

Nut Graphs: Maximally Extending Cores

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Abstract

A graph G is singular if there is a non-zero eigenvector v_0 in the nullspace of its adjacency matrix A . Then $Av_0 = 0$. The subgraph induced by the vertices corresponding to the non-zero components of v_0 is the **core** of G (w.r.t. v_0). The set whose members are the remaining vertices of G is called the **periphery** (w.r.t. v_0) and corresponds to the zero components of v_0 . The dimension of the nullspace of A is called the nullity of G .

This paper investigates **nut graphs** which are graphs of nullity one whose periphery is empty. It is shown that nut graphs of order n exist for each $n \geq 7$ and that among singular graphs nut graphs are characterized by their deck of spectra.

1 Introduction

The **adjacency matrix** $A(G) = A$ of a graph G having vertex set $\mathcal{V}(G) = \{v_1, v_2, \dots, v_n\}$ is an $n \times n$ symmetric matrix (a_{ij}) such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. A is also represented by $(R_1, R_2, \dots, R_n)^T$ where R_i is the i th row vector of A corresponding to vertex v_i . The **rank** of a graph G , denoted by $r(G)$, is the rank of its adjacency matrix A which is equal to $n(G) - \eta(G)$ where $n(G)$, $\eta(G)$ denote the order of G and the dimension of the nullspace of A (i.e. the nullity of A) respectively.

All the graphs we consider are **simple** (i.e. without multiple edges or loops) and the vertex set is labelled.

A graph G is said to be **singular** if its adjacency matrix A is a singular matrix. Then there exists a non-zero vector v_0 such that $Av_0 = 0$. Thus at least one of the eigenvalues of A is zero. Also v_0 is in the nullspace of A (which is the kernel of the linear transformation corresponding to A) and is called a **kernel eigenvector**.

For a graph G of nullity one, the nullspace of A is generated by v_0 since its dimension is one. Following [6], the subgraph of G induced by the vertices corresponding to the non-zero components of v_0 is called the **core** χ_t (sometimes referred to as the **support** in the literature) with respect to v_0 , where t is the number of vertices of the core called the **core-order**.

The set, the members of which are the remaining vertices of G , is called the **periphery** (w.r.t. v_0).

In [6], the structural features particular to the 61 singular graphs of nullity one, called minimal configurations, whose periphery induces the empty graph (with no edges) and where $|\mathcal{V}(\chi_t)| \leq 5$, were determined. It would be interesting to study the extent to which the vertices of the core spread through G . The indications were that the core is a **proper** subgraph of G . The same result holds for $|\mathcal{V}(\chi_t)| = 6$. However in [1] an example of a graph of core-order 9, where the core is G itself, is given and the question, whether there are other graphs with this property, that are substantially different in structure, is posed.

In this paper such graphs which we call nut graphs (having an empty periphery) are investigated. Certain structural features are studied and new structures presented. Furthermore it is shown that, for each integer $n \geq 7$, nut graphs of order n exist and that among singular graphs, nut graphs are characterized by their deck of spectra.

In related work [8], these results are used to show that for singular graphs the core-orders for which nut graphs occur determine the maximum weight of the vectors in a minimal basis for the nullspace of A .

Definition 1.1 *A graph G of order n is a **nut graph** if it is*

- i) singular with nullity one*
- ii) none of the components of a kernel eigenvector is zero.*

So in a nut graph, $G = \chi_n$ and the periphery \mathcal{P} is empty.

2 Structural features of nut graphs

The following lemmas describe five structural features of nut graphs.

Lemma 2.1 *A nut graph G is connected.*

Proof Let G be a nut graph of order n with adjacency matrix A and with kernel eigenvector $v_0 = (\alpha_1, \alpha_2, \dots, \alpha_t, \beta_1, \beta_2, \dots, \beta_s)^T$. Suppose it is disconnected with 2 components G_1 and G_2 . Then in block form,

$$A = \left(\begin{array}{c|c} A(G_1) & \mathbf{O} \\ \hline \mathbf{O} & A(G_2) \end{array} \right), \quad (1)$$

where \mathbf{O} is a zero matrix, and G_1 and G_2 are the components of G of order t and s respectively.

Now $A(v_0) = 0 \implies A(\alpha_1, \alpha_2, \dots, \alpha_t, 0, \dots, 0)^T = 0$
and $A(0, \dots, 0, \beta_1, \beta_2, \dots, \beta_s)^T = 0$.
 $\implies A$ has at least two linearly independent kernel eigenvectors.

This is a contradiction since a nut graph has nullity one. //

Lemma 2.2 *A nut graph G has no terminal vertices.*

Proof : Let u_k be a terminal vertex of a nut graph G with adjacency matrix A and kernel eigenvector $v_0 = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$, $\alpha_i \neq 0$, $i \in \{1, 2, \dots, n\}$. Then the corresponding row vector of A is $R_k = (a_{k1}, a_{k2}, \dots, a_{kn})^T$ where $a_{k,i} = 1$ for exactly one value $i \in 1, 2, \dots, t$, and zero for the rest. Thus $Av_0 = 0$ implies $\alpha_i = 0$, a contradiction. Thus R_k has at least two non-zero entries so that a core cannot have terminal vertices. //

Lemma 2.3 *A nut graph G is not bipartite.*

Proof: Let G be a nut graph of order n with kernel eigenvector $v_0 = (\alpha_1, \alpha_2, \dots, \alpha_t, \beta_1, \beta_2, \dots, \beta_s)^T$. Suppose that G is the bipartite graph $G(X, Y, \mathcal{E})$. Then in block form,

$$A = \left(\begin{array}{c|c} \mathbf{O} & B_1 \\ \hline B_2 & \mathbf{O} \end{array} \right), \quad (2)$$

where \mathbf{O} is a zero square matrix, and B_1 and B_2 describe the edges between X and Y .

Now $A(v_0) = 0 \implies A(\alpha_1, \alpha_2, \dots, \alpha_t, 0, \dots, 0)^T = 0$
and $A(0, \dots, 0, \beta_1, \beta_2, \dots, \beta_s)^T = 0$.
 $\implies A$ has at least two linearly independent kernel eigenvectors.

This is a contradiction since a nut graph has nullity one. //

Lemma 2.4 *Let G be a nut graph. A vertex $v \in \mathcal{V}(G)$ is not adjacent to just two adjacent vertices which have the same neighbours.*

Proof: Suppose v_1 is adjacent to the two vertices v_2 and v_3 which are adjacent and have the same neighbours. Then by applying relation $Av_0 = 0$, it follows that $\alpha_2 = \alpha_3 = 0$, a contradiction. Hence this configuration is inadmissible. //

Lemma 2.5 *Let G be a nut graph. None of the vertex deleted subgraphs $G - v$ is K_{n-1} .*

Proof: Suppose that the one vertex-deleted subgraphs of the nut graph G , say $G - v_n$, is the complete graph K_{n-1} . Further suppose that v_n has valency ρ , $1 \leq \rho \leq n - 1$. Without loss of generality, it may be assumed that v_n is adjacent to the vertices v_1, v_2, \dots, v_ρ . Then the adjacency matrix of G is of the form

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 & a_{1n} \\ 1 & 0 & 1 & 1 & \dots & 1 & a_{2n} \\ 1 & 1 & 0 & 1 & \dots & 1 & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad (3)$$

where $a_{1n} = a_{2n} = \dots = a_{\rho n} = 1$ and $a_{\rho+1,n} = a_{\rho+2,n} = \dots = a_{nn} = 0$.

Then if $v_0 = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$, $\alpha_i \neq 0$, $i \in \{1, 2, \dots, n\}$, the first ρ rows in the matrix equation $Av_0 = 0$ are

$$\begin{array}{cccccccccc} \alpha_2 & + & \alpha_3 & + & \alpha_4 & + & \dots & + & \alpha_\rho & + & \alpha_{\rho+1} & + & \dots & + & \alpha_n & = & 0 \\ \alpha_1 & + & \alpha_3 & + & \alpha_4 & + & \dots & + & \alpha_\rho & + & \alpha_{\rho+1} & + & \dots & + & \alpha_n & = & 0 \\ \alpha_1 & + & \alpha_2 & + & \alpha_4 & + & \dots & + & \alpha_\rho & + & \alpha_{\rho+1} & + & \dots & + & \alpha_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_1 & + & \alpha_2 & + & \alpha_3 & + & \dots & + & \alpha_{\rho-1} & + & \alpha_{\rho+1} & + & \dots & + & \alpha_n & = & 0 \end{array}$$

Hence

$$\alpha_1 = \alpha_2 = \alpha_3 = \dots = \alpha_\rho. \quad (4)$$

The last row in the matrix equation $Av_0 = 0$ is

$$\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_\rho = 0. \quad (5)$$

From (5) and (6) it follows that $\alpha_i = 0$, $i = 1, 2, \dots, \rho$. Hence, some of the components of the kernel eigenvector v_0 are zero, a contradiction since G is a nut graph.//

3 Small Nut Graphs

Theorem 3.1 *Let G be a nut graph. Then $\circ(G) \geq 7$ and the number of edges m is at least 8. There is only one nut graph with 7 vertices and 8 edges.*

Proof: The graphs of order less than 7 that have nullity one, that are connected, non-bipartite and without terminal vertices are as follows [2, 3]:

- 1 of order 4,

- 2 of order 5 and
- 17 of order 6.

By direct checking, it is established that each of these graphs has core number less than the order of the graph and so is not a nut graph.

Let G be a nut graph of order $n \geq 7$. Since it is connected $m \geq n - 1$. Equality is ruled out since connected graphs with $n - 1$ edges are trees which have terminal vertices.

Suppose now that $m = n$. Then G is a circuit (which is not a nut graph) or has terminal vertices. Hence $m > n$.

Among the graphs of order 7 with 8 edges, 4 have nullity one. Of these 4 graphs, only the graph X_1 shown in Fig. 1 is a nut graph. Thus this graph is the smallest nut graph. This completes the proof. //

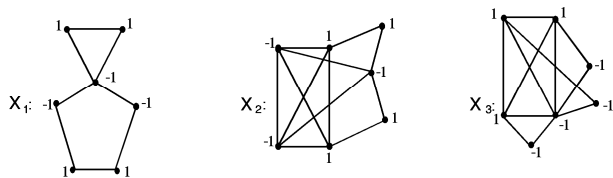


Fig.1

Theorem 3.2 *There are 3 nut graphs of order 7 shown in Fig.1.*

4 Construction of larger nut graphs

4.1 Inserting 4 vertices into an edge of a nut graph

Starting with a nut graph G with 2 adjacent vertices u, v , a larger nut graph G^* is constructed by inserting 4 vertices a, b, c, d in the edge uv , as shown in Fig.2. The components of a kernel eigenvector of G^* corresponding to the vertices u, a, b, c, d, v of the subgraph P_6 taken in order are $f(u), f(v), -f(u), -f(v), f(u), f(v)$ where $f(u)$ and $f(v)$ are the components, corresponding to the vertices u and v , respectively, of a kernel eigenvector v_0 of G . The components corresponding to the vertices of G remain unchanged.

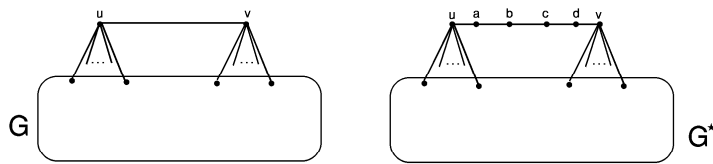


Fig.2

Lemma 4.1 *The graph G^* just constructed is a nut graph.*

In block form this is $\left(\begin{array}{c|c} A_n & \mathbf{0} \\ \hline \mathbf{0} & I_4 \end{array} \right)$ where $\mathbf{0}$ is a zero square matrix, I_4 is the identity matrix of order 4 and A_n denotes the rows and columns of $A(G)$.

It is clear that the rank of $A(G^*)$ is 4 more than the rank of $A(G)$. Thus the nullity of G^* is one and so G^* is a nut graph.

4.2 Addition of one vertex

Starting with a nut graph G with a kernel eigenvector $v_0 = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$, a vertex v is added joining a set of vertices \mathcal{I} such that the sum of their corresponding α_i vanishes. Another vertex u of G which is not adjacent to any of the vertices of \mathcal{I} is joined to each vertex of \mathcal{I} . This construction, is shown in Fig. 3 for $|\mathcal{I}| = 2$. If the nullity of the constructed graph G' is one then a nut graph G' of order $\circ(G) + 1$, is produced such that the components of the kernel eigenvector corresponding to the vertex v is equal but of opposite sign to that of u , whereas the components corresponding to the other vertices remain unchanged.

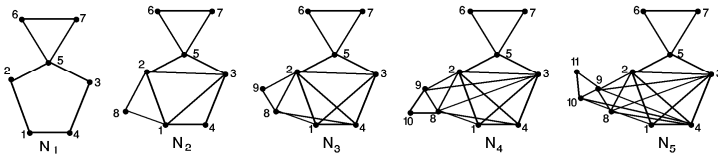


Fig.3

Starting with $N_1 = X_1$ shown in Fig.1, N_2 is constructed by joining each of the vertices v_3 and v_8 to v_1 and to v_2 . A nut graph of order 8 is obtained. A nut graph N_3 of order 9 is constructed by starting with N_2 and joining each of the vertices v_4 and v_9 to v_2 and to v_8 . A nut graph N_4 of order 10 is constructed by starting with N_3 and joining each of the vertices v_3 and v_{10} to v_8 and to v_9 .

The 4 nutgraphs N_1, N_2, N_3, N_4 will be used in the proof to show for which orders nut graphs exist.

Repeating the procedure for constructing N_1, N_2, N_3, N_4 , another nut graph N_5 of order 11 is constructed by starting with N_4 and joining each of the vertices v_4 and v_{11} to v_9 and to v_{10} . However, starting with N_5 and joining each of the vertices v_3 and v_{12} to v_{10} and to v_{11} does not yield a nut graph since the graph produced has nullity 2.

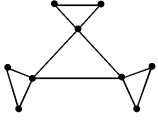


Fig.6

It is now shown that this construction yields nut graphs. The adjacency matrix of such a graph G of order $3n$ can be written as

$$A(G) = \left(\begin{array}{cccc|cccccc} & & & & 1 & 1 & 0 & & & \dots & & 0 \\ & & & & 0 & 0 & 1 & 1 & 0 & & \dots & & 0 \\ & & & & & & \ddots & & \ddots & & \ddots & & \\ & & & & 0 & & \dots & & & & 0 & 1 & 1 & 0 & 0 \\ & & & & 0 & & \dots & & & & \dots & & 0 & 1 & 1 \\ \hline 1 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & 0 & & \dots & & & & 0 \\ 1 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 & & \dots & & & & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & & & & \dots & & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & & & & \dots & & 0 \\ & & & \ddots & & & & & \ddots & & \ddots & & & & \\ & & & & & & & & & & & & \ddots & & \\ 0 & \dots & & 1 & 0 & & & & \dots & & & & 0 & 0 & 1 \\ 0 & \dots & & 1 & 0 & & & & \dots & & & & 0 & 1 & 0 \end{array} \right),$$

where C_n represents the rows and columns of C_n . (The horizontal lines in the matrix were introduced to render the pattern of the entries more obvious).

Using elementary row and column operations a matrix L of the same rank as $A(G)$ is obtained.

$$L = \left(\begin{array}{ccc|c} & & & \pm 1 \\ & & & \vdots \\ & I_{3n-1} & & \pm 1 \\ \hline 0 & \dots & & 0 \end{array} \right).$$

5 Order of nut graphs

Theorem 5.1 *There exist nut graphs of order n for each $n \geq 7$.*

Proof:

The 4 nutgraphs N_1, N_2, N_3, N_4 (constructed in section 4.2) and the constant value of the nullity on inserting 4 vertices (proved in section 4.1) ensure that there are nut graphs for all $n \geq 7$.//

This result is the basis of the study of the maximum weight of the vectors in a minimal basis for the nullspace of the adjacency matrix of a singular graph in [8, 9].

6 Deck of spectra of nut graphs

Let $\text{Adj}(A)$ denote the **adjoint** of A , which is the matrix (A_{ij}) where A_{ij} is the cofactor corresponding to a_{ij} of A .

Lemma 6.1 *Let G be a singular graph with adjacency matrix A . If $u_i = (A_{i1}, \dots, A_{in})$ is a non-zero row vector of $\text{Adj}(A)$, then u_i^T is a kernel eigenvector.*

Proof: The result follows since $\text{Det}(A) = 0$, and therefore $\forall i \in \{1, 2, \dots, n\}$,

$$A \begin{pmatrix} A_{i1} \\ A_{i2} \\ A_{i3} \\ \vdots \\ A_{in} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (6)$$

Lemma 6.2 *Let G be a graph with adjacency matrix A . Then $\eta = 1 \implies \text{rank}(\text{Adj}(A)) = 1$.*

Proof: It is shown that each row of $\text{Adj}(A)$ is a multiple of a non-zero row of $\text{Adj}(A)$. If $\eta(G) = 1$ then $r(G) = o(G) - 1$ and the determinant of at least one submatrix of order $n - 1$ of A is not zero. So at least one row vector of the adjoint of A is not zero. By Lemma 1, every non-zero row vector of $\text{Adj}(A)$ is a kernel eigenvector and is unique (up to multiples), since the nullspace of G has dimension one. //

Lemma 6.3 *For $\eta = 1$ the one-dimensional nullspace of A is generated by $u_k = (A_{k1}, \dots, A_{kn})$, any non-zero row vector of the adjoint.*

Proof: This follows from Lemmas 1 and 2 since the non-zero rows of the adjoint are multiples of each other. //

Theorem 6.1 *A nut graph has a deck of nonsingular vertex-deleted subgraphs.*

Proof: Let G be a nut graph with adjacency matrix A and adjoint $\text{Adj}(A) = (A_{ij})$. Since $\eta = 1$, $\text{rank}(\text{Adj}(A)) = 1$. The core of G is χ_n . So there exists $k \in \{1, 2, \dots, n\}$ such that, $u_k^T = (A_{k1}, A_{k2}, \dots, A_{kn})^T$ is a kernel eigenvector no component of which is zero. Since $\text{Adj}(A)$

is symmetrical, each row is non-zero and a rational multiple of u_k . Thus $A_{ki} \neq 0, \forall i, k$. Hence each diagonal entry A_{ii} of $\text{Adj}(A)$, is non-zero. On the other hand A_{ii} is equal to the determinant of the adjacency matrix of the vertex-deleted subgraph $G - v_i$. The determinant of the adjacency matrix of a singular graph is equal to zero. Thus none of the vertex-deleted subgraphs of G is singular. //

Theorem 6.2 *A graph with nullity one having a deck of nonsingular vertex-deleted subgraphs is a nut graph.*

Proof: Let G be a graph with adjacency matrix A and adjoint $\text{Adj}(A) = (A_{ij})$. The determinant of a vertex-deleted subgraph $G - v_i$ of G is A_{ii} which is therefore non-zero. Hence every row of the adjoint is a non-zero vector. By Lemma 6.1, every non-zero row of the adjoint is a kernel eigenvector of G . Because $\eta(G) = 1$, all rows of the adjoint are rational multiples of each other by Lemma 6.2. Hence each entry of $\text{Adj}(A)$ is non-zero and therefore all components of a kernel eigenvector in the one-dimensional nullspace of A are non-zero. //

From Theorems 6.1 and 6.2 the following results are deduced:

Theorem 6.3 *If a graph G has nullity one, then G is a nut graph if and only if none of the spectra in the deck of spectra of G has a zero eigenvalue.*

Corollary 6.1 *If a graph G has a deck of nonsingular vertex-deleted subgraphs, then G is either non-singular or is a nut graph.*

Finally Theorem 6.1 is applied to deduce another structural feature of nut graphs.

Lemma 6.4 *If e is an edge of the nut graph G such that $G - e$ has two disjoint components A, B , then neither A nor B is bipartite.*

Proof Let $e = uv$, such that u is a vertex of A and v is a vertex of B . Suppose that B is bipartite. Then B is a component of $G - u$, whereas $B - v$ is a component of $G - v$. Either B or $B - v$ possesses an odd number of vertices; hence either B or $B - v$ is singular. Therefore either $G - u$ or $G - v$ is singular. Therefore, by Theorem 6.1, G is not a nut graph. //

It is observed that this property agrees with the structure of the nut graphs in sections 4.3 and 4.4.

7 Coefficient of λ

In [7] it is proved that, for a graph G with nullity one, the square of the norm of the principal kernel eigenvector v_0 (the one with integral components the g.c.d. of which is one) divides \mathcal{L} ,

the coefficient of λ in the characteristic polynomial of $A(G)$. A minimal configuration is a singular graph with nullity one, the periphery of which induces the empty graph and where the deletion of a vertex of the periphery increases the nullity. A conjecture was presented which said that for a minimal configuration, examples of which are shown in Fig.7, $|\mathcal{L}| = \|v_0\|^2$. Nut graphs are minimal configurations and counter examples to the above conjecture are found among nut graphs. In fact for nut graph N_1 , $mf = \frac{|\mathcal{L}|}{\|v_0\|^2} = 1$ but for nut graphs N_2, N_3, N_4 and N_5 , $mf = \frac{|\mathcal{L}|}{\|v_0\|^2} \neq 1$. Indeed the investigation of the significance of mf is still an open problem.

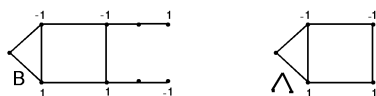


Fig.7

8 Acknowledgement:

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