Bi-fractional transforms in phase space

Sanfo David AGYO

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Faculty of Engineering and Informatics
School of Electrical Engineering and Computer Science

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Sanfo D. Agyo

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Abstract

The displacement operator is related to the displaced parity operator through a two-dimensional Fourier transform. Both operators are important operators in phase space and the trace of both with respect to the density operator gives the Wigner functions (displaced parity operator) and Weyl functions (displacement operator). The generalisation of the parity-displacement operator relationship considered here is called the bi-fractional displacement operator, $\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)$. Additionally, the bi-fractional displacement operators lead to the novel concept of bi-fractional coherent states.

The generalisation from Fourier transform to fractional Fourier transform can be applied to other phase space functions. The case of the Wigner-Weyl function is considered and a generalisation is given, which is called the bi-fractional Wigner functions, $\mathcal{H}(\alpha, \beta; \theta_\alpha, \theta_\beta)$. Furthermore, the $Q-$function and $P-$function are also generalised to give the bi-fractional $Q-$functions and bi-fractional $P-$functions respectively. The generalisation is likewise applied to the Moyal star product and Berezin formalism for products of non-commutating operators. These are called the bi-fractional Moyal star product and bi-fractional Berezin formalism.

Finally, analysis, applications and implications of these bi-fractional transforms to the Heisenberg uncertainty principle, photon statistics and future applications are discussed.

Dedication

I would like to dedicate this thesis first to God almighty who gave me the grace to work hard. Also to my wife and friend Mayowa Agyo for all her kisses and support. To the loving memory of my Dad who was/is my mentor and for motivating me towards a PhD. To my Mama, for the insomnia this research has caused you, on bended knees at every waking thought of me. To my siblings and all my family members for the support and encouragement. Finally to my supervisors, Prof A. Vourdas for being a father figure and motivating supervisor, as well as Dr Ci, for the help especially in MATLAB and for both his sea-calming words and his Midas touch.

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other University. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration, except where specifically indicated in the text. Part of this research has been published in the following work:

Journals

- Agyo, S., Lei, C., and Vourdas, A. (2015b). *Interpolation between phase space quantities with bifractional displacement operators.* Physics Letters A, 379(4):255-260.
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List of notations

Notation	
$\hbar$	Planck's constant
m	Mass of a particle
$\hat{x}$	Position operator
$\hat{p}$	Momentum operator
$L_n(x)$	Laguerre polynomial
$H_n(x)$	Hermite polynomial
$K(\alpha, \beta; \theta_\alpha)$	Kernel of the fractional Fourier transform
$\widehat{W}(\alpha, \beta\|\Theta)$	Weyl function
$W(\alpha, \beta\|\Theta)$	Wigner function
$\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta\|\Theta)$	Bi-fractional displacement operator
$W(\alpha, \beta; \theta_\alpha, \theta_\beta\|\Theta)$	Bi-fractional Wigner function
$\mathcal{G}_S(\alpha - \alpha', \beta - \beta'\|\theta_\alpha, \theta_\beta)$	Bi-fractional distance between bi-fractional coherent states
$\mathcal{B}(z, w^*; \theta_\alpha, \theta_\beta\|\Theta)$	Bi-fractional Berezin formalism

List of abbreviations

HW	Heisenberg Weyl Group
SU(1,1)	Special Unitary Group (1,1)
SU(2)	Special Unitary Group (2)
GRIN	Graded Index Media
LCT	Linear Canonical Transform

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Chapter 1

Introduction

1.1 Background

Quantum mechanics is one of the theories of nature; it explains nature at the microscopic level. Theories governed by Newtonian physics are frequently termed classical mechanics or theories, but theories based on quantum principles are referred to as Quantum mechanics. While a classical system can be localised, quantum mechanics considers the probability of localisation and obeys Heisenberg's uncertainty principle. There are two common perspectives with respect to quantum mechanics, Schrödinger and Wigner-Weyl [22].

Schrödinger's view of quantum mechanics is based on the probability of finding a particle thus stating that it is impossible to locate a particle in both position and momentum. The distribution of these probabilities is determined by wave-functions. The Schrödinger equation is a linear partial differential equation that describes a time evolution of the wave-function. A complementary approach is the Wigner-Weyl perspective which helps in calculating the classical limit for $\hbar \to 0$ in obtaining classical mechanics. The classical limit is helpful in understanding the non-commutativity property of quantum mechanics with applications to Moyal star product and Berezin formalism for two non-commutating operators. The Wigner-Weyl perspective leads to the Wigner function and Weyl function. The Wigner function also has its drawbacks as it is not a proper probability distribution (quasi- or semi-probability distribution) since it is not positive in every case. More notes and explanation on the Wigner-Weyl formalism can be found in [27].

For the context of this thesis, the focus is on Wigner's perspective to quantum mechanics. Furthermore, the Wigner function and its generalisation will be widely applied in phase space methods.

Two important operators in phase space methods, are the displacement operator and the parity operator [18, 39, 73]. They are related to each other through a two-dimensional Fourier transform. In this work, the two Fourier transforms are replaced with two fractional Fourier transforms [10, 51, 57, 70], leading to new unitary operators called the bi-fractional displacement operators $\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)$ [5]. Both the displacement operator and parity operator are special cases of these more general operators, $\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)$. The bi-fractional displacement operators, $\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)$ which are introduced with an interpolation motivation are elements of the group $G = HW \rtimes SU(1,1)$ and contains both displacements and squeezing transformations [6, 74]. However, the general element of G cannot always be written in the form $\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)$. The proof of this is rather complex, and is presented in section (3.6.1.4).

The bi-fractional displacement operators can further be used to get new generalisations called the bi-fractional coherent states, bi-fractional Wigner functions, and bi-fractional Moyal star product.

The ordinary coherent states are well known as the eigenstates of the displacement operator [6, 45, 62] and they play a central role in phase space methods [64, 80]. Various generalisations of these ordinary coherent states have been studied, especially in connection with groups like $SU(2)$, $SU(1,1)$, etc. These concepts ($SU(2)$, $SU(1,1)$) are not considered in this work as the generalisation treated is in context of the bi-fractional coherent states over $(\theta_\alpha, \theta_\beta)$.

The new bi-fractional coherent states are different from the ordinary coherent state and are derived by acting the bi-fractional displacement operators on the vacuum state. This is an improvement on the previous method of deriving the Glauber coherent state which is by acting the displacement operator on the vacuum state. In this new formalism, these bi-fractional coherent states are different for each pair of $(\theta_\alpha, \theta_\beta)$, such that the Glauber coherent state is a special case of the bi-fractional coherent states when $(\theta_\alpha = \theta_\beta = \frac{\pi}{2})$.

Using the bi-fractional displacement operators, the bi-fractional Wigner functions, $\mathcal{H}(\alpha, \beta; \theta_\alpha, \theta_\beta)$ are introduced. Both the Wigner and Weyl functions are special cases of these more general functions for $(\theta_\alpha, \theta_\beta)$ equals $(0,0)$ and $(\frac{\pi}{2}, \frac{\pi}{2})$ respectively. Examples of different functions for different angles of $(\theta_\alpha, \theta_\beta)$ are calculated numerically. These interpolations between displaced parity-displacement operators for the bi-fractional displacement operators and those between Wigner-Weyl for bi-fractional Wigner function provide a deeper insight into the phase space formalism.

1.2 Motivation

There are quite a number of cases of generalisations in phase space methods. The characteristic function, for example, generalises and interpolates between the P−function, Q−function and Wigner functions [37]. Another form of generalisation is the Fourier transform to fractional Fourier transform [53, 57]. Since many phase space formulations are known to be related through Fourier transform, a generalisation to the fractional Fourier transform will lead to novel quantities. Such generalisation can be applied to different aspects of phase space methods which before now were limited to the confines of Fourier transform. Other areas where fractional Fourier generalisations have been very efficient include signal analysis, image encryption and noise reduction [8, 9, 29, 35, 42, 57, 57, 70, 79]. Thus, the impact of the fractional Fourier transform is explored with respect to the Wigner and Weyl functions which are related through a two-dimensional Fourier transform. This generalisation gives other functions in phase space which are neither Wigner nor Weyl functions. These functions are referred to as 'bi-fractional Wigner functions'. It is to be noted that interpolation for characteristic functions is due to a constant with values $0, 1, -1$; however in this case it is based on two angles $\theta_\alpha, \theta_\beta$. The Wigner and Weyl functions are at $(\theta_\alpha = \theta_\beta = \frac{\pi}{2})$ and $(\theta_\alpha = \theta_\beta = 0)$ respectively. The bi-fractional Wigner functions give a wider spectrum of possibilities, $[0, 2\pi]$ for both angles $\theta_\alpha = \theta_\beta$ as compared to the characteristic function which is limited to three possibilities.

Further consideration is given to the displacement operator and parity operator, related through a Fourier transform and they are likewise generalised. The generalisation is extended to the Glauber coherent state to give the bi-fractional coherent states, with each pair of $\theta_\alpha, \theta_\beta$ creating a family of coherent states. Of great importance is the fact that a new paradigm is introduced by the interpolation between phase space quantities which can lead to in-between concepts or intermediary studies at fractional levels of these phase space quantities. Thus, an introduction to these functions available at intermediaries of $\theta_\alpha, \theta_\beta$ is formulated. For such intermediaries, these functions are some sort of superposition of both Wigner and Weyl functions. And for the case of operators, superpositions of displaced parity and displacement operators.

Also, this generalisation naturally extends to other phase space functions which are studied in the Berezin formalism and Moyal star product. The study of the Berezin formalism and Moyal star product helps in advancing the research in classical limits which fuels better understanding of quantum theory. Other areas that could benefit from this research include quantum physics, quantum information theory, quantum cryptography, quantum computing, quantum chemistry.

1.3 Aims and Objectives

The following details the novel aspects of the thesis:

1. Introduce the concept of bi-fractional transforms in phase space by generalising the displaced parity operators and displacement operators to give a family of operators called the bi-fractional displacement operators over $(\theta_\alpha, \theta_\beta)$.

2. To extend this concept of bi-fractional transformations to phase space functions and give new fractional Fourier transform generalisation of the Wigner, $Q-$ and $P-$ functions over $(\theta_\alpha, \theta_\beta)$. These each give a family of functions over $(\theta_\alpha, \theta_\beta)$, called the bi-fractional Wigner functions, bi-fractional $Q-$functions and bi-fractional $P-$functions.

3. To give analysis of these functions in the context of Heisenberg's uncertainty principle and photon statistics (bunching and anti-bunching) which helps in deeper understanding of the quantum nature of light.

4. Explain the physical implication of bi-fractional Wigner function with respect to noise and correlation. Both noise and correlation are independent concepts considered before now in terms of the Wigner and Weyl functions respectively. However in the context of the bi-fractional Wigner function they are duals of the same concept and interpolation between both is possible.

5. Take numerical examples of these interpolations especially the bi-fractional Wigner functions and show the interpolation effect of intermediaries between Wigner

functions at ($\theta_\alpha = \theta_\beta = \frac{\pi}{2}$) and Weyl functions at ($\theta_\alpha = \theta_\beta = 0$). Analysis is carried out for special cases when $\theta_\alpha = \theta_\beta$ and when both angles are not equal.

1.4 Overview of Thesis

In Chapter 2, a brief review of quantum mechanics in infinite Hilbert space is presented; stating various background concepts such as Dirac delta functions, Fourier transform, Weyl and Wigner functions, Q-function, P-function and Moyal star product.

Chapter 3 introduces the fractional Fourier transforms, bi-fractional displacement operators and bi-fractional coherent states. The bi-fractional displacement operators do not form a group, and the concept of groupoid is used in studying them. Key properties like the marginal property and other properties of the bi-fractional displacement operator are analytically derived. For the bi-fractional coherent states, the analyticity property is given and a new formulation for the bi-fractional distance between bi-fractional coherent states is proved. The Bargmann function of special bi-fractional coherent states are treated and numerical analysis is performed for these examples. Further analysis shows the change in uncertainty and also the second correlation function for different angles of $(\theta_\alpha, \theta_\beta)$

In Chapter 4, the concepts of the bi-fractional displacement operators and bi-fractional coherent states are extended to other state space functions to give the bi-fractional Wigner functions, bi-fractional $Q-$functions and bi-fractional $P-$functions. The bi-fractional Laplacian is stated as a new generalisation of the conventional Laplacian and this is used to get the Berezin formalism for two operators in the context of bi-fractional coherent states. Similarly, fractional generalisations to the bi-fractional Moyal star product for two operators with respect to bi-fractional Wigner functions is derived. The proof is given so that the bi-fractional Moyal star product reduces to the ordinary Moyal star product of Chapter 2.

In Chapter 5, possible applications of bi-fractional transformations to other phase space functions are proposed as future work and a conclusion is presented.

Chapter 2

Quantum mechanics in Hilbert space

2.1 Introduction

This chapter begins with a detailed introduction to basic formalism of quantum mechanics. Quantum mechanics is considered in terms of phase space formulation. Therefore, considering the quantum harmonic oscillator on the basis of the position and momentum states on a phase space $\mathbb{R}\times\mathbb{R}$ with states on the Hilbert space $\mathcal{H}$. The harmonic oscillator provides a good introduction to the concept of the creation and annihilation operators. States in phase space formalism are considered including the number state, coherent state and squeezed state. We also discuss the Wigner as a quasi-probability distribution that contains information about the state of a system. Weyl function and its relationship to the Wigner are also observed. Finally, the Husimi Q- function and the Glauber-Sudarshan P-function are stated.

Various postulates [36, 38] have been developed for quantum mechanics and some of them include:-

- The state of a particle is represented by a ket vector in Hilbert space.

- Linear Hermitian operators are the equivalents of classical observables. Thus

$$x \to \hat{x}$$
$$p \to \hat{p} \tag{2.1}$$

  where x and p represent position and momentum and $\hat{x}$ and $\hat{p}$ are the momentum and position operators.

- Measurement of the observable with respect to an operator $\hat{\Omega}$ will result in values Ω, such that

$$\hat{\Omega}\,|\psi\rangle = \Omega\,|\psi\rangle \tag{2.2}$$

  Ω representing the eigenvalue.

- If a system is described by the normalised function $|\psi\rangle$ then the average value of the observable corresponding to an operator Ω is

$$\langle\Omega\rangle = \langle\psi|\hat{\Omega}|\psi\rangle\,. \tag{2.3}$$

We note that in quantum mechanics, the phase space formalism is described by the position and momentum coordinates [37, 47].

2.2 Quantum harmonic oscillator

The position and momentum of a particle can be easily computed in classical mechanics at any given time; however in quantum mechanics this is done through the wave function $\psi(x,t)$. It has a statistical interpretation such that $|\psi(x,t)|^2$ gives the probability of finding the particle at position x at time t, such that $\int dx|\psi(x,t)|^2 = 1$. The vibration of the harmonic oscillator produces energy levels at equally spaced intervals and these energy intervals can be shown using the solution of Schrödinger′s equation,

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \frac{\hat{p}^2\psi(x,t)}{2m} + \widehat{V}\psi(x,t) \tag{2.4}$$

where $\hbar$ and m are constants representing Planck's constant and mass respectively. Also,

$$\begin{aligned} \widehat{V} &= \frac{1}{2}kx^2, \\ \hat{x} = x, \;\; \hat{p} &= -i\hbar\frac{\partial}{\partial x} \end{aligned} \tag{2.5}$$

are the potential energy, position operator and momentum operator respectively. In addition, k is the constant force. We note that the position and momentum operators do not commute. This is evident because quantum-mechanical operators do not in general commute. The position and momentum operators satisfy the relation

$$[\hat{x}, \hat{p}] = i\hbar \tag{2.6}$$

Upon separating the wave function in terms of position and time, the Schrödinger equation can be resolved into the eigenvalue problem. This is given in terms of the Hamiltonian and energy,

$$\widehat{H}\psi(x) = E\psi(x). \tag{2.7}$$

The Hamiltonian, $\widehat{H}$ and its Energy, E can be given as,

$$\widehat{H} = \frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2; \quad \omega = \sqrt{\frac{k}{m}}$$
$$\left(\frac{1}{2m}\hat{p}^2 + \frac{m\omega^2}{2}\hat{x}^2\right)\psi(x) = E\psi(x) \tag{2.8}$$

The Hamiltonian leads to the introduction of the creation ($\hat{a}$) and annihilation($\hat{a}^\dagger$) operators given as,

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i\hat{p}}{\sqrt{2m\hbar\omega}} \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i\hat{p}}{\sqrt{2m\hbar\omega}}, \tag{2.9}$$

where,

$$\hat{a}^\dagger\hat{a} = \frac{m\omega x^2}{2\hbar} + \frac{\hat{p}^2}{2\hbar m\omega} + \frac{i}{2\hbar}[\hat{x},\hat{p}] = \frac{\widehat{H}}{\hbar\omega} - \frac{1}{2} \tag{2.10}$$

Assuming $m = \omega = \hbar = 1$, the simplified form is shown to be

$$\hat{a}^\dagger = \frac{\hat{x} - i\hat{p}}{\sqrt{2}}, \quad \hat{a} = \frac{\hat{x} + i\hat{p}}{\sqrt{2}} \tag{2.11}$$

The creation and annihilation operators lead to the number operator $\hat{n} = \hat{a}^\dagger a$, and thus the Hamiltonian written in terms of the number operator as,

$$\widehat{H} = \hbar\omega(\hat{n} + 1/2). \tag{2.12}$$

The electromagnetic energy eigenstates are called the Fock or number states which will be discussed later, and given as a ket representation, $|n\rangle$. The ground state of an electromagnetic system is defined as the state with the lowest energy. The ground state can be defined as,

$$\langle x|0\rangle = \pi^{-1/4} e^{-\frac{x^2}{2}}, \tag{2.13}$$

and the definition for a wave function with respect to position state and momentum state is,

$$\psi(x) = \langle x|\psi\rangle\,. \tag{2.14}$$

Also the annihilation operator acts on the vacuum state to give,

$$\hat{a}\,|0\rangle = 0. \tag{2.15}$$

A collection of properties of the ladder operators for various ordering is well documented in the literature [66]. One of these properties is the commutation relation,

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{a}, \hat{a}^\dagger\hat{a}] = \hat{a}, \quad [\hat{a}^\dagger, \hat{a}^\dagger\hat{a}] = -\hat{a}^\dagger, \tag{2.16}$$

proving that the ladder operators do not commute.

2.3 States of the harmonic oscillator

2.3.1 Position and momentum states

In phase space methods the position and momentum operators can be defined through their eigenstates

$$\hat{x}\,|x\rangle = x\,|x\rangle\,, \qquad \hat{p}\,|p\rangle = p\,|p\rangle\,, \tag{2.17}$$

where x and p represent the eigenvalues for each case. The eigenstate of the position and momentum operator form a complete set such that,

$$\int dx\,|x\rangle\langle x| = \int dp\,|p\rangle\langle p| = \mathbb{1}. \tag{2.18}$$

With the following inner product resulting to,

$$\begin{aligned}\langle x|x'\rangle &= \delta(x - x'),\\ \langle p|p'\rangle &= \delta(p - p'),\\ \langle x|p\rangle &= (2\pi)^{-1/2} e^{ixp}.\end{aligned} \tag{2.19}$$

where δ denotes the Dirac delta function. Therefore using Eqs. (2.18, 2.19), any arbitrary state $|\psi\rangle$ can be represented in terms of the position or momentum state

$$\int dx\,|x\rangle\langle x|\psi\rangle = \int dp\,|p\rangle\langle p|\psi\rangle. \tag{2.20}$$

Thus, the wave function of the momentum state can be represented as the Fourier transform of the position state. We will discuss the Fourier transform extensively later on.

$$\widetilde{\psi}(p) = \int dx\psi(x)\exp[-ixp]. \tag{2.21}$$

This is true because of Eq. (2.19). The normalised position and momentum representation with respect to the vacuum state,

$$\langle x|0\rangle = \frac{1}{\pi^{1/4}}e^{-\frac{x^2}{2}}, \qquad \langle p|0\rangle = \frac{1}{\pi^{1/4}}e^{-\frac{p^2}{2}}. \tag{2.22}$$

2.3.2 Number state

The creation operator is used to obtain the excited state also called the number state,

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^{\dagger})^{n}|0\rangle . \tag{2.23}$$

It is easily proved from Eqs. (2.26, 2.16),

$$\begin{gathered}[\hat{a},(\hat{a}^{\dagger})^{n}] = \hat{a}(\hat{a}^{\dagger})^{n} - (\hat{a}^{\dagger})^{n}\hat{a} = n(\hat{a}^{\dagger})^{n-1} \\ |n\rangle = \frac{\hat{a}^{\dagger}|n-1\rangle}{\sqrt{n}} = \frac{\hat{a}^{\dagger}}{\sqrt{n}}\frac{\hat{a}^{\dagger}|n-2\rangle}{\sqrt{n-1}}.............. = \frac{(\hat{a}^{\dagger})^{n}}{\sqrt{n!}}|0\rangle .\end{gathered} \tag{2.24}$$

The solution to Eq. (2.7) is given by the Gauss-Hermite functions ($H_n(x)$), which is a position representation of the number state,

$$\langle x|n\rangle = \psi_{n}(x) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{2^{n}n!}}H_{n}(x)e^{-\frac{x^2}{2}}, \tag{2.25}$$

where $H_n(x)$ is the Hermite polynomial. Using the Dirac notation, certain properties of the a and $a^{\dagger}$ can be shown, by acting on the number state,

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \qquad \hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle \tag{2.26}$$

A full list of properties for different permutations of $\hat{a}$ and $\hat{a}$ acting on the number state is given in [66]. Given that the wavefunction is orthogonal, then,

$$\delta_{n,m} = \langle n|m\rangle = \left\{\begin{array}{ll} 0, & n \neq m \\ 1, & n = m \end{array}\right\}.$$

This leads to the closure relation of the wavefunction for the harmonic oscillator which is complete

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = \mathbb{1}. \tag{2.27}$$

2.4 The density operator

The density operator is used in describing microscopic systems [21], and in the context of quantum mechanics, it conceptually describes the quantum state instead of using the state vector. The density operator encodes all the information of a quantum system. For a particle described by the state ψ, the probability density of locating the particle at position x is given as,

$$P = \langle x|\psi\rangle \langle\psi|x\rangle = |\psi(x)|^2, \tag{2.28}$$

where $\hat{\Theta} = |\psi\rangle \langle\psi|$ and for pure state has the following properties,

1	Projection	$\hat{\Theta}^2 = \Theta$
2	Hermiticity	$\hat{\Theta}^\dagger = \Theta$
3	Normalisation	$\mathrm{Tr}[\hat{\Theta}] = 1$
4	Positivity	$\hat{\Theta} \geq 0$

Table 2.1 Properties of density operator for pure states

Properties 1 and 2 in the Table (2.1) are the projector and hermiticity of the density operator. For a normalised density operator, property 3 is satisfied,

$$\langle\Theta\rangle = \mathrm{Tr}[\hat{\Theta}^2] = \mathrm{Tr}[\hat{\Theta}] = 1. \tag{2.29}$$

Due to the fact that probability must range from 0 to 1, property 4 is proved because the eigenvalues of $\hat{\Theta}$ must be greater than or equal to zero.

A system is said to be pure if there is complete knowledge about that system. Consider an ensemble of objects where all the states are in the same state, then the ensemble is said to be pure. A mixed state is the probabilistic combination of various pure states, where all the objects in the ensemble are in the same state and thus can be written as a weighted or convex sum,

$$\hat{\Theta} = \sum_{k=1}^{N} p_k \left|\psi_k\right\rangle \left\langle\psi_k\right| . \tag{2.30}$$

with $|\psi_k\rangle$ is the pure state and p_k represents the weight which satisfies the condition,

$$0 < p_k < 1 \qquad \sum_{k=1}^{N} p_k = 1, \tag{2.31}$$

indicating that a mixed state is far from a pure state. A pure state contains only quantum noise while a mixed state has both quantum and classical noise. For more discussion on pure and mixed state, refer to [63].

2.5 Displacement and displaced parity operators

2.5.1 Displacement operators

In phase-space formalism, the displacement operator moves a particle across the position and momentum axis. It is a unitary operator and is given by,

$$\widehat{D}(z) = \exp(za^\dagger - z^* a). \tag{2.32}$$

The displacement operator can also be written, using $z = \alpha + i\beta$

$$\widehat{D}(\alpha,\beta) = \exp(i\sqrt{2}\beta\hat{x} - i\sqrt{2}\alpha\hat{p}). \tag{2.33}$$

The displacement operator is not hermitian,

$$\widehat{D}^{\dagger}(\alpha,\beta) = [\widehat{D}(\alpha,\beta)]^{*} = \widehat{D}(-\alpha,-\beta), \tag{2.34}$$

and the product of two such operators becomes

$$\widehat{D}(\alpha_1,\beta_1)\widehat{D}(\alpha_2,\beta_2) = \widehat{D}(\alpha_1+\alpha_2,\beta_1+\beta_2)\exp[i(\beta_1\alpha_2 - \alpha_1\beta_2)]. \tag{2.35}$$

It can also show that with a phase factor such as $\widehat{D}(\alpha,\beta,\gamma) = D(\alpha,\beta)e^{i\gamma}$, where $\alpha,\beta,\gamma \,\epsilon\, \mathbb{R}$, then $\widehat{D}(\alpha,\beta,\gamma)$ forms a representation of the Heisenberg Weyl group. This is obvious by confirming the closure, identity, inverse and associative properties given as

Properties of displacement operators	
Identity	$\widehat{D}(0,0,0) = \mathbb{1}$
Closure	$\widehat{D}(\alpha_1,\beta_1,\gamma_1)\widehat{D}(\alpha_2,\beta_2,\gamma_2) = \widehat{D}(\alpha_3,\beta_3,\gamma_3)$
Inverse	$\widehat{D}^{\dagger}(\alpha_1,\beta_1,\gamma_1)\widehat{D}(\alpha_1,\beta_1,\gamma_1) = \mathbb{1}$
Associative	$\widehat{D}(\alpha_1,\beta_1,\gamma_1)[\widehat{D}(\alpha_2,\beta_2,\gamma_2)\widehat{D}(\alpha_3,\beta_3,\gamma_3)] = [\widehat{D}(\alpha_1,\beta_1,\gamma_1)\widehat{D}(\alpha_2,\beta_2,\gamma_2)]\widehat{D}(\alpha_3,\beta_3,\gamma_3)$

Table 2.2 Properties of displacement operator

The displacement operator has marginal properties so that integration with respect to position (α) gives the momentum component (β),

$$\frac{1}{\sqrt{2}\pi}\int d\alpha\; \widehat{D}(\alpha,\beta) = \left|\beta/\sqrt{2}\right\rangle\left\langle -\beta/\sqrt{2}\right| \tag{2.36}$$

And with respect to momentum (β) gives the position component,

$$\frac{1}{\sqrt{2}\pi}\int d\beta\ \widehat{D}(\alpha,\beta)=\left|\alpha/\sqrt{2}\right\rangle\left\langle-\alpha/\sqrt{2}\right| \tag{2.37}$$

And with repect to both position and momentum we get the parity operator,

$$\frac{1}{2\pi}\int d\alpha d\beta\ \widehat{D}(\alpha,\beta)=\widehat{P}(0,0)=\int d\alpha\left|\alpha\right\rangle\left\langle-\alpha\right| \tag{2.38}$$

where $\widehat{P}(0,0)$ is the parity operator which is introduce later in this work as a special case of the fractional Fourier transform. The marginal properties in Eq. (2.36) are proved using the following important relations,

$$\begin{aligned} e^{-i\beta\hat{x}}\left|x\right\rangle &= e^{-i\beta x}\left|x\right\rangle, \qquad e^{-i\beta\hat{x}}\left|p\right\rangle=\left|p-\beta\right\rangle \\ e^{-i\alpha\hat{p}}\left|x\right\rangle &= \left|x+\alpha\right\rangle, \qquad e^{-i\alpha\hat{p}}\left|p\right\rangle=e^{-i\alpha p}\left|p\right\rangle \end{aligned} \tag{2.39}$$

The displacement operator helps provide structured matrices which have found use in diverse fields to obtain computational improvements [67]. For instance in data approximation, Vandermonde matrices are used, Cauchy matrices are used for error-correction in coding theory and Toeplitz matrices for image restoration in image processing. We give the matrix elements of the displacement operator with respect to the position and momentum states [18],

$$\begin{aligned} \langle x|\widehat{D}(\alpha,\beta)|p\rangle &= \sqrt{2\pi}\exp\left[i(xp-\sqrt{2}\alpha p+\sqrt{2}\beta x-\alpha\beta)\right] \\ \langle p|\widehat{D}(\alpha,\beta)|x\rangle &= \sqrt{2\pi}\exp\left[i(-xp-\sqrt{2}\alpha p+\sqrt{2}\beta x+\alpha\beta)\right]. \end{aligned} \tag{2.40}$$

With respect to single states,

$$\langle x|\widehat{D}(\alpha,\beta)|x'\rangle = \exp[i\beta(x+x')/\sqrt{2}] \times \delta(x-x'-\sqrt{2}\alpha)$$
$$\langle p|\widehat{D}(\alpha,\beta)|p'\rangle = \exp[-i\beta(p+p')/\sqrt{2}] \times \delta(p-p'-\sqrt{2}\beta) \qquad (2.41)$$

Both Eqs. (2.40, 2.41) invariably lead to the relation for the trace of the displacement operator [1]

$$\mathrm{Tr}[\widehat{D}(\alpha,\beta)] = \pi\delta(\alpha)\delta(\beta); \quad \mathrm{Tr}[\widehat{D}(\alpha,\beta)\widehat{D}^{\dagger}(\alpha',\beta')] = \pi\delta(\alpha-\alpha')\delta(\beta-\beta'). \qquad (2.42)$$

The displacement operator can also act on other operators. This is done using the Baker-Campbell-Hausdorff operator relation for two operators given as,

$$e^{\hat{A}}\hat{B}e^{\hat{A}} = \hat{B} + \frac{[\hat{A},\hat{B}]}{1!} + \frac{[\hat{A},[\hat{A},\hat{B}]]}{2!} + \frac{[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]}{3!}.......$$
$$e^{-\hat{A}}\hat{B}e^{\hat{A}} = \hat{B} - \frac{[\hat{A},\hat{B}]}{1!} + \frac{[\hat{A},[\hat{A},\hat{B}]]}{2!} - \frac{[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]}{3!}........ \qquad (2.43)$$

Thus with respect to the creation and annihilation operators the relation is shown to be,

$$\widehat{D}(\alpha,\beta)\hat{a}^{\dagger}\widehat{D}^{\dagger}(\alpha,\beta) = \hat{a}^{\dagger} - z^{*}, \qquad \widehat{D}^{\dagger}(\alpha,\beta)\hat{a}^{\dagger}\widehat{D}(\alpha,\beta) = \hat{a}^{\dagger} + z^{*},$$
$$\widehat{D}(\alpha,\beta)\hat{a}\widehat{D}^{\dagger}(\alpha,\beta) = \hat{a} - z, \qquad \widehat{D}^{\dagger}(\alpha,\beta)\hat{a}\widehat{D}(\alpha,\beta) = \hat{a} + z, \qquad (2.44)$$

where $z = \alpha + i\beta$. Furthermore, the matrix elements of the displacement operator can be given with respect to number states.

$$\langle m|\widehat{D}(\alpha,\beta)|n\rangle = \left(\frac{n!}{m!}\right)^{1/2} (\alpha+i\beta)^{m-n} \exp\left(-\frac{|\alpha+\beta|^2}{2}\right) L_n^{m-n}(|\alpha^2+\beta^2|^2) \qquad (2.45)$$

where L_n^{m-n} represents the Laguerre polynomial. The proof of the formula $\langle m|\widehat{D}(\alpha,\beta)|n\rangle$

is shown in Appendix A.

2.5.2 Parity and displaced parity operators

The parity operation is a transformation about a point (commonly the origin) such that $\widehat{P}(0,0)\left|z\right\rangle = \left|-z\right\rangle$ and is hermitian $\widehat{P}(0,0) = \widehat{P}(0,0)^\dagger$ and $\widehat{P}(0,0)\widehat{P}(0,0)^\dagger = \mathbb{1}$. In this light, the parity operator acts on the coherent state and is defined as,

$$\widehat{P}(0,0)\left|z\right\rangle = \left|-z\right\rangle, \quad \widehat{P}(0,0) = e^{i\pi a^\dagger a} = \sum_N (-1)^N \left|N\right\rangle \left\langle N\right|, \tag{2.46}$$

from which it is easily proved that,

$$\widehat{P}(0,0)\hat{a}\widehat{P}(0,0)^\dagger = -a, \quad \widehat{P}(0,0)^\dagger \hat{a}^\dagger \widehat{P}(0,0) = -a^\dagger. \tag{2.47}$$

It also causes a rotation in the $x-p$ plane [23] as shown,

$$\begin{aligned} \widehat{P}(0,0)\left|x\right\rangle = \left|-x\right\rangle, \quad \widehat{P}(0,0)\left|p\right\rangle = \left|-p\right\rangle \\ \widehat{P}(0,0)\hat{x}\widehat{P}(0,0)^\dagger = -\hat{x}, \quad \widehat{P}(0,0)\hat{p}\widehat{P}(0,0)^\dagger = -\hat{p} \end{aligned} \tag{2.48}$$

Furthermore, the displaced parity operator [18] which is a parity transformation around the point $z := (\alpha, \beta)$ is defined as

$$\begin{aligned} \widehat{P}(\alpha,\beta) &= D(\alpha,\beta)\widehat{P}(0,0)D^\dagger(\alpha,\beta) \\ &= D[2(\alpha,\beta)]\widehat{P}(0,0) \\ &= \widehat{P}(0,0)D^\dagger[2(\alpha,\beta)]. \end{aligned} \tag{2.49}$$

It will be shown later that the displaced parity operator can be used to derive the Wigner function. The displaced parity operators also has the property that,

$$\widehat{P}(\alpha_1,\beta_1)\widehat{P}(\alpha_2,\beta_2) = \widehat{D}[2(\alpha_1-\alpha_2),2(\beta_1-\beta_2)]\exp\left[4i(\alpha_2\beta_1-\alpha_1\beta_2)\right], \tag{2.50}$$

which combines with the displacement operator to form additional group multiplication properties such that,

$$\begin{aligned}\widehat{D}(\alpha_1,\beta_1)\widehat{P}(\alpha_2,\beta_2) &= \widehat{P}\left[\frac{1}{2}\alpha_1+\alpha_2,\frac{1}{2}\beta_1+\beta_2\right]\exp\left[i(\alpha_2\beta_1-\alpha_1\beta_2)\right]\\ \widehat{P}(\alpha_1,\beta_1)\widehat{D}(\alpha_2,\beta_2) &= \widehat{P}\left[\alpha_1-\frac{1}{2}\alpha_2,\beta_1-\frac{1}{2}\beta_2\right]\exp\left[i(\alpha_2\beta_1-\alpha_1\beta_2)\right]\end{aligned} \tag{2.51}$$

Also, the parity operator $\widehat{P}(\alpha,\beta)$, can acts on the position and momentum operators to give,

$$\begin{aligned}\widehat{P}(\alpha,\beta)\hat{x}\widehat{P}^{\dagger}(\alpha,\beta) &= -\hat{x}+2\sqrt{2}\alpha\\ \widehat{P}(\alpha,\beta)\hat{p}\widehat{P}^{\dagger}(\alpha,\beta) &= -\hat{p}+2\sqrt{2}\beta\end{aligned} \tag{2.52}$$

The marginal properties of the parity operator are given as,

$$\begin{aligned}\frac{\sqrt{2}}{\pi}\int d\alpha\widehat{P}(\alpha,\beta) &= |\sqrt{2}\beta\rangle\ \langle\sqrt{2}\beta|\\ \frac{\sqrt{2}}{\pi}\int d\beta\widehat{P}(\alpha,\beta) &= |\sqrt{2}\alpha\rangle\ \langle\sqrt{2}\alpha|\\ \frac{2}{\pi}\int d\alpha d\beta\widehat{P}(\alpha,\beta) &= \mathbb{1}\end{aligned} \tag{2.53}$$

2.6 The uncertainty relation

Heisenberg in his paper [41] formulated the uncertainty principle for two non-commutating operators. For two non-commutative operators given as,

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}, \tag{2.54}$$

the uncertainty principle states that it is not possible to measure the corresponding values of both operators with precise accuracy. The more accurate the measurement of one observable, the less accurate the measurement of the other. The concept of the uncertainty principle has contributed to the development of quantum mechanics. Considering two symmetric operators $\hat{A}$ and $\hat{B}$, with mean or expectation values given as,

$$\begin{aligned} \langle A \rangle &= \langle \psi | \hat{A} | \psi \rangle \\ \langle B \rangle &= \langle \psi | \hat{B} | \psi \rangle \end{aligned} \tag{2.55}$$

Then the standard deviation of both operators is given,

$$\Delta_A \Delta_B \geq \frac{1}{2} | \langle [A, B] \rangle | \tag{2.56}$$

where,

$$\Delta_A = (\langle A^2 \rangle - \langle A \rangle^2)^{1/2} \geq 0, \quad \Delta_B = (\langle B^2 \rangle - \langle B \rangle^2)^{1/2} \geq 0 \tag{2.57}$$

Similarly, for two physical operators, position and momentum, they satisfy minimum uncertainty

$$\Delta_x \Delta_p \geq \frac{1}{2} \left| \langle [x, p] \rangle \right| = \frac{1}{2} \tag{2.58}$$

It can also be shown that the ground state, $|0\rangle$, satisfies minimum uncertainty,

$$\begin{aligned}
\langle x \rangle_0 &= \langle 0|\hat{x}|0\rangle = \frac{1}{\sqrt{2}} \langle 0|\hat{a} + \hat{a}^\dagger|0\rangle = 0 \\
\langle p \rangle_0 &= \langle 0|\hat{p}|0\rangle = \frac{1}{\sqrt{2}i} \langle 0|\hat{a} - \hat{a}^\dagger|0\rangle = 0 \\
\langle x^2 \rangle_0 &= \langle 0|\hat{x}^2|0\rangle = \frac{1}{2} \langle 0|(\hat{a} + \hat{a}^\dagger)^2|0\rangle = \frac{1}{2} \\
\langle p^2 \rangle_0 &= \langle 0|\hat{p}^2|0\rangle = -\frac{1}{2} \langle 0|(\hat{a} - \hat{a}^\dagger)^2|0\rangle = \frac{1}{2}
\end{aligned} \tag{2.59}$$

where $\hat{a}\,|0\rangle = \langle 0|\,\hat{a}^\dagger = 0$ and Eq. (2.16) were used. It is trivial to see that the final result gives minimum uncertainty of $\frac{1}{2}$. Thus,the position and momentum of a particle cannot be accurately measured at the same time.

Another state that meets the minimum uncertainty is the coherent state which will now be discussed in detail.

2.7 Coherent states

Coherent states are obtained by acting the displacement operator on the vacuum state,

$$\widehat{D}(z)\,|0\rangle = |z\rangle\,. \tag{2.60}$$

Otherwise, they are defined as,

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle\,. \tag{2.61}$$

The coherent states are normalised,

$$\langle z|z\rangle = e^{-|z|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{z^n z^{*m}}{\sqrt{n!m!}}\langle m|n\rangle = e^{-|z|^2}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!} = 1, \tag{2.62}$$

where both n and m are number states. Using Eqs. (2.27, 2.23), the coherent state is shown to be,

$$|z\rangle = \langle 0|z\rangle\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle, \quad \langle 0|z\rangle = e^{-\frac{1}{2}|z|^2}. \tag{2.63}$$

The prove of Eq. (2.60) is obtained by substituting Eq. (2.23) into Eq. (2.61) to get,

$$|z\rangle = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{n!}(a^\dagger)^n|0\rangle = e^{-\frac{1}{2}|z|^2}e^{za^\dagger}|0\rangle. \tag{2.64}$$

Indeed, using the Baker-Campbell-Hausdorff relation for two non-commutating operators,

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}; \quad e^{\hat{A}}e^{\hat{B}} = e^{\hat{B}}e^{\hat{A}}e^{[\hat{A},\hat{B}]} \tag{2.65}$$

satisfied only when $[\hat{A},\hat{B}] \neq 0$ and $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$, then the coherent state is further shown to be,

$$|z\rangle = e^{-\frac{1}{2}|z|^2}e^{za^\dagger}|0\rangle = e^{za^\dagger - z^* a}|0\rangle = \widehat{D}(z)|0\rangle. \tag{2.66}$$

$\widehat{D}(z)$ was earlier defined in Eq. (2.60) as the displacement operator.

Coherent states are eigenstates of the annihilation operators,

$$\hat{a}|z\rangle = z|z\rangle \tag{2.67}$$

From Eq. (2.26), it is easily seen that, $\langle n|\hat{a}|z\rangle = \sqrt{n+1}\,\langle n+1|z\rangle$ and since it also acts on the coherent states, prove the overlap of the number and coherent state as,

$$\langle n|z\rangle = \frac{z}{\sqrt{n}}\,\langle n-1|z\rangle\,; \quad \langle n| = \frac{z}{\sqrt{n}}\,\langle n-1| = \frac{z}{\sqrt{n}}\frac{z}{\sqrt{n-1}}\,\langle n-2|\,..........\frac{z^n}{\sqrt{n!}}\,\langle 0|$$
$$\langle n|z\rangle = \frac{z^n}{\sqrt{n!}}\,\langle 0|z\rangle \tag{2.68}$$

More so, the matrix element of the displacement operator with respect to number state $|n\rangle$ and coherent state $|w\rangle$ can be derived as,

$$\langle n|\widehat{D}(z)|w\rangle = \frac{(z+w)^n}{\sqrt{n!}}\exp\left[\frac{1}{2}\left[(zw^* - z^*w) - |z+w|^2\right]\right] \tag{2.69}$$

Interestingly as mentioned earlier, the coherent state satisfies minimum uncertainty and can be shown with respect to the position operators,

$$\begin{aligned}(\Delta_x)_z^2 &= \left[\langle z|\hat{x}^2|z\rangle - (\langle z|\hat{x}|z\rangle)^2\right]\\ &= \frac{1}{2}\left[\langle z|(\hat{a}+\hat{a}^\dagger)^2|z\rangle - (\langle z|(\hat{a}+\hat{a}^\dagger)|z\rangle)^2\right]\end{aligned} \tag{2.70}$$

By using the relations $\hat{a}\,|z\rangle = z\,|z\rangle$ and $\langle z|\,\hat{a}^\dagger = \langle z|\,z^*$, the uncertainty for the coherent states is,

$$\begin{aligned}(\Delta_x)_z^2 &= \frac{1}{2}\left[\left(\langle z|\hat{a}\hat{a}|z\rangle + \langle z|\hat{a}\hat{a}^\dagger|z\rangle + \langle z|\hat{a}^\dagger\hat{a}|z\rangle + \langle z|\hat{a}^\dagger\hat{a}^\dagger|z\rangle\right) - \left(\langle z|(\hat{a}+\hat{a}^\dagger)|z\rangle\right)^2\right]\\ &= \frac{1}{2}\left[\left(\langle z|\hat{a}\hat{a}|z\rangle + \langle z|[\hat{a},\hat{a}^\dagger]|z\rangle + 2\,\langle z|\hat{a}^\dagger\hat{a}|z\rangle + \langle z|\hat{a}^\dagger\hat{a}^\dagger|z\rangle\right) - \left(\langle z|(\hat{a}+\hat{a}^\dagger)|z\rangle\right)^2\right]\\ &= \frac{1}{2}\left[z^2 + 1 + 2|z|^2 + {z^*}^2 - (z+z^*)^2\right] = \frac{1}{2}\end{aligned} \tag{2.71}$$

Carrying out the standard deviation with respect to the momentum state, the uncertainty product can be found to be,

$$(\Delta_x)_z^2(\Delta_p)_z^2 = \frac{1}{4}. \tag{2.72}$$

This interesting property of the coherent state makes it very useful in quantum mechanics.

2.7.1 Properties of coherent state

Having established the fact that coherent states have minimum expectation values, moreover, they are quantum states which are close to classical states because of their expectation values. More details about the features and classical versus quantum properties of coherent states are clearly defined in [37]. More properties of coherent states are explored in this section. One key property is that the overlap of two coherent states is not orthogonal,

$$\begin{aligned}
\langle\gamma|z\rangle &= e^{-\frac{1}{2}|\gamma|^2-\frac{1}{2}|z|^2} \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\gamma^*)^n z^m}{\sqrt{n!m!}} \langle n|m\rangle \\
&= e^{-\frac{1}{2}|\gamma|^2-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{(\gamma^* z)^n}{n!} \\
&= \exp\left[-\frac{1}{2}|\gamma|^2 - \frac{1}{2}|z|^2 + z\gamma^*\right] \\
&= \exp\left[-\frac{1}{2}|\gamma - z|^2 + \frac{1}{2}(z\gamma^* - z^*\gamma)\right].
\end{aligned} \tag{2.73}$$

This result can be used to state the distance between two coherent states,

$$D_s(\gamma, z) = |\langle\gamma|z\rangle|^2 = \exp\left[-(|\gamma|^2 + |z|^2) + z\gamma^* + \gamma z^*\right] = \exp\left[-|\gamma - z|^2\right]. \tag{2.74}$$

This shows clearly that two coherent states are not orthogonal but can be almost so if $|\gamma - z|$ is extremely large. Another property is the completeness relation proved for example in [37] and given as,

$$\int d^2z \, |z\rangle \langle z| = 1, \tag{2.75}$$

where $z = z_R + iz_I$. Because of the completeness property, any state can be written in terms of the coherent state. In particular, any coherent state can be expanded using other coherent states,

$$|z\rangle = \int d^2\gamma \, |\gamma\rangle \langle\gamma|z\rangle = \int d^2\gamma \, \exp\left[-\frac{1}{2}|\gamma|^2 - \frac{1}{2}|z|^2 + \gamma^* z\right] |\gamma\rangle \,. \tag{2.76}$$

This makes the coherent state overcomplete. Furthermore, the position space and momentum-space representation of the coherent state is given by [18],

$$\begin{aligned} \langle x|z\rangle &= \left(\frac{1}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}|z|^2 - \frac{1}{2}z^2 - \frac{1}{2}x^2 + \sqrt{2}xz\right], \\ \langle p|z\rangle &= \left(\frac{1}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}|z|^2 - \frac{1}{2}z^2 - \frac{1}{2}p^2 - i\sqrt{2}pz\right]. \end{aligned} \tag{2.77}$$

2.8 Squeezed states

A quantum state is squeezed if any of its quadratures has a standard deviation that falls below the coherent state value of $\frac{1}{2}$ [69]. The coherent and vacuum states are not squeezed. The result according to Heisenberg's uncertainty principle is that the uncertainty in one quadrature if squeezed below $\frac{1}{2}$ invariably leads to a stretch in the other quadrature. It is known that squeezed quadrature states may not need be minimum-uncertainty state [69].

The unitary squeezing operator is given as,

$$\hat{S}(\zeta) = \exp\left[\frac{1}{4}\zeta\hat{a}^2 - \frac{1}{4}\zeta^*\hat{a}^{\dagger^2}\right]$$
$$\zeta = re^{i\phi}, \quad r > 0, \quad -\pi < \phi \leq \pi \tag{2.78}$$

The operator is unitary, $\hat{S}^\dagger(\zeta) = \hat{S}^{-1}(\zeta) = \hat{S}(-\zeta)$ and acts on the creation and annihilation operators to give the transformation,

$$\hat{S}(\zeta)\hat{a}\hat{S}^\dagger(\zeta) = \cosh\left(\frac{1}{2}r\right)\hat{a} + e^{-i\phi}\sinh\left(\frac{1}{2}r\right)\hat{a}^\dagger,$$
$$\hat{S}(\zeta)\hat{a}^\dagger\hat{S}^\dagger(\zeta) = e^{i\phi}\sinh\left(\frac{1}{2}r\right)\hat{a} + \cosh\left(\frac{1}{2}r\right)\hat{a}^\dagger. \tag{2.79}$$

Similarly, from Eq. (2.79), the squeezing operators based on Eq. (2.43) can act on the displacement operator,

$$\hat{S}(\zeta)\widehat{D}(z)\hat{S}^\dagger(\zeta) = \widehat{D}(z_\phi) \Longleftrightarrow \hat{S}(\zeta)\widehat{D}(z) = \widehat{D}(z_\zeta)\hat{S}(\zeta)$$
$$z_\phi \equiv \cosh\left(\frac{1}{2}r\right)z + e^{-i\phi}\sinh\left(\frac{1}{2}r\right)z^*. \tag{2.80}$$

Two other interesting properties of the squeezing operator is that it can act on the vacuum and number states respectively.

$$\hat{S}(\zeta)|0\rangle = |\zeta\rangle$$
$$\hat{S}(\zeta)\widehat{D}(z)|n\rangle = \hat{S}(\zeta)|n;z\rangle = |n;z,\zeta\rangle \tag{2.81}$$

The first relation in Eq. (2.81) gives the squeezed vacuum and the second is the displaced and squeezed number states. From Eq. (2.80) it is obvious that the squeezing operator and displacement operators do not commute and for special case where $n = 0$, Eq. (2.81) becomes the squeezed coherent state, $\hat{S}(\zeta)\widehat{D}(z)|0\rangle$.

2.9 Fourier transform

In quantum physics, the concept of wave-particle duality is well known [33, 65] with waves delocalised and particles localised. Both concepts are necessary for quantum mechanics to hold. This dual relationship is extended to other dual concepts such energy-time, and position-momentum. The modality for this extension is the Fourier transform based on the uncertainty principle. A Fourier transformation [64, 80] maps a one-dimensional time signal into a one-dimensional frequency function of the signal spectrum [7]. The transform of a function $f(x)$ and its inverse are given as,

$$\begin{aligned} \widetilde{F}(p) &= \int dx f(x) e^{-ixp} \\ f(x) &= \frac{1}{2\pi} \int dp \widetilde{F}(p) e^{ixp}. \end{aligned} \tag{2.82}$$

If the wave-function of a particle with respect to momentum states is $\langle p|\psi\rangle$, then following Eqs. (2.18,2.19), the Fourier transform can be shown as

$$\langle p|\psi\rangle = \int dx \langle p|x\rangle \langle x|\psi\rangle = \widetilde{\psi}(p) = \int dx \psi(x) e^{-ixp} \tag{2.83}$$

Thus the wavefunction with respect to momentum is related to the wavefunction with respect to position using the Fourier transform. The inverse relation of the Fourier transform is possible using the Dirac delta function with the properties given in [57].

2.9.1 Properties of Fourier transform

2.9.1.1 Linearity

If a linear combination of functions $g(x) = \alpha f(x) + \beta h(x)$ is defined, where α and β are constants, then the Fourier transform is given by,

$$\begin{aligned}\widetilde{G}(p) &= \alpha \int dx\, f(x) e^{-ipx} + \beta \int dx\, h(x) e^{-ipx} \\ &= \alpha \widetilde{F}(p) + \beta \widetilde{H}(p)\end{aligned} \tag{2.84}$$

2.9.1.2 Conjugation

Defining the Fourier transform as an operator, $\mathcal{F}$, then the conjugation property can be shown as,

$$\mathcal{F}(f(x)) = \widetilde{F}(p); \quad \mathcal{F}(f^*(x)) = \widetilde{F}^*(-p) \tag{2.85}$$

The Fourier transform can also be viewed as a transform of rotation across the quadrants of the $x - p$ (position-momentum) plane,

$$\begin{aligned}\mathcal{F}[f(x)] = \widetilde{F}(p); &\quad \mathcal{F}[\widetilde{F}(p)] = f(-x) \\ \mathcal{F}[f(-x)] = \widetilde{F}(-p); &\quad \mathcal{F}[\widetilde{F}(-p)] = f(x)\end{aligned} \tag{2.86}$$

2.9.1.3 Shift and scaling

For Fourier transform of a shifted function $g(x) = f(x - \alpha)$, is

$$\widetilde{G}(p) = \int dx f(x-\alpha) e^{-ipx} = e^{-ip\alpha} \widetilde{F}(p) \tag{2.87}$$

Here we use the substitution $x_1 = x - \alpha$.

Likewise for a scaled function $h(x) = f(x/\alpha)$, then the Fourier transform is given using $b = x/\alpha$,

$$\widetilde{H}(p) = \alpha \int dbf(b)e^{-ip\alpha b} = \alpha\widetilde{F}(\alpha p) \tag{2.88}$$

2.9.1.4 Differentiation

For a differentiable function $g(x) = f'(x)$, the Fourier transform is computed using integration by parts so that,

$$\widetilde{G}(p) = \int dxf'(x)e^{-ipx} = -ip\widetilde{F}(p). \tag{2.89}$$

The differential property of the Fourier transform is used in solving differential equations with applications in electrical circuits.

2.10 Phase-space distributions

2.10.1 Weyl function

The Weyl function [18, 22] is a correlation function. The Weyl function is given with respect to the displacement operator, $\widehat{D}(\alpha, \beta)$ and matrix elements of the operator in position and momentum,

$$\begin{aligned}\widetilde{W}(\alpha,\beta|\hat{\Theta}) &= \mathrm{Tr}[\hat{\Theta}\widehat{D}(\alpha,\beta)] \\ &= \int dx \left\langle x - \frac{\alpha}{\sqrt{2}} \middle| \hat{\Theta} \middle| x + \frac{\alpha}{\sqrt{2}} \right\rangle e^{i\sqrt{2}\beta x}. \\ &= \int dp \left\langle p - \frac{\beta}{\sqrt{2}} \middle| \hat{\Theta} \middle| p + \frac{\beta}{\sqrt{2}} \right\rangle e^{-i\sqrt{2}\alpha p}. \end{aligned} \tag{2.90}$$

The Weyl transform converts an operator into a function of α and β. A key feature of the Weyl function is the fact that the product of two Weyl functions gives the trace of the operators,

$$\begin{aligned}\int d\alpha d\beta \widetilde{W}(\alpha,\beta|\hat{\Theta}_1)\widetilde{W}(\alpha,\beta|\hat{\Theta}_2) &= \int d\alpha d\beta dx dx' \left\langle x - \frac{\alpha}{\sqrt{2}} \middle| \hat{\Theta}_1 \middle| x + \frac{\alpha}{\sqrt{2}} \right\rangle \\ &\times \left\langle x' - \frac{\alpha}{\sqrt{2}} \middle| \hat{\Theta}_2 \middle| x' + \frac{\alpha}{\sqrt{2}} \right\rangle e^{i\sqrt{2}\beta(x-x')} \\ &= \sqrt{2}\pi \int d\alpha dx \left\langle x - \frac{\alpha}{\sqrt{2}} \middle| \hat{\Theta}_1 \middle| x + \frac{\alpha}{\sqrt{2}} \right\rangle \\ &\times \left\langle -x - \frac{\alpha}{\sqrt{2}} \middle| \hat{\Theta}_2 \middle| -x + \frac{\alpha}{\sqrt{2}} \right\rangle. \end{aligned} \tag{2.91}$$

Performing a change of variables, $u = x + \alpha/\sqrt{2}$ and $v = x - \alpha/\sqrt{2}$, leads to,

$$\int d\alpha d\beta \widetilde{W}(\alpha,\beta|\hat{\Theta}_1)\widetilde{W}(\alpha,\beta|\hat{\Theta}_2) = \mathrm{Tr}[\hat{\Theta}_1\hat{\Theta}_2]. \tag{2.92}$$

The Weyl function is a correlation function since its displacements are taken in both position and momentum.

2.10.2 Wigner function

The uncertainty relation forbids the possibility of having a standard probability distribution; thus the use of a quasi-probability distribution such as the Wigner distribution function [13, 22, 31, 50, 77]. From Eq. (2.83), $|\widetilde{\psi}(p)|^2$ gives the probability in the position variable. However it would be advantageous to have a function that simultaneously shows the probability with respect to both position and momentum. The Wigner distribution is also commonly used in quadratic time-frequency representations [7, 50].

The Wigner function for an operator $\hat{\Theta}$ is given as,

$$\begin{aligned} W(\alpha,\beta|\Theta) &= \mathrm{Tr}[\hat{\Theta}\widehat{P}(\alpha,\beta)] \\ &= \int dx \left\langle \frac{\alpha}{\sqrt{2}} + x \middle| \hat{\Theta} \middle| \frac{\alpha}{\sqrt{2}} - x \right\rangle e^{-i\sqrt{2}\beta x}. \\ &= \int dp \left\langle \frac{\beta}{\sqrt{2}} + p \middle| \hat{\Theta} \middle| \frac{\beta}{\sqrt{2}} - p \right\rangle e^{-i\sqrt{2}\alpha p}. \end{aligned} \tag{2.93}$$

Having used $\widehat{P}(\alpha,\beta) = D(\alpha,\beta)\widehat{P}(0,0)$ as the definition for the parity operator to derive the Wigner function above. From Eq. (2.93), it is easily proved that the Wigner function is always real-valued but not necessarily positive and preserves both position and momentum distributions. Most important is the fact that it satisfies the marginal properties with respect to position

$$\begin{aligned} \frac{1}{\sqrt{2}\pi} \int d\beta W(\alpha,\beta|\Theta) &= \int dx\, d\beta \left\langle \frac{\alpha}{\sqrt{2}} + x \middle| \hat{\Theta} \middle| \frac{\alpha}{\sqrt{2}} - x \right\rangle e^{-i\sqrt{2}\beta x} \\ &= \left\langle \frac{\alpha}{\sqrt{2}} \middle| \hat{\Theta} \middle| \frac{\alpha}{\sqrt{2}} \right\rangle, \end{aligned} \tag{2.94}$$

and with respect to momentum

$$\frac{1}{\sqrt{2}\pi}\int d\alpha W(\alpha,\beta|\Theta) = \int dp\, d\alpha \left\langle \frac{\beta}{\sqrt{2}} + p \middle| \hat{\Theta} \middle| \frac{\beta}{\sqrt{2}} - p \right\rangle e^{-i\sqrt{2}\alpha p}$$
$$= \left\langle \frac{\beta}{\sqrt{2}} \middle| \hat{\Theta} \middle| \frac{\beta}{\sqrt{2}} \right\rangle, \tag{2.95}$$

as well as with respect to both position and momentum,

$$\frac{1}{\pi}\int d\alpha d\beta W(\alpha,\beta|\Theta) = \mathrm{Tr}[\hat{\Theta}]. \tag{2.96}$$

The marginal properties show the projection in phase space, so that projection in x−axis gives the probability distribution in p, and the projection on the p−axis gives the distribution in x.

The Wigner distribution can roughly be considered as the spread of the position and momentum of a particle and is useful because the uncertainty principle prohibits the simultaneous position and momentum calculation of a particle's position and momentum. Thus, with the Wigner function of a particle, the expectation value with respect to position and momentum can be derived.

2.10.2.1 Examples of Wigner function

We give examples of the Wigner function of various states as shown in figure 2.1 The Wigner functions for the vacuum state, $|0\rangle$ and coherent states, $|z\rangle$ are given as,

$$W(\alpha,\beta;|0\rangle) = \exp\left[\frac{1}{2}(\alpha^2+\beta^2)\right]$$
$$W(\alpha,\beta;|z\rangle) = \exp\left[-\left(\frac{\alpha}{\sqrt{2}} - \sqrt{2}z_r\right)^2 - \left(\frac{\beta}{\sqrt{2}} - \sqrt{2}z_i\right)^2\right] \tag{2.97}$$

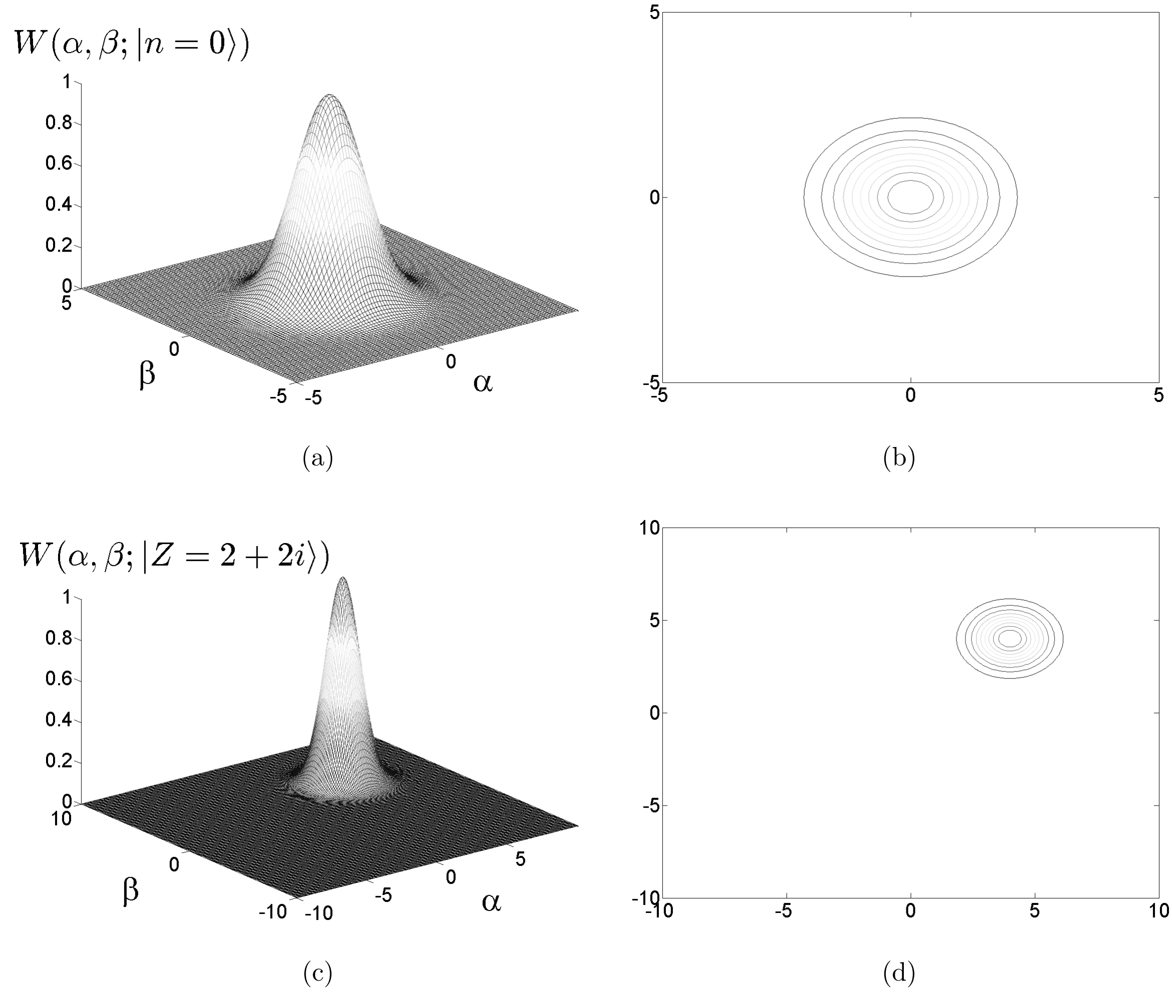


Fig. (2.1) Examples of Wigner functions. (a),(b) Vacuum state $|0\rangle$; (c),(d) Coherent state $|2+2i\rangle$

Fig. (2.1) shows the Wigner function of a coherent state, $|Z=2+2i\rangle$ and also for a vacuum state, $|n=0\rangle$ with minimum uncertainty. We can likewise derive the Wigner function for number states. We begin with the relation by considering an arbitrary operator, Θ,

$$\Theta = \sum_{n,m} \Theta_{nm} |n\rangle \langle m| \tag{2.98}$$

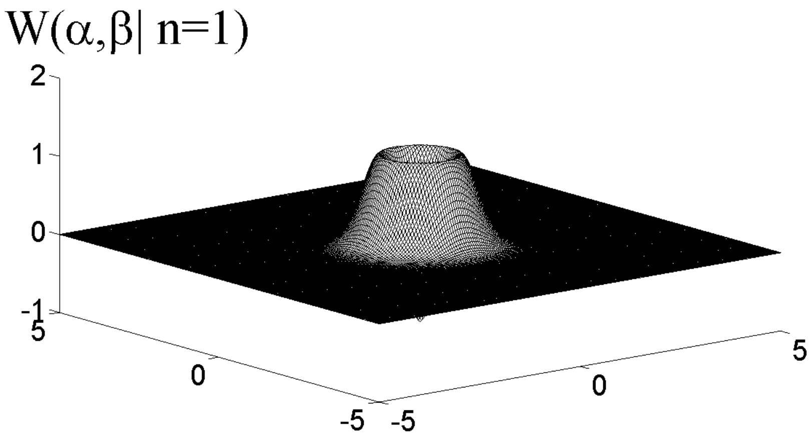


Fig. (2.2) $W(\alpha,\beta)$(Wigner function) of number states for the state of Eq. (2.112), $n = 1$

Then the Wigner function from Eq. (2.93), can be written as,

$$W(\alpha,\beta) = \mathrm{Tr}[\hat{\Theta}\widehat{P}(\alpha,\beta)] = \sum_{n,m}^{\infty} \Theta_{nm} W_{mn}(\alpha,\beta) \tag{2.99}$$

Then using Eqs. $(2.49, 2.98, 2.45)$, the Wigner function for a number state is shown to be

$$\begin{aligned} W_{mn}(\alpha,\beta) &= \langle m|\widehat{P}(\alpha,\beta)|n\rangle \\ &= (-1)^n \left(\frac{n!}{m!}\right)^{1/2} [2(\alpha + i\beta)]^{m-n} \exp{(-2\alpha^2 - 2\beta^2)} L_n^{m-n}(4\alpha^2 + 4\beta^2) \end{aligned} \tag{2.100}$$

The Wigner function for the number states above, for a special case, $W(\alpha,\beta|n = 0)$ is shown in Fig. (2.1a).

The Wigner and Weyl functions are related through a 2-dimensional Fourier transform. It is evident from Eq. (2.93) and Eq. (2.90) that,

$$W(\alpha,\beta) = \int d\lambda d\gamma \widetilde{W}(\lambda,\gamma) \exp[i(\beta\lambda - \gamma\alpha)] \tag{2.101}$$

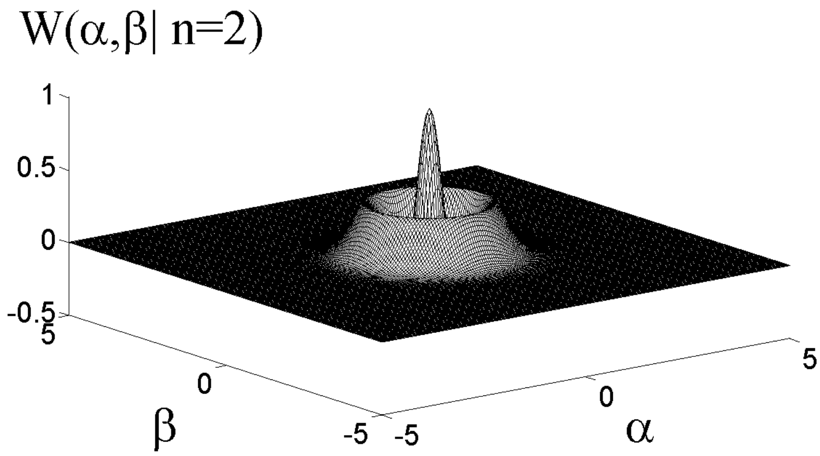


Fig. (2.3) $W(\alpha, \beta)$(Wigner function) of number states for the state of Eq. (2.112), n = 2

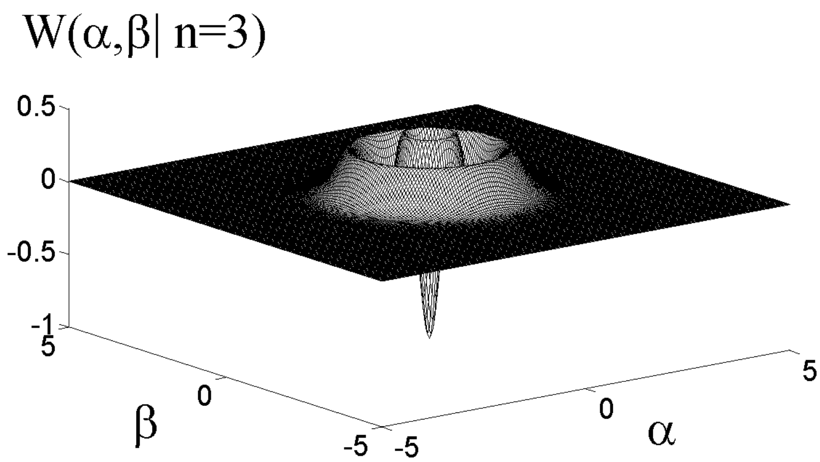


Fig. (2.4) $W(\alpha, \beta)$(Wigner function) of number states for the state of Eq. (2.112), n = 3

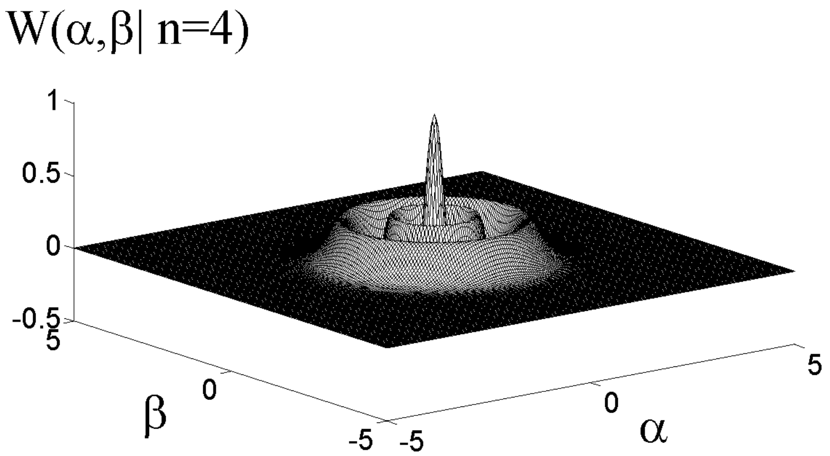


Fig. (2.5) $W(\alpha,\beta)$(Wigner function) of number states for the state of Eq. (2.112), $n = 4$

Marginal properties also exist for $[W(\alpha,\beta)]^2$ and $|\widetilde{W}(\alpha,\beta)|^2$.

$$\int [W(\alpha,\beta|\Theta)]^2 d\beta = \sqrt{2}\pi \int dx \left| \left\langle \frac{\alpha}{\sqrt{2}} + x \middle| \hat{\Theta} \middle| \frac{\alpha}{\sqrt{2}} - x \right\rangle \right|^2$$
$$\int |\widetilde{W}(\alpha,\beta|\Theta)|^2 d\beta = \sqrt{2}\pi \int dx \left| \left\langle x - \frac{\alpha}{\sqrt{2}} \middle| \hat{\Theta} \middle| x + \frac{\alpha}{\sqrt{2}} \right\rangle \right|^2 \tag{2.102}$$

And thus,

$$\frac{1}{\pi} \int d\alpha d\beta [W(\alpha,\beta|\Theta)]^2 = \frac{1}{\pi} \int d\alpha d\beta |\widetilde{W}(\alpha,\beta|\Theta)|^2 = \mathrm{Tr}[\hat{\Theta}^2] \leq 1 \tag{2.103}$$

However, for a pure state $\hat{\Theta} = |t\rangle\langle t|$, both $[W(\alpha,\beta)]^2$ and $|\widetilde{W}(\alpha,\beta)|^2$ can be viewed as probability densities. Such that from Eq. (2.102),

$$\int d\beta [W(\alpha,\beta)]^2 = \sqrt{2}\pi \int dx \left| t\left(\frac{\alpha}{\sqrt{2}} + x\right) \right|^2 \left| t\left(\frac{\alpha}{\sqrt{2}} - x\right) \right|^2$$
$$\int d\beta |\widetilde{W}(\alpha,\beta)|^2 = \sqrt{2}\pi \int dx \left| t\left(x - \frac{\alpha}{\sqrt{2}}\right) \right|^2 \left| t\left(x + \frac{\alpha}{\sqrt{2}}\right) \right|^2 \tag{2.104}$$

For such a case Eq. (2.103) equals 1.

2.10.3 Sudarshan-Glauber and Husimi functions

We can represent the density operator which describes a quantum state in terms of the coherent state

$$\hat{\Theta} = \int d\alpha d\beta d\alpha' d\beta' \langle \alpha, \beta|\hat{\Theta}|\alpha', \beta'\rangle |\alpha, \beta\rangle \langle \alpha', \beta'| \tag{2.105}$$

Having used $z = \alpha + i\beta$ and Eq. (2.75).

Alternatively, writing Θ in terms of the Sudarshan-Glauber $P-$function as,

$$\hat{\Theta} = \int d\alpha d\beta P(\alpha, \beta|\Theta) |\alpha, \beta\rangle \langle \alpha, \beta| \tag{2.106}$$

The equation above is the diagonal form of the density operator [37], and the $P-$function is a probability distribution [37]. The $P-$function is a quasi-probability distribution in that for some quantum states which are non-classical, the $P-$function is more singular than the Dirac delta function or negative. For states where the $P(\alpha, \beta) \geq 0$ or no more singular than the Dirac delta function, such states are termed classical states. The coherent states are quasi-mechanical with $P-$functions equal to the Dirac delta function.

We can obtain the $P-$function [52] as,

$$\langle -\alpha', -\beta'|\hat{\Theta}|\alpha', \beta'\rangle = \int d\alpha d\beta P(\alpha, \beta) \langle -\alpha', -\beta'|\alpha, \beta\rangle \langle \alpha, \beta|\alpha', \beta'\rangle$$
$$P(\alpha, \beta) = \frac{e^{-(\alpha^2+\beta^2)}}{\pi^2} \int d\alpha' d\beta' \langle -\alpha', -\beta'|\hat{\Theta}|\alpha', \beta'\rangle \exp[2i(\beta\alpha' - \alpha\beta')] \tag{2.107}$$

Another concept in phase-space probability distribution is the $Q-$function or Husimi function [37],which is the expectation value of the density operator with respect to the coherent state,

$$Q(\alpha,\beta|\Theta) = \frac{1}{\pi}\langle\alpha,\beta|\hat{\Theta}|\alpha,\beta\rangle \tag{2.108}$$

The $Q-$function is positive for all quantum states and like the $P-$function is normalised,

$$\int d\alpha d\beta Q(\alpha,\beta) = \int d\alpha d\beta P(\alpha,\beta) = 1 \tag{2.109}$$

The $Q-$function, $P-$function and Wigner function are related through,

$$\begin{aligned}
Q(\alpha,\beta) &= \frac{1}{\pi}\int d\alpha' d\beta' P(\alpha',\beta')\langle\alpha,\beta|\alpha',\beta'\rangle\langle\alpha',\beta'|\alpha,\beta\rangle \\
&= \frac{1}{\pi}\int d\alpha' d\beta' P(\alpha',\beta') D_{\mathcal{C}}(\alpha,\beta|\alpha',\beta') \\
&= \frac{1}{\pi}\int d\alpha' d\beta' W(\alpha',\beta')(\langle\alpha,\beta|\alpha',\beta'\rangle\langle\alpha',\beta'|\alpha,\beta\rangle)^2 \\
&= \frac{1}{\pi}\int d\alpha' d\beta' W(\alpha',\beta')(D_{\mathcal{C}}(\alpha,\beta|\alpha',\beta'))^2
\end{aligned} \tag{2.110}$$

We note that the overlap of two coherent states is shown in Eq. (2.73) and write the distance between two coherent states,

$$\begin{aligned}
D_{\mathcal{C}}(\alpha,\beta|\alpha',\beta') &= \langle\alpha,\beta|\alpha',\beta'\rangle\langle\alpha',\beta'|\alpha,\beta\rangle = |\langle\alpha,\beta|\alpha',\beta'\rangle|^2 \\
&= \exp[-|(\alpha+i\beta)-(\alpha'+i\beta')|^2] \\
&= \exp[-(\alpha^2+\beta^2+\alpha'^2+\beta'^2)+2(\alpha\alpha'+\beta\beta')]
\end{aligned} \tag{2.111}$$

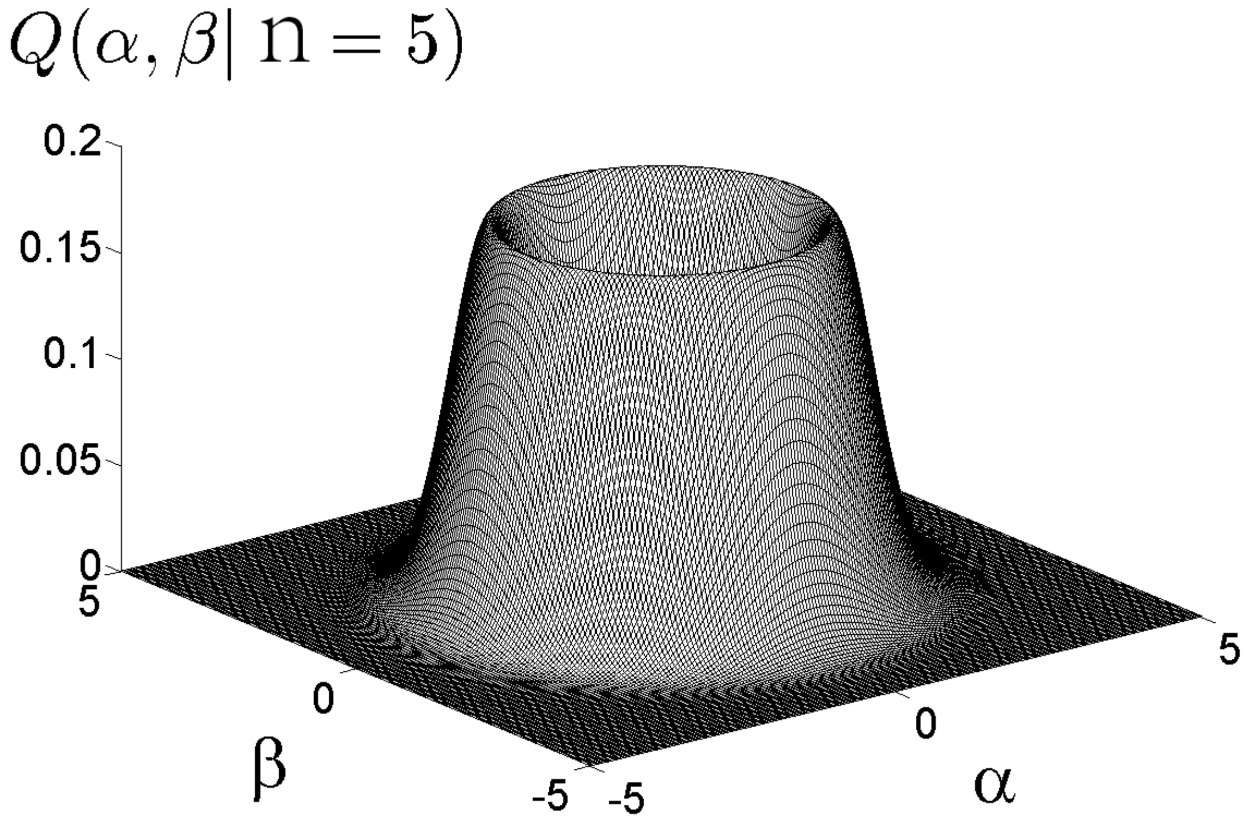


Fig. (2.6) $Q(\alpha, \beta)$(Q-function) of number states for the state of Eq. (2.112) with $n = 5$

The $Q-$function for the number state is given as,

$$Q(\alpha, \beta|n) = \frac{1}{\pi} \langle \alpha, \beta|n\rangle \langle n|\alpha, \beta\rangle = \frac{z^n z^{*n}}{n!} \exp\left[-|z|^2\right]. \tag{2.112}$$

The $Q-$function of the coherent state is the distance between two coherent states of Eq. (2.74). An example of the $Q-$function for number state, $n = 5$ is given in figure 2.6.

2.11 Moyal star formalism

Moyal proved [12, 55] that for arbitrary states, $|\gamma\rangle, |\zeta\rangle, |\epsilon\rangle, |\delta\rangle$

$$\frac{1}{\pi}\int d\alpha d\beta \langle\gamma|\widehat{D}^{\dagger}(\alpha,\beta)|\delta\rangle\langle\epsilon|\widehat{D}(\alpha,\beta)|\zeta\rangle = \langle\gamma|\zeta\rangle\langle\epsilon|\delta\rangle \tag{2.113}$$

The relation is easily proved using Eqs. (2.40, 2.19). The detailed proof is given in Appendix A. A similar relation has also been proved for the displaced parity operator [18],

$$\frac{4}{\pi}\int d\alpha d\beta \langle\gamma|\widehat{P}(\alpha,\beta)|\delta\rangle\langle\epsilon|\widehat{P}(\alpha,\beta)|\zeta\rangle = \langle\gamma|\zeta\rangle\langle\epsilon|\delta\rangle. \tag{2.114}$$

Moyal also proved that any operator can be written in terms of the displacement operators so that,

$$\Theta = \frac{1}{\pi}\int d\alpha d\beta \;\; \mathrm{Tr}[\widehat{D}^{\dagger}(\alpha,\beta)\Theta]\widehat{D}(\alpha,\beta). \tag{2.115}$$

This is easily proved using Eq. (2.113) above,

$$\begin{aligned}\langle N|\Theta|M\rangle &= \frac{1}{\pi}\sum_{K}\sum_{T}\int d\alpha d\beta \;\; \langle K|\widehat{D}^{\dagger}(\alpha,\beta)|T\rangle\langle T|\Theta|K\rangle \\ &\times \; \langle N|\widehat{D}(\alpha,\beta)|M\rangle \\ &= \sum_{K}\sum_{T}\delta(K-M)\delta(T-N)\langle T|\Theta|K\rangle \\ &= \langle N|\Theta|M\rangle. \end{aligned} \tag{2.116}$$

Having shown from Eq. (2.115) that an operator can be written in terms of the Weyl function, it can also be written in terms of the Wigner function [18] such that,

$$\Theta = \frac{2}{\pi}\int d\alpha d\beta \;\; W(\alpha,\beta|\Theta)\widehat{P}(\alpha,\beta). \tag{2.117}$$

It can be extended to two operators so that,

$$\Theta_1\Theta_2 = \frac{4}{\pi^2}\int d\alpha' d\beta' d\alpha d\beta \, W(\alpha,\beta|\Theta_1)\, W(\alpha',\beta'|\Theta_2) \times \widehat{P}(\alpha,\beta)\widehat{P}(\alpha',\beta'). \tag{2.118}$$

Using properties of Eq. (2.42), the trace of both operators leads to,

$$\mathrm{Tr}(\Theta_1\Theta_2) = \frac{1}{\pi}\int d\alpha d\beta \, W(\alpha,\beta|\Theta_1)\, W(\alpha,\beta|\Theta_2) \tag{2.119}$$

The relation in Eq. (2.118) leads to the Moyal star formalism [12, 55] which gives the Wigner function of two non-commuting operators in phase space,

$$W(\alpha,\beta|\Theta_1\Theta_2) = \frac{4}{\pi^2}\int d\alpha' d\beta' d\alpha'' d\beta'' W(\alpha+\alpha',\beta+\beta'|\Theta_1) W(\alpha+\alpha'',\beta+\beta''|\Theta_2) \times \exp\left[2i(\alpha'\beta'' - \beta'\alpha'')\right] \tag{2.120}$$

Further studies on the Moyal star product are given in [18].

Chapter 3

Fractional Fourier transform in phase space

3.1 Introduction to fractional Fourier transform

The fractional Fourier transform is defined as a generalisation of the normal Fourier transform thus championing improvements and generalisations in areas where Fourier transform was applied. Since its introduction [70], it has found applications in signal processing [8, 57], quantum optics [57, 70], image analysis [9, 42], encryption [29, 35, 79] and quantum cryptography [78]. In time frequency analysis it is interpreted as a rotation in time-frequency plane and can show characteristics of the system from the time and frequency domains [68].

The Fourier transform was introduced by Jean Baptiste Fourier in 1807 while solving a heat conduction problem. It has been known to have limitations like its inability to obtain time-frequency characteristics for non-stationary signals. The fractional Fourier transform is one of the proposed solutions for such problems [68]. Further in this chapter, the fractional Fourier transform is extended to quantum mechanics in generalising the coherent states and Wigner function. Also, it can be applied to optics

[59] and even recently to the squeezing operator [30].

The fractional Fourier transform can be defined as both a transform and as an operator. As a transform it is shown by starting with the two dimensional non-separable linear canonical transform (LCT) which is the 2-dimensional counterpart of the 1-dimensional linear canonical transform (LCT) [34]

$$\mathcal{F}^{(A,B,U,V)}(\alpha,\beta) = \frac{1}{2\pi det\mathbf{B}} \int d\lambda d\gamma \exp\left[\frac{i\left[(-b_{22}\alpha + b_{12}\beta)\lambda + (b_{21}\alpha - b_{11}\beta)\gamma\right]}{det\mathbf{B}}\right] \times \exp\left[\frac{i(k_1\alpha^2 + k_2\alpha\beta + k_3\beta^2)}{2det\mathbf{B}} + \frac{i(p_1\lambda^2 + p_2\lambda\gamma + p_3\gamma^2)}{2det\mathbf{B}}\right] f(\lambda,\gamma), \tag{3.1}$$

where $det\mathbf{B} \neq 0$ and **A,B,U,V** are matrices,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}.$$

The values of k_1, k_2, k_3, p_1, p_2 and p_3 are given as,

$$\begin{aligned} k_1 &= v_{11}b_{22} - v_{12}b_{21}, \;\; k_2 = 2(-v_{11}b_{12} + v_{12}b_{11}), \;\; k_3 = -v_{21}b_{12} + v_{22}b_{11} \\ p_1 &= a_{11}b_{22} - a_{21}b_{12}, \;\; p_2 = 2(a_{12}b_{22} - a_{22}b_{12}), \;\; p_3 = -a_{12}b_{21} + a_{22}b_{11} \end{aligned} \tag{3.2}$$

With the following constraints to be satisfied,

$$
\begin{aligned}
&A^TU = U^TA, \quad B^TV = V^TB, \quad A^TV - U^TB = \mathbb{I} \\
&a_{11}v_{11} + a_{21}v_{21} - (b_{11}u_{11} + b_{21}u_{21}) = 1, \quad a_{12}v_{12} + a_{22}v_{22} - (b_{12}u_{12} + b_{22}u_{22}) = 1 \\
&a_{11}u_{12} + a_{21}u_{22} = a_{12}u_{11} + a_{22}u_{21}, \quad b_{11}v_{12} + b_{21}v_{22} = b_{12}v_{11} + b_{22}v_{21}, \\
&a_{11}v_{12} + a_{21}v_{22} = u_{11}b_{12} + u_{21}b_{22}, \quad a_{12}v_{11} + a_{22}v_{21} = u_{12}b_{11} + u_{22}b_{21}.
\end{aligned} \tag{3.3}
$$

The reduction of the 2-dimensional linear canonical transform (LCT) to the 2-dimensional fractional Fourier transform is shown

$$
\widetilde{F}(\alpha,\beta) = \int d\lambda d\gamma K(\alpha,\lambda;\theta_\alpha)K(\beta,\gamma;\theta_\beta)f(\lambda,\gamma), \tag{3.4}
$$

where,

$$
\begin{aligned}
&a_{12} = a_{21} = b_{12} = b_{21} = u_{12} = u_{21} = v_{12} = v_{21} = 0, \\
&a_{11} = v_{11} = -\cos\theta_\alpha, b_{11} = u_{11} = -\sin\theta_\alpha, a_{22} = v_{22} = \cos\theta_\beta, b_{22} = -u_{22} = \sin\theta_\beta \\
&K(\alpha,\lambda;\theta_\alpha) = \left[\frac{(1 - i\cot\theta_\alpha)}{2\pi}\right]^{1/2} \exp\left[\frac{-i(\alpha^2 + \lambda^2)\cot\theta_\alpha}{2} + \frac{i\alpha\lambda}{\sin\theta_\alpha}\right], \\
&K(\beta,\gamma;\theta_\beta) = \left[\frac{(1 - i\cot\theta_\beta)}{2\pi}\right]^{1/2} \exp\left[\frac{-i(\beta^2 + \gamma^2)\cot\theta_\beta}{2} + \frac{i\beta\gamma}{\sin\theta_\beta}\right].
\end{aligned} \tag{3.5}
$$

Likewise, it reduces to 2-dimensional Fourier transform, when $b_{11} = b_{22} = 1, u_{11} = u_{22} = -1$, others = 0,

$$
\widetilde{F}(\alpha,\beta) = \frac{1}{2\pi}\int d\lambda d\gamma \exp\left[-i(\alpha\lambda + \beta\gamma)\right]f(\lambda,\gamma). \tag{3.6}
$$

In terms of an operator, the fractional Fourier operator in 1-dimension is given by

$$\mathfrak{F}(\theta;\alpha) = \exp\left[\frac{i\theta}{2}(\alpha^2 - \partial_\alpha^2 + 1)\right]; \quad 0 \leq \theta < 2\pi. \tag{3.7}$$

Acting with a function $f(\alpha)$ leads to

$$\mathfrak{F}(\theta;\alpha)[f(\alpha)] = \int d\gamma K(\alpha,\gamma;\theta) f(\gamma),$$
$$K(\alpha,\gamma;\theta) = \left[\frac{1 + i\cot\theta}{2\pi}\right]^{1/2} \exp\left[\frac{-i(\alpha^2+\gamma^2)\cot\theta}{2} + \frac{i\alpha\gamma}{\sin\theta}\right]. \tag{3.8}$$

The proof of Eq. (3.8) is well known and included in Appendix A. It is called the kernel of the fractional Fourier transform.

3.2 Properties of the kernel of fractional Fourier transform

Of note is that the kernel of the fractional Fourier transform, $K(-\alpha,\lambda;\theta_\alpha)$ is symmetric along the axis (α and λ) so that,

$$K(-\alpha,\lambda;\theta_\alpha) = K(\alpha,-\lambda;\theta_\alpha); \;\; K(-\alpha,-\lambda;\theta_\alpha) = K(\alpha,\lambda;\theta_\alpha). \tag{3.9}$$

However, it is not symmetric with respect to the angle

$$K(\alpha,\lambda;-\theta_\alpha) \neq K(-\alpha,-\lambda;\theta_\alpha); \;\; K(\alpha,\lambda;-\theta_\alpha) \neq K(-\alpha,\lambda;\theta_\alpha). \tag{3.10}$$

For special cases of different rotations of the angle in 1-diemnsion

$$K(\alpha,\gamma;\theta) = \left\{ \begin{array}{ll} \left[\frac{1+i\cot\theta}{2\pi}\right]\exp\left[\frac{-i(\alpha^2+\gamma^2)\cot\theta}{2} + \frac{i\alpha\gamma}{\sin\theta}\right], \forall\theta : \theta \neq n\pi & \\ \delta(\alpha-\gamma), & \text{if } \theta = 2n\pi \\ \delta(\alpha+\gamma), & \text{if } \theta = (2n+1)\pi \end{array} \right\} \tag{3.11}$$

and in the case $\theta = \pi/2$, the kernel becomes the Fourier transform $K\left(\alpha,\gamma;\frac{\pi}{2}\right) = \exp(i\alpha\gamma)/(2\pi)^{1/2}$. Therefore $\mathfrak{F}(\pi/2;\alpha)$ is the Fourier operator and

$$\mathfrak{F}\left(\frac{\pi}{2};\alpha\right)[f(\alpha)] = \frac{1}{(2\pi)^{1/2}} \int d\gamma \exp(i\alpha\gamma) f(\gamma). \tag{3.12}$$

In addition, $\mathfrak{F}(\pi;\alpha)$ is the parity operator. In this case $K(\alpha,\gamma;\pi) = \delta(\alpha+\gamma)$. For a detailed description of the properties of the fractional Fourier transform as well as its transform with respect to different signals, is given in [54, 56, 57].

An important property of the fractional Fourier transform kernel which is widely used in this work is the index additivity given as,

$$\int d\gamma K(\alpha,\gamma;\theta_\alpha)K(\gamma,\beta;\theta_\beta) = K(\alpha,\beta;\theta_\alpha+\theta_\beta). \tag{3.13}$$

The proof of this property is given in Appendix A. The kernel also has the obvious inverse property that,

$$[K(\alpha,\gamma;\theta_\alpha)]^\dagger = K(-\alpha,-\gamma;-\theta_\alpha) = K(\gamma,\alpha;-\theta_\alpha). \tag{3.14}$$

The Kernel has two optical interpretations, one as a propagation through GRIN (Graded Index media) medium [48, 53, 54] and the second as the rotation in the position-momentum plane [75].

3.3 Examples of fractional Fourier transform of different waveforms

The fractional Fourier transform of different waveforms is known [58] and in the Figs. (3.1a, 3.1b) below some examples are given. Two numerical examples for the triangular and square functions (tripuls(x,1) and rectpuls(x,3)). Each case shows the gradual transform of the angle between 0 and 1 with the angle defined as $\theta = \frac{a\pi}{2}$. At $a = 0$, there is no transformation, and then there is a gradual transformation towards a sinc-function as the angle approaches $\frac{\pi}{2}$ when $a = 1$.

3.4 Fractional Fourier operator

The 1-dimensional fractional Fourier transform can also be given as an operator in terms of the number operator [23],

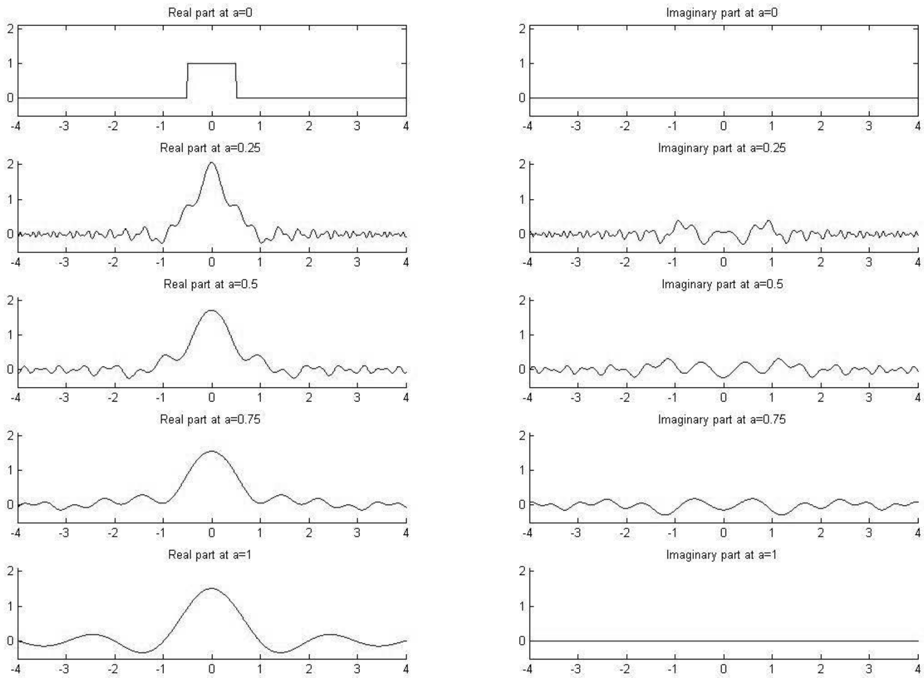
$$\begin{aligned} \mathcal{K}(\theta) &= \exp\left(i\theta n\right); \quad n = \hat{a}^{\dagger}\hat{a}, \\ \mathcal{K}(\theta)\left|x\right\rangle &= \left|x;\theta\right\rangle; \quad \mathcal{K}(\theta)\left|p\right\rangle = \left|p;\theta\right\rangle. \end{aligned} \tag{3.15}$$

This operator can also show that it creates a rotation as earlier mentioned rotating both position ($\hat{x}$) and momentum ($\hat{p}$) operators as,

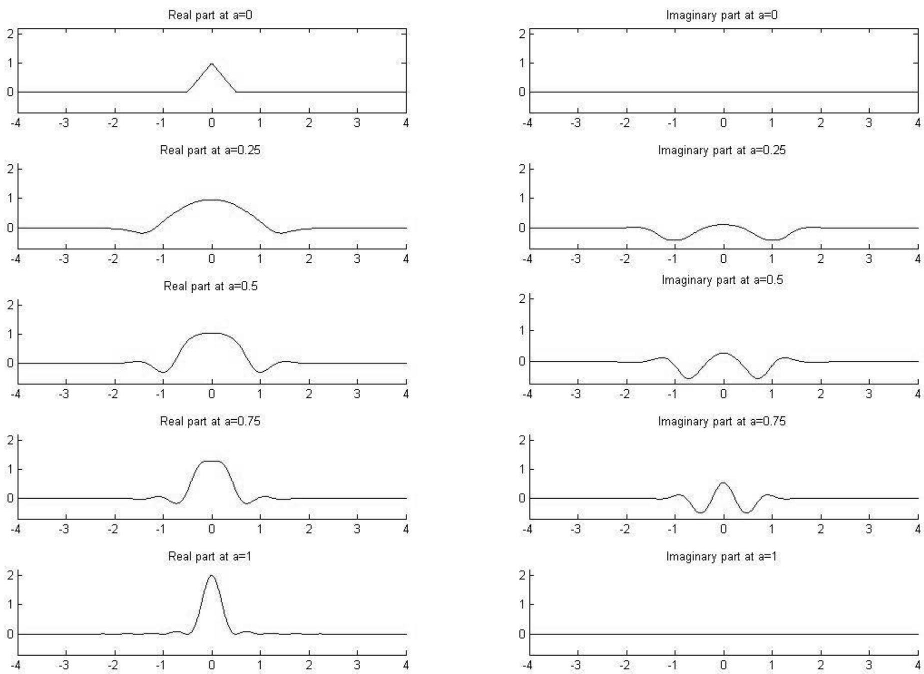
$$\begin{aligned} \hat{x}_{\theta} &\equiv \mathcal{K}(\theta)\hat{x}\mathcal{K}^{\dagger}(\theta) = \hat{x}\cos\theta + \hat{p}\sin\theta, \\ \hat{p}_{\theta} &\equiv \mathcal{K}(\theta)\hat{p}\mathcal{K}^{\dagger}(\theta) = -\hat{x}\sin\theta + \hat{p}\cos\theta. \end{aligned} \tag{3.16}$$

Acting the fractional Fourier operator on both sides of the displacement operator gives,

$$\mathcal{K}(\theta)\widehat{D}(\alpha,\beta)\mathcal{K}^{\dagger}(\theta) = \widehat{D}(\alpha\cos\theta - \beta\sin\theta, \beta\cos\theta + \alpha\sin\theta). \tag{3.17}$$



(a)



(b)

Fig. (3.1) (a): Rectangular pulse wave rectpuls(x,1), (b): Triangular pulse wave tripuls(x,2)

Of particular interest is the equivalence shown between the operator and the continuous kernel of the fractional Fourier transform [23]

$$\begin{aligned}\langle p|x;\theta\rangle = K(x,p;\theta) &= \left(\frac{1}{\pi}\right)^{1/2} \sum_{N=0}^{\infty} e^{iN\theta} \frac{H_N(x)H_N(p)}{2^N N!} \exp\left[\frac{1}{2}(x^2+p^2)\right] \\ &= \left[\frac{1-i\cot\theta}{2\pi}\right]^{1/2} \exp\left[\frac{-i(x^2+p^2)\cot\theta}{2} + \frac{ixp}{\sin\theta}\right]. \end{aligned} \tag{3.18}$$

3.5 Non-orthogonal plane in the $(\theta_\alpha, \theta_\beta)$ axes

In many formulas throughout this thesis the factor $\cos(\theta_\alpha - \theta_\beta)$ appears. The following arguments show the Jacobian nature of this factor, and also gives some quantities used in this thesis in terms of coordinates in a non-orthogonal frame. This is shown by considering an orthogonal frame $\alpha - \beta$, and a non-orthogonal frame $\alpha' - \beta'$ as shown in Fig. (3.2).

The 'bi-fractional transform' in the present context is to rotate the x-axis by an angle θ_α, and the y-axis by an angle θ_β, and change variables from α, β to α', β'. Let (α_0, β_0) and (α'_0, β'_0) be the coordinates of a point in these two frames, correspondingly. With elementary trigonometry, the (α'_0, β'_0) in terms of (α_0, β_0) can be expressed as follows

$$\begin{aligned} &\alpha'_0 = G_1\alpha_0 + G_2\beta_0; \quad \beta'_0 = G_3\alpha_0 + G_4\beta_0 \\ &G_1 = \frac{1}{\cos\theta_\alpha} - \frac{\tan\theta_\alpha \sin\theta_\beta}{\cos(\theta_\alpha - \theta_\beta)}; \quad G_2 = \frac{\sin\theta_\beta}{\cos(\theta_\alpha - \theta_\beta)} \\ &G_3 = -\frac{\sin\theta_\alpha}{\cos(\theta_\alpha - \theta_\beta)}; \quad G_4 = \frac{\cos\theta_\alpha}{\cos(\theta_\alpha - \theta_\beta)}. \end{aligned} \tag{3.19}$$

Therefore the Jacobian corresponding to this change of variables is

$$\frac{\partial(\alpha'_0, \beta'_0)}{\partial(\alpha_0, \beta_0)} = G_1G_4 - G_2G_3 = \frac{1}{\cos(\theta_\alpha - \theta_\beta)}. \tag{3.20}$$

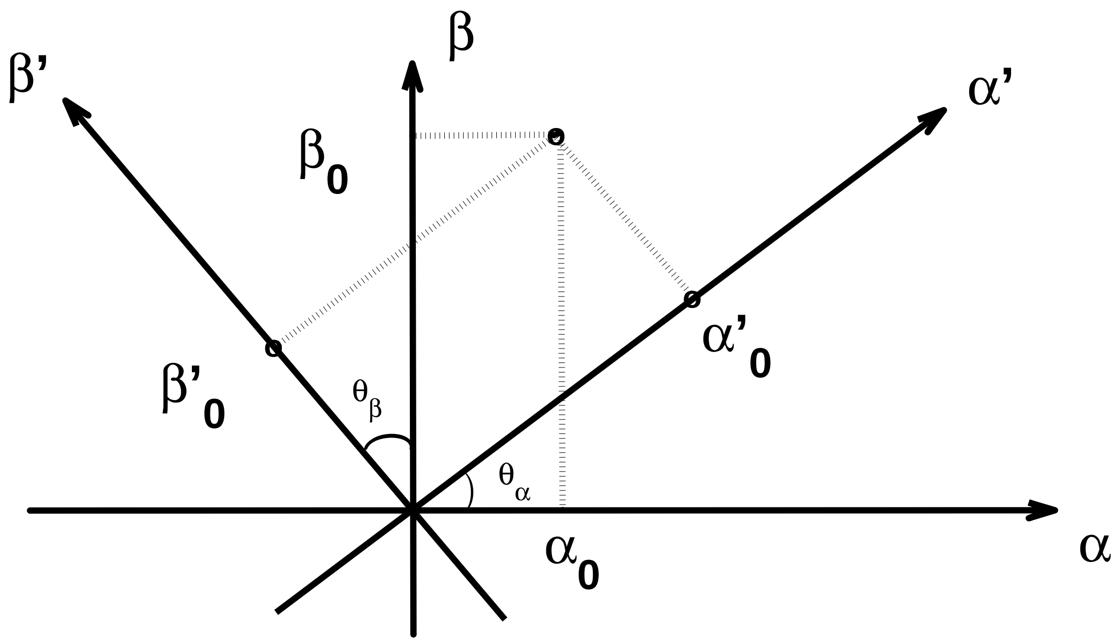


Fig. (3.2) The orthogonal and non-orthogonal axes

The distance of the point (α', β') from the origin is given in terms of the coordinates in the non-orthogonal frame is given by

$$[D_s(\alpha', \beta'|\theta_\alpha, \theta_\beta)]^2 = (\alpha')^2 + (\beta')^2 + 2\alpha'\beta' \sin(\theta_\alpha - \theta_\beta). \tag{3.21}$$

3.6 Bi-fractional displacement operators

The displaced parity operator is given as,

$$\widehat{P}(\alpha, \beta) = \widehat{D}\left(\frac{\alpha}{2}, \frac{\beta}{2}\right) \widehat{P}(0,0) \widehat{D}^\dagger \left(\frac{\alpha}{2}, \frac{\beta}{2}\right) = \widehat{D}(\alpha, \beta) \widehat{P}(0,0). \tag{3.22}$$

It is related to the displacement operator through a two-dimensional Fourier transform [18] written in this case as an interpolation along angles $(\frac{\pi}{2}, \frac{\pi}{2})$

$$\begin{aligned}\widehat{P}(\alpha,\beta) &= \frac{1}{2\pi}\int d\alpha' d\beta' \widehat{D}(\alpha',\beta') \exp\left[i(\beta\alpha' - \beta'\alpha)\right] \\ &= \int d\alpha' d\beta' K\left(\beta,\alpha';\frac{\pi}{2}\right) K\left(\alpha,-\beta';\frac{\pi}{2}\right)\widehat{D}(\alpha',\beta'). \end{aligned} \tag{3.23}$$

Since the displaced parity operator can be given in terms of the kernel of the Fourier transform, then it can be generalised to the fractional Fourier transform. The generalisation of the displaced parity operator is the following unitary operator which is called bi-fractional displacement operator [5] given as,

$$\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) = |\cos(\theta_\alpha - \theta_\beta)|^{1/2}\int d\alpha' d\beta'\, K\left(\beta,\alpha';\theta_\beta\right) K\left(\alpha,-\beta';\theta_\alpha\right)\widehat{D}(\alpha',\beta'). \tag{3.24}$$

Furthermore, the inverse function derived by taking the fractional Fourier transform of both sides is given as,

$$\widehat{D}(\alpha',\beta') = |\cos(\theta_\alpha - \theta_\beta)|^{-1/2}\int d\alpha d\beta K\left(\beta,\alpha';-\theta_\beta\right) K\left(\alpha,-\beta';-\theta_\alpha\right)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta). \tag{3.25}$$

Eq. (3.24) is derived by replacing the two Fourier transforms in Eq. (3.23) with two fractional Fourier transforms. The two fractional Fourier transforms use the variables α', β' which are related to position and momentum and are dual to each other. The two-dimensional fractional Fourier transform is not a direct generalisation of a one-dimensional fractional Fourier transform. It is therefore not a trivial generalisation.

This is further evident from the crucial role of the pre-factor $|\cos(\theta_\alpha - \theta_\beta)|^{\frac{1}{2}}$ in the proof of unitarity in section (3.6.1.1). Since $K(x, y; \theta + \pi) = K(x, -y; \theta)$ it follows that

$$\begin{aligned}
&\mathcal{O}(\alpha, \beta; \theta_\alpha + \pi, \theta_\beta) = \mathcal{O}(-\alpha, \beta; \theta_\alpha, \theta_\beta),\\
&\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta + \pi) = \mathcal{O}(\alpha, -\beta; \theta_\alpha, \theta_\beta).
\end{aligned} \tag{3.26}$$

Therefore taking $(\theta_\alpha, \theta_\beta) \in \Theta = [0, \pi) \times [0, \pi) - \mathcal{L}$, where $\mathcal{L}$ is the lines $\theta_\alpha - \theta_\beta = \pm\frac{\pi}{2}$. The pre-factor in Eq. (3.24) is zero for such a case and thus we avoid such cases in numerical work.

3.6.1 Properties of the bi-fractional operator

3.6.1.1 Unitarity

The bi-fractional displacement operator is unitary. Because the integral is not a straight forward generalisation of a 1-dimensional case, a pre-factor is present which is necessary for unitarity. In order to prove unitarity, the following relation is used

$$\begin{aligned}
&\widehat{D}(\alpha', \beta')\widehat{D}(\alpha'', \beta'') = \widehat{D}(\alpha' + \alpha'', \beta' + \beta'')\exp[i(\beta'\alpha'' - \alpha'\beta'')],\\
&[\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)]^\dagger = \mathcal{O}(-\alpha, -\beta; -\theta_\alpha, -\theta_\beta).
\end{aligned} \tag{3.27}$$

Performing a change of variables and evaluating the integral below,

$$\begin{aligned}
\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)\mathcal{O}(-\alpha, -\beta; -\theta_\alpha, -\theta_\beta) = \cos(\theta_\alpha - \theta_\beta) &\int d\alpha' d\beta' d\alpha'' d\beta'' K(\beta, \alpha'; \theta_\beta) K(\alpha, -\beta'; \theta_\alpha)\\
&\times K(-\alpha, -\beta''; -\theta_\alpha) K(-\beta, \alpha''; -\theta_\beta) D(\alpha', \beta') D(\alpha'', \beta'')\\
&= \mathbb{1}
\end{aligned} \tag{3.28}$$

This unitarity is proved below as,

$$\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)]^\dagger = \int d\alpha' d\beta' d\alpha'' d\beta'' K\left(\beta,\alpha';\theta_\beta\right) K\left(\alpha,-\beta';\theta_\alpha\right)$$
$$\times K\left(-\beta,\alpha'';-\theta_\beta\right) K\left(-\alpha,-\beta'';-\theta_\alpha\right) D(\alpha',\beta')D(\alpha'',\beta''), \tag{3.29}$$

where each of the kernels of the fractional Fourier transform can be given,

$$\begin{aligned}
K\left(\beta,\alpha';\theta_\beta\right) &= \left[\frac{1+i\cot\theta_\beta}{2\pi}\right]^{1/2} \exp\left[\frac{-i(\alpha'^2+\beta^2)\cot\theta_\beta}{2} + \frac{i\alpha'\beta}{\sin\theta_\beta}\right], \\
K\left(\alpha,-\beta';\theta_\alpha\right) &= \left[\frac{1+i\cot\theta_\alpha}{2\pi}\right]^{1/2} \exp\left[\frac{-i(\alpha^2+\beta'^2)\cot\theta_\alpha}{2} - \frac{i\alpha\beta'}{\sin\theta_\alpha}\right], \\
K\left(-\beta,\alpha'';-\theta_\beta\right) &= \left[\frac{1-i\cot\theta_\beta}{2\pi}\right]^{1/2} \exp\left[\frac{i(\alpha''^2+\beta^2)\cot\theta_\beta}{2} + \frac{i\alpha''\beta}{\sin\theta_\beta}\right], \\
K\left(-\alpha,-\beta'';-\theta_\alpha\right) &= \left[\frac{1-i\cot\theta_\alpha}{2\pi}\right]^{1/2} \exp\left[\frac{i(\alpha^2+\beta''^2)\cot\theta_\alpha}{2} - \frac{i\alpha\beta''}{\sin\theta_\alpha}\right].
\end{aligned} \tag{3.30}$$

Substituting each of the kernels yields,

$$\begin{aligned}
&\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)]^\dagger \\
=&\frac{1}{4\pi^2\sin\theta_\alpha\sin\theta_\beta}\int d\alpha' d\alpha'' d\beta' d\beta'' \exp\left[\frac{i(\alpha''^2-\alpha'^2)\cot\theta_\beta + i(\beta''^2-\beta'^2)\cot\theta_\alpha}{2}\right] \\
&\times\exp\left[\frac{i\beta(\alpha''+\alpha')}{\sin\theta_\beta} - \frac{i\alpha(\beta''+\beta')}{\sin\theta_\alpha}\right] D(\alpha'+\alpha'',\beta'+\beta'')\exp\left[i(\beta'\alpha''-\alpha'\beta'')\right]
\end{aligned} \tag{3.31}$$

The next step is to change variables by making the following substitutions,

$\alpha_s = \alpha''+\alpha'$, $\alpha_d = \alpha''-\alpha'$, $\beta_s = \beta''+\beta'$, $\beta_d = \beta''-\beta'$,

and use the Jacobian such that,

$$d\alpha' d\alpha'' = \left|\frac{\partial(\alpha',\alpha'')}{\partial(\alpha_s,\alpha_d)}\right| d\alpha_s d\alpha_d = \frac{1}{4} d\alpha_s d\alpha_d. \tag{3.32}$$

By using the relation in Eq. (3.27), Eq. (3.31) is resolved to,

$$
\begin{aligned}
&\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)]^\dagger \\
&= \frac{1}{16\pi^2\sin\theta_\alpha\sin\theta_\beta}\int d\alpha_s d\alpha_d d\beta_s d\beta_d \;\; D(\alpha_s,\beta_s) \\
&\times\exp\left[\frac{i(\alpha_s\alpha_d\cot\theta_\beta+\beta_s\beta_d\cot\theta_\alpha)}{2}+\frac{i\alpha_s\beta}{\sin\theta_\beta}-\frac{i\alpha\beta_s}{\sin\theta_\alpha}+i\left(\frac{\alpha_d\beta_s-\alpha_s\beta_d}{2}\right)\right] \\
&= \frac{1}{16\pi^2\sin\theta_\alpha\sin\theta_\beta}\int d\alpha_s d\beta_s d\beta_d \;\; D(\alpha_s,\beta_s) \\
&\times\exp\left[\frac{i\beta_s\beta_d\cot\theta_\alpha}{2}+\frac{i\alpha_s\beta}{\sin\theta_\beta}-\frac{i\alpha\beta_s}{\sin\theta_\alpha}-\frac{i\alpha_s\beta_d}{2}\right]4\pi\delta(\beta_s+\alpha_s\cot\theta_\beta) \\
&= \frac{1}{4\pi\sin\theta_\alpha\sin\theta_\beta}\int d\alpha_s \;\; D(\alpha_s,-\alpha_s\cot\theta_\beta) \\
&\times\exp\left[\frac{i\alpha_s\beta}{\sin\theta_\beta}+\frac{i\alpha\alpha_s\cot\theta_\beta}{\sin\theta_\alpha}\right]\frac{4\pi\delta(\alpha_s)}{1+\cot\theta_\alpha\cot\theta_\beta} \\
&= \frac{1}{\sin\theta_\alpha\sin\theta_\beta(1+\cot\theta_\alpha\cot\theta_\beta)} = \frac{1}{\cos(\theta_\alpha-\theta_\beta)}
\end{aligned}
\tag{3.33}
$$

Since this pre-factor cannot be factorised as a function of θ_α times a function of θ_β, proves the fact that the variables α',β' are dual quantum variables. In the case $\theta_\alpha-\theta_\beta=\frac{\pi}{2}$ the integral of Eq. (3.24) diverges, and in numerical work this is avoided.

3.6.1.2 Interpolation between displacement and parity operators

The following are special cases of the interpolation of the bi-fractional operator sweeping between both the displacement operators and parity operator for special cases of $\theta_\alpha=\theta_\beta$,

$$\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) = \left\{ \begin{array}{l} \widehat{D}(\beta,-\alpha), \text{ if } \theta_\alpha = \theta_\beta = 0 \\ \\ \widehat{P}(\alpha,\beta), \text{ if } \theta_\alpha = \theta_\beta = \frac{\pi}{2} \\ \\ \widehat{D}(-\beta,\alpha), \text{ if } \theta_\alpha = \theta_\beta = \pi \end{array} \right\}. \tag{3.34}$$

For the scope of this work, only special cases are considered. For $\theta_\alpha \neq \theta_\beta$, the bi-fractional operator gives us some other operators which are still open to research.

3.6.1.3 Marginal properties for $\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$

The bi-fractional displacement operator, $\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$ also has marginal properties which for special cases of ($\theta_\alpha = \theta_\beta = \frac{\pi}{2}$), these properties reduce to the marginal properties of the displaced parity operator.

Integration of $\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$ with respect to α gives

$$\begin{aligned} &\int d\alpha\, \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) \\ &\quad = |\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}} \int d\alpha' d\beta' K\left(\beta,\alpha';\theta_\beta\right) K\left(0,\beta';\frac{\pi}{2}-\theta_\alpha\right) \widehat{D}(\alpha',\beta'). \end{aligned} \tag{3.35}$$

The proof is given below,

$$
\begin{aligned}
&\int d\alpha\, \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) \\
&= |\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}} \left[\frac{1+i\cot\theta_\alpha}{2\pi}\right]^{\frac{1}{2}} \int d\alpha' d\beta' K\left(\beta,\alpha';\theta_\beta\right) \\
&\times \exp\left[\frac{-i\beta'^2\cot\theta_\alpha}{2}\right] D(\alpha',\beta') \int d\alpha \exp\left[-\alpha^2\left(\frac{i\cot\theta_\alpha}{2}\right)-\alpha\left(\frac{i\beta'}{\sin\theta_\alpha}\right)\right] \\
&= |\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}} \left[1-i\tan\theta_\alpha\right]^{\frac{1}{2}} \int d\alpha' d\beta' K\left(\beta,\alpha';\theta_\beta\right) \exp\left[\frac{i\beta'^2\tan\theta_\alpha}{2}\right] D(\alpha',\beta') \\
&= |\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}} \int d\alpha' d\beta' K\left(\beta,\alpha';\theta_\beta\right) K\left(0,\beta';\frac{\pi}{2}-\theta_\alpha\right) D(\alpha',\beta') \qquad (3.36)
\end{aligned}
$$

Having used the relation

$$
K\left(0,\beta';\frac{\pi}{2}-\theta\right) = \left[1-i\tan\theta\right]^{\frac{1}{2}} \exp\left[\frac{i\beta'^2\tan\theta}{2}\right] \qquad (3.37)
$$

The result shows that the integration affects only the kernel with respect to α. Integration of $\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$ with respect to β gives a similar result following very similar steps as that with respect to α,

$$
\begin{aligned}
&\int d\beta\, \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) \\
&\quad = |\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}} \int d\alpha' d\beta' K\left(0,\alpha';\frac{\pi}{2}-\theta_\beta\right) K\left(\alpha,-\beta';\theta_\alpha\right) \widehat{D}(\alpha',\beta') \qquad (3.38)
\end{aligned}
$$

Finally, with respect to both α and β gives

$$
\begin{aligned}
\int d\alpha d\beta \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) &= |\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}} \int d\alpha' d\beta' K\left(0,\alpha';\frac{\pi}{2}-\theta_\beta\right) K\left(0,\beta';\frac{\pi}{2}-\theta_\alpha\right) \\
&\quad \times \widehat{D}(\alpha',\beta') \qquad (3.39)
\end{aligned}
$$

The effect is a combination of results from Eqs. (3.35, 3.38).

3.6.1.4 Bi-fractional operators as special elements of the group G of squeezing and displacement transformations

Another form of the bi-fractional displacement operator given in Eq. (3.24) using the Baker-Campbell-Hausdorff operator relation of Eq. (2.65) as,

$$\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) = \exp(i\phi)\exp\left[i\tau(\hat{p}-\tan\theta_\alpha\hat{x}+\sigma)^2\right]\exp\left(i\frac{\hat{x}^2}{\cot\theta_\alpha}-i\frac{\sqrt{2}\alpha\hat{x}}{\cos\theta_\alpha}\right),$$
$$\tau = \frac{\cos\theta_\alpha\sin\theta_\beta}{\cos(\theta_\alpha-\theta_\beta)};\quad \sigma = \frac{\alpha}{\sqrt{2}\cos\theta_\alpha}-\frac{\beta}{\sqrt{2}\sin\theta_\beta},$$
$$\phi = -\frac{1}{2}(\theta_\alpha+\theta_\beta)-\frac{1}{2}(\alpha^2\cot\theta_\alpha+\beta^2\cot\theta_\beta)+\frac{\alpha^2}{\sin 2\theta_\alpha}. \tag{3.40}$$

The proof is given by performing integration in Eq. (3.24) as follows.

$$\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) = |\cos(\theta_\alpha-\theta_\beta)|^{1/2}\int d\alpha' d\beta' K(\beta,\alpha';\theta_\beta)\, K(\alpha,-\beta';\theta_\alpha)\, D(\alpha',\beta')$$
$$= \mathcal{R}_1\int d\alpha' d\beta' \exp\left[-\frac{i\cot\theta_\beta}{2}\alpha'^2-\frac{i\cot\theta_\alpha}{2}\beta'^2+\frac{i\beta\alpha'}{\sin\theta_\beta}-\frac{i\alpha\beta'}{\sin\theta_\alpha}+i\sqrt{2}\beta'\hat{x}-i\sqrt{2}\alpha'\hat{p}\right]$$
$$= \mathcal{R}_1\int d\alpha' L(\alpha') \tag{3.41}$$

where

$$L(\alpha') = \exp\left[-\frac{i\cot\theta_\beta}{2}\alpha'^2+\frac{i\beta\alpha'}{\sin\theta_\beta}-i\sqrt{2}\alpha'\hat{p}\right]\int d\beta' \exp\left[-\frac{i\cot\theta_\alpha}{2}\beta'^2-\frac{i\alpha\beta'}{\sin\theta_\alpha}+i\sqrt{2}\beta'\hat{x}+i\alpha'\beta'\right]$$
$$\mathcal{R}_1 = |\cos(\theta_\alpha-\theta_\beta)|^{1/2}\left[\frac{1+i\cot\theta_\alpha}{2\pi}\right]^{1/2}\left[\frac{1+i\cot\theta_\beta}{2\pi}\right]^{1/2}\exp\left[-\frac{i}{2}(\alpha^2\cot\theta_\alpha+\beta^2\cot\theta_\beta)\right]. \tag{3.42}$$

The relation for two operators in Eq. (2.65) was used for these calculations.

Gaussian integration produces,

$$\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) = \mathcal{R}_2\left\{\int d\alpha' \exp\left[-\frac{i\cot\theta_\beta}{2}\alpha'^2 + \frac{i\beta\alpha'}{\sin\theta_\beta} - i\sqrt{2}\alpha'\hat{p}\right]\exp\left(i\frac{2^{3/2}\hat{x}\alpha' + \alpha'^2}{2\cot\theta_\alpha} - i\frac{\alpha\alpha'}{\cos\theta_\alpha}\right)\right\}$$
$$\times \exp\left(i\frac{\hat{x}^2}{\cot\theta_\alpha} - i\frac{\sqrt{2}\alpha\hat{x}}{\cos\theta_\alpha}\right)$$
$$= \mathcal{R}_2\left\{\int d\alpha' \exp\left[-\frac{i\cot\theta_\beta}{2}\alpha'^2 + \frac{i\beta\alpha'}{\sin\theta_\beta} - i\sqrt{2}\alpha'\hat{p} + i\frac{2^{3/2}\hat{x}\alpha' + \alpha'^2}{2\cot\theta_\alpha} - i\frac{\alpha\alpha'}{\cos\theta_\alpha} - i\frac{\alpha'^2}{\cot\theta_\alpha}\right]\right\}$$
$$\times \exp\left(i\frac{\hat{x}^2}{\cot\theta_\alpha} - i\frac{\sqrt{2}\alpha\hat{x}}{\cos\theta_\alpha}\right), \tag{3.43}$$

where

$$\mathcal{R}_2 = \mathcal{R}_1\left(\frac{2\pi}{i\cot\theta_\alpha}\right)^{1/2}\exp\left(i\frac{\alpha^2}{\sin 2\theta_\alpha}\right). \tag{3.44}$$

From this follows Eq. (3.40).

Here we note that the operators $\hat{x}^2$, $\hat{p}^2$, $\hat{x}\hat{p}+\hat{p}\hat{x}$, $\hat{x}$, $\hat{p}$, $\mathbb{1}$, form a closed structure under commutation, and therefore

$$Y(a_1,a_2,a_3,a_4,a_5,a_6) = \exp[a_1\hat{x}^2 + a_2\hat{p}^2 + a_3(\hat{x}\hat{p}+\hat{p}\hat{x}) + a_4\hat{x} + a_5\hat{p} + a_6\mathbb{1}], \tag{3.45}$$

form a group G. The displacement transformations

$$Y_d(a_4,a_5,a_6) = \exp(a_4\hat{x} + a_5\hat{p} + a_6\mathbb{1}), \tag{3.46}$$

form a representation of the Heisenberg Weyl group HW, which is a normal subgroup of G. Also the squeezing transformations

$$Y_s(a_1,a_2,a_3) = \exp[a_1\hat{x}^2 + a_2\hat{p}^2 + a_3(\hat{x}\hat{p}+\hat{p}\hat{x})] \tag{3.47}$$

form a representation of the $SU(1,1)$ group, which is a subgroup of G. Every element of G can be written as a product of an element of HW and an element of $SU(1,1)$. The intersection of these two subgroups of G, contains only the unit operator. G is the semidirect product of the Heisenberg Weyl group HW of displacements, by the $SU(1,1)$ group of squeezing transformations:

$$G = HW \rtimes SU(1,1). \tag{3.48}$$

The operators $\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)$ depend on four parameters, and they are special cases of the operators $Y(a_1, a_2, a_3, a_4, a_5, a_6)$. But clearly the general element $Y(a_1, a_2, a_3, a_4, a_5, a_6)$ which depends on six parameters cannot always be written as $\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)$ which depends on four parameters.

3.6.1.5 Bi-fractional displacement operators in different sets

The bi-fractional displacement operator can be generalised for special cases of $(\theta_\alpha + \phi_\alpha, \theta_\beta + \phi_\beta)$ using Eq. (3.13). The bi-fractional displacement operators in different sets, $\mathcal{S}_\mathcal{O}(\theta_\alpha, \theta_\beta)$ and $\mathcal{S}_\mathcal{O}(\theta_\alpha + \phi_\alpha, \theta_\beta + \phi_\beta)$ are related by fractional Fourier transform,

$$\begin{aligned} \mathcal{O}(\alpha, \beta; \theta_\alpha + \phi_\alpha, \theta_\beta + \phi_\beta) = \frac{|\cos(\theta_\alpha + \phi_\alpha - \theta_\beta - \phi_\beta)|^{1/2}}{|\cos(\theta_\alpha - \theta_\beta)|^{1/2}} \\ \times \int d\alpha' d\beta' K(\beta, \beta'; \phi_\beta) K(\alpha, \alpha'; \phi_\alpha) \mathcal{O}(\alpha', \beta'; \theta_\alpha, \theta_\beta). \end{aligned} \tag{3.49}$$

The proof is given by taking two special cases of bi-fractional displacement operators. Each case is a phase shift in the angles, so that $\theta_\alpha \to (\theta_\alpha + \phi_\alpha)$,

$$\mathcal{O}(\alpha,\beta;\theta_\alpha+\phi_\alpha,\theta_\beta) = \frac{|\cos(\theta_\alpha+\phi_\alpha-\theta_\beta)|^{1/2}}{|\cos(\theta_\alpha-\theta_\beta)|^{1/2}} \int d\alpha' K\left(\alpha,\alpha';\phi_\alpha\right)\mathcal{O}(\alpha',\beta;\theta_\alpha,\theta_\beta), \tag{3.50}$$

and for $\theta_\beta \to (\theta_\beta + \phi_\beta)$, it leads to

$$\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta+\phi_\beta) = \frac{|\cos(\theta_\alpha+\phi_\beta-\theta_\beta)|^{1/2}}{|\cos(\theta_\alpha-\theta_\beta)|^{1/2}} \int d\beta' K\left(\beta,\beta';\phi_\beta\right)\mathcal{O}(\alpha,\beta';\theta_\alpha,\theta_\beta). \tag{3.51}$$

Combining both of them gives Eq. (3.49).

3.6.1.6 Groupoid of transformations from $\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$

The bi-fractional displacement operators show squeezing properties and do not form a group and so the groupoids are proposed to describe their mathematical structure. The groupoid is an algebraic structure with partial function. Alternatively, a groupoid is a small category in which every morphism is an isomorphism [44]. The groupoid [19, 76] is a weaker structure, designed for ‘variable symmetries’. Groups are special cases of groupoids [19, 76]. Applications of groupoids include non-commutative geometry [28, 46] and quantum tomography [43].

Unlike a group, the groupoid does not have the closure property. Consider two base ($\mathfrak{B}$) sets, t_1, t_2 being the start and target of a map, then a groupoid with two maps is defined,

$$T_1(x) = t_1 \quad T_2(x) = t_2 \quad t_1, t_2 \in \mathfrak{B} \tag{3.52}$$

where t_1 and t_2 represent source and target of x. We can consider x as an 'arrow' which starts at t_1 and ends at t_2. It does however have a partial associative multiplication x_1x_2, defined only in the case that $T_2(x_1) = T_1(x_2)$. It also has an involution ('inverse') property, $x \to x^{-1}$; $[x^{-1}]^{-1} = x$, and has $L_x = xx^{-1}$ and $R_x = x^{-1}x$ called left and right identities which are in general different. The identities are such that, $L_x x = xR_x = x$ with a base set $\mathcal{B}$, is isomorphic to the set of all left identities and to the set of all right identities.

Considering the set of transformations with the base set, $\mathfrak{B}$ defined as,

$$\mathfrak{B} = \{\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) \,|\, \alpha,\beta \in \mathbb{R},\ \theta_\alpha,\theta_\beta \in [0,2\pi],\ \theta_\alpha \neq \theta_\beta \pm \tfrac{\pi}{2}\}, \tag{3.53}$$

and also the map

$$\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta) : \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) \longrightarrow \mathcal{O}(\gamma,\lambda;\phi_\alpha,\phi_\beta), \tag{3.54}$$

where

$$\mathcal{O}(\gamma,\lambda;\phi_\alpha,\phi_\beta) = \frac{|\cos(\phi_\alpha-\phi_\beta)|^{\frac{1}{2}}}{|\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}}} \int d\alpha d\beta K(\beta,\lambda;\phi_\beta-\theta_\beta)\, K(\alpha,\gamma;\phi_\alpha-\theta_\alpha)\, \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta). \tag{3.55}$$

Eq. (3.55) is a generalised version of Eq. (3.24), which in the present notation is the map

$$\mathcal{M}(\alpha',\beta';0,0|\alpha,\beta;\theta_\alpha,\theta_\beta) : \mathcal{O}(\alpha',\beta';0,0) \longrightarrow \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta). \tag{3.56}$$

The compatibility between the two, is shown in the first part of the proof below.

In the special case that $\phi_\alpha = \theta_\alpha$ and $\phi_\beta = \theta_\beta$, the $K(\beta,\lambda;0)$ and $K(\alpha,\gamma;0)$ are

Dirac delta functions and

$$\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\theta_\alpha,\theta_\beta) : \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta) \;\longrightarrow\; \mathcal{O}(\gamma,\lambda;\theta_\alpha,\theta_\beta). \tag{3.57}$$

Next, considering the following notation for the composition

$$\begin{aligned}&[\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha\phi_\beta)\circ\mathcal{M}(\gamma,\lambda;\phi_\alpha,\phi_\beta|\epsilon,\zeta;\psi_\alpha,\psi_\beta)][\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)]\\&=\mathcal{M}(\gamma,\lambda;\phi_\alpha,\phi_\beta|\epsilon,\zeta;\psi_\alpha,\psi_\beta)[\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)]\end{aligned} \tag{3.58}$$

The set $\{\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)\}$ is a connected groupoid with base set $\mathcal{B}$ (in Eq. (3.58), and with composition as multiplication. The inverse of $\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_2|\gamma,\lambda;\phi_1,\phi_2)$ is

$$[\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)]^{-1} = \mathcal{M}(\gamma,\lambda;\phi_\alpha,\phi_\beta|\alpha,\beta;\theta_\alpha,\theta_\beta). \tag{3.59}$$

The left and right identities are $\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\alpha,\beta;\theta_\alpha,\theta_\beta)$ and $\mathcal{M}(\gamma,\lambda;\phi_\alpha,\phi_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)]$. The proof consists of the following three parts:

(1) The first part shows that the following compatibility relation holds

$$\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)\circ\mathcal{M}(\gamma,\lambda;\phi_\alpha,\phi_\beta|\epsilon,\zeta;\psi_\alpha,\psi_\beta) = \mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\epsilon,\zeta;\psi_\alpha,\psi_\beta), \tag{3.60}$$

and start with the relations

$$\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)[\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)] = \mathcal{O}(\gamma,\lambda;\phi_\alpha,\phi_\beta)$$
$$= \frac{|\cos(\phi_\alpha-\phi_\beta)|^{\frac{1}{2}}}{|\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}}}\int d\alpha d\beta K(\beta,\lambda;\phi_\beta-\theta_\beta)K(\alpha,\gamma;\phi_\alpha-\theta_\alpha)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta), \tag{3.61}$$

and

$$\mathcal{M}(\gamma,\lambda;\phi_\alpha,\phi_\beta|\epsilon,\zeta;\psi_\alpha,\psi_\beta)[\mathcal{O}(\gamma,\lambda;\phi_\alpha,\phi_\beta)] = \mathcal{O}(\epsilon,\zeta;\psi_\alpha,\psi_\beta)$$
$$= \frac{|\cos(\psi_\alpha-\psi_\beta)|^{\frac{1}{2}}}{|\cos(\phi_\alpha-\phi_\beta)|^{\frac{1}{2}}}\int d\gamma d\lambda K(\lambda,\zeta;\psi_\beta-\phi_\beta)K(\gamma,\epsilon;\psi_\alpha-\phi_\alpha)\mathcal{O}(\gamma,\lambda;\phi_\alpha,\phi_\beta). \tag{3.62}$$

Inserting Eq. (3.61) into Eq. (3.62) produces

$$\mathcal{M}(\gamma,\lambda;\phi_\alpha,\phi_\beta|\epsilon,\zeta;\psi_\alpha,\psi_\beta)[\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)[\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)]]$$
$$= \frac{|\cos(\psi_\alpha-\psi_\beta)|^{\frac{1}{2}}}{|\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}}}\int d\gamma d\lambda K(\lambda,\zeta;\psi_\beta-\phi_\beta)K(\gamma,\epsilon;\psi_\alpha-\phi_\alpha)$$
$$\times d\alpha d\beta K(\beta,\lambda;\phi_\beta-\theta_\beta)K(\alpha,\gamma;\phi_\alpha-\theta_\alpha)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta). \tag{3.63}$$

The compatibility relation of Eq. (3.60) holds because using Eq. (3.13) shows that Eq. (3.63) reduces to

$$\begin{aligned}
&\mathcal{M}(\gamma,\lambda;\phi_\alpha,\phi_\beta|\epsilon,\zeta;\psi_\alpha,\psi_\beta)[\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)[\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)]]\\
&=\frac{|\cos(\psi_\alpha-\psi_\beta)|^{\frac{1}{2}}}{|\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}}}\int d\alpha d\beta K(\beta,\zeta;\psi_\beta-\theta_\beta)K(\alpha,\epsilon;\psi_\alpha-\theta_\alpha)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\\
&=\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\epsilon,\zeta;\psi_\alpha,\psi_\beta)[\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)]. \qquad (3.64)
\end{aligned}$$

(2) For the inverse, it is easily seen that it is an involution:

$$\{[\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)]^{-1}\}^{-1}=\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta). \qquad (3.65)$$

The left and right identities are shown to be

$$\begin{aligned}
&\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)\circ\mathcal{M}(\gamma,\lambda;\phi_\alpha,\phi_\beta|\alpha,\beta;\theta_\alpha,\theta_\beta)=\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\alpha,\beta;\theta_\alpha,\theta_\beta)\\
&\mathcal{M}(\gamma,\lambda;\phi_\alpha,\phi_\beta|\alpha,\beta;\theta_\alpha,\theta_\beta)\circ\mathcal{M}(\alpha,\beta;\theta_\alpha,\theta_\beta|\gamma,\lambda;\phi_\alpha,\phi_\beta)=\mathcal{M}(\gamma,\lambda;\phi_1,\phi_2|\gamma,\lambda;\phi_\alpha,\phi_\beta).
\end{aligned} \qquad (3.66)$$

(3) The above two parts show that $\mathcal{M}$ is a groupoid. In fact it is a connected groupoid because any two elements $\mathcal{O}(\gamma,\lambda;\phi_\alpha,\phi_\beta)$, $\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$ in a set, are related through Eq. (3.55).

3.7 Bi-fractional coherent states

Coherent states have been studied extensively in the literature for a long time [6, 45, 62], and they play a central role in phase space methods in quantum mechanics [64, 80]. Various generalisations of coherent states have also been studied, especially in connection with groups like $SU(2)$, $SU(1,1)$, etc. Acting with the bi-fractional displacement operators on the vacuum creates various classes of generalised coherent states (one for each pair $(\theta_\alpha, \theta_\beta)$), called the bi-fractional coherent states. They are squeezed states, and were studied briefly [5] as a topic because of their role in interpolations of different quantities in phase space methods.

Given a pair $(\theta_\alpha, \theta_\beta)$ the set of 'bi-fractional coherent states' is given as:

$$\begin{aligned}\mathcal{S}_{\mathcal{C}}(\theta_\alpha, \theta_\beta) &\equiv \{|\alpha, \beta; \theta_\alpha, \theta_\beta\rangle = \mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)\,|0\rangle\,; \alpha, \beta \in \mathbb{R}\} \\ &= |\cos(\theta_\alpha - \theta_\beta)|^{1/2} \int d\alpha' d\beta'\, K(\beta, \alpha'; \theta_\beta)\, K(\alpha, -\beta'; \theta_\alpha)\, |\alpha', \beta'\rangle\,. \end{aligned} \tag{3.67}$$

Also in the special case of $\theta_\alpha = \theta_\beta = 0$ one has

$$|\alpha, \beta; 0, 0\rangle = \mathcal{O}(\alpha, \beta; 0, 0)\,|0\rangle = \widehat{D}(\beta, -\alpha)\,|0\rangle\,, \tag{3.68}$$

and for $\theta_\alpha = \theta_\beta = \frac{\pi}{2}$

$$\left|\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{2}\right\rangle = \mathcal{O}\left(\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{2}\right)|0\rangle = \widehat{D}(\alpha, \beta)\,|0\rangle\,. \tag{3.69}$$

Therefore in these special one obtains the standard Glauber coherent states,

$$|-\beta, \alpha; 0, 0\rangle = \left|\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{2}\right\rangle = |\alpha, \beta\rangle\,. \tag{3.70}$$

Since the bi-fractional operator is unitary, the coherent states in the set $\mathcal{S}_C(\theta_\alpha, \theta_\beta)$ are Glauber coherent states with respect to the operators

$$d(\theta_\alpha, \theta_\beta) = \mathcal{O}(0,0;\theta_\alpha,\theta_\beta)\hat{a}[\mathcal{O}(0,0;\theta_\alpha,\theta_\beta)]^\dagger; \quad d^\dagger(\theta_\alpha, \theta_\beta) = \mathcal{O}(0,0;\theta_\alpha,\theta_\beta)\hat{a}^\dagger[\mathcal{O}(0,0;\theta_\alpha,\theta_\beta)]^\dagger. \tag{3.71}$$

Yet they have novel non-trivial properties with respect to $\hat{a}, \hat{a}^\dagger$. The coherent states in the set $\mathcal{S}_C(\theta_\alpha, \theta_\beta)$ satisfy the resolution of the identity

$$\frac{1}{2\pi\cos(\theta_\alpha - \theta_\beta)} \int d\alpha d\beta \, |\alpha, \beta; \theta_\alpha, \theta_\beta\rangle \langle\alpha, \beta; \theta_\alpha, \theta_\beta| = \mathbb{1}, \tag{3.72}$$

which in later sections will be elaborated. From this follows that an arbitrary state $|g\rangle$ can be written as

$$|g\rangle = \int d\alpha d\beta \, |\alpha, \beta; \theta_\alpha, \theta_\beta\rangle \, g(\alpha, \beta; \theta_\alpha, \theta_\beta); \quad g(\alpha, \beta; \theta_\alpha, \theta_\beta) = \frac{1}{2\pi} \langle\alpha, \beta; \theta_\alpha, \theta_\beta | \, g\rangle. \tag{3.73}$$

3.7.1 Properties of the bi-fractional coherent states

In this section, some of the properties of the bi-fractional coherent states are stated.

3.7.1.1 Bi-fractional coherent states in different sets

Having considered the property of bi-fractional displacement operators in Eq. (3.49), this can also be extended to the coherent states. The coherent states can be described in different sets $\mathcal{S}_{\mathcal{C}}(\theta_\alpha + \phi_\alpha, \theta_\beta + \phi_\beta)$ which are related to the coherent states in the set $\mathcal{S}_{\mathcal{C}}(\theta_\alpha, \theta_\beta)$ by fractional Fourier transform through the fractional Fourier transform

$$\begin{aligned} |\alpha, \beta; \theta_\alpha + \phi_\alpha, \theta_\beta + \phi_\beta\rangle = & \frac{|\cos(\theta_\alpha + \phi_\alpha - \theta_\beta - \phi_\beta)|^{1/2}}{|\cos(\theta_\alpha - \theta_\beta)|^{1/2}} \\ & \times \int d\alpha' d\beta' K(\beta, \beta'; \phi_\beta) \, K(\alpha, \alpha'; \phi_\alpha) \, |\alpha', \beta'; \theta_\alpha, \theta_\beta\rangle \, . \end{aligned} \tag{3.74}$$

Special cases of this relation with respect to each of the angles is given as

$$\left|\alpha,\beta;\theta_\alpha+\frac{\pi}{2},\theta_\beta\right\rangle = \frac{|\cos(\theta_\alpha+\frac{\pi}{2}-\theta_\beta)|^{1/2}}{\sqrt{2\pi}|\cos(\theta_\alpha-\theta_\beta)|^{1/2}}\int d\alpha'\exp\left(i\alpha\alpha'\right)|\alpha',\beta;\theta_\alpha,\theta_\beta\rangle$$
$$\left|\alpha,\beta;\theta_\alpha,\theta_\beta+\frac{\pi}{2}\right\rangle = \frac{|\cos(\theta_\alpha-\theta_\beta-\frac{\pi}{2})|^{1/2}}{\sqrt{2\pi}|\cos(\theta_\alpha-\theta_\beta)|^{1/2}}\int d\beta'\exp\left(i\beta\beta'\right)|\alpha,\beta';\theta_\alpha,\theta_\beta\rangle\,. \tag{3.75}$$

As explained earlier, in order to avoid divergencies it is required that $\theta_\alpha-\theta_\beta\neq\pm\frac{\pi}{2}$.

3.7.1.2 Analyticity property of bi-fractional coherent states

The coherent states discussed in Eq. (3.67) can be represented as,

$$\widehat{D}(\alpha,\beta)|0\rangle = \exp\left[-\frac{1}{2}(\alpha^2+\beta^2)\right]\sum_{n=0}^{\infty}(\alpha+i\beta)^n(n!)^{-1/2}|n\rangle_n \tag{3.76}$$

The coherent state is obviously analytic with respect to $z^*=\alpha-i\beta$. The bi-fractional coherent state on the other hand cannot be written exclusively in terms of $z=\alpha+i\beta$, because it interpolates between both z and z^*. For the sake of convenience, the bi-fractional coherent state is given in complex notation, w so that,

$$|\alpha,\beta;\theta_\alpha,\theta_\beta\rangle \equiv |w(\theta_\alpha,\theta_\beta)\rangle\,;\quad w(\theta_\alpha,\theta_\beta)=\frac{\alpha E_\alpha+i\beta E_\beta}{\cos(\theta_\alpha-\theta_\beta)}$$
$$E_\alpha=i\exp(-i\theta_\alpha);\quad E_\beta=i\exp(-i\theta_\beta) \tag{3.77}$$

We show that $w((\theta_\alpha,\theta_\beta)$ is a linear combination of z and z^*. We write the bi-fractional displacement operator in complex notation and show that,

$$|\cos(\theta_\alpha-\theta_\beta)|^{\frac{1}{2}}\int\frac{d^2\zeta}{2}A(w,\zeta)\frac{\partial}{\partial\zeta^*}[\exp(\zeta\hat{a}^\dagger-\zeta^*\hat{a})|0\rangle]=0, \tag{3.78}$$

and integration by parts gives

$$|\cos(\theta_\alpha - \theta_\beta)|^{\frac{1}{2}} \int \frac{d^2\zeta}{2}\left[\frac{\partial}{\partial\zeta^*}A(w,\zeta)\right]\exp(\zeta a^\dagger - \zeta^*\hat{a})\,|0\rangle = 0. \tag{3.79}$$

The necessary proof which will lead to the proof of analyticity will be shown to be,

$$\frac{\partial}{\partial w^*}A(w,\zeta) = \frac{\partial}{\partial\zeta^*}A(w,\zeta). \tag{3.80}$$

The analyticity of Glauber coherent states is proved by taking $\partial_{z^*}\left[\exp\left(\frac{1}{2}|z|^2\right)\right]$ and in this case the bi-fractional coherent can be written as a combination of w and w^*,

$$\begin{aligned}\exp\left(-\frac{1}{2}[B(w^*)^2 + \Gamma|w|^2]\right)|w;\theta_\alpha,\theta_\beta\rangle &= H(w,w^*)\,|w;\theta_\alpha,\theta_\beta\rangle\\ H(w,w^*) = \exp&\left(\frac{1}{2}\left[|w(\theta_\alpha,\theta_\beta)|^2 - Bw^*(\theta_\alpha,\theta_\beta)^2\right]\right)\end{aligned} \tag{3.81}$$

where B and Γ are constants that need to be determined for obtaining analyticity. The analyticity in this case is that it depends only on w, and does not depend on w^*. The proof begins with,

$$\exp\left(-\frac{1}{2}[B(w^*(\theta_\alpha,\theta_\beta)^2 + \Gamma|w(\theta_\alpha,\theta_\beta)|^2]\right)|w(\theta_\alpha,\theta_\beta)\rangle$$

$$= |\cos(\theta_\alpha - \theta_\beta)|^{\frac{1}{2}} \int d^2\zeta A(w,\zeta)\exp(\zeta a^\dagger - \zeta^*\hat{a})\,|0\rangle\,,$$

$$A(w,\zeta) = K(\nu,\zeta_R;\theta_\beta)\,K(\mu,-\zeta_I;\theta_\alpha)\exp\left[-\frac{1}{2}Bw(\theta_\alpha,\theta_\beta)*^2 + \frac{1}{2}|w(\theta_\alpha,\theta_\beta)|^2 - \frac{1}{2}|\zeta|^2\right],$$

$$\mu = \frac{1}{2}(E_\alpha^* w(\theta_\alpha,\theta_\beta) + E_\alpha w(\theta_\alpha,\theta_\beta)^*); \qquad \nu = \frac{1}{i2}(E_\beta^* w(\theta_\alpha,\theta_\beta) - E_\beta w(\theta_\alpha,\theta_\beta)^*), \tag{3.82}$$

And further perform,

$$\frac{\partial}{\partial w^*}A(w,\zeta) = A(w,\zeta)\left[-\frac{i\cot\theta_\beta}{4}(|E_\beta|^2 w(\theta_\alpha,\theta_\beta) - E_\beta^2 w^*(\theta_\alpha,\theta_\beta)) - \frac{E_\beta \zeta_R}{2\sin\theta_\beta}\right]$$
$$+ A(w,\zeta)\left[-\frac{i\cot\theta_\alpha}{4}(|E_\alpha|^2 w(\theta_\alpha,\theta_\beta) - E_\alpha^2 w^*(\theta_\alpha,\theta_\beta)) - i\frac{E_\alpha \zeta_I}{2\sin\theta_\alpha}\right]$$
$$+ A(w,\zeta)\left[-Bw(\theta_\alpha,\theta_\beta)^* + \frac{1}{2}w(\theta_\alpha,\theta_\beta)\right] \tag{3.83}$$

And also,

$$\frac{\partial}{\partial \zeta^*}A(w,\zeta) = A(w,\zeta)\left[-\frac{iE_\beta\zeta_R}{2\sin\theta_\beta} + \frac{1}{4\sin\theta_\beta}(E_\beta^* w(\theta_\alpha,\theta_\beta) - E_\beta w^*(\theta_\alpha,\theta_\beta))\right]$$
$$+ A(w,\zeta)\left[-\frac{iE_\alpha\zeta_I}{2\sin\theta_\alpha} + \frac{1}{4\sin\theta_\alpha}(E_\alpha^* w(\theta_\alpha,\theta_\beta) - E_\alpha w^*(\theta_\alpha,\theta_\beta))\right]. \tag{3.84}$$

Comparing coefficients in Eqs. (3.83, 3.84) gives the values for E_α and E_β given in Eq. (3.77). B and Γ can also be solved as

$$\Gamma = \frac{1}{2}; \quad B = \frac{1}{4}[\exp(-i2\theta_\alpha) - \exp(-i2\theta_\beta)] \tag{3.85}$$

Inserting Eq. (3.80) into Eq. (3.79) leads to the important relation,

$$\frac{\partial}{\partial w^*}\left[\exp\left(-\frac{1}{2}[B(w^*(\theta_\alpha,\theta_\beta))^2 + \Gamma|w(\theta_\alpha,\theta_\beta)|^2]\right)|w;\theta_\alpha,\theta_\beta\rangle\right] = 0 \tag{3.86}$$

This proves the analyticity of bi-fractional coherent states.

3.7.1.3 Bi-fractional resolution of identity

Due to the interpolation of the bi-fractional displacement operator, two different coherent states can be defined. The ordinary Glauber coherent state are special cases where $\theta_\alpha = \theta_\beta = 0$. However for a special case, $\alpha = \beta = 0$, another type of bi-fractional coherent states which is referred to as, 'O-coherent states',

$$|\alpha, \beta; \theta_\alpha, \theta_\beta\rangle_O = \mathcal{O}(0, 0; \theta_\alpha, \theta_\beta) |\alpha, \beta\rangle \tag{3.87}$$

These 'O-coherent states' are eigenstates of the annihilation operator,

$$\begin{aligned} \mathcal{E}(\theta_\alpha, \theta_\beta) &= \mathcal{O}(0, 0; \theta_\alpha, \theta_\beta)\hat{a}[\mathcal{O}(0, 0; \theta_\alpha, \theta_\beta)]^\dagger, \\ [\mathcal{E}(\theta_\alpha, \theta_\beta)]^\dagger &= \mathcal{O}(0, 0; \theta_\alpha, \theta_\beta)\hat{a}^\dagger[\mathcal{O}(0, 0; \theta_\alpha, \theta_\beta)]^\dagger. \end{aligned} \tag{3.88}$$

And they obey the resolution of identity,

$$\frac{1}{2\pi} \int |\alpha, \beta; \theta_\alpha, \theta_\beta\rangle_{O\,O} \langle\alpha, \beta; \theta_\alpha, \theta_\beta| = \mathbb{1}. \tag{3.89}$$

For the bi-fractional coherent state of Eq. (3.67), these states are not the eigenstate of the annihilation operator, but still obey the resolution of identity,

$$\frac{1}{2\pi \cos(\theta_\alpha - \theta_\beta)} \int |\alpha, \beta; \theta_\alpha, \theta_\beta\rangle \langle\alpha, \beta; \theta_\alpha, \theta_\beta| = \mathbb{1}. \tag{3.90}$$

It can also be written in the complex form as,

$$\int \frac{d^2w}{2\pi} |w(\theta_\alpha, \theta_\beta)\rangle \langle w(\theta_\alpha, \theta_\beta)| = \mathbb{1}. \tag{3.91}$$

A special case of the bi-fractional displacement operators, when $\alpha = \beta = 0$, acting on the displacement operator is given as,

$$|\alpha,\beta;\theta_\alpha,\theta_\beta\rangle_O = |-\beta\cos\theta_\alpha - \alpha\sin\theta_\alpha, \alpha\cos\theta_\beta - \beta\sin\theta_\beta;\theta_\alpha,\theta_\beta\rangle \exp(iX)$$
$$X = \frac{1}{4}(\beta^2-\alpha^2)[\sin 2\theta_\beta - \sin 2\theta_\alpha] + \alpha\beta(\cos^2\theta_\alpha - \cos^2\theta_\beta)$$
$$\mathcal{O}(0,0;\theta_\alpha,\theta_\beta)\widehat{D}(\alpha,\beta) = \mathcal{O}(-\beta\cos\theta_\alpha - \alpha\sin\theta_\alpha, \alpha\cos\theta_\beta - \beta\sin\theta_\beta;\theta_\alpha,\theta_\beta). \tag{3.92}$$

It is easily proved by first taking $\mathcal{O}(0,0;\theta_\alpha,\theta_\beta)\widehat{D}(\alpha,\beta)$, so that,

$$\begin{aligned}\mathcal{O}(0,0;\theta_\alpha,\theta_\beta)\widehat{D}(\alpha,\beta) &= \mathcal{O}(-\beta\cos\theta_\alpha - \alpha\sin\theta_\alpha, \alpha\cos\theta_\beta - \beta\sin\theta_\beta;\theta_\alpha,\theta_\beta)\exp(iX)\\ &= |\cos(\theta_\alpha-\theta_\beta)|^{1/2}\int d\alpha' d\beta' \exp\left[-\frac{i}{2}(\alpha'^2\cot\theta_\beta + \beta'^2\cot\theta_\alpha)\right]\\ &\times \widehat{D}(\alpha+\alpha',\beta+\beta')\exp[i\alpha\beta' - i\alpha'\beta].\end{aligned} \tag{3.93}$$

Changing variables and acting on a vacuum state results in,

$$\begin{aligned}\mathcal{O}(0,0;\theta_\alpha,\theta_\beta)|\alpha,\beta\rangle &= |\cos(\theta_\alpha-\theta_\beta)|^{1/2}\exp(iX)\int d\gamma d\lambda K(\gamma,\alpha\cos\theta_\beta - \beta\cos\theta_\beta;\theta_\beta)\\ &\times K(-\lambda,-\beta\cos\theta_\alpha - \alpha\sin\theta_\alpha;\theta_\alpha)|\gamma,\lambda\rangle,\end{aligned} \tag{3.94}$$

thus proving Eq. (3.92). Similarly, the effect of acting the displacement operator on both sides of the bi-fractional parity operator [4] is studied.

$$\begin{aligned}\widehat{D}(\gamma,\xi)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\widehat{D}^\dagger(\gamma,\xi) &= \mathcal{O}(\alpha+2\gamma\sin\theta_\alpha,\beta+2\xi\sin\theta_\beta;\theta_\alpha,\theta_\beta)\\ &\times\exp[i(2\alpha\gamma\cos\theta_\alpha + \gamma^2\sin 2\theta_\alpha)]\\ &\times\exp[i(2\beta\xi\cos\theta_\beta + \xi^2\sin 2\theta_\beta)]\end{aligned} \tag{3.95}$$

The proof is given as follows,

$$\widehat{D}(\gamma,\xi)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\widehat{D}(-\gamma,-\xi) = |\cos(\theta_\alpha-\theta_\beta)|^{1/2}\int d\alpha' d\beta'\; K(\beta,\alpha';\theta_\beta)\,K(\alpha,-\beta';\theta_\alpha)$$
$$\times\,\widehat{D}(\gamma,\xi)\widehat{D}(\alpha',\beta')\widehat{D}(-\gamma,-\xi). \tag{3.96}$$

Using Eq. (2.35), the relation for three displacement operators,

$$\widehat{D}(\gamma,\xi)\widehat{D}(\alpha',\beta')\widehat{D}(-\gamma,-\xi) = \widehat{D}(\alpha',\beta')\exp[2i\xi\alpha' - 2i\gamma\beta']. \tag{3.97}$$

Expanding each of the kernels of the fractional Fourier transform,

$$\widehat{D}(\gamma,\xi)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\widehat{D}(-\gamma,-\xi)$$
$$= \left[\frac{1+i\cot\theta_\alpha}{2\pi}\right]^{1/2}\left[\frac{1+i\cot\theta_\beta}{2\pi}\right]^{1/2}|\cos(\theta_\alpha-\theta_\beta)|^{1/2}\int d\alpha' d\beta'\widehat{D}(\alpha,\beta')$$
$$\times\exp\left[-\frac{i(\beta+2\xi\sin\theta_\beta)^2\cot\theta_\beta}{2} - \frac{i\alpha'^2\cot\theta_\beta}{2} + \frac{i\alpha'(\beta+2\xi\sin\theta_\beta)}{\sin\theta_\beta} + 2i\beta\xi\cos\theta_\beta + i\xi^2\sin 2\theta_\beta\right]$$
$$\times\exp\left[\frac{i(\alpha+2\gamma\sin\theta_\alpha)^2\cot\theta_\alpha}{2} - \frac{i\beta'^2\cot\theta_\alpha}{2} - \frac{i\beta'(\alpha+2\gamma\sin\theta_\alpha)}{\sin\theta_\alpha} + 2i\alpha\gamma\cos\theta_\alpha + i\gamma^2\sin 2\theta_\alpha\right].$$
$$\tag{3.98}$$

Then using Eq. (3.24) gives,

$$\widehat{D}(\gamma,\xi)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\widehat{D}(-\gamma,-\xi) = |\cos(\theta_\alpha - \theta_\beta)|^{1/2}\int d\alpha' d\beta' \widehat{D}(\alpha,\beta')K(J_1,\alpha';\theta_\beta)$$

$$\times K(J_2,-\beta';\theta_\alpha)\exp[2i\beta\xi\cos\theta_\beta + i\xi^2\sin 2\theta_\beta + 2i\alpha\gamma\cos\theta_\alpha + i\gamma^2\sin 2\theta_\alpha]$$

$$= \mathcal{O}(J_1,J_2;\theta_\alpha,\theta_\beta)\exp[2i\beta\xi\cos\theta_\beta + i\xi^2\sin 2\theta_\beta + 2i\alpha\gamma\cos\theta_\alpha + i\gamma^2\sin 2\theta_\alpha]$$

$$J_1 = \beta + 2\xi\sin\theta_\beta;\quad J_2 = \alpha + 2\gamma\sin\theta_\alpha \tag{3.99}$$

This proves Eq. (3.95).

3.8 Bi-fractional distance in phase space

The overlap of two Glauber coherent states has already been given in Eq. (2.73). The equivalent is given in terms of bi-fractional coherent states,

$$\langle\alpha,\beta;\theta_\alpha,\theta_\beta|\alpha',\beta';\theta_\alpha,\theta_\beta\rangle = \exp\left\{-\frac{1}{2}\frac{D_S(\alpha-\alpha',\beta-\beta'|\theta_\alpha,\theta_\beta)}{\cos^2(\theta_\alpha-\theta_\beta)}\right\}$$
$$\times\exp\left[\frac{i}{2}(\alpha'^2-\beta'^2-\alpha^2+\beta^2)\tan(\theta_\alpha-\theta_\beta)+\frac{i(\alpha\beta'-\alpha'\beta)}{\cos(\theta_\alpha-\theta_\beta)}\right]$$
$$D_S(\alpha-\alpha',\beta-\beta'|\theta_\alpha,\theta_\beta) = (\alpha-\alpha')^2 + 2(\alpha-\alpha')(\beta-\beta')\sin(\theta_\alpha-\theta_\beta) + (\beta-\beta')^2, \tag{3.100}$$

where $D_S(\alpha-\alpha',\beta-\beta'|\theta_\alpha,\theta_\beta)$ is the distance between two points, in terms of coordinates in a non-orthogonal frame, given in Eq. (3.21). The denominator $\cos^2(\theta_\alpha-\theta_\beta)$ can be understood as a Jacobian as variables are changed from an orthogonal frame to variables in a non-orthogonal frame (see section 3.5). Eq. (3.100) is proved using

Eq. (3.24) for the two bi-fractional coherent states:

$$\langle \alpha, \beta; \theta_\alpha, \theta_\beta | \alpha', \beta'; \theta_\alpha, \theta_\beta \rangle = R \int d\gamma d\lambda d\gamma' d\lambda' \exp\left[-\frac{i}{2}(\gamma'^2 - \gamma^2)\cot\theta_\beta + i(\beta\gamma + \beta'\gamma')\csc\theta_\beta \right]$$

$$\times \exp\left[-\frac{i}{2}(\lambda'^2 - \lambda^2)\cot\theta_\alpha - i(\alpha\lambda + \alpha'\lambda')\csc\theta_\alpha \right] \langle 0|\hat{D}(\gamma,\lambda)\hat{D}(\gamma',\lambda')|0\rangle \qquad (3.101)$$

where,

$$R = [4\pi^2 \sin\theta_\alpha \sin\theta_\beta]^{-1} \exp\left[\frac{i}{2}(\alpha^2\cot\theta_\alpha + \beta^2\cot\theta_\beta - \alpha'^2\cot\theta_\alpha - \beta'^2\cot\theta_\beta) \right]. \qquad (3.102)$$

Then variables are changed, perform integrating that gives Dirac delta functions and reduce Eq. (3.101) to Eq. (3.100).

Consequently, the overlap can be written in complex states $|w(\theta_\alpha, \theta_\beta)\rangle$ and $|v(\theta_\alpha, \theta_\beta)\rangle$ so that,

$$\begin{aligned}\langle w(\theta_\alpha,\theta_\beta)|v(\theta_\alpha,\theta_\beta)\rangle &= \exp\left[-\frac{1}{2}|w(\theta_\alpha,\theta_\beta)|^2 - \frac{1}{2}|v(\theta_\alpha,\theta_\beta)|^2 + w^*(\theta_\alpha,\theta_\beta)v(\theta_\alpha,\theta_\beta) \right] \\ &\times \exp\left[\frac{i}{2}[C^* w(\theta_\alpha,\theta_\beta)^2 + C w^*(\theta_\alpha,\theta_\beta)^2]\tan(\theta_\alpha - \theta_\beta) \right] \\ &\times \exp\left[\frac{i}{2}[C^* v(\theta_\alpha,\theta_\beta)^2 + C v^*(\theta_\alpha,\theta_\beta)^2]\tan(\theta_\alpha - \theta_\beta) \right] \qquad (3.103)\end{aligned}$$

where,

$$C = \frac{1}{4}[\exp(-i2\theta_\alpha) + \exp(-i2\theta_\beta)], v = \frac{\alpha' E_\alpha + i\beta' E_\beta}{\cos(\theta_\alpha - \theta_\beta)} \qquad (3.104)$$

The overlap of the two bi-fractional coherent states (Eqs. (3.100, 3.103)) easily leads to the bi-fractional distance $\mathcal{G}_S(\alpha - \alpha', \beta - \beta' | \theta_\alpha, \theta_\beta)$,

$$\mathcal{G}_S(\alpha - \alpha', \beta - \beta' | \theta_\alpha, \theta_\beta) = \exp\left[-\frac{D_S(\alpha - \alpha', \beta - \beta' | \theta_\alpha, \theta_\beta)}{\cos^2(\theta_\alpha - \theta_\beta)} \right] \qquad (3.105)$$

And can also be given in the complex notation as,

$$\mathcal{G}_S(w, v|\theta_\alpha, \theta_\beta) = \exp(-|w(\theta_\alpha, \theta_\beta) - v(\theta_\alpha, \theta_\beta)|^2) \tag{3.106}$$

The first form of the bi-fractional distance is given in complex notation of $w(\theta_\alpha, \theta_\beta)$ and $v(\theta_\alpha, \theta_\beta)$ and the second is derived by using Eqs. (3.77, 3.104). As shown above, the proof is easier if done in the non-complex notations of α, β.

3.9 Bargmann representation of bi-fractional coherent state $|\alpha, \beta; \theta_\alpha, 0\rangle$

The Bargmann representation [11, 73] for a normalised state $|g\rangle$

$$\mathfrak{B}(\alpha, \beta) = g\left(\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{2}\right) \exp\left[\frac{1}{2}(\alpha^2 + \beta^2)\right]. \tag{3.107}$$

is a Bargmann function with respect to the Glauber coherent states $\left|\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{2}\right\rangle$.

We define the bargmann function for a generalised coherent state as,

$$\mathfrak{B}(\gamma, \zeta; \theta_\alpha, \theta_\beta) = \left\langle \gamma, \zeta; \frac{\pi}{2}, \frac{\pi}{2} \middle| \alpha, \beta; \theta_\alpha, \theta_\beta \right\rangle \exp\left[\frac{1}{2}(\gamma^2 + \zeta^2)\right] \tag{3.108}$$

Taking an example, we calculate the Bargmann functions $\mathfrak{B}(\gamma, \zeta; \theta_\alpha, 0)$ for the coherent states by taking a special case with only $\theta_\beta = 0$. The special case considered is for convenience of calculating the overlap of bi-fractional coherent states with one state being the ordinary coherent states ($\theta_\alpha = \frac{\pi}{2}, \theta_\beta = \frac{\pi}{2}$). Previously, the general case of the overlap of the bi-fractional coherent states was introduced and here a special case of the bi-fractional coherent state is taken,

$$\begin{aligned} |\alpha, \beta; \theta_\alpha, 0\rangle &= |\cos\theta_\alpha|^{1/2} \int d\alpha' d\beta' \delta\left(\beta - \alpha'\right) K\left(\alpha, -\beta'; \theta_\alpha\right) \left|\alpha', \beta'; \frac{\pi}{2}, \frac{\pi}{2}\right\rangle \\ &= |\cos\theta_\alpha|^{1/2} \int d\beta' K\left(\alpha, -\beta'; \theta_\alpha\right) \left|\beta, \beta'; \frac{\pi}{2}, \frac{\pi}{2}\right\rangle, \end{aligned} \tag{3.109}$$

and give the overlap as,

$$\begin{aligned} \left\langle \gamma, \zeta; \frac{\pi}{2}, \frac{\pi}{2} \middle| \alpha, \beta; \theta_\alpha, 0 \right\rangle &= |\cos\theta_\alpha|^{1/2} \int d\beta' K\left(\alpha, -\beta'; \theta_\alpha\right) \left\langle \gamma, \zeta; \frac{\pi}{2}, \frac{\pi}{2} \middle| \beta, \beta'; \frac{\pi}{2}, \frac{\pi}{2} \right\rangle \\ &= |\cos\theta_\alpha|^{1/2} \int d\beta' K\left(\alpha, -\beta'; \theta_\alpha\right) \\ &\quad \times \exp\left[-\frac{1}{2}(\gamma^2 + \zeta^2 + \beta'^2 + \beta^2) + (\gamma - i\zeta)(\beta + i\beta')\right]. \end{aligned} \tag{3.110}$$

From this follows that the Bargmann function $\mathfrak{B}(\gamma, \zeta; \theta_\alpha, 0)$ for the coherent state $|\alpha, \beta; \theta_\alpha, 0\rangle$ is

$$\begin{aligned} \mathfrak{B}(\gamma, \zeta; \theta_\alpha, 0) &= |\cos\theta_\alpha|^{1/2} \int d\beta' K\left(\alpha, -\beta'; \theta_\alpha\right) \left\langle \gamma, \zeta; \frac{\pi}{2}, \frac{\pi}{2} \middle| \beta, \beta'; \frac{\pi}{2}, \frac{\pi}{2} \right\rangle \exp\left[\frac{1}{2}(\gamma^2 + \zeta^2)\right] \\ &= |\cos\theta_\alpha|^{1/2} \int d\beta' K\left(\alpha, -\beta'; \theta_\alpha\right) \\ &\quad \times \exp\left[-\frac{1}{2}(\beta'^2 + \beta^2) + (\gamma - i\zeta)(\beta + i\beta')\right]. \end{aligned} \tag{3.111}$$

The result is a Gaussian integral and taking $\Gamma = \gamma - i\zeta$ is

$$\begin{aligned}
\mathfrak{B}(\Gamma;\theta_\alpha,0) &= |\cos\theta_\alpha|^{1/2}\exp(A\Gamma^2 + B\Gamma + \Sigma)\\
A &= -\frac{1}{2(1+i\cot\theta_\alpha)}\\
B &= \beta + \frac{\alpha}{\sin\theta_\alpha(1+i\cot\theta_\alpha)}\\
\Sigma &= -\frac{1}{2}(\beta^2+\alpha^2).
\end{aligned} \tag{3.112}$$

This result should be compared and contrasted with the Bargmann function for squeezed coherent state, $\hat{S}(\Gamma)\widehat{D}(z)|0\rangle$. which is,

$$\begin{aligned}
\mathfrak{B}(\Gamma) &= (1-|a|^2)^{1/4}\exp\left(\frac{1}{2}a\Gamma^2 + b\Gamma + c\right)\\
a &= -\tanh\left(\frac{r}{2}\right)e^{-i\phi}\\
b &= z(1-|a|^2)^{1/2}\\
c &= -\frac{1}{2}a^*z^2 - \frac{1}{2}|z|^2.
\end{aligned} \tag{3.113}$$

If in Eq. (3.113), a is replaced with $2A$, b with B, and c with Σ, Eq. (3.112) is obtained. This shows the squeezing property of the bi-fractional coherent state.

3.10 Statistical properties of the coherent states $|\alpha,\beta;\theta_\alpha,0\rangle$

Given the Bargmann representation of the coherent state $|\alpha,\beta;\theta,0\rangle$, the wavefunction [73] for such a state can be calculated. For the coherent states $|\alpha,\beta;\theta_\alpha,0\rangle$ the wavefunction $F(x)$ is calculated using the relations

$$F(x) = \pi^{-3/4} \exp\left(-\frac{x^2}{2}\right) \int dp\, \mathfrak{B}((x + ip)\sqrt{2}; \theta_\alpha, 0) \exp(-p^2), \tag{3.114}$$

and the Bargmann function of Eq. (3.112). Analytical computation results in,

$$\begin{aligned} F(x) &= |\cos\theta_\alpha|^{1/2} \pi^{-1/4} \left(\frac{1}{1+2A}\right)^{1/2} \exp\left(\frac{\kappa x^2 + 2^{3/2} B x + \lambda}{2 + 4A}\right) \\ \kappa &= 2A - 1; \quad \lambda = 2\Sigma + 4A\Sigma - B^2, \end{aligned} \tag{3.115}$$

where A, B, Σ have been given in Eq. (3.112).

3.10.1 Uncertainty relation

Having already defined the uncertainty principle in section (2.6), it is also known that Heisenberg's uncertainty can be calculated with the Wigner function [24, 72]. The position and momentum operators with respect to α and β are denoted as, $\hat{\alpha}$ and $\hat{\beta}$ and show that,

$$\begin{aligned} \langle \alpha^T \rangle &= \mathrm{Tr}[\hat{\alpha_x}^T \Theta] = 2 \int d\alpha\, \alpha^T \langle \alpha | \Theta | \alpha \rangle = \int d\alpha d\beta\, \alpha^T\, W(\alpha, \beta), \\ \langle \beta^T \rangle &= \mathrm{Tr}[\hat{\beta_x}^T \Theta] = 2 \int d\beta\, \alpha^T \langle \beta | \Theta | \beta \rangle = \int d\alpha d\beta\, \beta^T\, W(\alpha, \beta), \end{aligned} \tag{3.116}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are eigenstates for position and momentum.

Eq. (3.116) above, can easily be derived using Eqs. (2.94, 2.95) with the position and momentum operators acting as in Eq. (2.17). However, since the wavefunction has been defined in Eq. (3.114), then the uncertainty is given as,

$$\Delta_{xx} = \langle x^2 \rangle - \langle x \rangle^2\,; \quad \langle x^n \rangle = \int dx\; x^n |F(x)|^2$$

$$\Delta_{pp} = \langle p^2 \rangle - \langle p \rangle^2\,; \quad \langle p^n \rangle = \int dx\; [F(x)]^* \hat{p}^n F(x)$$

$$\Delta_{xp} = \left\langle \frac{1}{2}(xp + px) \right\rangle - \langle x \rangle \langle p \rangle$$

$$\left\langle \frac{1}{2}(xp + px) \right\rangle = -\frac{i}{2} + \int dx[F(x)]^* x(-i\partial_x)F(x), \tag{3.117}$$

and similarly for Δp. The Robertson-Schrödinger form of the uncertainty relation is given as,

$$\Delta_{xx}\Delta_{pp} - \left[\left\langle \frac{xp + px}{2} \right\rangle - \langle x \rangle \langle p \rangle\right]^2 \geq \frac{1}{4}. \tag{3.118}$$

Similarly, $F(p)$ can be calculated from $F(x)$ shown in Eq. (3.115),

$$\langle p \rangle = \int dx\; F^*(x)(-i\delta_x)F(x) = \int -i\left(\frac{kx + \sqrt{2}\beta}{1 + 2A}\right) F^*(x)F(x)dx$$

$$\langle p^2 \rangle = \int dx\; F^*(x)(-i\delta_x)^2 F(x) = \int -\left[\left(\frac{kx + \sqrt{2}\beta}{1 + 2A}\right)^2 + \frac{k}{1 + 2A}\right] F^*(x)F(x)dx$$

$$\left\langle \frac{xp + px}{2} \right\rangle = \int dx\; F^*(x)\left[\frac{x(-i\delta_x) + (-i\delta_x x)}{2} F(x)dx\right] = \int F^*(x)[2\hat{x}\hat{p} - i]F(x)dx \tag{3.119}$$

As expected for squeezed states, numerical calculations showed that for all angles

θ_α, the Robertson-Schrödinger relation gives

$$\Delta_{xx}\Delta_{pp} - \Delta_{xp}^2 = \frac{1}{4} \tag{3.120}$$

The result shows the diversity in the interaction between $(\theta_\alpha, \theta_\beta)$. Considering a special case where $\theta_\beta = 0$ and yet the uncertainty relation depends only on β and not on α or θ_α. This result is shown in Fig. (3.3), Δ_{pp} is plotted as a function of θ_α. Similarly, it is worth noting that,

$$\Delta_{xp}^2 = \frac{1}{2}\Delta_{pp} - \frac{1}{4}. \tag{3.121}$$

The result is confirmed analytically by simplifying Eq. (3.114),

$$F(x) = 2^{-\frac{1}{2}}\pi^{-\frac{3}{4}}\left[\frac{2\pi(1 + i\cot\theta_\alpha)}{i\cot\theta_\alpha}\right]^{\frac{1}{2}} \exp(Rx^2 + 2^{\frac{3}{2}}Sx + T),$$
$$R = 2i - \cot\theta_\alpha; \;\; S = \beta\cot\theta_\alpha - i\left(\beta + \frac{\alpha}{\sin\theta_\alpha}\right); \;\; T = -2\beta^2\cot\theta_\alpha + i\left(\alpha^2 + \beta^2 + \frac{2\alpha\beta}{\sin\theta_\alpha}\right),$$

and calculating $|F(x)|^2$ and results to,

$$|F(x)|^2 = \pi^{-\frac{1}{2}}\sec\theta_\alpha \exp\left(-x^2 + 2\sqrt{2}\beta x - 2\beta^2\right). \tag{3.122}$$

From this the following is derived:

$$\int |F(x)|^2 dx = \frac{1}{\cos\theta_\alpha}$$
$$\langle x\rangle = \beta\sqrt{2}; \quad \langle x^2\rangle = 2\beta^2 + \frac{1}{2}; \quad \Delta_{xx} = \frac{1}{2}. \tag{3.123}$$

3.10.2 Second order correlation

Using a Taylor expansion the Bargmann function $\mathfrak{B}(\Gamma;\theta_\alpha,0)$ of Eq. (3.112), with $\alpha = \beta = 2$, can be expressed as,

$$\mathfrak{B}(\Gamma;\theta_\alpha,0) = \sum_{n=0}^{\infty} \frac{a_n \Gamma^n}{\sqrt{n!}}. \tag{3.124}$$

Numerically the series is truncated at 30 and can be shown that in this case $\sum |a_n|^2 = 0.999$. The number of photons and second order correlation can be calculated using the formulas

$$\langle n^\nu \rangle = \sum_{n=0}^{30} n^\nu |a_n|^2; \quad g^{(2)} = \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2}. \tag{3.125}$$

In Fig. (3.3), $\langle n \rangle$ and $g^{(2)}$ are shown as a functions of θ_α. It is seen that for $\theta_\alpha < 0.8$ there is antibunching ($g^{(2)} < 1$).

3.11 Discussion

This chapter provides a deeper insight into the phase space method, by considering parity and displacement operators, which are all important tools in phase space methods. The parity operators are a two-dimensional Fourier transform of the displacement operators (Eq. (2.101)). Replacing the two Fourier transforms, with two fractional Fourier transforms, results in the bi-fractional displacement operators $\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$. The importance of the pre-factor $|\cos(\theta_\alpha - \theta_\beta)|^{1/2}$ in these operators was highlighted and shown that the $\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$ are special cases of the squeezing operators in section (3.6.1.4). Nevertheless, it was stated that these bi-fractional displacement operators in the context of this thesis are used for interpolation in phase space.

Acting with the bi-fractional displacement operators on the vacuum state gives the bi-fractional coherent states, which are special cases of squeezed states. Furthermore, the uncertainties and statistical properties of these states were calculated. Coherent states are all known to be analytic [73], and the analyticity property and resolution of identity were studied detailing that although the coherent states are analytic with respect to z, the bi-fractional coherent states (due to its bi-fractional nature) is analytic with respect to a combination of z and z^* symbolised as the term w. This non-intuitive property of the bi-fractional coherent state adds novelty to this work and gives a new perspective to the study of coherent states.

The distance between two Glauber coherent states is well known [73] and the bi-fractional distance introduced in section (3.8) is the generalised form of that between two Glauber coherent states [73]. In section (3.5), the relevance of the pre-factor, $\cos(\theta_\alpha - \theta_\beta)^{1/2}$ of Eqs. (3.24, 3.74) was explored; this pre-factor will be also seen later in the bi-fractional Wigner function of Chapter 4.

In explaining the analysis in Fig. (3.3), the concepts of bunching and anti-bunching need to be considered. In thermal light field for example, photon distribution is not completely random [61], instead there is a time distribution between photons, this concept is known as bunching. On the other hand, photon anti-bunching is characteristic when photons are more equally spaced [81]. Anti-bunching exibits sub-Poisson photon statistics; that is a photon number distribution for which the variance is less than the mean [81]. Bunching in thermal light is a classical property with super-Poisson characteristics while anti-bunching is non-classical [71]. Coherent states for instance have Poisson characteristics which is obvious from the details in Fig. (2.1) and yields random photon spacing.

One other way of detecting bunching or anti-bunching is by using the second order correlation, which is defined as the probability of detecting two simultaneous photons, normalised by the probability of detecting two photons [49]. The relation for second order correlation, $g^{(2)}$, in Eq. (3.125), for bunching $g^{(2)} \geq 1$ and for anti-bunching $g^{(2)} < 1$ [49]. In Fig. (3.3), it can be seen that for certain values of θ_α, there is anti-bunching but predominantly for most values of θ_α there is bunching.

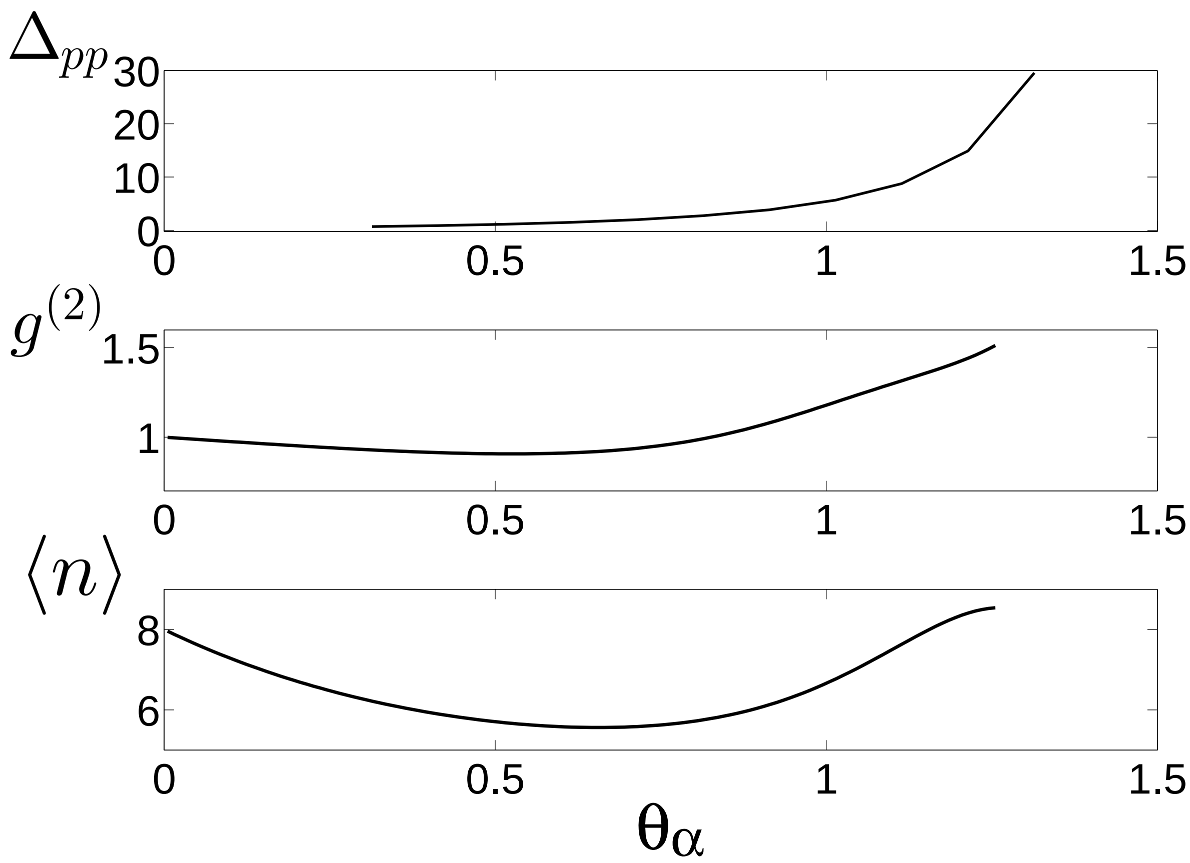


Fig. (3.3) The uncertainty Δ_{pp}, the $g^{(2)}$ and the average number of photons $\langle n \rangle$ as a function of θ_α (in rads), for the coherent states $|2, 2; \theta_\alpha, 0\rangle$

Chapter 4

Application to bi-fractional transforms

4.1 Introduction

In Chapter 3, the concept of fractional Fourier transforms in phase space methods was pointed out and its importance in quantum mechanics. The generalisation of both the displacement operator into the bi-fractional displacement operator and the bi-fractional coherent states were explained. In this chapter, the same principle will be applied by generalising the Wigner function into the bi-fractional Wigner function and give the marginal properties of the bi-fractional Wigner functions. Other phase space quantities will be reviewed and the concept of the fractional Fourier transform will be applied to them, including the Moyal star formalism [12, 55] for two non-commutating operators and the Berezin formalism [14–17] which as explained earlier helps in giving a deeper understanding of the nature of a quantum particle. In each case the novelty of the work is stressed in terms of the interpolation of these quantities beyond the conventional cases of Fourier transforms. The new formalisms are called the bi-fractional Berezin formalism and bi-fractional Moyal star product over angles

$(\theta_\alpha, \theta_\beta)$. Previously known concepts of the Berezin formalism and Moyal star product are special cases of their bi-fractional counterparts at $(\theta_\alpha = \theta_\beta = \frac{\pi}{2})$.

An important aspect which is not pursued in detail is the bi-fractional $P-$ function, however the building block deriving the bi-fractional $P-$function by introducing the bi-fractional distance between two bi-fractional coherent states. Furthermore, the bi-fractional $Q-$function is introduced in Eq. (4.19).

4.2 Interpolations between Wigner and Weyl functions

Taking the trace of the bi-fractional displacement operators with an operator gives the bi-fractional Wigner functions. Wigner and Weyl functions are special cases of the bi-fractional Wigner functions, and this is followed by introducing the bi-fractional Q-functions, and possible extensions to bi-fractional $P-$functions.

4.2.1 Bi-fractional Wigner functions

The Weyl and Wigner functions have been defined earlier in Eqs. (2.90, 2.93). From Eq. (2.102) it follows that the Wigner and Weyl functions are related through the two-dimensional Fourier transform [24, 25], discussed previously in Eq. (2.101),

$$W(\alpha, \beta|\Theta) = \frac{1}{2\pi} \int d\alpha' d\beta' \widetilde{W}(\alpha', \beta'|\Theta) \exp\left[i(\beta\alpha' - \beta'\alpha)\right]. \quad (4.1)$$

In analogy with Eq. (3.24) the bi-fractional Wigner function is defined from Eq. (3.24),

$$\begin{aligned}\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta) &= \mathrm{Tr}[\Theta\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)]\\ &= |\cos(\theta_\alpha-\theta_\beta)|^{1/2}\int d\alpha' d\beta' K\left(\beta,\alpha';\theta_\beta\right)K\left(\alpha,-\beta';\theta_\alpha\right)\widetilde{W}(\alpha',\beta'|\Theta).\end{aligned} \tag{4.2}$$

$\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)$ includes both the Wigner and Weyl functions as special cases:

$$\begin{aligned}\mathcal{H}\left(\alpha,\beta;0,0|\Theta\right) &= \widetilde{W}(\beta,-\alpha|\Theta)\\ \mathcal{H}\left(\alpha,\beta;\frac{\pi}{2},\frac{\pi}{2}\middle|\Theta\right) &= W(\alpha,\beta|\Theta)\\ \mathcal{H}\left(\alpha,\beta;\pi,\pi|\Theta\right) &= \widetilde{W}(-\beta,\alpha|\Theta).\end{aligned} \tag{4.3}$$

In previous work, a single fractional Fourier transform concept was derived between Wigner and Weyl functions [23] and this gives generalised Wigner functions that depend on one angle. The new formalism described here for the Wigner function uses a double fractional Fourier transform (bi-fractional Wigner functions) that depends on two angles, $(\theta_\alpha,\theta_\beta)$.

4.3 Marginal properties for bi-fractional Wigner function

The bi-fractional Wigner function, $\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)$ discussed in section (4.2.1) has obvious marginal properties which are similar to those given in section (3.6.1.3). Marginal properties also hold for $\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2$, so that integration of $|\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2$

with respect to α gives,

$$\int d\alpha|\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2 = \sqrt{2}\pi|\cos(\theta_\alpha-\theta_\beta)|\int dx \times\left|\int d\alpha'\left\langle x-\frac{\alpha'}{\sqrt{2}}\right|\Theta\left|x+\frac{\alpha'}{\sqrt{2}}\right\rangle K\left(\beta,\alpha';\theta_\beta\right)\right|^2. \tag{4.4}$$

The proof is derived by using Eq. (4.2) which gives,

$$\int d\alpha|\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2 = |\cos(\theta_\alpha-\theta_\beta)|\int d\alpha' d\beta' K\left(\beta,\alpha';\theta_\beta\right)K\left(\alpha,-\beta';\theta_\alpha\right)\widetilde{W}(\alpha',\beta'|\Theta) \times\int d\alpha'' d\beta'' d\alpha K\left(-\beta,\alpha'';-\theta_\beta\right)K\left(-\alpha,-\beta'';-\theta_\alpha\right)\widetilde{W}(\alpha'',\beta''|\Theta) \tag{4.5}$$

Using Eq. (3.13), integration with respect to α gives a delta function, and then integration with respect β'' gives

$$\int d\alpha\left|\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)\right|^2 = |\cos(\theta_\alpha-\theta_\beta)|\int d\alpha' d\alpha'' d\beta' K\left(\beta,\alpha';\theta_\beta\right)K\left(-\beta,\alpha'';-\theta_\beta\right) \times\widetilde{W}(\alpha',\beta'|\Theta)\widetilde{W}(\alpha'',-\beta'|\Theta) \tag{4.6}$$

Substituting the Weyl function of Eq. (2.90),

$$\int d\alpha\left|\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)\right|^2 = |\cos(\theta_\alpha-\theta_\beta)|\int d\alpha' d\alpha'' d\beta' K\left(\beta,\alpha';\theta_\beta\right)K\left(-\beta,\alpha'';-\theta_\beta\right) \times\int dx\left\langle x-\frac{\alpha'}{\sqrt{2}}\right|\Theta\left|x+\frac{\alpha'}{\sqrt{2}}\right\rangle\exp(i\sqrt{2}\beta' x) \int dy\left\langle y-\frac{\alpha''}{\sqrt{2}}\right|\Theta\left|y+\frac{\alpha''}{\sqrt{2}}\right\rangle\exp(-i\sqrt{2}\beta' y) \tag{4.7}$$

Further integration with respect to β' gives a Dirac delta function, and changing variables, $x \to x - \frac{\alpha'}{\sqrt{2}}$ and $y \to y - \frac{\alpha''}{\sqrt{2}}$, produces,

$$\begin{aligned}\int d\alpha|\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta)|^2 &= \sqrt{2}\pi|\cos(\theta_\alpha-\theta_\beta)|\int d\alpha' dx\left\langle x-\frac{\alpha'}{\sqrt{2}}\right|\Theta\left|x+\frac{\alpha'}{\sqrt{2}}\right\rangle \\ &\times K(\beta,\alpha';\theta_\beta)\int d\alpha''\left\langle x-\frac{\alpha''}{\sqrt{2}}\right|\Theta\left|x+\frac{\alpha''}{\sqrt{2}}\right\rangle K(-\beta,\alpha'';-\theta_\beta) \\ &= \sqrt{2}\pi|\cos(\theta_\alpha-\theta_\beta)|\int dx\left|\int d\alpha'\left\langle x-\frac{\alpha'}{\sqrt{2}}\right|\Theta\left|x+\frac{\alpha'}{\sqrt{2}}\right\rangle K(\beta,\alpha';\theta_\beta)\right|^2\end{aligned} \tag{4.8}$$

The second marginal property is derived by integration of $|\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2$ with respect to β, this gives

$$\begin{aligned}\int d\beta|\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2 &= \sqrt{2}\pi|\cos(\theta_\alpha-\theta_\beta)|\int dp \\ &\times\left|\int d\beta'\left\langle p-\frac{\beta'}{\sqrt{2}}\right|\Theta\left|p+\frac{\beta'}{\sqrt{2}}\right\rangle K(\alpha,-\beta';\theta_\alpha)\right|^2\end{aligned} \tag{4.9}$$

The proof of Eq. (4.9) is similar to that above with respect of α.

Integration of $|\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2$ with respect to both α and β gives

$$\int d\alpha d\beta|\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2 = \pi|\cos(\theta_\alpha-\theta_\beta)|\mathrm{Tr}(\Theta^2) \tag{4.10}$$

The proof is derived by using Eqs. (4.4, 4.9) so that,

$$\int d\alpha d\beta |\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2 = \lambda \int d\alpha d\beta d\alpha' d\beta' d\alpha'' d\beta''$$
$$\times \exp\left[\frac{i}{2}(\alpha''^2 - \alpha'^2)\cot\theta_\beta + \frac{i}{2}(\beta''^2 - \beta'^2)\cot\theta_\alpha\right]$$
$$\times \exp\left[\frac{i\beta(\alpha' + \alpha'')}{\sin\theta_\beta} - \frac{i\alpha(\beta' + \beta'')}{\sin\theta_\alpha}\right]$$
$$\times \mathrm{Tr}[\Theta D(\alpha',\beta')]\mathrm{Tr}[\Theta D(\alpha'',\beta'')] \tag{4.11}$$

where

$$\lambda = |\cos(\theta_\alpha - \theta_\beta)| \left[\frac{1 + i\cot\theta_\alpha}{2\pi}\right]^{1/2} \left[\frac{1 - i\cot\theta_\alpha}{2\pi}\right]^{1/2} \left[\frac{1 + i\cot\theta_\beta}{2\pi}\right]^{1/2} \left[\frac{1 - i\cot\theta_\beta}{2\pi}\right]^{1/2}$$
$$= \frac{|\cos(\theta_\alpha - \theta_\beta)|}{4\pi^2 \sin\theta_\alpha \sin\theta_\beta} \tag{4.12}$$

Integration over α, β gives δ-functions, and results in Eq. (4.10).

4.4 Numerical implementations

As an example, consider $\Theta = |s\rangle\langle s|$ where $|s\rangle$ is the following superposition of two Glauber coherent states:

$$|s\rangle = \mathcal{N}[|\alpha_0,\beta_0\rangle + |-\alpha_0,-\beta_0\rangle] \tag{4.13}$$

where $\mathcal{N}$ is a normalisation factor. In this case

$$\Theta_A = \mathcal{N}^2[|\alpha_0,\beta_0\rangle\langle\alpha_0,\beta_0| + |-\alpha_0,-\beta_0\rangle\langle-\alpha_0,-\beta_0|]$$
$$\Theta_C = \mathcal{N}^2[|\alpha_0,\beta_0\rangle\langle-\alpha_0,-\beta_0| + |-\alpha_0,-\beta_0\rangle\langle\alpha_0,\beta_0|]$$
$$\Theta = \Theta_A + \Theta_C \tag{4.14}$$

where Θ_A and Θ_C are the 'auto-part' and the 'cross-part' of the density matrix, and the normalisation constant is $\mathcal{N}=\frac{1}{2+2e^{-2|\alpha_0+\beta_0|^2}}$. Inserting this in Eq. (4.2) produces the 'auto-part' and the 'cross-part' of the $\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta)$, shown with A and C in the figures.

For the Wigner function, the auto-terms are implemented as,

$$\begin{aligned}\Theta_A &= \int dx\,\langle x|\alpha_0,\beta_0\rangle\langle\alpha_0,\beta_0| - x + \sqrt{2}\alpha\rangle\, e^{-i\sqrt{2}\beta x + i\alpha\beta} \\ &+ \int \langle x| - \alpha_0,-\beta_0\rangle\langle-\alpha_0,-\beta_0| - x + \sqrt{2}\alpha\rangle\, e^{-i\sqrt{2}\beta x+i\alpha\beta}dx \\ &= \mathcal{N}^2\left[\exp\left(-\frac{\beta^2}{2} - \frac{\alpha^2}{2} + 2\alpha\alpha_0 - 2\alpha_0\right) + \exp\left(-\frac{\beta^2}{2} - \frac{\alpha^2}{2} - 2\alpha\alpha_0 - 2\alpha_0^2\right)\right]\end{aligned} \tag{4.15}$$

and the cross terms,

$$\begin{aligned}\Theta_C &= \int dx\,\langle x|\alpha_0,\beta_0\rangle\langle-\alpha_0,-\beta_0| - x + \sqrt{2}\alpha\rangle\, e^{-i\sqrt{2}\beta x+i\alpha\beta} \\ &+ \int dx\,\langle x| - \alpha_0,-\beta_0\rangle\langle\alpha_0,\beta_0| - x + \sqrt{2}\alpha\rangle\, e^{-i\sqrt{2}\beta x+i\alpha\beta} \\ &= \mathcal{N}^2\left[\exp\left(-\frac{\beta^2}{2} - \frac{\alpha^2}{2} - 2i\beta\alpha_0\right) + \exp\left(-\frac{\beta^2}{2} - \frac{\alpha^2}{2} + 2i\beta\alpha_0\right)\right] \\ &= \mathcal{N}^2\left[\exp\left[-\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right]\cos\left(2\beta\alpha_0\right)\right]\end{aligned} \tag{4.16}$$

The Weyl function is calculated,

$$
\begin{aligned}
\Theta_A &= \mathcal{N}^2 \left[\left(|Z\rangle \langle Z| \right) + \left(|-Z\rangle \langle -Z| \right) \right] \\
&= \int \langle x|\alpha_0, \beta_0\rangle \langle \alpha_0, \beta_0|x + \sqrt{2}\alpha\rangle \, e^{i\sqrt{2}\beta x + i\alpha\beta} dx \\
&+ \int \langle x| - \alpha_0, -\beta_0\rangle \langle -\alpha_0, -\beta_0|x + \sqrt{2}\alpha\rangle \, e^{i\sqrt{2}\beta x + i\alpha\beta} dx \\
&= N^2 \left[\exp\left(-\frac{\beta^2}{2} - \frac{\alpha^2}{2} + i2\alpha_0\beta \right) + \exp\left(-\frac{\beta^2}{2} - \frac{\alpha^2}{2} - i2\alpha_0\beta \right) \right]
\end{aligned} \tag{4.17}
$$

and the cross terms,

$$
\begin{aligned}
\Theta_C &= \mathcal{N}^2 (|\alpha_0, \beta_0\rangle \langle -\alpha_0, -\beta_0| + |-\alpha_0, -\beta_0\rangle \langle \alpha_0, \beta_0|) \\
&= \int \langle x|\alpha_0, \beta_0\rangle \langle -\alpha_0, -\beta_0|x + \sqrt{2}\alpha\rangle \, e^{i\sqrt{2}\beta x + i\alpha\beta} + \int \langle x| - \alpha_0, -\beta_0\rangle \langle \alpha_0, \beta_0|x + \sqrt{2}\alpha\rangle \, e^{i\sqrt{2}\beta x + i\alpha\beta} dx \\
&= \exp\left[-\frac{\beta^2}{2} - \frac{\alpha^2}{2} + 2\alpha\alpha_0 - 2\alpha^2 \right] + \exp\left[-\frac{\beta^2}{2} - \frac{\alpha^2}{2} - 2\alpha\alpha_0 - 2\alpha^2 \right]
\end{aligned}
$$

The plots given in Figs. (4.1, 4.2, 4.4, 4.5, 4.6, 4.7, 4.8) include, the real Weyl function $\mathcal{H}(\alpha, \beta; 0, 0)$, real Wigner function $\mathcal{H}\left(\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{2}\right)$ and other bi-fractional combinations of varying angles across the continuum of $(\theta_\alpha, \theta_\beta)$ given in $\mathcal{H}\left(\alpha, \beta; \frac{\pi}{4}, \frac{\pi}{4}\right)$, $\mathcal{H}\left(\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{4}\right)$, $\mathcal{H}\left(\alpha, \beta; \frac{\pi}{4}, \frac{\pi}{2}\right)$, $\mathcal{H}\left(\alpha, \beta; 0, \frac{\pi}{4}\right)$ and $\mathcal{H}\left(\alpha, \beta; \frac{\pi}{4}, 0\right)$ correspondingly, for $\alpha_0 = 2$ and $\beta_0 = 0$.

In the Weyl function of Fig. (4.1) the auto-terms are in the middle and the cross-terms are in the 'wings', while in the Wigner function of Fig. (4.2), the cross-terms are in the middle and the auto-terms are in the wings. Moving from the Wigner function to the Weyl function, the auto-terms move from the wings to the centre. The movement of the auto-terms and cross terms is quite undefined in the intermediate functions as shown in the figures with both terms in the middle and the wings. It is

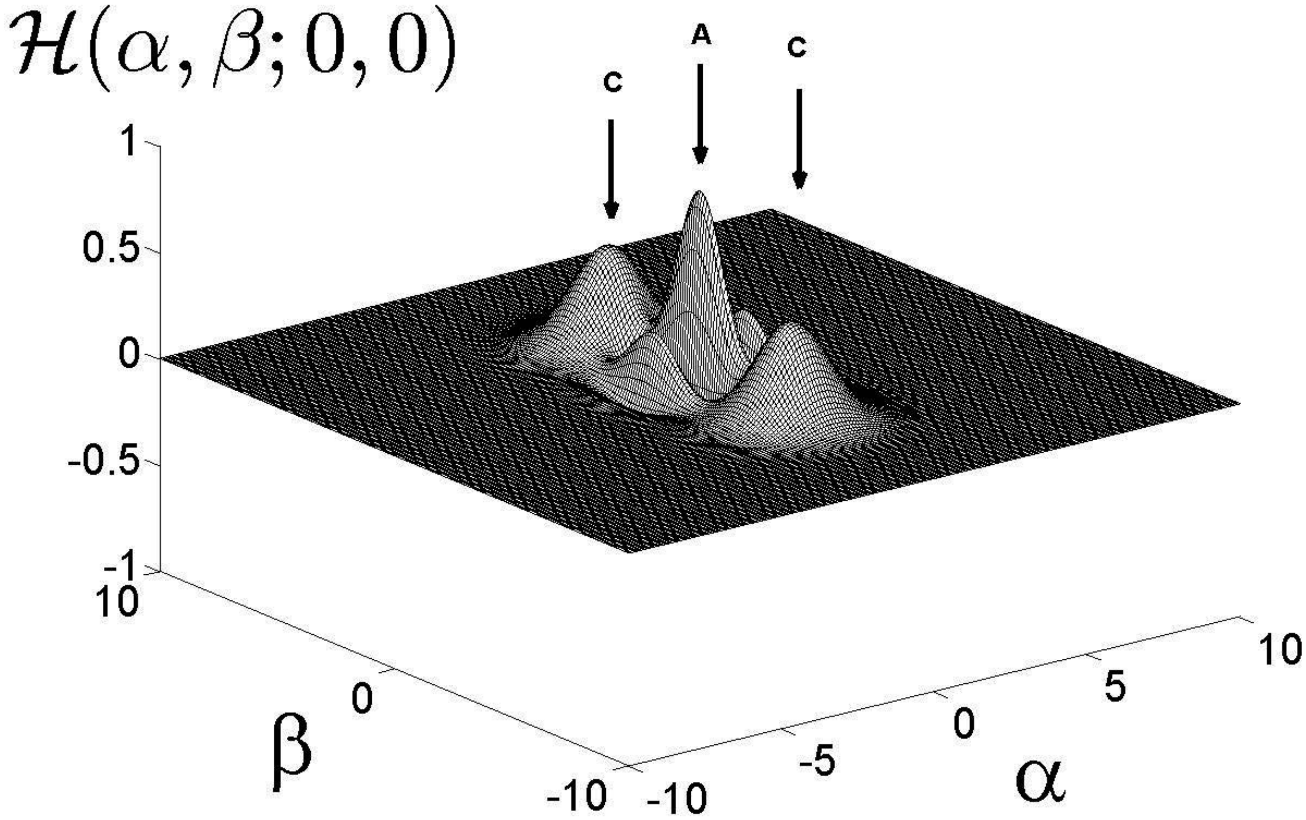


Fig. (4.1) $\mathcal{H}(\alpha,\beta;0,0)$(Weyl function) for the state of Eq. (4.2) with $\alpha = 2$ and $\beta = 0$. The arrows indicate the autoparts (A) and cross-parts (C)

also worth noting that the effect of θ_α and θ_β are different as indicated by Fig. (4.7) and Fig. (4.8). Therefore for special cases where either θ_α or θ_β equals zero and the other angle is between 0 and $\frac{\pi}{2}$, varying results were observed. More questions can also be considered on analysis of the uncertainty relation where it is not a function of α but of β when θ_β = 0. Arrows are used in the figures indicate the auto-terms (A) and the cross-terms (C).

Another consideration is that for all cases where θ_α = θ_β, the graphs are symmetric.

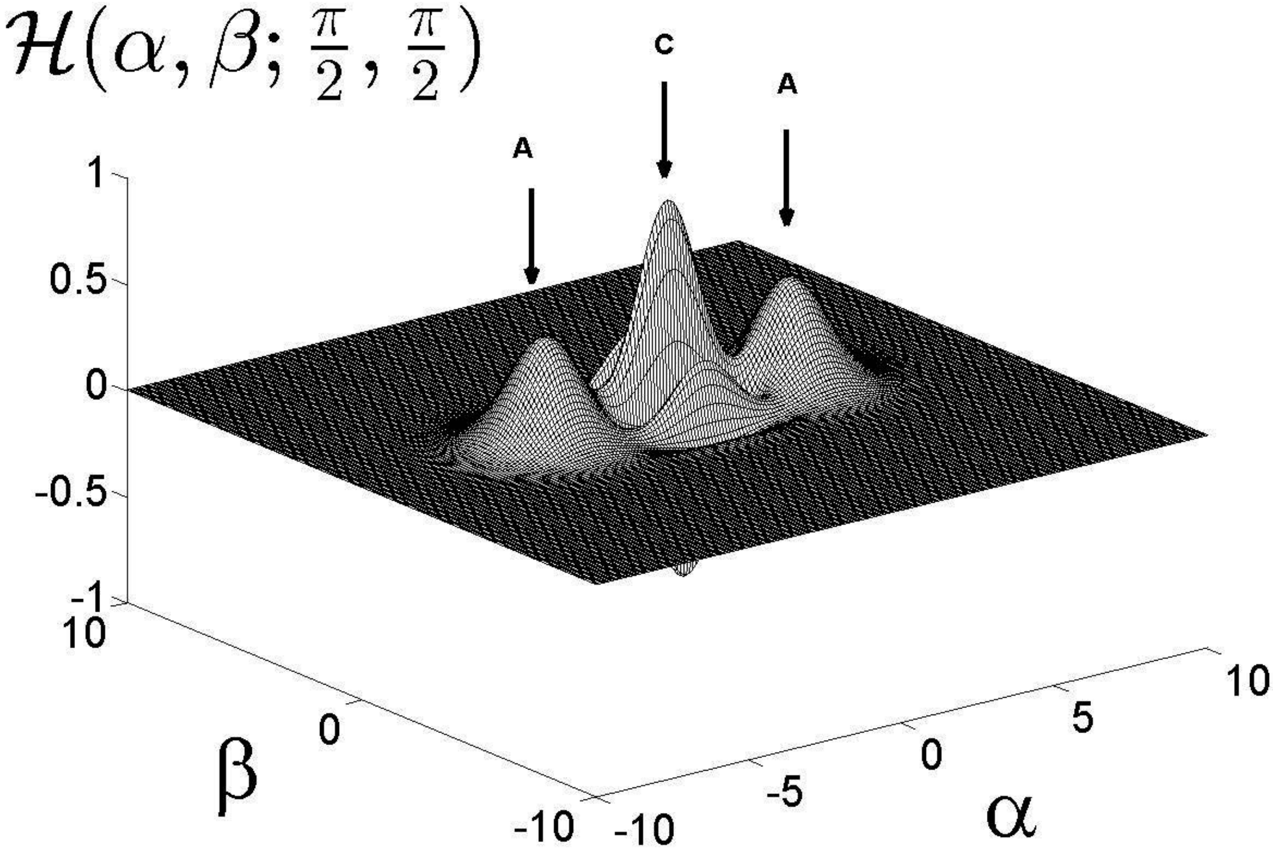


Fig. (4.2) $\mathcal{H}(\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{2})$(Wigner function) for the state of Eq. (4.2) with $\alpha = 2$ and $\beta = 0$. The arrows indicate the autoparts (A) and cross-parts (C)

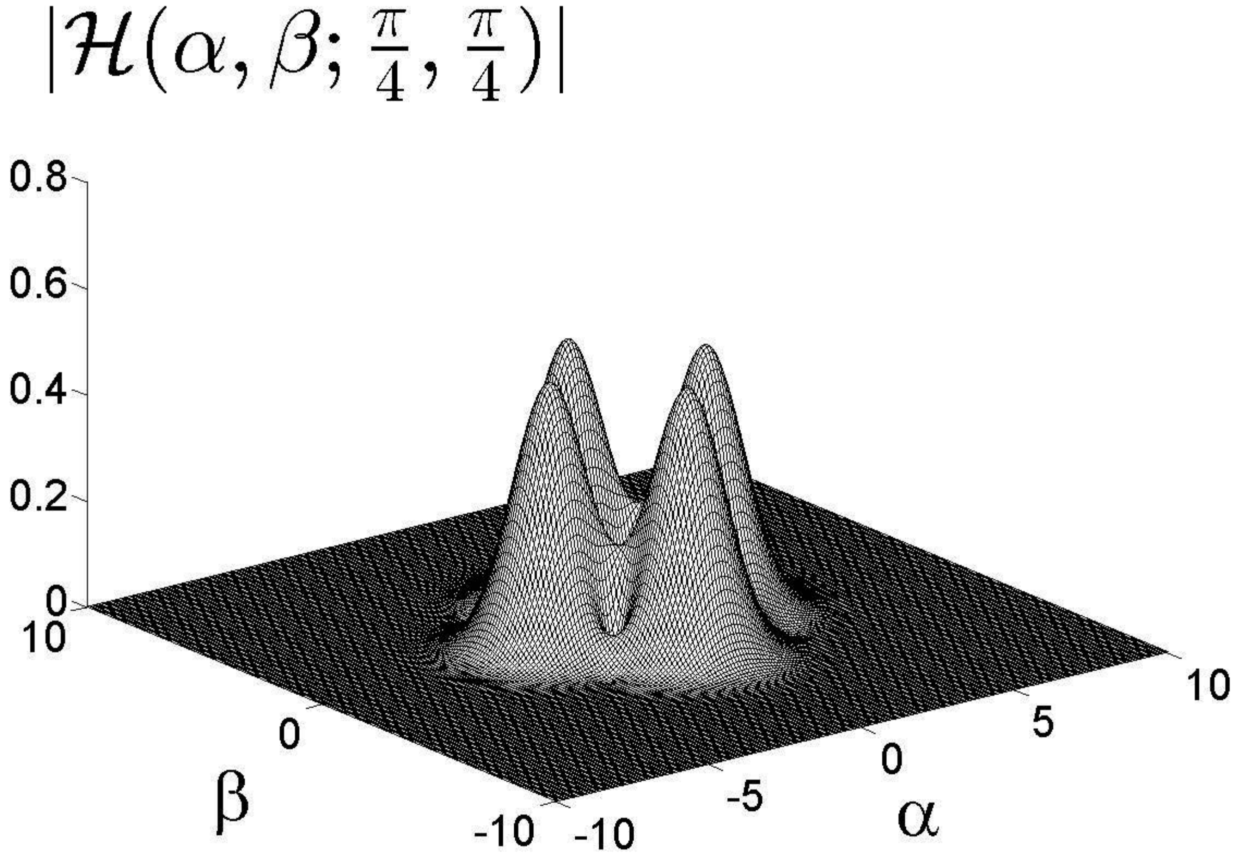


Fig. (4.3) $|\mathcal{H}(\alpha,\beta;\frac{\pi}{4},\frac{\pi}{4})|$ for the state of Eq. (4.2) with $\alpha = 1.8$ and $\beta = 0$.

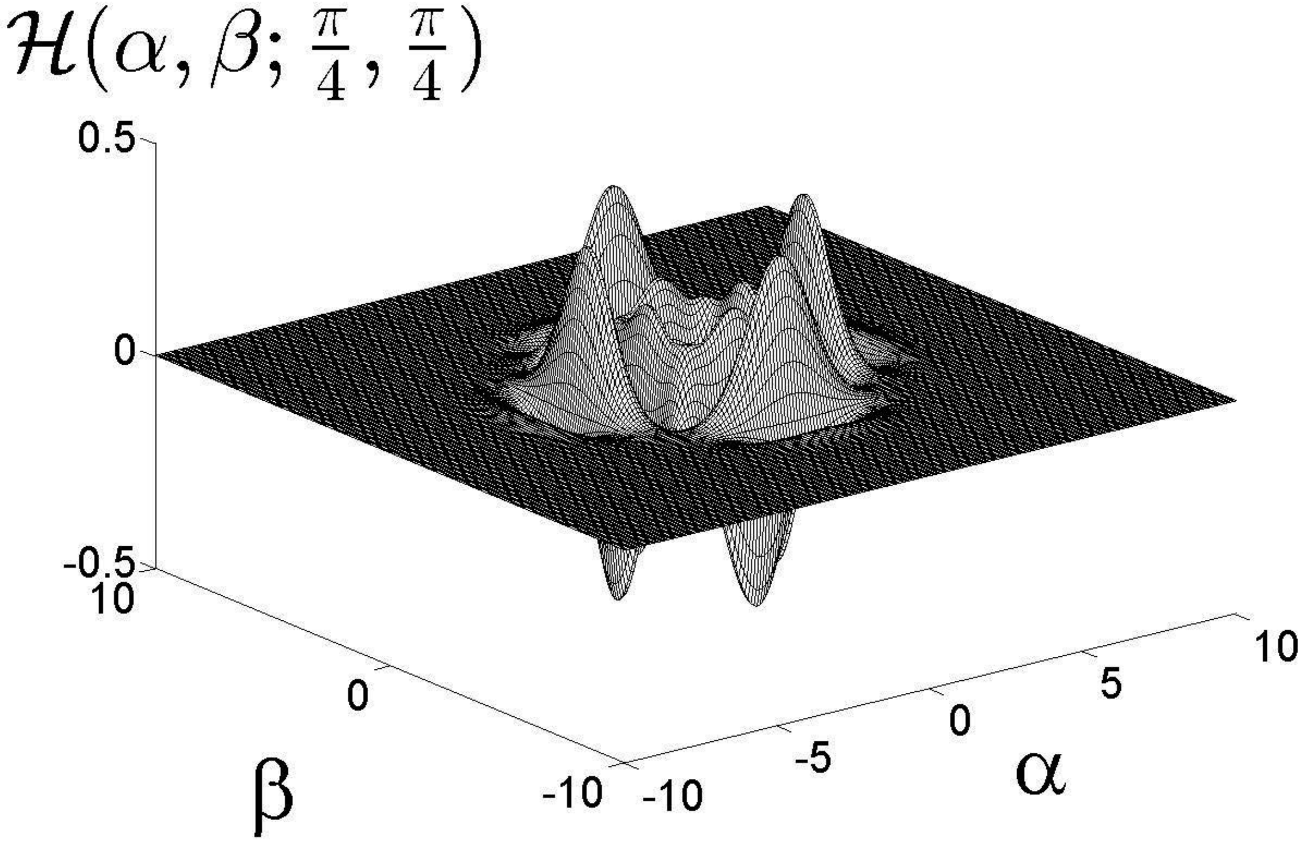


Fig. (4.4) $\mathcal{H}(\alpha, \beta; \frac{\pi}{4}, \frac{\pi}{4})$ for the state of Eq. (4.2) with $\alpha = 2$ and $\beta = 0$.

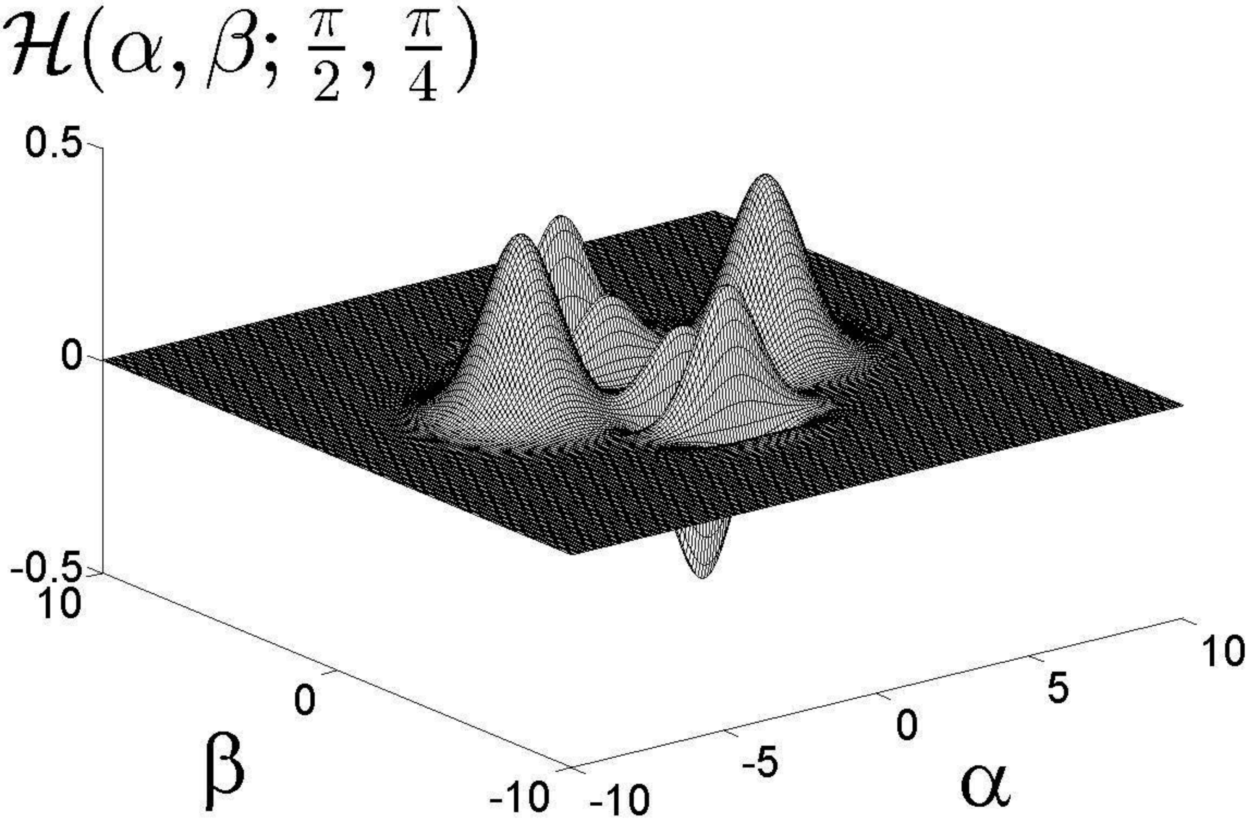


Fig. (4.5) $\mathcal{H}(\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{4})$ for the state of Eq. (4.2) with $\alpha = 2$ and $\beta = 0$.

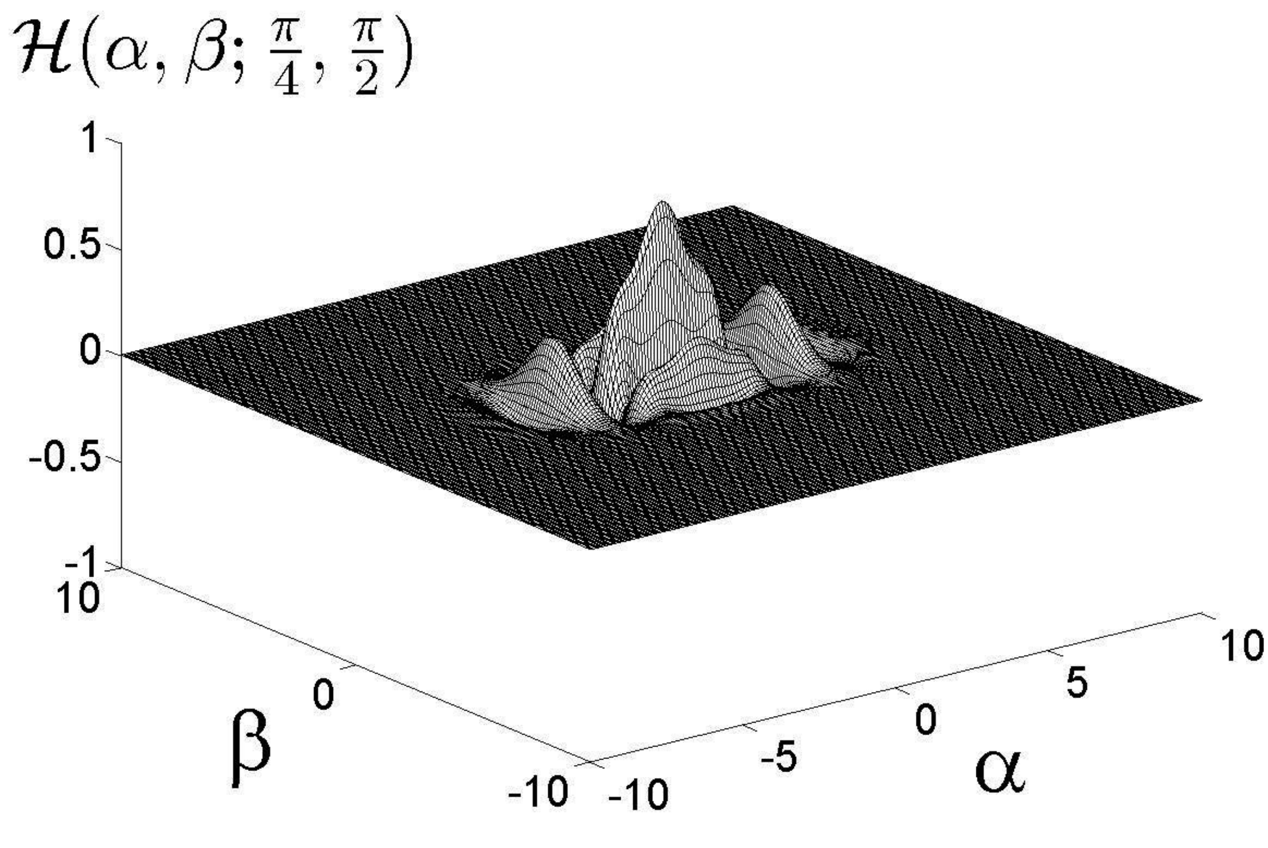


Fig. (4.6) $\mathcal{H}(\alpha,\beta;\frac{\pi}{4},\frac{\pi}{2})$ for the state of Eq. (4.2) with $\alpha = 2$ and $\beta = 0$.

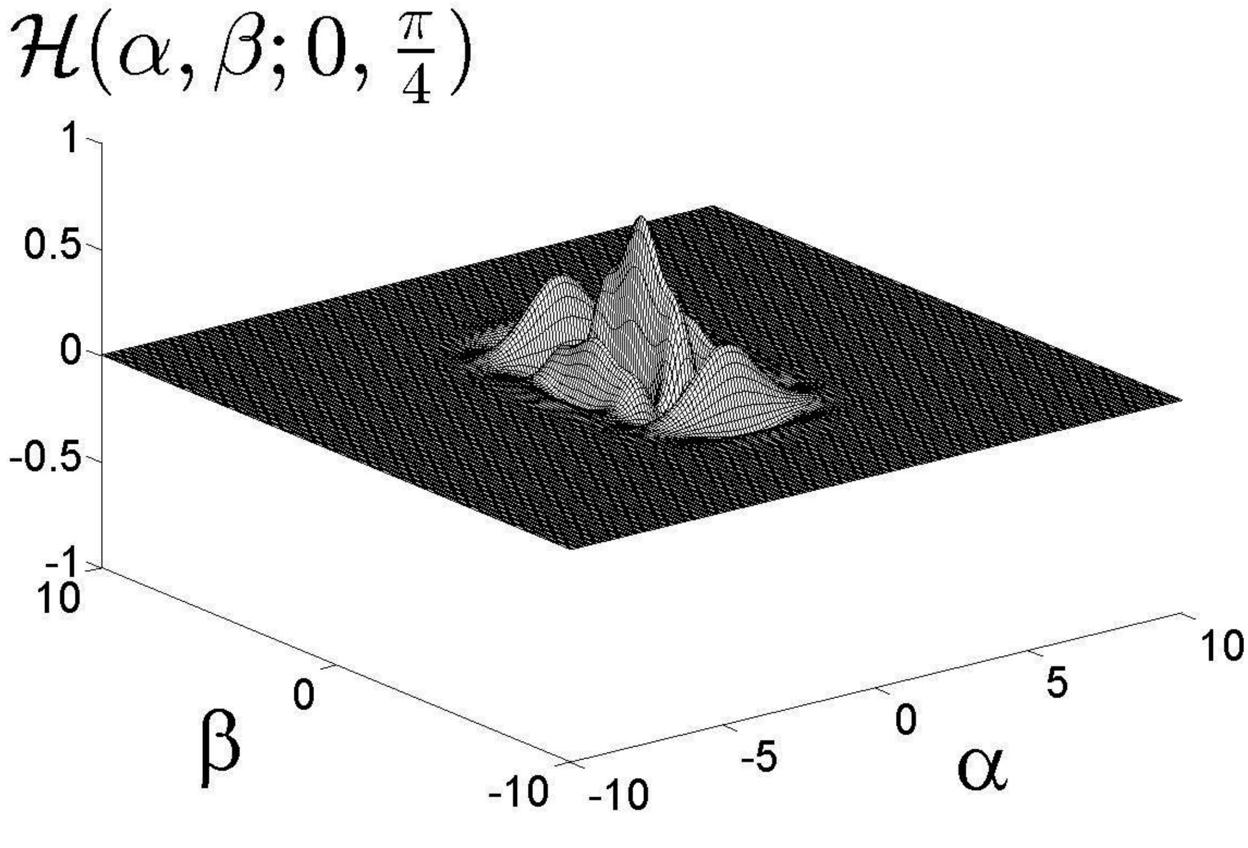


Fig. (4.7) $\mathcal{H}(\alpha, \beta; 0, \frac{\pi}{4})$ for the state of Eq. (4.2) with $\alpha = 2$ and $\beta = 0$.

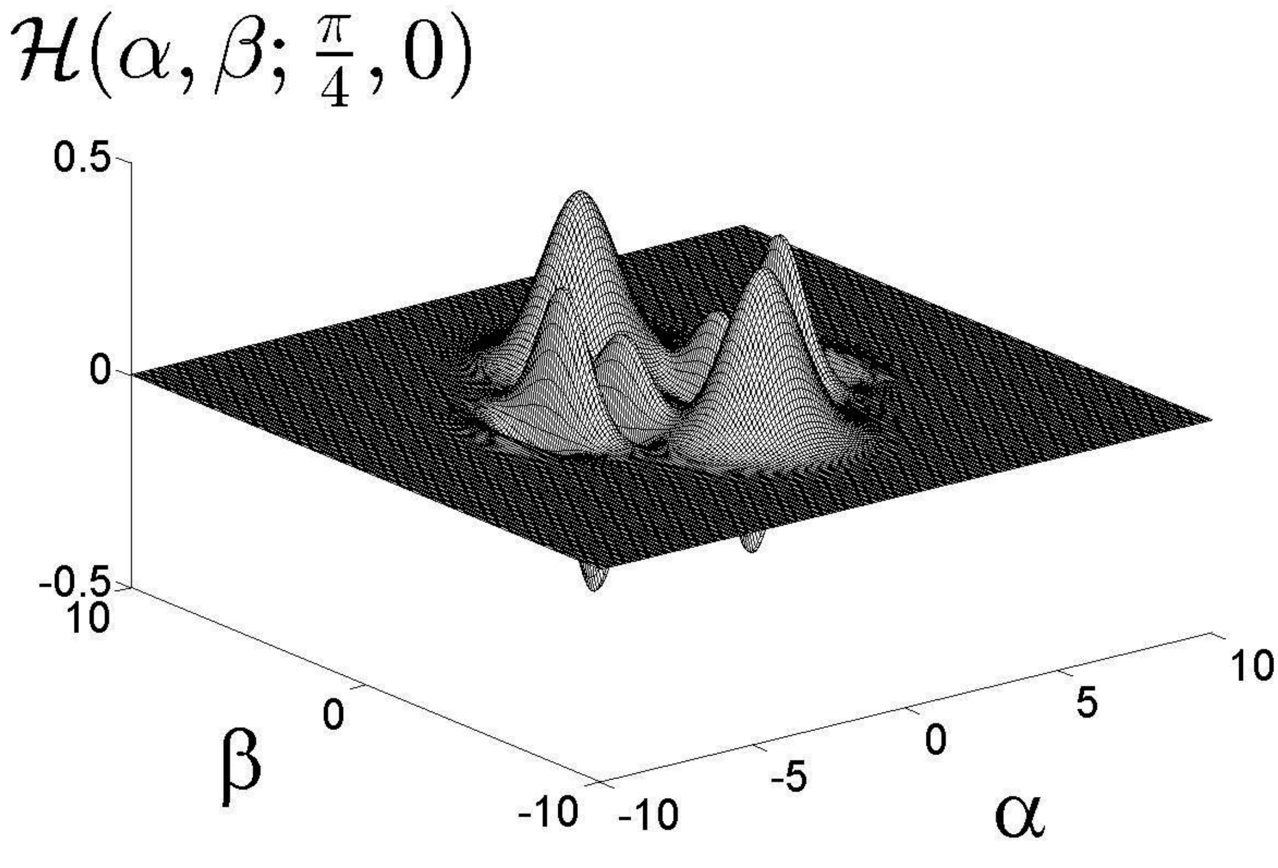


Fig. (4.8) $\mathcal{H}(\alpha, \beta; \frac{\pi}{4}, 0)$ for the state of Eq. (4.2) with $\alpha = 2$ and $\beta = 0$.

4.5 Bi-fractional Q-functions and bi-fractional P-functions

Q-function (or Husimi function) for a trace class operator Θ, is defined as

$$Q(\alpha, \beta|\Theta) = \langle \alpha, \beta| \Theta |\alpha, \beta\rangle , \tag{4.18}$$

and define bi-fractional Q-functions with respect to the bi-fractional coherent states as follows:

$$Q(\alpha, \beta; \theta_\alpha, \theta_\beta|\Theta) = \langle \alpha, \beta; \theta_\alpha, \theta_\beta| \Theta |\alpha, \beta; \theta_\alpha, \theta_\beta\rangle . \tag{4.19}$$

Using the resolution of the identity in Eq. (3.72) gives the relation,

$$\frac{1}{2\pi} \int d\alpha d\beta\, Q(\alpha, \beta; \theta_\alpha, \theta_\beta|\Theta) = \mathrm{Tr}\Theta. \tag{4.20}$$

Clearly

$$Q(\alpha, \beta; 0, 0|\Theta) = Q(\beta, -\alpha|\Theta); \quad Q\left(\alpha, \beta; \frac{\pi}{2}, \frac{\pi}{2}\middle|\Theta\right) = Q(\alpha, \beta|\Theta). \tag{4.21}$$

The bi-fractional Q-functions are generalisations of the Q-functions.

The bi-fractional P-function $P(\alpha, \beta; \theta_\alpha, \theta_\beta|\Theta)$ is also introduced with respect to the bi-fractional coherent states, as

$$\Theta = \frac{1}{\pi} \int d\alpha\, d\beta P(\alpha, \beta; \theta_\alpha, \theta_\beta|\Theta) |\alpha, \beta; \theta_\alpha, \theta_\beta\rangle \langle \alpha, \beta; \theta_\alpha, \theta_\beta| . \tag{4.22}$$

In the case $\theta_\alpha = \theta_\beta = \frac{\pi}{2}$, it reduces to the ordinary P-function $P(\alpha, \beta)$.

Using Eq. (3.110), $Q(\alpha, \beta; \theta_\alpha, 0|\Theta)$ is calculated and various instances are shown

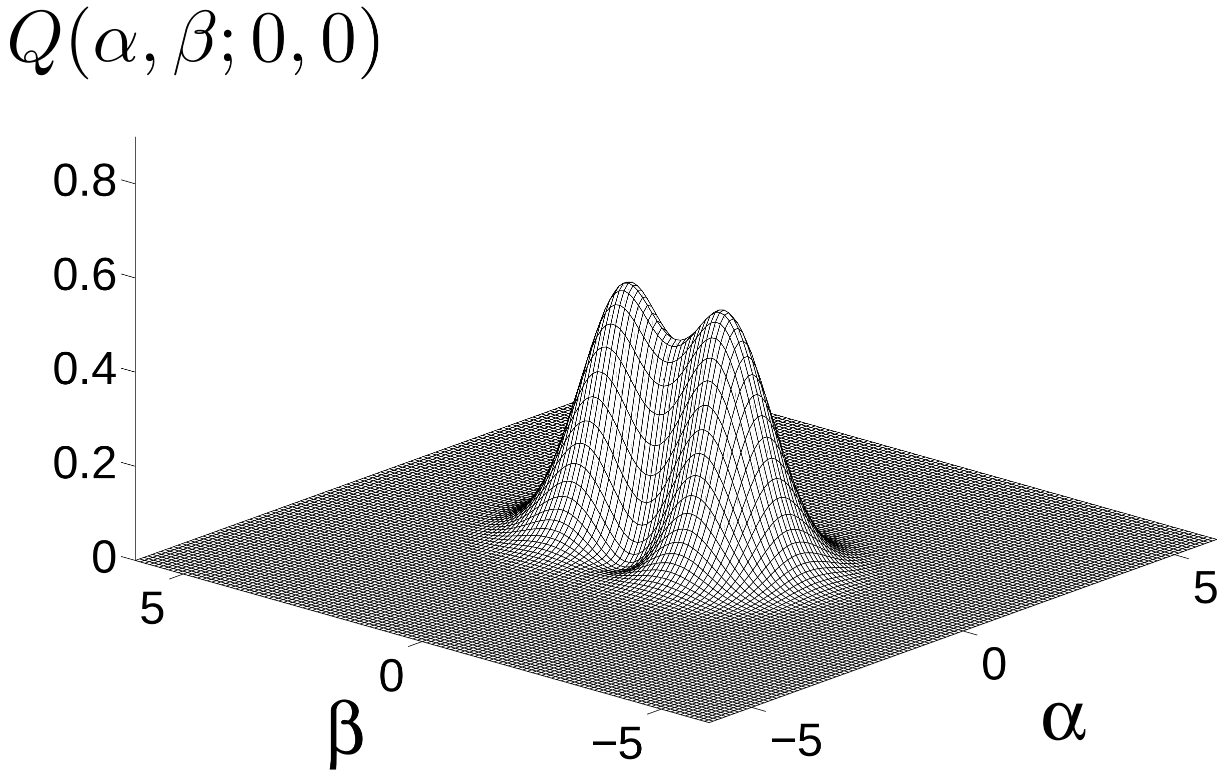


Fig. (4.9) The Q-function $Q(\alpha, \beta; 0, 0)$ for the state of Eq. (4.13) with $\alpha_0 = 1.2$ and $\beta_0 = 0$

in Fig. (4.9, 4.10), presenting $Q(\alpha, \beta; 0, 0|\Theta)$ and $Q\left(\alpha, \beta; \frac{\pi}{4}, 0|\Theta\right)$ respectively for the state of Eq. (4.13) with $\alpha_0 = 1.2$ and $\beta_0 = 0$. The result show the rotation of $Q(\alpha, \beta; \theta_\alpha, 0|\Theta)$, as θ_α changes.

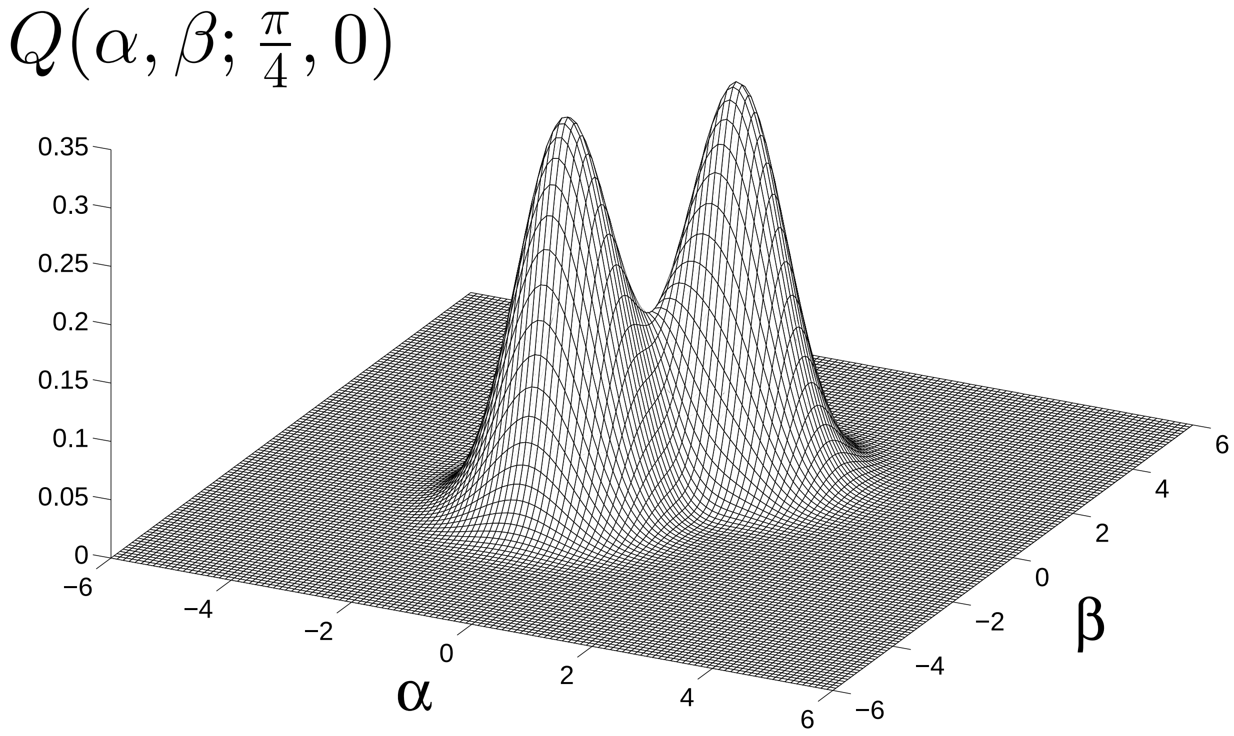


Fig. (4.10) The bi-fractional Q-function $Q\left(\alpha, \beta; \frac{\pi}{4}, 0\right)$ for the state of Eq. (4.13) with $\alpha_0 = 1.2$ and $\beta_0 = 0$

4.6 Bi-fractional Moyal star formalism

The Moyal star formalism in phase space has already been explained in section (2.11). An extension is given by defining the bi-fractional equivalent which is called, the 'bi-fractional Moyal formalism'. The starting point is by giving the generalised form of the Moyal equation given for arbitrary states $|\gamma\rangle, |\zeta\rangle, |\epsilon\rangle, |\delta\rangle$,

$$\frac{1}{\pi\cos(\theta_\alpha - \theta_\beta)} \int d\alpha d\beta \langle\gamma|\mathcal{O}^\dagger(\alpha, \beta; \theta_\alpha, \theta_\beta)|\delta\rangle \langle\epsilon|\mathcal{O}(\alpha, \beta; \theta_\alpha, \theta_\beta)|\zeta\rangle = \langle\gamma|\zeta\rangle \langle\epsilon|\delta\rangle . \quad (4.23)$$

The proof is relatively obvious using the matrix properties of the displacement operators (Eq. (2.40)) and resolution of identity property (Eq. (2.18)),

$$\frac{1}{\pi\cos(\theta_\alpha-\theta_\beta)}\int d\alpha d\beta\, \langle\gamma|\mathcal{O}^\dagger(\alpha,\beta;\theta_\alpha,\theta_\beta)|\delta\rangle\, \langle\epsilon|\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)|\zeta\rangle$$
$$=\frac{1}{\pi}\int d\alpha d\beta d\alpha' d\beta' d\alpha'' d\beta''\, \langle\gamma|\,\widehat{D}(\alpha',\beta')\,|\delta\rangle\, \langle\epsilon|\,\widehat{D}(\alpha'',\beta'')\,|\zeta\rangle$$
$$\times K(-\beta,\alpha';-\theta_\beta)K(-\alpha,-\beta';-\theta_\alpha)K(\beta,\alpha'';\theta_\beta)K(\alpha,-\beta'';\theta_\alpha)$$
$$=\frac{1}{\pi}\int d\alpha'' d\beta''\, \langle\gamma|\,D^\dagger(\alpha'',\beta'')\,|\delta\rangle\, \langle\epsilon|\,D(\alpha'',\beta'')\,|\zeta\rangle = \langle\gamma|\zeta\rangle\, \langle\epsilon|\delta\rangle\,. \tag{4.24}$$

The last part of Eq. (4.24) above has already been proved in Eq. (2.113). Furthermore, the operator, Θ which was previously given in terms of the displacement operator is now replaced with the bi-fractional displacement operators,

$$\Theta = \frac{1}{\pi\cos(\theta_\alpha-\theta_\beta)}\int d\alpha d\beta\ \ \mathrm{Tr}[\mathcal{O}^\dagger(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)]\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$$
$$= \frac{1}{\pi\cos(\theta_\alpha-\theta_\beta)}\int d\alpha d\beta\ \ \mathcal{H}^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta). \tag{4.25}$$

The proof is similar to that of Eq. (4.24) shown above and is given by,

$$\frac{1}{\pi\cos(\theta_\alpha-\theta_\beta)}\int d\alpha d\beta\ \ \mathcal{H}^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$$
$$= \frac{1}{\pi\cos(\theta_\alpha-\theta_\beta)}\int d\alpha d\beta\ \ \mathrm{Tr}[\mathcal{O}^\dagger(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)]\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)$$
$$=\frac{1}{\pi}\int d\alpha d\beta d\alpha' d\beta' d\alpha'' d\beta''\ \ \mathrm{Tr}[\widehat{D}^\dagger(\alpha',\beta')\Theta]\widehat{D}(\alpha'',\beta'')K(-\beta,\alpha';-\theta_\beta)K(-\alpha,-\beta';-\theta_\alpha)$$
$$\times K(\beta,\alpha'';\theta_\beta)K(\alpha,-\beta'';\theta_\alpha) = \frac{1}{\pi}\int d\alpha d\beta\ \ \mathrm{Tr}[\widehat{D}^\dagger(\alpha,\beta)\Theta]\widehat{D}(\alpha,\beta) = \Theta. \tag{4.26}$$

The last equality is already known as shown in Eq. (2.115). The proof can also be shown using Eq. (4.23) and taking the matrix elements on both sides of the operator,

$$\langle z|\Theta|w\rangle = \frac{1}{\pi\cos(\theta_\alpha - \theta_\beta)} \int d\alpha d\beta\ \mathcal{H}^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)\, \langle z|\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)|w\rangle\,, \tag{4.27}$$

and using the resolution of identity it can also be shown that,

$$\mathcal{H}^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta) = \mathrm{Tr}\left[\Theta\mathcal{O}^\dagger(\alpha,\beta;\theta_\alpha,\theta_\beta)\right] = \frac{1}{\pi^2}\int d^2z\ d^2u\, \langle z|\Theta|u\rangle\, \langle u|\mathcal{O}^\dagger(\alpha,\beta;\theta_\alpha,\theta_\beta)|z\rangle\,. \tag{4.28}$$

Thus combining Eqs. (4.28, 4.27, 4.23) proves Eq. (4.25). It further leads to the case of two operators, $\Theta_1\Theta_2$,

$$\begin{aligned}\Theta_1\Theta_2 &= \frac{1}{[\pi\cos(\theta_\alpha - \theta_\beta)]^2}\int d\alpha' d\beta' d\alpha d\beta\ \mathrm{Tr}[\mathcal{O}^\dagger(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_1]\\ &\times \mathrm{Tr}[\mathcal{O}^\dagger(\alpha',\beta';\theta_\alpha,\theta_\beta|\Theta_2]\ \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\ \mathcal{O}(\alpha',\beta';\theta_\alpha,\theta_\beta)\\ &= \frac{1}{[\pi\cos(\theta_\alpha - \theta_\beta)]^2}\int d\alpha' d\beta' d\alpha d\beta\ \mathcal{H}^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_1)\ \mathcal{H}^*(\alpha',\beta';\theta_\alpha,\theta_\beta|\Theta_2)\\ &\times \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\ \mathcal{O}(\alpha',\beta';\theta_\alpha,\theta_\beta).\end{aligned} \tag{4.29}$$

Correspondingly, the trace of a product of two operators can be given as,

$$\mathrm{Tr}(\Theta_1\Theta_2) = \frac{1}{\pi\cos(\theta_\alpha - \theta_\beta)}\int d\alpha d\beta\ \mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_1)\mathcal{H}^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_2). \tag{4.30}$$

The proof is by using Eq. (4.2),

$$
\begin{aligned}
&\frac{1}{\pi\cos(\theta_\alpha-\theta_\beta)}\int d\alpha d\beta \mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_1)\mathcal{H}^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_2)\\
&=\frac{1}{\pi\cos(\theta_\alpha-\theta_\beta)}\int d\alpha' d\beta' d\alpha'' d\beta'' \widetilde{W}(\alpha',\beta'|\Theta_1)\widetilde{W}(\alpha'',\beta''|\Theta_2)\\
&\times K(-\beta,\alpha';-\theta_\beta)K(-\alpha,-\beta';-\theta_\alpha)K(\beta,\alpha'';\theta_\beta)K(\alpha,-\beta'';\theta_\alpha). \qquad (4.31)
\end{aligned}
$$

Using Eq. (3.13) gives Dirac delta functions, which give

$$
\begin{aligned}
&=\frac{1}{\pi}\int d\alpha'' d\beta''\, \widetilde{W}(-\alpha'',-\beta''|\Theta_1)\widetilde{W}(\alpha'',\beta''|\Theta_2)\\
&=\frac{1}{\pi}\int d\alpha'' d\beta''\, \left\langle x+\frac{\alpha''}{\sqrt{2}}\right|\Theta_1\left|x-\frac{\alpha''}{\sqrt{2}}\right\rangle\left\langle y-\frac{\alpha''}{\sqrt{2}}\right|\Theta_2\left|x+\frac{\alpha''}{\sqrt{2}}\right\rangle e^{i\sqrt{2}\beta''(y-x)}. \qquad (4.32)
\end{aligned}
$$

Integrating with respect to β'', and changing variables, reduces to,

$$
\begin{aligned}
&=\sqrt{2}\int d\alpha'' dx\, \left\langle x+\frac{\alpha''}{\sqrt{2}}\right|\Theta_1\left|x-\frac{\alpha''}{\sqrt{2}}\right\rangle\left\langle x-\frac{\alpha''}{\sqrt{2}}\right|\Theta_2\left|x+\frac{\alpha''}{\sqrt{2}}\right\rangle\\
&=\int dk\,\langle k|\Theta_1\Theta_2|k\rangle = \mathrm{Tr}[\Theta_1\Theta_2]. \qquad (4.33)
\end{aligned}
$$

Given the $\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_1)$ and $\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_2)$ of two operators Θ_1,Θ_2, the following proposition gives the $\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_1\Theta_2)$ of their product which is the bi-fractional Moyal star product,

$$
\begin{aligned}
&\mathcal{H}(\epsilon,\zeta;\theta_\alpha,\theta_\beta|\Theta_1\Theta_2)\\
&=\frac{1}{\pi[\cos(\theta_\alpha-\theta_\beta)]^2}\int d\alpha' d\beta' d\alpha\, d\beta\, d\gamma\, d\lambda\, d\gamma'\, d\lambda'\, \mathcal{H}^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_1)\, \mathcal{H}^*(\alpha',\beta';\theta_\alpha,\theta_\beta|\Theta_2)\\
&\times K(\beta,\gamma;\theta_\beta)K(\alpha,-\lambda;\theta_\alpha)K(\beta',\gamma';\theta_\beta)K(\alpha',-\lambda';\theta_\alpha)K(\zeta,-(\gamma+\gamma');\theta_\beta)K(\epsilon,\lambda+\lambda';\theta_\alpha)\\
&\times \exp[i\lambda\gamma'-i\gamma\lambda']. \qquad (4.34)
\end{aligned}
$$

The bi-fractional Wigner function of Eq. (4.2) can be applied to the product of two operators,

$$\mathcal{H}(\epsilon,\zeta;\theta_\alpha,\theta_\beta|\Theta_1\Theta_2) = |\cos(\theta_\alpha-\theta_\beta)|^{1/2}\int d\epsilon'\, d\zeta'\, K(\zeta,\epsilon';\theta_\beta)K(\epsilon,-\zeta';\theta_\alpha) \times \mathrm{Tr}[\widehat{D}(\epsilon',\zeta')\Theta_1\Theta_2] \tag{4.35}$$

The main aim of the preceeding proofs for bi-fractional Moyal star formalism is to show that Eq. (4.34) equals Eq. (4.35). For a case of two operators, from Eq. (4.25) it is obvious that

$$\Theta_1\Theta_2 = \frac{1}{[\pi\cos(\theta_\alpha-\theta_\beta)]^2}\int d\alpha' d\beta' d\alpha d\beta\, \mathcal{H}^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_1)\, \mathcal{H}^*(\alpha',\beta';\theta_\alpha,\theta_\beta|\Theta_2) \times\, \mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\, \mathcal{O}(\alpha',\beta';\theta_\alpha,\theta_\beta). \tag{4.36}$$

Therefore,

$$\begin{aligned}\mathcal{H}(\epsilon,\zeta;\theta_\alpha,\theta_\beta|\Theta_1\Theta_2) &= \frac{[\cos(\theta_\alpha-\theta_\beta)]^{-3/2}}{2\pi^3}\int d\alpha' d\beta' d\alpha d\beta\; d\epsilon' d\zeta' \mathcal{H}^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_1)\\ &\times \mathcal{H}^*(\alpha',\beta';\theta_\alpha,\theta_\beta|\Theta_2)\mathrm{Tr}[\mathcal{O}(\alpha,\beta;\theta_\alpha,\theta_\beta)\mathcal{O}(\alpha',\beta';\theta_\alpha,\theta_\beta)\widehat{D}(\epsilon',\zeta')]\\ &\times K(\zeta,\epsilon';\theta_\beta)K(\epsilon,-\zeta';\theta_\alpha)\\ &= \frac{[\cos(\theta_\alpha-\theta_\beta)]^{-3/2}}{2\pi^3}\int d\alpha' d\beta' d\alpha d\beta\; d\epsilon' d\zeta' H^*(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta_1)\\ &\times \mathcal{H}^*(\alpha',\beta';\theta_\alpha,\theta_\beta|\Theta_2)K(\beta,\gamma;\theta_\beta)K(\alpha,-\lambda;\theta_\alpha)K(\beta',\gamma';\theta_\beta)K(\alpha',-\lambda';\theta_\alpha)\\ &\times \mathrm{Tr}[\widehat{D}(\gamma,\lambda)\widehat{D}(\gamma',\lambda')\widehat{D}(\epsilon',\zeta')]K(\zeta,\epsilon';\theta_\beta)K(\epsilon,-\zeta';\theta_\alpha).\end{aligned} \tag{4.37}$$

Using a generalised form of Eq. (2.42) for three displacement operators gives

$$\mathrm{Tr}[\widehat{D}(\gamma,\lambda)\widehat{D}(\gamma',\lambda')\widehat{D}(\epsilon',\zeta')] = \pi\delta(\epsilon'+\gamma+\gamma')\delta(\zeta'+\lambda+\lambda')\exp[i(\lambda\gamma'-\gamma\lambda'). \quad (4.38)$$

Inserting this in Eq. (4.37) proves Eq. (4.35).

4.7 Bi-fractional Berezin formalism

The Berezin formalism is well known [11, 14–17, 40, 73] and quite similar to the Moyal star product in that it analytically represents an operator. It can further be shown that the Berezin product of two operators can be expanded as a Taylor series to give classical terms and quantum corrections.

The proof begins by giving the relation of the bi-fractional Laplacian relevant for deriving the Berezin formalism,

$$\begin{aligned}\frac{1}{2\pi\cos(\theta_\alpha-\theta_\beta)}\int d\alpha' d\beta' F(\alpha',\beta')K\exp\left[-\frac{K[D_s(\alpha-\alpha',\beta-\beta'|\theta_\alpha,\theta_\beta)]}{\cos^2(\theta_\alpha-\theta_\beta)}\right]\\ =\frac{1}{2}\left[\exp\left(\frac{\Delta_{(\alpha,\beta|\theta_\alpha,\theta_\beta)}}{4K}\right)F(\alpha,\beta)\right],\end{aligned} \quad (4.39)$$

which reduces for a special case of $\theta_\alpha=\theta_\beta=\frac{\pi}{2}$ to,

$$\begin{aligned}\frac{1}{2\pi}\int d\alpha' d\beta' F(\alpha',\beta')K\exp\left[-K\left[\mathcal{G}_S\left(\alpha-\alpha',\beta-\beta'\middle|\frac{\pi}{2},\frac{\pi}{2}\right)\right]\right]\\ =\frac{1}{2}\left[\exp\left(\frac{\Delta_{(\alpha,\beta|\frac{\pi}{2},\frac{\pi}{2})}}{4K}\right)F(\alpha,\beta)\right].\end{aligned} \quad (4.40)$$

The relation $\mathcal{G}_S\left(\alpha-\alpha',\beta-\beta'\middle|\frac{\pi}{2},\frac{\pi}{2}\right)$ is a special case of the distance between two bi-fractional coherent states given in Eq. (3.100). The bi-fractional Laplacian $\Delta_{(\alpha,\beta|\theta_\alpha,\theta_\beta)}$

is given as,

$$\Delta_{(\alpha,\beta|\theta_\alpha,\theta_\beta)} = \frac{\partial^2}{\partial^2\alpha} + \frac{\partial^2}{\partial^2\beta} - 2\frac{\partial^2}{\partial\alpha\partial\beta}\sin(\theta_\alpha - \theta_\beta). \tag{4.41}$$

The bi-fractional Laplacian reduces to the normal Laplacian for special cases of $\theta_\alpha = \theta_\beta = \frac{\pi}{2}$. The relation of Eq. (4.39), can also be given in complex terms, w, w^*, with respect to the complex bi-fractional distance and equivalently the bi-fractional Laplacian as,

$$\frac{1}{2\pi}\int d^2wF(w,w^*)K\exp\left[-K|w(\theta_1,\theta_2) - z(\theta_1,\theta_2)|^2\right] = \frac{1}{2}\left[\exp\left(\frac{\Delta_{(z,z^*|\theta_1,\theta_2)}}{4K}\right)F(z,z^*)\right],$$

$$\Delta_{(z,z^*|\theta_1,\theta_2)} = 4\frac{\partial^2}{\partial z\partial z^*} - 2i\left[\frac{\partial^2}{\partial^2 z} - \frac{\partial^2}{\partial^2 z^*}\right]\sin(\theta_1 - \theta_2). \tag{4.42}$$

The relation given in Eq. (4.39) and Eq. (4.40) is easily proved by taking the fractional Fourier transform and Fourier transforms respectively. The full proof is given in Appendix A. Thus, the bi-fractional Berezin function is defined as,

$$\mathcal{B}(z,w^*;\theta_\alpha,\theta_\beta|\Theta) = \exp\left[\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - zw^*\right]\langle z^*(\theta_\alpha,\theta_\beta)|\Theta|w^*(\theta_\alpha,\theta_\beta)\rangle. \tag{4.43}$$

It is a generalisation of the Berezin formalism [73] and is analytic with respect to $w^*(\theta_\alpha,\theta_\beta)$ and $z(\theta_\alpha,\theta_\beta)$. Previous work on the Berezin formalism showed the analyticity with respect to the special cases of $w^*(\theta_\alpha,\theta_\beta)$ and $z(\theta_\alpha,\theta_\beta)$ for $\theta_\alpha = \theta_\beta = \frac{\pi}{2}$ and in this sense this formalism is generalised.

Furthermore, the Berezin function for the product of two operators can be shown to be

$$\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_1\Theta_2) = \int d^2w\, \mathcal{G}_S\left(z,w|\theta_\alpha,\theta_\beta\right)\mathcal{B}(z,w^*;\theta_\alpha,\theta_\beta|\Theta_1)\mathcal{B}(w,z^*;\theta_\alpha,\theta_\beta|\Theta_2)$$

$$\mathcal{G}_S\left(z,w|\theta_\alpha,\theta_\beta\right) = |\left\langle z(\theta_\alpha,\theta_\beta)|w(\theta_\alpha,\theta_\beta)\right\rangle|^2. \tag{4.44}$$

Furthermore, using Eq. (4.42) it can be shown that,

$$\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_1\Theta_2) = \frac{1}{2}\left[\exp\left(\Delta_{\frac{(\zeta,\zeta^*|\theta_\alpha,\theta_\beta)}{4}}\right)\mathcal{B}(z,\zeta^*;\theta_\alpha,\theta_\beta|\Theta_1)\mathcal{B}(\zeta,z^*;\theta_\alpha,\theta_\beta|\Theta_1)\right]_{\zeta=z} \tag{4.45}$$

Then using Taylor's expansion it can be shown that,

$$\begin{aligned}\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_1\Theta_2) =&\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_1)\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_2)\\ &+\frac{\partial\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_1)}{2\partial z^*}\frac{\partial\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_2)}{\partial z}\\ &+\left[i\sin(\theta_\alpha-\theta_\beta)\frac{\partial^2\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_1)\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_2)}{4\partial z^{2*}}\right]\\ &-\left[i\sin(\theta_\alpha-\theta_\beta)\frac{\partial^2\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_1)\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_2)}{4\partial z^{2}}\right]+....\end{aligned} \tag{4.46}$$

It is to be noted that for semi-classical studies, $\hbar^{1/2}$ is attached to each of the derivatives and in the limit where $\hbar\to 0$

$$\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_1\Theta_2) \approx \mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_1)\mathcal{B}(z,z^*;\theta_\alpha,\theta_\beta|\Theta_2) \tag{4.47}$$

This means that commutativity is achieved when all quantum corrections are removed and a classical result is obtained.

4.8 Interpolating quantum noise and correlations

For a density matrix Θ, the Wigner function $W(\alpha,\beta|\Theta)$ and the Weyl function $\widetilde{W}(\alpha,\beta|\Theta)$ were defined in Eqs. (2.93, 2.90). The Wigner function quantifies noise, and the Weyl function quantifies correlations in a quantum system. In depicting correlation, the Weyl function integrates a wavefunction with its displacement in phase space. The Wigner function based on its width describes quantum noise and classical noise in both the position and momentum. The α,β in the Wigner function are position and momentum, while the α,β in the Weyl function are position and momentum increments, related to correlations.

The bi-fractional Wigner function $\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)$ becomes relevant because it interpolates between correlations and noise. Therefore, just as the Wigner function of Eq. (2.93) gives a position-momentum duality, the bi-fractional Wigner function also gives a 'correlation-noise duality'. For special cases of $\theta_\alpha,\theta_\beta$ approaching zero, the bi-fractional Wigner function approaches the Weyl function which is correlations-related. For a case of $(\theta_\alpha,\theta_\beta)$ approaching $\pi/2$, it approaches the Wigner function which leads to uncertainties.

For general cases of $\theta_\alpha,\theta_\beta$ the $\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)$ interpolates between both the Wigner and Weyl functions, and quantifies noise-correlations duality.

For special cases of Eq. (4.2) for $\theta_\alpha=\theta_\beta=0$ and $\theta_\alpha=\theta_\beta=\frac{\pi}{2}$ which gives the Wigner and Weyl functions respectively, from Eq. (2.103) it was shown that,

$$\frac{1}{\pi}\int d\alpha d\beta\left|\mathcal{H}(\alpha,\beta;\frac{\pi}{2},\frac{\pi}{2}|\Theta)\right|^2=\frac{1}{\pi}\int d\alpha d\beta|\mathcal{H}(\alpha,\beta;0,0|\Theta)|^2 d\alpha d\beta=\mathrm{Tr}(\Theta^2). \quad (4.48)$$

Uncertainty quantities can be introduced with respect to the bi-fractional Wigner function as follows,

$$\langle\langle\alpha^n\rangle\rangle = \frac{1}{\pi \mathrm{Tr}(\Theta^2)} \int \alpha^n |\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2 d\alpha d\beta; \quad \Delta_\alpha(\theta_\alpha,\theta_\beta) = \left[\langle\langle\alpha^n\rangle\rangle - (\langle\langle\alpha\rangle\rangle)^2\right]^{1/2},$$
$$\langle\langle\beta^n\rangle\rangle = \frac{1}{\pi \mathrm{Tr}(\Theta^2)} \int d\alpha d\beta\, \beta^n |\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta|\Theta)|^2; \quad \Delta_\beta(\theta_\alpha,\theta_\beta) = \left[\langle\langle\beta^n\rangle\rangle - (\langle\langle\beta\rangle\rangle)^2\right]^{1/2}. \tag{4.49}$$

For the special case of Wigner and Weyl functions of Eqs. (2.90, 2.93) these quantities are known [72]. It has been shown [72] that for pure states they are the usual uncertainties, but for mixed states they are different. For $(\theta_\alpha, \theta_\beta)$ approaching zero, $\Delta_\alpha(\theta_\alpha,\theta_\beta)$ and $\Delta_\beta(\theta_\alpha,\theta_\beta)$ quantify correlations in position and momentum, and if $(\theta_\alpha,\theta_\beta)$ are close to $\pi/2$, they quantify noise.

A general case of $\Delta_\alpha(\theta_\alpha,\theta_\beta)\Delta_\beta(\theta_\alpha,\theta_\beta)$ is given in [72] and proved that $\Delta_\alpha(\frac{\pi}{2},\frac{\pi}{2})\Delta_\beta(0,0) \geq \frac{1}{2}$. In the case of arbitrary angles considered here, the inequality falls apart, these special cases will be considered. As an example, considering a special case of $\Delta_\alpha(\frac{\pi}{2},\theta_\beta)\Delta_\beta(0,0)$ as a function of θ_β in Fig. (4.11), for the quantum state described with the density matrix

$$\Theta = \frac{1}{2}[|\alpha_0,\beta_0\rangle\langle\alpha_0,\beta_0| + |-\alpha_0,-\beta_0\rangle\langle-\alpha_0,-\beta_0|]; \quad |\alpha_0,\beta_0\rangle = \widehat{D}(\alpha_0,\beta_0)|0\rangle; \quad \alpha_0 = 2; \quad \beta_0 = 0. \tag{4.50}$$

More so, the plot of $\Delta_\alpha(\frac{\pi}{4},\frac{\pi}{4})\Delta_\beta(0,0)$ as a function of p is given in Fig. (4.12), for the quantum state described with the density matrix

$$\Theta = p|\alpha_0,\beta_0\rangle\langle\alpha_0,\beta_0| + (1-p)|-\alpha_0,-\beta_0\rangle\langle-\alpha_0,-\beta_0|; \quad 0 \leq p \leq 1; \quad \alpha_0 = 2; \quad \beta_0 = 0. \tag{4.51}$$

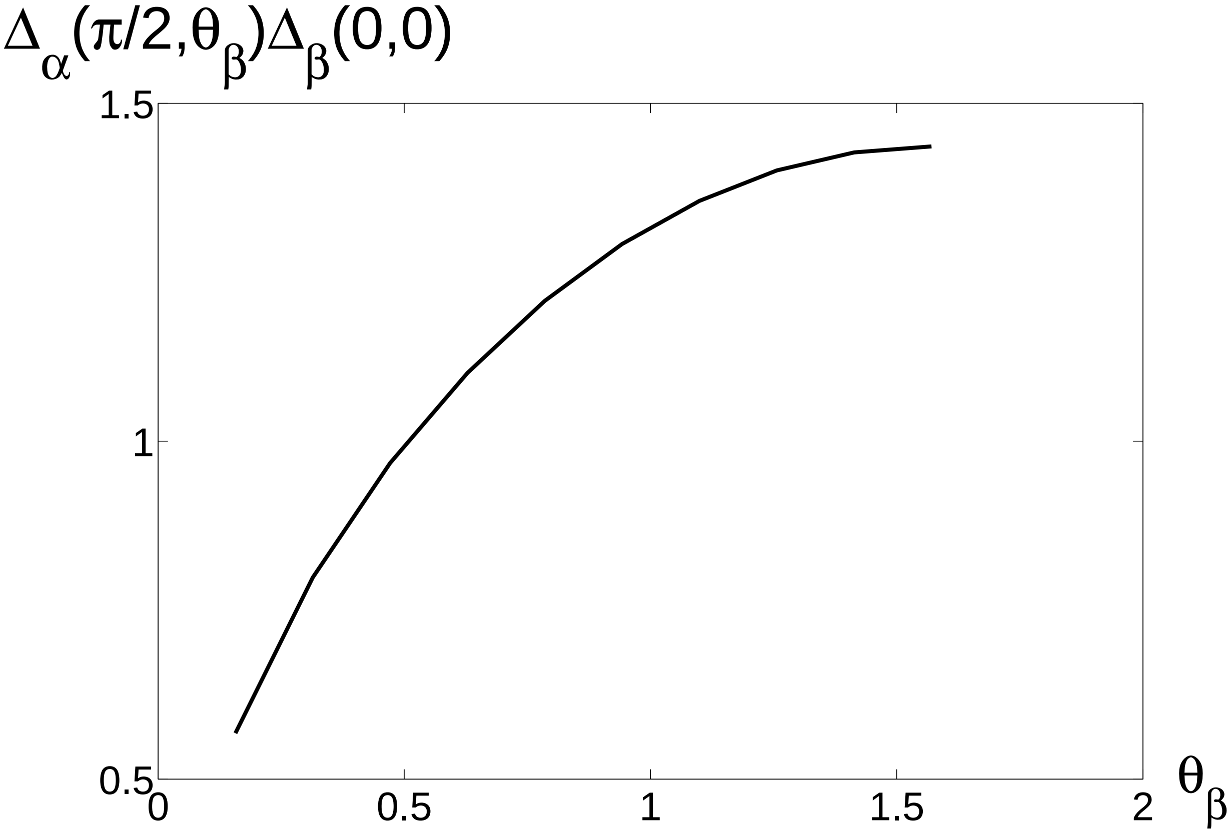


Fig. (4.11) The uncertainty product $\Delta_\alpha(\frac{\pi}{2}, \theta_\beta)\Delta_\beta(0,0)$ using the density matrix of Eq. (4.50) for $\alpha_0 = 2$; $\beta_0 = 0$ as a function of θ_β

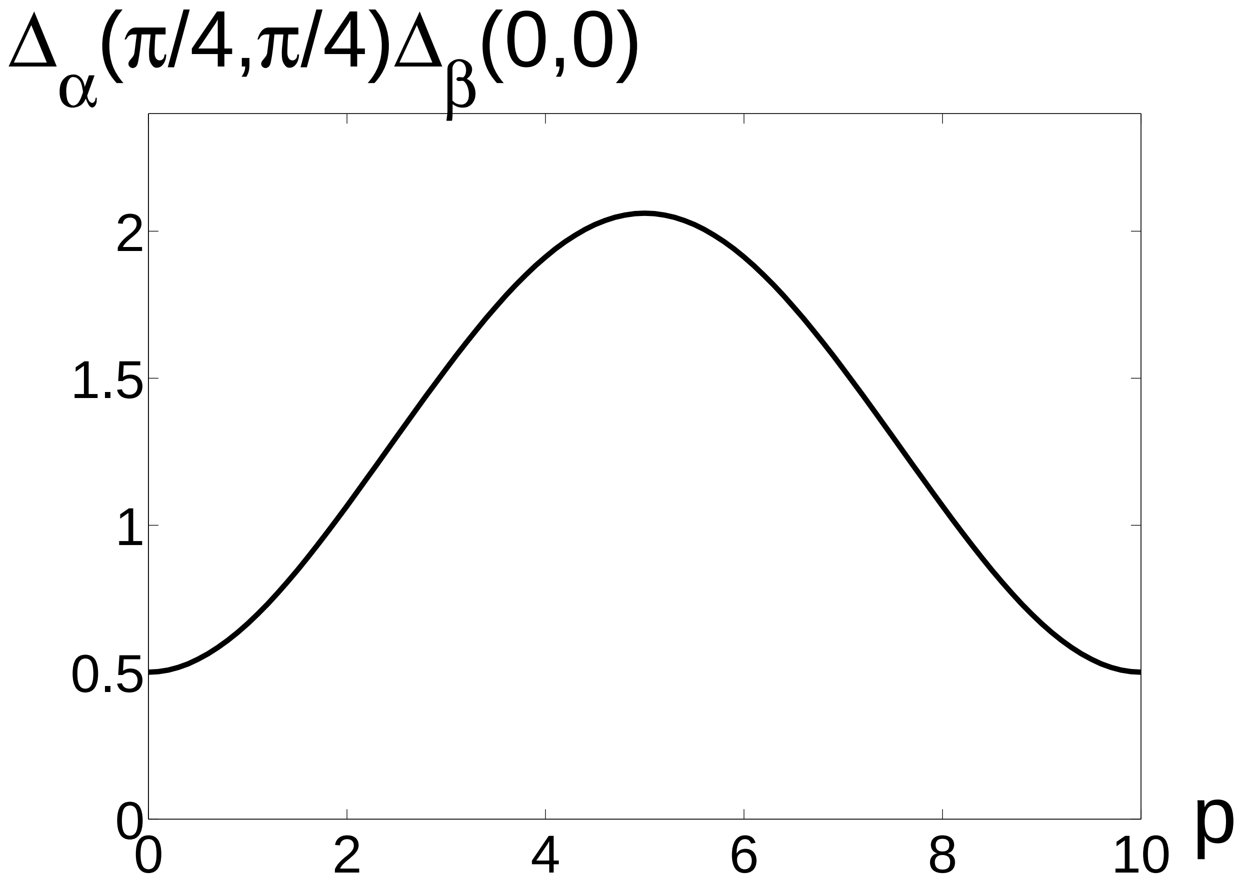


Fig. (4.12) The uncertainty product $\Delta_{\alpha}(\frac{\pi}{4},\frac{\pi}{4})\Delta_{\beta}(0,0)$ using the density matrix of Eq. (4.51) for $\alpha_0 = 2$; $\beta_0 = 0$ as a function of p

4.9 Discussion

In this chapter the bi-fractional displacement operators were applied to define different phase space functions like the Wigner functions, $Q-$function and $P-$function by deriving generalisations of these functions for different $(\theta_{\alpha}, \theta_{\beta})$. Using the bi-fractional displacement operators the bi-fractional Wigner functions was introduced in Eq. (4.2). Both the Wigner and Weyl functions are special cases of this more general function. Examples of these functions have been given in Figs. (4.1, 4.2, 4.5, 4.6, 4.7, 4.8). Examples of the bi-fractional Q-functions were also given in Figs. (4.9, 4.10).

The Wigner function for a single coherent gives a non-negative gaussian which is classical as seen in Fig. (2.1). However for bi-fractional Wigner function, $\mathcal{H}(\alpha,\beta;\frac{\pi}{2},\frac{\pi}{2})$

of Eq. (4.2) in Fig. (4.2) the auto terms which are non-negative and the superposition cross terms which have negative values meaning they cannot be understood in classical terms. The opposite is the case in the Weyl function in Fig. (4.1) where the auto terms become negatives in the middle and the cross terms go to the wings. Of special interest is the bi-fractional Wigner function, $\mathcal{H}(\alpha, \beta; \frac{\pi}{4}, \frac{\pi}{4})$, where the two cross and auto terms all have negative values. For this case the whole Wigner function is quantum with no classicality to it. This new function could be useful in quantum analysis as it gives a full quantum picture.

Chapter 5

Conclusion and future work

Quantum mechanics rules have been conventionally used to study the microscopic world [21]. Quantum mechanics laws like superposition, entanglement and teleportation are counter-intuitive. One of those rules commonly considered in many texts and also in this thesis is Heisenberg's uncertainty principle [20, 60] that has been directly related to position-momentum and energy-time.

In this thesis phase space methods [31, 37] were considered, with position and momentum variables in a way that fully respects Heisenberg's principle of $\Delta x \Delta p \geq \frac{\pi}{2}$ [32]. Furthermore, obeying the minimum uncertainty principle are the coherent states [6, 45, 62] derived by acting the displacement operator on a vacuum state.

The displacement operator was considered as well as parity operator and shown that they are related through a two dimensional Fourier transform. It was also stated that the Wigner functions and Weyl functions are also related through a two-dimensional Fourier transform. Wigner functions are relevant because Heisenberg's uncertainty principle forbids the possibility of a standard probability distribution since one cannot exactly measure with accuracy the position and momenta of a particle. Wigner functions which are semi probability distributions with negative values are used in quantum mechanics. The bi-fractional equivalent of these functions were given and we also included the bi-fractional Moyal star product and bi-fractional Berezin formalism for the product of two operators.

For further work, other relations which were not mentioned in this work can be considered and generalised. Of particular interest for future work is a special case of the bi-fractional Wigner function, $\mathcal{H}(\alpha, \beta; \frac{\pi}{4}, \frac{\pi}{4}|\Theta)$ which is intermediate between the Wigner and the Weyl because of the angles ($\theta_\alpha = \theta_\beta = \frac{\pi}{4}$). Any formulation of this function and possible properties could give interesting results which have not been covered in this work. Some other concepts to be considered for future work are listed below.

5.1 Other properties of bi-fractional coherent states

Conventionally, the coherent states overlap with the position and momentum states as given in Eq. (2.77); a further generalisation would be giving instances and implications of such with respect to the bi-fractional coherent states. The overlap of the bi-fractional coherent states with position and momentum states are given,

$$\begin{aligned} \langle x|\alpha, \beta; \theta_\alpha, \theta_\beta\rangle &= |\cos(\theta_\alpha - \theta_\beta)|^{1/2} \int d\alpha' d\beta' \, K\left(\beta, \alpha'; \theta_\beta\right) K\left(\alpha, -\beta'; \theta_\alpha\right) \langle x|\alpha', \beta'\rangle \,, \\ \langle p|\alpha, \beta; \theta_\alpha, \theta_\beta\rangle &= |\cos(\theta_\alpha - \theta_\beta)|^{1/2} \int d\alpha' d\beta' \, K\left(\beta, \alpha'; \theta_\beta\right) K\left(\alpha, -\beta'; \theta_\alpha\right) \langle x|\alpha', \beta'\rangle \,. \end{aligned} \tag{5.1}$$

It would be interesting to further investigate the implications of these overlap, give possible applications and show their physical meaning.

5.2 Tomography of the bi-fractional Wigner function

Quantum tomography of the Wigner function has earlier been studied in the literature [23] using the Radon transform along a line of a single angle given as,

$$\begin{aligned} \mathcal{T}(q,\theta) &= \int dxdp\, \mathcal{H}\left(\alpha,\beta;\frac{\pi}{2},\frac{\pi}{2}\right)\delta(x\sin\theta - p\cos\theta - q) \\ &= \int duW(q\sin\theta + u\cos\theta, -q\cos\theta + u\sin\theta). \end{aligned} \tag{5.2}$$

A proposed work would be to derive the Radon transform for the bi-fractional Wigner function this time along two different angles, $(\theta_\alpha, \theta_\beta)$. This would generalise the Radon transform of Eq. (5.2) given above to produce the bi-fractional radon transform. It is a known fact that by substituting the Wigner function for a function $f(x)$, and changing variables, the radon transform is related to the fractional Fourier transform so that

$$\begin{aligned} \mathcal{T}(q,\theta) &= \int dxdp\, \mathcal{H}\left(\alpha,\beta;\frac{\pi}{2},\frac{\pi}{2}\right)\delta(x\sin\theta - p\cos\theta - q) \\ &= \frac{2}{\pi\sin\theta}\int dydy' f(y)\exp\left[\frac{i\lambda y}{\sin\theta} - \frac{i(\lambda^2+y^2)}{2\tan\theta}\right] f^*(y)\exp\left[-\frac{i\lambda y'}{\sin\theta} + \frac{i(\lambda^2+y'^2)}{2\tan\theta}\right] \\ &= 4\int f(y)K(\lambda,y;\theta)f^*(y')K^\dagger(\lambda,y';\theta). \end{aligned} \tag{5.3}$$

A formalism that considers the bi-fractional Wigner function, $\mathcal{H}(\alpha,\beta;\theta_\alpha,\theta_\beta)$ could give more insight into the concept of the bi-fractional Wigner function.

5.3 Application to the extended phase space

The extended phase space formalism of $x - p - X - P$, was introduced in [26] and the Wigner function to that respect was given as,

$$\begin{aligned} W_e(x,p,X,P) &= (2\pi)^2 \int dx'dp' W\left(x + \frac{1}{2}x', p + p'\right) W\left(x - \frac{1}{2}x', p - p'\right) \exp[i(Xp' - Px')] \\ &= \int dX'dP' \widehat{W}^*\left(X + \frac{1}{2}X', P + P'\right) \widehat{W}\left(X - \frac{1}{2}X', P - P'\right) \exp[i(X'p - P'x)], \end{aligned} \tag{5.4}$$

where $W\left(x + \frac{1}{2}x', p + p'\right)$ is the Wigner function and $\widehat{W}^*\left(X + \frac{1}{2}X', P + P'\right)$ the Weyl function. Similarly, all the Fourier transforms can be changed to fractional Fourier transforms as made possible in Eq. (4.2) and extend to the bi-fractional extended Wigner function. Extra relations such as the marginal properties can be derived. Another proposition will be to carry out the Radon transform on both the extended Wigner function and the bi-fractional extended Wigner functions.

5.4 Application to the characteristic function

The characteristic function gives a generalisation of P−function, Q−function and Wigner functions [2, 3] in terms of a Fourier transform,

$$R(\alpha', \beta'; s) = \int d\alpha d\beta\, W(\alpha, \beta) \exp[i\beta'\alpha - i\alpha'\beta] \exp\left[\frac{s(\alpha^2 + \beta^2)}{2}\right]. \tag{5.5}$$

Of note is that the generalisation from Fourier transform in this case to fractional Fourier transform is not straight-forward.

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Appendix A

Equations and proofs

Matrix elements of displacement operators with respect to number states

We begin the proof by showing that,

$$\begin{aligned}\widehat{D}(z)\,|z_1\rangle = \widehat{D}(z)\widehat{D}(z_1)\,|0\rangle = \widehat{D}(z+z_1)\exp\left[\frac{1}{2}(zz_1^* - z^* z_1)\right] \\ = |z+z_1\rangle \exp\left[\frac{1}{2}(zz_1^* - z^* z_1)\right]\end{aligned} \tag{A.1}$$

Therefore we can show that,

$$\begin{aligned}\langle n|\widehat{D}(z)|z_1\rangle = \langle n|z+z_1\rangle \exp\left[\frac{1}{2}(zz_1^* - z^* z_1)\right] \\ \frac{(z+z_1)^n}{\sqrt{N!}}\exp\left[\frac{1}{2}(zz_1^* - z^* z_1)\right]\langle 0|z+z_1\rangle = \frac{(z+z_1)^n}{\sqrt{n!}}\exp\left[\frac{1}{2}(zz_1^* - z^* z_1) - \frac{1}{2}|z+z_1|^2\right]\end{aligned} \tag{A.2}$$

Alternatively, we can write $\langle n|\widehat{D}(z)|z_1\rangle$ in terms of $\langle N|\widehat{D}(z)|m\rangle$ such that,

$$\begin{aligned}\langle n|\widehat{D}(z)|z_1\rangle &= \langle n|\widehat{D}(z)e^{\frac{-|z_1|}{2}}\sum_{m=0}^{\infty}\frac{z_1^m}{\sqrt{m!}}|m\rangle \\ &= e^{\frac{-|z_1|}{2}}\sum_{m=0}^{\infty}\frac{z_1^m}{\sqrt{m!}}\langle n|\widehat{D}(z)|m\rangle \end{aligned} \tag{A.3}$$

Then comparing Eq. $(A.2)$ and Eq. $(A.3)$, we get that,

$$\frac{(z+z_1)^n}{\sqrt{n!}}\exp\left[\frac{1}{2}(|z|^2-|z_1|^2)-z^*z_1\right] = e^{\frac{-|z_1|}{2}}\sum_{m=0}^{\infty}\frac{z_1^m}{\sqrt{m!}}\langle n|\widehat{D}(z)|m\rangle \tag{A.4}$$

Taking $t = \frac{z_1}{z}$, we can write the generating function of the Laguerre polynomial, $L_M^{n-m}(x)$ as,

$$\begin{aligned}(1+t)^n e^{-tx} &= \sum_{m=0}^{\infty} L_M^{n-m}(x)t^m \\ (1+t)^n e^{-t|z|^2} &= e^{\frac{1}{2}|z|^2}\sum_{m=0}^{\infty}\frac{\sqrt{n!}}{\sqrt{m!}}\langle n|\widehat{D}(z)|m\rangle\, z^{m-n}\frac{z_1^m}{z^m}\end{aligned} \tag{A.5}$$

And then comparing the two equations we get that,

$$\langle n|\widehat{D}(z)|m\rangle = \frac{\sqrt{n!}}{\sqrt{m!}}z^{n-m}e^{-\frac{1}{2}|z|^2}L_m^{n-m}(|z|^2) \tag{A.6}$$

Fractional Fourier transform

In order to prove Eq. (3.18), we consider the number eigenstate wavefunctions

$$
\begin{aligned}
&u_N(x) = \frac{1}{\pi^{1/4}} \frac{1}{(2^N N!)^{1/2}} H_N(x) \exp\left(-\frac{1}{2}x^2\right) \\
&\sum_{N=0}^{\infty} u_N(x) u_N(y) = \delta(x-y) \\
&\mathfrak{F}(\theta; x) u_N(x) = \exp(iN\theta) u_N(x)
\end{aligned}
\tag{A.7}
$$

where $H_N(x)$ is the Hermite polynomial. We have that,

$$
\begin{aligned}
\mathfrak{F}(\theta; x)[f(x)] = \mathfrak{F}(\theta; x) \int \delta(x-y) f(y) dy &= \mathfrak{F}(\theta; x) \int \sum_{N=0}^{\infty} u_N(x) u_N(y) f(y) dy \\
&= \int \left[\sum_{N=0}^{\infty} \exp(iN\theta) u_N(x) u_N(y)\right] f(y) dy
\end{aligned}
\tag{A.8}
$$

We can then define the fractional Fourier transform as a linear transformation using the kernel $\Delta(x, y; \theta)$ such that

$$
\begin{aligned}
\Delta(x, y; \theta) &= \sum_{N=0}^{\infty} \exp(iN\theta) u_N(x) u_N(y) \\
&= \sum_{N=0}^{\infty} \frac{1}{\sqrt{\pi}\ 2^N N!} \exp(iN\theta) H_N(x) H_N(y) \exp\left(-\frac{1}{2}x^2\right) \exp\left(-\frac{1}{2}y^2\right)
\end{aligned}
\tag{A.9}
$$

Using a formula from Mehler for the integral representation of the Hermite polynomials given as,

$$
\sum_{N=0}^{\infty} \frac{1}{\sqrt{\pi}\ 2^N N!} \exp(iN\theta) H_N(x) H_N(y) = \frac{1}{\sqrt{\pi}\sqrt{1-e^{2i\theta}}} \exp\left[\frac{2xy\ e^{i\theta} - e^{2i\theta}\ (x^2+y^2)}{1-e^{2i\theta}}\right]
\tag{A.10}
$$

Using the the Euler's relation $e^{i\theta} = \cos\theta + i\sin\theta$, we have that,

$$\frac{1}{\sqrt{\pi}\sqrt{1-e^{2i\theta}}} = \frac{1}{\sqrt{\pi(1-\cos^2\theta+\sin^2\theta+2i\sin\theta\cos\theta)}} = \left[\frac{1-i\cot\theta}{2\pi}\right]^{\frac{1}{2}} \tag{A.11}$$

And further prove that,

$$\begin{aligned}
&\exp\left[\frac{2xy\ e^{i\theta} - e^{2i\theta}\ (x^2+y^2)}{1-e^{2i\theta}}\right]\exp\left(-\frac{1}{2}x^2 - \frac{1}{2}y^2\right) \\
&= \exp\left[\frac{2xy(\cos\theta+i\sin\theta) - (x^2+y^2)[\cos 2\theta + i\sin 2\theta]}{1-(\cos 2\theta + i\sin 2\theta)}\right] \\
&= \exp\left[\frac{ixy}{\sin\theta} - \frac{i(x^2+y^2)\cot\theta}{2}\right]
\end{aligned}$$

This proves Eq. (3.18).

Additivity property of fractional Fourier transform

The property of Eq. (3.13) can be proved using Eq. (3.8), such that,

$$\begin{aligned}
&\int dyK(x,y;\theta_1)K(y,z;\theta_2) = \left[\frac{1+i\cot\theta_1}{2\pi}\right]^{\frac{1}{2}}\left[\frac{1+i\cot\theta_2}{2\pi}\right]^{\frac{1}{2}} \\
&\times \int dy \exp\left[-\frac{i(\cot\theta_1+\cot\theta_2)y^2}{2} - \frac{ix^2\cot\theta_1}{2} - \frac{iz^2\cot\theta_2}{2} + \frac{ixy}{\sin\theta_1} + \frac{iyz}{\sin\theta_2}\right] \\
&= R\int \exp\left[-\frac{iy^2(\cot\theta_1+\cot\theta_2)}{2} + iy\left(\frac{x}{\sin\theta_1} + \frac{z}{\sin\theta_2}\right)\right] \\
&= R\left[\frac{-2i\pi}{\cot\theta_1+\cot\theta_2}\right]\exp\left[\frac{-(x^2+z^2)\cos(\theta_1+\theta_2)+2ixz}{2(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)}\right] \\
&= \left[\frac{1+i\cot(\theta_1+\theta_2)}{2\pi}\right]^{\frac{1}{2}}\exp\left[\frac{-i(x^2+z^2)\cot(\theta_1+\theta_2)}{2} + \frac{ixz}{\sin(\theta_1+\theta_2)}\right]
\end{aligned} \tag{A.12}$$

where,

$$R = \left[\frac{(1+i\cot\theta_1)(1+i\cot\theta_2)}{4\pi^2}\right]^{\frac{1}{2}} \exp\left[-i\left(\frac{x^2\cot\theta_1 + z^2\cot\theta_2}{2}\right)\right] \tag{A.13}$$

Thus we have proved Eq. (3.13), since Eq. $(A.12)$ is $K(x,z;\theta_1+\theta_2)$.

Moyal star product

The proof given by Moyal [55] is shown,

$$\begin{aligned}
&\frac{1}{\pi}\int d\alpha d\beta \langle\gamma| D^\dagger(\alpha,\beta)|\delta\rangle\langle\epsilon| D(\alpha,\beta)|\zeta\rangle \\
&= \frac{1}{\pi}\int d\alpha d\beta\, dx\, dp\, dx' dp' \langle\gamma|x\rangle\langle x|D^\dagger(\alpha,\beta)|p\rangle\langle p|\delta\rangle\langle\epsilon|p'\rangle\langle p'|D(\alpha,\beta)|x'\rangle\langle x'|\zeta\rangle \\
&= \frac{1}{\pi}\int d\alpha d\beta\, dx\, dp\, dx' dp' \langle\gamma|x\rangle\langle x'|\zeta\rangle\langle\epsilon|p'\rangle\langle p|\delta\rangle\langle x|D^\dagger(\alpha,\beta)|p\rangle\langle p'|D(\alpha,\beta)|x'\rangle \\
&= \frac{1}{\pi}\int dx\, dp\, dx' dp' \langle\gamma|x\rangle\langle x'|\zeta\rangle\langle\epsilon|p'\rangle\langle p|\delta\rangle \\
&\times \frac{1}{2\pi}\exp[i(x'p'-xp)]\int d\alpha\, e^{i\sqrt{2}\alpha(p'-p)}\int d\beta\, e^{i\sqrt{2}\beta(x-x')} \\
&= \int dx\, dp \langle\gamma|x\rangle\langle x|\zeta\rangle\langle\epsilon|p\rangle\langle p|\delta\rangle \\
&= \langle\gamma|\zeta\rangle\langle\epsilon|\delta\rangle
\end{aligned} \tag{A.14}$$

Laplacian for Berezin formalism

We give the proof that leads to the Laplacian as,

$$\begin{aligned}
\frac{1}{\pi}\int d\alpha' d\beta' F(\alpha',\beta') K \exp\left[-K\left[D_s\left(\alpha-\alpha',\beta-\beta'\middle|\frac{\pi}{2},\frac{\pi}{2}\right)\right]\right] \\
= \left[\exp\left(\frac{\Delta_{(\alpha,\beta|\frac{\pi}{2},\frac{\pi}{2})}}{4K}\right)F(\alpha,\beta)\right]
\end{aligned} \tag{A.15}$$

where,

$$\Delta_{(\alpha,\beta|\frac{\pi}{2},\frac{\pi}{2})} = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}; \quad K = 1 \tag{A.16}$$

Taking the Fourier transform of both sides we have that,

$$\frac{1}{\pi}\int d\alpha' d\beta' d\alpha d\beta F(\alpha',\beta') \exp\left[-\left[D_s\left(\alpha-\alpha',\beta-\beta'\middle|\frac{\pi}{2},\frac{\pi}{2}\right)\right]\right]\exp[i\alpha\beta' + i\alpha'\beta]$$
$$= \int d\alpha d\beta \exp\left[\frac{1}{4}\left(\frac{\partial^2}{\partial^2\alpha} + \frac{\partial^2}{\partial^2\beta}\right)\right] F(\alpha,\beta)\exp[i\alpha\beta' + i\alpha'\beta] \tag{A.17}$$

Then using,

$$\int e^{ixp}\,\frac{\partial}{\partial x}^{n}[f(x)]dx = (-ip)^n\mathcal{F}(p) \tag{A.18}$$

Using Taylor's expansion, we show that,

$$\exp\left[\frac{1}{4}\left(\frac{\partial^2}{\partial^2\alpha} + \frac{\partial^2}{\partial^2\beta}\right)\right]\exp[i\alpha\beta' + i\alpha'\beta] = 1 + \frac{(-\alpha'^2 - \beta'^2)}{4} + \left[\frac{(-\alpha'^2 - \beta'^2)}{4}\right]^2 + \ldots.. \tag{A.19}$$

Thus the RHS becomes,

$$= \int d\alpha d\beta \exp\left[-\frac{1}{4}\left(\alpha'^2 + \beta'^2\right)\right]\exp[i\alpha\beta' + i\alpha'\beta]F(\alpha,\beta) \tag{A.20}$$

And taking the gaussian intergral in the LHS with respect to α and β, we show that it's the same as Eq. $(A.20)$.