



University of Dundee

Semiclassical regularization of Vlasov equations and wavepackets for nonlinear Schrödinger equations

Athanassoulis, Agissilaos

Published in:
Nonlinearity

DOI:
[10.1088/1361-6544/aa9a86](https://doi.org/10.1088/1361-6544/aa9a86)

Publication date:
2018

Document Version
Peer reviewed version

[Link to publication in Discovery Research Portal](#)

Citation for published version (APA):

Athanassoulis, A. (2018). Semiclassical regularization of Vlasov equations and wavepackets for nonlinear Schrödinger equations. *Nonlinearity*, 31(3), 1045-1072. <https://doi.org/10.1088/1361-6544/aa9a86>

General rights

Copyright and moral rights for the publications made accessible in Discovery Research Portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from Discovery Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain.
- You may freely distribute the URL identifying the publication in the public portal.

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Semiclassical regularization of Vlasov equations and wavepackets for nonlinear Schrödinger equations

Agissilaos Athanassoulis *

Tuesday 3rd October, 2017

Abstract

We consider the semiclassical limit of nonlinear Schrödinger equations with initial data that are well localized in both position and momentum (non-parametric wavepackets). We recover the Wigner measure of the problem, a macroscopic phase-space density which controls the propagation of the physical observables such as mass, energy and momentum. Wigner measures have been used to create effective models for wave propagation in random media, quantum molecular dynamics, mean field limits, and the propagation of electrons in graphene. In nonlinear settings, the Vlasov-type equations obtained for the Wigner measure are often ill-posed on the physically interesting spaces of initial data. In this paper we are able to select the measure-valued solution of the 1+1 dimensional Vlasov-Poisson equation which correctly captures the semiclassical limit, thus finally resolving the non-uniqueness in the seminal result of [Zhang, Zheng & Mauser, *Comm. Pure Appl. Math.* (2002) 55, doi:10.1002/cpa.3017]. The same approach is also applied to the Vlasov-Dirac-Benney equation with small wavepacket initial data, extending several known results.

MSC subject classification: 81S30; 35Q55; 81Q20; 81R30

Keywords: nonlinear Schrödinger equation, semiclassical asymptotics, wavepackets, Wigner measure

Contents

1	Introduction	1
2	Statement of the main results	6
3	Wigner measures and the new functional framework	10
4	Background results	12
5	Proof of the main results	17

1 Introduction

1.1 The problem

A well known asymptotic problem for nonlinear Schrödinger equations

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon - F(|\psi^\varepsilon|^2)\psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon \in H^1(\mathbb{R}^n) \quad (1)$$

*a.athanassoulis@dundee.ac.uk, Department of Mathematics, University of Dundee, UK

is to describe the evolution of *macroscopic observables*, such as

$$\begin{aligned}
\text{mass} & \quad m(x, t) = |\psi^\varepsilon(x, t)|^2, \\
\text{momentum} & \quad j(k, t) = \varepsilon^n |\widehat{\psi}^\varepsilon(\varepsilon k, t)|^2, \\
\text{kinetic energy} & \quad E_{kin}(x, t) = |\nabla \psi^\varepsilon(x, t)|^2
\end{aligned} \tag{2}$$

when $\varepsilon \rightarrow 0$. Variations of this problem arise in many different physical contexts, including quantum molecular dynamics [3], mean field limits [13, 33, 34], wave propagation over large (geophysical) distances [48, 50], the formation of rogue waves [26] and the study of graphene [27, 28]. We will use the term semiclassical to describe this regime [17, 36, 37]; other terms used in the literature are zero-dispersion limit [52], high frequency limit [31], and geometric optics [20, 21].

While numerical solution of (1) becomes increasingly expensive as $\varepsilon \rightarrow 0$, it often turns out that we can recover approximations to the observables with $O(1)$ cost, i.e. with complexity independent of ε . This can be achieved by taking a quadratic transform of (1), namely the Wigner transform (WT)

$$W^\varepsilon(x, k, t) = W^\varepsilon[\psi(t)](x, k) = \int_y e^{-2\pi i k \cdot y} \psi(x + \frac{\varepsilon y}{2}, t) \overline{\psi}(x - \frac{\varepsilon y}{2}, t) dy, \tag{3}$$

leading to the nonlinear Wigner equation

$$\begin{aligned}
\partial_t W^\varepsilon + 2\pi k \cdot \nabla_x + i \mathcal{F}_{y \rightarrow k}^{-1} \left[\frac{V(x + \frac{\varepsilon y}{2}, t) - V(x - \frac{\varepsilon y}{2}, t)}{\varepsilon} \mathcal{F}_{k' \rightarrow y} [W^\varepsilon(x, k', t)] \right] &= 0, \\
V(x, t) &= F \left(\int_\xi W^\varepsilon(x, \xi, t) \right).
\end{aligned} \tag{4}$$

This is essentially a second moment of (1), and it has two important properties. First of all, equation (4) has a meaningful (formal, for now) limit as $\varepsilon \rightarrow 0$, namely the Vlasov-type equation

$$\partial_t W^0 + 2\pi k \cdot \nabla_x W^0 - \frac{1}{2\pi} \nabla_x V \cdot \nabla_k W^0 = 0, \quad V(x, t) = F \left(\int_\xi W^0(x, \xi, t) \right). \tag{5}$$

Moreover, the *Wigner measure*, i.e. the limit of the Wigner transform

$$W^0 = \lim_{\varepsilon \rightarrow 0} W^\varepsilon \tag{6}$$

controls macroscopic observables in weak sense [44, 31], e.g.

$$\begin{aligned}
\text{mass} & \quad m(x, t) = \int_k W^0(x, k, t) dk + o(1), \\
\text{momentum} & \quad j(k, t) = \int_x W^0(x, k, t) dx + o(1), \\
\text{kinetic energy} & \quad E_{kin}(x, t) = 4\pi^2 \int_k |k|^2 W^0(x, k, t) dk + o(1),
\end{aligned} \tag{7}$$

as $\varepsilon \rightarrow 0$. A self-contained discussion of Wigner measures, including the sense of convergence and the systematic extraction of observables, can be found in Section 3.

This technique has been established for a wide variety of wave problems, including Schrödinger [3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 31, 33, 34, 44, 47, 54], Dirac [27, 28], and acoustic [10, 46], elastic and Maxwell equations with smooth, random or periodic coefficients [14, 31, 48].

A key trade-off between this approach and WKB-type expansions [20, 21, 35, 36, 37, 40, 52] is that we no longer try to approximate ψ^ε , but only the observables, through the Wigner measure. In return, we get an elegant and widely applicable model, including in many cases the painless resolution of caustics. This can be seen as a semiclassical regularization and continuation of the WKB system past the formation of caustics, by the introduction of a novel sense of solution [39]. Moreover, approximations of $\psi^\varepsilon(t)$ are often destroyed by nonlinear effects at a much faster rate than macroscopic approximation for $W^\varepsilon[\psi^\varepsilon(t)]$; this can be seen very clearly in the discussion after Theorem 2.3.

Another important advantage of the Wigner measures approach is that it does not require the wavefunction to be approximated by some parametric ansatz such as WKB or coherent states. This non-parametric character of Wigner measures is crucial for noisy problems where the data of interest are known to not be of WKB or other explicit parametric forms, not even initially [48, 50, 51]. In fact, the second-moment character of the Wigner transform makes it a particularly powerful tool for stochastic problems, and it has played a key role in the recent understanding of self-averaging in wave propagation in random media [10, 47]. In the same context, the Wigner transform seems to be the appropriate generalization of the spectral density for harmonizable (non-stationary) processes [45].

Infinite systems of Schrödinger equations can be treated with Wigner measures using the same formalism. This aspect is crucial in certain fields such as statistical physics; a far-from-exhaustive list of references is [8, 13, 33, 34] and the references therein. It must be also noted that infinite systems of nonlinear Schrödinger equations (often referred to as “mixed states”) are attracting intense attention recently [25, 42], following recent fundamental advances in harmonic analysis [30]. In fact, in the context of Wigner measures, mixed states lead to simpler problems as they lead to initial data W_0^0 in Sobolev spaces, or even in spaces of analytic functions. This is elaborated e.g. in [16, 44]. In this work we will focus on pure states only, i.e. we will always start from a single nonlinear Schrödinger equation (1).

While for many classes of problems the Wigner measures approach is worked out, key questions are still open in many interesting problems, such as systems with eigenvalue crossings [27, 28], nonsmooth [3, 4, 5, 6], and nonlinear problems. In nonlinear problems in particular, the limit Vlasov-type equation (5) is typically not well-posed for measures. For example, in the seminal work by Zhang, Zheng & Mauser [54], it is shown that if we start with the 1-dimensional Schrödinger-Poisson equation,

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon - \frac{b}{2}\int_y|x-y||\psi^\varepsilon(y,t)|^2dy\psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon \in H^1(\mathbb{R}^n) \quad (8)$$

its Wigner measure $W^0 = \lim_{\varepsilon \rightarrow 0} \overline{W^\varepsilon}[\psi^\varepsilon]$ satisfies (in an appropriate weak sense [53]) the 1 + 1-dimensional Vlasov-Poisson equation with initial data $W_0^0 = \lim_{\varepsilon \rightarrow 0} W^\varepsilon[\psi_0^\varepsilon]$. However, the notion of solution used for the Vlasov-Poisson equation is so weak that uniqueness is lost. The question of determining the correct weak solution for the semiclassical limit has been the subject of numerical investigation [38], but it is still not settled. Theorem 2.1 answers this question for any wavepacket initial data.

More recently, Bardos & Besse in the breakthrough paper [12] showed that, under appropriate conditions, in the case of the defocusing cubic nonlinearity

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon - b|\psi^\varepsilon|^2\psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon \in H^1(\mathbb{R}^n) \quad (9)$$

the Wigner measure indeed satisfies the resulting Vlasov-Dirac-Benney equation

$$\partial_t W^0 + 2\pi k \cdot \nabla_x + \frac{b}{2\pi} \nabla_x V \cdot \nabla_k W^0 = 0, \quad V(x, t) = \int_\xi W^0(x, \xi, t). \quad (10)$$

However this equation is known to be ill-posed on any Sobolev space [11], and at the moment there is no sense of measure-valued solutions.

Thus, the picture that emerges for nonlinear problems can be described as follows: in many cases a Vlasov equation can be derived and justified, i.e. it can be shown that the Wigner measure does satisfy it. However this is only the first step towards approximating the evolution of the Wigner measure in time, as the Vlasov equation may be ill-posed. Indeed, as we saw, neither uniqueness nor stability can be taken for granted. In this paper we construct an approximation to the Wigner measure for wavepackets evolving under some common nonlinearities for long times, thus extracting the correct weak solution for the semiclassical limit. By *wavepacket* we mean any function which is well localized both on space and Fourier space.

More specifically: our main results, stated in detail in Section 2, can be described as *the formulation of linearizability conditions for the Wigner measure*, i.e. conditions under which the Wigner measure completely ignores the nonlinearity. This is significant because it includes problems where the wavefunction ψ^ε is known to exhibit $O(1)$ nonlinear effects. Thus a regime can be quantified where the wavefunction is not linearizable, but the Wigner measure is. This highlights a subtle “stability of macroscopic observables” that is *not derived*

from the stability of the nonlinear Schrödinger equation on the level of the wavefunction, and can be accessed only by working directly at the level of the Wigner transform.

On a more technical level, the proofs are based on two key features:

- (i) a well-chosen frame of reference;

this allows us to fully exploit the symmetries of the Schrödinger equation in conjunction with the the space-and-Fourier-localization of wavepackets without being bound to a parametric ansatz. This non-parametric approach allows a very large class of initial data, stronger nonlinearities and longer timescales than the state of the art (which is recalled in detail in Section 2). Moreover,

- (ii) we introduce a novel functional framework which enables us to have quantitative estimates for the Wigner measure in a nonlinear context.

The basic notions and standard framework for the Wigner measure were introduced in [31, 44], and are briefly recalled in Section 3. However, the standard functional framework doesn't seem to be always ideal for work in nonlinear problems – even less so for power nonlinearities. This is reflected in the lack of a stand-alone stability theory for the nonlinear Wigner equation (4) for power nonlinearities, as well as in the recent papers introducing different functional frameworks for the Wigner measure adapted to specific problems [33, 34]. The \mathbf{A}^{-1} framework we introduce, discussed in some detail in Sections 1.2 and 3, has the advantages that it is directly comparable to the standard \mathcal{A}' setting; it is very simple to work with; and it demonstrably works very well for wavepackets. More broadly, a well-posedness theory for the nonlinear Wigner equation in \mathbf{A}^{-1} seems possible in the future, and could impact other nonlinear problems beyond wavepackets as well.

Structure of the paper: The main results are stated in Section 2. Comparisons with existing results are also given, including evidence for strong nonlinear behaviour of the wavefunction. The proofs of the main results can be found in Section 5, while auxiliary results are stated and proved in Sections 3 and 4.

1.2 Notations and Definitions

We will use standard multi-index notations. The Fourier transform normalization will be

$$\hat{f}(k) = \int_{x \in \mathbb{R}^n} e^{-2\pi i k \cdot x} f(x) dx.$$

Because of the particular manipulations necessary in this work, we will keep track of variable names under Fourier transforms with the notation

$$\begin{aligned} \hat{f}(k) &= \mathcal{F}_{x \rightarrow k}[f] = \int_{x \in \mathbb{R}^n} e^{-2\pi i k \cdot x} f(x) dx, \\ \hat{f}(X, K) &= \mathcal{F}_{x, k \rightarrow X, K}[f] = \int_{x, k \in \mathbb{R}^n} e^{-2\pi i [x \cdot X + k \cdot K]} f(x, k) dx dk, \\ \mathcal{F}_{k \rightarrow K}[f] &= \int_{k \in \mathbb{R}^n} e^{-2\pi i k \cdot K} f(x, k) dk. \end{aligned}$$

The convention $\hat{X} := \{f \mid \hat{f} \in X\}$ will be used for brevity.

We will also use the Wiener-Sobolev spaces \mathbf{A}^s :

Definition 1.1 (The Wiener-Sobolev spaces \mathbf{A}^s). *For $s \geq 0$, we will denote with $\mathbf{A}^s(\mathbb{R}^n)$ the Banach space of functions generated by the norm*

$$\|\phi\|_{\mathbf{A}^s(\mathbb{R}^n)} := \int_{y \in \mathbb{R}^n} (1 + |y|^s) |\hat{\phi}(y)| dy.$$

In phase-space this becomes

$$\|\phi\|_{\mathbf{A}^s(\mathbb{R}^{2n})} := \int_{X,K \in \mathbb{R}^n} \left(1 + \sqrt{|X|^2 + |K|^2}\right)^s |\widehat{\phi}(X, K)| dy.$$

When $s > 0$, we will denote the dual of \mathbf{A}^s by \mathbf{A}^{-s} , i.e.

$$\|\phi\|_{\mathbf{A}^{-s}} = \sup_{\|\psi\|_{\mathbf{A}^s}=1} |\langle \phi, \psi \rangle|.$$

Remark 1.2. When $s = 0$ we recover the standard Wiener algebra, $\|\phi\|_{\mathbf{A}^0} = \|\widehat{\phi}\|_{L^1}$. Its dual space will be denoted as

$$(\mathbf{A}^0)' = \widehat{L}^\infty = \{f : \|\widehat{f}\|_{L^\infty} < \infty\}.$$

The Wiener algebra has been used for semiclassical analysis in other nonlinear contexts, for example [19, 41]. Here we will use heavily A^{-1} on phase-space as a convenient setting for quantitative approximation of Wigner measures. The mechanics behind this choice can be seen in the proof of Lemma 5.1.

Definition 1.3. We will denote by \mathcal{T}_z the translation operator

$$\mathcal{T}_z f(x) = f(x + z), \tag{11}$$

and by \mathcal{M}_z the modulation operator

$$\mathcal{M}_z f(x) = e^{-2\pi i z \cdot x} f(x). \tag{12}$$

Definition 1.4. Let $\psi \in H^1 \cap \widehat{H}^1$ be a wavefunction with unit mass, i.e. $\|\psi\|_{L^2} = 1$. We will denote

$$\mu_x(\psi) := \int_x x |\psi|^2 dx, \quad \mu_k(\psi) := \varepsilon \int_k k |\widehat{\psi}|^2 dk, \tag{13}$$

and read μ_x as the mean position and μ_k as the mean (rescaled) momentum of the wavefunction ψ . Moreover, we will denote

$$\sigma_x^2(\psi) := \int_x (x - \mu_x(\psi))^2 |\psi|^2 dx, \quad \sigma_k^2(\psi) := \varepsilon^2 \int_k \left(k - \frac{\mu_k(\psi)}{\varepsilon}\right)^2 |\widehat{\psi}|^2 dk, \tag{14}$$

and read σ_x^2 as the variance in position and σ_k^2 as the variance in (rescaled) momentum of the wavefunction ψ .

The variances $\sigma_x^2(\psi)$, $\sigma_k^2(\psi)$ are the only measures of space and Fourier localization that we use to develop our non-parametric wavepacket analysis, as captured in the following

Definition 1.5 (Semiclassical family of wavepackets). Let $\{\psi^\varepsilon\}_{\varepsilon \in (0,1)} \subseteq H^1 \cap \widehat{H}^1$, $\|\psi^\varepsilon\|_{L^2} = 1 \quad \forall \varepsilon$. Then if

$$\lim_{\varepsilon \rightarrow 0} \mu_x(\psi^\varepsilon) = x_0 \in \mathbb{R}^n, \quad \lim_{\varepsilon \rightarrow 0} \mu_k(\psi^\varepsilon) = k_0 \in \mathbb{R}^n,$$

and

$$\lim_{\varepsilon \rightarrow 0} \sigma_x(\psi^\varepsilon) = 0, \quad \lim_{\varepsilon \rightarrow 0} \sigma_k(\psi^\varepsilon) = 0,$$

we will say that $\{\psi^\varepsilon\}$ is a semiclassical family of wavepackets with mean position x_0 and mean (rescaled) momentum k_0 .

However we can use the term wavepacket more broadly¹, to mean any function $\psi^\varepsilon \in H^1 \cap \widehat{H}^1$ with $\|\psi^\varepsilon\|_{L^2} = 1$ and

$$\sigma_x(\psi^\varepsilon) + \sigma_k(\psi^\varepsilon) = o(1). \tag{15}$$

¹Often one has in mind a concrete problem, where ε is a parameter with a fixed value – “small” with regard to other meaningful quantities, but not tending to 0. Accordingly, there is a single initial datum ψ_0^ε . For example, $\varepsilon \approx \frac{1}{2000}$ is mentioned as a reference value of ε in molecular dynamics in [2]. Our main results can be applied in such a context as well, and will give a bound on the nonlinear effects on the Wigner measure. The judgment of whether the fixed number $\sigma_x(\psi^\varepsilon) + \sigma_k(\psi^\varepsilon)$ is “small” (and thus whether ψ_0^ε can be considered a wavepacket) will have to be made with regard to other parameters of the physical problem.

Equation (15) holds for all standard classes of parametric wavepackets, such as coherent states and squeezed states, as well as less common parametric classes like chirp wavepackets, i.e. localized functions with quadratic oscillations, cf. Lemmata 4.10 and 4.12. In any case, this fully non-parametric notion of wavepacket based on equation (15) is really quantified by Corollary 5.2, where it is shown that

$$\|W^\varepsilon[\psi^\varepsilon](x, k) - \delta(x - \mu_x(\psi^\varepsilon), k - \mu_k(\psi^\varepsilon))\|_{\mathbf{A}^{-1}} \leq 2\pi \left(\sigma_x(\psi^\varepsilon) + \sigma_k(\psi^\varepsilon) \right).$$

The Banach space \mathbf{A}^{-1} , specified in Definition 1.1, contains δ -functions.

It must be noted that, when working on the appropriate frame of reference, the variances $\sigma_x^2(\psi)$, $\sigma_k^2(\psi)$ take a very simple form:

Observation 1.6. *If a wavefunction ψ is centered via a Galilean transform, i.e. if*

$$u = \mathcal{M}_{\frac{\mu_k(\psi)}{\varepsilon}} \mathcal{T}_{\mu_x(\psi)} \psi, \quad (16)$$

then one readily computes

$$\mu_x(u) = \mu_k(u) = 0, \quad \sigma_x(\psi) = \sigma_x(u) = \|xu\|_{L_x^2}, \quad \sigma_k(\psi) = \sigma_k(u) = \frac{1}{2\pi} \|\varepsilon \nabla u\|_{L^2}. \quad (17)$$

The uncertainty principle [29] means that we cannot make both of $\sigma_x(\psi)$, $\sigma_k(\psi)$ arbitrarily small at the same time, e.g.

$$\sigma_x(\psi) \sigma_k(\psi) \geq \frac{\varepsilon \|\psi\|_{L^2(\mathbb{R})}^2}{4\pi}. \quad (18)$$

While only gaussian coherent states saturate the uncertainty principle, equation (15) outlines a much broader class. Squeezed states, a class of wavepackets generalizing coherent states, are properly introduced in Definition 4.9. Chirp wavepackets are introduced in Definition 4.11.

2 Statement of the main results

2.1 Wigner measures for wavepackets

Theorem 2.1 (1-dimensional defocusing Schrödinger-Poisson equation). *Let $\psi^\varepsilon(t)$ be the solution of*

$$i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon - \frac{b}{2} \int_y |x - y| |\psi^\varepsilon(y, t)|^2 dy \psi^\varepsilon = 0, \quad \psi^\varepsilon(t = 0) = \psi_0^\varepsilon \in \mathcal{S}(\mathbb{R}), \quad \|\psi_0^\varepsilon\|_{L^2} = 1 \quad (19)$$

for some $b > 0$. Then

$$\|W^\varepsilon[\psi^\varepsilon(t)] - \delta(x - \mu_x(\psi_0^\varepsilon) - 2\pi t \mu_k(\psi_0^\varepsilon), k - \mu_k(\psi_0^\varepsilon))\|_{\mathbf{A}^{-1}} \leq 2\pi(1 + t) \left[\sigma_k(\psi_0^\varepsilon) + \sqrt{\frac{b}{2\pi} \sigma_x(\psi_0^\varepsilon)} \right] + 2\pi \sigma_x(\psi_0^\varepsilon).$$

The proof is given in Section 5.1.

Thus the Wigner transform for any wavepacket, i.e. any initial data ψ_0^ε so that $\sigma_x(\psi_0^\varepsilon) + \sigma_k(\psi_0^\varepsilon) = o(1)$, remains close to a δ -function. Moreover, despite the fact that the nonlinear effects on ψ^ε are of $O(1)$, the Wigner measure is not affected by the nonlinearity. In that sense we can say that the Wigner measure satisfies the Vlasov-Poisson equation

$$\partial_t W^0 + 2\pi k \cdot \nabla_x W^0 + \frac{b}{4\pi} \nabla_x \int_{y, \xi} |x - y| W^0(y, \xi, t) dy d\xi \cdot \nabla_k W^0 = 0, \quad W_0^0 = \delta(x - x_0, k - k_0), \quad (20)$$

if the nonlinear term is completely dropped, which is precisely what happens if we interpret it naively².

²Indeed, if $W^0(x, k) = \delta(x_0, k_0)$, then

$$\begin{aligned} \nabla_x \int_{y, \xi} |x - y| W^0(y, \xi, t) dy d\xi \cdot \nabla_k W^0 &= \nabla_x \int_{y, \xi} |x - y| \delta(y - x_0, \xi - k_0) dy d\xi \cdot \nabla_k \delta(x - x_0, k - k_0) = \\ &= \nabla_x \int_y |x - y| \delta(y - x_0) dy \cdot \nabla_k \delta(x - x_0, k - k_0) = \text{sign}(x - x_0) \nabla_k \delta(x - x_0, k - k_0). \end{aligned}$$

Now observe that $\text{sign}(x - x_0)$ evaluated on x_0 is 0; moreover $\nabla_k \delta(x - x_0, k - k_0)$ evaluated on any (x, k) with $x \neq x_0$ is 0.

Moreover, Theorem 2.1 remains valid for a timescale much longer than the usual $\log \frac{1}{\varepsilon}$ Ehrenfest time-scale [15, 23]. This can be made precise for squeezed states initial data in terms of the following

Corollary 2.2 (Squeezed states for the 1-dimensional Schrödinger-Poisson equation). *Let*

$$\psi_0^\varepsilon = \varepsilon^{-\frac{n\beta}{2}} a\left(\frac{x-x_0}{\varepsilon^\beta}\right) e^{\frac{2\pi i k_0 \cdot (x-x_0)}{\varepsilon}}, \quad 0 < \beta < 1,$$

be a squeezed state as in Definition 4.9, and let

$$i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon - \frac{b}{2} \int_y |x-y| |\psi^\varepsilon(y,t)|^2 dy \psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon. \quad (21)$$

Then there exists a constant C independent of ε, t so that

$$\|W^\varepsilon[\psi^\varepsilon(t)] - \delta(x-x_0 - 2\pi k_0 t, k-k_0)\|_{\mathcal{A}^{-1}} < (1+t)C \left(\varepsilon^{\frac{\beta}{2}} + \varepsilon^{\frac{1-\beta}{2}} \right).$$

The same approach can be applied to power nonlinearities as well:

Theorem 2.3 (Defocusing power nonlinearities). *Let $\psi^\varepsilon(t)$ be the solution of*

$$i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon - b(\varepsilon) |\psi^\varepsilon|^{2\sigma} \psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon \in \mathcal{S}(\mathbb{R}^n), \quad \|\psi_0^\varepsilon\|_{L^2} = 1 \quad (22)$$

for some $b = b(\varepsilon) > 0$. Moreover, let C_*^{GN} be the sharp constant of the Gagliardo-Nirenberg inequality, see Lemma 4.8 for details. If for some $\eta > 0$

$$\sigma_x(\psi_0^\varepsilon) < \eta, \quad \sigma_k(\psi_0^\varepsilon) < \eta, \quad \sigma_k^{\frac{n\sigma}{2}}(\psi_0^\varepsilon) \sqrt{\frac{b(\varepsilon) C_*^{GN} (2\pi)^{n\sigma-2}}{\varepsilon^{n\sigma} (2\sigma+2)}} < \eta,$$

then

$$\|W^\varepsilon[\psi^\varepsilon(t)] - \delta(x - \mu_x(\psi_0^\varepsilon) - 2\pi t \mu_k(\psi_0^\varepsilon), k - \mu_k(\psi_0^\varepsilon))\|_{\mathcal{A}^{-1}} < 2\pi(3+2t)\eta.$$

The proof is given in Section 5.2.

Allowing $b(\varepsilon) = B\varepsilon^\gamma = o(1)$ and $O(1)$ initial data, is equivalent to considering small initial data and $b = B = O(1)$, through the rescaling

$$\begin{aligned} i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon - \varepsilon^\gamma B |\psi^\varepsilon|^{2\sigma} \psi^\varepsilon = 0 \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon &\Leftrightarrow \\ \Leftrightarrow i\varepsilon \partial_t \Psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \Psi^\varepsilon - B |\Psi^\varepsilon|^{2\sigma} \Psi^\varepsilon = 0 \quad \Psi^\varepsilon(t=0) = \varepsilon^{\frac{\gamma}{2\sigma}} \psi_0^\varepsilon. \end{aligned}$$

Here we keep the normalization $\|\psi_0^\varepsilon\|_{L^2} = 1$ so that $W^\varepsilon[\psi^\varepsilon]$ scales correctly (i.e. so that the Wigner measure exists and is not zero).

Note that even for these weakly-nonlinear problems, nonlinear effects are known to appear on the level of the wavefunction [17, 20, 21] and the semiclassical limit for wavepackets was heretofore not known. For example, in [17] a model of Bose-Einstein condensates is studied, namely equation (22) with

$$n = 3, \quad \sigma = 1, \quad b(\varepsilon) = \varepsilon^2 > 0. \quad (23)$$

It is shown therein that instabilities on the level of the wavefunction are possible for special localized initial data. This negative result builds upon initial data of the form $\psi_0^\varepsilon = \varepsilon^{-\frac{n}{2}} a\left(\frac{x-x_0}{\varepsilon}\right)$, which are localized in space but not in the Fourier variable. In another problem with $b(\varepsilon) = \varepsilon^2, \sigma = \frac{2}{n}$, it has even been shown that the Wigner measure can be discontinuous in time [18].

It is natural to ask if for some particularly convenient initial data, like coherent states, the semiclassical limit for (23) is known. For coherent states, the state of the art is [23]. The main result of [23] can be summarized as follows: assume

$$|b(\varepsilon)| = O(\varepsilon^{1+\frac{n\sigma}{2}}), \quad (24)$$

and the initial wavefunction ψ_0^ε is a coherent state

$$\psi_0^\varepsilon(x) = \varepsilon^{-\frac{n}{4}} a\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right) e^{\frac{2\pi i k_0(x-x_0)}{\varepsilon}}, \quad a \in \mathcal{S}(\mathbb{R}^n), \quad \|a\|_{L^2} = 1, \quad x_0, k_0 \in \mathbb{R}^n. \quad (25)$$

Then this parametric form is preserved, in the sense that there exists a coherent-state approximate solution of (22),

$$\|\psi^\varepsilon(x, t) - \varepsilon^{-\frac{n}{4}} a\left(\frac{x-X(t)}{\sqrt{\varepsilon}}, t\right) e^{\frac{2\pi i K(t)(x-X(t))}{\varepsilon} + i\theta(t)}\|_{L^2} = o(1), \quad (26)$$

where $X(t)$, $K(t)$, $\theta(t)$, $a(x, t)$, satisfy simple ε -independent equations. Moreover, this is valid for timescales

$$t = O(\log \log \frac{1}{\varepsilon}).$$

A corollary of [23] is that for $|b(\varepsilon)| \geq \varepsilon^{1+\frac{n\sigma}{2}}$ nonlinear effects on $\psi^\varepsilon(t)$ are of $O(1)$.

Equation (26) provides a lot of information for the problem, but at the cost of a rather weak nonlinearity, i.e. assumption (24), excluding many physically relevant problems. In particular, the nonlinearity (23) is too strong for the result of [23]. Moreover, in most realistic settings the values of ε range between 10^{-2} and 10^{-6} , so this would lead to short timescales as well since, for the natural logarithm, $\log \log 10^6 \approx 2.6$.

In other words, it was not known heretofore whether we can have any control of the observables in the problem described by the scaling (23) for wavepacket initial data; not even for coherent state initial data. To answer this question one observes that Theorem 2.3 implies the following

Corollary 2.4 (Squeezed states for defocusing power nonlinearities). *Let*

$$\psi_0^\varepsilon = \varepsilon^{-\frac{n\beta}{2}} a\left(\frac{x-x_0}{\varepsilon^\beta}\right) e^{\frac{2\pi i k_0(x-x_0)}{\varepsilon}}, \quad 0 < \beta < 1,$$

be a squeezed state as in Definition 4.9, and let

$$i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon - \varepsilon^\gamma |\psi^\varepsilon|^{2\sigma} \psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon. \quad (27)$$

Then there exists a constant C independent of ε , t so that

$$\|W^\varepsilon[\psi^\varepsilon(t)] - \delta(x-x_0-2\pi k_0 t, k-k_0)\|_{\mathbf{A}^{-1}} < (1+t)C\left(\varepsilon^\beta + \varepsilon^{1-\beta} + \varepsilon^{\frac{\gamma-n\sigma\beta}{2}}\right).$$

Setting $\gamma = 2$, $n = 3$ in Corollary 2.4 above means we recover the setting of (23). Then, if ψ_0^ε is a squeezed state with $\beta < \frac{2}{3}$ it follows that $W^\varepsilon[\psi^\varepsilon(t)]$ evolves linearly as long as $t \cdot (\varepsilon^\beta + \varepsilon^{1-\frac{3\beta}{2}}) = o(1)$.

Remark 2.5. *If we view Corollary 2.4 as a linearizability result, it is interesting to compare with what is known for linearizability on the level of the wavefunction for the defocusing NLS. Theorem 1.1 in [22] requires $\gamma \geq n\sigma$ for linearizability (in an appropriate sense) to hold on the level of the wavefunction. Here, taking advantage of the wavepacket character of our initial data, we can relax the condition to $\gamma > n\sigma\beta$, for $\beta \in (0, 1)$.*

We can apply this approach to focusing power nonlinearities as well:

Theorem 2.6 (Focusing power nonlinearities). *Let $\psi^\varepsilon(t)$ be the solution of*

$$i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon - b(\varepsilon) |\psi^\varepsilon|^{2\sigma} \psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon \in \mathcal{S}(\mathbb{R}^n), \quad \|\psi_0^\varepsilon\|_{L^2} = 1 \quad (28)$$

for some $b = b(\varepsilon) < 0$, and for $n\sigma = 1$. If for some $\eta > 0$

$$\sigma_x(\psi_0^\varepsilon) < \eta, \quad \sigma_k(\psi_0^\varepsilon) < \eta, \quad \frac{|b(\varepsilon)|}{\varepsilon} \frac{C_*^{GN}}{2\pi(4+\frac{4}{n})} + \sqrt{\frac{|b(\varepsilon)|}{\varepsilon} \frac{C_*^{GN}}{2\pi(4+\frac{4}{n})}} \sqrt{\frac{|b(\varepsilon)|}{\varepsilon} \frac{C_*^{GN}}{2\pi(4+\frac{4}{n})} + \frac{\sigma_k(\psi_0^\varepsilon)}{2\pi}} < \eta,$$

then

$$\|W^\varepsilon[\psi^\varepsilon(t)] - \delta(x - \mu_x(\psi_0^\varepsilon) - 2\pi t \mu_k(\psi_0^\varepsilon), k - \mu_k(\psi_0^\varepsilon))\|_{\mathbf{A}^{-1}} < 2\pi(3+2t)\eta.$$

The proof is given in Section 5.3.

Remark 2.7. *The restriction $n\sigma = 1$ has to do with working out explicitly the upper bound in the technical Lemma 4.7. It can be removed at the cost of a less explicit statement. Indeed, if instead of $n\sigma = 1$ we assume*

$$n\sigma < 2, \quad |b(\varepsilon)| = O(\varepsilon^\gamma), \quad \gamma > n\sigma,$$

then it can be shown that

$$\|W^\varepsilon[\psi^\varepsilon(t)] - \delta(x - \mu_x(\psi_0^\varepsilon) - 2\pi t\mu_k(\psi_0^\varepsilon), k - \mu_k(\psi_0^\varepsilon))\|_{\mathbf{A}^{-1}} = O\left(\sigma_x(\psi_0^\varepsilon) + \sigma_k(\psi_0^\varepsilon) + \varepsilon^{\frac{\gamma-n\sigma}{2-n\sigma}}\right) (1+t). \quad (29)$$

The proof of this version of the result is given in Section 5.4.

Now let us compare Theorem 2.6 to the state of the art. The aforementioned result of [23] applies in the same way to focusing and defocusing problems. Theorem 2.6 allows for stronger focusing nonlinearities, longer timescales, and of course more general initial data. This can be seen clearly in the following

Corollary 2.8 (Squeezed states for focusing nonlinearities). *Let*

$$\psi_0^\varepsilon = \varepsilon^{-\frac{\beta}{2}} a\left(\frac{x-x_0}{\varepsilon^\beta}\right) e^{\frac{2\pi i k_0 \cdot (x-x_0)}{\varepsilon}}, \quad 0 < \beta < 1,$$

be a squeezed state as in Definition 4.9, $n\sigma = 1$, and

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon + \varepsilon^\gamma|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon. \quad (30)$$

Then there exists a constant C independent of ε , t so that

$$\|W^\varepsilon[\psi^\varepsilon(t)] - \delta(x - x_0 - 2\pi k_0 t, k - k_0)\|_{\mathbf{A}^{-1}} < (1+t)C\left(\varepsilon^\beta + e^{1-\beta} + \varepsilon^{\gamma-1} + \varepsilon^{\frac{\gamma-\beta}{2}}\right)$$

Thus, for any $\gamma > 1$ control of the Wigner measure is obtained, as opposed to $\gamma \geq \frac{3}{2}$ in [23].

Remark 2.9. *The counterparts of Corollaries 2.2, 2.4, 2.8 for chirp-wavepacket initial data,*

$$\psi_0^\varepsilon(x) = \varepsilon^{-\frac{n\beta}{2}} a\left(\frac{x-x_0}{\varepsilon^\beta}\right) e^{\frac{2\pi i k_0 \cdot (x-x_0)}{\varepsilon}} e^{\frac{iz \cdot (x-x_0)^2}{\varepsilon}}$$

can be readily established by making use of Lemma 4.12.

2.2 Idea of the proofs

The idea behind the proofs for all of the main results follows the same general steps, bringing together several different ideas, and adjusting the details as needed for each nonlinearity:

Step 1: Go to the appropriate frame of reference. The nonlinearities we work with are Galilean invariant. In that context, we use a frame of reference that centers the initial data

$$u_0^\varepsilon(x) = \mathcal{M}_{\frac{\mu_k(\psi_0^\varepsilon)}{\varepsilon}} \mathcal{T}_{\mu_x(\psi_0^\varepsilon)} \psi_0^\varepsilon = \psi_0^\varepsilon(x + x_0) e^{-2\pi i \frac{\mu_k(\psi_0^\varepsilon) \cdot x}{\varepsilon}}, \quad (31)$$

and work on problem (1) through

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon - F(|u^\varepsilon|^2)u^\varepsilon = 0, \quad u^\varepsilon(t=0) = u_0^\varepsilon. \quad (32)$$

The Galilean invariance of (1) (recalled in Lemmata 4.4, 4.4) means that $\psi^\varepsilon(x, t)$ is related to $u^\varepsilon(x, t)$ through

$$\psi^\varepsilon(x, t) = u^\varepsilon(x - vt - x_0, t) e^{i\left(\frac{v \cdot (x-x_0)}{\varepsilon} - \frac{v \cdot v}{2\varepsilon}\right)}, \quad v = 2\pi\mu_k(\psi_0^\varepsilon), \quad x_0 = \mu_x(\psi_0^\varepsilon).$$

Step 2: Show that if $\sigma_x(\psi_0^\varepsilon)$, $\sigma_k(\psi_0^\varepsilon)$ are small, then $\sigma_x(u^\varepsilon(t))$, $\sigma_k(u^\varepsilon(t))$ are also small. By state of the art methods for nonlinear Schrödinger equations, one can obtain bounds for $\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2}$ in terms of $\|\varepsilon\nabla u_0^\varepsilon\|_{L^2}$.

Then we proceed to bound $\|xu^\varepsilon(t)\|_{L^2}$ by appropriate functions of $\|xu_0^\varepsilon\|_{L^2}$, $\|\varepsilon\nabla u_0^\varepsilon\|_{L^2}$. From this we conclude that $\sigma_x(u^\varepsilon(t))$, $\sigma_k(u^\varepsilon(t))$ are bounded by appropriate functions of $\sigma_x(u_0^\varepsilon) = \sigma_x(\psi_0^\varepsilon)$, $\sigma_k(u_0^\varepsilon) = \sigma_k(\psi_0^\varepsilon)$.

Working out the details in each case determines the constants and, crucially, the timescales for which this bound is useful.

Step 3: Conclude that $W^\varepsilon[u^\varepsilon(t)] \approx \delta(x, k)$, quantify the rate and timescale of convergence, and go back to the initial frame of reference to obtain the result for $W^\varepsilon[\psi^\varepsilon(t)]$. The previous step is exploited through Corollary 5.2 to complete the proof.

Every effort has been made to state and prove regularity results, bootstrap arguments etc in a self-contained way in Sections 3 and 4. That way Section 5 is devoted to presenting coherently how the different pieces fit together, without being sidetracked by various technical details. The engine behind the proofs is Lemma 5.1 and its Corollary 5.2, which translate H^1 and \widehat{H}^1 estimates to convergence results for the Wigner measure. It is through Lemma 5.1 that the new functional framework, introduced in detail in Section 3 below, makes the results of this paper possible.

3 Wigner measures and the new functional framework

The Wigner transform (WT) can be seen as a sesquilinear transform

$$W^\varepsilon : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^{2n}) : f, g \mapsto W^\varepsilon[f, g],$$

defined as

$$W^\varepsilon[f, g](x, k) = \int_{y \in \mathbb{R}^n} e^{-2\pi i k y} f\left(x + \frac{\varepsilon y}{2}\right) \bar{g}\left(x - \frac{\varepsilon y}{2}\right) dy. \quad (33)$$

One easily checks the following elementary properties [7, 9]:

$$\begin{aligned} f, g \in L^2(\mathbb{R}^n) &\Rightarrow W^\varepsilon[f, g] \in L^2(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n}), \\ f, g \in H^1(\mathbb{R}^n) \cap \widehat{H}^1(\mathbb{R}^n) &\Rightarrow W^\varepsilon[f, g] \in H^1(\mathbb{R}^{2n}) \cap \widehat{H}^1(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n}), \\ f, g \in \mathcal{S}(\mathbb{R}^n) &\Rightarrow W^\varepsilon[f, g] \in \mathcal{S}(\mathbb{R}^{2n}). \end{aligned} \quad (34)$$

Often the quadratic version is used, in which case we denote

$$W^\varepsilon[f] := W^\varepsilon[f, f].$$

The WT $W^\varepsilon[f]$ describes the quadratic observables of f through

$$\int_{x, k \in \mathbb{R}^n} W^\varepsilon[f](x, k) \phi(x, k) dx dk = \int_{x \in \mathbb{R}^n} \overline{f(x)} \phi(x, \varepsilon \nabla_x) f(x) dx$$

where $\phi(x, \varepsilon \nabla_x)$ is the Weyl pseudodifferential operator with symbol $\phi(x, k)$ ³ [31, 44]. Thus weak approximations of $W^\varepsilon[f]$ can provide information for the quadratic observables of f – but not for its point values.

The most fruitful application of the ε -dependent WT is to an ε -dependent family of functions, $\{\psi^\varepsilon\}_\varepsilon$. Under appropriate conditions, it is known that $W^\varepsilon[\psi^\varepsilon]$ converges in weak-* sense to a probability measure W^0 on \mathbb{R}^{2n} as $\varepsilon \rightarrow 0$ [31, 44]; W^0 is then called the Wigner measure (WM) of the family of functions $\{\psi^\varepsilon\}_\varepsilon$. Intuitively, the WM keeps track of the limits of the observables of ψ^ε as $\varepsilon \rightarrow 0$ through

$$\lim_{\varepsilon \rightarrow 0} \int_{x \in \mathbb{R}^n} \overline{\psi^\varepsilon(x)} \phi(x, \varepsilon \nabla_x) \psi^\varepsilon(x) dx = \int_{x, k \in \mathbb{R}^n} W^0(x, k) \phi(x, k) dx dk$$

while the family $\{\psi^\varepsilon\}_\varepsilon$ itself has no meaningful limit (typically $\lim_{\varepsilon \rightarrow 0} \psi^\varepsilon = 0$ in the sense of distributions).

³We can think of $\phi \in \mathcal{S}(\mathbb{R}^{2n})$ for now; the point of this discussion is to motivate a judicious selection of the space where ϕ will be taken to belong.

The framework developed in [44] for the weak-* convergence of the WT towards the WM is based on the algebra of test functions \mathcal{A} , generated by the norm $\|\phi\|_{\mathcal{A}} := \|\mathcal{F}_{k \rightarrow K}[\phi](x, K)\|_{L_K^1 L_x^\infty}$. A back-of-the-envelope calculation explains the selection of this norm in the following sense: Let $\|\psi^\varepsilon\|_{L^2} = 1$, then

$$\begin{aligned} \int_{x, k \in \mathbb{R}^n} W^\varepsilon[\psi^\varepsilon](x, k) \phi(x, k) dx dk &= \int_{x, k, y \in \mathbb{R}^n} e^{-2\pi i k y} \psi^\varepsilon(x + \frac{\varepsilon y}{2}) \overline{\psi^\varepsilon}(x - \frac{\varepsilon y}{2}) \phi(x, k) dx dk = \\ &= \int_{x, y \in \mathbb{R}^n} \psi^\varepsilon(x + \frac{\varepsilon y}{2}) \overline{\psi^\varepsilon}(x - \frac{\varepsilon y}{2}) \int_{k \in \mathbb{R}^n} e^{-2\pi i k y} \phi(x, k) dk dx dy \quad \Rightarrow \\ \Rightarrow |\langle W^\varepsilon[\psi^\varepsilon], \phi \rangle| &\leq \|\psi^\varepsilon(x + \frac{\varepsilon y}{2}) \overline{\psi^\varepsilon}(x - \frac{\varepsilon y}{2})\|_{L_y^\infty L_x^1} \|\mathcal{F}_{k \rightarrow y}[\phi]\|_{L_y^1 L_x^\infty}, \end{aligned} \quad (35)$$

where of course

$$\|\psi^\varepsilon(x + \frac{\varepsilon y}{2}) \overline{\psi^\varepsilon}(x - \frac{\varepsilon y}{2})\|_{L_y^\infty L_x^1} = \sup_y \int_{x \in \mathbb{R}^n} |\psi^\varepsilon(x + y) \overline{\psi^\varepsilon}(x - y)| dx = 1.$$

Thus the set $\{W^\varepsilon[\psi^\varepsilon]\}_\varepsilon$ is uniformly bounded in the dual of \mathcal{A} , \mathcal{A}' , and hence weak-* compact by virtue of the Banach-Alaoglu Theorem. By extracting a subsequence in ε if necessary, the WM W^0 is now well defined. It is known that W^0 is in fact a non-negative finite measure [44], hence the term Wigner measure is justified.

Finding ways to metrize the weak-* limit

$$\langle W^0, \phi \rangle = \lim_{\varepsilon \rightarrow 0} \langle W^\varepsilon, \phi \rangle \quad \forall \phi \in \mathcal{A}$$

is important in itself, as it could yield better control on uniqueness questions, and of course help quantify the rate of convergence. One might think that since W^0 is a probability measure, W^ε would naturally be seen converge to W^0 in some Banach space of measures. However, for $\psi^\varepsilon \in L^2(\mathbb{R}^n)$, $W^\varepsilon = W^\varepsilon[\psi^\varepsilon] \in L^2(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ may not even be in $L^1(\mathbb{R}^{2n})$ [49]. In that case, $\int W^\varepsilon dx dk = \|\psi^\varepsilon\|_{L^2}^2$ in Cauchy-principal-value sense, but W^ε does not define a finite measure at all. By using a Fourier based norm, as in Definition 1.1, we go around this integrability question, and let the Fourier transform absorb any improper integrals. Moreover it must be noted that the new spaces \mathbf{A}^s are in fact closely related with the space \mathcal{A} , as can be seen in the following

Lemma 3.1 (Consistency of \mathcal{A} , \mathbf{A}^0 and \mathbf{A}^1). *For every ϕ in the Schwartz class of test functions $\mathcal{S}(\mathbb{R}^n)$*

$$\|\phi\|_{\mathcal{A}} \leq \|\phi\|_{\mathbf{A}^0} \leq \|\phi\|_{\mathbf{A}^1}.$$

Proof: Simply observe that, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\phi\|_{\mathcal{A}} = \|\mathcal{F}_{k \rightarrow K}[\phi]\|_{L_K^1 L_x^\infty} = \int \sup_{K \in \mathbb{R}^n} |\mathcal{F}_{k \rightarrow K}[\phi](x, K)| dK \leq \int \int |\mathcal{F}_{x, k \rightarrow X, K}[\phi](X, K)| dX dK = \|\widehat{\phi}\|_{L^1} = \|\phi\|_{\mathbf{A}^0}.$$

□

This leads to the following

Lemma 3.2. *For any $\|\psi^\varepsilon\|_{L^2} = 1$,*

$$\|W^\varepsilon[\psi^\varepsilon]\|_{\mathbf{A}^{-1}} = \|W^\varepsilon[\psi^\varepsilon]\|_{\widehat{L}^\infty} = 1.$$

Proof: First of all, recall that $\widehat{L}^\infty = (\mathbf{A}^0)'$. Now by virtue of equation (35) we get

$$|\langle W^\varepsilon[\psi^\varepsilon], \phi \rangle| \leq \sup_s \|\psi^\varepsilon(x + \frac{\varepsilon s}{2}) \overline{\psi^\varepsilon}(x - \frac{\varepsilon s}{2})\|_{L^1} \|\mathcal{F}_{k \rightarrow y}[\phi]\|_{L_y^1 L_x^\infty} \leq \|\psi^\varepsilon\|_{L^2}^2 \|\mathcal{F}_{x, k \rightarrow z, y}[\phi]\|_{L_{z, y}^1} = \|\phi\|_{\mathbf{A}^0}.$$

This shows $\|W^\varepsilon[\psi^\varepsilon]\|_{\mathcal{F}L^\infty} \leq 1$; equality follows by selecting $\phi_R = e^{-\pi R(x^2 + k^2)}$, and taking $\sup_{R \rightarrow 0} |\langle W^\varepsilon[\psi^\varepsilon], \phi_R \rangle|$ (observe that $\|\phi_R\|_{\mathbf{A}^0} = 1$).

The estimate $\|W^\varepsilon[\psi^\varepsilon]\|_{\mathbf{A}^{-1}} \leq 1$ follows in the same way. To show that $\|W^\varepsilon[\psi^\varepsilon]\|_{\mathbf{A}^{-1}} = 1$ it suffices to take ϕ_R as before, and compute $\|\phi_R\|_{\mathbf{A}^1} = 1 + CR^{\frac{3n}{2}}$. □

In other words, the norms \mathbf{A}^{-1} , \widehat{L}^∞ are correctly scaled to capture the Wigner measure as $\varepsilon \rightarrow 0$. We will be working mainly in \mathbf{A}^{-1} , that is the admissible observables will be those operators with Weyl symbols $\phi \in \mathbf{A}^1$. Technically, this is a slightly smaller class of observables than the class \mathcal{A} introduced in [44].

4 Background results

4.1 Background on Schrödinger equations

4.1.1 Well-posedness and conservation of energy

The 1-dimensional Schrödinger-Poisson problem has certain special features. One is that 1-dimensional Poisson kernel, $|x|$, grows at infinity. This means that the standard methods for $V(x, t) = \int_y K(x-y)|\psi^\varepsilon(y, t)|^2 dy$ with kernels $K \in L^\infty + L^p$ [24] cannot be used off-the-shelf. Because of that feature, the nonlinear potential

$$V(x, t) = \frac{b}{2} \int_y |x-y| |\psi^\varepsilon(y, t)|^2 dy \quad (36)$$

has nontrivial behavior at infinity,

$$\lim_{x \rightarrow \pm\infty} \frac{d}{dx} V(x, t) = \mp \frac{b}{2} \|\psi^\varepsilon(x, t)\|_{L^2}^2.$$

We will use the approach of [54], and modify it to also control the moments of the solution:

Theorem 4.1 (Solutions for the 1-dimensional Schrödinger-Poisson equation). *Consider the Cauchy problem*

$$i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon - \frac{b}{2} \int_y |x-y| |\psi^\varepsilon(y, t)|^2 dy \psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon \in H^2(\mathbb{R}) \cap \widehat{H}^1(\mathbb{R}), \quad \|\psi_0^\varepsilon\|_{L^2} = 1. \quad (37)$$

This problem has a unique, global-in-time solution in $H^2(\mathbb{R}) \cap \widehat{H}^1(\mathbb{R})$ which conserves mass

$$\|\psi^\varepsilon(t)\|_{L^2} = \|\psi_0^\varepsilon\|_{L^2} = 1 \quad (38)$$

and energy

$$\frac{\varepsilon^2}{2} \|\nabla \psi^\varepsilon(t)\|_{L^2}^2 + \frac{b}{4} \int_{x,y} |x-y| |\psi^\varepsilon(x, t)|^2 |\psi^\varepsilon(y, t)|^2 dx dy = \frac{\varepsilon^2}{2} \|\nabla \psi_0^\varepsilon\|_{L^2}^2 + \frac{b}{4} \int_{x,y} |x-y| |\psi_0^\varepsilon(x)|^2 |\psi_0^\varepsilon(y)|^2 dx dy. \quad (39)$$

Moreover,

$$\left\| \varepsilon \frac{d}{dx} \psi^\varepsilon(t) \right\|_{L^2} \leq \left\| \varepsilon \frac{d}{dx} \psi_0^\varepsilon \right\|_{L^2} + |b| |t| \quad (40)$$

and

$$\|x \psi^\varepsilon(t)\|_{L^2} \leq \|x \psi_0^\varepsilon\|_{L^2} + \int_{\tau=0}^t \left\| \varepsilon \frac{d}{dx} \psi^\varepsilon(\tau) \right\|_{L^2} d\tau. \quad (41)$$

Proof: By virtue of Theorem B.1 of [54], $\psi_0^\varepsilon \in H^2$, $\|\psi_0^\varepsilon\|_{L^2} = 1$ implies the existence of a global solution $\psi^\varepsilon \in L^\infty([0, T], H^2)$ for any $T > 0$, satisfying in addition

$$\|\psi_0(t)\|_{L^2} = 1, \quad \varepsilon^2 \|\psi^\varepsilon(t)\|_{H^2} \leq C(1 + \varepsilon^2 \|\psi_0^\varepsilon\|_{H^2}) e^{Ct^2}$$

for some absolute constant C . We will now work on this solution to prove equations (39), (40), (41).

Denote for brevity $V(x, t)$ the nonlinear potential as in equation (36). $V(x, t)$ is the solution of

$$\Delta V(x, t) = b |\psi^\varepsilon(x, t)|^2; \quad (42)$$

equation (36) yields

$$\frac{d}{dx} V(x, t) = \frac{b}{2} \left(\int_{y=x}^{\infty} |\psi^\varepsilon(y, t)|^2 dy - \int_{y=-\infty}^x |\psi^\varepsilon(y, t)|^2 dy \right), \quad (43)$$

and therefore, using the conservation of mass,

$$\left| \frac{d}{dx} V(x, t) \right| \leq \frac{|b|}{2} \|\psi^\varepsilon(t)\|_{L^2}^2 = \frac{|b|}{2} \|\psi_0^\varepsilon\|_{L^2}^2. \quad (44)$$

Now, following the steps of the proof of Lemma 2.1 of [54], we check that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\varepsilon \frac{d}{dx} \psi^\varepsilon(t)\|_{L^2}^2 &= -\varepsilon \operatorname{Im} \left[\langle \psi^\varepsilon \frac{d}{dx} V, \frac{d}{dx} \psi^\varepsilon \rangle \right] \leq \frac{|b|}{2} \|\psi_0^\varepsilon\|_{L^2}^3 \|\varepsilon \frac{d}{dx} \psi^\varepsilon(t)\|_{L^2} \Rightarrow \\ &\Rightarrow \frac{d}{dt} \|\varepsilon \frac{d}{dx} \psi^\varepsilon(t)\|_{L^2} \leq |b| \|\psi_0^\varepsilon\|_{L^2}^3. \end{aligned} \quad (45)$$

Thus equation (40), which is essentially equation (2.8) of [54], follows. Similarly,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|x\psi^\varepsilon(t)\|_{L^2}^2 &= \operatorname{Re} \left[\frac{i\varepsilon}{2} \langle x\Delta\psi^\varepsilon(t), x\psi^\varepsilon(t) \rangle \right] = \operatorname{Re} \left[i\varepsilon \langle x\psi^\varepsilon(t), \frac{d}{dx} \psi^\varepsilon(t) \rangle \right] \leq \|\varepsilon \nabla \psi^\varepsilon(t)\|_{L^2} \|x\psi^\varepsilon\|_{L^2} \Rightarrow \\ &\Rightarrow \frac{d}{dt} \|x\psi^\varepsilon(t)\|_{L^2} \leq \|\varepsilon \nabla \psi^\varepsilon(t)\|_{L^2}. \end{aligned} \quad (46)$$

Equation (41) follows.

Now there is enough regularity to justify the conservation of energy by standard arguments [24]. \square

Well-posedness for the nonlinear Schrödinger equation with power nonlinearities on H^1 is exhaustively well studied [24]. Here we briefly recall the relevant results in the semiclassical scaling, and outline how control of moments (\widehat{H}^1 norm) follows.

Theorem 4.2 (Solutions for the NLS with power nonlinearities). *Consider , i.e. the Cauchy problem*

$$i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon - b|\psi^\varepsilon|^{2\sigma} \psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon \in H^1(\mathbb{R}^n) \quad (47)$$

either in the **energy sub-critical defocusing regime**,

$$b > 0, \quad 0 < \sigma < \frac{2}{(n-2)_+}; \quad (48)$$

or in the **mass sub-critical focusing regime**,

$$b < 0, \quad 0 < \sigma < \frac{2}{n}. \quad (49)$$

Then there is a unique, global-in-time solution in H^1 which conserves mass

$$\|\psi^\varepsilon(t)\|_{L^2} = \|\psi_0^\varepsilon\|_{L^2} \quad (50)$$

and energy

$$\frac{\varepsilon^2}{2} \|\nabla \psi^\varepsilon(t)\|_{L^2}^2 + \frac{b}{\sigma+1} \|\psi^\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+2} = \frac{\varepsilon^2}{2} \|\nabla \psi_0^\varepsilon\|_{L^2}^2 + \frac{b}{\sigma+1} \|\psi_0^\varepsilon\|_{L^{2\sigma+2}}^{2\sigma+2} \quad (51)$$

Theorem 4.3 (Moments under power nonlinearities). *Let ψ^ε be the solution of*

$$i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon - b|\psi^\varepsilon|^{2\sigma} \psi^\varepsilon = 0, \quad \psi^\varepsilon(t=0) = \psi_0^\varepsilon \in H^1(\mathbb{R}^n) \cap \widehat{H}^1(\mathbb{R}^n). \quad (52)$$

Then

$$\|x\psi^\varepsilon(t)\|_{L^2} \leq \|x\psi_0^\varepsilon\|_{L^2} + \int_{\tau=0}^t \|\varepsilon \frac{d}{dx} \psi^\varepsilon(\tau)\|_{L^2} d\tau. \quad (53)$$

Proof: This follows in exactly the same way as in Theorem 4.1. More specifically, one directly computes

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|x\psi^\varepsilon(t)\|_{L^2}^2 &= \operatorname{Re} \left[\frac{i\varepsilon}{2} \langle x\Delta\psi^\varepsilon(t), x\psi^\varepsilon(t) \rangle \right] = \operatorname{Re} \left[i\varepsilon \langle x\psi^\varepsilon(t), \frac{d}{dx} \psi^\varepsilon(t) \rangle \right] \leq \|\varepsilon \nabla \psi^\varepsilon(t)\|_{L^2} \|x\psi^\varepsilon\|_{L^2} \Rightarrow \\ &\Rightarrow \frac{d}{dt} \|x\psi^\varepsilon(t)\|_{L^2} \leq \|\varepsilon \nabla \psi^\varepsilon(t)\|_{L^2}. \end{aligned} \quad (54)$$

The result follows. \square

4.1.2 Galilean invariance

Lemma 4.4 (Galilean invariance). *Let ψ satisfy*

$$i\varepsilon\partial_t\psi + \frac{\varepsilon^2}{2}\Delta\psi - b\int_y K(x-y)|\psi(y,t)|^2dy\psi = 0, \quad \psi(t=0) = \psi_0 \in L^2(\mathbb{R}^n). \quad (55)$$

For any $x_0, v \in \mathbb{R}^n$, and denote

$$u(x,t) = \psi(x+vt+x_0,t)e^{-i\left(\frac{v\cdot x}{\varepsilon} + \frac{|v|^2}{2\varepsilon}t\right)}. \quad (56)$$

Then u satisfies

$$i\varepsilon\partial_tu + \frac{\varepsilon^2}{2}\Delta u - b\int_y K(x-y)|u(y,t)|^2dyu = 0, \quad u(t=0) = u_0 = \psi_0(x+x_0)e^{-i\frac{v\cdot x}{\varepsilon}} \in L^2(\mathbb{R}^n) \quad (57)$$

and

$$W^\varepsilon[u(t)](x,k) = W^\varepsilon[\psi(t)]\left(x+vt+x_0, k + \frac{v}{2\pi}\right). \quad (58)$$

Similarly, if Ψ satisfies

$$i\varepsilon\partial_t\Psi + \frac{\varepsilon^2}{2}\Delta\Psi - b|\psi|^{2\sigma}\Psi = 0, \quad \Psi(t=0) = \Psi_0 \in L^2(\mathbb{R}^n). \quad (59)$$

and

$$U(x,t) = \Psi(x+vt+x_0,t)e^{-i\left(\frac{v\cdot x}{\varepsilon} + \frac{|v|^2}{2\varepsilon}t\right)}, \quad (60)$$

then

$$i\varepsilon\partial_tU + \frac{\varepsilon^2}{2}\Delta U - b|U|^{2\sigma}U = 0, \quad U(t=0) = U_0 = \Psi_0(x+x_0)e^{-i\frac{v\cdot x}{\varepsilon}} \in L^2(\mathbb{R}^n). \quad (61)$$

Moreover,

$$W^\varepsilon[U(t)](x,k) = W^\varepsilon[\Psi(t)]\left(x+vt+x_0, k + \frac{v}{2\pi}\right). \quad (62)$$

Proof: See [32] for the transformation of equation (59), i.e. for equations (60), (61).

Equation (62) follows by the elementary computation

$$\begin{aligned} W^\varepsilon[U(t)] &= W^\varepsilon[\Psi(x+vt+x_0,t)e^{-i\left(\frac{v\cdot x}{\varepsilon} + \frac{|v|^2}{2\varepsilon}t\right)}] = \\ &= \int_y e^{-2\pi ik\cdot y}\Psi\left(x + \frac{\varepsilon y}{2} + vt + x_0, t\right)e^{-i\left(\frac{v\cdot(x+\frac{\varepsilon y}{2})}{\varepsilon} + \frac{|v|^2}{2\varepsilon}t\right)}\overline{\Psi}\left(x - \frac{\varepsilon y}{2} + vt + x_0, t\right)e^{i\left(\frac{v\cdot(x-\frac{\varepsilon y}{2})}{\varepsilon} + \frac{|v|^2}{2\varepsilon}t\right)}dy = \\ &= \int_y e^{-2\pi i\left(k+\frac{v}{2\pi}\right)\cdot y}\Psi\left(x + \frac{\varepsilon y}{2} + vt + x_0, t\right)\overline{\Psi}\left(x - \frac{\varepsilon y}{2} + vt + x_0, t\right)dy = W^\varepsilon[\Psi(t)]\left(x+vt+x_0, k + \frac{v}{2\pi}\right). \end{aligned}$$

The proof for the Schrödinger-Poisson nonlinearity is essentially the same. \square

Lemma 4.5 (Center of mass and conservation of momentum). *Let ψ satisfy*

$$i\varepsilon\partial_t\psi + \frac{\varepsilon^2}{2}\Delta\psi - \frac{b}{2}\int_x |x-y||\psi(y,t)|^2dy\psi = 0, \quad \psi(t=0) = \psi_0 \in \mathcal{S}(\mathbb{R}) \quad (63)$$

Then

$$\frac{d}{dt}\mu_x(\psi(t)) = 2\pi\mu_k(\psi_0), \quad \frac{d}{dt}\mu_k(\psi(t)) = 0.$$

Similarly, if Ψ satisfies

$$i\varepsilon\partial_t\Psi + \frac{\varepsilon^2}{2}\Delta\Psi - b|\psi|^{2\sigma}\Psi = 0, \quad \Psi(t=0) = \Psi_0 \in \mathcal{S}(\mathbb{R}^n), \quad (64)$$

then

$$\frac{d}{dt}\mu_x(\Psi(t)) = 2\pi\mu_k(\Psi_0), \quad \frac{d}{dt}\mu_k(\Psi(t)) = 0.$$

Proof: We compute

$$\begin{aligned}\frac{d}{dt}\mu_x(\psi) &= \frac{d}{dt}\langle x\psi, \psi \rangle = \frac{i\varepsilon}{2} (\langle x\Delta\psi, \psi \rangle - \langle x\psi, \Delta\psi \rangle) = \\ &= \frac{i\varepsilon}{2} (\langle \psi, \nabla\psi \rangle - \langle \nabla\psi, \psi \rangle) = i\varepsilon\langle \nabla\psi, \psi \rangle = 2\pi\mu_k(\psi(t)).\end{aligned}$$

Moreover, denoting $V(x, t) = \frac{b}{2} \int_y |x - y| |\psi(y, t)|^2 dy$ the nonlinear potential we have

$$\begin{aligned}\frac{d}{dt}\mu_k(\psi) &= \varepsilon \frac{d}{dt}\langle k\hat{\psi}, \hat{\psi} \rangle = \frac{\varepsilon}{2\pi i} \frac{d}{dt}\langle \nabla\psi, \psi \rangle = \frac{1}{\pi} \operatorname{Re}\langle \nabla\psi, V\psi \rangle = \\ &= \frac{1}{2\pi} \int_x V(x, t) (\overline{\psi}\nabla\psi + \psi\nabla\overline{\psi}) dx = \frac{1}{2\pi} \int_x V(x, t) \nabla|\psi(x, t)|^2 dx\end{aligned}$$

and now we complete the computation by observing

$$\begin{aligned}\int_x V(x, t) \nabla|\psi(x, t)|^2 dx &= \frac{b}{2} \int_x \int_y |x - y| |\psi(y, t)|^2 dy \nabla|\psi(x, t)|^2 dx = \\ &= -\frac{b}{2} \int_x \int_y \operatorname{sign}(x - y) |\psi(y, t)|^2 dy |\psi(x, t)|^2 dx = 0.\end{aligned}$$

The proof for power nonlinearities follows along the same steps. □

4.2 Inequalities

Observation 4.6. For any $a, b, q > 0$

$$(a + b)^q \leq C(a^q + b^q) \quad \text{for} \quad C = \begin{cases} 2^{q-1}, & q \geq 1, \\ 1, & 0 < q \leq 1 \end{cases}$$

Lemma 4.7 (Algebraic bound). Let $f(t) \in C([0, \infty), [0, \infty))$, $0 < A, B$, $0 < \theta < 1$ and

$$f(t) \leq A + Bf^\theta(t).$$

Then $f(t)$ is bounded by the largest positive solution of,

$$x - Bx^\theta - A = 0. \tag{65}$$

In the case $\theta = \frac{1}{2}$,

$$f(t) \leq A + \frac{B^2}{2} + \frac{B\sqrt{B^2 + 4A}}{2}.$$

In the case where A, B depend on ε and we have

$$A \ll B \quad \text{as } \varepsilon \rightarrow 0,$$

then one easily checks that the largest positive solution of (65) has to be of the form

$$f_{max} \leq B^{\frac{1}{1-\theta}} (1 + o(1)).$$

In the case where A, B depend on ε and we have

$$B \ll A \quad \text{as } \varepsilon \rightarrow 0,$$

then one easily checks that the largest positive solution of (65) has to be of the form

$$f_{max} \leq A(1 + o(1)).$$

Proof: Since $b\sqrt{t}$ grows more slowly than t when $t \rightarrow \infty$, it is clear that $f(t)$ is bounded above. Moreover the maximum value f_{max} will satisfy (65); indeed if for some value f

$$f < A + B\sqrt{f}$$

this means that a somewhat larger value f would still be possible.

Thus we need to compute the largest solution of (65); if $\theta = \frac{1}{2}$ this is achieved by solving the quadratic equation

$$\left(\sqrt{f_{max}}\right)^2 - B\sqrt{f_{max}} - A = 0.$$

When $A \ll B$ the problem becomes

$$x - Bx^\theta - A = 0 \quad \Rightarrow \quad \frac{x}{B} - x^\theta = o(1) \quad \Rightarrow \quad x^{\theta-1}(1 + o(1)) = B^{-1},$$

and the computation for $B \ll A$ is analogous. □

Lemma 4.8. *Let*

$$f \in H^1(\mathbb{R}^n), \quad \|f\|_{L^2(\mathbb{R}^n)} = 1, \quad \sigma \in \left(0, \frac{2}{(n-2)_+}\right).$$

Then

$$\|f\|_{L^{2\sigma+2}(\mathbb{R}^n)}^{2\sigma+2} \leq C_*^{GN} \|\nabla f\|_{L^2(\mathbb{R}^n)}^{n\sigma}.$$

Remark: This is a special case of the Gagliardo-Nirenberg L^2 -gradient inequality, cf. [1, 24, 32]. The sharp constant is known; indeed if we denote $C_{q,p,n}^{GN}$ the sharp constant for the Gagliardo-Nirenberg inequality, $\|f\|_{L^p(\mathbb{R}^n)} \leq C_{q,p,n}^{GN} \|\nabla f\|_{L^2(\mathbb{R}^n)}^\theta \|f\|_{L^q(\mathbb{R}^n)}^{1-\theta}$, [1], then $C_*^{GN}(n, \sigma) := (C_{2,2\sigma+2,n}^{GN})^{2\sigma+2}$.

4.3 Computations for concrete wavepackets

Definition 4.9. *Let*

$$a \in \mathcal{S}(\mathbb{R}^n), \quad \|a\|_{L^2} = 1, \quad \mu_x(a) = \mu_k(a) = 0,$$

$\beta \in (0, 1)$. *The function*

$$\psi_0^\varepsilon(x) = \varepsilon^{-\frac{n\beta}{2}} a\left(\frac{x-x_0}{\varepsilon^\beta}\right) e^{\frac{2\pi i k_0 \cdot (x-x_0)}{\varepsilon}}$$

will be called a squeezed state with envelope a and rate of concentration β .

Standard computations yield the following

Lemma 4.10. *Let*

$$\psi_0^\varepsilon(x) = \varepsilon^{-\frac{n\beta}{2}} a\left(\frac{x-x_0}{\varepsilon^\beta}\right) e^{\frac{2\pi i k_0 \cdot (x-x_0)}{\varepsilon}}$$

be a squeezed state with envelope a and rate of concentration β . Then

$$\|\psi_0^\varepsilon\|_{L^2} = 1, \quad \mu_x(\psi_0^\varepsilon) = x_0, \quad \mu_k(\psi_0^\varepsilon) = k_0, \quad \sigma_x(\psi_0^\varepsilon) = O(\varepsilon^\beta) \quad \sigma_k(\psi_0^\varepsilon) = O(\varepsilon^{1-\beta}).$$

Other classes of wavepackets can also be of interest:

Definition 4.11. *Let*

$$a \in \mathcal{S}(\mathbb{R}^n), \quad \|a\|_{L^2} = 1, \quad \mu_x(a) = \mu_k(a) = 0,$$

$\beta \in (0, 1)$, $0 \neq z \in \mathbb{R}$. *The function*

$$\psi_0^\varepsilon(x) = \varepsilon^{-\frac{n\beta}{2}} a\left(\frac{x-x_0}{\varepsilon^\beta}\right) e^{\frac{2\pi i k_0 \cdot (x-x_0)}{\varepsilon}} e^{\frac{iz \cdot (x-x_0)^2}{\varepsilon}}$$

will be called a chirp wavepacket with envelope a , rate of concentration β and quadratic rate of oscillation z .

Direct computations in the spirit of Lemma 4.10 yield the following

Lemma 4.12. *Let*

$$\psi_0^\varepsilon(x) = \varepsilon^{-\frac{n\beta}{2}} a\left(\frac{x-x_0}{\varepsilon^\beta}\right) e^{\frac{2\pi i k_0 \cdot (x-x_0)}{\varepsilon}} e^{\frac{iz \cdot (x-x_0)^2}{\varepsilon}}$$

be a chirp wavepacket with envelope a , rate of concentration β and quadratic rate of oscillation z . Then

$$\|\psi_0^\varepsilon\|_{L^2} = 1, \quad \mu_x(\psi_0^\varepsilon) = x_0, \quad \mu_k(\psi_0^\varepsilon) = k_0, \quad \sigma_x(\psi_0^\varepsilon) = O(\varepsilon^\beta) \quad \sigma_k(\psi_0^\varepsilon) = O(\varepsilon^{1-\beta} + \varepsilon^\beta).$$

Proof: The only difference from Lemma 4.10 is in the computation of $\sigma_k(\psi_0^\varepsilon)$:

$$\begin{aligned} \sigma_k^2(\psi_0^\varepsilon) &= \varepsilon^2 \int_k (k - \frac{k_0}{\varepsilon})^2 |\widehat{\psi_0^\varepsilon}(k)|^2 dk = \frac{\varepsilon^2}{4\pi^2} \int_x |\nabla \left(\varepsilon^{-\frac{n\beta}{2}} a\left(\frac{x}{\varepsilon^\beta}\right) e^{\frac{iz \cdot x^2}{\varepsilon}} \right)|^2 dx \leq \\ &\leq \frac{\varepsilon^{2-n\beta-2\beta}}{4\pi^2} \int_{x_j=1}^n \sum a_j^2\left(\frac{x}{\varepsilon^\beta}\right) dx + \frac{\varepsilon^{-n\beta} iz}{2\pi^2} \int_{x_j=1}^n \sum x_j^2 |a\left(\frac{x}{\varepsilon^\beta}\right)|^2 dx = O(\varepsilon^{2(1-\beta)}) + O(\varepsilon^{2\beta}). \end{aligned}$$

□

5 Proof of the main results

5.1 Proof of Theorem 2.1

By virtue of Lemma 4.4, the solution of the problem

$$i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon - \frac{b}{2} \int_y |x-y| |u^\varepsilon(y,t)|^2 dy u^\varepsilon = 0 \quad u^\varepsilon(t=0) = u_0^\varepsilon = \psi_0^\varepsilon(x + \mu_x(\psi_0^\varepsilon)) e^{-2\pi i \frac{\mu_k(\psi_0^\varepsilon) \cdot x}{\varepsilon}} \quad (66)$$

is related to ψ^ε through

$$u^\varepsilon(x,t) = \psi^\varepsilon(x + vt + x_0, t) e^{-i\left(\frac{v \cdot x}{\varepsilon} + \frac{v \cdot v \cdot t}{2\varepsilon}\right)}, \quad v = 2\pi \mu_k(\psi_0^\varepsilon), \quad x_0 = \mu_x(\psi_0^\varepsilon). \quad (67)$$

By virtue of Lemma 4.5 and by the construction of u_0^ε ,

$$\begin{aligned} \mu_x(u^\varepsilon(t)) &= \mu_k(u^\varepsilon(t)) = 0, \\ \sigma_x(\psi^\varepsilon(t)) &= \sigma_x(u^\varepsilon(t)) = \|xu^\varepsilon(t)\|_{L_x^2}, \quad \sigma_k(\psi^\varepsilon(t)) = \sigma_k(u^\varepsilon(t)) = \frac{1}{2\pi} \|\varepsilon \nabla u^\varepsilon(t)\|_{L^2}. \end{aligned} \quad (68)$$

For now we will work with equation (66), and ultimately transfer our results to $W^\varepsilon[\psi^\varepsilon(t)]$.

By virtue of the conservation of energy (39), we have

$$\begin{aligned} \frac{\varepsilon^2}{2} \|\nabla u^\varepsilon(t)\|_{L^2}^2 &\leq \frac{\varepsilon^2}{2} \|\nabla u^\varepsilon(t)\|_{L^2}^2 + \frac{b}{4} \int_{x,y} |x-y| |u^\varepsilon(x,t)|^2 |u^\varepsilon(y,t)|^2 dx dy = \\ &= \frac{\varepsilon^2}{2} \|\nabla u_0^\varepsilon\|_{L^2}^2 + \frac{b}{4} \int_{x,y} |x-y| |u_0^\varepsilon(x)|^2 |u_0^\varepsilon(y)|^2 dx dy \leq \frac{\varepsilon^2}{2} \|\nabla u_0^\varepsilon\|_{L^2}^2 + \frac{b}{2} \int_x |x| |u_0^\varepsilon(x)|^2 dx \leq \\ &\leq \frac{\varepsilon^2}{2} \|\nabla u_0^\varepsilon\|_{L^2}^2 + \frac{b}{2} \|xu_0^\varepsilon\|_{L^2}. \end{aligned}$$

The triangle inequality $|x-y| \leq |x| + |y|$ was also used; also recall that $b > 0$. Thus by virtue of Observation 4.6 we have

$$\|\varepsilon \nabla u^\varepsilon(t)\|_{L^2} \leq \|\varepsilon \nabla u_0^\varepsilon\|_{L^2} + \sqrt{b \|xu_0^\varepsilon\|_{L^2}}. \quad (69)$$

Moreover, by virtue of equation (41),

$$\|xu^\varepsilon(t)\|_{L^2} \leq \|xu_0^\varepsilon\|_{L^2} + t \left(\|\varepsilon \nabla u_0^\varepsilon\|_{L^2} + \sqrt{b \|xu_0^\varepsilon\|_{L^2}} \right). \quad (70)$$

Recalling equation (68), we can recast equations (69), (70) as

$$\sigma_k(\psi^\varepsilon(t)) = \sigma_k(u^\varepsilon(t)) \leq \sigma_k(u_0^\varepsilon) + \sqrt{\frac{b}{2\pi} \sigma_x(u_0^\varepsilon)}, \quad \sigma_x(\psi^\varepsilon(t)) = \sigma_x(u^\varepsilon(t)) \leq \sigma_x(u_0^\varepsilon) + t \left(\sigma_k(u_0^\varepsilon) + \sqrt{\frac{b}{2\pi} \sigma_x(u_0^\varepsilon)} \right),$$

and finally

$$\sigma_k(\psi^\varepsilon(t)) + \sigma_x(\psi^\varepsilon(t)) \leq \sigma_k(u_0^\varepsilon)(1+t) + \sigma_x(u_0^\varepsilon) + \sqrt{\frac{b}{2\pi}}(1+t)\sqrt{\sigma_x(u_0^\varepsilon)}.$$

The proof is complete by recalling that

$$\mu_x(\psi^\varepsilon(t)) = \mu_x(\psi_0^\varepsilon) + 2\pi t \mu_k(\psi_0^\varepsilon), \quad \mu_k(\psi^\varepsilon(t)) = \mu_k(\psi_0^\varepsilon),$$

by virtue of Lemma 4.5, and then applying Corollary 5.2. □

5.2 Proof of Theorem 2.3

In exact analogy to what we did before, the solution of the problem

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon - b|u^\varepsilon|^{2\sigma} u^\varepsilon = 0 \quad u^\varepsilon(t=0) = u_0^\varepsilon = \psi_0^\varepsilon(x + \mu_x(\psi_0^\varepsilon))e^{-2\pi i \frac{\mu_k(\psi_0^\varepsilon)}{\varepsilon} \cdot x} \quad (71)$$

is related to ψ^ε through

$$u^\varepsilon(x, t) = \psi^\varepsilon(x + vt + x_0, t)e^{-i\left(\frac{v \cdot x}{\varepsilon} + \frac{v \cdot v}{2\varepsilon}t\right)}, \quad v = 2\pi\mu_k(\psi_0^\varepsilon), \quad x_0 = \mu_x(\psi_0^\varepsilon). \quad (72)$$

Again, by virtue of Lemma 4.5 and by the construction of u_0^ε ,

$$\begin{aligned} \mu_x(u^\varepsilon(t)) &= \mu_k(u^\varepsilon(t)) = 0, \\ \sigma_x(\psi^\varepsilon(t)) &= \sigma_x(u^\varepsilon(t)) = \|xu^\varepsilon(t)\|_{L_x^2}, \quad \sigma_k(\psi^\varepsilon(t)) = \sigma_k(u^\varepsilon(t)) = \frac{1}{2\pi}\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2}. \end{aligned} \quad (73)$$

By virtue of the conservation of energy, equation (51),

$$\frac{\varepsilon^2}{2}\|\nabla u^\varepsilon(t)\|_{L^2}^2 \leq \frac{\varepsilon^2}{2}\|\nabla u_0^\varepsilon\|_{L^2}^2 + \frac{b}{\sigma+1}\|u_0^\varepsilon\|_{L^{2\sigma+2}}^{2\sigma+2} \leq \frac{1}{2}\|\varepsilon\nabla u_0^\varepsilon\|_{L^2}^2 + \frac{b}{\sigma+1}C_*^{GN}\|\nabla u_0^\varepsilon\|_{L^2}^{n\sigma},$$

where in the last step we used the Gagliardo-Nirenberg inequality, Lemma 4.8. Using Observation 4.6, this becomes

$$\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2} \leq \|\varepsilon\nabla u_0^\varepsilon\|_{L^2} + \sqrt{\varepsilon^{-n\sigma} b \frac{C_*^{GN}}{2\sigma+2}} \|\varepsilon\nabla u_0^\varepsilon\|_{L^2}^{\frac{n\sigma}{2}}$$

Moreover, equation (53) of Theorem 4.3 implies that

$$\|xu^\varepsilon(t)\|_{L^2} \leq \|xu_0^\varepsilon\|_{L^2} + t \left(\|\varepsilon\nabla u_0^\varepsilon\|_{L^2} + \sqrt{\varepsilon^{-n\sigma} b \frac{C_*^{GN}}{2\sigma+2}} \|\varepsilon\nabla u_0^\varepsilon\|_{L^2}^{\frac{n\sigma}{2}} \right).$$

Collecting the last two equations, and recalling equation (73), we have

$$\sigma_x(u^\varepsilon(t)) + \sigma_k(u^\varepsilon(t)) \leq \sigma_x(u_0^\varepsilon) + (1+t)\sigma_k(u_0^\varepsilon) + (1+t) \left(\sigma_k^{\frac{n\sigma}{2}}(u_0^\varepsilon) \sqrt{\frac{b}{\varepsilon^{n\sigma}} \frac{C_*^{GN}(2\pi)^{n\sigma-2}}{2\sigma+2}} \right)$$

The proof is complete by recalling that

$$\mu_x(\psi^\varepsilon(t)) = \mu_x(\psi_0^\varepsilon) + 2\pi t\mu_k(\psi_0^\varepsilon), \quad \mu_k(\psi^\varepsilon(t)) = \mu_k(\psi_0^\varepsilon),$$

by virtue of Lemma 4.5, and then applying Corollary 5.2. \square

5.3 Proof of Theorem 2.6

In exact analogy to what we did before, the solution of the problem

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon - b|u^\varepsilon|^{2\sigma} u^\varepsilon = 0 \quad u^\varepsilon(t=0) = u_0^\varepsilon = \psi_0^\varepsilon(x + \mu_x(\psi_0^\varepsilon))e^{-2\pi i \frac{\mu_k(\psi_0^\varepsilon)}{\varepsilon} \cdot x} \quad (74)$$

is related to ψ^ε through

$$u^\varepsilon(x, t) = \psi^\varepsilon(x + vt + x_0, t)e^{-i\left(\frac{v \cdot x}{\varepsilon} + \frac{v \cdot v}{2\varepsilon}t\right)}, \quad v = 2\pi\mu_k(\psi_0^\varepsilon), \quad x_0 = \mu_x(\psi_0^\varepsilon). \quad (75)$$

Again, by virtue of Lemma 4.5 and by the construction of u_0^ε ,

$$\begin{aligned} \mu_x(u^\varepsilon(t)) &= \mu_k(u^\varepsilon(t)) = 0, \\ \sigma_x(\psi^\varepsilon(t)) &= \sigma_x(u^\varepsilon(t)) = \|xu^\varepsilon(t)\|_{L_x^2}, \quad \sigma_k(\psi^\varepsilon(t)) = \sigma_k(u^\varepsilon(t)) = \frac{1}{2\pi}\|\varepsilon\nabla u^\varepsilon(t)\|_{L^2}. \end{aligned} \quad (76)$$

By virtue of the conservation of energy, equation (51),

$$\begin{aligned} \frac{\varepsilon^2}{2} \|\nabla u^\varepsilon(t)\|_{L^2}^2 &= \frac{\varepsilon^2}{2} \|\nabla u_0^\varepsilon\|_{L^2}^2 + \frac{b}{\sigma+1} \|u_0^\varepsilon\|_{L^{2\sigma+2}}^{2\sigma+2} + \frac{|b|}{\sigma+1} \|u^\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \leq \\ &\leq \frac{\varepsilon^2}{2} \|\nabla u_0^\varepsilon\|_{L^2}^2 + \frac{|b|}{\sigma+1} \|u^\varepsilon(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \leq \frac{1}{2} \|\varepsilon \nabla u_0^\varepsilon\|_{L^2}^2 + \frac{|b|}{\sigma+1} C_*^{GN} \|\nabla u^\varepsilon(t)\|_{L^2}^{n\sigma}, \end{aligned}$$

where in the last step we used the Gagliardo-Nirenberg inequality, Lemma 4.8. Using Observation 4.6, this becomes

$$\|\varepsilon \nabla u^\varepsilon(t)\|_{L^2} \leq \|\varepsilon \nabla u_0^\varepsilon\|_{L^2} + \sqrt{\varepsilon^{-n\sigma} |b(\varepsilon)| \frac{C_*^{GN}}{2\sigma+2}} \|\varepsilon \nabla u^\varepsilon(t)\|_{L^2}^{\frac{n\sigma}{2}}. \quad (77)$$

Since $\frac{n\sigma}{2} = \frac{1}{2}$, Lemma 4.7 applies to $f(t) = \|\varepsilon \nabla u^\varepsilon(t)\|_{L^2}$, yielding

$$\|\varepsilon \nabla u^\varepsilon(t)\|_{L^2} \leq \|\varepsilon \nabla u_0^\varepsilon\|_{L^2} + \frac{|b(\varepsilon)| C_*^{GN}}{\varepsilon(4 + \frac{4}{n})} + \frac{1}{2} \sqrt{\frac{|b(\varepsilon)| C_*^{GN}}{\varepsilon(2 + \frac{2}{n})}} \sqrt{\frac{|b(\varepsilon)| C_*^{GN}}{\varepsilon(2 + \frac{2}{n})} + 4\|\varepsilon \nabla u_0^\varepsilon\|_{L^2}} \quad (78)$$

For brevity we will denote

$$\mathcal{K} := \frac{|b(\varepsilon)| C_*^{GN}}{\varepsilon(4 + \frac{4}{n})} + \frac{1}{2} \sqrt{\frac{|b(\varepsilon)| C_*^{GN}}{\varepsilon(2 + \frac{2}{n})}} \sqrt{\frac{|b(\varepsilon)| C_*^{GN}}{\varepsilon(2 + \frac{2}{n})} + 4\|\varepsilon \nabla u_0^\varepsilon\|_{L^2}} \quad (79)$$

Moreover, equation (53) of Theorem 4.3 implies that

$$\|xu^\varepsilon(t)\|_{L^2} \leq \|xu_0^\varepsilon\|_{L^2} + t \left(\|\varepsilon \nabla u_0^\varepsilon\|_{L^2} + \mathcal{K} \right).$$

Collecting the last two equations, and recalling equation (76), we have

$$\sigma_x(u^\varepsilon(t)) + \sigma_k(u^\varepsilon(t)) \leq \sigma_x(u_0^\varepsilon) + \sigma_k(u_0^\varepsilon)(1+t) + \mathcal{K} \frac{1+t}{2\pi}.$$

The proof is complete by recalling that

$$\mu_x(\psi^\varepsilon(t)) = \mu_x(\psi_0^\varepsilon) + 2\pi t \mu_k(\psi_0^\varepsilon), \quad \mu_k(\psi^\varepsilon(t)) = \mu_k(\psi_0^\varepsilon),$$

by virtue of Lemma 4.5, and then applying Corollary 5.2. \square

5.4 Proof of Remark 2.7

We follow the proof of Theorem 2.6 up to equation (77), i.e. up to the estimation of $\|\varepsilon \nabla u^\varepsilon(t)\|_{L^2}$ by virtue of Lemma 4.7. Observe that equation (77) is of the form

$$f(t) \leq A + B f^\theta(t), \quad A = \|\varepsilon \nabla u_0^\varepsilon\|_{L^2}, \quad B = O(\varepsilon^{\frac{\gamma-n\sigma}{2}}), \quad \theta = \frac{n\sigma}{2}, \quad f(t) = \|\varepsilon \nabla u^\varepsilon(t)\|_{L^2}.$$

Thus we can proceed in each of the following cases:

Case 1: $A \ll B$ i.e. $\|\varepsilon \nabla u_0^\varepsilon\|_{L^2} = o(\varepsilon^{\frac{\gamma-n\sigma}{2}})$. Then Lemma 4.7 yields

$$\|\varepsilon \nabla u^\varepsilon(t)\|_{L^2} = O(\varepsilon^{\frac{\gamma-n\sigma}{2}}).$$

Thus if $\gamma > n\sigma$ one can resume the proof of Theorem 2.6 with $\mathcal{K} = O(\varepsilon^{\frac{\gamma-n\sigma}{2}})$.

Case 2: $B \ll A$ i.e. $\varepsilon^{\frac{\gamma-n\sigma}{2}} = o(\|\varepsilon \nabla u_0^\varepsilon\|_{L^2})$. Then Lemma 4.7 yields

$$\|\varepsilon \nabla u^\varepsilon(t)\|_{L^2} = O(\|\varepsilon \nabla u_0^\varepsilon\|_{L^2}).$$

Essentially one resumes the proof of Theorem 2.6 with $\mathcal{K} = O(\|\varepsilon \nabla u_0^\varepsilon\|_{L^2})$.

Observe that since we are interested in wavepackets, i.e. $\|\varepsilon \nabla u_0^\varepsilon\|_{L^2} = o(1)$, $\gamma > n\sigma$ appears to be virtually a necessary condition in this case as well. \square

5.5 The concentration estimates

Lemma 5.1 (Concentration of Wigner transforms to $\delta(x, k)$ for Schwartz functions). *Let $u \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$\|W^\varepsilon[u] - \|u\|_{L^2}^2 \cdot \delta(x, k)\|_{\mathbf{A}^{-1}} \leq \|u\|_{L^2} (2\pi \|xu\|_{L^2} + \varepsilon \|\nabla u\|_{L^2}).$$

Proof: For brevity we will denote $W^\varepsilon(x, k) = W^\varepsilon[u](x, k)$, and X, K the Fourier dual variables to x, k . Naturally, the idea of the proof will be to work on the Fourier dual of the variables in which the Lemma is stated, namely we will use the fact that

$$|\langle W^\varepsilon - \|u\|_{L^2}^2 \cdot \delta(x, k), \phi \rangle| = \left| \langle \widehat{W}^\varepsilon(X, K) - \|u\|_{L^2}^2 \widehat{\phi}, \widehat{\phi} \rangle \right|.$$

In what follows we will use the elementary computation

$$\widehat{W}^\varepsilon(X, K) = \mathcal{F}_{(x,k) \rightarrow (X,K)}[W^\varepsilon(x, k)] = \int_x e^{-2\pi i x \cdot X} u(x - \frac{\varepsilon K}{2}) \overline{u(x + \frac{\varepsilon K}{2})} dx. \quad (80)$$

Now observe that, for any $j \in \{1, \dots, n\}$,

$$\begin{aligned} \partial_{K_j} \widehat{W}^\varepsilon(X, K) &= \partial_{K_j} \int_x e^{-2\pi i x \cdot X} u(x - \frac{\varepsilon K}{2}) \overline{u(x + \frac{\varepsilon K}{2})} dx = \\ &= \frac{\varepsilon}{2} \int_x e^{-2\pi i x \cdot X} \left[u(x - \frac{\varepsilon K}{2}) \partial_{x_j} \overline{u(x + \frac{\varepsilon K}{2})} - \overline{u(x + \frac{\varepsilon K}{2})} \partial_{x_j} u(x - \frac{\varepsilon K}{2}) \right] dx \Rightarrow \\ &\Rightarrow |\partial_{K_j} \widehat{W}^\varepsilon(X, K)| \leq \varepsilon \|\nabla u\|_{L^2} \|u\|_{L^2}, \end{aligned} \quad (81)$$

where we used the fact that

$$\left| \int_x e^{-2\pi i x \cdot X} u(x - \frac{\varepsilon K}{2}) \overline{v(x + \frac{\varepsilon K}{2})} dx \right| \leq \|u\|_{L^2} \|v\|_{L^2}$$

by virtue of the Cauchy-Schwartz inequality.

On the other hand, using once again equation (80),

$$\begin{aligned} \frac{i}{\pi} \partial_{X_j} \widehat{W}^\varepsilon(X, K) &= 2 \int_x e^{-2\pi i x \cdot X} x_j u(x - \frac{\varepsilon K}{2}) \overline{u(x + \frac{\varepsilon K}{2})} dx = \\ &= \int_x e^{-2\pi i x \cdot X} \left[(x - \frac{\varepsilon K}{2}) u(x - \frac{\varepsilon K}{2}) \overline{u(x + \frac{\varepsilon K}{2})} + u(x - \frac{\varepsilon K}{2}) (x + \frac{\varepsilon K}{2}) \overline{u(x + \frac{\varepsilon K}{2})} \right] dx \Rightarrow \\ &\Rightarrow |\partial_{X_j} \widehat{W}^\varepsilon(X, K)| \leq 2\pi \|u\|_{L^2} \|x_j u\|_{L^2}. \end{aligned} \quad (82)$$

Combining equations (81) and (82) it follows that

$$\|\nabla_{X,K} \widehat{W}^\varepsilon(X, K)\|_{L_{X,K}^\infty} \leq \|u\|_{L^2} (2\pi \|xu\|_{L^2} + \varepsilon \|\nabla_x u\|_{L^2}). \quad (83)$$

Finally, observe that

$$\widehat{W}^\varepsilon(0, 0) = \|u\|_{L^2}^2, \quad (84)$$

e.g. by evaluating equation (80) at $(X, K) = (0, 0)$. Now we Taylor expand $\widehat{W}^\varepsilon(X, K)$ around $(0, 0)$ to obtain

$$\left| \widehat{W}^\varepsilon(X, K) - \|u\|_{L^2}^2 \right| \leq |(X, K)| \cdot \|\nabla_{X,K} \widehat{W}^\varepsilon\|_{L^\infty} \leq \sqrt{|X|^2 + |K|^2} \|u\|_{L^2} (2\pi \|xu\|_{L^2} + \varepsilon \|\nabla u\|_{L^2}). \quad (85)$$

The proof is completed by integrating against any \mathbf{A}^1 test function ϕ ,

$$\begin{aligned} |\langle W^\varepsilon - \|u\|_{L^2}^2 \cdot \delta(0, 0), \phi \rangle| &= \left| \langle \widehat{W}^\varepsilon(X, K) - \|u\|_{L^2}^2 \widehat{\phi}, \widehat{\phi} \rangle \right| \leq \\ &\leq \|u\|_{L^2} (2\pi \|xu\|_{L^2} + \varepsilon \|\nabla u\|_{L^2}) \int_{X,K} \sqrt{|X|^2 + |K|^2} \widehat{\phi}(X, K) |dXdK| \leq \\ &\leq \|u\|_{L^2} (2\pi \|xu\|_{L^2} + \varepsilon \|\nabla u\|_{L^2}) \|\phi\|_{\mathbf{A}^1}. \end{aligned} \quad (86)$$

□

Corollary 5.2 (Concentration of Wigner transforms to $\delta(\mu_x(\psi), \mu_k(\psi))$ for Sobolev functions). *Let $\psi \in H^1 \cap \widehat{H}^1$, $\|\psi\|_{L^2} = 1$. Then*

$$\|W^\varepsilon[\psi] - \delta(x, k)\|_{\mathbf{A}^{-1}} \leq 2\pi\|x\psi\|_{L^2} + \varepsilon\|\nabla\psi\|_{L^2}, \quad (87)$$

and more generally

$$\|W^\varepsilon[\psi] - \delta(x - \mu_x(\psi), k - \mu_k(\psi))\|_{\mathbf{A}^{-1}} \leq 2\pi\left(\sigma_x(\psi) + \sigma_k(\psi)\right). \quad (88)$$

Proof: The proof of the Corollary consists of two parts: first, we check that the arguments in the proof of Lemma 5.1 still work for $H^1 \cap \widehat{H}^1$ wavefunctions. Then we apply a Galilean transform to obtain concentration on any point of phase-space.

Since $\psi \in H^1(\mathbb{R}^n) \cap \widehat{H}^1(\mathbb{R}^n)$, recall that $W^\varepsilon[\psi] \in H^1(\mathbb{R}^{2n}) \cap \widehat{H}^1(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ by virtue of equation (34). Moreover, equations (81) and (82) mean that $\widehat{W}^\varepsilon[\psi] \in W^{1,\infty}(\mathbb{R}^{2n})$. Therefore the Taylor expansion of equation (85) makes sense as a Taylor expansion in $W^{1,\infty}(\mathbb{R}^{2n})$ [43], and equation (87) follows.

In order to prove equation (88), let us call u the “centered version of ψ ,”

$$u(x) = \mathcal{M}_{\frac{\mu_k(\psi)}{\varepsilon}} \mathcal{T}_{\mu_x(\psi)} \psi = \psi(x + \mu_x(\psi)) e^{-2\pi i \frac{\mu_k(\psi) \cdot x}{\varepsilon}};$$

by construction $\mu_x(u) = \mu_k(u) = 0$. Now observing that

$$\sigma_x(\psi) = \sigma_x(u) = \|xu\|_{L^2}, \quad \sigma_k(\psi) = \sigma_k(u) = \frac{\varepsilon}{2\pi} \|\nabla u\|_{L^2},$$

equation (87) implies that

$$\|W^\varepsilon[u] - \delta(x, k)\|_{\mathbf{A}^{-1}} \leq 2\pi\left(\sigma_x(\psi) + \sigma_\mu(\psi)\right). \quad (89)$$

Moreover,

$$\begin{aligned} W^\varepsilon[u(t)] &= W^\varepsilon[\psi(x + \mu_x(\psi)) e^{-i \frac{2\pi\mu_k(\psi) \cdot x}{\varepsilon}}] = \\ &= \int_y e^{-2\pi i k \cdot y} \psi(x + \frac{\varepsilon y}{2} + \mu_x(\psi)) e^{-i \frac{2\pi\mu_k(\psi) \cdot (x + \frac{\varepsilon y}{2})}{\varepsilon}} \overline{\psi}(x - \frac{\varepsilon y}{2} + \mu_x(\psi)) e^{i \frac{2\pi\mu_k(\psi) \cdot (x - \frac{\varepsilon y}{2})}{\varepsilon}} dy = \\ &= \int_y e^{-2\pi i (k + \mu_k(\psi)) \cdot y} \psi(x + \frac{\varepsilon y}{2} + \mu_x(\psi)) \overline{\psi}(x - \frac{\varepsilon y}{2} + \mu_x(\psi)) dy = W^\varepsilon[\psi](x + \mu_x(\psi), k + \mu_k(\psi)) \end{aligned}$$

and thus (89) means

$$\begin{aligned} &\|W^\varepsilon[\psi](x + \mu_x(\psi), k + \mu_k(\psi)) - \delta(x, k)\|_{\mathbf{A}^{-1}} \leq 2\pi\left(\sigma_x(\psi) + \sigma_k(\psi)\right) \Leftrightarrow \\ &\Leftrightarrow \|W^\varepsilon[\psi](x, k) - \delta(x - \mu_x(\psi), k - \mu_k(\psi))\|_{\mathbf{A}^{-1}} \leq 2\pi\left(\sigma_x(\psi) + \sigma_k(\psi)\right). \end{aligned}$$

□

Acknowledgement

I would like to thank the anonymous referees for their detailed comments and insightful suggestions that helped improve the paper substantially.

References

- [1] M. Agueh, “Gagliardo-Nirenberg inequalities involving the gradient L^2 -norm”, **Comptes Rendus de l’Académie des Sciences - Series I - Mathematics** 346 (2008) pp. 757–762
- [2] L. Ambrosio, G. Friesecke & J. Giannoulis, “Passage from quantum to classical molecular dynamics in the presence of Coulomb interactions”, **Communications in Partial Differential Equations** 35-8 (2010) pp. 1490-1515.

- [3] L. Ambrosio, A. Figalli, G. Friesecke, J. Giannoulis & T. Paul, “Semiclassical limit of quantum dynamics with rough potentials and well posedness of transport equations with measure initial data”, **Communications on Pure and Applied Mathematics** 64-9 (2011) pp. 1199–1242
- [4] A. Athanassoulis, Th. Katsaounis & I. Kyza, “Regularized semiclassical limits: linear flows with infinite Lyapunov exponents”, **Communications in Mathematical Sciences** 14-7 (2016) pp. 1821-1858
- [5] A. Athanassoulis & T. Paul, “Strong and weak semiclassical limits for some rough Hamiltonians”, **Mathematical Models and Methods in Applied Sciences** 12-22 (2012) 1250038
- [6] A. Athanassoulis & T. Paul, “On the selection of the classical limit for potentials with BV derivatives”, **Journal of Dynamics and Differential Equations**, 25-1 (2013) pp. 33–47
- [7] A. Athanassoulis & T. Paul, “Strong phase-space semiclassical asymptotics”, **SIAM Journal on Mathematical Analysis**, 43-5 (2011) pp. 2116-2149
- [8] A. Athanassoulis, T. Paul, F. Pezzotti & M. Pulvirenti, “Strong Semiclassical Approximation of Wigner Functions for the Hartree Dynamics”, **Rendiconti Lincei - Matematica e Applicazioni** 22-4 (2011) pp. 525-552
- [9] A. Athanassoulis, N.J. Mauser & T. Paul, “Coarse-scale representations and smoothed Wigner transforms”, **Journal de Mathématiques Pures et Appliquées** 91-3 (2009) pp. 296-338
- [10] G. Bal, L. Ryzhik & T. Komorowski, “Kinetic limits for waves in a random medium”, **Kinetic and Related Models** 3 (2010) pp. 529-644
- [11] C. Bardos & N. Besse, “About a Variant of the 1d Vlasov equation, dubbed “Vlasov-Dirac-Benney Equation”, **Séminaire Laurent Schwartz - EDP et applications** vol. 2012-13, pp. 1-21
- [12] C. Bardos & N. Besse, “Semi-classical limit of an infinite dimensional system of nonlinear Schrödinger equations”, **Bulletin of the Institute of Mathematics Academia Sinica New Series** 11-1 (2016) pp. 43-61
- [13] N. Benedikter, M. Porta, C. Saffirio & B. Schlein, “From Hartree dynamics to the Vlasov equation”, **Archive for Rational Mechanics and Analysis**, 221-1 (2015) pp. 273-334.
- [14] L. Borcea, G. Papanicolaou & C. Tsogka, “Asymptotics for the space-time Wigner transform with applications to imaging”, in **Stochastic Differential Equations: Theory and Applications (in Honor of Prof. Boris L. Rozovskii)**, Interdisciplinary Mathematical Sciences vol. 2, (2007) pp. 91-112 World Scientific
- [15] A. Bouzouina & D. Robert, “Uniform semiclassical estimates for the propagation of quantum observables”, **Duke Mathematical Journal**, 111-2 (2002) pp. 223-252.
- [16] F. Brezzi & P. Markowich, “The three-dimensional Wigner-Poisson problem: existence, uniqueness and approximation”, **Mathematical Models and Methods in Applied Sciences**, 14 (1991) pp. 35-61
- [17] N. Burq & M. Zworski, “Instability for the Semiclassical Non-linear Schrödinger Equation”, **Communications in Mathematical Physics** 260-1 (2005) pp. 45-58
- [18] R. Carles, C. Fermanian-Kammerer, N. Mauser & H. P. Stimming, “On the time evolution of Wigner measures for Schrödinger equations” **Communications on Pure and Applied Analysis** 8-2 (2009)
- [19] R. Carles, E. Dumas & C. Sparber, “Multiphase Weakly Nonlinear Geometric Optics for Schrödinger Equations” **SIAM Journal on Mathematical Analysis** 42-1 (2010) pp. 489-518
- [20] R. Carles, “Geometric optics and instability for semi-classical Schrödinger equations”, **Archive for Rational Mechanics and Analysis** 183-3 (2007) pp. 525-553
- [21] R. Carles, “Geometric optics with caustic crossing for some nonlinear Schrödinger equations”, **Indiana University Mathematics Journal** 49-2 (2000) pp. 475-552
- [22] R. Carles, C. Fermanian-Kammerer & I. Gallagher, “On the role of quadratic oscillations in nonlinear Schrödinger equations”, **Journal of Functional Analysis** 203-2 (2003) pp. 453-493.
- [23] R. Carles & C. Fermanian-Kammerer, “Nonlinear Coherent States and Ehrenfest Time for Schrödinger Equation”, **Communications in Mathematical Physics** 301-2 (2010) pp. 443-472
- [24] T. Cazenave, “Semilinear Schrödinger equations”, Courant Lecture Notes **American Mathematical Society** Providence RI (2003)
- [25] T. Chen, Y. Hong & N. Pavlović, “Global Well-posedness of the NLS System for infinitely many fermions”, **Archive for Rational Mechanics and Analysis**, 224-1 pp. 91-123
- [26] G. El, E. Khamis & A. Tovbis, “Dam break problem for the focusing nonlinear Schrödinger equation and the generation of rogue waves”, **Nonlinearity** 29-9 (2017) 2798

- [27] A. Faraj & S. Jin, “The Landau-Zener transition and the surface hopping method for the 2D Dirac equation for graphene”, **Communications in Computational Physics**, 21-2 (2017) pp. 313-357
- [28] C. Fermanian-Kammerer & F. Méhats, “A kinetic model for the transport of electrons in a graphene layer”, **Journal of Computational Physics**, 327 (2016) pp. 450-483
- [29] G. Folland & A. Sitaram, “The uncertainty principle: A mathematical survey”, **The Journal of Fourier Analysis and Applications** 3-3 (1997) pp. 207-238
- [30] R. Frank, M. Lewin, E. Lieb & R. Seiringer, “Strichartz inequality for orthonormal functions”, **Journal of the European Mathematical Society** 16-7 (2014) pp. 1507-1526
- [31] P. Gérard, P. Markowich, N. Mauser & F. Poupaud, “Homogenization limits and Wigner transforms”, **Communications on Pure and Applied Mathematics** 50-4 (1997) pp. 323-379
- [32] J. Ginibre, “Introduction aux équations de Schrödinger non linéaires”, **Edition Paris-Onze Paris** (1994)
- [33] F. Golse, C. Mouhot, & T. Paul, “On the Mean-Field and Classical Limits of Quantum Mechanics” **Communications in Mathematical Physics** 343 (2016) pp. 165-205.
- [34] F. Golse & T. Paul, “The Schrödinger Equation in the Mean-Field and Semiclassical Regime”, **Archive for Rational Mechanics and Analysis**, 223-1 (2017) pp. 57-94
- [35] E. Grenier, “Semiclassical limit of the nonlinear Schrödinger equation in small time”, **Proceedings of the American Mathematical Society** 126-2 (1998) pp. 523-530
- [36] R. Jenkins & K. McLaughlin, “The semi-classical limit of focusing NLS for a family of square barrier initial data”, **Communications on Pure and Applied Mathematics** 67-2 (2014) pp. 246-320
- [37] S. Jin, D. Levermore & D. McLaughlin, “The Semiclassical Limit of the Defocusing NLS Hierarchy”, **Communications on Pure and Applied Mathematics** 52-5 (1999) pp. 613-654
- [38] S. Jin, X. Liao & X. Yang, “The Vlasov-Poisson equations as the semiclassical limit of the Schrödinger-Poisson equations: A numerical study”, **Journal of Hyperbolic Differential Equations** 5-3 (2008) pp. 569-587
- [39] S. Jin & X. Li, “Multi-phase computations of the semiclassical limit of the Schrödinger equation and related problems: Whitham vs Wigner” **Physica D: Nonlinear Phenomena** 182-1 (2003) pp. 46-85
- [40] S. Kamvissis, K. McLaughlin & P. Miller, “Semiclassical Soliton Ensembles for the Focusing Nonlinear Schrödinger Equation”, **Princeton University Press** Princeton (2003)
- [41] D. Lannes, “High-frequency nonlinear optics: from the nonlinear Schrödinger approximation to ultrashort-pulses equations”, **Proceedings of the Royal Society of Edinburgh: Section A Mathematics** 141-2 (2011) pp. 253–286
- [42] M. Lewin & J. Sabin, “The Hartree Equation for Infinitely Many Particles I. Well-Posedness Theory”, **Communications in Mathematical Physics** 334-1 (2015) pp. 117-170
- [43] E. Lieb & M. Loss, “Analysis” 2nd Ed. **American Mathematical Society** Providence RI (2001)
- [44] P. Lions & T. Paul, “Sur les mesures de Wigner”, **Revista Matemática Iberoamericana** 9-3 (1993) pp. 553-618
- [45] W. Martin & P. Flandrin, “Wigner-Ville spectral analysis of nonstationary processes”, **IEEE Transactions on Acoustics, Speech and Signal Processing** 33-6 (1985) pp. 1461-1470
- [46] L. Miller, “Refraction of high-frequency waves density by sharp interfaces and semiclassical measures at the boundary”, **Journal de mathématiques pures et appliquées** 79-3 (2000) pp. 227-269
- [47] O. Pinaud, “Classical Limit for a System of Non-Linear Random Schrödinger Equations”, **Archive for Rational Mechanics Analysis**, 209-1 (2013) pp. 321-364
- [48] L. Ryzhik, G. Papanicolaou & J. Keller, “Transport equations for elastic and other waves in random media”, **Wave Motion** 24-4 (1996) pp. 327-370
- [49] B. Simon, “They Weyl transform and L^p functions on phase-space”, **Proceedings of the American Mathematical Society**, 116-4 (1992) pp. 1045-1047
- [50] P. Smit & T. Janssen, “The Evolution of Inhomogeneous Wave Statistics through a Variable Medium”, **Journal of the American Meteorological Society** 43-8 (2013) pp. 1741-1758
- [51] P. Smit, R. Bland, T. Janssen & B. Laughlin, “Remote sensing of nearshore wave interference” **Journal of Geophysical Research: Oceans** 121-5 (2016) pp. 34093421
- [52] A. Tovbis, S. Venakides & X. Zhou, “On semiclassical (zero dispersion limit) solutions of the focusing nonlinear Schrödinger equation”, **Communications on Pure and Applied Mathematics** 57-7 (2004) pp. 877–985

- [53] Y. Zheng & A. Majda, “Existence of global weak solutions to one-component Vlasov-Poisson and Fokker-Planck-Poisson systems in one space dimension with measures as initial data”, **Communications on Pure and Applied Mathematics** 48-10 (1994) pp. 1365-1401
- [54] P. Zhang, Y. Zheng & N. Mauser, “The Limit from the Schrödinger-Poisson to the Vlasov-Poisson Equations with General Data in One Dimension”, **Communications on Pure and Applied Mathematics** 55-5 (2002) pp. 582-632