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Mathematical aspects of an exactly solvable inflationary model

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Abstract. The inflationary scenario is nowadays the most natural and simple way to solve the problems of the standard Friedmann-Lemaître-Robertson-Walker cosmology. Generally, inflation is driven by models of gravity minimally coupled to a scalar field ϕ rolling on a suitable potential $V(\phi)$. We discuss a particular model presenting two scenarios: the vacuum universe and the matter-filled universe, in order to find exact general isotropic and homogeneous cosmological solutions displaying an inflationary behavior at early times and a power-law expansion at late times.

1. Introduction

The inflationary scenario is based on the hypothesis that a period of accelerated expansion has taken place right after the Big Bang [1, 3]. It is today the widely adopted paradigm to interpret the high isotropy and homogeneity at large scales, the spatial flatness of our universe and the presence of about 24% of dark matter of unknown origin [4], and provides explanations of the formations of structures in the universe [5, 6].

The most common realization of inflation is based on models of gravity minimally coupled to a scalar field (the inflaton) with a self-interaction potential. These models can be classified according to the form of the potential [4]: in particular, among exponential potentials [7, 8] and even more general form [9, 10], double exponential potentials are the source of brane solutions of Einstein-scalar gravity, called domain walls, which can be analytically continued into Friedmann-Lemaître-Robertson-Walker cosmological solutions [11, 12] from the same model but with opposite scalar potential [13].

Inspired by this observation, we discuss the mathematical aspects of a model [12] of gravity minimally coupled to a scalar field ϕ defined by the action

$$I = \int \sqrt{-g} \left[\frac{R}{16\pi G} - 2(\partial\phi)^2 - V(\phi) \right] d^4x, \quad (1)$$

where $R = g^{\rho\rho} R_{\rho\rho}$ is the trace of the Ricci tensor $R_{\mu\nu}$ and g is the determinant of the metric tensor $g_{\mu\nu}$. Indeed, in this model we use natural unities such that $c = \hbar = 1$. The scalar potential

$$V(\phi) = \frac{2\lambda^2}{3\gamma} \left(e^{2\sqrt{3}\beta\phi} - \beta^2 e^{2\sqrt{3}\phi/\beta} \right) \quad (2)$$



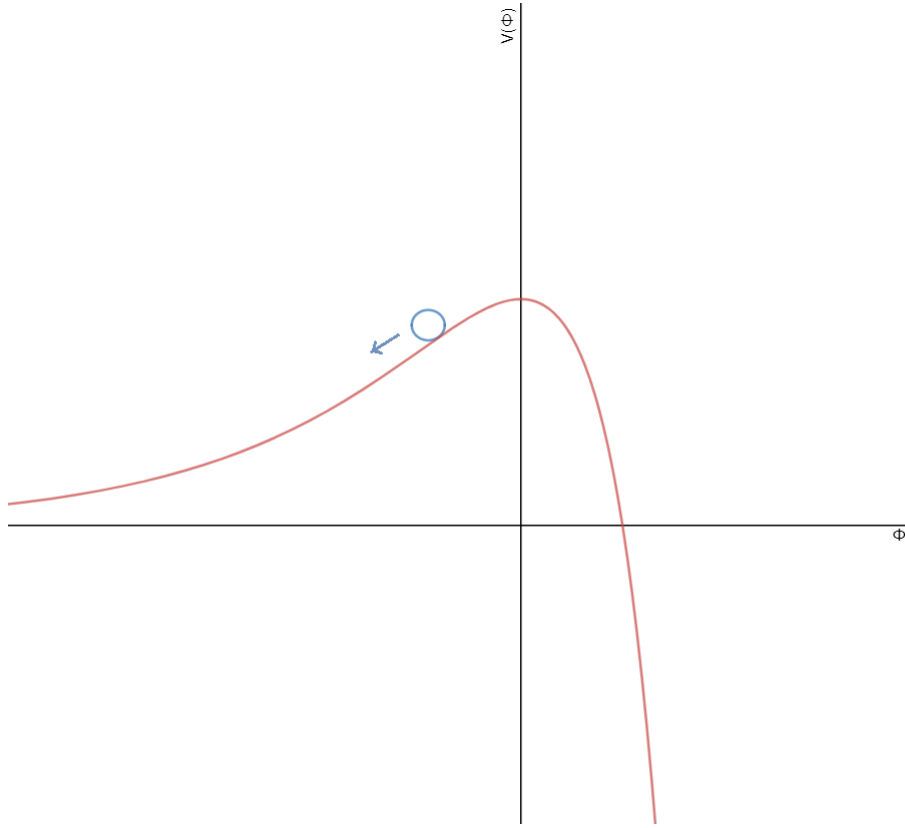


Figure 1. Plot of the potential (2) (in red) for $\lambda = 1$, $\beta = 1/\sqrt{3}$, with the inflaton (in blue) rolling on it.

depends on two parameters λ and β , with $\gamma = 1 - \beta^2$. The potential (2), for particular values of its parameters, is shown in Fig. 1. In general, the model is characterized by an exact solution in which the scale factor expands exponentially at early times and then behaves as a power law. Moreover, it admits a duality for $\beta \rightarrow \frac{1}{\beta}$ and for this reason we shall limit ourselves to the discussion of the case $0 < \beta^2 < 1$.

2. General solutions

We are interested in general isotropic and homogeneous cosmological solutions, with flat spatial sections, that we parametrize as

$$ds^2 = -e^{2a(\tau)}d\tau^2 + e^{2b(\tau)}d\Omega^2, \quad \phi = \phi(\tau), \quad (3)$$

with τ a time variable.

With this parametrization, defining in the gauge $a = 3b$ two new variables $\Psi = a + \sqrt{3}\beta\phi$, $\chi = a + \frac{\sqrt{3}}{\beta}\phi$, we obtain the following exactly integrable vacuum equations

$$\ddot{\Psi} = \lambda^2 e^{2\Psi}, \quad \ddot{\chi} = \lambda^2 e^{2\chi}, \quad \dot{\Psi}^2 - \beta^2 \dot{\chi}^2 = \lambda^2 (e^{2\Psi} - \beta^2 e^{2\chi}). \quad (4)$$

These equations are invariant under time reversal, $\tau \rightarrow -\tau$. The equations (4) admit first integrals

$$\dot{\Psi}^2 = \lambda^2 e^{2\Psi} + Q_1, \quad \dot{\chi}^2 = \lambda^2 e^{2\chi} + Q_2 \quad (5)$$

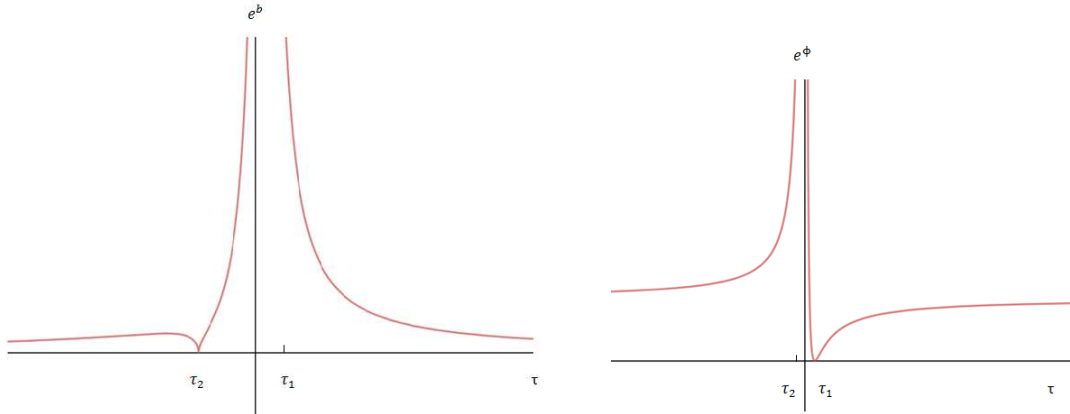


Figure 2. The solutions with $Q_i = 0$ for $e^{b(\tau)}$ and $e^{\phi(\tau)}$ when $\tau_1 \neq \tau_2$.

with Q_1 and Q_2 integration constants that satisfy the constraint $Q_1 = \beta^2 Q_2$. In this way, we can formally simplify the solutions of the system (4).

The solutions of the equations (4) are summarised in table 1, while the solution $Q_i = 0$ is depicted in Fig. 2.

Table 1. Solutions of the equations (4) depending on the sign of Q_i ($i = 1, 2$), with $Q_1 = \beta^2 Q_2$ and $\tau_1 \neq \tau_2$ where τ_1, τ_2 are integration constants. We define $q^2 = Q_2 = Q_1/\beta$.

$Q_i > 0$	$Q_i < 0$	$Q_i = 0$
$e^{2b} = \frac{q^2}{\lambda^2} \frac{ \sinh [q (\tau - \tau_2)] ^{2\beta^2/\gamma}}{ \beta^{-1} \sinh [\beta q (\tau - \tau_1)] ^{2/\gamma}}$	$e^{2b} = \frac{q^2}{\lambda^2} \frac{ \sin [q (\tau - \tau_2)] ^{2\beta^2/\gamma}}{ \beta^{-1} \sin [\beta q (\tau - \tau_1)] ^{2/\gamma}}$	$e^{2b} = \frac{1}{\lambda^2} \frac{ \tau - \tau_2 ^{2\beta^2/\gamma}}{ \tau - \tau_1 ^{2/\gamma}}$
$e^{2\sqrt{3}\phi/\beta} = \left \frac{\sinh [\beta q (\tau - \tau_1)]}{\beta \sinh [q (\tau - \tau_2)]} \right ^{2/\gamma}$	$e^{2\sqrt{3}\phi/\beta} = \left \frac{\sin [\beta q (\tau - \tau_1)]}{\beta \sin [q (\tau - \tau_2)]} \right ^{2/\gamma}$	$e^{2\sqrt{3}\phi/\beta} = \left \frac{\tau - \tau_1}{\tau - \tau_2} \right ^{2/\gamma}$

We note that the parameter $q = \sqrt{Q_1}/\beta$ only sets the scale of the time parameter τ , while the physics interpretation is more transparent by changing the parametrization (3) defining the cosmic time t as $dt = \pm e^a d\tau$:

$$ds^2 = -dt^2 + e^{2b(t)} d\Omega^2, \quad (6)$$

but the solutions in table 1 cannot be written in terms of elementary functions of t , except when $Q_i = 0$, $\tau_1 = \tau_2$. In this case, one obtains an expanding universe with $e^{2b} = e^{2\lambda(t-t_0)/3}$ and $\phi = 0$.

In the general case, to describe the smooth transition from inflationary expansion to a later power-law behavior, we consider those solutions that behave exponentially for $t \rightarrow -\infty$ and as a power law for $t \rightarrow \infty$. These are obtained for $Q_i = 0$, $\tau_1 \neq \tau_2$, as shown in Fig. 3, and correspond to an initial configuration where the scalar field is in the unstable equilibrium configuration at the top of the potential and then rolls down to $\phi = \infty$, that is $e^{2b} \sim e^{2\lambda t/3}$, $e^{2\sqrt{3}\phi/\beta} \sim \text{const}$ for $t \rightarrow -\infty$ and $e^{2b} \sim t^{2/3\beta^2}$, $e^{2\sqrt{3}\phi/\beta} \sim t^{-2/\beta^2}$ for $t \rightarrow \infty$.

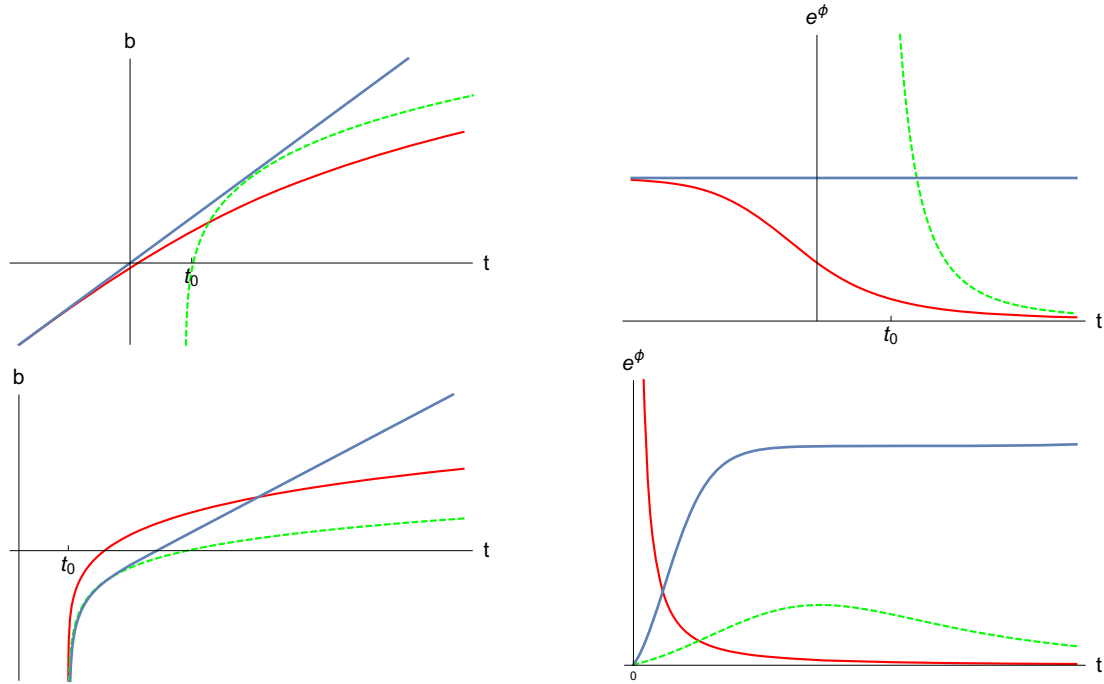


Figure 3. The solutions with $Q_i = 0$ (top panels) and with $Q_i \neq 0$ (bottom panels) in terms of the cosmic time t . The blue lines correspond to the solutions with $\tau_1 = \tau_2$ (top panels) and with $\tau_1 \neq \tau_2$ (bottom panels), while the continuous and the dashed curve (in red and green, respectively) are the expanding branches of the $\tau_1 = \tau_2$ (top panels) and of the $\tau_1 \neq \tau_2$ solutions (bottom panels).

3. Solutions with matter

Let us introduce matter in the form of a perfect fluid. The continuity equation $\dot{\rho} + (3p + \rho) \dot{a} = 0$, integrated for the equation of state $p = \omega\rho$ (with $\omega \geq -1$), gives $\rho = \rho_0 e^{-(1+\omega)a}$, where ρ_0 is an integration constant. The Friedmann equations (4) become

$$\ddot{\Psi} = \lambda^2 e^{2\Psi} + \frac{3}{4} \rho_0 (1 - \omega) e^{(1-\omega)(\Psi - \beta^2 \chi)/\gamma}, \quad \ddot{\chi} = \lambda^2 e^{2\chi} + \frac{3}{4} \rho_0 (1 - \omega) e^{(1-\omega)(\Psi - \beta^2 \chi)/\gamma} \quad (7)$$

subject to the constraint

$$\dot{\Psi}^2 - \beta^2 \dot{\chi}^2 = \lambda^2 (e^{2\Psi} - \beta^2 e^{2\chi}) + \frac{3}{2} \rho_0 \gamma e^{(1-\omega)(\Psi - \beta^2 \chi)/\gamma}. \quad (8)$$

To investigate their properties, we put them in the form of a dynamical system, defining

$$X = \frac{1}{2\gamma(1-\omega)} (\dot{\Psi} - \beta^2 \dot{\chi}), \quad Y = \dot{\chi}, \quad Z = \lambda e^{\chi}, \quad W = \sqrt{\lambda} e^{(\Psi - \beta^2 \chi)/2\gamma}. \quad (9)$$

With this choice of variables, we obtain a first-order differential system:

$$\dot{X} = \frac{\alpha}{2} W^2 - \frac{\beta^2}{2\gamma} Z^2 + \frac{1}{2\gamma} W^{4\gamma(1-\omega)} Z^{2\beta^2}, \quad \dot{Y} = Z^2 + \alpha W^2, \quad \dot{Z} = YZ, \quad (10)$$

Table 2. Classification of equilibrium points of dynamical system associated to the equations (7)

z^2	y	w^2	v^2	Eigenvalues	Kind of stability
0	$y_0 = 2(1 \pm \beta^{-1})$	0	0	$(0, 0, y_0)$	unstable
0	2	$\frac{2(1-\omega)}{\alpha}$	0	$(\omega - 1, \omega - 1, 1 - \omega)$	unstable
$\frac{4\gamma^2}{\beta^4}$	$-\frac{2\gamma}{\beta^2}$	0	0	$\left(\frac{2\gamma}{\beta^2}, \frac{2\gamma}{\beta^2}, \frac{4\gamma}{\beta^2}\right)$	stable
4	2	0	$\rightarrow 4$	$(-2, 2, -4)$	unstable
0	0	0	$\rightarrow 4\gamma^2$	$(-2\gamma, -2\gamma, -2\gamma)$	stable

with $\alpha = \frac{3\rho_0}{4\lambda}$ and W implicitly defined as

$$\frac{2\alpha}{1-\omega}W^2 + \frac{1}{\gamma}Z^{2\beta^2}W^{4\gamma(1-\omega)} = \frac{\beta^2}{\gamma}Z^2 + 4\gamma X^2 + 4\beta^2 XY - \beta^2 Y^2. \quad (11)$$

The global properties of the solutions of the system (10) can be deduced from the study of its behavior near the critical points at finite distance or at infinity in the phase space.

In particular, the critical points at finite distance lie on two straight lines on the plane $Z = 0$, where $e^{2b} \sim t^{2/3}$, $e^{2\sqrt{3}\phi/\beta} \sim t^{2/\beta}$ as $t \rightarrow 0$.

The remaining critical points lie on the surface at infinity of the phase space. They can be studied by defining new variables $u = 1/X$, $y = Y/X$, $z = Z/X$, $w = W/X$ and considering the limit $u \rightarrow 0$. This is attained for $\tau \rightarrow \tau_0$, where τ_0 is a finite constant. For simplicity, we also define $v := z^{\beta^2} w^{2\gamma/(1-\omega)} u^{-\gamma(1+\omega)/(1-\omega)}$. One finds the critical points classified in table 2.

4. A more general form of the inflation potential

We discuss in this section a generalization of the potential (2) with a positive parameter $h \neq 1$, so that

$$V(\phi) = \frac{2\lambda^2}{3\gamma} \left(e^{2\sqrt{3}\beta h\phi} - \beta^2 e^{2\sqrt{3}h\phi/\beta} \right). \quad (12)$$

The action (1) and the parametrization (3) are unchanged. Defining two new variables $\tilde{\Psi} = \sqrt{3}\beta h\phi + 3b$, $\tilde{\chi} = \frac{\sqrt{3}h\phi}{\beta} + 3b$, the time-reversal invariant vacuum equations read

$$\ddot{\Psi} = \frac{\lambda^2}{\gamma} [(1 - h^2\beta^2) e^{2\Psi} - \beta^2 (1 - h^2) e^{2\chi}], \quad \ddot{\chi} = \frac{\lambda^2}{\gamma} [(1 - h^2) e^{2\Psi} - (\beta^2 - h^2) e^{2\chi}] \quad (13)$$

with $(1 - h^2\beta^2)\beta^2\dot{\chi}^2 + (\beta^2 - h^2)\dot{\Psi}^2 - 2\beta^2(1 - h^2)\dot{\chi}\dot{\Psi} = \lambda^2 h^2 \gamma (e^{2\Psi} - \beta^2 e^{2\chi})$.

This system is no longer exactly integrable: as in the case of matter, we study the related dynamical system and we can find the critical points in table 3.

By comparing the $h \neq 1$ phase space with the one corresponding to $h = 1$, we see that the numerical behaviour of the solutions can be varied depending on the value of h . This allows the possibility of obtaining more realistic solutions for the expansion of the universe, as discussed in [14].

5. Conclusion

In order to describe explicitly the transition from the exponential inflation to the late time power-law expansion of the universe we have introduced an exactly solvable model of gravity

Table 3. Classification of equilibrium points of dynamical system associated to the equations (13)

z^2	y	w^2	Eigenvalues	Kind of stability
0	$y_0 = \frac{\beta^2 - h^2}{\beta^2(1-h^2)}$	0	$(0, 0, y_0)$	unstable
0	y_0	$-\frac{\gamma}{\beta^2}y_0$	$\left(-y_0, -y_0, \frac{h^2\gamma}{\beta^2(1-h^2)}\right)$	unstable
$\frac{\gamma}{1-h^2\beta^2}$	$\frac{1-h^2}{1-\beta^2h^2}$	0	$\left(-1, -1, -\frac{2}{\gamma}(1-h^2\beta^2)\right)$	stable
1	1	1	$\left(-1, -1, -\frac{2}{\gamma}(1-h^2\beta^2)\right)$	unstable

minimally coupled to a scalar field subject to a doubly-exponential potential, whose solution corresponds to the decay from an initial configuration where the scalar field is in an unstable vacuum state. The aim of our mathematical investigation has been to study the general solutions of such model in vacuum and in the presence of matter. It turns out that the exponential solution corresponds to very specific initial conditions, while generic solutions present instead power-law inflation. The physical relevance of the solutions depends on the possibility that the inflationary mechanism provided by our model gives correct predictions on the evolution of perturbations and on observable cosmological data. This is possible if one generalizes the potential as in (12). Although in the general case it is not possible to find exact solutions, the behavior of the solutions is qualitatively similar to the previous case, except that the new parameter allows for the possibility of obtaining more realistic predictions. Some exact solutions however are still available for specific initial conditions, which essentially coincide with those found in the previous case.

References

- [1] Starobinski A 1980 *Phys. Lett. B* **91** 99
- [2] Guth A 1981 *Phys. Rev. D* **23** 347
- [3] Linde A 1982 *Phys. Lett. B* **108** 389
- [4] Fré P, Sagnotti A and Sorin A 2013 *Nucl. Phys. B* **877** 1028
- [5] Lyth D and Liddle A 2009 *The Primordial Density Perturbation* (Cambridge: Cambridge University Press)
- [6] Bassett B, Tsujikawa S and Wands D 2006 *Rev. Mod. Phys.* **78** 537
- [7] Lucchin F and Matarrese S 1985 *Phys. Rev. D* **32** 1316
- [8] Salopek D and Bond J 1990 *Phys. Rev. D* **32** 3936
- [9] Ellis G and Madsen M 1991 *Class. Quantum Grav.* **8** 667
- [10] Chervon S, Zhuravlev V and Shchigolev V 1997 *Phys. Lett. B* **398** 269
- [11] Cadoni M and Ciulu M 2014 *J. High Energy Phys.* JHEP05(2014)089
- [12] Mignemi S and Pintus N 2015 *Gen. Relativ. Grav.* **47** 51
- [13] Skenderis K and Townsend P 2006 *Phys. Rev. Lett.* **96** 191301
- [14] Cadoni M, Franzin E and Mignemi S 2016 *Eur. Phys. J. C* **76** 483