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[GTN LXIII:5] POLYNOMIAL RECONSTRUCTION FOR CERTAIN SUBCLASSES OF DISCONNECTED GRAPHS

Jane Coates, Josef Lauri, and Irene Sciriha[1]
Department of Mathematics
Faculty of Science
University of Malta
Msdia MSD 20 80, MALTA
<coates_jane@yahoo.com>
<josef.lauri@um.edu.mt>
<irene.sciriha-aquilina@um.edu.mt>

Abstract

The reconstruction conjecture (RC) and the polynomial reconstruction problem (PRP) are two open problems in algebraic graph theory. They have been resolved successfully for a number of different classes and subclasses of graphs. This paper offers proofs for a positive conclusion for polynomial reconstruction of the following three subclasses of the class of disconnected graphs. These subclasses are disconnected graphs with two unicyclic components, bidegreed disconnected graphs with regular components, and disconnected graphs with a wheel as one component.

1. Introduction

The graphs considered in this paper are simple graphs, that is graphs with no loop or multiple edge. A graph defined as $G = (V(G), E(G))$, consisting of two disjoint sets: a non-empty set $V(G)$ the elements of which are called vertices and a set $E(G)$ of unordered pairs of distinct elements from $V(G)$ whose elements are called edges. Two vertices in the same edge are said to be *adjacent*. If every pair of vertices of G are adjacent, then G is called a *complete graph*. A complete graph of order n is denoted by K_n. Deleting a vertex v and all the edges incident on v from G generates the *subgraph* $G - v$ of G. The *adjacency matrix* of a labelled graph G, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the (0,1)-symmetric matrix $\mathbf{A}(G) = (a_{ij})$ of order $n \times n$ whose ij^{th} entry, a_{ij}, is 1 if v_i is adjacent to v_j and 0 otherwise.

The *characteristic polynomial* of a graph G is the characteristic polynomial of its adjacency matrix $\mathbf{A}(G)$ and is denoted by $\phi(G, \lambda)$. Let $\mathbf{I}_n$ denote the $n \times n$ identity matrix. Then,

$$\phi(G, \lambda) = \det(\lambda \mathbf{I}_n - \mathbf{A}(G)) = \sum_{i=0}^{n} a_i \lambda^i,$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ denote the coefficients of the characteristic polynomial of the graph G. From the definition of the characteristic polynomial $a_n = 1$ and for a graph with no loop, $a_{n-1} = 0$.

Solving the equation $\phi(G, \lambda) = 0$ gives the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}(G)$; these are also called the *eigenvalues* of G. For a particular λ_i, there is at least one non-zero vector $\mathbf{x}$, called an *eigenvector*, in the eigenspace of λ_i, that satisfies the equation $\mathbf{A}(G)\mathbf{x} = \lambda_i \mathbf{x}$. Because $\mathbf{A}(G)$ is a real and symmetric matrix, all the eigenvalues are real and can be ordered $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. The multiset of eigenvalues of $\mathbf{A}(G)$ consisting of the eigenvalues and their multiplicities is called the *spectrum* of G and is denoted $\Lambda(G)$. The spectrum of a graph is invariant under a change in labelling of the vertices of G. The *polynomial deck* of G is the multiset of the characteristic polynomials of all vertex-deleted subgraphs of G and is denoted by $PD(G)$.

The *degree* of a vertex v_i in a graph, denoted by ρ_i, is the number of edges incident on v_i. A graph is said to be *regular* if all its vertices have the same degree. If every vertex of a graph has either degree ρ_1 or degree ρ_2 ($\rho_1 \neq \rho_2$) then the graph is called a *bidegreed graph*.

A connected 2-regular graph of order n is called a *cycle graph* and is denoted by C_n. A graph that has precisely one cycle is called a *unicyclic graph*. Removing an edge from C_n gives a *path graph* of length $n - 1$, denoted by P_n. A *path* can be defined as a sequence of distinct vertices $v_1, v_2, \ldots, v_{k+1}$ and edges

[1] Corresponding author.

$e_1, e_2, \ldots, e_k$, where $e_i = \{v_i, v_{i+1}\}$. Adding a vertex v to C_{n-1} and connecting this to all the other vertices (i.e., inserting a *dominant vertex*) gives the *wheel graph* on n vertices, denoted by W_n. Examples of these three types of graph with order six are shown in Figure 1.

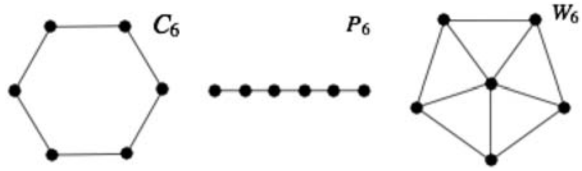

Figure: The graphs C_6, P_6, and W_6.

Ulam's famous *reconstruction conjecture* **[1]** has given rise to many variant problems. One such problem is the *polynomial reconstruction problem* (PRP) that was posed by Cvetković and independently by Schwenk. It was later outlined by Gutman and Cvetković in **[2]**. This problem considers reconstructing the characteristic polynomial of a graph G from its polynomial deck, $PD(G)$. The problem can be stated as follows:

> **Polynomial Reconstruction Problem:** Is it true, that for $n > 2$ the characteristic polynomial $\phi(G, \lambda)$ of a graph G of order n is uniquely determined by the collection of characteristic polynomials $\phi(G - v_i, \lambda)$ of the vertex-deleted subgraphs $G - v_i$ of G, $i = 1, 2, \ldots, n$? □

The lower bound on the order n of the graph G in this formulation of the polynomial reconstruction problem is required because the two graphs K_2 and $\overline{K_2}$ on two vertices are the only two non-isomorphic graphs, known to date, that have the same polynomial deck but different characteristic polynomials. In this paper, a graph G is said to be *polynomial reconstructible* if for every graph H such that $PD(G) = PD(H)$, then $\phi(G, \lambda) = \phi(H, \lambda)$.

The polynomial reconstruction problem is still open in general, as is Ulam's reconstruction conjecture. However the polynomial reconstruction problem has been shown to be true for certain classes of graphs. For example, it has been shown to be true for trees **[3]**, certain subclasses of the class of graphs with terminal vertices **[4]**, certain subclasses of disconnected graphs **[5]** and for unicyclic graphs **[6]**.

The class of disconnected graphs is one that warrants attention, since if the polynomial deck can yield the information that the graph is disconnected, then the characteristic polynomial can be reconstructed from $PD(G)$. Currently it is not yet known whether the property of a graph being disconnected can be determined from the polynomial deck of the graph. If this graph property of disconnectedness can be determined from the polynomial deck then this would also help tremendously when searching for possible counterexamples to the polynomial reconstruction problem. Since it is conceivable that there may exist a connected graph and a disconnected graph that have the same polynomial decks, such a pair of graphs could possibly be a counterexample pair to the polynomial reconstruction problem. A *counterexample pair*, denoted by (G, H), consists of two graphs G and H having characteristic polynomials $\phi(G, \lambda) \neq \phi(H, \lambda)$ and $PD(G) = PD(H)$.

Before proving the main results of this paper we present additional results and techniques. In Section 2, graph properties and other invariants that can be determined from the polynomial deck are outlined. In Section 3, the counterexample technique is considered; that is, a pair of graphs that offer a potential counterexample pair to the polynomial reconstruction problem are studied, and necessary conditions are determined for such a pair of graphs with the aim of contradicting some combinatorial property of the pair of graphs. The implication would then be that the pair of graphs considered are polynomial reconstructible. In Section 4, the main results of this paper are proved. Here, three subclasses of disconnected graphs are proved to be polynomial reconstructible: (1) graphs having unicyclic components, (2) bidegreed graphs with two regular components, and (3) graphs with a wheel as one component.

2. Polynomial Reconstructible Graph Properties and Invariants

In this section, graph properties and invariants are outlined and shown to be reconstructible from the polynomial deck. These properties and invariants may lead to a positive outcome to the polynomial reconstruction problem for certain classes of graphs.

The following theorem was originally proved by F.H. Clarke in **[7]**.

Theorem 2.1: The derivative of the characteristic polynomial of a graph G is given by

$$\phi'(G, \lambda) = \sum_{i=1}^{n} \phi(G - v_i, \lambda) . \tag{1}$$ ■

Hence, integrating $\phi'(G, \lambda)$ gives all terms, except the constant term a_0, of the characteristic polynomial of the graph G from its polynomial deck. If an eigenvalue of G, λ_i, is known, then $\phi(G, \lambda_i) = 0$ and so a_0 can be determined, in which case G is polynomial reconstructible. This proves Lemma 2.2:

Lemma 2.2: If the polynomial deck $PD(G)$ of a graph reveals an eigenvalue of G, then G is polynomial reconstructible. ■

Another extremely useful theorem when considering the polynomial reconstruction problem is the *interlacing theorem*. This can be formulated as follows (see **[8]**, page 37).

Theorem 2.3 (Interlacing Theorem): Let G be a graph of order n and let $v \in V(G)$. Let the spectrum of G be $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ and let the spectrum of $G - v$ be $\{\mu_1, \mu_2, \ldots, \mu_{n-1}\}$, where $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_{n-1}$. Then the eigenvalues of $G - v$ interlace those of G; that is, $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \ldots \ge \mu_{n-1} \ge \lambda_n$. ■

Corollary 2.4: If one of the polynomials of $G - v$ has a repeated root, then this root is also an eigenvalue of G and the characteristic polynomial of G is reconstructible. ■

Certain graph properties and invariants of G can be determined from the polynomial deck and we now outline and prove some of these by means of a series of lemmas.

Lemma 2.5: The degree sequence $\boldsymbol{d}(G)$, consisting of the degrees of all the vertices, of a graph G is polynomial reconstructible.

Proof: The degree sequence, $\boldsymbol{d}(G)$, of G is $\{\rho_i\}$, where ρ_i is the degree of vertex v_i of G. The degree ρ_i is the number of edges removed when v_i is deleted from G and is the difference between the coefficients $-\lambda^{n-2}$ in $\phi(G, \lambda)$ and $-\lambda^{n-3}$ in $\phi(G - v_i, \lambda)$ obtained by integrating Equation **(1)**. ■

Theorem 2.6 [9]: The length of a shortest odd cycle of a graph G and the number of such cycles can be determined from the polynomial deck $PD(G)$. ■

Corollary 2.7 [9]: The number of triangles in a graph G of order n is equal to $-a_{n-3}/2$ and can therefore be reconstructed from the polynomial deck $PD(G)$. ■

Lemma 2.8 [9]: The number N_k^i of closed walks of length k that start and end at a vertex v of a graph G of order n can be calculated from the collection $PD(G)$ for $k = 0, 1, \ldots, n-1$. ■

We now discuss the *spectral moments* of a graph G of order n with adjacency matrix $\mathbf{A}(G)$ and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. For $k \in \mathbb{N}$, the k^{th} spectral moment of G is the sum S_k of the k^{th} powers of the n eigenvalues. Note that this is equal to the trace, $\operatorname{tr}(\mathbf{A}^k)$ of the k^{th} power ($\mathbf{A}^k(G)$) of the adjacency matrix $\mathbf{A}(G)$. It is also well known that this is equal to the number of closed walks of length k in G.

Corollary 2.9 [9]: The spectral moments $S_0, S_1, \ldots, S_{n-1}$ of a graph G of order n can be determined from the polynomial deck $PD(G)$. ■

Theorems 2.13 and 2.15, that we state and prove later, show that the number of quadrangles and pentagons in a graph are polynomial reconstructible. Theorem 2.13 is taken from **[9]**, but before proving that the number of quadrangles is reconstructible, a lemma is needed to prove that $S_4 = 2m + 2p + 8q$, where m is the number edges, p is the number of pairs of incident edges, and q is the number of quadrangles in a graph G. A proof of this lemma is not provided or referenced in **[9]** and is proved here by applying results in **[10]** to *Newton's recurrence*.

Theorem 2.15 can be found in **[9]** and here we offer a different proof. In **[9]** this result is first proved by stating the equation $S_5 = 10m + 10s + 30t$, where m is the number of edges, s is the number of triangles with a pendant edge, and t is the number of triangles of G; and then showing that each of the terms can be determined from the polynomial deck, thus obtaining the number of pentagons of G from the polynomial deck. The proof we offer below correlates the number of closed walks of length five to the fifth spectral moment, thus determining the number of pentagons of G.

Before proving Lemma 2.11, additional definitions and the *Harary–Sachs Coefficient Theorem* (Theorem 2.10) are required. Theorem 2.10 was originally proven by Sachs in **[11]** and in the literature, in particular in chemical graph theory, it is referred to as the Sachs Coefficient Theorem. In **[12]** Gutman remarks that this should be called the Harary–Sachs Coefficient Theorem since this is an immediate consequence of Harary's results in **[13]**.

Definition 1 [13]: A *basic figure* for a graph G is a spanning subgraph of G whose components consist only of cycles or K_2 graphs. The number of components and cycles in a basic figure B are denoted by $k(B)$ and $c(B)$, respectively. ■

Theorem 2.10 [12]:

Let $\phi(G, \lambda) = \sum_{h=0}^{n} a_h \lambda^h$ be the characteristic polynomial of a graph G then,

for $h = n$, $a_h = 1$, and for $h \le n-1$, $a_h = \sum_{B \in B_{n-h}} (-1)^{k(B)} 2^{c(B)}$,

where the sum is over all basic figures B that have exactly $n-h$ vertices in the set B_{n-h} and that are subgraphs contained in the graph G. ■

Lemma 2.11: The fourth spectral moment, S_4, of a graph G is equal to $2m + 4p + 8q$, where m is the number of edges, p is the number of pairs of incident edges, and q is the number of quadrangles in G.

Proof: From Newton's recurrence (see **[14]**, pages 31–33), we obtain

$$S_1 + a_{n-1} = 0, \text{ for } \phi(G, \lambda) \Rightarrow S_1 = 0,$$

$$S_2 + a_{n-1}S_1 + 2a_{n-2} = 0, \text{ for } \phi(G, \lambda) \Rightarrow S_2 = -2a_{n-2} = 2m,$$

$$S_3 + a_{n-1}S_2 + a_{n-2}S_1 + 3a_{n-3} = 0, \text{ for } \phi(G, \lambda) \Rightarrow S_3 = -3a_{n-3} = 6t,$$

$$S_4 + a_{n-1}S_3 + a_{n-2}S_2 + a_{n-3}S_1 + 4a_{n-4} = 0, \text{ for } \phi(G, \lambda) \Rightarrow S_4 = 2m^2 - 4a_{n-4}.$$

From Theorem 2.10,

$$a_{n-4} = \sum_{B \in B_4} (-1)^{k(B)} 2^{c(B)},$$

where $k(B)$ is the number of components of B and $c(B)$ is the number of cycles in B. For $B \in B_4$, there are two types of basic figures on four vertices; namely, $2K_2$ and C_4.

For $2K_2$, $k = 2$ and $c = 0$; for C_4, $k = 1$ and $c = 1$. Hence, $a_{n-4} = (-1)^2 2^0 k_2 + (-1)^1 2^1 |C_4|$, where k_2 is the number of subgraphs isomorphic to $2K_2$ in G (that is, the number of disjoint edge pairs). The number of disjoint edge pairs in a graph can be evaluated by determining the number of ways one can choose any two distinct edges from the graph G and then subtracting the number of subgraphs isomorphic to P_3 in G. The number of ways of choosing two distinct edges from a graph with size m is $\binom{m}{2}$. For each vertex v_i with degree ρ_i, there are $\binom{\rho_i}{2}$ pairs of adjacent edges. Hence,

$$k_2 = \binom{m}{2} - \sum_{i=1}^{n} \binom{\rho_i}{2}.$$

Therefore,

$$a_{n-4} = k_2 - 2|C_4| = \frac{m(m-1)}{2} - \sum_{i=1}^{n} \binom{\rho_i}{2} - 2|C_4|.$$

Substituting this in the Newton formula result for S_4 gives

$$S_4 = 2m^2 - 4\left[\frac{m(m-1)}{2} - \sum_{i=1}^{n} \binom{\rho_i}{2} - 2|C_4|\right].$$

Let

$$p = \sum_{i=1}^{n} \binom{\rho_i}{2} \text{ and } q = |C_4|,$$

then $S_4 = 2m^2 - 2m(m-1) + 4p + 8q = 2m + 4p + 8q$, which completes the proof. ■

From Newton's recurrence, the following result is immediate.

Corollary 2.12: The coefficients $a_n, a_{n-1}, \ldots, a_1$ and the spectral moments $S_{n-1}, S_{n-2}, \ldots, S_1$ for a graph G of order n can be determined from the polynomial deck $PD(G)$. ■

Theorem 2.13 [9]: The number of quadrangles in a graph G of order n can be determined from the polynomial deck $PD(G)$.

Proof: Using the notation in Lemma 2.11, $m = \sum \rho_i$, $p = \sum \binom{\rho_i}{2}$. Hence, by Lemma 2.5, both m and p can be determined from $PD(G)$.

By Corollary 2.12, $S_4 = 2m^2 - 4a_{n-4}$. Consequently, S_4 can be determined from $PD(G)$.

Using Lemma 2.11, q can be determined from $PD(G)$. ■

Lemma 2.14: The number of triangles s with a pendant edge in a graph G with order n is given by

$$s = \sum_{i=1}^{n} N_3^i(\rho_i - 2).$$

Proof: Consider a vertex $v_i \in V(G)$ and suppose that v_i is a vertex of a triangle subgraph of G with at least one pendant edge. Then, the number of pendant edges incident on v_i is $(\rho_i - 2)$, where the factor of two removes the two edges that belong to the triangle subgraph being considered. By Lemma 2.8, the number of triangle subgraphs is N_3^i. Hence, summing over all $v_i \in V(G)$ gives the result. ■

Theorem 2.15 [9]: The number of pentagons in G can be determined from the polynomial deck $PD(G)$.

Proof: The fifth spectral moment S_5 of G is equal to the number of closed walks of length five. The subgraphs of G that permit closed walks of length five are the pentagon C_5, the triangle C_3, and C_3 with a pendant edge. A C_5 subgraph gives a total of ten closed walks of length five, a C_3 subgraph gives a total of 30 closed walks of length five and C_3 with a pendant edge gives a total of ten closed walks of length five. Let $|C_5|$ be the number of C_5 subgraphs in G, s be the number of C_3 subgraphs with a pendant edge, and $|C_3|$ be the number of C_3 subgraphs. Summing gives $S_5 = 10|C_5| + 10s + 30|C_3|$.

From Lemma 2.14, $s = \sum_{i=1}^{n} N_3^i(\rho_i - 2)$. Thus, s, S_5, and $|C_3|$ are all reconstructible from $PD(G)$.

Hence, $|C_5|$ is also reconstructible from $PD(G)$. ■

3. Properties of a Counterexample Pair

Recall that two graphs G and H are said to be a *counterexample pair*, denoted by (G, H), if the characteristic polynomials $\phi(G, \lambda) \neq \phi(H, \lambda)$ but $PD(G) = PD(H)$. Some properties that a counterexample pair must have are listed below. These results are taken from **[4]**. Unless otherwise stated, G and H are taken to be graphs in a counterexample pair.

Lemma 3.1: The characteristic polynomials of G and H differ only in the constant term a_0. ■

Hence, for a counterexample pair (G, H) the constant terms of the characteristic polynomials are related by $a_0(G) = a_0(H) + \Delta a_0$, where $\Delta a_0 \in \mathbb{Z} - \{0\}$.

Lemma 3.2 [4]: G and H have no eigenvalue in common. ■

Lemma 3.3 [4]: No polynomial in $PD(G)$ has a repeated eigenvalue. ■

The *Perron–Frobenius Theorem* concerns specific spectral properties of connected graphs. This theorem is one of the most important theorems in spectral graph theory and was proved independently by G. Frobenius in **[15]** and O. Perron in **[16]**. We state the theorem as in **[8]** (see Appendix A).

> **Theorem 3.4 (Perron–Frobenius Theorem):** If G is a connected graph with at least two vertices then its largest eigenvalue $\lambda_{\max}$ is a simple root of $\phi(G, \lambda)$ and there exists an eigenvector $\boldsymbol{x}$ corresponding to $\lambda_{\max}$ all of whose entries are positive. If λ is any other eigenvalue of G then $-\lambda_{\max} \le \lambda \le \lambda_{\max}$ and the deletion of any edge e from G decreases the largest eigenvalue; that is, $\lambda_{\max}(G-e) < \lambda_{\max}(G)$. ■

From Theorem 3.4, a number of results related to the polynomial reconstruction problem follow easily. One such result is the following lemma.

> **Lemma 3.5 [4]:** The two graphs G and H in a counterexample pair are not both disconnected.

Proof: The maximum eigenvalue (also called the *index*) of a disconnected graph is the maximum eigenvalue of a component and this appears in the polynomial deck corresponding to a vertex- deleted subgraph containing the same component. If G and H have the same deck and both are disconnected graphs then they will have the same maximum eigenvalue, which contradicts Lemma 3.2. ■

In an attempt to expand the notion of a counterexample pair to the polynomial reconstruction problem, one can ask whether a counterexample pair (G_1, G_2) could generate infinite families of counterexample pairs $(G_1 \cup H, G_2 \cup H)$ in the form of disjoint unions with an arbitrary graph H. From Lemma 3.5 the answer is in the negative. On the other hand, if Tutte's remarkable result, in **[17]**; namely, that the deck $\{G - v_i : v_i \in V(G)\}$ consisting of the one-vertex deleted subgraphs leads to the unique characteristic polynomial of the adjacency matrix of G, is taken into account then two non-isomorphic graphs G_1 and G_2 in a counterexample pair with the same polynomial deck arise from different polynomial decks $\{G_1 \cup H, G_2 \cup H\}$ for an arbitrary graph H. Moreover, since $\phi(G_1, \lambda) \ne \phi(G_2, \lambda)$, the infinite family $G_1 \cup H, G_2 \cup H$ for an arbitrary graph H does not lead to pairs of graphs with the same polynomial deck.

4. Polynomial Reconstruction of Subclasses of Disconnected Graphs

Disconnected graphs with more than two components and with two components of unequal order were proved to be polynomial reconstructible in **[3]**.

> **Theorem 4.1 [3]:** Let $G = G_1 \cup G_2$ be a disconnected graph of order $n \ge 3$ and with exactly two connected components G_1 and G_2. If $|V(G_1)| \ne |V(G_2)|$, then G is polynomial reconstructible. ■

> **Theorem 4.2 [3]:** Let $G = G_1 \cup G_2 \cup \ldots \cup G_k$ be a disconnected graph with k connected components $G_1, G_2, \ldots, G_k$ for $k > 2$. Then, G is polynomial reconstructible. ■

> **Corollary 4.3:** If G is a disconnected graph with an odd number of vertices then G is polynomial reconstructible. ■

From these results, the only remaining class of disconnected graphs that have not yet been determined to be polynomial reconstructible are those disconnected graphs that have precisely two components of equal order. We now prove that disconnected graphs (1) that have precisely two unicyclic components, (2) that are bidegreed with regular components, and (3) that have a wheel as a component are polynomial reconstructible.

4.1. Disconnected Graphs with two Unicyclic Components

In **[6]**, the polynomial reconstruction problem is solved for the class of *unicyclic graphs*. Here, that result is used to prove that disconnected graphs having two unicyclic components are also polynomial reconstructible.

Theorem 4.4: Let $G = G_1 \cup G_2$ be a disconnected graph of order n whose components are both unicyclic graphs. Then G is polynomial reconstructible.

Proof: Suppose that G is not polynomial reconstructible. Then there exists a graph H such that (G, H) is a counterexample pair to the polynomial reconstruction problem. The order of each component of G is equal to $n/2$ since, if the components of G do not have equal orders then G is polynomial reconstructible by Theorems 4.1 and 4.2. Because each component is unicyclic, then the number of edges of each component is $m(G_1) = m(G_2) = n/2$, and thus, $m(G) = m(G_1) + m(G_2) = n$.

By Lemma 3.5, the graph H must be connected and have the same order and number of edges as G. Hence, by Lemma 3.1, $m(H) = n$. This can only be the case if H is a unicyclic graph but then from **[6]**, H is uniquely polynomial reconstructible.

Hence, (G, H) cannot be a counterexample pair and G is polynomial reconstructible. ■

4.2. Bidegreed Disconnected Graphs with Regular Components

In **[18]**, the *edge reconstruction conjecture*, another variant of Ulam's reconstruction conjecture, was solved positively for *bidegreed graphs*. This prompted us to study this class of graphs in the polynomial reconstruction problem setting. Here, the polynomial reconstruction problem is shown to hold for bidegreed disconnected graphs with regular components.

Theorem 4.5: Let $G = G_1 \cup G_2$ be a disconnected graph of order n such that G_1 is regular of degree ρ_1 and G_2 is regular of degree ρ_2 ($\rho_1 \neq \rho_2$). Then, G is polynomial reconstructible.

Proof: Suppose that there exists a graph H such that (G, H) is a counterexample pair. Since G is disconnected then, from Lemma 3.5, H must be a connected graph. Furthermore, from Lemma 3.1, the characteristic polynomials of G and H differ only in the constant term.

Since G_1 is regular of degree ρ_1, then its maximum eigenvalue $\lambda_{max}(G_1) = \rho_1$. Similarly, $\lambda_{max}(G_2) = \rho_2$. Because G is disconnected, then $\lambda_{max}(G) = \rho_1$ or ρ_2, as noted in the proof of Lemma 3.5. Suppose, without loss of generality, that $\lambda_{max}(G) = \rho_1$; that is, $\rho_1 \geq \rho_2$.

Now, $\lambda_{max}(H) \leq \rho_{max}(H)$, where $\rho_{max}(H)$ is the maximum degree of H, because H is a connected graph. Since, by Lemma 2.5, G and H have the same degree sequence, then $\rho_{max}(H) = \rho_1$. However, by Lemma 3.1, the characteristic polynomials of G and H differ by a constant, and therefore, it follows that $\lambda_{max}(H) > \lambda_{max}(G) = \rho_1$. Hence, $\rho_{max}(G) = \rho_{max}(H) = \rho_1 \geq \lambda_{max}(H) > \lambda_{max}(G) = \rho_1$. This is a contradiction. Therefore, G is polynomial reconstructible. ■

4.3. Disconnected Graphs with a Wheel as a Component

Because the existence of a *dominant vertex* in a graph G can be determined from the polynomial deck $PD(G)$ (by Lemma 2.5), the class of graphs having a component with a dominant vertex warrants study. Here, we show that the class of disconnected graphs that have a wheel as a component are polynomial reconstructible. We prove this using the property that the wheel component has a dominant vertex.

Theorem 4.6: Let $G = G_1 \cup G_2$ be a disconnected graph of order n and let one of the components be a wheel. Then, G is polynomial reconstructible.

Proof: Suppose, without loss of generality, that G_1 is the component that is a wheel and let v be the dominant vertex of this component. Then the polynomial corresponding to $G - v$ in $PD(G)$ is the product of the characteristic polynomial of a cycle and the characteristic polynomial of the component G_2. Because, for $r \geq 3$, every cycle, C_r, has a repeated root, then by Corollary 2.4, G is polynomial reconstructible. ■

5. Conclusion

Spectral properties that arise from the class of disconnected graphs can be sufficient to determine the polynomial reconstruction of certain subclasses of the class of disconnected graphs. For example, in **[5]** a number of subclasses of disconnected graphs are shown to be polynomial reconstructible. In this paper the list of subclasses of disconnected graphs that are polynomial reconstructible has been extended to include those that have exactly two unicyclic components, bidegreed disconnected graphs with regular components, and disconnected graphs that have a wheel as a component.

The polynomial reconstruction problem for the class of disconnected graphs remains open in general. If the property of disconnectedness of a graph can be determined from its polynomial deck then this class will immediately be polynomial reconstructible, since the maximum eigenvalue that appears in the polynomial deck $PD(G)$ is also an eigenvalue of G. However, to date it has not been shown that this property can be determined from the polynomial deck. Were connectedness to be proved then not only would this give a positive result for the class of disconnected graphs but it would also rule out a disconnected graph being one of the graphs in a counterexample pair. Hence, a number of connected graphs that require a disconnected graph to form a counterexample pair would also be shown to be polynomial reconstructible.

References

[1] S.M. Ulam; *A Collection of Mathematical Problems*, Wiley Interscience (1960).

[2] I. Gutman and D.M. Cvetković; The reconstruction problem for characteristic polynomials of graphs, *Univ. Beograd. Publ. Elektrotehn. Fak.*, pages 45–48 (1975).

[3] D. Cvetković and M. Lepović; Seeking counterexamples to the reconstruction conjecture for the characteristic polynomial of graphs and a positive result, *Bulletin T. CXVI de l'Académie Serbe des Sciences et des Arts*, **23** (1998).

[4] I. Sciriha; Polynomial reconstruction and terminal vertices, *Linear Algebra and its Applications*, **356**, 145–156 (2002).

[5] I. Sciriha and J.M. Formosa; On polynomial reconstruction of disconnected graphs, *Utilitas Mathematica*, **64**, 33–44 (2003).

[6] S.K. Simić and Zoran Stanić; The polynomial reconstruction of unicyclic graphs is unique, *Linear and Multilinear Algebra*, **55(1)**, 35–43 (2007).

[7] F.H. Clarke; A graph polynomial and its applications, *Discrete Mathematics*, **3**, 305–315 (1972).

[8] D. Cvetković, P. Rowlinson, and S. Simić; Eigenspaces of graphs, *Encyclopedia of Mathematics and its Applications*, **66**, Cambridge (1997).

[9] D. Cvetković; On the reconstruction of the characteristic polynomial of a graph, *Discrete Mathematics*, **212**, 45–52 (1999).

[10] A. Farrugia; *On Main Eigenvalues and Walks of Graphs*, Master's Thesis, University of Malta (2004).

[11] H. Sachs; Beziehungen zwischen den in einem Graphen enthaltenen Kreisen und seinem charakteristiscehn Polynom, *Publ. Math (Debrecen)*, **11**, 119–134 (1963).

[12] I. Gutman; Rectifying a misbelief: Frank Harary's role in the discovery of the coefficient-theorem in chemical graph theory, *J. Mathematical Chemistry*, **16**, 73–78 (1994).

[13] F. Harary; The determinant of the adjacency matrix of a graph, *SIAM Review*, **4(3)**, 202–210 (1962).

[14] C. Zerafa; *The Spectrum of the Adjacency Matrix of Graphs and Regular Subgraphs*, Master's Thesis, University of Malta (2009).

[15] G. Frobenius; Über matrizen aus positiven elementen, *Sitzber. Akad. Wiss, Berlin Phys. Math Kl.*, 471–476 (1908).

[16] O. Perron; Zur theorie der matrices, *Mathematische Annalen*, **64**, 248–263 (1907).

[17] W.T. Tutte; All the king's horses (a guide to reconstruction), *Graph Theory and Related Topics* (1979).

[18] W.J. Myrold, M.N. Ellingham, and D.G. Hoffman; Bidegreed graphs are edge reconstructible, *J. Graph Theory*, **11**, 281–302 (1987).

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