

Research Article

On the Spectrum of Threshold Graphs

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The antiregular connected graph on r vertices is defined as the connected graph whose vertex degrees take the values of $r - 1$ distinct positive integers. We explore the spectrum of its adjacency matrix and show common properties with those of connected threshold graphs, having an equitable partition with a minimal number r of parts. Structural and combinatorial properties can be deduced for related classes of graphs and in particular for the minimal configurations in the class of singular graphs.

1. Introduction

A graph $G = G(\mathcal{U}, \mathcal{E})$ of order n has a labelled vertex set $\mathcal{U} = \{1, 2, \dots, n\}$ containing n vertices and a set \mathcal{E} of m edges consisting of unordered pairs of the vertices. When a subset \mathcal{U}_1 of \mathcal{U} is deleted, the edges incident to \mathcal{U}_1 are also deleted. The subgraph $G - \mathcal{U}_1$ of G is said to be an *induced* subgraph of G . The subgraph of G obtained by deleting a particular vertex v is simply denoted by $G - v$. The *cycle* and the *complete graph* on n vertices are denoted by C_n and K_n , respectively.

The graphs we consider are *simple*, that is, without loops or multiple edges. We use bold face, say \mathbf{G} , to denote the 0-1-adjacency matrix of the graph bearing the same name G , where the ij th entry of the symmetric matrix \mathbf{G} is 1 if $\{i, j\} \in \mathcal{E}$ and 0 otherwise. We note that the graph G is determined, up to isomorphism, by \mathbf{G} . The adjacency matrix \mathbf{G}^C of the complement G^C of G is $\mathbf{J} - \mathbf{I} - \mathbf{G}$, where each entry of \mathbf{J} is one and \mathbf{I} is the identity matrix. The degree of a vertex i is the number of nonzero entries in the i th row of \mathbf{G} .

The disconnected graph with two components G_1 and G_2 is their *disjoint union*, denoted by $G_1 \cup G_2$. For $r \geq 2$, the graph rG is the disconnected graph with r components, where each component is isomorphic to G . The *join* $G_1 \nabla G_2$ of G_1 and G_2 is $(G_1^C \cup G_2^C)^C$.

For the linear transformation \mathbf{G} , the n real numbers $\{\lambda\}$ satisfying $\mathbf{G}\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{R}^n$ are said to be *eigenvalues* of \mathbf{G} and form the spectrum of G . They are the

solutions of the *characteristic polynomial* $\phi(G, \lambda)$ of \mathbf{G} , defined as the polynomial $\det(\lambda \mathbf{I} - \mathbf{G})$ in λ . The subspace $\ker \mathbf{G}$ of \mathbf{R}^n that maps to zero under \mathbf{G} is said to be the *nullspace* of \mathbf{G} . A graph G is said to be *singular* of *nullity* η if the dimension of $\ker(\mathbf{G})$ is η . The nonzero vectors, $x \in \mathbf{R}^n$, in the nullspace, termed *kernel eigenvectors* of G , satisfy $\mathbf{G}x = \mathbf{0}$. We note that the multiplicity of the eigenvalue zero is η . If there exists a kernel eigenvector of \mathbf{G} with no zero entries, then G is said to be a *core graph*. The cycle C_4 on four vertices is a core graph of nullity two with a kernel eigenvector $(1, 1, -1, -1)^t$ for the usual labelling of the vertices round the cycle. A core graph of nullity one is said to be a *nut graph* [1]. A *minimal configuration* for a particular core, to be defined formally in Section 6, is intuitively a graph of nullity one with a minimal number of vertices and edges for that core.

The distinct eigenvalues $\mu_1, \mu_2, \dots, \mu_p$, $1 \leq p \leq n$, which have an associated eigenvector *not* orthogonal to \mathbf{j} (the vector with each entry equal to one) are said to be *main*. We denote the remaining distinct eigenvalues by μ_{p+1}, \dots, μ_s , $s \leq n$, and refer to them as *nonmain*. By the Perron-Frobenius theorem [2, page 6] the maximum eigenvalue of the adjacency matrix of a connected graph has an associated eigenvector (termed the Perron vector) with all its entries positive. Therefore, at least one eigenvalue of a graph is main.

A *cograph*, or complement-reducible graph, is a graph that can be generated from the single-vertex graph K_1 by complementation and disjoint union. *Threshold graphs* are a subclass of cographs. They were first introduced in 1977 by Chvátal and Hammer in connection with the equivalence between set packing and knapsack problems [3] and independently, in the same year, by Henderson and Zalcstein for parallel systems in computer programming [4]. It is surprising that they kept being rediscovered in different contexts leading to several equivalent definitions. The most useful for our purposes are two, given below: one in terms of their forbidden induced subgraphs and the other in terms of their degree sequence [5, 6]. For the latter definition, the *graph partition* Π of $2m$ into *parts* equal to the vertex degrees $\{\rho_1, \rho_2, \dots, \rho_n\}$ is needed. The array of boxes $F(\Pi)$, known as a *Ferrers/Young diagram* for the monotonic nonincreasing sequence $\Pi = \{\rho_1, \rho_2, \dots, \rho_n\}$ consists of n rows of ρ_i boxes as i runs successively from 1 to n . Threshold graphs are characterized by a particular shape of the Ferrers/Young diagram (see Figure 4), which will be described in Section 3.4.

Definition 1.1. (i) A *threshold graph* is a graph with no induced subgraphs isomorphic to any of the following subgraphs on four vertices: the path P_4 , the cycle C_4 and the two copies $2K_2$ of the complete graph K_2 on two vertices. It is said to be P_4 -, C_4 -, and $2K_2$ -free.

Equivalently, (ii) if the monotonic nonincreasing degree sequence, $\Pi = \{\rho_1, \rho_2, \dots, \rho_n\}$, of a graph G is represented by the rows of a Ferrers/Young diagram $F(\Pi)$, where the length of the principal square of $F(\Pi)$ is $f(\Pi)$ and the lengths $\{\pi_k^* : 1 \leq k \leq f(\Pi)\}$ of the columns of $F(\Pi)$ satisfy $\pi_k^* = \rho_k + 1$, then G is said to be a *threshold graph* [7, Lemma 7.23].

If the parts of a threshold graph partition of $2m$ are all equal, then the graph is regular and corresponds to the complete graph. If, on the other hand, there are as many distinct sizes of the parts of a threshold graph partition of $2m$ as possible, then the graph is said to be *antiregular*. Recall that at least two vertices in a graph have the same degree.

Definition 1.2. An *antiregular graph* on r vertices is defined as a threshold graph whose vertex degrees take as many different values as possible, that is, $r - 1$ distinct nonnegative integral values.

Definition 1.3. The partition $\mathcal{U}_1 \dot{\cup} \mathcal{U}_2 \dot{\cup} \dots \dot{\cup} \mathcal{U}_r$ of the vertex set \mathcal{U} of a graph G is said to be an *equitable partition* if, for all $i, j \in \{1, 2, \dots, r\}$, the number of neighbours in \mathcal{U}_j of a vertex in \mathcal{U}_i depends only on the choice of i and j .

The overall aim of this paper is to explore the spectrum of its adjacency matrix and show common properties with those of connected threshold graphs, having an equitable partition with a minimal number r of parts.

The paper is organised as follows. In Section 2, cographs are reviewed and made use of in Section 3 to determine a particular representation of a threshold graph that has earned it the name of *nested split graph*. We also present various other representations that are used selectively to simplify our proofs. In Section 4, a procedure that transforms the Ferrers/Young diagram into the adjacency matrix of the threshold graph for a particular vertex labelling is given. The structures of the graph and of its underlying antiregular graph are also compared.

Our main results are as follows.

- (i) In Section 5, the Ferrers/Young diagram comes in use to explore the nullspace of a threshold graph.
- (ii) In Section 6, we show that all minimal configurations on at least five vertices have the subgraph P_4 induced.
- (iii) We show in Section 7 that the spectrum of a connected threshold graph G and its underlying antiregular graph show common characteristics. All the eigenvalues other than 0 and -1 are main and each main eigenvalue contributes to the number of walks. Moreover, the spectrum of its quotient graph G/Π consists precisely of the main eigenvalues of G . The characteristic polynomial of G/Π is reducible over the integers (i.e., it has polynomial factors) for certain threshold graphs G .
- (iv) We end with a discussion, in Section 8, on the variation in the sign pattern of the spectrum as vertices are added to a threshold graph to produce another threshold graph.

2. Cographs

A cograph is the union or the join of subgraphs of the form $(\dots((r_1 K_1)^C \dot{\cup} (r_2 K_1))^C \dot{\cup} \dots \dot{\cup} (r_s K_1)^C)$, where $r_i \in \mathbb{Z}^+ \cup \{0\}$, for all i . Therefore, the family of cographs is the smallest class of graphs that includes K_1 and is closed under complementation and disjoint union. It is well known that no cograph on at least four vertices has P_4 as an induced subgraph [8]. In fact cographs can also be characterized as P_4 -free graphs.

Cographs have received much attention since the 1970s. They were discovered independently by many authors including Jung [9] in 1978, Lerchs [10] in 1971 and, Seinsche [11] and Sumner [12], both in 1974. For a more detailed treatment of cographs, see [8].

Connected graphs, which are $2K_2$ -, P_4 -, and C_4 -free, necessarily have a *dominating vertex*, that is, a vertex adjacent to all the other vertices of the graph. Thus, all connected threshold graphs have a dominating vertex.

By construction, a connected cograph also has a dominating vertex. Therefore, its complement has at least one isolated vertex. A necessary condition for a connected graph to have a connected complement is that it has P_4 as an induced subgraph [7, Theorem 1.19]. The set of cographs and the class of graphs with a connected complement are disjoint as sets.

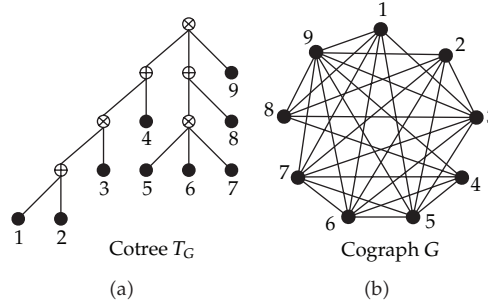


Figure 1: Cotree T_G for the cograph G .

However, if the graph H is $P_4 \cup K_1$, then both H and H^C have P_4 -induced. Thus there exist connected graphs that are neither P_4 free nor have a connected complement.

Recall that $G_1 \nabla G_2 = (G_1^C \cup G_2^C)^C$. Hence, cographs are also characterized as the smallest class of graphs that includes K_1 and is closed under join and disjoint union. On this definition of cographs, the proofs in [13], of the result that cographs are polynomial reconstructible from the deck of characteristic polynomials of the one-vertex deleted subgraphs, are based.

A cograph can be represented uniquely by a *cotree*, as explained in [14] and later in [13]. Figure 1 shows the cotree T_G of the cograph G . The vertices \oplus , \otimes , and \bullet of a cotree represent the disjoint union, the join, and the vertices of the cograph, respectively. For simplicity we say that the terminal vertices of T_G are vertices of G . The cotree T_G is a rooted tree and only the terminal vertices represent the cograph vertices. An interior vertex \oplus or \otimes of T_G represents the subgraph of G induced by its terminal successors. The immediate successors of \oplus can be cograph vertices \bullet or \otimes . Similarly the immediate successors of \otimes can be cograph vertices \bullet or \oplus . Therefore, the interior vertices of T_G on a (oriented) path descending from the root to a terminal vertex of T_G are a sequence of alternating \otimes and \oplus .

3. Representations of Threshold Graphs

In this section we present some of the various representations of threshold graphs. Collectively, they provide a wealth of information that determine combinatorial properties of these graphs. We start with the cotree representation as in the previous section. There are certain restrictions on the structure of a cotree in the case when a cograph is a threshold graph.

We give a proof to the following result quoted in [13].

Lemma 3.1. *If a cograph G is also a threshold graph, then each interior vertex of T_G has at most one interior vertex as an immediate successor.*

Proof. A threshold graph G is P_4 -free and therefore is a cograph which can be represented by a cotree T_G . Note that P_4 cannot be represented as a cotree. In a threshold graph, there are no induced subgraphs isomorphic to C_4 or to $2K_2$. Therefore, the configurations in Figure 2(a) representing C_4 and 2(b) representing $2K_2$ as cotrees are not allowed in the cotree T_G corresponding to a threshold graph G . We deduce that the number of interior vertices which are immediate successors of an interior vertex is less than two, as required. \square

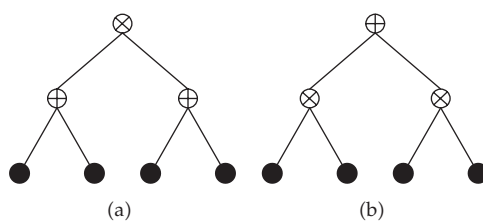


Figure 2: Representations of C_4 and $2K_2$ in a cotree.

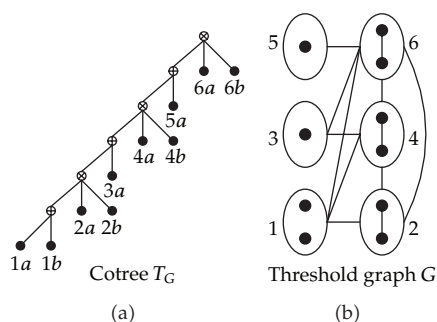


Figure 3: Cotree T_G and the nested structure of the threshold graph G .

We now present various other representations of threshold graphs that are used in the proofs that follow.

3.1. Cotrees of Nested Split Graphs

A *caterpillar* is a tree in which the removal of all terminal vertices (i.e., those of degree 1) gives a path. The following result follows immediately from Lemma 3.1.

Corollary 3.2. *The cotree of a threshold graph is a caterpillar.*

The vertex set of a *split graph* is partitioned into two subsets, one of which is a *clique* (inducing a complete subgraph) and the other a *coclique* or an independent set (inducing the empty graph with no edges). Because of its structure, a threshold graph is also referred to as a *nested split graph*.

The first vertex labelling (which we will refer to as Lab1) of a threshold graph is according to its construction. Starting from K_1 (vertex 1a), the graph in Figure 3 is ((((((((($(K_1 \cup K_1) \nabla K_1$) ∇K_1) $\cup K_1$) ∇K_1) ∇K_1) $\cup K_1$) ∇K_1) ∇K_1) coded as (((((($(K_1 \cup K_1) \nabla_2 K_1$) $\cup K_1$) $\nabla_2 K_1$) $\cup K_1$) $\nabla_2 K_1$) to avoid repetitions of successive joins or unions. Therefore, according to the vertex labelling in Figure 3, G is (((((((((1a \cup 1b) ∇ 2a) ∇ 2b) \cup 3a) ∇ 4a) ∇ 4b) \cup 5a) ∇ 6a) ∇ 6b). The cotree T_G represents the threshold graph G drawn next to it in a way so as to emphasise the nested split graph structure of G , where the circumscribed vertices labelled 1 represent the subgraph induced by the vertices 1a and 1b, and similarly for the other circumscribed subsets of vertices.

In T_G , the terminal vertices $\{\bullet\}$ which are immediate successors of a vertex \otimes form a clique (inducing a complete subgraph) whereas those immediately succeeding a vertex \oplus form a coclique (inducing a subgraph without edges). A line in G joining R and S , which are circumscribed cliques or cocliques, means that each vertex of R is adjacent to each vertex of S .

3.2. Minimal Equitable Partition of the Vertex Set

Our labelling of the r parts in the equitable partition of the vertices of a connected threshold graph $C(a_1, a_2, \dots, a_r)$ follows the addition of the vertices in the construction in order, namely, $((\dots(\dot{\cup}_{a_1} K_1 \nabla_{a_2} K_1) \dot{\cup}_{a_3} K_1) \nabla \dots \nabla_{a_r} K_1)$ according to the coded representation of the graph in Figure 3. Then, the nested structure of the threshold graph becomes clear. The parts are cliques or cocliques of size a_i for $1 \leq i \leq r$. For a minimal value of r , Π is said to be a *nondegenerate* equitable partition for the *nondegenerate* representation $C(a_1, a_2, \dots, a_r)$. All other equitable partitions of the vertex set are refinements of Π with a larger number of parts, when an equitable partition and the corresponding representation $C(a_1, a_2, \dots, a_r)$ are said to be *degenerate*. Unless otherwise stated we will assume that equitable vertex partitions and representations are nondegenerate. In particular $a_1 \neq 1$.

According to our labelling convention (Lab1) for $C(a_1, a_2, \dots, a_r)$ as in Figure 3, a threshold graph G whose cotree T_G has root \otimes is connected. If r is even, then a_1 is associated with a coclique, whereas, for r odd, a_1 is associated with a clique. It follows that the monotonic non-increasing vertex degree sequence of G will be associated with $a_r, a_{r-2}, \dots, a_2, a_1, a_3, \dots, a_{r-1}$ in that order if r is even and $a_r, a_{r-2}, \dots, a_1, a_2, a_4, \dots, a_{r-1}$ in that order if r is odd. By convention therefore, for a nondegenerate equitable partition, $a_i \geq 1$ for $2 \leq i \leq r-1$ and $a_1 \geq 2$. According to this representation, the graph of Figure 3 has the nondegenerate representation $C(2, 2, 1, 2, 1, 2)$.

3.3. The Binary Code of a Threshold Graph

For the purposes of inputting an n -vertex-threshold graph to be processed in a computer program, the graph is encoded as a string of $n-1$ bits. The graph is represented as a sequence of 0 and 1 entries where 0 represents the *addition* of an isolated vertex and 1 represents the *addition* of a dominating vertex in the construction of the graph, starting from K_1 , as described above.

The graph of Figure 3 is encoded as (011011011).

3.4. Degree Sequence

The last representation of a threshold graph that we now give is constructed from the degree sequence. Following Definition 1.1(ii), let $F(\Pi)$ be the Ferrers/Young diagram (Figure 4) for the nonincreasing degree sequence giving a vertex partition $\Pi = \{\rho_1, \rho_2, \dots, \rho_n\}$ of $2m$ for an n -vertex graph. The largest principal square of boxes in $F(\Pi)$ is termed the *Durfee* square and $f(\Pi)$ denotes the size of the Durfee square (i.e., the length of a side of the Durfee square). A graph is graphical if and only if $\pi_i^* \geq \rho_i + 1$ for $1 \leq i \leq f(\Pi)$ [15].

It is well known that there exist nonisomorphic graphs with the same degree sequence. A graph determined, up to isomorphism, by its degree sequence is said to be a *unigraph*.

Lemma 3.3 (see [7, Theorem 7.30]). *A threshold graph is a unigraph.*

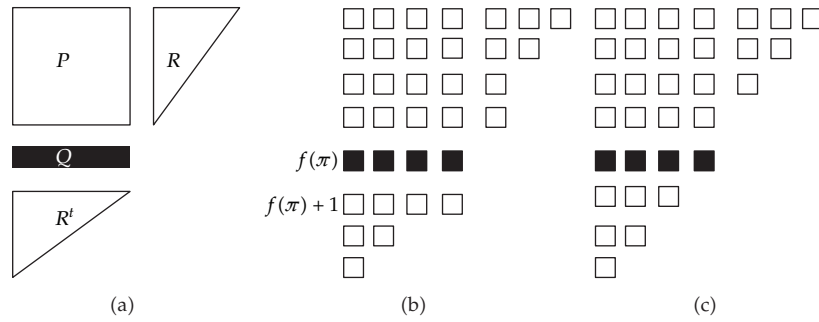


Figure 4: The Ferrers/Young diagram for a threshold graph.

The degree sequence Π of a threshold graph also produces a particular structure of the Ferrers/Young diagram $F(\Pi)$, shown in Figure 4.

Lemma 3.4 (see [7]). *For a threshold graph, $F(\Pi)$ consists of four blocks P , Q , R , and its transpose R^t , where P is the Durfee square, Q is the $(f(\Pi) + 1)$ th row of $F(\Pi)$ of length $f(\Pi)$, and R is the array of boxes left after removing the Durfee square from the first $f(\Pi)$ rows of $F(\Pi)$.*

4. The Structure of Threshold Graphs

An interesting algorithm was presented in [15] to construct a threshold graph. The adjacency list $adjList$ of the graph, that is the list of neighbours of each vertex, is in fact obtained by filling in the boxes of the i th row in $F(\Pi)$ with consecutive integers starting from 1, but skipping i . By Lemma 3.3, $F(\Pi)$ gives a unique threshold graph, up to isomorphism and therefore provides a canonical vertex labelling. We now present a procedure to produce the adjacency matrix of the labelled threshold graph corresponding to $adjList$ from $F(\Pi)$. We note that this gives us the second labelling, Lab2, in order of the nonincreasing degree sequence and therefore different from Lab1 used for $C(a_1, a_2, \dots, a_r)$.

Theorem 4.1. *The $n \times n$ adjacency matrix \mathbf{G} of a threshold graph G is obtained from its Ferrers/Young diagram $F(\Pi)$, representing the degree sequence of a n -vertex graph, as follows. The i th box is inserted in each i th row and filled with a zero entry. The rest of the existing boxes are filled with the entry 1. Boxes are now inserted so that a $n \times n$ array of boxes is obtained. Each of the remaining empty boxes is filled with zero. The $n \times n$ array of 0-1-numbers obtained is the adjacency matrix \mathbf{G} .*

The rows and columns of the adjacency matrix constructed in Theorem 4.1 are indexed according to the nonincreasing degree sequence. If, for a threshold graph, each of the boxes of the i th row in $F(\Pi)$ is filled with i to obtain $H(\Pi)$, then the adjacency list $adjList$ of the graph is just a rearrangement of the entries of $H(\Pi)$ since, by Definition 1.1, $\pi_k^* = \rho_k + 1$. Due to the shape of the nonzero part, the adjacency matrix is said to have “a stepwise” form [16, 17].

4.1. The Antiregular Graph

The antiregular graph A_r may be considered to be the smallest threshold graph for an equitable vertex partition having a given number $(r - 1)$ of parts.

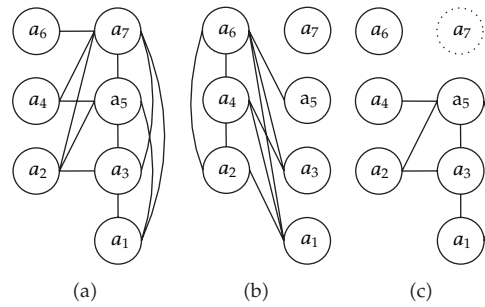


Figure 5: The threshold graph $G = C(a_1, a_2, \dots, a_7)$, G^C and $G - v$ for $a_7 = 1$.

Definition 4.2. An *antiregular graph* A_r on r vertices is a graph whose vertex degrees take the values of $r - 1$ distinct (nonnegative) integers.

We shall use the r -vertex connected antiregular graph A_r with the largest number $(r - 1)$ of parts in its equitable partition, having degenerate representation $C(1, 1, \dots, 1)$ using Lab1. Any part can be expanded to produce a threshold graph $C(a_1, a_2, \dots, a_r)$, taking care to preserve the nested split structure. The connected antiregular graph A_r with degenerate equitable partition into r parts is adopted as the *underlying graph* of a connected threshold graph for an equitable vertex partition with r parts.

Lemma 4.3. An induced subgraph H of $G = C(a_1, a_2, \dots, a_r)$ is $C(b_1, b_2, \dots, b_r)$, where $0 \leq b_i \leq a_i$ for $1 \leq i \leq r$.

Proof. When $a_i \neq b_i$ for at least one value of i , to produce H , vertices are deleted from the part of size a_i in the equitable partition of \mathcal{U}_G . This procedure produces an induced subgraph at each stage and it is repeated until b_i is reached for each i . \square

The threshold graph $C(1, 1, \dots, 1)$ having r parts, where each part is of size 1, is the degenerate form of A_r . Its nondegenerate form, consistent with the cotree representations of threshold graphs, is $C(2, 1, \dots, 1, 1)$ having $r - 1$ parts, with only the first part of size 2. As an immediate consequence of Lemma 4.3 we have the following.

Corollary 4.4. The connected antiregular graph $C(2, 1, 1, \dots, 1)$, having $r - 1$ parts, with degenerate representation $C(1, 1, 1, \dots, 1)$, having r parts, is an induced subgraph of $C(a_1, a_2, \dots, a_r)$ where $1 \leq a_i$ for $2 \leq i \leq r$ and $a_1 \geq 2$.

On taking the complement of $C(a_1, a_2, \dots, a_r)$ or on deleting a dominating vertex when $a_r = 1$, a disconnected graph is obtained (see Figures 5 and 6).

Proposition 4.5. Let v be the dominating vertex of A_r . Then, (i) $A_r - v$ is $K_1 \cup A_{r-2}$ and (ii) $A_r^C = K_1 \cup A_{r-1}$.

Figures 5 and 6, respectively, show the threshold graphs with underlying A_7 and A_8 , their complements, and the v -deleted subgraphs when v is the only dominating vertex. The corresponding representations of A_7 and A_8 are $C(2, 1, 1, 1, 1, 1)$ and $C(2, 1, 1, 1, 1, 1, 1)$, respectively.

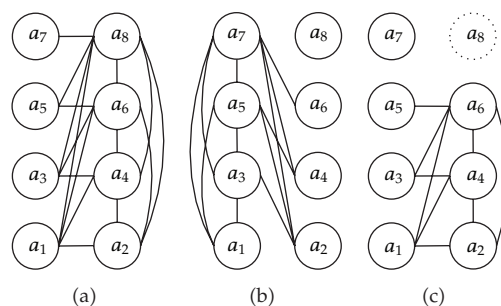


Figure 6: The threshold graph $G = C(a_1, a_2, \dots, a_8)$, G^C and $G - v$ for $a_8 = 1$ (Lab1).

Proposition 4.6. *The binary codes for the connected antiregular graphs A_{2k} and A_{2k+1} are, respectively, the $(2k - 1)$ -string $(1\ 0\ 1\ 0\ \dots\ 1)$ and the $2k$ -string $(0\ 1\ 0\ \dots\ 1)$ with alternating 0 and 1 entries.*

Since the binary code follows the construction of A_r algorithmically, we have the following.

Corollary 4.7. *The construction of connected antiregular graphs is as follows: for $k \in \mathbb{Z}^+$:*

$$A_{2k} = (\dots (K_1 \nabla K_1) \dot{\cup} K_1) \nabla \dots) \nabla K_1,$$

$$A_{2k+1} = (\dots (K_1 \dot{\cup} K_1) \nabla K_1) \dot{\cup} \dots) \nabla K_1.$$

4.2. The Complement of a Threshold Graph

The complement of a connected threshold graph $C(a_1, a_2, \dots, a_r)$ is disconnected and is denoted by $D(a_1, a_2, \dots, a_r)$ (see Figure 3). The following result is deduced from the construction of the complement.

Proposition 4.8 (see [18]). *The cotree T_{G^C} of the complement $G^C = D(a_1, a_2, \dots, a_r)$ of $G = C(a_1, a_2, \dots, a_r)$ is obtained from T_G by changing the interior vertices from \otimes to \oplus and viceversa.*

Corollary 4.9. *The complement G^C of the connected threshold graph $C(a_1, a_2, a_3, \dots, a_r)$, is the disconnected threshold graph $D(a_1, a_2, \dots, a_r)$ isomorphic to $C(a_1, a_2, \dots, a_{r-1}) \dot{\cup} a_r K_1$.*

Proof. Since $C(a_1, a_2, a_3, \dots, a_r)$ is connected, its cotree has \otimes as a root. Therefore, by Proposition 4.8, the cotree $D(a_1, a_2, \dots, a_r)$ has \oplus as a root, and therefore it has coclique K_{a_r} . □

Proposition 4.10. *The binary string coding of the threshold graph $C(a_1, a_2, \dots, a_{2k})$, with the underlying graph A_{2k} , is the $2k$ -string $(0^{a_1-1}\ 1^{a_2}\ \dots\ 1^{a_{2k}})$ of 0 and 1 entries. (The superscripts denote repetition; 1^{a_i} denotes the substring $111\dots$ with 1 repeated a_i times).*

Similarly the binary string coding of the threshold graph $C(a_1, a_2, \dots, a_{2k+1})$, with underlying graph A_{2k+1} , is the $2k + 1$ -string $(1^{a_1-1}\ 0^{a_2}\ \dots\ 1^{a_{2k+1}})$.

5. The Nullity of Threshold Graphs

A pair of *duplicate* vertices of a graph are nonadjacent and have common neighbours, whereas a pair of *coduplicate* vertices are adjacent and have common neighbours. The rows of the adjacency matrix corresponding to duplicate vertices are identical and for those of coduplicate vertices k and h , the k th and h th rows differ only in the k th and h th entries. It follows that both duplicates and coduplicates produce the eigenvector with only two nonzero entries, namely, 1 and -1 , at positions corresponding to the pair of vertices, with corresponding eigenvalue 0 and -1 , respectively.

Remark 5.1. In this section we adopt the vertex labelling Lab2 of a threshold graph induced by the Ferrers/Young diagram in accordance with the procedure to form the “stepwise” adjacency matrix presented in Theorem 4.1.

A graph with duplicates is often considered as having repeated vertices and therefore redundant properties. We call the induced subgraph of a graph obtained by removing repeated vertices *canonical*.

Theorem 5.2. *An upper bound for the nullity $\eta(G)$ of the adjacency matrix of a threshold graph is $n - f(\Pi) - 1$.*

Proof. When the adjacency matrix \mathbf{G} is obtained from *adjList*, the first $f(\Pi)$ rows are shifted so that none of them is repeated. The first $f(\Pi) + 1$ labelled vertices form a clique and hence the rank $rk(\mathbf{G})$ of the adjacency matrix \mathbf{G} of the n -vertex G which is $n - \eta(G)$ is at least $f(\Pi) + 1$. \square

The bound in Theorem 5.2 is reached, for instance, by the threshold graphs $C(f(\Pi) + 1)$ (the complete graph) and by $C(f(\Pi) + 1, f(\Pi))$.

Theorem 5.3. *Let G be a threshold graph on n vertices, with Durfee square size $f(\Pi)$ and nullity $\eta(G)$. If $n > 2f(\Pi)$, then G has duplicate vertices.*

Proof. The last $n - f(\Pi)$ rows of $F(\Pi)$ are not affected by the introduction of the zero diagonal when constructing \mathbf{G} as in Theorem 4.1. Hence, duplicates may only occur among the last $n - f(\Pi)$ labelled vertices. If G were to have no duplicate vertices, then the last $n - f(\Pi)$ rows of \mathbf{G} need to be all different. Since the $f(\Pi)$ th row is $f(\Pi)$ long, then, by a form of the pigeon-hole principle, the largest number n of vertices possible for the graph to have no duplicates is $2f(\Pi)$. Therefore if $n > 2f(\Pi)$, G has at least one pair of duplicate vertices. \square

A threshold graph may have duplicate vertices even if $n < 2f(\Pi)$. We note again that a kernel eigenvector corresponding to duplicate vertices has only two nonzero entries. This prompts the question: can a kernel eigenvector of the threshold graph have more than two nonzero entries? The answer is in the negative as we will now see.

Theorem 5.4. *The nullity $\eta(G)$ of a threshold graph G is the number of vertices removed to obtain a canonical graph.*

Proof. Let H be the canonical graph obtained from G by removing all the duplicate vertices. Let us say that the number of vertices removed is t . Since the reflection in the first column of the adjacency matrix \mathbf{H} of H is in row echelon form, then the rows of \mathbf{H} after the $f(\Pi)$ th is in strict “stepwise” form. Hence, the columns of \mathbf{H} are linearly independent. Now if

the t vertices are added to H in turn to obtain G again, then the nullity increases by one at each stage, contributing to the nullspace of the graph obtained, a kernel eigenvector (with exactly two nonzero entries) while preserving the existing ones. We deduce that there are only t linear combinations among the rows of \mathbf{G} arising from the repeated rows in the last $n - f(\Pi)$ rows. Therefore, the nullity of G is t . Moreover, a kernel eigenvector cannot have more than two nonzero entries. \square

In the proof of Theorem 5.4, the following result becomes evident.

Corollary 5.5. *If a threshold graph is singular, then no kernel eigenvector has more than two nonzero entries.*

Note that any repeated rows in the first $f(\Pi)$ rows of $F(\Pi)$ give coduplicates. Also $f(\Pi)$ is the degree of a vertex in the first part of the equitable partition of the threshold graph defined by $C(a_1, a_2, a_3, \dots, a_r)$ for Lab1. For A_r , this corresponds to the $\lfloor (r+1)/2 \rfloor$ th degree in the monotonic nonincreasing sequence of distinct degrees (the $\lfloor (r+1)/2 \rfloor$ th vertex for labeling Lab2).

That an antiregular graph has exactly one pair of either duplicates or coduplicates follows from its construction.

Theorem 5.6.

- (i) *An antiregular graph A_{2k-1} on an odd number of vertices has a duplicate vertex.*
- (ii) *An antiregular graph A_{2k} on an even number of vertices has a coduplicate vertex.*

Proof. The graph A_r is $C(2, 1, 1, \dots, 1)$. Therefore if r is even, it has a clique of two and hence a pair of coduplicate vertices. On the other hand, if r is odd, then it has a coclique of two, producing a pair of duplicate vertices. \square

To obtain the number of duplicate and coduplicate vertices in a threshold graph, we count the number of vertices to be removed from G and G^C , respectively, to obtain a canonical graph.

Theorem 5.7. *A threshold graph with nondegenerate representation $C(a_1, a_2, a_3, \dots, a_r)$, where r is even, has*

- (i) $\sum_{k=1}^{r/2} (a_{2k-1} - 1)$ duplicate vertices,
- (ii) $\sum_{k=1}^{r/2} (a_{2k} - 1)$ coduplicate vertices.

For odd r , $C(a_1, a_2, a_3, \dots, a_r)$ has

- (i) $\sum_{k=1}^{(r-1)/2} (a_{2k} - 1)$ duplicate vertices,
- (ii) $\sum_{k=1}^{(r+1)/2} (a_{2k-1} - 1)$ coduplicate vertices.

6. Minimal Configurations

Most of the information to determine the grounds for a labelled graph G to be singular is encoded in the nullspace $\ker(\mathbf{G})$ of its adjacency matrix \mathbf{G} (i.e., in $\ker(\mathbf{G}) := \{\mathbf{x} : \mathbf{G}\mathbf{x} = \mathbf{0}\}$). The support of a kernel eigenvector \mathbf{x} in $\ker(\mathbf{G})$ is the set of vertices corresponding to the nonzero

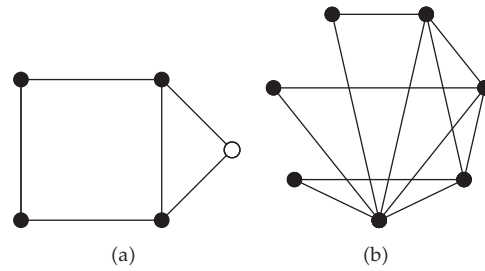


Figure 7: Two minimal configurations: P_5^C and a nut graph.

entries. These vertices induce a subgraph termed *the core of G with respect to \mathbf{x}* . Therefore a core of G with respect to \mathbf{x} is a core graph in its own right. The size of the support is said to be the *core order* [19].

Definition 6.1 (see [19]). Let F be a core graph on at least two vertices, with nullity $s \geq 1$ and a kernel eigenvector \mathbf{x}_F having no zero entries. If a graph N , of nullity one, having \mathbf{x}_F as the nonzero part of the kernel eigenvector, is obtained by adding $s - 1$ independent vertices, whose neighbours are vertices of F , then N is said to be a *minimal configuration* (MC) with core (F, \mathbf{x}_F) .

Hence, an MC with core (F, \mathbf{x}_F) is a connected singular graph of nullity one having a minimal number of vertices and edges for the core F , satisfying $\mathbf{F}\mathbf{x}_F = \mathbf{0}$. The MCs may be considered as the “atoms” of a singular graph [19, 20]. The smallest MC is P_3 corresponding to a pair of duplicates. For core order three, the only MC is P_3 . The number of MCs increases fast for higher core order (see e.g., [21]). Figure 7 shows two graphs, (a) P_5^C , the only MC with core C_4 and (b) a nut graph of order seven [1].

A basis for the nullspace $\ker(\mathbf{G})$ of the adjacency matrix \mathbf{G} of a graph G of nullity $\eta > 1$ can take different forms. We choose a *minimal basis* B_{\min} for the nullspace of \mathbf{G} , that is, a basis having a minimal total number of nonzero entries in its vectors [19, 22].

Such a minimal basis for $\ker(\mathbf{G})$ has the property that the corresponding monotonic non-decreasing sequence of core orders (termed *the core order sequence*) is unique and lexicographically placed first in a list of bases for $\ker(\mathbf{G})$, also ordered according to the non-increasing core orders. Moreover, for all i , the i th entry of the core order sequence for B_{\min} , does not exceed the i th entry of any other core order sequence of the graph. We say that the vectors in B_{\min} define a *fundamental system of cores* of G , consisting of a collection of cores of minimal core order corresponding to a basis of linearly independent nullspace vectors [23]. The significance of MCs can be gauged from the next result.

Theorem 6.2 (see [19, 20]). *Let H be a singular graph of nullity η . There exist η MCs which are subgraphs of H whose core vertices are associated with the nonzero entries of the η vectors in a minimal basis of the nullspace of \mathbf{H} .*

To give an example supporting Theorem 6.2, Figure 8 shows a six-vertex graph of nullity two and two MCs corresponding to a fundamental system of cores found as subgraphs.

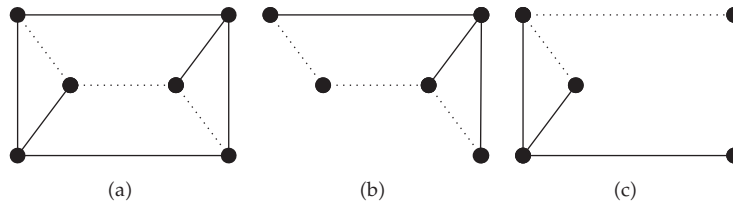


Figure 8: A graph of nullity two with two MCs as subgraphs.

From Theorem 5.4, the following result follows immediately.

Corollary 6.3. *The only MC found in a threshold graph as a subgraph is P_3 .*

Corollary 6.3 has been generalized to cographs in [24]; that is, in cographs, only P_3 (corresponding to duplicate vertices) may be found as an MC corresponding to a vector in B_{\min} . Therefore it is sufficient to have just P_4 as a forbidden subgraph for a graph to have only core order two contributing to the nullity.

Theorem 6.4. *All MCs with core order at least three have P_4 as an induced subgraph.*

Proof. Suppose an MC is P_4 -free. Then, it is a cograph. Therefore, the only MC to contribute to the nullity is P_3 of core order two. We deduce that all other MCs, which have core order at least three, are not cographs. \square

Since P_4 is self-complementary, it follows that the complement of an MC with core order at least three also has P_4 as an induced subgraph. Figures 8(b) and 8(c) show P_4 as an induced subgraph (dotted edges) of the MC P_5^C .

The second largest eigenvalue of P_4 is the golden section $\sigma := (\sqrt{5}-1)/2$. By interlacing, we obtain the following result.

Theorem 6.5. *The second largest eigenvalue of an MC $\neq P_3$ is bounded below by σ .*

The only MC for which the bound is known to be strict is the seven-vertex nut graph of Figure 7.

7. The Main Characteristic Polynomial

The main eigenvalues of a graph G are closely related to the number of walks in G . The product of those factors of the minimum polynomial of G , corresponding to the main eigenvalues only, has interesting properties.

Definition 7.1. The polynomial $M(G, x) := \prod_{i=1}^p (x - \mu_i)$ whose roots are the main eigenvalues of the adjacency matrix of a graph G is termed the *main characteristic polynomial*.

For a proof of the following result, see [25], for instance.

Lemma 7.2 (see [25], rowmain). *The main characteristic polynomial $M(G, x) = x^p - c_0x^{p-1} - c_1x^{p-2} - \dots - c_{p-2}x - c_{p-1}$ has integer coefficients c_i , for all $i, 0 \leq i \leq p - 1$.*

7.1. The Main Eigenvalues of Antiregular Graphs

Recall that A_r has exactly one pair of either duplicates or coduplicates.

Theorem 7.3. *All eigenvalues of A_r other than 0 or -1 are main.*

Proof. Let $\text{Prop}(r)$ be all eigenvalues of A_r , other than 0 or -1 , are main. We prove $\text{Prop}(r)$ by induction on r .

- (i) $\text{Prop}(2)$ refers to K_2 whose only nonmain eigenvalue is -1 . $\text{Prop}(3)$ refers to P_3 whose only nonmain eigenvalue is 0.

This establishes the base cases.

- (ii) Assume that $\text{Prop}(r)$ is true for all $r \leq k$. Therefore for a nonmain eigenvalue λ other than 0 or -1 , $\mathbf{A}_r \mathbf{x} = \lambda \mathbf{x}$ implies $\mathbf{x} = 0$ for $r \leq k$.
- (iii) Consider A_{k+1} and let \mathbf{A}_{k+1} be its adjacency matrix.

For the case when $k + 1$ is odd and A_{k+1} is connected, let $\mathbf{A}_{k+1} \mathbf{x} = \lambda \mathbf{x}$ for an eigenvalue λ and $\mathbf{x} = (x_1, x_2, \dots, x_{k+1}) \neq 0$. It follows that, for $1 \leq q \leq f(\Pi)$, $\sum_{i=1}^{k+2-q} x_i = (1 + \lambda)x_q$ and, for $f(\Pi) + 1 \leq q \leq k + 1$, $\sum_{i=1}^{k+2-q} x_i = (\lambda)x_q$. Similar equations are obtained for the case when $k + 1$ is even.

The eigenvalue λ is nonmain if and only if $\mathbf{j}^t \mathbf{x} = 0$, whence $\lambda = -1$ or $\lambda = 0$ or $x_1 = x_2 = 0$.

If v (labelled 1) is the dominating vertex of A_{k+1} , then, by Proposition 4.5, $A_{k+1} - v = K_1 \dot{\cup} A_{k-1}$.

If $x_1 = x_2 = 0$, then \mathbf{x} restricted to A_{k-1} is an eigenvector for the same eigenvalue λ . Therefore, by the induction hypothesis $\mathbf{x} = 0$. Hence, $\lambda = -1$ or $\lambda = 0$. The result follows by induction on r . \square

7.2. The Main Eigenvalues of Threshold Graphs

By Theorem 7.3, all eigenvalues of A_r that are not 0 or -1 are main. We show that this is still the case for a threshold graph $C(a_1, a_2, \dots, a_r)$ having $a_1 \geq 2$ and $a_i \geq 1$ for $2 \leq i \leq r$ obtained from the degenerate form $A_r = C(1, 1, \dots, 1)$ by adding duplicates and/or coduplicates.

Lemma 7.4. *A graph has the same number of main eigenvalues as its complement.*

Proof. Let \mathbf{G}^C be the adjacency matrix of the complement of a graph G and \mathbf{J} the matrix with each entry equal to one. Then, $\mathbf{G} + \mathbf{G}^C = \mathbf{J} - \mathbf{I}$. Now λ is a nonmain eigenvalue of G if and only if $\mathbf{J} \mathbf{x} = 0$. Hence, \mathbf{G} and \mathbf{G}^C share the same eigenvectors only for nonmain eigenvalues. \square

Theorem 7.5. *Let G be a threshold graph. All eigenvalues, other than 0 or -1 , are main.*

Proof. Let G be $C(a_1, a_2, \dots, a_r)$, $a_1 \geq 2$, $a_i \geq 1$ for $2 \leq i \leq r$. Let the proposition $\text{Prop}(r)$ be all eigenvalues of $C(a_1, a_2, \dots, a_r)$ other than 0 or -1 , are main. We prove $\text{Prop}(r)$ by induction on r .

- (i) If $G = C(a_1, a_2)$, $a_1 \geq 2$, $a_2 \geq 1$, then G is not regular. Hence, the number of main eigenvalues is at least two. The other distinct eigenvalues, 0 and/or -1 ,

are nonmain. By Theorem 5.7, G has at least $n - 2$ nonmain eigenvalues equal to 0 or -1 . Thus, the number of main eigenvalues of G is two. This establishes the base case, namely, Prop(2).

(ii) The induction hypothesis is as follows: assume that Prop(k) is true.

(iii) We show that this is also true for a nondegenerate $H = C(a_1, a_2, \dots, a_{k+1})$.

The complement \overline{H} of H is $C(a_1, a_2, \dots, a_k) \dot{\cup} a_{k+1}K_1$. By Lemma 7.4, H and \overline{H} have the same number of main eigenvalues. One of the a_{k+1} isolated vertices in \overline{H} contributes to the number of main eigenvalues. By the induction hypothesis, $C(a_1, a_2, \dots, a_k)$ has k main eigenvalues and $\sum_{i=1}^k (a_i - 1)$ nonmain eigenvalues. Hence, H has $k+1$ main eigenvalues. The result follows by induction on r . \square

We deduce immediately a spectral property of a threshold graph and its underlying antiregular graph.

Corollary 7.6. *The nondegenerate threshold graph $C(a_1, a_2, \dots, a_r)$ and its underlying A_r have r and $r - 1$ main eigenvalues, respectively.*

An equitable partition $\Pi := \mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_r$ of the vertex set of a graph satisfies $\mathbf{G}\mathbf{X} = \mathbf{X}\mathbf{Q}$, where \mathbf{X} is the $n \times r$ indicator matrix whose i th column is the characteristic 0-1-vector associated with the i th part, containing $|\mathcal{U}_i|$ entries equal to 1. The matrix \mathbf{Q} turns out to be the adjacency matrix of the quotient graph G/Π (also known as divisor).

Lemma 7.7. *The main part of the spectrum of \mathbf{G} is included in the spectrum of \mathbf{Q} .*

Proof. Let λ be a main eigenvalue of G . Then, $\mathbf{G}\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{j}^t\mathbf{x} \neq 0$. Since $\mathbf{G}\mathbf{X} = \mathbf{X}\mathbf{Q}$, $\lambda\mathbf{x}^t\mathbf{X} = \mathbf{x}^t\mathbf{G}\mathbf{X} = (\mathbf{x}^t\mathbf{X})\mathbf{Q}$ so that $\lambda(\mathbf{X}^t\mathbf{x}) = \mathbf{Q}(\mathbf{X}^t\mathbf{x})$. Thus, the eigenvalue λ of G is also an eigenvalue of \mathbf{Q} , provided that $\mathbf{X}^t\mathbf{x} \neq 0$. Indeed this is the case when λ is a main eigenvalue, since $\mathbf{x}^t \cdot \mathbf{X}, \mathbf{j} = \mathbf{x}^j \neq 0$. Thus, the main part of the spectrum of G is contained in the spectrum of \mathbf{Q} . \square

We now show that the main part of the spectrum of $G = C(a_1, a_2, \dots, a_r)$ is precisely the spectrum of \mathbf{Q} . Consider the equitable vertex partition Π for $G = C(a_1, a_2, \dots, a_r)$ as outlined in Section 3.2.

Theorem 7.8. *Let the threshold graph $G = C(a_1, a_2, a_3, \dots, a_r)$ have η duplicates, $\bar{\eta}$ coduplicates, and an equitable partition Π corresponding to the parts $\{a_i\}$. Let \mathbf{Q} be the adjacency matrix of the quotient graph G/Π . Then, $\phi(G, \lambda) = \lambda^\eta(1 + \lambda)^{\bar{\eta}}\phi(\mathbf{Q}, \lambda)$, where $\phi(\mathbf{Q})$ is the main characteristic polynomial $M(G, \lambda)$ of G .*

Proof. The vertex labelling Lab1 is used. Let the vertices be labelled in order starting from those corresponding to a_1 , followed by those for a_2 and so on. If \mathbf{X} is the $n \times r$ indicator matrix whose i th column is the characteristic 0-1-vector associated with a_i containing exactly a_i nonzero entries (each equal to 1), then $\mathbf{G}\mathbf{X} = \mathbf{X}\mathbf{Q}$, where \mathbf{Q} is $r \times r$. Now, by Theorem 7.5, in a threshold graph, 0 and -1 are the only nonmain eigenvalues and these correspond to duplicates and coduplicates, respectively. Therefore, the number of main eigenvalues of \mathbf{G} is exactly r . Since the main spectrum of \mathbf{G} is contained in the spectrum of \mathbf{Q} and \mathbf{Q} is $r \times r$, then the roots of $\phi(\mathbf{Q})$ are the main eigenvalues of G . \square

We give an example to clarify the procedure. Consider the threshold graph $G = C(2, 2, 1, 2, 1, 2)$ (Lab1), of Figure 3. We use the adjacency matrix \mathbf{G} and indicator matrix \mathbf{X} , indexed according to Lab2:

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (7.1)$$

The rows of \mathbf{Q} are the distinct rows of \mathbf{GX} . Therefore,

$$\mathbf{Q} = \begin{pmatrix} 1 & 2 & 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 & 0 \\ 2 & 2 & 1 & 2 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (7.2)$$

Its spectrum is 7.16, 0.892, 0.448, -1.40, -1.59, -2.50, which is precisely the main part of the spectrum of G .

For $\ell \geq 0$, the entries of $\mathbf{G}^\ell \mathbf{j}$ give the number of walks of length ℓ from each vertex v of G . The $n \times k$ matrix whose ℓ th column is $\mathbf{G}^{\ell-1} \mathbf{j}$ is denoted by \mathbf{W}_k . The dimension of the subspace $\text{ColSp}(\mathbf{W}_k)$ generated by the columns of \mathbf{W}_k is the rank of \mathbf{W}_k .

Theorem 7.9 (see [26]). *For a graph with p main eigenvalues, the rank, $\dim(\text{ColSp}(\mathbf{W}_k))$, of the $n \times k$ matrix $\mathbf{W}_k = (\mathbf{j}, \mathbf{G}\mathbf{j}, \mathbf{G}^2\mathbf{j}, \dots, \mathbf{G}^{k-1}\mathbf{j})$ is p , for $k \geq p$.*

The columns $\mathbf{j}, \mathbf{G}\mathbf{j}, \mathbf{G}^2\mathbf{j}, \dots, \mathbf{G}^{p-1}\mathbf{j}$ are a maximal set of linearly independent vectors in $\text{ColSp}(\mathbf{W}_k)$. Thus, the first p columns provide all the information on the number of walks from each vertex of any length [27].

Definition 7.10. The matrix $\mathbf{W}_p = (\mathbf{j}, \mathbf{G}\mathbf{j}, \mathbf{G}^2\mathbf{j}, \dots, \mathbf{G}^{p-1}\mathbf{j})$ of rank p is said to be the *walk matrix* \mathbf{W} .

Note that \mathbf{W} has the least number of columns for a walk matrix \mathbf{W}_k to reach the maximum rank possible which is p . From Corollary 7.6, $C(a_1, a_2, a_3, \dots, a_r)$ has r main eigenvalues.

Theorem 7.11. *The rank of the walk matrix of $C(a_1, a_2, a_3, \dots, a_r)$ is r .*

The number of walks of length k can be expressed in terms of the main eigenvalues [28, page 46].

Theorem 7.12. *The number w_k of walks of length k starting from any vertex of G is given by*

$$w_k = \sum_{i=1}^p c'_i \mu_i^k, \quad (7.3)$$

where $c'_i \in \mathbb{R} \setminus \{0\}$ is independent of k for each i and $\mu_1, \mu_2, \dots, \mu_p$ are the main eigenvalues of \mathbf{G} .

Since 0 is never a main eigenvalue of $C(a_1, a_2, a_3, \dots, a_r)$, it follows that all the main eigenvalues of $C(a_1, a_2, a_3, \dots, a_r)$ contribute to the number of walks.

7.3. Cases of Reducible Main Polynomial

By Theorem 7.3, only one eigenvalue of A_r is not main. Recall that the minimal equitable vertex partition of $G = C(a_1, a_2, a_3, \dots, a_r)$ satisfies $\mathbf{GX} = \mathbf{XQ}$, where \mathbf{Q} is the adjacency matrix of the quotient graph G/Π and $\phi(\mathbf{Q}, \lambda) = M(G, \lambda)$, the main characteristic polynomial $M(G, \lambda)$ of G .

We note that for many threshold graphs $\phi(\mathbf{Q}, \lambda)$ is irreducible over the integers. For example the only eigenvalue of $A_8 = C(1, 1, 1, 1, 1, 1, 1, 1)$ (in degenerate form) which is not main is -1 and $M(A_8, x) = (1 - 7x + 9x^2 + 15x^3 - 13x^4 - 15x^5 - x^6 + x^7)$, which is irreducible.

Now we add vertices to the degenerate form $A_8 = C(1, 1, 1, 1, 1, 1, 1, 1)$. If we add a vertex to the first part, to obtain $G_1 = C(2, 1, 1, 1, 1, 1, 1, 1)$, a negative eigenvalue (not -1) and 0 appear. The eigenvalue -1 is lost and $M(G_1, x) = (2 - 12x + 6x^2 + 40x^3 - 40x^5 - 20x^6 + x^8)$. When a vertex is added to the third part to obtain $G_3 = C(2, 2, 1, 1, 1, 1, 1, 1)$, the eigenvalue -1 is retained while the zero eigenvalue appears and $M(G_3, x) = (2 - 12x + 12x^2 + 22x^3 - 16x^4 - 18x^5 - x^6 + x^7)$. In both these latter two cases $\phi(\mathbf{Q}, \lambda)$ is irreducible over the integers. Now when a vertex is added to the seventh part to obtain $G_7 = C(2, 1, 1, 1, 1, 2, 1)$, the eigenvalue -1 is retained while the zero eigenvalue appears. In this case, however, $M(G_7, x) = (x^2 + 2x - 1)(x^5 - 3x^4 - 9x^2 + 3x^3 + 8x - 2)$, and therefore it is reducible over the integers.

This is also the case for some instances of the threshold graphs $C(d, 1, t)$ when the cubic polynomial $\phi(\mathbf{Q}, \lambda)$ has an integer as a root and therefore is reducible. The divisor \mathbf{Q} is $\begin{pmatrix} d-1 & 0 & t \\ 0 & 0 & t \\ d & 1 & t-1 \end{pmatrix}$ with characteristic polynomial $\phi(\mathbf{Q}, \lambda) = -t + dt + \lambda - d\lambda - 2t\lambda + 2\lambda^2 - d\lambda^2 - t\lambda^2 + \lambda^3$.

If λ is 0, 2 or 3, there are no integral values of t and d satisfying the polynomial $\phi(\mathbf{Q}, \lambda)$. If $\lambda = 1$, the graph either for $t = 3$ and $d = 8$ or for $t = 4$ and $d = 6$ satisfies it. Also for $\lambda = -2$ either the graph for $t = 3$ and $d = 5$, or $t = 4$ and $d = 3$, or $t = 6$ and $d = 2$ satisfies it, while for $\lambda = -3$, the graph for $t = 7$ and $d = 40$ satisfies it.

8. Sign Pattern of the Spectrum of a Threshold Graph

We conclude with a note on the distribution of the eigenvalues of a threshold graph. In [29] it was remarked that an antiregular graph has a *bipartite character*, that is, the number r^- of negative eigenvalues is equal to the number r^+ of positive ones. We denote the number of zero eigenvalues by η .

8.1. The Spectrum of A_r

For $n \geq 4$, A_r is not bipartite. Therefore, $-\lambda_{\min} \neq \lambda_{\max}$. The proof of the next result is by induction on the order of the antiregular graph. We will need the following evident fact.

Lemma 8.1. *To transform A_r to A_{r+1} (according to the labelling (Lab2) of the stepwise adjacency matrix),*

- (i) *a vertex duplicate to the $\lfloor (r+1)/2 \rfloor$ th is added for even r ,*
- (ii) *a vertex coduplicate to the $\lfloor (r+1)/2 \rfloor$ th is added for odd r .*

Theorem 8.2. $r^+ = r^-$ for A_r .

Proof. The proof is by induction on r .

The spectra of the three smallest antiregular graphs, $\text{Sp}(A_1) = \{0\}$, $\text{Sp}(A_2) = \{-1, 1\}$, and $\text{Sp}(A_3) = \{-\sqrt{2}, 0, \sqrt{2}\}$, establish the base cases.

Assume that the theorem is true for A_k .

We prove it true for A_{k+1} .

If A_k is singular, then it has a duplicate vertex and k is odd. By the induction hypothesis $r^+ = r^-$.

If, on the other hand, A_k is nonsingular, then A_k has a coduplicate vertex and k is even. Again the nonzero eigenvalues satisfy $r^+ = r^-$.

We apply Lemma 8.1, using Lab2. For odd k , if a vertex w , coduplicate to the $\lfloor (r+1)/2 \rfloor$ th vertex, is added to A_k , then only one of the duplicate vertices of A_k will have w as a neighbour in A_{k+1} . The zero eigenvalue of A_k vanishes and the eigenvalue -1 is introduced for A_{k+1} . By the Perron Frobenius theorem adding edges to a graph ($A_k \cup K_1$) increases the maximum eigenvalue. Therefore, by interlacing, the number of positive eigenvalues increases by one. Since the new coduplicate vertex w contributes the new eigenvalue -1 to the spectrum, it follows that $r^+ = r^-$ will be satisfied in A_{k+1} . By interlacing, adding a duplicate vertex to any graph retains the number of positive and negative eigenvalues and adds 0 to the spectrum. For even k , if a vertex w , duplicate to the $\lfloor (r+1)/2 \rfloor$ th vertex, is added, then a duplicate vertex is added to the graph, retaining $r^+ = r^-$.

The result follows by induction on n . □

8.2. The Spectrum of a Threshold Graph

In this section, we shall represent the antiregular graph A_r by the degenerate form $C(1, 1, \dots, 1)$. As in Section 4, any part can be expanded to produce a threshold graph $C(a_1, a_2, \dots, a_r)$. We need the following evident facts regarding the effect on the distribution of the spectrum of the adjacency matrix when a vertex is added.

Lemma 8.3. *If on adding a vertex to a graph (i) the multiplicity of an eigenvalue λ_0 of the adjacency matrix increases, then, by interlacing, the number $n^-(\lambda_0)$ of eigenvalues less than λ_0 and the number $n^+(\lambda_0)$ greater than λ_0 remain the same; (ii) the multiplicity of an eigenvalue λ_0 of the adjacency matrix decreases, then by interlacing, each of the numbers $n^+(\lambda_0)$ and $n^-(\lambda_0)$ increases by one.*

We shall write n^+ for $n^+(0)$ and n^- for $n^-(0)$.

First we see an application of Lemma 8.3(i) using Lab1. For even r , if one of the even indexed a_i , for $i \geq 2$, of $C(a_1, a_2, a_3, \dots, a_r)$ is increased, then a coduplicate of a vertex is added. This forces η and n^+ to remain unchanged while each of n^- and the multiplicity $m(-1)$ of the eigenvalue -1 increases by one. If the odd indexed a_i , for some $i \geq 1$, is increased, then a duplicate of a vertex is added forcing n^+ and n^- to remain unchanged.

Similarly, for odd r , if the even indexed a_i , for some $i \geq 2$, is increased, then a duplicate of a vertex is added. This forces n^- and n^+ to remain unchanged while η increases by one. If the odd indexed a_i , for some $i \geq 3$, is increased, then a coduplicate of a vertex is added forcing n^+ and η to remain unchanged.

The case for even r and $a_1 > 1$ is the same as for odd r with $a_1 = 1$ (Lab1). Taking $C(a_1, a_2, a_3, \dots, a_r)$ for odd r with $a_1 = 1$ and expanding to $C(a_1, a_2, a_3, \dots, a_r)$ with $a_1 > 1$ gives the unique case where η decreases by one and $m(-1)$ increases by one. Since η decreases by one, by Lemma 8.3(ii), each of n^+ and n^- increases by one, the latter corresponding to the increase in the multiplicity of the eigenvalue -1 . We have proved the following result.

Theorem 8.4. *If the threshold graph $C(a_1, a_2, \dots, a_r)$ is transformed to another threshold graph by increasing exactly one of the a_i s by one, then*

$$\left\{ \begin{array}{ll} \text{if a duplicate is added,} & \text{then } n^- \text{ and } n^+ \text{ are unchanged} \\ & \text{and } \eta \text{ increases;} \\ \text{if a coduplicate is added, and if } r \text{ is even} & \text{then } \eta \text{ and } n^+ \text{ are unchanged} \\ \text{or if } r \text{ is odd and } a_i \geq 3 \text{ or if } r \text{ is odd and } a_1 > 1, & \text{and } n^- \text{ increases;} \\ \text{if a coduplicate is added, and if } r \text{ is odd and } a_1 = 1 & \text{then } n^- \text{ and } n^+ \text{ increase} \\ & \text{and } \eta \text{ decreases.} \end{array} \right. \tag{8.1}$$

9. Conclusion

The simple graphic appeal of the Ferrers/Young diagram $F(\Pi)$, with rows representing the degree sequence of a n -vertex threshold graph has been instrumental to obtain interesting results on the nullity and structure of the graph. The shape of $F(\Pi)$ has been also used to determine the nature of the eigenvalues as main or nonmain.

Let \mathbf{D} be the diagonal entries whose nonzero entries are the vertex degrees for some labelling of the vertices. Like the adjacency matrix \mathbf{A} , the Laplacian $\mathbf{D} - \mathbf{A}$ also gives a wealth of information about the graph. It is well known that the class of graphs for which the

Laplacian spectrum and the conjugate degree sequence π^* (i.e., the lengths of the columns of $F(\Pi)$) coincide is exactly the class of threshold graphs [30, Chapter 10]. The Grone-Merris Conjecture, asserting that the spectrum of the Laplacian matrix of a finite graph is majorized by the conjugate degree sequence of the graph, has been recently proved by Bai [31].

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