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Rigid body motion measurements with Fourier lensless holography

N. Bolognini, H. J. Rabal, E. E. Sicre, and M. Garavaglia

Centro de Investigaciones Opticas, Casilla de Correo 124, 1900 La Plata, Republic of Argentina.
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A recently proposed[1] holographic and moiré technique for in-plane motion measurements is generalized to certain cases of rigid body motion measurements.

In this case, a three-recording-plates set, in a trihedral arrangement attached to an object, registers the interference between spherical and plane coherent wave fronts. The interference fringes are plane sections of a paraboloid revolution family with a common axis parallel to the plane wave propagation direction; it also contains the point source originating the spherical wave. If, between two such exposures, the arrangement is translated or rotated in the space, the two overlapping interference patterns registered will produce low frequency moiré patterns. If the movement is small (less than the innermost Fresnel ring), the moiré pattern consists of families of equally spaced straight lines, the spacing and orientation of which are related to the magnitude and orientation of the displacement or rotation. So, as in the case of in-plane motions of the photographic plate, these moiré fringes can be used to determine the magnitude and direction of the displacement or the angles of rotation. In addition, each family of moiré fringes gives rise, in the Fourier plane, to two bright spots separated by a distance proportional to the magnitude of the displacement or rotation. Also, a slanted image of the pupil limiting the plane wave, reconstructed in the Fourier plane, is modulated by Young's fringes, whose interfringe is also reciprocally proportional to the magnitude of the movement.

First, let us consider the intensity moiré pattern produced by the overlapping of the two interference patterns in each of the three recording plates of the trihedral arrangement. In general, the transmittance of the developed plates will be

$$t \alpha C_1 + C_2 \cos\left[\frac{k}{2}(|\mathbf{r}-\mathbf{r}_0| - |\mathbf{r}'-\mathbf{r}_0|) - \tfrac{1}{2}(\mathbf{k}\cdot\mathbf{r} - \mathbf{k}\cdot\mathbf{r}')\right] \times \cos\left[\frac{k}{2}(|\mathbf{r}-\mathbf{r}_0| + |\mathbf{r}'-\mathbf{r}_0|) - \tfrac{1}{2}(\mathbf{k}\cdot\mathbf{r} + \mathbf{k}\cdot\mathbf{r}')\right] \quad (1)$$

where C_1 and C_2 are related to the amplitudes of the spherical and plane waves, $\mathbf{k}$ is the wave number vector, and $\mathbf{r}$ and $\mathbf{r}'$ are position vectors before and after the movement of the trihedral. $\mathbf{r}_0$ is a constant position vector from the origin of the coordinates system to the reference point source, as shown in Fig. 1. The position vectors $\mathbf{r}$ and $\mathbf{r}'$ are related by

$$\mathbf{r}' = A\mathbf{r} + \mathbf{b}, \quad (2)$$

where A is the rotation matrix and $\mathbf{b}$ is the translation vector, both characterizing the trihedral movement.

Assuming $|\mathbf{r}| \ll |\mathbf{r}_0|$, the transmittance t given by (1) becomes

$$t \alpha C_1 + C_2 \cos\left\{\frac{k}{4r_0}[r^2 - r'^2 - 2\mathbf{r}_0\cdot(\mathbf{r}-\mathbf{r}')] - \tfrac{1}{2}(\mathbf{k}\cdot\mathbf{r} - \mathbf{k}\cdot\mathbf{r}')\right\} \cos\left\{\frac{k}{4r_0}[r^2 + r'^2 - 2\mathbf{r}_0\cdot(\mathbf{r}+\mathbf{r}') + 2r_0] - \tfrac{1}{2}(\mathbf{k}\cdot\mathbf{r} + \mathbf{k}\cdot\mathbf{r}')\right\} \quad (3)$$

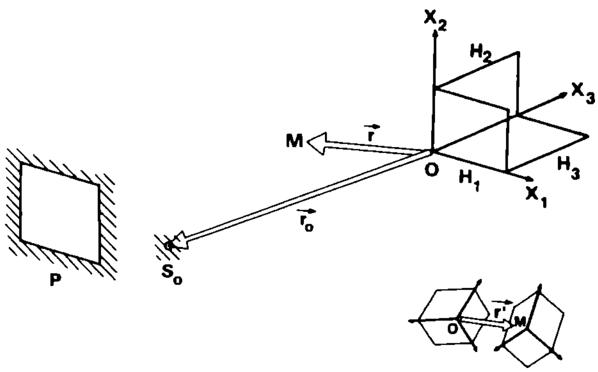

Fig. 1. Scheme of the experimental setup: H_1, H_2, and H_3 are three recording plates in a trihedral arrangement, $\mathbf{r}$ denotes a point position vector on one of the plates, $\mathbf{r}'$ indicates the position vector of the same point after the trihedral motion, and $\mathbf{r}_0$ is the position vector of the reference source S_0. P is an empty pupil which diffracts an approximately plane wave.

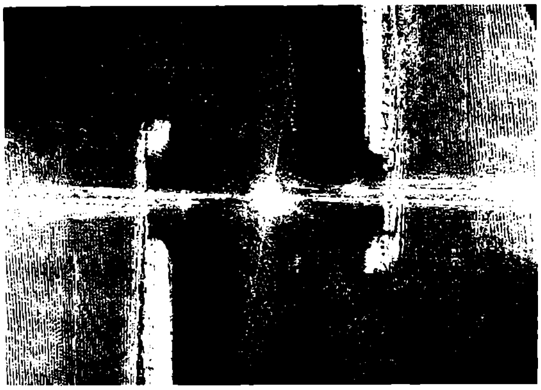

Fig. 2. Positive reproduction of the Fourier plane, for the case of a rotation of 2° about an axis perpendicular to the plane defined by $x_1 + x_2 + x_3 = 1$. At both sides of the zero order two symmetrical spots can be observed. Also Young's fringes in the reconstructed pupil appear.

The last equation consists of a high frequency family of curves modulated by a low frequency one represented by the first cosine. The moiré fringes condition of maxima will be given by

$$[(A - I)\mathbf{r}] \cdot \mathbf{h} - A\mathbf{r} \cdot \mathbf{b} = \frac{2\P n r_0}{k} + \frac{b^2}{2} - \mathbf{b} \cdot \mathbf{h}, \tag{4}$$

where Eq. (2) has been used, I denotes the identity matrix, n is an integer number, and $\mathbf{h}$ is a known position vector given by

$$\mathbf{h} = \mathbf{r}_0 + \frac{r_0}{k}\mathbf{k}.$$

Rewriting Eq. (4), it becomes

$$\sum_{j=1}^{3} B_j x_j = \frac{2\P r_0 n}{k} + \frac{b^2}{2} - \mathbf{b} \cdot \mathbf{h}, \tag{5}$$

where x_j are the components of $\mathbf{r}$, and the coefficients B_j are $B_j = \hat{x}_j' \cdot (\mathbf{h} - \mathbf{b}) - h_j$, $\hat{x}_j'$ being the versors that characterize the trihedral orientation after rotation.

That is, in each of the three planes, the moiré fringe pattern consists of a family of parallel lines, as in the case of in-plane motions,[1] that are the intersections between the three recording plates and a family of equally spaced planes given by Eq. (5).

A straightforward calculation shows that the spacing Δ_{ij} and the slope m_{ij} of the moiré fringes in the (i,j)-plane are

$$\Delta_{ij} = \frac{4\P r_0}{k} \times \frac{1}{\sqrt{[\hat{x}_i' \cdot (\mathbf{h} - \mathbf{b}) - h_i]^2 + [\hat{x}_j' \cdot (\mathbf{h} - \mathbf{b}) - h_j]^2}}, \tag{6}$$

$$m_{ij} = -\frac{\hat{x}_i' \cdot (\mathbf{h} - \mathbf{b}) - h_i}{\hat{x}_j' \cdot (\mathbf{h} - \mathbf{b}) - h_j}. \tag{7}$$

When the developed plates are separately Fourier transformed in the conventional way,[2] the moiré fringes act as a low frequency linear grating producing two bright spots symmetrically located to the zero order, the separation d_{ij} between them being related to the spacing Δ_{ij} as

$$d_{ij} = \frac{\lambda D}{\Delta_{ij}},$$

where λ is the reconstruction wavelength and D is the distance between the hologram and the Fourier plane, when the former is illuminated with a spherical converging beam.

Returning to (1), it may be written as follows:

$$\begin{aligned} t\alpha C_1 &+ C_2\,[\exp(-i\mathbf{k} \cdot \mathbf{r}) \exp(ik|\mathbf{r} - \mathbf{r}_0|) \\ &+ \exp(-i\mathbf{k} \cdot \mathbf{r}') \exp(ik|\mathbf{r}' - \mathbf{r}_0|)] \\ &+ C_2[\exp(i\mathbf{k} \cdot \mathbf{r}) \exp(-ik|\mathbf{r} - \mathbf{r}_0|) \\ &+ \exp(i\mathbf{k} \cdot \mathbf{r}') \exp(-ik|\mathbf{r}' - \mathbf{r}_0|]. \end{aligned} \tag{8}$$

Fourier transforming Eq. (8) and taking into account only one diffracted order, the amplitude distribution in the Fourier plane will be

$$U^{(+1)} = CF\left\{\exp\left[\frac{ik}{2r_0}(\mathbf{r} - \mathbf{h})^2\right] + \exp(i\alpha)\exp\left[\frac{ik}{2r_0}(A\mathbf{r} + \mathbf{b} - \mathbf{h})^2\right]\right\}, \tag{9}$$

where C and α are constants, and $F\{\ \}$ denotes the Fourier transform. As $F\{\ \}$ operates over a bidimensional transparency, all the position vectors are two-coordinate dependent.

In the case of an infinitesimal rotation, Eq. (9) becomes

$$U^{(+1)} = CF\left\{\exp\left[\frac{ik}{2r_0}(\mathbf{r} - \mathbf{h})^2\right] * [\delta(\mathbf{r}) + \exp(i\alpha)\delta(\mathbf{r} - \mathbf{v})]\right\}, \tag{10}$$

in which α is a constant phase delay, * denotes the convolution product, and $\mathbf{v}$ is related to the translation $\mathbf{b}$ and the rotation $A(\phi,\psi,\theta)$, with ϕ,ψ,θ being the rotation's Euler angles, as

$$\mathbf{v} = A^{+}(\mathbf{h} - \mathbf{b}), \tag{11}$$

where + indicates transposition.

As shown in Eq. (10), the reconstructed pupil in the Fourier plane will be modulated by Young's fringes, the spacing of which is given by the reciprocal of the modulus of $\mathbf{v}$.

Returning to either Eq. (6) or Eq. (7) we can conclude that in certain cases, it is possible by this method to study 3-D motions. If we deal with only a rigid body translation or a very small rotation, the movement can be determined. In other cases, additional information is needed for determining motion parameters.

The measurements of Δ_{ij} can be made, provided that the two overlapping interference patterns registered produce at least two moiré fringes on the holographic plate. Therefore, this condition establishes the smallest displacement that can be measured by this technique, so limiting the sensitivity. On the other hand, the accuracy mainly depends on the precision of the Δ_{ij} measurements.

Figure 1 shows the experimental setup, and Fig. 2 shows experimental results for the case of a small rotation about an axis that lies in the direction of light propagation.

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