big|_{\delta=0^\circ} = \left(1+\sin\varphi_e\big|_{\delta=0^\circ}\right)/\left(1-\sin\varphi_e\big|_{\delta=0^\circ}\right)$. $\varphi_e\big|_{\delta=0^\circ}$ is the friction

394 angle obtained from the sample with $\delta=0^\circ$ under conventional triaxial extension, as shown in Fig. 23(c).

395   4.3.3   Determination of ρ

396   ρ is required to capture the variation of shear strength as δ increases from 0° to 90°. Samples with 0°<δ<90°

397   as shown in Fig. 23(d) are tested under triaxial compression. The obtained internal friction angle $\left.\varphi_{c}\right|_{\delta}$ is used to

398   calculate ρ, and the expression can be transformed from Eq. (24).

399
$$\rho = \frac{2}{\left(m_{\mathrm{I}} - m_{\mathrm{II}}\right)} \left(\sqrt{\frac{\eta_0 + \eta_0 \Omega_3 - \left.\eta_c\right|_\delta}{3\eta_0\Omega_3}} - m_{\mathrm{II}} \right) \tag{33}$$

400   In Eq. (33), $\left.\eta_c\right|_\delta = \left.\left(q_\alpha/p\right)_c\right|_\delta = 3\left(\left.R_c\right|_\delta - 1\right)/\left(\left.R_c\right|_\delta + 2\right)$, where $\left.R_c\right|_\delta = \left(1 + \sin\left.\varphi_c\right|_\delta\right)/\left(1 - \sin\left.\varphi_c\right|_\delta\right)$. m_{I} and m_{II} can

401   be obtained from Eq. (15) based on $\left.R_c\right|_\delta$.

402   5   VERIFICATION

403   An experimental database has been compiled to study the coupled effects of δ and b on the peak shear

404   strength behaviour of transversely isotropic soils when ω=90° and to verify the proposed criterion. The database

405   comprises Toyoura sand (Lam & Tatsuoka, 1988), Nevada sand (Lade *et al.*, 2014) and Leighton Buzzard sand

406   (Yang *et al.*, 2016). Material parameters are determined based on the determined method of this paper and listed in

407   Table 2.

408   *5.1   Toyoura sand*

409   Air-pluviated Toyoura sand was tested by Lam & Tatsuoka (1988) to study the effects of b and initial

410   anisotropic fabric on the peak shear strength. Drained tests in triaxial compression (b=0), plane strain (b≈0.3) and

411   triaxial extension (b=1) were performed when ω=90° and δ=0°, 30°, 60° and 90°. The shear strength data for

412   σ_3=98 kPa are summarized and found to be strongly influenced by b and δ. Five test results circled in Fig. 24 are

413   used to determine the material parameters listed in Table 2, where the two points labelled 4 are the same test

414   results displayed at different locations. Points 1 and 4 are used to determine η_0 and Ω_3. Subsequently, α can be

415   obtained from point 5 by combining values of η_0 and Ω_3. Four points, i.e., 1, 2, 3 and 4, are used to determine ρ.

416   The evolution of the TI-NUSC curves with δ for Toyoura sand is shown in Fig. 24. The test data and predicted

417   curve on the deviatoric plane are compared in Fig. 25 for the case of $\delta=0°$. The isotropic NUSC strength curve

418   with the same $\varphi_c|_{\delta=0°}$ is also shown. The availability of the TI-NUSC is verified by comparing the test data with

419   the predicted strength surface and curve in Fig. 24 and Fig. 25. The comparison shows that the peak shear strength

420   variation rules of Toyoura sand can be well captured by the proposed strength criterion.

421   *5.2 Nevada sand*

422   A total of 44 drained torsion shear tests on fine Nevada sand were performed by Lade *et al.* (2014) with an

423   initial effective confining stress of 100 kPa and with 25 combinations of constant b and δ. The open circles shown

424   in Fig. 26 are the original data from Figs. 8-12 in the original paper (Lade *et al.*, 2014). If there is more than one

425   point around the target δ- and b-values, the average of the internal friction angles is taken for comparison, as

426   indicated by the solid squares in Fig. 26. The 25 test points obtained are shown by the red cube in Fig. 27. Four

427   test results are sufficient to determine the material parameters. However, additional test data are needed because of

428   the discrete nature of the test results. Ten test points that are circled and numbered as 1-10 in Fig. 27 are used to

429   determine the material parameters. Points 1 and 5 are used to determine η_0 and Ω_3. After that, point 6 together

430   with 1 and point 10 together with 5 are used to determine α. The triaxial compression test points 1, 2, 3, 4 and 5

431   can be used to obtain $\rho|_{b=0}=1.08$ as listed in Table 2, and the triaxial extension test points 7, 8, and 9 together with

432   η_0, Ω_3 and α can be used to determine $\rho|_{b=1}=2.224$. Thus, the average value $\rho=1.652$ can be obtained to capture the

433   variation of the internal friction angle with δ. Fig. 27 and Fig. 28 give a comprehensive comparison between the

434   test data and predicted curves. The effects of δ and b on φ of transversely isotropic Nevada sand shown in Fig. 27

435   can be well captured by the proposed criterion. The effects of δ and b on peak shear strength can be captured by

436   the evolution of the strength curve, as shown in Fig. 28. Fig. 29 shows a comparison between the test data and the

437   strength curves on the deviatoric plane predicted by the TI-NUSC and the isotropic NUSC. The overall

438   satisfactory performance of the TI-NUSC in predicting the strength rules for Nevada sand can be observed at each

439   δ value.

5.3 Leighton Buzzard sand

441   A series of hollow cylinder torsional shear tests were performed by Yang *et al.* (2016) on dense Leighton

442 Buzzard sand with various combinations of δ and b. The angle δ of the samples were 0°, 15°, 30°, 60°, 75° and

443 90°. The intermediate principal stress coefficient b was set as 0, 0.2, 0.5 and 1 with a constant mean effective

444 stress of p=200 kPa. The test results are shown by dots in Fig. 30-Fig. 32. Seven points circled and numbered as

445 1-7 in Fig. 30 are used to determine the material parameters. Points 1 and 6 are used to determine η_0 and Ω_3.

446 Points 2, 3, 4 and 5 together with η_0 and Ω_3 can determine ρ. α is determined by points 1 and 7. The determined

447 material parameters are listed in Table 2. The comparison between the test data and the φ-δ-b surface predicted by

448 the TI-NUSC is shown in Fig. 30. The effects of δ and b on peak shear strength are captured by the evolution of

449 the TI-NUSC strength curve as shown in Fig. 31. The variation rules that the strength parameter φ and the peak

450 shear strength decrease initially decreases and then increases when b is constant are well captured as shown in Fig.

451 30 and Fig. 31. The clearer comparisons between the test data and the strength curves predicted by the TI-NUSC

452 and the NUSC with $\varphi_c|_{\delta=0°}$ =37.23° on the deviatoric plane are shown in Fig. 32. The predicted surfaces and

453 curves can properly capture the coupled effects of b and δ in a unified way as shown by comparisons with

454 experimental data.

455 6   CONCLUSIONS

456   In this paper, the failure mechanism of soils was introduced and analysed for transversely isotropic soils. The

457 essences of the failure mechanism, i.e., the direction of the mobilized plane and the critical value that the

458 shear-normal stress ratio acting on that plane can reach are all affected by the microstructure of transversely

459 isotropic soils. Based on the understanding of the failure mechanism, the effects of the directionality of

460 transversely isotropic soils on shear strength rules were analysed. And further, a comprehensive and unified 3D

461 transversely isotropic strength parameter was proposed. The proposed strength parameter can reflect the variation

462 of the shear strength with the direction angle of the bedding plane. The TI-NUSC was then established by

combining the 3D transversely isotropic strength parameter with the NUSC. Two primary strength characteristics

that the asymmetry of the strength curve with respect to the three principal stress axes and variability of shear

strength with the bedding plane direction angle δ can thus be reflected in a unified way.

Four material parameters are included in the TI-NUSC. These material parameters have clear physical

meanings and can be easily determined by conventional triaxial tests. The established TI-NUSC was verified

favourably against available test data of transversely isotropic soils. Furthermore, the strength surface is

continuous, smooth and convex and could be used to construct elastoplastic constitutive models for transversely

isotropic soils. This mobilized plane approach for establishing transversely isotropic strength criteria could also be

suitable for developing strength criteria for other geomaterials, like concrete or rock.

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REFERENCES

Abelev, A. V., and Lade, P. V. (2004). Characterization of failure in cross-anisotropic soils. *Journal of Engineering Mechanics, ASCE*. **130**, No. 5, 599-606.

Casagrande, A., and Carillo, N. (1944). Shear failure of anisotropic materials. *Boston Society of Civil Engineering Journal*. **31**, No. 2, 74-87.

Chang, C. S., and Bennett, K. (2017). Micromechanical modeling for the deformation of sand with noncoaxiality between the stress and material axes. *Journal of Engineering Mechanics*. **143**, No. 1, 1-15.

Dafalias, Y. F., Papadimitriou, A. G., and Li, X. S. (2004). Sand plasticity model accounting for inherent fabric anisotropy. *Journal of Engineering Mechanics, ASCE*. **130**, No. 11, 1319-1333.

Du, X. L., Lu, D. C., Gong, Q. M., and Zhao, M. (2010). Nonlinear unified strength criterion for concrete under three-dimensional stress states. *Journal of Engineering Mechanics, ASCE*. **136**, No. 1, 51-59.

Fu, P. C., and Dafalias, Y. F. (2011). Study of anisotropic shear strength of granular materials using DEM simulation. *International Journal for Numerical and Analytical Methods in Geomechanics*. **35**, No. 10, 1098-1126.

Gao, Z. W., Zhao, J. D., and Yao, Y. P. (2010). A generalized anisotropic failure criterion for geomaterials. *International Journal of Solids and Structures*. **47**, No. 22-23, 3166-3185.

Gao, Z. W., and Zhao, J. D. (2012). Efficient approach to characterize strength anisotropy in soils. *Journal of Engineering Mechanics, ASCE*. **138**, No. 12, 1447-1456.

Imam, S. M. R., Chan, D. H., Robertson, P. K., Morgenstern, N. R., and Abelev, A. V. (2002). Effect of anisotropic

495   yielding on the flow liquefaction of loose sand. *Soils and Foundations*. **42**, No. 2, 33-44.

496   Kirkgard, M. M., and Lade, P. V. (1993). Anisotropic three-dimensional behavior of a normally consolidated clay.

497   *Canadian Geotechnical Journal*. **30**, No. 5, 848-858.

498   Kong, Y. X., Zhao, J. D., and Yao, Y. P. (2013). A failure criterion for cross-anisotropic soils considering microstructure.

499   *Acta Geotechnica*. **8**, No. 6, 665-673.

500   Lade, P. V. (2007). Modeling failure in cross-anisotropic frictional materials. *International Journal of Solids and*

501   *Structures*. **44**, No. 16, 5146-5162.

502   Lade, P. V. (2008). Failure criterion for cross-anisotropic soils. *Journal of Geotechnical and Geoenvironmental*

503   *Engineering, ASCE*. **134**, No. 1, 117-124.

504   Lade, P. V., Rodriguez, N. M., and Van Dyck, E. J. (2014). Effects of Principal Stress Directions on 3D Failure

505   Conditions in Cross-Anisotropic Sand. *Journal of Geotechnical and Geoenvironmental Engineering, ASCE*. **140**, No.

506   2, 1-12.

507   Lam, W., and Tatsuoka, F. (1988). Effects of initial anisotropic fabric and σ2 on strength and deformation characteristics

508   of sand. *Soils and Foundations*. **28**, No. 1, 89-106.

509   Li, X. S., and Dafalias, Y. F. (2002). Constitutive Modeling of Inherently Anisotropic Sand Behavior. *Journal of*

510   *Geotechnical and Geoenvironmental Engineering*. **128**, No. 10, 868-880.

511   Liu, M. D., and Indraratna, B. N. (2011). General strength criterion for geomaterials including anisotropic effect.

512   *International Journal of Geomechanics, ASCE*. **11**, No. 3, 251-261.

513   Lu, D. C. (2006). *A constitutive model for soils considering complex stress paths based on the generalized nonlinear*

514   *strength theory*. PhD thesis, Beihang University, Beijing, China (in Chinese)

515   Lu, D. C., Ma, C., Du, X. L., Jin, L., and Gong, Q. M. (2017). Development of a new nonlinear unified strength theory

516   for geomaterials based on the characteristic stress concept. *International Journal of Geomechanics, ASCE*. **17**, No. 2,

517   04016058.

518   Lü, X. L., Huang, M. S., and Andrade, J. E. (2016). Strength criterion for cross-anisotropic sand under general stress

519   conditions. *Acta Geotechnica*. **11**, No. 6, 1339-1350.

520   Lü, X. L., Huang, M. S., and Qian, J. G. (2011). The onset of strain localization in cross-anisotropic soils under true

521   triaxial condition. *Soils and Foundations*. **51**, No. 4, 693-700.

522   Ma, C., Lu, D. C., Du, X. L., and Zhou, A. N. (2017). Developing a 3D elastoplastic constitutive model for soils: a new

523   approach based on characteristic stress. *Computers and Geotechnics*. **86**, 129-140.

524   Matsuoka, H., and Nakai, T. (1974). Stress-deformation and strength characteristics of soil under three different

525   principal stresses. *Proceedings of the Japan Society of Civil Engineers*. **232**, 59-70.

526   Matsuoka, H., Junichi, H., and Kiyoshi, H. (1984). Deformation and failure of anisotropic sand deposits. *Soil*

527   *Mechanics and Foundation Engineering*. **32**, No. 11, 31-36.

528   Mortara, G. (2010). A yield criterion for isotropic and cross-anisotropic cohesive-frictional materials. *International*

529   *Journal for Numerical and Analytical Methods in Geomechanics*. **34**, No. 10, 953-977.

530   Mroz, Z., and Maciejewski, J. (2002). Failure criteria of anisotropically damaged materials based on the critical plane

531   concept. *International Journal for Numerical and Analytical Methods in Geomechanics*. **26**, No. 4, 407-431.

532   Nishimura, S., Minh, N. A., and Jardine, R. J. (2007). Shear strength anisotropy of natural London Clay. *Géotechnique*.

533   **57**, No. 1, 49-62.

534   Oboudi, M., Pietruszczak, S., and Razaqpur, A. G. (2016). Description of inherent and induced anisotropy in granular

535   media with particles of high sphericity. *International Journal of Geomechanics, ASCE*. **16**, No. 4, 04016006.

536   Oda, M., Koishikawa, I., and Higuchi, T. (1978). Experimental study of anisotropic shear strength of sand by plane

537   strain test. *Soils and Foundations*. **18**, No. 1, 25-38.

538   Pietruszczak, S., and Guo, P. J. (2013). Description of deformation process in inherently anisotropic granular materials.

539   *International Journal for Numerical and Analytical Methods in Geomechanics*. **37**, No. 5, 478-490.

540   Pietruszczak, S., and Mroz, Z. (2000). Formulation of anisotropic failure criteria incorporating a microstructure tensor.

541   *Computers and Geotechnics*. **26**, No. 2, 105-112.

542 Pietruszczak, S., and Mroz, Z. (2001). On failure criteria for anisotropic cohesive-frictional materials. *International*
543   *Journal for Numerical and Analytical Methods in Geomechanics*. **25**, No. 5, 509-524.

544 Rodriguez, N. M., and Lade, P. V. (2013). Effects of principal stress directions and mean normal stress on failure
545   criterion for cross-anisotropic sand. *Journal of Engineering Mechanics, ASCE*. **139**, No. 11, 1592-1601.

546 Shire, T., O'Sullivan, C., Barreto, D., and Gaudray, G. (2013). Quantifying stress-induced anisotropy using inter-void
547   constrictions. *Géotechnique*. **63**, No. 1, 85-91.

548 Su, S. F., and Liao, H. J. (1999). Effect of strength anisotropy on undrained slope stability in clay. *Géotechnique*. **49**, No.
549   2, 215-230.

550 Sun, D. A., Matsuoka, H., Yao, Y. P., and Ishii, H. (2004). An anisotropic hardening elastoplastic model for clays and
551   sands and its application to FE analysis. *Computers and Geotechnics*. **31**, No. 1, 37-46.

552 Tatsuoka, F., Nakamura, S., Huang, C., and Tani, K. (1990). Strength anisotropy and shear band direction in plane strain
553   tests of sand. *Soils and Foundations*. **30**, No. 1, 35-54.

554 Tobita, Y. (1988). Yield condition of anisotropic granular materials. *Soils and Foundations*. **28**, No. 2, 113-126.

555 Wang, G. S., Lu, D. C., Du, X. L., and Zhou, X. (2018). Dynamic multiaxial strength criterion for concrete based on
556   strain rate – dependent strength parameters. *Journal of Engineering Mechanics*. **144**, No. 5, 4018018.

557 Wood, D. M. (1990). Soil behaviour and critical state soil mechanics. Cambridge University Press, Cambridge.

558 Xiao, Y., Liu, H. L., and Yang, G. (2012). Formulation of cross-anisotropic failure criterion for granular material.
559   *International Journal of Geomechanics*. **12**, No. 2, 182-188.

560 Yang, L. T., Li, X., Yu, H. S., and Wanatowski, D. (2016). A laboratory study of anisotropic geomaterials incorporating
561   recent micromechanical understanding. *Acta Geotechnica*. **11**, No. 5, 1111-1129.

562 Yang, Z. X., Li, X. S., and Yang, J. (2008). Quantifying and modelling fabric anisotropy of granular soils.
563   *Géotechnique*. **58**, No. 4, 237-248.

564 Yao, Y. P., Lu, D. C., Zhou, A. N., and Zou, B. (2004). Generalized non-linear strength theory and transformed stress
565   space. *Science in China Ser. E Engineering & Materials Science*. **47**, No. 6, 691-709.

566 Yao, Y. P., Tian, Y., and Gao, Z. W. (2017). Anisotropic UH model for soils based on a simple transformed stress method.
567   *International Journal for Numerical and Analytical Methods in Geomechanics*. **41**, No. 1, 54-78.

568 Yao, Y. P., and Kong, Y. X. (2012). Extended UH model: three-dimensional unified hardening model for anisotropic
569   clays. *Journal of Engineering Mechanics, ASCE*. **138**, No. 7, 853-866.

570 Zdravkovic, L., Potts, D. M., and Hight, D. W. (2002). The effect of strength anisotropy on the behaviour of
571   embankments on soft ground. *Géotechnique*. **52**, No. 6, 447-457.

NOTATION

$\mathbf{A^{2D}}$	2D microstructure tensor
$\mathbf{A}$	3D microstructure tensor
a_1, a_2, a_3	eigenvalues of the microstructure tensor
$\Omega_1, \Omega_2, \Omega_3$	eigenvalues of the deviatoric microstructure tensor
$\mathbf{D}$	normal vector of the bedding plane
d_1, d_2, d_3	direction cosines of the normal vector $\mathbf{D}$
δ, ω, θ	direction angle of the normal vector $\mathbf{D}$ in the $Oxyz$ principal stress space
$\mathbf{N^{2D}}$	normal vector of the mobilized plane in the Oyz stress space
$\tilde{\mathbf{N}}^{2D}$	normal vector of the mobilized plane in the $O\tilde{y}\tilde{z}$ physical space
$\tilde{n}_1^{2D}, \tilde{n}_3^{2D}$	direction cosines of $\tilde{\mathbf{N}}^{2D}$ in the $O\tilde{y}\tilde{z}$ physical space
$\mathbf{N}$	normal vector of the mobilized plane in the $Oxyz$ stress space
n_x, n_y, n_z	direction cosines of $\mathbf{N}$ in the $Oxyz$ stress space
$\tilde{\mathbf{N}}$	normal vector of the mobilized plane in the $O\tilde{x}\tilde{y}\tilde{z}$ physical space
$\tilde{n}_x, \tilde{n}_y, \tilde{n}_z$	direction cosines of $\tilde{\mathbf{N}}$ in the $O\tilde{x}\tilde{y}\tilde{z}$ physical space
ζ	angle between the bedding plane and mobilized plane
$\sigma_x, \sigma_y, \sigma_z$	principal stresses
I_1, I_2, I_3	first, second and third stress invariants
$\bar{I}_1, \bar{I}_2, \bar{I}_3$	first, second and third stress invariants in the in the transitional stress space
τ, σ	shear and normal stress acting on the mobilized plane
p	mean principal stress ($p=1/3(\sigma_1+\sigma_2+\sigma_3)$)
q	deviatoric stress ($q = 1/\sqrt{2}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}$)

M_f failure stress ratio

R principal stress ratio ($R=\sigma_1/\sigma_3$)

b intermediate principal stress coefficient ($b=(\sigma_2-\sigma_3)/(\sigma_1-\sigma_3)$)

φ internal friction angle

φ_c internal friction angle under triaxial compressive conditions

φ_e internal friction angle under triaxial extensive conditions

η^{2D} 2D transversely isotropic strength parameter

η_0^{2D} average value of η^{2D}

η 3D transversely isotropic strength parameter

η_0 average value of a_1, a_2 and a_3

η_c stress ratio under triaxial compressive conditions

η_e stress ratio under triaxial extensive conditions

α triaxial extension–compression strength ratio

ρ distribution coefficient

Φ generalized material parameter

Θ generalized direction angle

Table 1

Table 1 Material parameter values for demonstration

$\varphi_c\|_{\delta=0°}$	$\varphi_c\|_{\delta=90°}$	η_0	Ω_3
	40°	1.559	-0.140
	37.5°	1.490	-0.075
35°	35°	1.418	0
	32.5°	1.344	0.086
	30°	1.267	0.186

Table 2

Table 2 Values of the material parameters for the TI-NUSC

Soils	η_0	Ω_3	α	ρ
Toyoura sand (Lam & Tatsuoka, 1988)	1.613	0.085	0.365	1.222
Nevada sand (Lade *et al.*, 2014)	1.476	0.111	0.564	1.080
Leighton Buzzard sand (Yang *et al.*, 2016)	1.359	0.172	0.449	1.595

Mobilized plane 1

$\mathbf{N^{2D}} \left(\mathbf{\tilde{N}^{2D}} \right)$

$45° + \varphi_\delta / 2$

$\tilde{z}$

$\mathbf{D}$

z

$\zeta = \zeta_1$

δ

$\sqrt{\sigma_1}$

A

ζ_2

$\sqrt{\sigma_3}$

C

O

y

$\tilde{y}$

Mobilized plane 2

Bedding plane

σ_3

σ_1

σ_1

σ_3

- Oda et al. (1978)
- Matsuoka et al. (1984)
- Tatsuoka et al. (1990)

$\eta_0 = 1.067, \Omega_3 = 0.126$

$\eta_0 = 0.969, \Omega_3 = 0.132$

$\eta_0 = 1.005, \Omega_3 = 0.191$

Direction angle, δ : degrees

Friction angle, $\varphi|_\delta$: degrees

$\eta_0 = 1.267$, $\Omega_3 = 0.186$

$\rho = 0$

$\rho = 2$

$\eta = \eta_0[1 + \Omega_3(1 - 3m_{II}^2)]$

$\eta = \eta_0[1 + \Omega_3(1 - 3m_I^2)]$

1.42

1.09

3D strength parameter, η

Direction angle, δ : degrees

σ_z

σ_x

σ_y

O

p

NUSC

TI-NUSC

$\delta = 0°$

D

y

z

x

σ_z

I

II

Sector I

σ_1

σ_3

σ_2

Sector II

III

σ_2

σ_3

σ_1

Sector III

σ_y

σ_x

σ_3

σ_2

σ_1

$p = 200\ \text{kPa}$

$\Omega_3 = 0.186$

η_0 increase

σ_z

σ_y

σ_x

$\varphi_c|_{\delta=0} = 35°$

$\varphi_c|_{\delta=90} = 30°$

$p = 200 \text{ kPa}$
$\eta_0 = 1.267$

σ_z

σ_y

σ_x

$\Omega_3 = 0.2$
$\Omega_3 = 0.1$
$\Omega_3 = 0$
$\Omega_3 = -0.1$
$\Omega_3 = -0.2$

TI-NUSC

$\alpha = 1$

$\alpha = 2/3$

$\alpha = 1/3$

$\alpha = 0$

σ_z

σ_y

σ_x

I

II

III

I

II

III

Sector II

σ_3

σ_2

σ_1

$b = 0$

$b = 0.5$

$b = 1$

Direction angle, δ : degrees

Friction angle, φ : degrees

Friction angle, ϕ : degrees

Direction angle, δ : degrees

$b = 0$

$b = 0.5$

$b = 1$

Sector I

Sector VI

σ_1
σ_2
σ_3

Direction angle, δ :degrees

Friction angle, φ :degrees

$b = 0$
$b = 0.5$
$b = 1$

Fig. 1 Strength evolution with b and δ at $p = 200$ kPa

Toyoura Sand
TI-NUSC
NUSC

σ_z
150
100
50

$\varphi_{cz} = 41.76°$
$I_1 = 294$ kPa

σ_y
σ_x

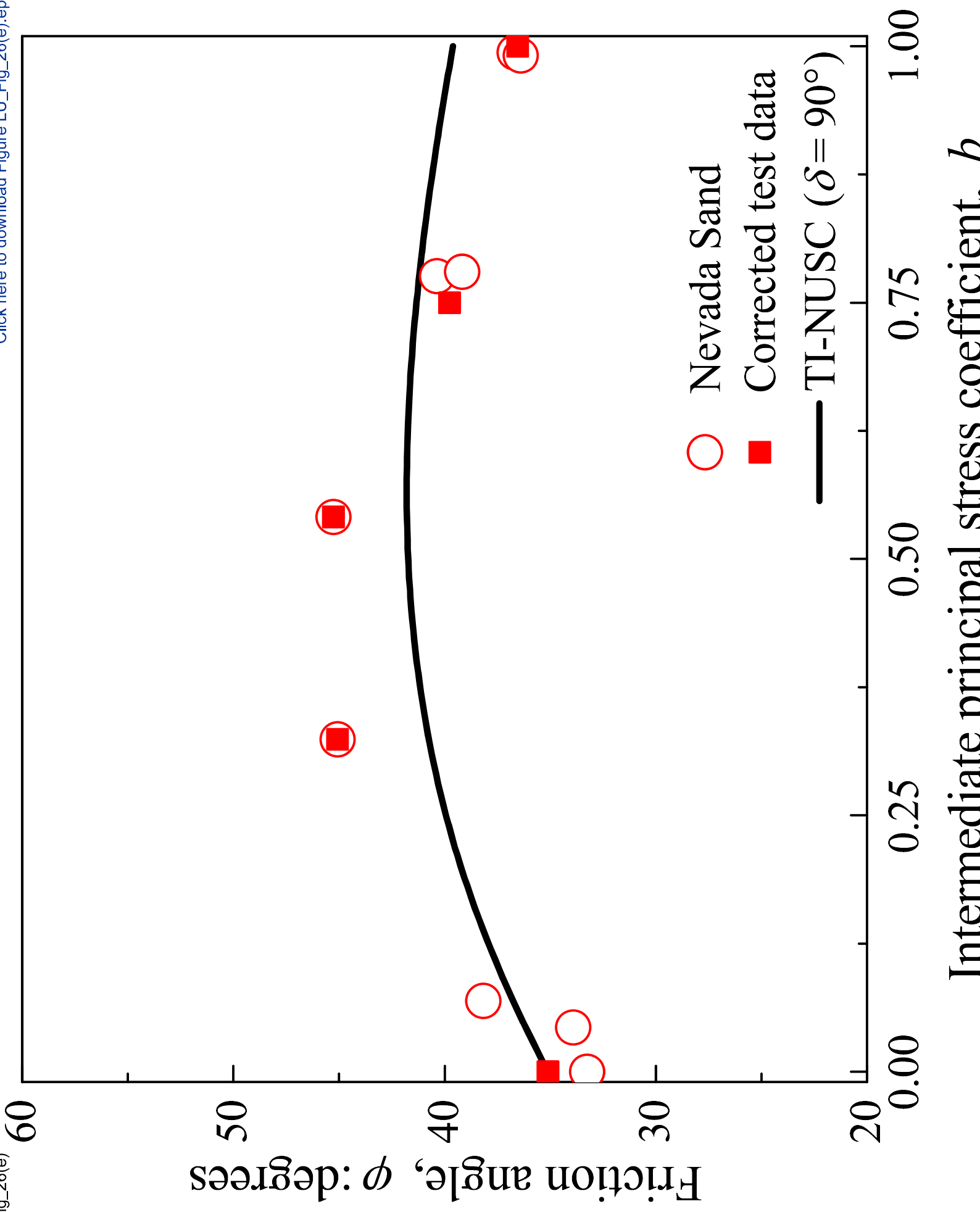Friction angle, φ :degrees

Intermediate principal stress coefficient, b

○ Nevada Sand
■ Corrected test data
— TI-NUSC ($\delta = 22.5°$)

$\varphi_{cz} = 38.95°$
$I_1 = 300$ kPa

$\delta = 0°$
$\delta = 90°$

σ_z
σ_y
σ_x

I II III IV V VI

$\delta = 0°$ $\square$ $\delta = 90°$

$\delta = 0°$ ■
$\delta = 90°$ □
TI-NUSC
NUSC

LU_Fig_S2(a)