Contests with Ambiguity

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Abstract

The paper examines contests where players perceive ambiguity about their opponents' strategies and determine how perceptions of ambiguity and attitudes to ambiguity affect equilibrium choice. Behaviour in our contest is affected by pessimistic and optimistic traits. Which of these traits dominates determines the relationship between the equilibrium under ambiguity and behaviour where contenders have expected utility preferences. Our model can explain experimental results such as overbidding and overspreading relative to Nash predictions.

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'Events, dear boy, events.'

Response to the question what does a prime minister most fear? Attributed to Harold Macmillan.

1 Introduction

The quote above illustrates how political life can be affected by unexpected shocks. Many uncertainties in the political arena arise from the strategic behaviour of political actors. A notable example of such interactions are electoral competitions which are often modelled as contests (Tullock, 1980). This paper studies how contests are affected by the possibility of unusual events, which we model by assuming that participants' beliefs about the behaviour of others are ambiguous. Ambiguity refers to uncertainties for which it is impossible or difficult to assign precise probabilities.

Many important economic interactions can be represented as contests where participants expend resources to obtain a single or multiple prizes and both winners and losers forfeit the resources expended during the competition. The relevant resources in the case of a political election are the campaign expenditures. Success is more likely the greater the amount that a candidate spends. However, it is a decreasing function of campaign expenditures by his/her opponents. Theoretical and empirical advances in the literature have led to a better understanding of the strategic forces and trade-offs in these and other contest-like environments and to recommendations for improving upon economic, political, and social outcomes. However, there is a stark dissonance between a number of standard theoretical results and the evidence which jeopardises the practical import of the theory.

We propose that these phenomena may be a response to ambiguity. Specifically we argue that participants in a contest may perceive ambiguity about their opponents' actions and study how perceptions of ambiguity and attitudes to ambiguity affect equilibrium behaviour. In his pioneering study, Ellsberg (1961) argued that individuals will exhibit behaviour that reveals preferences which differentiate between risk (known probabilities) and ambiguity (unknown probabilities). The prevalence of Ellsberg-type behaviour in experimental and naturally occurring settings has stimulated efforts to develop and axiomatise alternative models of decision-making.[1]

Our primary motivation is that it may be intrinsically difficult for contest participants

[1] For reviews of the literature on ambiguity, see Etner et al. (2012) and Trautmann et al. (2015).

to attach unique probabilities to the behaviour of other contenders. In other words, participants in many real-world contests may perceive ambiguity about their opponents' choices. Thus, we argue that the new explanation is plausible and, hence, modeling of these types of contests should reflect sensitivity of participants to ambiguity. Moreover, understanding how ambiguity affects behaviour sheds new light on actual expenditure and winning patterns in contest-like environments.

Ambiguity is likely when the resolution of uncertainty depends on:

- rare events for which little or no historical data is available;
- the behaviour of other people, which is intrinsically difficult to predict;
- new or advanced technology.

Some or all of these factors are present in many of the situations in which contest theory is applied as the following examples illustrate. For all of these and other real-world situations, no two contests are exactly alike and consequently any information about past behaviour is a very imperfect predictor of actions in future contests. Thus, one might expect ambiguity to persist over time.

1.1. Research and Development All of these factors can be present in patent races. Research and development, almost by definition, involves discovering something which was previously unknown. Thus historical data is at most of limited help. The outcome will often depend on complex and/or advanced technology and finally it relies on human beings having good ideas.

1.2. Political Campaigns Politics is frequently upset by unusual developments as the opening quotation illustrates. The outcome of electoral competitions depends on the behaviour of other people and upsets are not infrequent. These may be both from external events, the behaviour of opponents, or even actions taken by allies.[2] Historical data may be of little use if new parties or issues have emerged since the last election.[3]

[2]In the case of the 2016 US presidential election, there was substantial ambiguity in the beginning and midst of the campaign about the positions and actions that Donald Trump would be taking.

[3]For instance, international terrorism was not a prominent issue in elections in western countries before 2001.

Consider a competition where multiple candidates expend resources to win a political office. A candidate will condition her/his actions to win the election on expectations of what her/his opponents might do. The standard approach to analyzing this strategic environment hinges upon the assumption that each contender's strategy is conditioned on predictions of the opponents' choices represented by a unique probability distribution and that the equilibrium beliefs are correct. That is, contestants are assumed to behave according to the prescriptions of Nash equilibrium. In reality, a contender may entertain multiple scenarios about the strategies that will be employed by her political opponents. For example, under one scenario a contestant's opponents pursue a relatively negative campaign with a relatively large likelihood while under a different scenario the likelihood of a negative campaign by the opponents is relatively small. In other words, a contender may be unable to assign a unique probability to each course of action by her/his opponents.[4]

1.3. War and Conflict Military conflict is also highly ambiguous. Wars can often be viewed as unique events, since they have numerous idiosyncratic features. The outcome depends on human behaviour. A collapse in morale can result in a large army being defeated by a weaker opponent. New technology is frequently an important factor in war. Commanders often have imperfect information about the progress of a battle. Offensive actions are much more likely to succeed if they contain a large element of surprise. This creates considerable ambiguity for the other side.

1.4. Litigation Consider a litigation process where the opposing sides spend resources to affect the outcome in their favour. Does a party to a litigation process have a 'clear' idea, in probabilistic sense, about the strategy that will be followed by the opponent? For many cases that are not settled prior to going to court and are not commonplace, a considerable amount of ambiguity may be present about strategies that will be followed by the opposing side and this is likely to affect the litigating sides' actual behaviour.[5]

[4]In addition to predicting the opponents' behaviour, candidates also need to take into account the behavior of potential voters. This may also involve considerable degree of ambiguity from the perspective of contenders and pundits alike. Although we don't explicitly model this type of ambiguity, the techniques in the present paper can be rather easily extended to address this scenario.

[5]The probability of a favourable verdict may also be ambiguous because the litigating parties are likely to have little information about the disposition of the judicial body rendering the verdict.

We develop a model where contenders perceive ambiguity about strategies that are used by their opponents. We prove existence of equilibrium, following which we study the comparative statics of ambiguity and ambiguity-attitude. Comparative statics results are important since they enable us to find out what difference ambiguity makes. On a technical level, our paper contributes to the literature on monotone comparative statics since we extend the previous literature to games that have neither strategic substitutes nor strategic complements (Milgrom and Roberts, 1990). The paper also investigates how equilibrium under ambiguity is related to behaviour where contenders have expected utility preferences.

1.5. Organization of the paper The following section reviews the experimental evidence on contests. Our model is introduced in section 3. In Section 5 we specialise to the case of two types of players, which enables us to perform detailed comparative statics. Finally Section 6 concludes. The appendix contains proofs of those results not proved in the text.

2 Experimental Evidence

Recent experimental research on contests reveals that average expenditure to win the prize is significantly higher than the Nash prediction (commonly referred to as overbidding) and the variance of expenditure across experimental subjects is considerable (over-spreading).[6] In some experiments, the extent of overbidding is so prominent that the average earnings are negative.[7] A number of possible rationalizations have been put forth. Explanations of overbidding include hypotheses that experimental subjects

- derive a non-monetary utility from winning, on top of monetary incentives to win a prize (Sheremeta, 2010, Chen et al., 2011),
- exhibit behaviour sensitive to the experimental design (Chowdhury et al., 2017),
- have spiteful preferences and inequality aversion (Herrmann and Orzen, 2008; Bartling et al., 2009),
- have a predisposition to make mistakes (Potters et al., 1998, Lim et al., 2014),

[6] See Dechenaux et al. (2015) for an extensive survey of the experimental research on contests.

[7] There are, however, a couple of exceptions to overbidding (Shupp et al., 2013, Godoy et al., 2015). We relate some of these experimental results to our theoretical predictions after introducing the latter.

- rely on non-linear probability weighting to make their bids (Baharad and Nitzan, 2008, Duffy and Matros, 2012), and
- exhibit loss aversion (Kong, 2008).

Differences in these behavioural traits can also explain, at least in theory, a part of the large variation in expenditure of experimental subjects. The overspreading has also been linked to variation in risk aversion and demographic characteristics of experimental subjects.

We believe that ambiguity may be relevant for explaining experimental research on contests. Consider a typical experiment testing predictions for a game where experimental subjects acquire lottery tickets and a participant's probability of winning is equal to the ratio of the number of tickets (s)he has purchased to the total number of tickets sold. A subject in this type of experiment is likely to be uncertain about the number of lottery tickets that will be purchased by the other participants. (S)he may entertain a range of possibilities for the number of tickets that are bought by her opponents. Furthermore, it is not at all clear that (s)he will assign a unique probability to each of these possibilities. (S)he may very well contemplate a set of likelihoods for some of the prospects. In other words, the subject's beliefs may be ambiguous. The subjects may not only perceive ambiguity about the opponents' possible play but may also exhibit sensitivity to this ambiguity. An optimistic player (or, equivalently, an ambiguity-loving decision-maker) will expect her opponents to buy a relatively small number of tickets. In contrast, a pessimist (or an ambiguity-averse decision-maker) will expect her opponents to buy a relatively large number of tickets. As a result, an increase in the magnitude of ambiguity may have very different effects on pessimistic and optimistic contenders. The model in the present paper can explain overbidding and overspreading, relative to the Nash prediction, which are commonly observed in experimental studies of contests.

3 The model

Consider a contest with $n \geq 2$ players. To improve her chances of winning the prize each contestant $i \in \{1, 2, ..., n\}$ chooses action $x_i \in X_i = [\underline{x}_i, \bar{x}_i]$, where $\underline{x}_i \geq 0$ and $\infty > \bar{x}_i > 0$. On occasion, we will refer to these actions as effort or expenditure invested in the contest.

The bounds on effort levels may reflect institutional constraints. For example, in many presidential elections candidates receive public funds to compete. These serve as a lower bound on the amount that the candidates will spend on their election campaigns. The bounds may also be subjective where they represent the opponents' beliefs about a player's potential choices, rather than physical restrictions on the permissible values of contest expenditures.[8] For many competitive environments, it is sensible to expect that players will not anticipate that all of their opponents will choose zero effort; $\underline{x}_i > 0$ for some or even all i. Thus, even though zero bids are allowed players may believe that their opponents will choose strictly positive expenditure levels.

It is equally reasonable to expect that contenders may believe that their opponents' expenditures will not exceed a certain finite upper bound. The assumption that the expenditure to win the contest may be constrained from above also accounts for possibilities of budget-constrained participants and for possible exogenous restrictions on the level of expenditures in the contest (e.g., an upper threshold on expenditures set by a 'contest designer').[9] For practical reasons, in what follows we mainly focus on the interpretation of $(\underline{x}_1, ..., \underline{x}_n)$ and $(\bar{x}_1, ..., \bar{x}_n)$ as beliefs about possible bounds on the opponents' potential choices.

The cost of action x_i is given by x_i and incurred irrespective of the contest's outcome. The probability that contestant i receives the prize, the contest success function (CSF), is given by

$$p_i\left(x_i; \mathbf{x}_{-i}\right) = \begin{cases} \frac{h_i(x_i)}{\sum_{j=1}^{n} h_j(x_j)} & \text{if } \exists j \in \{1, ..., n\} \text{ such that } x_j > 0 \\ \frac{1}{n} & \text{if } x_j = 0 \text{ for all } j \in \{1, ..., n\} \end{cases}, \tag{1}$$

where $\mathbf{x}_{-i} \equiv (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ denotes the vector of action choices by all players except for contestant i. The set of strategy combinations of player i's opponents is denoted by $\mathbf{X}_{-i}$ and the set of strategy combinations of all players is denoted by $\mathbf{X}$. We also let $\mathbf{x}$ denote the vector of action choices by all participants in the contest; $\mathbf{x} \equiv (x_1, x_2, ..., x_n)$. The function $h_i(\cdot)$ $(i = 1, ..., n)$ is assumed to be increasing in its argument. Under this assumption, p_i is increasing in own action and decreasing in the actions of the opponents. It is also assumed

[8] For an early treatment of contests where expenditure must exceed some minimum level, see Schoonbeek and Kooreman (1997).

[9] Since resources are scarce, all of the participants in a contest will be budget constrained. However, for some or all contenders the budget constraint may be non-binding.

that $h_i(\cdot)$ is concave and twice-continuously differentiable and $h_i(0) = 0$ for all $i = 1, ..., n$. The assumption of concavity of $h_i(\cdot)$ implies that $p_i(x_i; \mathbf{x}_{-i})$ is concave for all $x_i > 0$ and all $\mathbf{x}_{-i}$.[10]

Contestant i's utility function is given by

$$U_i\left(x_i; \sum_{j \neq i} h_j(x_j)\right) = p_i(x_i; \mathbf{x}_{-i}) V_i - x_i, \tag{2}$$

where V_i denotes the value of the prize to contestant i. The assumption that the contenders are risk neutral is made primarily with the purpose of focusing on the effect of ambiguity aversion. Our results carry over to a more general setting with risk averse preferences under appropriate qualifying conditions.

The contest considered in the paper falls into a general category of aggregative games (Cornes and Hartley, 2011; Acemoglu and Jensen, 2013). The strategic interaction considered in the paper is a game with negative aggregate externalities (Eichberger et al., 2009), since the CSF in (1) is decreasing in the aggregate $\left[\sum_{j \neq i} h_j(x_j)\right]$ of the opponents' actions. The cross-partial derivative of contender i's utility function with respect to own and opponent k's actions is equal to

$$\frac{\partial^2 U_i\left(x_i; \sum_{j \neq i} h_j(x_j)\right)}{\partial x_i \partial x_k} = \frac{\partial^2 p_i(x_i; \mathbf{x}_{-i})}{\partial x_i \partial x_k} V_i,$$

where

$$\frac{\partial^2 p_i(x_i; \mathbf{x}_{-i})}{\partial x_i \partial x_k} = -h_i'(x_i) h_k'(x_k) \frac{\left[\sum_{j \neq i} h_j(x_j)\right] - h_i(x_i)}{\left[\sum_{j=1}^{n} h_j(x_j)\right]^3}.$$

It follows from this expression that the marginal benefit of own action $\frac{\partial p_i(x_i; \mathbf{x}_{-i})}{\partial x_i}$ is decreasing in opponent k's action when player i's opponents choose relatively large actions $\left(\sum_{j \neq i} h_j(x_j) > h_i(x_i)\right)$. However, it is increasing when the opponents choose relatively

[10] We maintain the assumption of a general function $h_i(\cdot)$ in this section. To streamline the presentation of our findings, Section 4 adopts the functional form $h_i(x_i) = x_i^{\beta}$ while Section 5 assumes simple lotteries with $h_i(x_i) = x_i$. To see that these assumptions are without loss of generality, note that by applying the transformation $h_i^{-1}(\cdot)$ to the choice variables, player i's objective function is transformed into an objective of a contender playing a simple lottery and having a non-linear cost function $h_i^{-1}(\cdot)$ (Cornes and Hartley, 2011).

small actions $\left(\sum_{j\neq i} h_j\left(x_j\right) < h_i\left(x_i\right)\right)$. In other words, when the aggregate of a player's opponents' efforts is sufficiently large, an increase in any opponent's effort will crowd out the player's effort (the player will partially give in). On the other hand, when the aggregate of a player's opponents' efforts is sufficiently small, the player will respond to an increase in any opponent's effort by increasing her effort (the player will keep up). Thus, this game does not globally exhibit either strategic complementarity or strategic substitutability (Bulow et al., 1985). Note also that when all players have the same function h_j and choose the same action, the strategies of the players are (local) strategic substitutes.

Suppose that the contenders perceive ambiguity about their opponents' choice of action. This ambiguity is represented by a capacity which reflects the weights a player places on different strategies of the opponents. A capacity is similar to a subjective probability with the exception that it may be non-additive. We restrict our attention to the case where the ambiguity for contestant i is represented by a neo-additive capacity[11] v_i defined on the set of the opponents' strategies $\mathbf{X}_{-i}$:

$$v_i\left(\varnothing\right)=0, v_i\left(\mathbf{X}_{-i}\right)=1, \text{ and } v_i\left(A\right)=\delta_i\left(1-\alpha_i\right)+\left(1-\delta_i\right)\pi_i\left(A\right) \text{ for all } \varnothing\subsetneqq A\subsetneqq\mathbf{X}_{-i}, \quad (3)$$

where $\alpha_i, \delta_i \in [0,1]$ and π_i is a standard probability distribution on $\mathbf{X}_{-i}$. Contestant i has some doubts that the probability distribution $\pi_i\left(\cdot\right)$ is the true probability distribution over the opponents' strategies and this ambiguity is reflected by the parameter δ_i. Parameter α_i characterises contestant i's ambiguity attitude. The support of a neo-additive capacity v_i is defined by $supp(v_i) = supp(\pi_i)$. We focus on neo-additive capacities because they offer a clear-cut separation of ambiguity perception from ambiguity attitude and allow for both ambiguity-averse and ambiguity-loving decision-makers. Moreover, contests have focal best and worst outcomes, i.e. winning and losing, which makes neo-additive capacities particularly suitable for analysis.

It is assumed that all participants in the contest have Choquet expected utility (CEU)

[11] A capacity v on a set S is a set function $v : S \to [0,1]$ such that $v\left(\varnothing\right) = 0, v\left(S\right) = 1$, and $v\left(A\right) \geq v\left(B\right)$ for any $A, B \subseteq S$ and $B \subseteq A$.

preferences (Schmeidler, 1989) with a neo-additive capacity (Chateauneuf et al., 2007):

$$W_i\left(x_i;\pi_i,\alpha_i,\delta_i\right)=\delta_i\left(1-\alpha_i\right)M_i\left(x_i\right)+\delta_i\alpha_i m_i\left(x_i\right)+\left(1-\delta_i\right)\int U_i\left(x_i;\sum_{j\neq i}h_j\left(x_j\right)\right)d\pi_i\left(\mathbf{x}_{-i}\right), \tag{4}$$

where

$$\begin{aligned}
M_i\left(x_i\right) &\equiv \max_{\mathbf{x}_{-i}\in\mathbf{X}_{-i}} U_i\left(x_i;\sum_{j\neq i}h_j\left(x_j\right)\right)=\frac{h_i\left(x_i\right)}{h_i\left(x_i\right)+\underline{Y}_{-i}}V_i-x_i,\\
m_i\left(x_i\right) &\equiv \min_{\mathbf{x}_{-i}\in\mathbf{X}_{-i}} U_i\left(x_i;\sum_{j\neq i}h_j\left(x_j\right)\right)=\frac{h_i\left(x_i\right)}{h_i\left(x_i\right)+\bar{Y}_{-i}}V_i-x_i,\\
\underline{Y}_{-i} &\equiv \sum_{j\neq i}h_j\left(\underline{x}_j\right)\text{ and }\bar{Y}_{-i}\equiv\sum_{j\neq i}h_j\left(\bar{x}_j\right).
\end{aligned}$$

The function $M_i\left(x_i\right)$ represents the best possible scenario of player i's opponents' choices for player i while $m_i\left(x_i\right)$ corresponds to the worst possible scenario.

A neo-additive capacity has the following intuitive interpretation and behavioural implications. A decision-maker with CEU preferences and a neo-additive capacity has subjective beliefs characterised by the additive probability distribution $\pi_i\left(\cdot\right)$ but lacks confidence in this belief. When $\delta_i=0$, the decision-maker is certain in her probabilistic assessment $\pi_i\left(\cdot\right)$ and, as a result, has expected utility preferences. In contrast, when $\delta_i>0$, (s)he will take into account the effect of her actions on the best and worst outcomes. The larger the parameter δ_i, the greater the weight that the decision-maker will place on these two extreme outcomes and the larger the deviation from the expected utility preferences. Thus, it is natural to interpret δ_i as measuring ambiguity, and we shall refer to it as the degree of ambiguity. The decision-maker's reaction to uncertainty about beliefs has optimistic and pessimistic traits. The optimistic trait is reflected by the weight on the best outcome $M_i\left(x_i\right)$, measured by $\delta_i\left(1-\alpha_i\right)$, while the pessimistic trait is given by the weight on the worst outcome $m_i\left(x_i\right)$, measured by $\delta_i\alpha_i$. Relatively high (low) values of α_i correspond to pessimistic (optimistic) attitudes to ambiguity. Thus, parameter α_i is referred to as the degree of pessimism (or degree of ambiguity aversion).

Substituting from (2) into (4), we obtain:

$$\begin{aligned} & W_i\left(x_i;\pi_i,\alpha_i,\delta_i\right) \\ = & \left[\delta_i\left(\frac{\left(1-\alpha_i\right)h_i\left(x_i\right)}{h_i\left(x_i\right)+\underline{Y}_{-i}}+\frac{\alpha_i h_i\left(x_i\right)}{h_i\left(x_i\right)+\bar{Y}_{-i}}\right)+\left(1-\delta_i\right)\int\left(\frac{h_i\left(x_i\right)}{h_i\left(x_i\right)+Y_{-i}}\right)d\pi_i\left(\mathbf{x}_{-i}\right)\right]V_i-x_i, \end{aligned}$$

where $Y_{-i}\equiv\sum_{j\neq i}h_j\left(x_j\right)$ .We assume that, given the structure of the game, the $\pi_i\left(\cdot\right)$s are determined endogenously while the degrees of optimism, α_i, and ambiguity, δ_i, are treated as exogenous parameters. A decision-maker's attitude towards ambiguity α_i is a personal trait akin to tastes in a standard consumer problem. Thus, it is reasonable to suppose that it is independent of the decision problem and exogenous influences.

Define the best-response correspondence of player i, given that her/his beliefs are represented by a neo-additive capacity v_i, by $R_i(v_i)=R_i\left(\pi_i,\alpha_i,\delta_i\right)\equiv \underset{x_i\in X_i}{argmax}W_i\left(x_i;\pi_i,\alpha_i,\delta_i\right)$. We adopt the following definition from Eichberger and Kelsey (2014), which is an extension of an earlier work in Dow and Werlang (1994):[12]

Definition 1 *(equilibrium under ambiguity) A vector of neo-additive capacities $\left(\hat{v}_1,\hat{v}_2,...,\hat{v}_n\right)$ is an Equilibrium Under Ambiguity (EUA) if for all $i=1,..,n$, $\varnothing\neq supp\left(\hat{v}_i\right)\subseteq \underset{j\neq i}{\times}R_j(\hat{v}_j)$. If $\hat{\mathbf{x}}_{-i}\in supp\left(\hat{v}_i\right)$ for all $i=1,2,..,n$, then $\left(\hat{x}_1,\hat{x}_2,...,\hat{x}_n\right)$ is called an equilibrium strategy profile. If $supp\left(\hat{v}_i\right)$ contains a single vector $\hat{\mathbf{x}}_{-i}$ for each player $i=1,2,...,n$, we will say that $\hat{\mathbf{x}}$ is a singleton equilibrium.*

Thus, an equilibrium is characterised by a capacity for each player. The support of this capacity consists of strategies that are best responses for the opponents. In the Appendix we prove that this strategic interaction has a singleton EUA:

Proposition 2 *The contest has a singleton EUA $\left(v_1^*,...,v_n^*\right)$ where $v_i^*=\delta_i\left(1-\alpha_i\right)+\left(1-\delta_i\right)\pi_i^*$,*

[12]For alternative approaches to analyzing strategic behavior under ambiguity, see, e.g. example, Lo (1996), Marinacci (2000), Eichberger and Kelsey (2000), Bade (2011), and Hanany et al. (2016).

$$\pi_i^*\left(\mathbf{x}_{-i}^*\right) = 1 \text{ for } i = 1,...,n,\ \mathbf{x}^* = \left(\phi_1\left(\sum_{j\neq 1} h_j\left(x_j^*\right)\right),...,\phi_n\left(\sum_{j\neq n} h_j\left(x_j^*\right)\right)\right),$$

$$\phi_i\left(\sum_{j\neq i} h_j\left(x_j\right)\right) = \begin{cases} \underline{x}_i, & \text{if } \frac{\partial Z_i\left(\underline{x}_i;\sum_{j\neq i} h_j(x_j)\right)}{\partial x_i} \leq 0 \\ \bar{x}_i, & \text{if } \frac{\partial Z_i\left(\bar{x}_i;\sum_{j\neq i} h_j(x_j)\right)}{\partial x_i} > 0 \\ \text{unique positive solution of} & \\ \frac{\partial Z_i(x_i,\sum_{j\neq i} h_j(x_j),\alpha_i,\delta_i)}{\partial x_i} = 0, & \text{otherwise} \end{cases} \tag{5}$$

and

$$Z_i(x_i,\sum_{j\neq i} h_j\left(x_j\right),\alpha_i,\delta_i) = \left[\delta_i\left(\left(1-\alpha_i\right)p_i\left(x_i;\underline{\mathbf{x}}_{-i}\right)+\alpha_i p_i\left(x_i;\bar{\mathbf{x}}_{-i}\right)\right)+\left(1-\delta_i\right)p_i\left(x_i;\mathbf{x}_{-i}\right)\right]V_i - x_i. \tag{6}$$

Proof. See Appendix. ■

This singleton equilibrium is the ambiguous equivalent of a pure strategy Nash equilibrium. There may also be non-singleton equilibria, in which there are two or more strategies in the support of the players' beliefs. These are the ambiguous analogues of mixed strategy Nash equilibria. We focus on the singleton equilibrium since, even in the absence of ambiguity, the interpretation of mixed equilibrium is problematic.[13] Given that (the analogies of) pure strategy equilibria always exist in our model, it is desirable to avoid these issues by confining attention to such equilibria.

We demonstrate in the proof of the above proposition that the payoff function can be written as (6) so that the contest under ambiguity is equivalent to a contest where player i's probability of winning the prize is equal to[14]

$$\delta_i\left(\left(1-\alpha_i\right)p_i\left(x_i;\underline{\mathbf{x}}_{-i}\right)+\alpha_i p_i\left(x_i;\bar{\mathbf{x}}_{-i}\right)\right)+\left(1-\delta_i\right)p_i\left(x_i;\mathbf{x}_{-i}\right)$$

[13] Since players are indifferent between all of the strategies to which they assign a positive probability they have no incentive to play the strategy which sustains the mixed equilibrium (Osborne and Rubinstein, 1994).

[14] The transformation of the probability of winning a contest in our paper is different from misperceptions about the probability of winning a contest that may be associated with the experimental design (Chowdhury et al., 2017). The two correspond to very different behavioral traits. A comparison is made difficult by the fact that the probability misperception story does not yet possess a theoretical underpinning of how exactly these misperceptions are formed and how they interact with other components of the model. Even if one were to formally model the misperception story, we believe that this theory would not yield predictions similar to our model. For example, we show below that overbidding can be a non-monotonic function of the number of players. In contrast, it seems that probability mispercetpions are likely to be monotonic in the number of contenders, yielding a monotonic relationship between overbidding and the number of contestants.

and the value of the prize is equal to V_i. The latter expression reveals that the incentives to invest in the contest come through three different channels; the optimistic scenario $\delta_i (1-\alpha_i) p_i \left(x_i; \underline{\mathbf{x}}_{-i}\right)$, the pessimistic scenario $\delta_i \alpha_i p_i (x_i; \bar{\mathbf{x}}_{-i})$, and the 'standard' scenario $(1-\delta_i) p_i (x_i; \mathbf{x}_{-i})$.

4 Symmetric case

We begin with an analysis of a symmetric contest where all of the players have the same value of the prize, the same contest success function, the same lower and upper bounds on contest expenditures, and the same degrees of pessimism:[15]

$$\begin{aligned} V_1 &= \cdots = V_n \equiv V,\ h_1(\cdot) = \cdots = h_n(\cdot) = h(\cdot), \\ \underline{x}_1 &= \cdots = \underline{x}_n \equiv \underline{x},\ \bar{x}_1 = \cdots = \bar{x}_n \equiv \bar{x}, \\ \alpha_1 &= \cdots = \alpha_n \equiv \alpha,\ \delta_1 = \cdots = \delta_n \equiv \delta. \end{aligned}$$

Suppose also that $h(x) = x^\beta$, where $\beta \leq 1$ (Tullock, 1967, 1980). There are a number of reasons we examine symmetric contests. First, they are more tractable. Second, they are more illustrative of how the degrees of ambiguity and ambiguity aversion affect behaviour in contests. Third, many experimental studies entail various symmetry assumptions and we are interested in juxtaposing our findings to the received experimental evidence.[16]

In a symmetric equilibrium, $x_1^* = ... = x_n^* \equiv x^*$. From (5), we obtain an implicit expression

[15] For space considerations, we mainly only focus on the comparative statics for the parameters that distinguish our framework from the received literature, namely, those that are associated with ambiguity about opponents' behavior.

[16] The assumption that all players have the same perception of ambiguity and attitude to ambiguity is a simplifying assumption which is relaxed in later sections of the paper.

for the unique interior symmetric equilibrium (when it exists)[17]:

$$F\left(x^{*},\alpha,\delta\right)\equiv\beta\left(n-1\right)\left(x^{*}\right)^{\beta-1}\left[\delta\left(\begin{array}{c}\left(1-\alpha\right)\frac{\underline{x}^{\beta}}{\left(\left(x^{*}\right)^{\beta}+\left(n-1\right)\underline{x}^{\beta}\right)^{2}}\\+\alpha\frac{\bar{x}^{\beta}}{\left(\left(x^{*}\right)^{\beta}+\left(n-1\right)\bar{x}^{\beta}\right)^{2}}\end{array}\right)+\left(1-\delta\right)\frac{1}{n^{2}\left(x^{*}\right)^{\beta}}\right]V-1=0. \tag{7}$$

4.1 Equilibrium effort and the comparative statics of ambiguity

In this section we find and compare the Nash equilibrium and the EUA. We then proceed to study the comparative statics of ambiguity and effort. The effect of the degree of ambiguity on the equilibrium effort is characterised in the following:

Lemma 3 *The equilibrium effort x^{*} under ambiguity is a decreasing function of the degree of ambiguity δ if and only if the Nash equilibrium effort $x^{N}\equiv\frac{\beta(n-1)}{n^{2}}V$ exceeds the equilibrium effort x^{*} under ambiguity:*

$$x^{N}\geq x^{*}. \tag{8}$$

Proof. See Appendix. ■

In the Appendix we also demonstrate that condition (8) holds if and only if

$$\frac{\left(1-\alpha\right)\underline{x}^{\beta}}{\left(\left(\beta\left(n-1\right)V\right)^{\beta}+n^{2\beta}\left(n-1\right)\underline{x}^{\beta}\right)^{2}}+\frac{\alpha\bar{x}^{\beta}}{\left(\left(\beta\left(n-1\right)V\right)^{\beta}+n^{2\beta}\left(n-1\right)\bar{x}^{\beta}\right)^{2}}\leq\frac{1}{n^{2\beta+2}\left(\beta\left(n-1\right)V\right)^{\beta}}, \tag{9}$$

which reveals how the model parameters affect the relationship between the degree of ambiguity δ and the equilibrium effort x^{*}. It also demonstrates that the equilibrium effort under ambiguity is a monotonic function of the degree of ambiguity. Under both inequality (8) and its reverse, an increase in ambiguity widens the gap between the equilibrium effort under ambiguity and the Nash prediction. If the ambiguity attitude of the contenders (and other parameters of the model) is such that the equilibrium effort under ambiguity exceeds the Nash prediction x^{N}, then an increase in the degree of ambiguity will widen this gap by

[17] Existence and uniqueness of an interior symmetric equilibrium is guaranteed by imposing restrictions on the lower and upper bounds for x and other parameters of the model. The comparative statics analysis is straightforward when either all of the players choose the lower bound on expenditure or all of the players choose the upper bound. For this reason, we focus on the interior solutions.

increasing the equilibrium effort under ambiguity. The dark-shaded curve in Figure 1 depicts this scenario for a parameterization of our model where the contenders are ambiguity seeking ($\alpha = 0.3$). The Nash equilibrium for this example is equal to 1.6 and the equilibrium effort under ambiguity continuously increases starting from this level as the degree of ambiguity changes from no ambiguity ($\delta = 0$) to the highest possible level of ambiguity ($\delta = 1$).

When $x^N \geq x^*$, an increase in ambiguity widens the gap by decreasing the equilibrium effort under ambiguity. The light-shaded curve in Figure 1 depicts this possibility for the case of ambiguity averse contenders ($\alpha = 0.9$). Note also that the contenders may be ambiguity loving but under-invest compared to the Nash equilibrium. It is easy to construct examples where the participants are ambiguity seeking but where x^* is decreasing in δ. More generally, it follows from (8) that if

$$\frac{\underline{x}^{\beta}}{\left(\left(\beta\left(n-1\right)V\right)^{\beta}+n^{2\beta}\left(n-1\right)\underline{x}^{\beta}\right)^{2}} > \frac{\bar{x}^{\beta}}{\left(\left(\beta\left(n-1\right)V\right)^{\beta}+n^{2\beta}\left(n-1\right)\bar{x}^{\beta}\right)^{2}} \tag{10}$$

then there is a threshold level of the degree of ambiguity aversion such that x^* is increasing in δ if and only if the participants have a degree of ambiguity aversion that is smaller than that threshold level. Conversely, if the reverse of inequality (10) holds then x^* is increasing in δ if and only if the participants have a degree of ambiguity aversion that is higher than some threshold level of ambiguity aversion.

It also follows from condition (9) that the equilibrium effort x^* either increases in the degree of ambiguity for all δ or decreases in the degree of ambiguity for all δ. Equivalently, whether the contenders under-invest or overinvest relative to the Nash equilibrium is independent of the degree of ambiguity $\delta \in (0, 1)$.

Finally, consider the case where $\beta = 1$. This is a simple lottery frequently explored in experimental studies. The parameter $\bar{x}$ may represent the subjects' endowment of experimental currency. Suppose also that the participants in the lottery believe that their opponents will buy at least $\underline{x} > 0$ lottery tickets. Under this scenario, a contender's equilibrium effort under ambiguity will exceed the Nash equilibrium if and only if

$$(1-\alpha)\frac{\underline{x}}{(V+n^2\underline{x})^2}+\alpha\frac{\bar{x}}{(V+n^2\bar{x})^2}-\frac{(n-1)}{n^4V}>0. \tag{11}$$

It follows from this expression that if the upper threshold $\bar{x}$ is relatively large compared to the lower threshold $\underline{x}$, then the equilibrium effort under ambiguity will exceed the Nash equilibrium if and only if the contenders are sufficiently optimistic. For example, when $V=10$, $\underline{x}=0.5$, $\bar{x}=20$, $n=5$, the inequality holds if and only if the degree of pessimism is less than 0.38. Formally, we use (11) to demonstrate the following[18]:

Proposition 4 *Consider a symmetric contest with* $\beta=1$, $\bar{x}=V$, *and* $\underline{x}=\theta V$. *Then*
(i) There exist $\bar{\alpha}$, $\underline{\theta}\in(0,1)$ *such that for all* $\alpha\in[0,\bar{\alpha})$ *and all* $\theta\in(\underline{\theta},1]$ *there exist* $\underline{n},\bar{n}\in\mathbb{N}$ *such that if the players' pessimism is equal to* α *and the number of players is between* $\underline{n}$ *and* $\bar{n}$, *then the equilibrium effort* x^* *under ambiguity will exceed the Nash equilibrium effort* x^N.
(ii) There exists a threshold value $\hat{n}\in\mathbb{N}$ *of the number of players such that* $x^N>x^*$ *for any contest with more than* $\hat{n}$ *players.*
(iii) There exists a threshold value $\hat{\alpha}\in[0,1)$ *of the degree of pessimism such that* $x^N>x^*$ *for any contest where the players' degree of pessimism exceeds* $\hat{\alpha}$.

Proof. See Appendix. ■

These findings are very intuitive. Relatively optimistic players expect their opponents to choose relatively low investments. When the number of opponents belongs to an intermediate range between $\underline{N}$ and $\bar{N}$, this results in higher incentives to invest. Consequently, the equilibrium effort under ambiguity is higher than the Nash equilibrium. This finding suggests that overinvestment observed in experimental settings may be due to subjects' optimistic attitudes. Combined with the recent experimental evidence (see, e.g., Halevy, 2007, and Ivanov, 2011) that a substantial share of experimental subjects exhibit optimism, this finding provides an explanation for overbidding observed in the lab.

The relationship between the number of contenders and overinvestment is also sensible. The potential for overinvestment comes from the optimistic channel and this channel is most

[18]This proposition extends to general symmetric contests. To economize on space, we have chosen to state it for simple lotteries only.

influential when the number of players is neither too small nor too large. Proposition 4 illustrates that both the mechanism through which the number of opponents affects equilibrium investments and the resultant comparative statics results for contests with non-linear probability weighting and our model are different. Baharad and Nitzan (p. 2055, 2008) find that for contests with relatively large numbers of contestants, ‘ ...the individuals tend to be optimistic, that is, the conceived winning probability is higher than the objective probability and this induces them to increase their effort sufficiently such that the contested rent is over-dissipated.’ The ‘optimistic’ trait in our model has a different interaction with the number of contestants.

As a numerical illustration of Proposition 4, consider the parameterization $V = \bar{x} = 100$ and $\theta = 0.05$. The EUA exceeds the Nash equilibrium under the following scenarios, for example; (a) when $\alpha = 0.1$, $x^* > x^N$ if and only if n is between 4 and 16, (b) when $\alpha = 0.2$, $x^* > x^N$ if and only if n is between 4 and 14, (c) when $\alpha = 0.3$, $x^* > x^N$ if and only if n is between 4 and 12, and (d) when $\alpha = 0.4$, $x^* > x^N$ if and only if n is between 5 and 9. Thus, as postulated by Proposition 4, the EUA exceeds the Nash equilibrium for intermediate values of the number of contenders. Furthermore, this range of intermediate values expands as the players become more optimistic. Note also that the range of values of the degree of pessimism α and the number of players n for which overbidding occurs will expand when the players’ beliefs about the lower threshold of their opponents’ expenditure increases (large θ).

4.2 Degree of pessimism and equilibrium effort

We now turn to the relationship between the equilibrium effort and the contenders’ pessimism. A change in the degree of pessimism shifts the weight between the pessimistic and optimistic channels, leaving the standard channel intact. The overall effect depends on which of these channels provides stronger incentives to invest in the contest. Formally, we have:

Proposition 5 *The equilibrium effort x^* under ambiguity will increase in the degree of pessimism α if and only if*

$$x^* > (n-1)^{\frac{1}{\beta}} \sqrt{\underline{x}\bar{x}}, \tag{12}$$

which holds if and only if

$$\frac{\delta}{\left(\sqrt{\underline{x}\bar{x}}\right)^{1-\beta}\left(\bar{x}^{\frac{\beta}{2}}+\underline{x}^{\frac{\beta}{2}}\right)^{2}}+\frac{(1-\delta)(n-1)}{n^{2}\left(\sqrt{\underline{x}\bar{x}}\right)}>\frac{(n-1)^{\frac{1}{\beta}}}{\beta V}. \tag{13}$$

Proof. See Appendix. ■

It follows immediately from (13) that:

Corollary 6 *The equilibrium effort x^* under ambiguity will decrease in the degree of pessimism α if at least one of the following conditions is satisfied:*
(i) the value of the prize V is sufficiently small,
(ii) the number of contestants n is sufficiently large,
(iii) the lower bound $\underline{x}$ on effort is sufficiently large, and
(iv) the upper bound $\bar{x}$ on effort is sufficiently large.

To gain intuition into the necessary and sufficient conditions for the equilibrium effort to be decreasing in the degree of pessimism, recall the three channels through which changes in the parameters affect the incentives to invest in the contest; the optimistic, pessimistic, and standard channels. Consider a decrease in the value of the prize V. The incentives to invest for all three channels will decrease. Moreover, the disincentives to invest associated with the pessimistic channel will be more prominent for a lower value of the prize. Hence, the equilibrium effort will be lower for contests with relatively pessimistic contenders and low value of the prize. Similar reasoning underlies part (ii) of Corollary 6. Consider now part (iv) of the Corollary (part (iii) has a similar intuition). A relatively pessimistic contender places most of the weight on the scenario where her opponents choose a relatively large expenditure, namely the upper bound $\bar{x}$. Part (iv) of the Corollary follows because an increase in $\bar{x}$ results in a decrease in the marginal benefit of own action and because this effect is stronger when the contender is relatively pessimistic.

5 The model with two types of contenders

Suppose, as in the previous section, that $\underline{x}_1 = \cdots = \underline{x}_n \equiv \underline{x}$ and $\bar{x}_1 = \cdots = \bar{x}_n \equiv \bar{x}$. But, in contrast to the preceding section, the contenders may differ in terms of their degrees of ambiguity δ, their degrees of ambiguity aversion α, and their values of the prize. There are two types of contenders; type-A contenders, who have a common degree of ambiguity δ_A and common degree of ambiguity aversion α_A, and type-B contenders, who have a common degree of ambiguity δ_B and common degree of ambiguity aversion α_B. The contenders $\{1, ..., m\}$ (for $1 \leq m \leq n-1$) are of type A while the remaining contenders are of type B. For simplicity, we assume that $h_1(\cdot) = \cdots = h_n(\cdot) = h(x) = x$, thus, restricting the setup in this section to simple lotteries. This assumption can be relaxed but at the expense of substantially cluttering the exposition.

When all contenders have the same value of the prize, $V_1 = \cdots = V_n \equiv V$, we will focus on an equilibrium where all contenders of the same type choose the same action; $x_1^* = ... = x_m^* \equiv x_A^*$ and $x_{m+1}^* = ... = x_n^* \equiv x_B^*$. From the first-order conditions in (5), the equilibrium actions (x_A^*, x_B^*) for an interior equilibrium in this case are implicitly given by the following system of equations[19]:

$$(n-1)\,\delta_A \left[\frac{(1-\alpha_A)\,\underline{x}}{(x_A+(n-1)\,\underline{x})^2} + \frac{\alpha_A \bar{x}}{(x_A+(n-1)\,\bar{x})^2}\right] + (1-\delta_A)\frac{(m-1)\,x_A+(n-m)\,x_B}{(mx_A+(n-m)\,x_B)^2} = \frac{1}{V}, \tag{14}$$

$$(n-1)\,\delta_B \left[\frac{(1-\alpha_B)\,\underline{x}}{(x_B+(n-1)\,\underline{x})^2} + \frac{\alpha_B \bar{x}}{(x_B+(n-1)\,\bar{x})^2}\right] + (1-\delta_B)\frac{mx_A+(n-m-1)\,x_B}{(mx_A+(n-m)\,x_B)^2} = \frac{1}{V}.$$

5.1 The model with two contenders

The assumption of two contenders allows for an informative graphical illustration of the comparative statics for ambiguity attitude and degree of ambiguity. We begin with the case where there is one player of each type, players A and B, and suppose, without loss of generality, that winning the contest is worth at least as much to player A as to player B. Hence, the value of the prize to the two individuals is given by $V_A = V$ and $V_B = \eta V$, where

[19] As in the previous section, an interior equilibrium materializes for certain conditions on the model parameters. A full analysis of corner solutions is available from the authors upon request.

$0 \leqslant \eta \leqslant 1$.

As a benchmark, we continue to use the Nash equilibrium which in this case is given by:

$$x_A^N = \frac{\eta V}{(\eta+1)^2} \text{ and } x_B^N = \frac{\eta^2 V}{(\eta+1)^2}. \tag{15}$$

Both x_A^N and x_B^N are increasing in η on the set of feasible values $0 \leqslant \eta \leqslant 1$. They take their maximum values at $\eta = 1$, in which case $x_A^N = x_B^N = \frac{V}{4}$. Thus, the symmetric contest produces the highest effort levels. This arises because competition is more intense in a symmetric contest. The marginal benefit of effort is greater when a player is level with his/her opponent than when (s)he is clearly ahead or clearly behind.

Assume that the set of feasible strategies may be written in the form $[\kappa V, \lambda V]$, where $0 < \kappa < \frac{1}{4}$ and $\lambda > \frac{1}{4}$.[20] Under this assumption, the symmetric Nash equilibrium is in the interior of the strategy sets. Comparing x_A^N, x_B^N, x_A^*, and x_B^*, we obtain:

Proposition 7 *Suppose that both players perceive a positive degree of ambiguity ($\delta_A, \delta_B > 0$). In an EUA both players will choose an effort lower than x_A^N, i.e. $x_i^* \leqslant \frac{\eta V}{(\eta+1)^2}$ for $i = A, B$.*

Proof. See Appendix. ■

Thus, in any EUA both players provide less effort than the Nash equilibrium effort level of the player with the highest value of the prize. Hence, ambiguity causes player A to choose lower effort than her Nash equilibrium level. In contrast, player B may provide more or less effort than her Nash level. It is straightforward to construct examples where the player with the lower valuation of the prize overbids compared to Nash.

Godoy et al. (2015) conducted a series of two-player laboratory contests and demonstrated that expenditure levels were higher in the treatment where contest expenditures are sequential and observable than in the treatment where expenditures were chosen simultaneously. This finding is consistent with the predictions of our model. It is natural to anticipate that players will not perceive much ambiguity in contests where expenditures are sequential and observable. In contrast, one may expect players to perceive significant ambiguity about

[20] The assumption $\kappa > 0$ is equivalent to the requirement $\underline{x}_i > 0$. For a motivation of the latter assumption, see the beginning of Section 3.

the opponent's action when the choices are made simultaneously. For the case of two players, such increase in ambiguity will result in lower expenditure according to Proposition 7.

It follows immediately from Proposition 7 that:

Corollary 8 *Suppose that both players perceive a positive degree of ambiguity ($\delta_A, \delta_B > 0$) and have the same value of the prize V. Then, in an EUA both players will make strictly less than the Nash equilibrium level of contributions $x_A^N = x_B^N = \frac{V}{4}$.*

The intuition behind this finding is as follows. Recall that the incentives to invest come through three channels. The pessimistic and optimistic channels produce incentives to invest that are lower than the incentives for the game without ambiguity. A complete pessimist believes that her opponent will choose a very large investment. In this case, the marginal product of the player's effort is relatively low since (s)he believes that (s)he will likely lose the contest unless (s)he invests a very large amount. The marginal product of effort is also relatively low for an optimist since there is only one opponent and optimism causes a player to overweight low effort from her opponent.[21] In this case, the player believes that (s)he can win the contest without much effort. Ambiguity causes the decision-maker to overweight both possibilities. As a result, the equilibrium expenditures under ambiguity are lower than in the Nash equilibrium. In the following section we show that, even with symmetric valuations, the EUA can exceed the Nash equilibrium in games with more than two contenders.

We now turn to the comparative statics. For space considerations, in the rest of the paper we focus on the case where all players have the same value of the prize.

In a contest with two contenders, player A's and B's best response functions, $\phi_A(x_B)$ and $\phi_B(x_A)$, for interior solutions are given by the unique solutions to the corresponding equations in (14) with $n = 2$ and $m = 1$. Lemma 12 in the Appendix summarises the monotonicity properties of the best-response functions. Figure 2 depicts these functions. For $x_i > x_j$ a marginal increase in the opponent's action x_j intensifies the competition leading to an increase in player i's effort. In contrast, for $x_j > x_i$ an increase in x_j reduces the intensity of the competition leading to a decrease in player i's effort. Player A's (player

[21] With multiple opponents, the optimistic channel may induce more effort from a player than under Nash.

B's) best response curve is downward (upward) sloping above the 45^0 degree line and upward (downward) sloping below it.

As a starting point in the comparison of equilibrium efforts we take a two-player symmetric contest where both players perceive the same degree of ambiguity δ and have the same degree of pessimism α. For such contests, we have:

Proposition 9 *Suppose* $\frac{1-\delta\alpha}{4\underline{x}} + \frac{\delta\alpha\bar{x}}{(\underline{x}+\bar{x})^2} > \frac{1}{V}$. *Then, the two-player symmetric contest has a unique equilibrium. Moreover, this equilibrium is symmetric and interior.*

Proof. See Appendix. ■

When $\frac{1-\delta\alpha}{4\underline{x}} + \frac{\delta\alpha\bar{x}}{(\underline{x}+\bar{x})^2} \leq \frac{1}{V}$, both players will choose the lowest possible expenditure $\underline{x}$. We sidestep this uninteresting case and instead focus on interior solutions in the rest of this section. It follows immediately from Lemma 3 and Corollary 8 that for a symmetric contest with two contenders, the unique equilibrium effort is a strictly decreasing function of the common degree of ambiguity δ. Proposition 5 in turn implies that for a symmetric two-contender contest the symmetric equilibrium will be a decreasing function of the common degree of pessimism if and only if $x^* < \sqrt{\underline{x}\bar{x}}$, which implies by Proposition 8 that if $\underline{x}\bar{x} > \frac{V^2}{16}$ then the symmetric equilibrium will be a decreasing function of the degree of pessimism. The effect of an increase in the degree of pessimism is to shift the decision weight from the best outcome to the worst. Under inequality $\underline{x}\bar{x} > \frac{V^2}{16}$, the extra weight on the worst outcome affects the marginal benefit more than the reduction of the weight on the best outcome.

We demonstrate in the Appendix (see Lemma 13) that the best response functions are single-peaked with the peak located at the unique symmetric equilibrium (see Figure 2 and the following proposition). Using this property of the best response functions, we can determine how changes in the degrees of ambiguity and attitudes to ambiguity of individual players affect behaviour starting from the symmetric environment. Let $x_A = x_B = x$ denote the equilibrium effort level for a symmetric contest with $\delta_A = \delta_B = \delta$ and $\alpha_A = \alpha_B = \alpha$ and let (x'_A, x'_B) denote the equilibrium for an asymmetric contest with $\delta_j = \delta < \delta_i = \delta'$ and $\alpha_A = \alpha_B = \alpha$, where $i, j \in \{A, B\}$ and $i \neq j$.[22] Figure 2 depicts the effect of an increase

[22] It is implicitly assumed that an increase in the degree of ambiguity from δ to δ' is relatively small. See the proof of the following proposition for more details.

in player A's degree of ambiguity on equilibrium behaviour starting from the symmetric scenario. As a result of this change, player A's best response curve shifts leftward while player B's best response curve remains unchanged. The EUA moves from point A to point A'. Formally, we have:

Proposition 10 *An increase in player i's ($i \in \{A, B\}$) degree of ambiguity starting from a symmetric contest will strictly decrease both players' equilibrium efforts; $x'_k < x_k$ for $k \in \{A, B\}$. Moreover, the resulting reduction in player i's effort will strictly exceed the reduction in player j's ($j \neq i$) effort; $x'_j > x'_i$.*

Proof. See Appendix. ■

An increase in ambiguity perceived by player i causes her to put more weight on the possibility that her opponent will choose very high expenditure $\bar{x}$ or very low expenditure $\underline{x}$. This decreases player i's perceived marginal benefit and, as a result, reduces her equilibrium effort. Since the competition from player i has become less intense, player $j \neq i$ responds by decreasing her effort as well. However, to stay ahead of her opponent in terms of having a higher probability of winning, player j reduces effort by less than player i. Thus, an increase in player i's degree of ambiguity renders a strategic advantage to player j and improves the latter player's payoff. Proposition 10 also implies that when the two contenders are involved in a rent-seeking activity, an increase in ambiguity perceived by either player will decrease the amount of rent dissipation. This may explain why in practice rent dissipation is not full, contrary to Tullock's predictions.

Consider the effect of changes in player i's degree of ambiguity aversion and now let (x'_A, x'_B) denote the equilibrium of an asymmetric contest with $\delta_A = \delta_B = \delta$ and $\alpha_j = \alpha < \alpha_i = \alpha'$, where $i, j \in \{A, B\}$ and $i \neq j$. Conducting analysis similar to that for the previous proposition, we obtain:[23]

Proposition 11 *An increase in player i's ($i \in \{A, B\}$) degree of ambiguity aversion starting from a symmetric contest will decrease both players' equilibrium efforts; $x'_k < x_k$ for $k \in$*

[23]The results for the scenarios where the players' values of the prize, initial degrees of ambiguity, and their initial attitudes to ambiguity are asymmetric are available from the authors upon request.

$\{A, B\}$. *Moreover, the resulting reduction in player i's effort will exceed the reduction in player j's ($j \neq i$) effort; $x'_j > x'_i$.*

5.2 The model with more than two players

Consider the case with m identical ambiguity averse contenders and $(n-m)$ identical ambiguity loving contenders. Figure 3 depicts the relationship between the number m of ambiguity averse contenders and the equilibrium expenditures of representative ambiguity averse and ambiguity loving contenders for a specific parameterization of our model. In this case, irrespective of the fraction of ambiguity averse players the equilibrium expenditure of ambiguity averse contenders is below the Nash equilibrium level (which is equal to 8) while the equilibrium expenditure of ambiguity loving contenders is above it. Note also that as the number of ambiguity averse players monotonically increases, both contender types increase their equilibrium expenditures.

The results reported in Figure 3 and, more generally, our findings in the previous sections demonstrate that perceptions and attitudes to ambiguity provide an explanation for overspreading frequently observed in experiments. A significant diversity of ambiguity attitudes, which is frequently observed in the lab (see, e.g., Halevy, 2007), can lead to considerable variations of actual investments in contests.

Contrasting these and Section 4's findings with those in Section 5.1, the reader will have noticed that there is a difference between the results for two-player contests and multi-player contests (by a multi-player contest we mean three or more players.). In two-player symmetric contests ambiguity always gives rise to effort levels below the Nash equilibrium. In contrast, in multi-player contests ambiguity can give rise to overbidding.

The difference can be explained intuitively as follows. Individual effort levels are mainly determined by the perceived marginal benefit of effort, which is in turn influenced by the intensity of competition. If a player is a long way ahead he has low marginal benefit since he is likely to win regardless of his/her own effort. Similarly an individual who is far behind will have a low marginal benefit, since (s)he is likely to lose whatever (s)he does. The highest marginal benefit comes in a roughly equal contest which gives rise to the most intense competition between the players.

In a multi-player contest each individual is effectively competing against the aggregate effort of all the others. Thus in a reasonably equal contest without ambiguity each individual perceives him/herself as being behind. For instance, if there are five similar players and each supplies $\frac{1}{5}$ of the total effort and thus has a relatively low (around 20%) chance of winning. Thus (s)he has a relatively low marginal benefit of effort. Now assume that a given player is ambiguity loving. Then (s)he will place positive decision weight on the possibility that his/her rivals will supply very low effort. This will increase his/her marginal benefit since it reduces the gap between the given individual's effort and the aggregate effort of the others. Suppose the given individual is very optimistic and takes this low level to be $\frac{1}{4}$ of the equilibrium effort. Then (s)he will perceive him/herself to be in a roughly equal contest with the aggregate of the other players. As a result, his/her marginal benefit of effort will be relatively high.

6 Conclusion

The paper has developed and analysed contests where contenders perceive ambiguity about strategies of their opponents. In addition to proving existence of equilibrium under ambiguity and exploring its uniqueness properties, we have investigated how the degree of ambiguity regarding other participants' strategies and preferences toward ambiguity affect equilibrium behaviour. The paper also established a relationship between the equilibrium under ambiguity and Nash equilibrium.

Our results suggest that relatively optimistic players tend to invest more than their pessimistic counterparts. Pessimists over-weight the event that their opponents will provide high effort which mutes incentives to expend resources. In contrast, optimists over-weight the scenario that opponents will choose low expenditures. In multi-player games the incentives to invest can be stronger when opponents choose relatively small expenditures than under the most pessimistic scenario. As a result, optimists invest more and have a higher chance of winning. This effect is especially pronounced for intermediate numbers of the opponents.

In the introduction, we have put forth a number of reasons why players in a contest may perceive ambiguity. One of the motivations stemmed from the uniqueness of many contests which may lead to ambiguous beliefs. But then one may argue that in such contexts players are unlikely to play any kind of equilibrium.[24] We have two arguments in favour of the combination of ambiguity and equilibrium. First, it provides a theoretical equivalent of a controlled experiment. If we introduce ambiguity but otherwise keep assumptions similar to a standard model then we know that any changes are due to the presence of ambiguity. In contrast, if we change two or more assumptions of a standard model it is less clear which of these is responsible for any new result.

A second argument in favour of the combination of ambiguity and equilibrium can be drawn from Milgrom and Roberts (1990). They show that in games with strategic complements many naive adjustment processes will lead the players to the equilibrium. An example would be playing a best response to the opponent's previous move. This adjustment may occur in real time. However, equally it may represent a thought process of the participants. If there is ambiguity in the same class of games, a similar adjustment process would result in convergence to the equilibrium. This is because what we refer to as the 'perturbed game' inherits the key property of strategic complementarity. Admittedly, contests do not satisfy Milgrom and Roberts' (1990) assumptions. However, we are reasonably confident that a similar result can be proved in this context. This is because the best response correspondences in contests are single peaked and the players' actions are strategic complements on the first part of the strategy space and strategic substitutes on the second part.

The paper developed in the paper uses the neo-additive model of ambiguity to represent beliefs of the contenders about the strategies of their opponents. These preferences satisfy both the axioms of CEU and Maxmin Expected Utility (MEU), which are two of the most commonly used models of ambiguity. The results in Eichberger and Kelsey (2014) suggest that our findings can be generalised to the class of CEU preferences with a Jaffray-Phillipe JP-capacity (Jaffray and Philippe, 1997). We also conjecture that the results in the present paper could be extended to include the class of smooth ambiguity preferences (Hanany et al., 2016). The present paper also opens the way for subsequent research on comparative statics

[24]We are indebted to Associate Editor Alan Beggs for bringing up this important point.

in a larger class of games. Our conjecture is that similar comparative static results could be proved for all 'well-behaved' games in which the (Nash) best response correspondence is single peaked. This would be similar to the way in which Eichberger and Kelsey (2014) generalise the model of public goods provision from Eichberger and Kelsey (2002). We leave all of these interesting theoretical explorations to future research.

The model yields a number of testable hypotheses. It would be informative to empirically investigate these in the lab and in the field. To elicit ambiguity perceptions and attitudes and their relationship to strategic behaviour, one could use a multi-stage procedure where in one of the stages the subjects' attitudes to ambiguity are elicited using an Ellsberg style experimental design while in the other stage these subjects strategically interact in a contest. An alternative is to directly elicit experimental subjects' beliefs about the strategies of their opponents and relate them to strategic choices. Finally, one could introduce ambiguity into an experimental setting by manipulating the identity of the opponent (Eichberger et al., 2008) and then examining whether this is associated with any significant changes in behaviour.

Supplementary material

Supplementary material is available on the OUP website. This is the online appendix.

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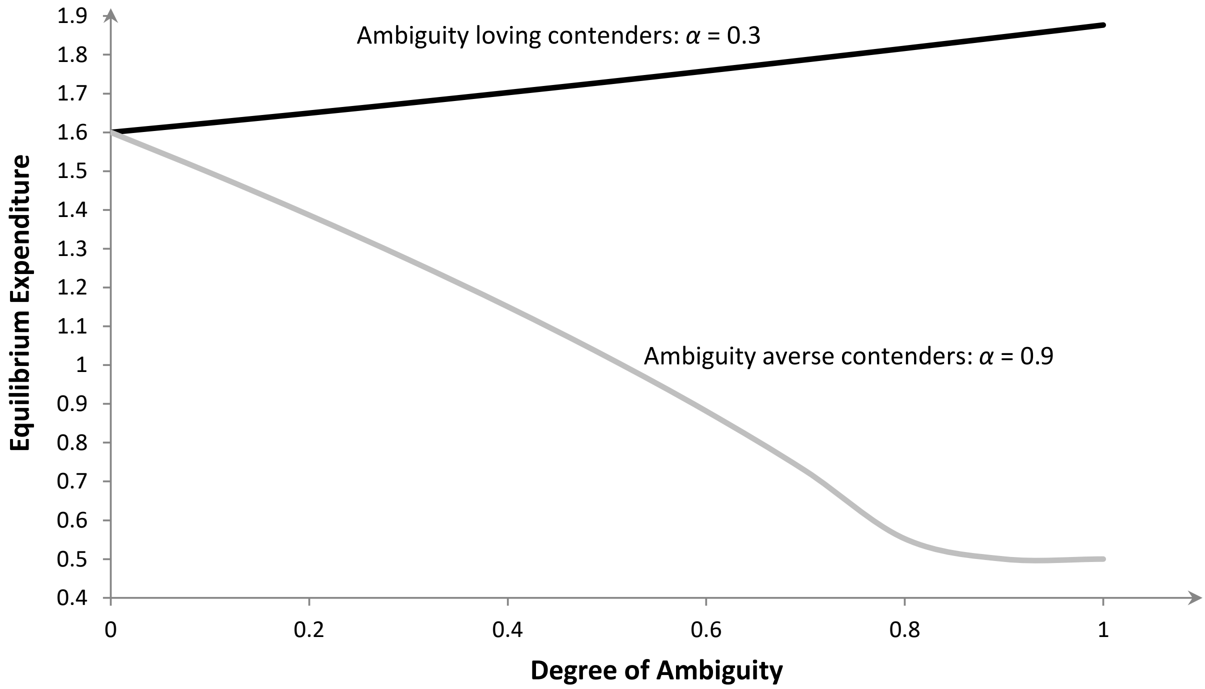

Figure 1. Equilibrium expenditure as a function of the degree of ambiguity

$\left(n = 5, V = 10, \beta = 1, \underline{x} = 0.5, \bar{x} = 10\right)$

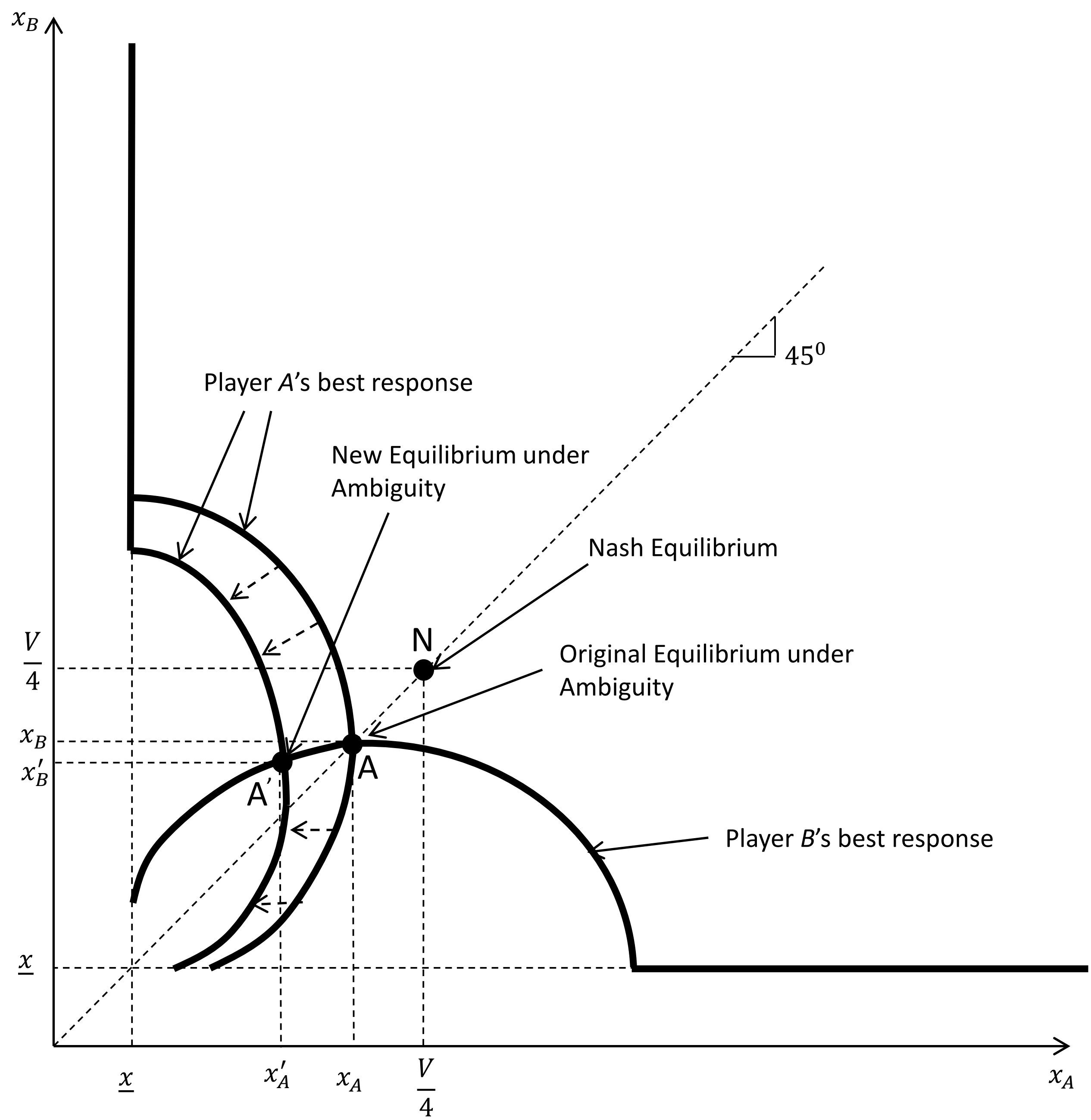

Figure 2. Player *A*'s perception of ambiguity and equilibrium efforts

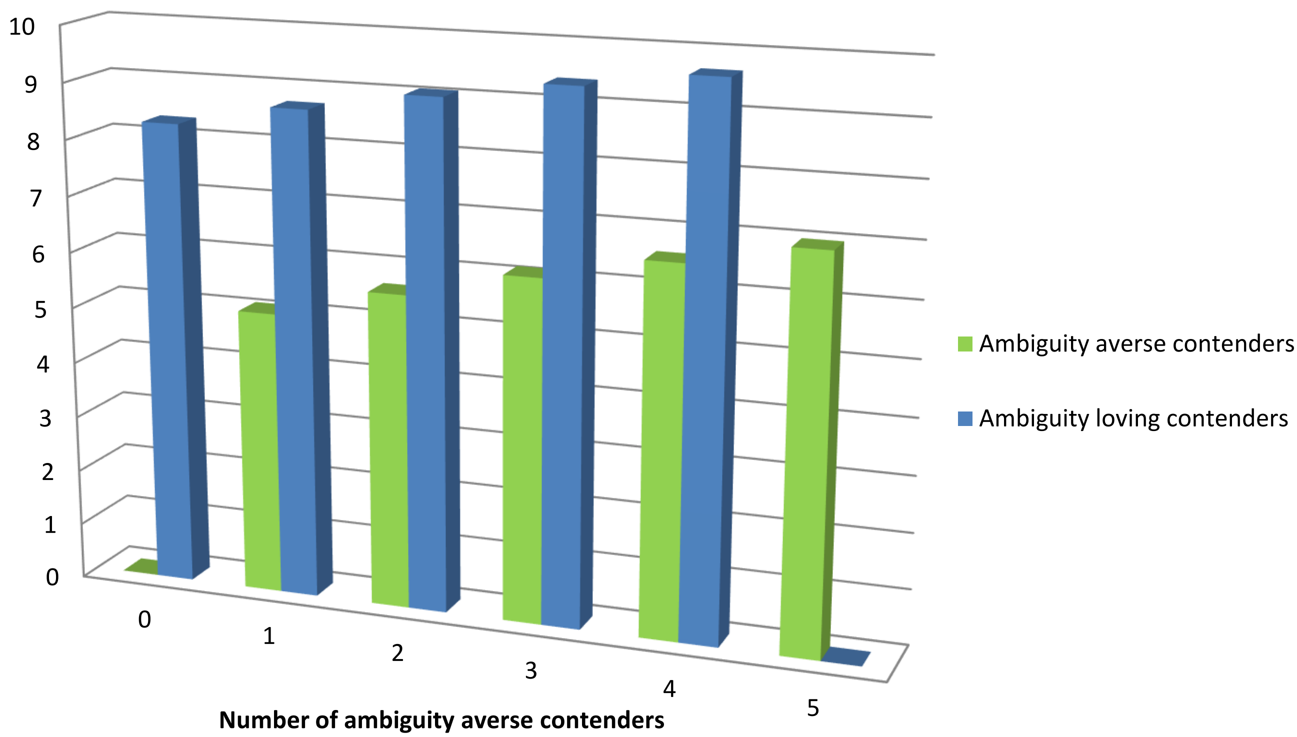

Figure 3. Equilibrium expenditures as a function of the number of ambiguity averse contenders: n = 5, V = 50, $\underline{x}$ = 3, $\delta_A = \delta_B = 0.3$, $\alpha_A = 0.7$, $\alpha_B = 0.3$.

Online Appendix
"Contests with Ambiguity" (by D. Kelsey and T. Melkonyan)

Our proof of Proposition 2 utilises the sufficient conditions for the existence of a pure strategy Nash equilibrium in Reny (1999). The analysis relies on the following definitions (see Reny, 1999 for more details). A game is called compact if each player's pure strategy set is non-empty and compact and each player has a bounded payoff function. A pair $(\mathbf{x}^*, \mathbf{u}^*) \in \mathbb{R}^n \times \mathbb{R}^n$ is in the closure of the graph of the vector payoff function if $\mathbf{u}^*$ is the limit of the vector of payoffs corresponding to some sequence of strategies converging to $\mathbf{x}^*$. Player i can secure a payoff of $\beta \in \mathcal{R}$ at $\mathbf{x} \in \mathbf{X}$ if there exists $\hat{x}_i \in X_i$, such that $Z_i\left(\hat{x}_i, \sum_{j \neq i} h_j\left(x'_j\right), \alpha_i, \delta_i\right) \geq \beta$ for all $\mathbf{x}'_{-i} \in \mathbf{X}_{-i}$ in some open neighbourhood of $\mathbf{x}_{-i}$. A game is better-reply secure if whenever $(\mathbf{x}^*, \mathbf{u}^*)$ is in the closure of the graph of its vector payoff function and $\mathbf{x}^*$ is not an equilibrium, some player i can secure a payoff strictly above u_i^* at $\mathbf{x}^*$. By Theorem 3.1 in Reny (1999), if the game is compact, quasi-concave, and better-reply secure, then it possesses a pure strategy Nash equilibrium.

Proof of Proposition 2. Consider the game

$$\Gamma(\boldsymbol{\delta}, \boldsymbol{\alpha}) = \left\langle (\mathbf{X}_i, \delta_i (1-\alpha_i) M_i(x_i) + \delta_i \alpha_i m_i(x_i) + (1-\delta_i) U_i(x_i; \mathbf{x}_{-i}))_{i=1,2,\ldots,n} \right\rangle,$$

where $\boldsymbol{\delta} \equiv (\delta_1, ..., \delta_n)$ and $\boldsymbol{\alpha} \equiv (\alpha_1, ..., \alpha_n)$. Note that $\Gamma(\boldsymbol{\delta}, \boldsymbol{\alpha})$ is a 'perturbed' game obtained from $G = \left\langle (\mathbf{X}_i, U_i(x_i; \mathbf{x}_{-i}))_{i=1,2,\ldots,n} \right\rangle$ by replacing $U_i(x_i; \mathbf{x}_{-i})$ with the function $\delta_i (1-\alpha_i) M_i(x_i) + \delta_i \alpha_i m_i(x_i) + (1-\delta_i) U_i(x_i; \mathbf{x}_{-i})$ for $i = 1, 2, ..., n$. Eichberger, Kelsey and Schipper (2009) establish a relationship between the sets of Nash equilibria of the perturbed game and Equilibria under Ambiguity for games with two players. Their arguments can be extended to games with an arbitrary number of players to show that for any pure strategy Nash equilibrium $(x_1^*, ..., x_n^*)$ of the perturbed game $\Gamma(\boldsymbol{\delta}, \boldsymbol{\alpha})$, there is a corresponding singleton EUA $(v_1^*, ..., v_n^*)$ of the game G with $v_i^* = \delta_i (1-\alpha_i) + (1-\delta_i) \pi_i^*$ and $\pi_i^*\left(\mathbf{x}_{-i}^*\right) = 1$ for $i = 1, ..., n$ (Eichberger and Kelsey, 2000, pp. 202-204). Moreover, the payoff function of the perturbed game can be written as (6). In light of the established relationship between the equilibria under ambiguity of the game G and Nash equilibria of the perturbed game, we examine the pure strategy equilibria of the latter game.

It follows from the strict concavity of $p_i(x_i; \mathbf{x}_{-i})$ in x_i for all $x_i > 0$ and all $\mathbf{x}_{-i}$ and the resultant strict concavity of the objective function in (6) that player i's best response function for $\sum_{j\neq i} h_j(x_j) > 0$ is given by (5). It is also true that player i's best response function is continuous for $\sum_{j\neq i} h_j(x_j) > 0$.

The perturbed game satisfies all of the conditions of Reny's (1999) Theorem 3.1. First, the perturbed game is compact because even if there is no upper limit on a player's action one can focus on an appropriately chosen compact subset of the real line. Second, the payoff functions of the players are bounded. Third, consider the concavity of the players' payoff functions. If at least one of player i's opponents chooses a strictly positive action player i's payoff function (6) is continuous and concave in own strategy. The only discontinuity occurs when all of the opponents choose inaction; $\mathbf{x}_{-i} = \mathbf{0}$. Note also that this case can materialise only when $\underline{\mathbf{x}}_{-i} = 0$. Under this scenario, player i's payoff function is given by

$$Z_i(x_i, 0, \alpha_i, \delta_i) = \begin{cases} [\delta_i(1-\alpha_i) + (1-\delta_i)]\frac{V_i}{n}, & \text{if } x_i = 0 \\ [\delta_i((1-\alpha_i) + \alpha_i p_i(x_i; \bar{\mathbf{x}}_{-i})) + (1-\delta_i)] V_i - x_i, & \text{if } x_i > 0 \end{cases},$$

which is discontinuous but concave. Thus, all players' payoff functions are concave, and hence quasi-concave, in own strategies.

It is only left to verify that the game is better-reply secure. Since the latter property is a weaker requirement than continuity, the condition for the game to be better-reply secure is satisfied at all points of continuity of a player's payoff function in own strategy, i.e. when $\sum_{j\neq i} h_j(x_j) > 0$ or when $\sum_{j\neq i} h_j(x_j) = 0$ and $x_i > 0$. Moreover, when $\mathbf{x} = \mathbf{0}$, player i's payoff function exhibits an upward jump and a strategy that slightly exceeds $x_i = 0$ can secure her a payoff that is greater than $Z_i(0, 0, \alpha_i, \delta_i) = [\delta_i(1-\alpha_i) + (1-\delta_i)]\frac{V_i}{n}$. Thus, the game is also better-reply secure, which concludes the proof of the existence of a pure strategy Nash equilibrium. ■

Proof of Lemma 3. From the implicit function theorem, $\frac{\partial x^*}{\partial \delta} = -\frac{\frac{\partial F(x^*,\alpha,\delta)}{\partial \delta}}{\frac{\partial F(x^*,\alpha,\delta)}{\partial x^*}}$. Differentiating (7) with respect to x^* and δ, respectively, and using (7) we obtain

$$\begin{aligned} \frac{\partial F(x^*, \alpha, \delta)}{\partial x^*} &= -\beta(n-1)V\left[\delta(x^*)^{\beta-2}\left(\begin{array}{c}\frac{(1-\alpha)\underline{x}\left((1+\beta)(x^*)^\beta+(1-\beta)(n-1)\underline{x}\right)}{\left((x^*)^\beta+(n-1)\underline{x}\right)^3} \\ +\frac{\alpha\bar{x}\left((1+\beta)(x^*)^\beta+(1-\beta)(n-1)\bar{x}\right)}{\left((x^*)^\beta+(n-1)\bar{x}\right)^3}\end{array}\right) + (1-\delta)\frac{1}{n^2(x^*)^2}\right] < 0, \\ \frac{\partial F(x^*, \alpha, \delta)}{\partial \delta} &= \frac{\beta(n-1)(x^*)^{\beta-1}}{\delta}\left[\frac{1}{V\beta(n-1)(x^*)^{\beta-1}} - \frac{1}{n^2(x^*)^\beta}\right]V. \end{aligned}$$

It follows from these expressions that $\frac{\partial F(x^*,\alpha,\delta)}{\partial \delta} > 0$ if and only if $x^* > \frac{\beta(n-1)}{n^2}V = x^N$.

The equivalence between (9) and condition $x^* > x^N$ follows from evaluating $F(\cdot, \alpha, \delta)$ at x^N :

$$F\left(x^N, \alpha, \delta\right) = \delta \left[n^{2\beta+2} \left(\beta \left(n-1\right) V\right)^{\beta} \left(\begin{array}{c} \frac{(1-\alpha)\underline{x}^{\beta}}{\left(\left(\beta(n-1)V\right)^{\beta}+n^{2\beta}(n-1)\underline{x}^{\beta}\right)^2} \\ +\frac{\alpha\bar{x}^{\beta}}{\left(\left(\beta(n-1)V\right)^{\beta}+n^{2\beta}(n-1)\bar{x}^{\beta}\right)^2} \end{array} \right) - 1 \right]. \tag{A.1}$$

■

Proof of Proposition 4. Inequality (11) can be written as

$$B\left(n, \theta\right) > A\left(n, \theta\right) \cdot \alpha, \tag{A.2}$$

where $A(n,\theta) \equiv \frac{\theta}{(1+n^2\theta)^2} - \frac{1}{(1+n^2)^2}$ and $B(n,\theta) \equiv \frac{\theta}{(1+n^2\theta)^2} - \frac{n-1}{n^4}$. We have that

$$\begin{aligned} A\left(n,\theta\right) &> 0 \Longleftrightarrow n^2\sqrt{\theta} > 1, \\ \frac{\partial A\left(n,\theta\right)}{\partial n} &= 4n\left(\frac{1}{\left(1+n^2\right)^3} - \frac{\theta^2}{\left(1+n^2\theta\right)^3}\right) < 0 \Longleftrightarrow \theta^{\frac{1}{3}}\left(n^2\theta^{\frac{1}{3}} - 1\right) > 1, \\ \frac{\partial^2 A\left(n,\theta\right)}{\partial n^2} &= \frac{4\theta^2\left(5n^2\theta - 1\right)}{\left(1+n^2\theta\right)^4} > 0 \Longleftrightarrow 5n^2\theta > 1. \end{aligned} \tag{A.3}$$

Hence, when n is relatively large, $A(n,\theta)$ is positive, decreasing, and convex in n. The asymptote of the graph of $A(n,\theta)$ as a function of n when the latter tends to infinity is vertical. Similarly, we have

$$\begin{aligned} B\left(n,\theta\right) &< 0 \Longleftrightarrow n\left(1+2\theta n^2+\theta^2 n^4\right) > 1+2\theta n^2+\theta\left(1+\theta\right)n^4, \\ \frac{\partial B\left(n,\theta\right)}{\partial n} &= -\frac{4\theta^2 n}{\left(1+\theta n^2\right)^3} + \frac{3n-4}{n^5} > 0 \Longleftrightarrow \left(3n-4\right)\left(1+\theta n^2\right)^3 > 4\theta^2 n^6, \\ \frac{\partial^2 B\left(n,\theta\right)}{\partial n^2} &= 4\left(\frac{\theta^2\left(5\theta n^2-1\right)}{\left(1+\theta n^2\right)^4} - \frac{3n-5}{n^6}\right) < 0 \Longleftrightarrow \left(3n-5\right)\left(1+\theta n^2\right)^4 > \theta^2 n^6\left(5\theta n^2-1\right). \end{aligned} \tag{A.4}$$

Hence, for sufficiently large n, $B(n,\theta)$ is negative, increasing, and concave. The asymptote of the graph of $B(n,\theta)$ as a function of n when the latter tends to infinity is vertical. Finally,

$$A\left(n,\theta\right) > B\left(n,\theta\right) \Longleftrightarrow n^4\left(n-2\right) + \left(2n^2+1\right)\left(n-1\right) > 0. \tag{A.5}$$

Both parts of the Proposition follow immediately from (A.2), (A.3), (A.4), and (A.5). ■

Proof of Proposition 5. From the implicit function theorem we have

$$\frac{\partial x^*}{\partial \alpha} = -\frac{\frac{\partial F(x^*,\alpha,\delta)}{\partial \alpha}}{\frac{\partial F(x^*,\alpha,\delta)}{\partial x^*}}.$$

It follows from the proof of Lemma 3 that $\frac{\partial F(x^*,\alpha,\delta)}{\partial x^*} < 0$. Differentiating (7) with respect to α, we obtain

$$\frac{\partial F(x^*,\alpha,\delta)}{\partial \alpha} = \beta\delta V(n-1)(x^*)^{\beta-1}\left(-\frac{\underline{x}^\beta}{\left((x^*)^\beta + (n-1)\underline{x}^\beta\right)^2} + \frac{\bar{x}^\beta}{\left((x^*)^\beta + (n-1)\bar{x}^\beta\right)^2}\right),$$

which yields the first part of the proposition. The equivalence between inequalities (12) and (13) is obtained by evaluating $F(\cdot,\alpha,\delta)$ at $(n-1)^{\frac{1}{\beta}}\sqrt{\underline{x}\bar{x}}$ and comparing the resulting expression to zero. ■

Lemma 12 *The slopes of the reaction functions at interior points ($\underline{x} < x_i, x_j < \bar{x}$) satisfy*

$$\frac{\partial \phi_i}{\partial x_j} > 0 \text{ if } x_i > x_j \text{ and } \frac{\partial \phi_i}{\partial x_j} < 0 \text{ if } x_i < x_j \text{ where } i,j \in \{A,B\} \text{ and } i \neq j.$$

Proof of Lemma 12. Application of the implicit function theorem to (14) yields

$$\frac{\partial \phi_i}{\partial x_j} = \frac{(x_i - x_j)(1-\delta_i)(x_i+\underline{x})^3(x_i+\bar{x})^3}{2\left[\delta_i(x_i+x_j)^3\left((1-\alpha_i)\underline{x}(x_i+\bar{x})^3 + \alpha_i\bar{x}(x_i+\underline{x})^3\right) + (1-\delta_i)x_j(x_i+\underline{x})^3(x_i+\bar{x})^3\right]},$$

which implies the lemma since all terms in the above expression except for $(x_i - x_j)$ are positive. ■

Proof of Proposition 7. First, note that the first order-conditions for an interior equilibrium are:

$$\delta_A\alpha_A\frac{\lambda V}{(x_A^* + \lambda V)^2} + \delta_A(1-\alpha_A)\frac{\kappa V}{(x_A^* + \kappa V)^2} + (1-\delta_A)\frac{x_B^*}{(x_A^* + x_B^*)^2} = \frac{1}{V}, \tag{A.6}$$

$$\delta_B\alpha_B\frac{\lambda V}{(\lambda V + x_B^*)^2} + \delta_B(1-\alpha_B)\frac{\kappa V}{(x_B^* + \kappa V)^2} + (1-\delta_B)\frac{x_A^*}{(x_A^* + x_B^*)^2} = \frac{1}{\eta V}. \tag{A.7}$$

We shall prove the result by contradiction. Suppose that there exists an EUA in which at least one of the players provides an effort greater than x_A^N. There are three cases to consider: Case 1: $x_A^* > \frac{\eta V}{(\eta+1)^2} = x_A^N$ and $x_B^* \geqslant \frac{\eta^2 V}{(\eta+1)^2} = x_B^N$.

We have

$$\frac{\lambda V}{\left(\lambda V+x_B^*\right)^2} \leqslant \frac{1}{V} \frac{\lambda}{\left(\lambda+\frac{\eta^2}{(\eta+1)^2}\right)^2} \leqslant \frac{1}{V} \frac{\lambda}{\left(\lambda+\frac{1}{4}\right)^2}<\frac{1}{V}<\frac{1}{\eta V}. \tag{A.8}$$

Similarly,

$$\frac{\kappa V}{\left(x_B^*+\kappa V\right)^2}<\frac{1}{\eta V}. \tag{A.9}$$

Finally,

$$\frac{x_A^*}{\left(x_A^*+x_B^*\right)^2} \leqslant \frac{x_A^*}{\left(x_A^*+x_B^N\right)^2}<\frac{x_A^N}{\left(x_A^N+x_B^N\right)^2}=\frac{1}{\eta V} \tag{A.10}$$

where the first inequality in (A.10) follows because $\frac{x_A^*}{\left(x_A^*+x_B\right)^2}$ is decreasing in x_B while the second inequality follows because $\frac{x_A}{\left(x_A+x_B^N\right)^2}$ is decreasing in x_A for $x_A \geqslant x_B$.
It then follows from (A.8), (A.9), and (A.10) that the left-hand-side of (A.6) is strictly smaller than $\frac{1}{\eta V}$ provided $\delta_B>0$. But this implies that the equilibrium condition (A.7) cannot be satisfied. Hence, there does not exist an EUA in this case.
Case 2: $x_A^*>\frac{\eta V}{(\eta+1)^2}=x_A^N$ and $x_B^N=\frac{\eta^2 V}{(\eta+1)^2}>x_B^*$. The proof is similar to case 1.
Case 3: $x_A^*<\frac{\eta V}{(\eta+1)^2}=x_A^N$ and $x_B^*>\frac{\eta^2 V}{(\eta+1)^2}=x_B^N$. The proof is similar to case 1.
Thus, there is no EUA in all three possible cases. The result follows. ■

Proof of Proposition 9. The derivative of a contestant's payoff with respect to own effort evaluated at a point where the contestants choose the same effort level x is given by

$$\left(\delta(1-\alpha) \frac{\underline{x}}{(x+\underline{x})^2}+\delta \alpha \frac{\bar{x}}{(x+\bar{x})^2}+\frac{(1-\delta)}{4 x}\right) V-1. \tag{A.11}$$

The expression in (A.11) evaluated at $x=\underline{x}$ is strictly positive under our assumption that $\frac{1-\delta\alpha}{4\underline{x}}+\frac{\delta\alpha\bar{x}}{(\underline{x}+\bar{x})^2}>\frac{1}{V}$. Moreover, it is is strictly negative at $x=\bar{x}$ since $\bar{x}>\frac{V}{4}>\underline{x}>0$. Hence, by the Intermediate Value Theorem there exists x for which the expression in (A.11) is equal to 0. Since the expression in (A.11) is strictly decreasing in x there can be only one value of $x \in(\underline{x}, \bar{x})$ for which it is equal to zero. This value of x is the effort level in the unique symmetric equilibrium. Moreover, by Lemma 13 this equilibrium is unique. ■

Lemma 13 *Let $i, j \in\{A, B\}$ and $i \neq j$. We have*

1. *$\phi_i\left(x_j\right) \gtreqless x_j$ as $x_j \lesseqgtr x^*$,*
2. *$\phi_i\left(x_j\right)$ is increasing (resp. decreasing) on $\left[\underline{x}, x^*\right]$, (resp. $\left[x^*, \bar{x}\right]$).*

Proof of Lemma 13. By definition of a symmetric equilibrium, $\phi_i\left(x^*\right)=x^*$. Now, note that since $\phi_i\left(x_j\right)$ attains its maximum at $x_j=x^*$, $\frac{\partial \phi_i\left(x^*\right)}{\partial x_j}=0$, there exists $\epsilon>0$ such that for $x_j \in [x^* - \epsilon, x^* + \epsilon]$, $\phi_i(x_j) > x_j$ if $x_j < x^*$ and $\phi_i(x_j) < x_j$ if $x_j > x^*$ (see the proof of the previous proposition for an argument demonstrating existence and uniqueness of x^*).

The claim is proved by contradiction. Suppose that there exists $\tilde{x}$, $\underline{x} < \tilde{x} < x^*$, with $\phi_i(\tilde{x}) < \tilde{x}$. Hence, by the Intermediate Value Theorem, there must exist $\hat{x}$, $\tilde{x} < \hat{x} < x^* - \epsilon$ such that $\phi_i(\hat{x}) = \hat{x}$. Hence, $\hat{x}$ is a symmetric equilibrium effort level. However, this contradicts uniqueness of the symmetric equilibrium (Proposition 9). Hence, no such $\tilde{x}$ can exist which, in turn, implies the claim. A similar argument demonstrates that $x_j > \phi_i(x_j)$ when $x_j > x^*$. Since $x_i > x_j$ for $x_j \in [\underline{x}, x^*]$, Lemma 12 implies that $\phi_i(x_j)$ is strictly increasing on this interval. Similarly, $\phi_i(x_j)$ is strictly decreasing for $x_j \in [x^*, \bar{x}]$. ■

Proof of Proposition 10. The marginal benefit of player i's effort is given by

$$\left(\delta_i \left[\frac{(1-\alpha_i)\,\underline{x}}{(x_i+\underline{x})^2} + \frac{\alpha_i \bar{x}}{(x_i+\bar{x})^2}\right] + (1-\delta_i)\,\frac{x_j}{(x_i+x_j)^2}\right) V.$$

It follows from this expression that the effect of change in δ_i on this marginal benefit is proportional to

$$-\left((1-\alpha_i)\left[\frac{x_j}{(x_i+x_j)^2} - \frac{\underline{x}}{(x_i+\underline{x})^2}\right] + \alpha_i \left[\frac{x_j}{(x_i+x_j)^2} - \frac{\bar{x}}{(x_i+\bar{x})^2}\right]\right) V,$$

which implies that as long as $x_j\bar{x} > x_i^2 > x_j\underline{x}$ an increase in δ_i will decrease marginal benefit of effort. Since the marginal cost of effort is constant, an increase in δ_i will result in a decrease in player i's effort when $x_j\bar{x} > x_i^2 > x_j\underline{x}$. Hence, for points around the symmetric equilibrium, the best response function will exhibit a decrease. By Lemma 12, $\phi_j(x_i)$ is increasing for $x_i \in [\underline{x}, x]$. Hence, as a result of the shift of player i's best response curve the equilibrium will move to a point where $x_k' < x_k$ for $k \in \{A, B\}$ and $x_j' > x_i'$. ■