Applicable Analysis and Discrete Mathematics, **1** (2007), 438–445.
Available electronically at **http://pefmath.etf.bg.ac.yu**

SOME PROPERTIES OF THE HOFFMAN-SINGLETON GRAPH

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The Hoffman-Singleton graph, with spectrum $7^{(1)}, 2^{(28)}, -3^{(21)}$, is characterized among regular graphs by a star complement for the eigenvalue 2, that is, by an induced subgraph of order 22 without 2 as an eigenvalue. Properties of other induced subgraphs are noted; in particular, the subgraph induced by vertices at distance 2 from a given vertex is the edge-disjoint union of three Hamiltonian cycles.

1. INTRODUCTION

The Hoffman-Singleton graph HS may be described as the unique Moore graph of degree 7 and diameter 2 [**2**], or as the unique 7-regular graph of order 50 with girth 5 [**1**, p. 189]. It may be constructed as follows, where a *heptad* is a set of seven triples which may be taken as the lines of a Fano plane whose points are $1, 2, 3, 4, 5, 6, 7$ [**6**, Section 5.9]. The vertices of HS are the 15 heptads in an orbit of the alternating group A_7 together with the 35 triples in $\{1, 2, 3, 4, 5, 6, 7\}$. There are edges in HS between disjoint triples, and between a heptad and each of its triples. It follows that HS has an induced subgraph $H_0 \cong K_{1,7}^{(2)}$ illustrated in Fig. 1, where the vertices of degree 1 and 7 are the 15 independent heptads. We note first that H_0 is a star complement for 2 in HS, in the sense of the following definition.

Let G be a finite graph of order n with an eigenvalue μ of multiplicity k. (Thus the corresponding eigenspace of a $(0, 1)$-adjacency matrix of G has dimension k.) A *star set* for μ in G is a set X of k vertices in G such that the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G (or in [**5**] a *μ-basic subgraph* of G). Star sets and star complements exist for any eigenvalue of any graph, and serve to explain the relation between graph structure and a single eigenvalue μ [**4**, Chapter 5].

2000 Mathematics Subject Classification. 05C50, 05C45.
Key Words and Phrases. Graph, eigenvalue, star complement, Hamiltonian cycle.

Now the spectrum of HS is $7^{(1)}, 2^{(28)}, -3^{(21)}$, while that of $K_{1,7}^{(2)}$ is $3^{(1)}, \sqrt{2}^{\,(6)}, 0^{(8)}, -\sqrt{2}^{\,(6)}, -3^{(1)}$. Thus H_0 is a star complement for 2 in HS.

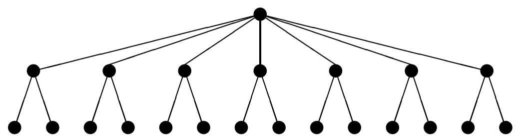

Fig. 1

The following result [**4**, Theorem 5.1.7] establishes the fundamental property of star complements: if X is a star set for μ in G, and if H is the star complement $G - X$, then G is determined by μ, H and the H-neighbourhoods of vertices in X. We shall use implicitly the fact that if $\mu \neq 0$ or -1 then these H-neighbourhoods are non-empty and distinct [**4**, Proposition 5.1.4].

Theorem 1.1. *Let X be a set of k vertices in the graph G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^T(\mu I - C)^{-1}B. \tag{1}$$

In this situation, the eigenspace of μ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$, where $\mathbf{x} \in I\!R^k$.

We take G to have vertex-set $V(G) = \{1, 2, \ldots, n\}$, and we write '$i \sim j$' to denote that vertices i and j are adjacent. We define a bilinear form on $\mathbb{R}^{n-k}$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T(\mu I - C)^{-1}\mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in I\!R^{n-k}).$$

Now equation (1) says that if B has columns $\mathbf{b}_1, \ldots \mathbf{b}_k$, then for all vertices i, j of X:

$$\langle \mathbf{b}_i, \mathbf{b}_j \rangle = \begin{cases} \mu & \text{if } i = j \\ -1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

Recall that μ is a *main* eigenvalue of G if the eigenspace $\mathcal{E}(\mu)$ is not orthogonal to the all-1 vector $\mathbf{j}_n$; and that in a connected r-regular graph, all eigenvalues other than r are non-main eigenvalues. If the conditions of Theorem 1.1 are satisfied, and if $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ is the standard basis of $\mathbb{R}^k$ then $\mathcal{E}(\mu)$ has a basis consisting of the vectors

$$\begin{pmatrix} \mathbf{e}_i \\ (\mu I - C)^{-1}B\mathbf{e}_i \end{pmatrix} \quad (i = 1, \ldots, k).$$

Since $B\mathbf{e}_i$ is the i-th column of B, we see that μ is a non-main eigenvalue if and only if

$$(3) \qquad \langle \mathbf{b}_i, \mathbf{j} \rangle = -1 \quad (i = 1, \ldots, k),$$

where $\mathbf{j} = \mathbf{j}_{n-k}$.

In Section 2, we discuss the addition of vertices to $K_{1,7}^{(2)}$ to obtain 2 as a non-main eigenvalue. This enables us to characterize HS as the only regular graph with $K_{1,7}^{(2)}$ as a star complement for 2. In Section 3 we discuss other induced subgraphs of HS; in particular, we note that the vertices at distance 2 from a given vertex induce a subgraph (of order 42) which is the edge-disjoint union of three Hamiltonian cycles.

Characterizations of other graphs by star complements are documented in the survey paper [**9**]. Many other properties of the Hoffman-Singleton graph are listed in [2, Section 13.1].

2. A CHARACTERIZATION OF HS

In this section we retain the notation of Theorem 1.1 and suppose that H is a star complement for 2 isomorphic to $K_{1,7}^{(2)}$. In this situation, with a natural ordering of vertices,

$$(4) \quad 20(2I - C)^{-1} = \left(\begin{array}{r|rrcr|rrrrcrr}
-4 & -4 & -4 & \cdots & -4 & -2 & -2 & -2 & -2 & \cdots & -2 & -2 \\
\hline
-4 & 16 & -4 & \cdots & -4 & 8 & 8 & -2 & -2 & \cdots & -2 & -2 \\
-4 & -4 & 16 & \cdots & -4 & -2 & -2 & 8 & 8 & \cdots & -2 & -2 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
-4 & -4 & -4 & \cdots & 16 & -2 & -2 & -2 & -2 & \cdots & 8 & 8 \\
\hline
-2 & 8 & -2 & \cdots & -2 & 14 & 4 & -1 & -1 & \cdots & -1 & -1 \\
-2 & 8 & -2 & \cdots & -2 & 4 & 14 & -1 & -1 & \cdots & -1 & -1 \\
-2 & -2 & 8 & \cdots & -2 & -1 & -1 & 14 & 4 & \cdots & -1 & -1 \\
-2 & -2 & 8 & \cdots & -2 & -1 & -1 & 4 & 14 & \cdots & -1 & -1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots \\
-2 & -2 & -2 & \cdots & 8 & -1 & -1 & -1 & -1 & \cdots & 14 & 4 \\
-2 & -2 & -2 & \cdots & 8 & -1 & -1 & -1 & -1 & \cdots & 4 & 14
\end{array}\right).$$

Here the blocks are determined by $\{u\} \,\dot\cup\, \Gamma_1(u) \,\dot\cup\, \Gamma_2(u)$, where u is the vertex of degree 7 in H and $\Gamma_i(u)$ is the set of vertices at distance i from u in H $(i = 1, 2)$. The graph $H - u$ consists of seven disjoint 2-claws, which we label $W_1, \ldots, W_7$, with central vertices $u_1, \ldots, u_7$ respectively. The remaining vertices of W_i are labeled s_i, t_i $(i = 1, \ldots, 7)$.

Lemma 2.1. *If* 2 *is a non-main eigenvalue of* $H + v$, *then*

(i) v *is not adjacent to* u,

(ii) v *is adjacent to just one vertex in* $\Gamma_1(u)$,

(iii) v *has just three neighbours in* $\Gamma_2(u)$,

(iv) *the four neighbours of* v *lie in four different claws of* H.

Proof. By equation (3), 2 is a non-main eigenvalue of $H + v$ if and only if the sum of entries of the rows of $20(2I - C)^{-1}$ indexed by the H-neighbourhood of v is -20. The sum of entries in row j is -60 if $j = u$, -20 if $j \in \Gamma_1(u)$, and 0 otherwise; statements (i) and (ii) follow.

We say that a claw W is of type $\alpha\beta$ in $H + v$ if v is adjacent to α vertices of degree 2 in W and β vertices of degree 1 in W ($\alpha = 0$ or 1, $\beta = 0, 1$ or 2). Further, v is of type *abcde* if $H + v$ has a, b, c, d, e claws of type $01, 02, 10, 11, 12$ respectively.

If $\mathbf{b}_v$ is the characteristic vector of the H-neighbourhood of v then

$$\begin{aligned} 20\mathbf{b}_v^T(2I - C)^{-1}\mathbf{b}_v = {} & (c + d + e)^2(-4) + (2b + a + d + 2e)^2(-1) \\ & + 2(c + d + e)(2b + a + d + 2e)(-2) + 40(b + e) \\ & + 20d + 40e + 15(a + d) + 20(c + d + e). \end{aligned}$$

(Here the first three summands are determined by a matrix with constant blocks, obtained from $20(2I - C)^{-1}$ by replacing 8 by -2, 16 by -4, 14 by -1 and 4 by -1; the remaining summands are the required correction terms.) We may write this equation in the form

$$20\mathbf{b}_v^T(2I - C)^{-1}\mathbf{b}_v = -q^2 + 10q + 5a + 20b + 25d + 60e, \tag{5}$$

where

$$q = a + 2b + 2c + 3d + 4e. \tag{6}$$

From equation (2), we have $\langle \mathbf{b}_v, \mathbf{b}_v \rangle = 2$, and equation (5) yields

$$15 + (q - 5)^2 = 5a + 20b + 25d + 60e, \tag{7}$$

Note that 5 divides q. Equations (6) and (7) may now be used to find all (nine) solutions for a, b, c, d, e and hence all $H + v$ ($v \not\sim u$) for which 2 is an eigenvalue. However, when 2 is a non-main eigenvalue, we have $c + d + e = 1$ from (ii). In this situation we have $q \in \{5, 10, 15\}$ since $a + b + c + d + e \leq 7$. For each of the possibilities $(c, d, e) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$, equations (6) and (7) yield simultaneous equations for a and b, and we find a unique solution $(a, b, c, d, e) = (3, 0, 1, 0, 0)$. Statements (iii) and (iv) follow. □

Next we investigate the intersection of the H-neighbourhoods $\Delta_H(v), \Delta_H(w)$ of two vertices v, w in X. By equation (2), $20\mathbf{b}_v^T(2I - C)^{-1}\mathbf{b}_w \in \{-20, 0\}$. Here the left-hand side is the sum $\sigma(v, w)$ of entries in a 4×4 submatrix M of $20(2I - C)^{-1}$ (see equation (4)). With a suitable labeling of the four vertices in $\Delta_H(v)$, M consists of four appropriate columns of the submatrix

$$(8)\quad N = \left(\begin{array}{c|ccccccc|cccccccccccccc} -4 & 16 & -4 & -4 & -4 & -4 & -4 & -4 & 8 & 8 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\ -2 & -2 & 8 & -2 & -2 & -2 & -2 & -2 & -1 & -1 & 14 & 4 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -2 & -2 & -2 & 8 & -2 & -2 & -2 & -2 & -1 & -1 & -1 & -1 & 14 & 4 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -2 & -2 & -2 & -2 & 8 & -2 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & 14 & 4 & -1 & -1 & -1 & -1 & -1 & -1 \end{array}\right).$$

Lemma 2.2. *Suppose that* 2 *is a non-main eigenvalue of* $H+v+w$ *of multiplicity* 2, *and that* $v \sim u_1$.

(i) *If* $w \sim u_1$ *then* $w \not\sim v$ *and* $H+v+w$ *has the form shown in Fig.* 2.

(ii) *If* $w \not\sim u_1$ *and* $w \not\sim v$ *then* $H+v+w$ *has one of the three forms shown in Figs.* 3, 4 *and* 5.

(iii) *If* $w \not\sim u_1$ *and* $w \sim v$ *then* $H+v+w$ *has the form shown in Fig.* 6.

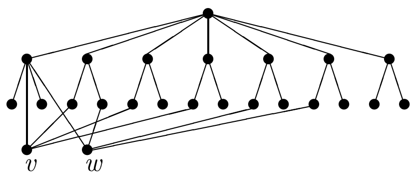

Fig. 2

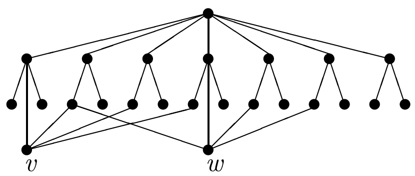

Fig. 3

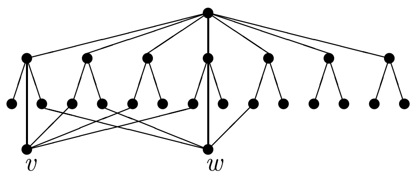

Fig. 4

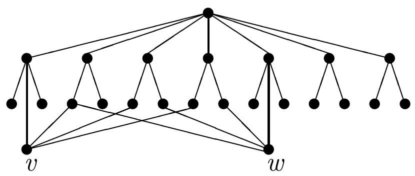

Fig. 5

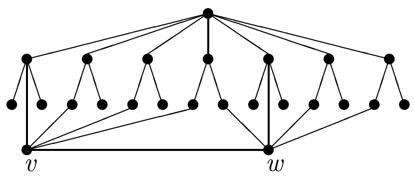

Fig. 6

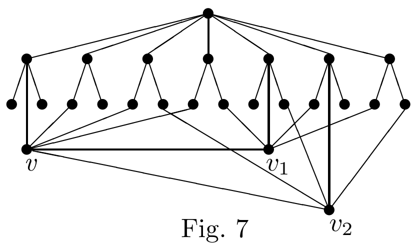

Fig. 7

Proof. Since w is of type 30100, $\sigma(v,w)$ is the sum of entries in one column from the second block of N and three columns from the third block (see equation (8)). Moreover, by Lemma 2.1(iv), the four vertices in $\Delta_H(w)$ lie in different claws.

For (i), the second column of N must be included; then $\sigma(v,w)=0$ and the four column sums are $10; 0, -5, -5$. In this case, $H+v+w$ has the form shown in Fig. 2.

For (ii), we know that the second column of N is excluded, and $\sigma(v,w)=0$.

Thus the column sums are $0; 10, -5, -5$ or $0; 5, 0, -5$ or $-10; 0, 0, 10$ or $0; 0, 0, 0$ or $-10; 5, 5, 0$. The fourth possibility is ruled out because $d = 0$ and the fifth is ruled out because $b = 0$. The remaining possibilities for $H + v + w$ are shown in Figs. 3, 4 and 5.

For (iii), the second column of N is excluded, and $\sigma(v, w) = -20$. Thus the column sums are $-10; 0, -5, -5$ and $H + v + w$ is as shown in Fig. 6. □

Theorem 2.3. *If G is a regular graph with $H \cong K_{1,7}^{(2)}$ as a star complement for the eigenvalue 2 then $G \cong HS$.*

Proof. Clearly G has degree at least 7, and so the eigenvalue 2 is a non-main eigenvalue of G. By Lemma 2.1(i), no vertex of the star set X is adjacent to u, and so G is 7-regular. For $i = 1, \ldots, 7$, let X_i be the set of 4 neighbours of u_i in X. By Lemma 2.2(i), each X_i is an independent set in G. It follows from Lemma 2.1(ii) that all vertices of G are at distance at most 2 from u, and that the sets $X_1. \ldots, X_7$ are pairwise disjoint. Hence $|X| = 28$ and G has order 50.

Our objective now is to show that G has girth at least 5. Clearly any possible 3-cycle or 4-cycle has a vertex in X; and by Lemma 2.1, such a cycle C has at least two vertices v, w in X. Suppose that C has exactly two vertices in X. By Lemma 2.2(iii), $v \not\sim w$, and so C is a 4-cycle; but this possibility is ruled out by Lemma 2.2(ii).

Now consider vertices v_1, v, v_2 in X such that $v_1 \sim v \sim v_2$. We may suppose that $\Delta_H(v) = \{u_1, s_2, s_3, s_4\}$ and $\Delta_H(v_1) = \{t_4, u_5, s_6, s_7\}$ (cf. Fig. 6). Suppose, by way of contradiction, that v_1, v, v_2 do not lie in different X_j. Then $v_2 \in X_5$, and so none of t_4, s_5, t_5 is a neighbour of v_2 (cf. Fig. 2). Since $v_2 \sim v$, none of s_1, t_1, s_4 is a neighbour of v_2 (cf. Fig. 6). Without loss of generality, $v_2 \sim t_6$ (cf. Fig. 2). Then none of s_6, s_7, t_7 is a neighbour of v_2. Without loss of generality, $v_2 \sim t_2$ (cf. Fig. 6). Then none of s_2, s_3, t_3 is a neighbour of v_2. Thus v_2 cannot have 4 neighbours in H, a contradiction.

Now we know that v_1, v, v_2 lie in different X_i, we may suppose that $v_2 \in X_6$. From Fig. 6 we see that not only $v_2 \not\sim s_4$ but also $v_2 \not\sim t_4$, for otherwise $\Delta_H(v_1) \cup \Delta_H(v_2)$ is contained in only four claws (namely W_4, W_5, W_6, W_7), contradicting Lemma 2.2. Without loss of generality, $v_2 \sim t_3$, and hence $\Delta_H(v_2) = \{t_3, t_5, u_6, t_7\}$. Now $H + v_1 + v_2$ has the form shown in Fig. 4. In particular, $v_1 \not\sim v_2$; therefore there are no 3-cycles in the graph induced by X, and hence no 3-cycles in G.

The graph $H + v + v_1 + v_2$ is shown in Fig. 7. This graph has no 4-cycles, and so to show that G has no 4-cycles, it suffices to show that there is no vertex $w \in X \setminus \{v\}$ such that w is adjacent to both v_1 and v_2. If w is such a vertex then both of $H + v_1 + w$ and $H + v_2 + w$ have the form shown in Fig. 6, and so none of W_5, W_6, W_7 contains a vertex of $\Delta_H(w)$. Hence $\Delta_H(v) \cup \Delta_H(w)$ is contained in only four claws (namely W_1, W_2, W_3, W_4), a contradiction.

We conclude that G has no 4-cycles. Since X contains adjacent vertices, G has girth 5. Since G has order 50, necessarily $G \cong HS$. □

3. SOME OTHER INDUCED SUBGRAPHS

Here we take $G = HS$ and retain the notation of Section 2, with $H = H_0$. Additionally, we let $\Delta_i(u)$ denote the set of vertices at distance i from u in G ($i = 1, 2$). (Note that $\Delta_1(u) = \Gamma_1(u)$.) We know that G is a transitive graph, and the stabilizer of the vertex u is S_7, with orbits $\{u\}, \Delta_1(u), \Delta_2(u)$ (of lengths $1, 7, 42$); moreover the subgraph G_2 induced by $\Delta_2(u)$ is the unique distance-regular graph with intersection array $\{6, 5, 1; 1, 1, 6\}$ [**2**,Theorem 13.1.1]. In answer to a question posed by S. Fiorini [private communication], we note here the following property of G_2.

Propostion 3.1. *The subgraph of HS induced by the vertices at distance* 2 *from a given vertex is the edge-disjoint union of three Hamiltonian cycles.*

Proof. Let $u = \{124, 235, 346, 457, 561, 672, 713\}$, so that we can take the vertex u_i in $\Gamma_1(u)$ to be the triple with elements $1\alpha^{i-1}, 2\alpha^{i-1}, 4\alpha^{i-1}$ ($i = 1, \dots, 7$), where α is the permutation (1234567). Then the neighbours of u_1 in $\Gamma_2(u)$ are

$$P = \{124, 135, 167, 236, 257, 347, 456\}, \quad Q = \{124, 136, 157, 237, 256, 345, 467\}.$$

Now we can check easily that G_2 is the edge-disjoint union of the following three 42-cycles:

$357, P\alpha^2, 247, 356, Q\alpha^3, 167, 234, 567, Q\alpha^2, 237, Q, 136, P\alpha, 145, 367,$
$245, Q\alpha^4, 127, 345, 126, Q\alpha^6, 467, 135, 246, 157, Q\alpha^5, 123, 456, Q\alpha, 134,$
$257, P\alpha^4, 147, 256, P\alpha^3, 146, P\alpha^6, 236, P, 347, 125, P\alpha^5, 357;$

$357, Q\alpha^4, 134, 567, 123, P\alpha^2, 167, P, 456, P\alpha^5, 234, Q\alpha^6, 145, 237, Q\alpha^3,$
$246, P\alpha^4, 345, P\alpha^6, 247, 135, P\alpha^3, 127, P\alpha, 347, 256, Q, 467, 125, 367,$
$Q\alpha, 157, 236, 147, 356, Q\alpha^5, 146, 257, 136, 245, Q\alpha^2, 126, 357;$

$357, 246, P\alpha, 567, P\alpha^6, 125, Q\alpha^3, 134, 256, P\alpha^2, 145, 236, Q\alpha^4, 467, 123,$
$P\alpha^4, 367, P\alpha^3, 234, 157, Q, 345, 167, 245, Q\alpha^5, 347, 126, Q\alpha, 247, 136,$
$P\alpha^5, 147, Q\alpha^2, 135, P, 257, Q\alpha^6, 356, 127, 456, 237, 146, 357.$

$\square$

The three 42-cycles in Proposition 3.1 were found by computer as follows. Let $v_{i1}, \dots, v_{i6}$ be the neighbours of u_i in G_2 ($i = 1, \dots, 6$). Each of the vertices $v_{11}, \dots, v_{16}$ is adjacent to 6 other vertices v_{ij} ($i \neq 1$); moreover the neighbourhoods of $v_{11}, \dots, v_{16}$ are disjoint, and so we have a subgraph $F \cong 6K_{1,6}$. We start with a spanning tree T for G_2 obtained by adding 5 edges to F, and construct 85 unicyclic graphs $U_1, \dots, U_{85}$ by adding to T each of the remaining 85 edges of G_2 in turn (cf. [**8**, Theorem 7.7]). Let Q_i be the unique cycle in U_i ($i = 1, \dots, 85$). We find a partition of $\{Q_1, \dots, Q_{85}\}$ into sets S_1, S_2, S_3 (of sizes 27, 28, 30) such that in the cycle space of G_2, the sum of cycles in S_i is a Hamiltonian cycle ($i = 1, 2, 3$).

Our final remarks concern the 28 vertices in $X = X_1 \,\dot\cup\, \cdots \,\dot\cup\, X_7$: these represent the triangles of a Fano plane (the triples not in the heptad u), and the subgraph they induce is therefore the Coxeter graph [**3**], with spectrum

$3^{(1)}, 2^{(8)}, (\sqrt{2}-1)^{(6)}, (-1)^{(7)}, (-\sqrt{2}-1)^{(6)}$. Since G_2 has spectrum $6^{(1)}, (-1)^{(6)}, 2^{(21)}, (-3)^{(14)}$ [**7**], the COXETER graph is a star complement for -3 in G_2. The corresponding star set is the independent set of 14 vertices in $\Gamma_2(u)$. Since -3 is an eigenvalue of HS of multiplicity 21, we can see also that HS has as a star complement for -3 a subgraph consisting of the COXETER graph and an isolated vertex (the subgraph induced by $\{u\} \dot{\cup} X$).

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