# HIGHLY TEMPERING INFINITE MATRICES

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Abstract. In this short note, it is proved the existence of infinite matrices that not only preserve convergence and limits of sequences but also convert every member of some dense vector space consisting, except for zero, of divergent sequences, into a convergent sequence.

## 1. Introduction

Summation methods of sequences possess applications in many branches of mathematics. If this method is carried out by an infinite matrix $A = (a_{ij})_{i,j\geq 1}$ then the $A$-transform of a sequence of scalars $x = (x_n)_{n\geq 1}$ is given by $Ax = \left(\sum_{j=1}^{\infty} a_{ij}x_j\right)_{i\geq 1}$, provided that every series $(Ax)_i := \sum_{j=1}^{\infty} a_{ij}x_j$ $(i = 1, 2, \dots)$ converges. Hence $A$ can be viewed as a linear operator $A : \mathcal{D}_A \to \omega$, where $\omega$ and $\mathcal{D}_A$ denote, respectively, the vector space $\mathbb{K}^{\mathbb{N}}$ of all scalar sequences ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathbb{N} := \{1, 2, \dots\}$) and the subspace of all sequences $x \in \omega$ for which $Ax \in \omega$, i.e., for which every series $(Ax)_i$ $(i \in \mathbb{N})$ converges.

Let $\mathcal{C}$ denote the set of convergent members of $\omega$. One of the problems in this area is to determine what matrices $A$ are "tempering" in the sense that they preserve not only convergence but also the limit or, in other words, what $A$'s satisfy $A^{-1}(\mathcal{C}) \supset \mathcal{C}$ and

$$A\text{-}\lim_{i\to\infty} := \lim_{i\to\infty} (Ax)_i = \lim_{i\to\infty} x_i$$

for all $x = (x_i)_{i\geq 1} \in \mathcal{C}$. This problem was completely solved by Toeplitz and Silverman, who proved that a matrix $A$ satisfies the latter properties if and only if the following three conditions are fulfilled (see for instance [8] or [10, Chap. 1]):

(1) $\sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} |a_{ij}| < \infty$,
(2) $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} = 1$, and
(3) $\lim_{i\to\infty} a_{ij} = 0$ for all $j \in \mathbb{N}$.

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Such matrices $A$ will be called *TS-matrices* from now on.

In this vein, one can wonder whether there are tempering matrices satisfying that they also convert some divergent (= non-convergent, in the whole text) sequence into a convergent one. As a matter of fact, this kind of matrices does exist. For instance, the TS-matrix $A_0 = (a_{ij})_{i,j\geq 1}$ defined by $a_{ij} = 1/i$ if $1 \leq j \leq i$ and $a_{ij} = 0$ if $i < j$, that is

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

sends the divergent sequence $((-1)^{n+1})_{n\geq 1}$ to the sequence $\left(\frac{1+(-1)^{n+1}}{2n}\right)_{n\geq 1}$, which converges to $0$ (see [6, pp. 64–65]).

Our purpose in this short note is to show the existence of "highly tempering" matrices, in the sense that they are TS-matrices converting many divergent sequences into convergent ones. As a result, this will be possible in a strong sense. Specifically, it will be proved that there are TS-matrices satisfying the last property for all members of some dense maximal dimensional vector subspace of $\omega$ consisting entirely (but for 0) of divergent sequences.

## 2. Divergent sequences that become convergent

In order to establish our result, it is convenient to fix some notation and to recall a number of facts from functional analysis and set theory (see, e.g., [7] and [9]), and also from the modern theory of lineability (see [2–4]). The set $\omega$ will be endowed with the topology product, that is, the one of coordenatewise convergence. Then $\omega$ becomes an F-space, that is, a completely metrizable topological vector space. It is separable because, for instance, the sequences in $c_{00}$ with entries in $\mathbb{Q}$ (the rationals) or $\mathbb{Q} + i\mathbb{Q}$ (according to $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, respectively) form a dense countable subset of $\omega$. Here $c_{00}$ denotes the family of sequences with finitely many nonzero entries. A standard application of Baire's category theorem shows that $\dim(\omega) = \mathfrak{c}$, the cardinality of the continuum.

A family $\{A_\lambda\}_{\lambda\in\Lambda}$ of infinite subsets of $\mathbb{N}$ is called *almost disjoint* (see, e.g., [5]) if

$$A_\lambda \cap A_\mu \text{ is finite whenever } \lambda \neq \mu.$$

The usual procedure to generate such a family is the following (see, e.g., [1]): denote by $(q_n)_{n\geq 1}$ an enumeration of the rational numbers. For every irrational $\alpha$, we choose a subsequence $(q_{n_k})_{k\geq 1}$ of $(q_n)_{n\geq 1}$ such that $\lim_{k\to\infty} q_{n_k} = \alpha$ and define

$A_\alpha := (n_k)_{k\geq 1}$. By construction, we obtain that $\{A_\alpha\}_{\alpha\in\mathbb{R}\setminus\mathbb{Q}}$ is an almost disjoint uncountable family of subsets of $\mathbb{N}$. A more general result, due to Sierpiński, can be found in [9, pp. 301–302].

Let $X$ be a vector space and $\mathcal{A}$, $\mathcal{B}$ be two subsets of $X$. Then we say that $\mathcal{A}$ is *stronger* than $\mathcal{B}$ whenever $\mathcal{A}+\mathcal{B}\subset\mathcal{A}$. The following assertion can be found in [2, Chap. 7].

**Lemma 2.1.** *Suppose that $X$ is a separable infinite dimensional F-space and that $\mathcal{A}$, $\mathcal{B}$ are subsets of $X$ satisfying the following properties:*

(i) *$\mathcal{A}$ is stronger than $\mathcal{B}$.*
(ii) *$\mathcal{A}\cap\mathcal{B}=\varnothing$.*
(iii) *$\mathcal{A}\cup\{0\}$ contains some vector space whose dimension equals* $\dim(X)$ $(=\mathfrak{c})$.
(iv) *$\mathcal{B}\cup\{0\}$ contains some dense vector subspace of $X$.*

*Then there exists a dense $\mathfrak{c}$-dimensional vector subspace of $X$ contained, except for $0$, in $\mathcal{A}$.*

We are now are ready to state our theorem. By a subsequence $\{n_k\}_{k\geq 1}$ we mean a strictly increasing sequence $n_1<n_2<\cdots<n_k<\cdots$ of natural numbers.

**Theorem 2.2.** *Assume that $A=(a_{ij})_{i,j\geq 1}$ is an infinite matrix over the scalar field $\mathbb{K}$ satisfying the following properties:*

(a) *For every $j\in\mathbb{N}$, the sequence $(a_{ij})_{i\geq 1}$ converges.*
(b) *There is a subsequence $(n_k)_{k\geq 1}\subset\mathbb{N}$ such that, for each one of its subsequences $(m_k)_{k\geq 1}\subset(n_k)_{k\geq 1}$ we have:*
    (b1) *the sequence $(a_{im_k})_{k\geq 1}$ is summable for every $i\in\mathbb{N}$, and*
    (b2) *the sequence $\left(\sum_{k=1}^\infty a_{im_k}\right)_{i\geq 1}$ converges.*

*Then there exists a dense $\mathfrak{c}$-dimensional vector subspace of $\omega$ all of whose nonzero members diverge but their $A$-transforms converge.*

*Proof.* Let $\mathcal{A}$ denote the family of all divergent sequences $x\in\omega$ such that $Ax$ is a well-defined convergent sequence, that is

$$\mathcal{A}:=A^{-1}(\mathcal{C})\setminus\mathcal{C}.$$

For every sequence $x=(x_n)_{k\geq 1}\in\omega$, we denote by $\sigma(x)$ its support, that is, $\sigma(x)=\{n\in\mathbb{N}:\,x_n\neq 0\}$. If we have finitely many sequences $x^1,x^2,\dots,x^p\in\omega\setminus c_{00}$ such that the family of their supports $\{\sigma(x^1),\dots,\sigma(x^p)\}$ is almost disjoint, then such sequences are linearly independent: indeed, if $(\lambda_1,\dots,\lambda_p)\in\mathbb{K}^p\setminus\{(0,\dots,0)\}$ and $\sum_{j=1}^p\lambda_j x^j=0$, with $x^j=(x_{j,n})_{n\geq 1}$ and, say, $\lambda_l\neq 0$, then we can choose $n_0\in\sigma(x^l)\setminus\bigcup_{\substack{j\neq k\\ j,k=1,\dots,p}}(\sigma(x_j)\cap\sigma(x_k))$, which yields $\lambda_l\,x_{l,n_0}=0$, so $x_{l,n_0}=0$, a contradiction. This shows the linear independence.

Pick an almost disjoint family $\{N_s\}_{s\in\Lambda}$ of infinite subsets of $\mathbb{N}$ with $\text{card}(\Lambda) = \mathfrak{c}$. Divide each set $N_s$ into two disjoint infinite sets, say $N_s = N_{1s} \cup N_{2s}$. Let $n_1 < n_2 < \cdots < n_k < \cdots$ be the sequence given by the assumption (a). Consider the almost disjoint family $\{(n_k)_{k\in N_s}\}_{s\in\Lambda}$. For each $s \in \Lambda$, we define the sequence $u^s = (u_{sm})_{m\geq 1} \in \omega$ by

$$u_{sm} = \begin{cases} 1, & \text{if } m = n_k \text{ with } k \in N_{1s} \\ -1, & \text{if } m = n_k \text{ with } k \in N_{2s} \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\sigma(u^s) = N_s$ and that $u_s$ diverges. Fix a finite family $\{s(1), \dots, s(p)\} \subset \Lambda\}$, with the $s(j)$'s pairwise different. The preceding paragraph shows that the sequences $u^{s(1)}, \dots, u^{s(p)}$ are linearly independent. Then the vector space $\mathcal{M} := \text{span}(\{u^s\}_{s\in\Lambda})$ is $\mathfrak{c}$-dimensional.

Fix $u \in \mathcal{M} \setminus \{0\}$. Then there is $(\lambda_1, \dots, \lambda_p) \in \mathbb{K}^p \setminus \{(0, \dots, 0)\}$ (with, say, $\lambda_l \neq 0$) such that $u = (u_m) := \sum_{j=1}^p \lambda_j u^{s(j)}$. The set

$$J := \bigcup_{\substack{j\neq k \\ j,k=1,\dots,p}} N_{s(j)} \cap N_{s(k)}$$

being finite, we may consider $\nu := \max J$. It is evident that, provided $k \in N_{s(l)} \cap \{\nu+1, \nu+2, \dots\}$, we get $u_{n_k} = \lambda_l$ ($u_{n_k} = -\lambda_l$) if $k \in N_{1s(l)}$ (if $k \in N_{2s(l)}$, resp.). To summarize, $u$ contains infinitely many 1's and infinitely many $-1$'s, hence it is divergent.

Now, $M \subset \mathcal{D}_A$ because $\mathcal{D}_A$ is a vector space and each $u^s \in \mathcal{D}_A$: indeed, the $i$th-coordinate of $Au_s$ is $\sum_{k=1}^\infty (a_{im_k} - a_{ip_k})$ for certain subsequences $(m_k)$, $(p_k)$ of $(n_k)$ (namely, $(m_k) = (n_l)_{l\in N_{1s}}$, $(p_k) = (n_l)_{l\in N_{2s}}$); so the convergence of the latter series is guaranteed by (b1). By a similar reason, in order to prove that $M \subset A^{-1}(\mathcal{C})$, it is enough to fix $s \in \Lambda$ as before and show that the sequence $(\sum_{k=1}^\infty (a_{im_k} - a_{ip_k}))_{i\geq 1}$ converges. This is guaranteed by (b2). Therefore condition (iii) of Lemma 2.1 (with $X := \omega$) is fulfilled. Conditions (ii) and (iv) are satisfied too if we define $\mathcal{B} := c_{00}$. Indeed, $c_{00} \subset \mathcal{C}$ and $c_{00}$ is itself a dense vector subspace of $\omega$.

It remains to prove condition (i) of Lemma 2.1. Firstly, observe that $c_{00} \subset \mathcal{D}_A$ since $c_{00}$ is algebraically generated by the basic sequences $e^j = (\delta_{jn})_{n\geq 1}$ ($j \in \mathbb{N}$) and each $Ae^j$ is simply the $j$th-column $(a_{ij})_{i\geq 1}$ of $A$. Moreover, these columns being convergent (due to (a)), we get by linearity that $Av$ converges for all $v \in c_{00}$. From the facts that $A$ acts linearly and the sum of a convergent sequence and a divergent one (of two convergent sequences, resp.) diverges (converges, resp.), we obtain that $\mathcal{A} + \mathcal{B} = (A^{-1}(\mathcal{C}) \setminus \mathcal{C}) + c_{00} \subset \mathcal{A}$, as required. Finally, Lemma 2.1 yields the conclusion. $\square$

Notice that *every TS-matrix* satisfies conditions (a) and (b1) of Theorem 2.2: indeed, each column tends to $0$ and (b1) is fulfilled with any subsequence $(n_k) \subset \mathbb{N}$ due to the absolute convergence of each series $\sum_{j=1}^{\infty} a_{ij}$ $(i \in \mathbb{N})$. Condition (b2) is satisfied by, for instance, the TS-matrix $A_0$ defined in Section 1 (take $n_k := k^2$; this would give $0$ as the limit for all sequences of (b2)), so concluding the existence of TS-matrices converting a large amount of divergent series in convergent ones.

Finally, one may wonder whether there are TS-matrices converting *every divergent sequence* into a convergent one. This is far from be true. As a matter of fact, given a TS matrix $A$, there is a sequence $x = (x_n) \in \omega$ with $x_n = \pm 1$ for all $n \in \mathbb{N}$ whose $A$-transform $A\,x$ diverges (see [6, pp. 66–67]).

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