UNIVERSITY of
BRADFORD

Library

The University of Bradford Institutional Repository

http://bradscholars.brad.ac.uk

Link to publisher version: *https://doi.org/10.1088/1742-6596/512/1/012036*

Citation: Vourdas A (2014) Quantum probabilities as Dempster-Shafer probabilities in the lattice of subspaces. Journal of Mathematical Physics. 55(8): 082107.

Subsystems of a finite quantum system and Bell-like inequalities

A. Vourdas

Department of Computing,
University of Bradford
Bradford BD7 1DP, UK

E-mail: `a.vourdas@bradford.ac.uk`

Abstract. The set of subsystems $\Sigma(m)$ of a finite quantum system $\Sigma(n)$ with variables in $\mathbb{Z}(n)$, together with logical connectives, is a Heyting algebra. The probabilities $\tau(m|\rho_n) = \mathrm{Tr}[\mathfrak{P}(m)\rho_n]$ (where $\mathfrak{P}(m)$ is the projector to $\Sigma(m)$) are compatible with associativity of the join in the Heyting algebra, only if the variables belong to the same chain. Consequently, contextuality in the present formalism, has the chains as contexts. Various Bell-like inequalities are discussed. They are violated, and this proves that quantum mechanics is a contextual theory.

1. Introduction

In recent work [1] we have studied the mathematical structure of the set of subsystems of a finite quantum system $\Sigma(n)$ with variables in $\mathbb{Z}(n)$ (the integers modulo n). A subsystem of $\Sigma(n)$, is another finite system $\Sigma(m)$, with $m|n$. In this case the variables of $\Sigma(m)$ take values in $\mathbb{Z}(m)$, which is a subgroup of $\mathbb{Z}(n)$. Also, all states of $\Sigma(m)$ are embedded in $\Sigma(n)$ as described below. We have shown that the set of subsystems of $\Sigma(n)$ with logical connectives, is a distributive lattice $\Lambda(\boldsymbol{\Sigma}_n)$. All finite distributive lattices are Heyting algebras[2, 3], and therefore $\Lambda(\boldsymbol{\Sigma}_n)$ is a Heyting algebra.

Probability theory needs the concepts of conjuction, disjunction and negation for its axioms. Kolmogorov probabilities are defined on a Boolean algebra (a powerset 2^{Ω}), where the intersection, union and complement play the role of conjuction, disjunction and negation. Let $q(E_i)$ be the Kolmogorov probability for event E_i. Then

$$q(E_1 \vee E_2) - q(E_1) - q(E_2) + q(E_1 \wedge E_2) = 0. \tag{1}$$

Refs [4, 5, 6] proved this equality, using the associativity property of the lattice. Alternatively, it is introduced as the axiom of additivity of probability (the last term generalizes it to non-exclusive events).

Quantum mechanics uses the orthomodular lattice of closed subspaces of a Hilbert space [7, 8, 9], which has various Boolean algebras as sublattices. Kolmogorov probabilities are defined on them, but the topic of contextuality and Bell inequalities [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], proves that there is inconsistency between the various parts of the full formalism.

Various aspects of contextuality have been studied in the literature. In a bipartite system with a space-like separation, it leads to non-locality which is a special form of contextuality.

In a single spin one particle, contextuality is proved through a logical contradiction. We are interested in another form of contextuality, which appears in the formalism of subsystems of a finite quantum system.

We consider the quantum probabilities, $\tau(m|\rho_n) = \text{Tr}[\mathfrak{P}(m)\rho_n]$, where $\mathfrak{P}(m)$ is the projector to the subsystem $\Sigma(m)$ and ρ_n is a density matrix of the system $\Sigma(n)$. In ref[1] we have shown that these probabilities are incompatible with the associativity of the join in the distributive lattice of subsystems $\Lambda(\mathbf{\Sigma}_n)$, because they do not obey the equality of Eq.(1), but they obey the inequality

$$\tau(m_1 \vee m_2|\rho_n) - \tau(m_1|\rho_n) - \tau(m_2|\rho_n) + \tau(m_1 \wedge m_2|\rho_n) \geq 0. \tag{2}$$

Only if the m_1, m_2 belong to the same chain, Eq(2) is valid as an equality. For this reason, chains play the role of contexts, in the present formalism.

Within a context (chain), the quantum probabilities $\tau(m|\rho_n)$ obey an equality analogous to Eq.(1), and they behave like Kolmogorov probabilities. But when we consider various contexts together, this equality and other important relations which are derived from it (e.g., Boole's inequality $\tau(m_1 \vee m_2|\rho_n) \leq \tau(m_1|\rho_n) + \tau(m_2|\rho_n)$ which is used in the proof of Bell-like inequalities), are no longer valid. In the present formalism, the term non-contextual quantum mechanics, means that the equality of Eq.(1) is assumed to be valid. The term contextual quantum mechanics, means that Eq.(1) is only valid within a context (chain).

Ref[19] has studied 'logical Bell inequalities' for Boolean variables. For Heyting variables, which are relevant in the present formalism, they acquire some extra terms, as discussed in [1]. The validity of these inequalities is based on the assumption that quantum mechanics is non-contextual theory, and their violation in experiments, proves that quantum mechanics is a contextual theory. In this paper we extend the work of [1], and present new Bell-like inequalities, together with examples which violate them. There are many technical details associated with the lattice structure of the set of subsystems, which are important for understanding why the Bell-like inequalities are violated, but which are not presented here (see [1]). In this paper we present a methodology for deriving novel Bell-like inequalities, starting from an equality in Boolean algebra, and using Boole's inequality for probabilities (which is valid in non-contextual quantum mechanics).

In section II we present briefly Heyting algebras in the present formalism, in order to eastablish the notation. In section III we discuss the set of subsystems of $\Sigma(n)$, and probabilities associated with projectors to these subsystems. In section IV we present various Bell-like inequalities and examples of their violation. We conclude in section V, with a discussion of our results.

2. Preliminaries

(1) $r|s$ or $r \prec s$ denotes that r is a divisor of s. $\text{GCD}(r, s)$ and $\text{LCM}(r, s)$ are the greatest common divisor and least common multiplier correspondingly, of the integers r, s. $\mathbb{D}(n)$ is the set of divisors of n.

(2) *Heyting and Boolean algebras:* A set A viewed as a lattice (i.e., with the operations $\vee$ and $\wedge$) is denoted as $\Lambda(A)$. Throughout the paper we have various finite distributive lattices and for simplicity we use the same symbols $\prec, \wedge, \vee, \neg$, for the 'partial order', 'meet', 'join' and 'negation', correspondingly. We also use the same symbols $\mathcal{O}$ and $\mathcal{I}$ for the smallest and greatest elements.

Every finite distributive lattice is a Heyting algebra. If $a \in \Lambda(A)$, then $\neg a$ is the largest element such that $a \wedge (\neg a) = \mathcal{O}$. The elements of a Heyting algebra obey the relations

$$a \vee (\neg a) \prec \mathcal{I}; \quad a \prec \neg\neg a \tag{3}$$

A special case of Heyting algebera are the Boolean algebras. In this case the relations of Eq.(3) are valid as equalities:

$$a \vee (\neg a) = \mathcal{I}; \quad a = \neg\neg a \tag{4}$$

The relation $a \vee (\neg a) = \mathcal{I}$ is the 'law of the excluded middle' and it does not hold in general in Heyting algebras.

(3) *The Heyting algebra of divisors of n:* The set $\mathbb{D}(n)$ with divisibility as partial order, and with

$$k \wedge m = \text{GCD}(k, m); \quad k \vee m = \text{LCM}(k, m) \tag{5}$$

is a finite distributive lattice and as such it is a Heyting algebra with $\mathcal{O} = 1$ and $\mathcal{I} = n$. We denote it as $\Lambda[\mathbb{D}(n)]$. $\neg k$ is the largest divisor of n, such that $k \wedge (\neg k) = 1$.

(4) *The Heyting algebra of subgroups of $\mathbb{Z}(n)$:* If $m \prec n$ then $\mathbb{Z}(m)$ is a subgroup of $\mathbb{Z}(n)$. The set $\mathfrak{Z}(n) = \{\mathbb{Z}(m) \mid m \in \mathbb{D}(n)\}$ of the subgroups of $\mathbb{Z}(n)$, with subgroup as partial order, and with

$$\mathbb{Z}(k \wedge m) = \mathbb{Z}(k) \wedge \mathbb{Z}(m); \quad \mathbb{Z}(k \vee m) = \mathbb{Z}(k) \vee \mathbb{Z}(m); \quad \neg\mathbb{Z}(k) = \mathbb{Z}(\neg k) \tag{6}$$

is the Heyting algebra $\Lambda[\mathfrak{Z}(n)]$, and it is isomorphic to $\Lambda[\mathbb{D}(n)]$. In this case $\mathcal{O} = \mathbb{Z}(1)$ and $\mathcal{I} = \mathbb{Z}(n)$. The elements of $\mathbb{Z}(m)$ can be embedded into a supergroup $\mathbb{Z}(k)$ (where $m \prec k \prec n$), by mapping $a \in \mathbb{Z}(m)$ into $\frac{ka}{m} \in \mathbb{Z}(k)$.

3. Subsystems of $\Sigma(n)$

$\Sigma(n)$ is a quantum system with variables in $\mathbb{Z}(n)$ and Hilbert space $H(n)$. $|X_n; r\rangle$ where $r \in \mathbb{Z}(n)$, is the basis of position states (the X_n in this notation is not a variable, but it simply indicates position states). Through a Fourier transform we get the basis of momentum states:

$$|P_n; r\rangle = n^{-1/2} \sum_{r,s} \omega_n(rs)|X_n; r\rangle.; \quad \omega_n(r) = \exp\left(\frac{i2\pi r}{n}\right) \tag{7}$$

For $m \prec k \prec n$, the $\Sigma(m)$ is a subsystem of $\Sigma(k)$. In this case the variables of $\Sigma(m)$ take values in $\mathbb{Z}(m)$, which is a subgroup of $\mathbb{Z}(k)$, and all states of $\Sigma(m)$ are embedded in $\Sigma(k)$ as follows:

$$\sum_{r=0}^{m-1} a_r |X_m; r\rangle \;\; \rightarrow \;\; \sum_{r=0}^{m-1} a_r |X_k; \frac{kr}{m}\rangle; \quad m \prec k. \tag{8}$$

The system $\Sigma(1)$ is physically trivial, and it consists of the 'vacuum' state $|X_1; 0\rangle = |P_1; 0\rangle$.

We define the projector to the system $\Sigma(m)$ (as embedded into a supersystem $\Sigma(k)$), as

$$\mathfrak{P}(m) = \sum_{r=0}^{m-1} |X_k; \frac{kr}{m}\rangle\langle X_k; \frac{kr}{m}|; \quad m \prec k; \quad m, k \in \mathbb{D}(n). \tag{9}$$

All these projectors commute with each other.

The set $\mathbf{\Sigma}_n$ of all subsystems of $\Sigma(n)$, with partial order 'subsystem', and with the logical connectives

$$\begin{aligned} \Sigma(m) \wedge \Sigma(k) &= \Sigma(m \wedge k) \\ \Sigma(m) \vee \Sigma(k) &= \Sigma(m \vee k) \\ \neg\Sigma(m) &= \Sigma(\neg m); \quad m, k \in \mathbb{D}(n) \end{aligned} \tag{10}$$

is the Heyting algebra $\Lambda(\boldsymbol{\Sigma}_n)$, and it is isomorphic to $\Lambda[\mathbb{D}(n)]$. In this case $\mathcal{O} = \Sigma(1)$ and $\mathcal{I} = \Sigma(n)$. The physical meaning of these connectives is discussed in [1]. In similar way we define the logical operations in the set $\mathbf{H}_n$ of the Hilbert spaces of the subsystems of $\Sigma(n)$. This is also a Heyting algebra isomorphic to $\Lambda[\mathbb{D}(n)]$ and $\Lambda(\boldsymbol{\Sigma}_n)$, which we denote as $\Lambda(\mathbf{H}_n)$.

In ref[1] we have proved that the space $H(m_1 \vee m_2)$ is given by

$$H(m_1 \vee m_2) = \mathrm{span}[H(m_1) \cup H(m_2)] \oplus S(m_1, m_2). \tag{11}$$

The $\mathrm{span}[H(m_1) \cup H(m_2)]$ contains all superpositions of states in $H(m_1)$ and $H(m_2)$. The $S(m_1, m_2)$ is orthogonal to it. The

$$\begin{aligned}
\mathfrak{T}(m_1, m_2) &= \mathfrak{P}(m_1) + \mathfrak{P}(m_2) - \mathfrak{P}(m_1 \wedge m_2) \\
\mathfrak{S}(m_1, m_2) &= \mathfrak{P}(m_1 \vee m_2) - \mathfrak{P}(m_1) - \mathfrak{P}(m_2) + \mathfrak{P}(m_1 \wedge m_2),
\end{aligned} \tag{12}$$

are projectors to the spaces $\mathrm{span}[H(m_1) \cup H(m_2)]$ and $S(m_1, m_2)$, correspondingly. In the special case that m_1, m_2 belong to the same chain, $H(m_1 \vee m_2) = \mathrm{span}[H(m_1) \cup H(m_2)]$.

We next define the quantum probabilities

$$\tau(m|\rho_n) = \mathrm{Tr}[\rho_n \mathfrak{P}(m)]; \qquad \sigma(m_1, m_2|\rho_n) = \mathrm{Tr}[\rho_n \mathfrak{S}(m_1, m_2)]; \qquad m, m_1, m_2 \in \mathbb{D}(n), \tag{13}$$

where ρ_n is density matrix describing the system $\Sigma(n)$. From Eq.(12), it follows that

$$\tau(m_1 \vee m_2|\rho_n) - \tau(m_1|\rho_n) - \tau(m_2|\rho_n) + \tau(m_1 \wedge m_2|\rho_n) = \sigma(m_1, m_2|\rho_n). \tag{14}$$

From this follows the inequality of Eq.(2). For variables in a chain, $\sigma(m_1, m_2|\rho_n) = 0$ for all density matrices, and the $\tau(m|\rho_n)$ obey the equality of Eq.(1). For this reason, contexts in the present formalism are the chains in the Heyting algebra.

For later use, we also define the

$$\widetilde{\mathfrak{P}}(m) = \mathfrak{P}(m) - \mathfrak{P}(1); \qquad \widetilde{\tau}(m|\rho_n) = \mathrm{Tr}[\rho_n \widetilde{\mathfrak{P}}(m)]. \tag{15}$$

3.1. Example

We consider the $\Lambda(\boldsymbol{\Sigma}_{12})$ which comprises of the subsystems of $\Sigma(12)$. The projectors to these subsystems are

$$\begin{aligned}
\mathfrak{P}(1) &= |X_{12}; 0\rangle\langle X_{12}; 0| \\
\mathfrak{P}(2) &= |X_{12}; 0\rangle\langle X_{12}; 0| + |X_{12}; 6\rangle\langle X_{12}; 6| \\
\mathfrak{P}(3) &= |X_{12}; 0\rangle\langle X_{12}; 0| + |X_{12}; 4\rangle\langle X_{12}; 4| + |X_{12}; 8\rangle\langle X_{12}; 8| \\
\mathfrak{P}(4) &= \sum_{\nu=0}^{3} |X_{12}; 3\nu\rangle\langle X_{12}; 3\nu| \\
\mathfrak{P}(6) &= \sum_{\nu=0}^{5} |X_{12}; 2\nu\rangle\langle X_{12}; 2\nu| \\
\mathfrak{P}(12) &= \mathbf{1}
\end{aligned} \tag{16}$$

In this case we have 3 maximal contexts:

$$\{\Sigma(12), \Sigma(6), \Sigma(3), \Sigma(1)\}; \quad \{\Sigma(12), \Sigma(6), \Sigma(2), \Sigma(1)\}; \quad \{\Sigma(12), \Sigma(4), \Sigma(2), \Sigma(1)\}. \tag{17}$$

In $\Sigma(12)$ we consider the state

$$\rho = |s\rangle\langle s|; \quad |s\rangle = \sum_{\nu=0}^{11} a_\nu |X_{12}; \nu\rangle; \quad \sum_{\nu=0}^{11} |a_\nu|^2 = 1, \tag{18}$$

and we find

$$\tau(12|\rho) = 1; \quad \tau(6|\rho) = \sum_{\nu=0}^{5} |a_{2\nu}|^2; \quad \tau(4|\rho) = \sum_{\nu=0}^{3} |a_{3\nu}|^2$$

$$\tau(3|\rho) = |a_0|^2 + |a_4|^2 + |a_8|^2; \quad \tau(2|\rho) = |a_0|^2 + |a_6|^2; \quad \tau(1|\rho) = |a_0|^2. \tag{19}$$

We next calculate the $\sigma(m_1, m_2)$ of Eq.(14). We find that

$$\begin{aligned}
\sigma(4,6|\rho) &= |a_1|^2 + |a_5|^2 + |a_7|^2 + |a_{11}|^2; \\
\sigma(4,3|\rho) &= |a_1|^2 + |a_2|^2 + |a_5|^2 + |a_7|^2 + |a_{10}|^2 + |a_{11}|^2; \\
\sigma(2,3|\rho) &= |a_2|^2 + |a_{10}|^2,
\end{aligned} \tag{20}$$

and that the rest $\sigma(m_1, m_2|\rho) = 0$.

4. Bell-like inequalities

Ref [19] has studied 'logical Bell inequalities' for Boolean variables. Their violation proves the contextual nature of quantum mechanics. In ref[1] we have generalized them for the case of Heyting variables. In this case, they acquire extra terms which we call 'Heyting factors' and which are zero in the case of Boolean variables.

In the present formalism:

- Quantum mechanics is a non-contextual theory, means that an equality analogous to Eq.(1) holds. In this case we easily prove Boole's inequality for probabilities:

$$\tau(m_1 \vee ... \vee m_N | \rho_n) \leq \tau(m_1 | \rho_n) + ... + \tau(m_N | \rho_n). \tag{21}$$

- Quantum mechanics is a contextual theory, means that an equality analogous to Eq.(1) is valid only for variables within a context (chain). When we consider various contexts together, then the inequality of Eq.(2) holds. In this case we cannot infer Boole's inequality for probabilities. An example where Boole's inequality is violated, is presented below.

There are many Bell-like inequalities, which are proved using Boole's inequality for probabilities and which would be valid if quantum mechanics were a non-contextual theory.

Proposition 4.1. *We assume that quantum mechanics is a non-contextual theory. If*

$$m_1 \wedge ... \wedge m_N = r; \quad m_1, ..., m_N \in \mathbb{D}(n) - \{1\}, \tag{22}$$

then

$$\sum_{i=1}^{N} \widetilde{\tau}(m_i | \rho_n) \leq N - \tau(r | \rho_n) - \sum_{i=1}^{N} f_i; \quad f_i = 1 - \tau(m_i \vee \neg m_i | \rho_n). \tag{23}$$

f_i *are 'Heyting factors' and are zero for Boolean variables.*

Proof. The proof has been presented in [1]. □

The following propositions start from an equality in Boolean algebra, and use Boole's inequality for probabilities, to derive a Bell-like inequality. There are many equalities in Boolean algebra which can be used in a similar way, and they will lead to various Bell-like inequalities. Below we give two of them. Violation of these inequalities, supports the fact that quantum mechanics is a contextual theory.

Proposition 4.2. *We assume that quantum mechanics is a non-contextual theory. Let* $q(A), q(B), q(C)$ *be probabilities corresponding to quantum events* A, B, C, *which are assumed to be elements of a Boolean algebra. Then*

$$q(A) \leq q(A \wedge B) + q[A \wedge (\neg B)] \tag{24}$$

$$q(A) \leq q(A \wedge C) + q(B \wedge C) + q[(\neg A) \wedge (\neg B) \wedge C] \tag{25}$$

Proof. Using the basic rules of Boolean algebra we prove that

$$A = (A \wedge B) \vee [A \wedge (\neg B)]$$
$$C = [C \wedge (A \vee B)] \vee \{C \wedge [\neg(A \vee B)]\} = [C \wedge A] \vee [C \wedge B] \vee [C \wedge (\neg A) \wedge (\neg B)] \tag{26}$$

Using Boole's inequality (in a non-contextual quantum mechanics) we get

$$q(A) = q\{(A \wedge B) \vee [A \wedge (\neg B)]\} \leq q(A \wedge B) + q[A \wedge (\neg B)], \tag{27}$$

and also

$$q(C) = q\{[C \wedge A] \vee [C \wedge B] \vee [C \wedge (\neg A) \wedge (\neg B)]\}$$
$$\leq q[C \wedge A] + q[C \wedge B] + q[C \wedge (\neg A) \wedge (\neg B)]. \tag{28}$$

$\square$

In our formalism, $q(A)$ will be the $\tau(m|\rho_n)$. It can be measured with the

$$M = a\mathfrak{P}(m) + b[\mathbf{1}_n - \mathfrak{P}(m)]. \tag{29}$$

We have to perform this von Neumann measurement on many systems in the state ρ_n, and count the number of times the system will collapse into a state which belongs entirely in $H(m)$.

4.1. Example

We consider the example discussed in section 3.1, and we show that Eqs.(21),(24),(25) are violated. An example which shows that Eq.(23) is violated, has been presented in [1]. Therefore quantum mechanics is a contextual theory.

(1) We consider Boole's inequality of Eq.(21), and we take $m_1 = 4$, $m_2 = 3$, $m_3 = 6$. In this case $m_1 \vee m_2 \vee m_3 = 12$ and the inequality becomes

$$\tau(6|\rho) + \tau(4|\rho) + \tau(3|\rho) \geq \tau(12|\rho). \tag{30}$$

We substitute the values from Eq.(19) and we get

$$\sum_{\nu=0}^{5} |a_{2\nu}|^2 + \sum_{\nu=0}^{3} |a_{3\nu}| + |a_0|^2 + |a_4|^2 + |a_8|^2 \geq 1 \tag{31}$$

which reduces to

$$2|a_0|^2 + |a_4|^2 + |a_6|^2 + |a_8|^2 \geq |a_1|^2 + |a_5|^2 + |a_7|^2 + |a_{11}|^2 \tag{32}$$

An example where this is violated is the case $a_0 = a_4 = a_6 = a_8 = 0$.

(2) We consider the inequality of Eq.(24). The events A, B are taken to be projections to the subsystems $\Sigma(12), \Sigma(4)$, correspondingly. Then

$$q(A) = \tau(12|\rho); \qquad q(B) = \tau(4|\rho). \tag{33}$$

The inequality becomes

$$\tau(12|\rho) \leq \tau(12 \wedge 4|\rho) + \tau[12 \wedge (\neg 4)|\rho], \tag{34}$$

which reduces to

$$\tau(12|\rho) \leq \tau(4|\rho) + \tau(3|\rho). \tag{35}$$

We substitute the values from Eq.(19) and we get

$$1 \leq \tau(4|\rho) + |a_0|^2 + |a_4|^2 + |a_8|^2 \tag{36}$$

An example where this is violated is the case $a_0 = a_4 = a_8 = 0$.

(3) We consider the inequality of Eq.(25). The events A, B, C are taken to be projections to the subsystems $\Sigma(12), \Sigma(4), \Sigma(3)$, correspondingly. Then

$$q(A) = \tau(12|\rho); \qquad q(B) = \tau(4|\rho); \qquad q(C) = \tau(3|\rho). \tag{37}$$

The inequality becomes

$$\tau(12|\rho) \leq \tau(12 \wedge 3|\rho) + \tau(4 \wedge 3|\rho) + \tau[(\neg 12) \wedge (\neg 4) \wedge 3|\rho], \tag{38}$$

which reduces to

$$\tau(12|\rho) \leq \tau(3|\rho) + \tau(1|\rho) + \tau(1|\rho). \tag{39}$$

We substitute the values from Eq.(19) and we get

$$1 \leq 3|a_0|^2 + |a_4|^2 + |a_8|^2 \tag{40}$$

An example where this is violated is the case $a_0 = a_4 = a_8 = 0$.

5. Discussion

We have studied the set of subsystems of a finite quantum system $\Sigma(n)$, as a distributive lattice (Heyting algebra). We have shown that the quantum probabilities $\tau(m|\rho_n)$ do not obey the equality of Eq.(1), which is intimately related to the associativity of the join in the lattice, but they obey the inequality of Eq.(2). Only if the variables belong to the same chain, Eq(2) is valid as an equality.

In the present formalism chains play the role of contexts. The $\tau(m|\rho_n)$ can be viewed as Kolmogorov probabilities only within a particular chain. When we consider various contexts together, Eq.(1) and other relations which are derived from it, are not valid. This is related to the fact that the space $H(m_1 \vee m_2)$ is larger than the space of superpositions $\text{span}[H(m_1) \cup H(m_2)]$ (Eq.(11)).

In this paper we extended the work of [1], and gave new Bell-like inequalities in proposition 4.2. They combine equalities in Boolean algebra, with Boole's inequality for probabilities. Boole's inequality follows from Eq.(2), but it does not follow from Eq.(1). Consequently, the inequalities in proposition 4.2 are valid for variables within a context (chain), but they are violated when variables take values in different contexts. More Bell-like inequalities can be proved with this methodology. In section 4.1 we gave examples which show that these inequalities are violated. This shows that quantum mechanics is a contextual theory.

The work studies contextuality within the formalism of subsystems of a finite quantum system.

[1] A. Vourdas, J. Math. Phys, 54, 082105 (2013)

[2] G. Birkhoff 'Lattice theory' (Amer. Math. Soc., Rhode Island, 1995)

[3] D.E. Rutherford, 'Introduction to lattice theory' (Oliver and Boyd, Edinburgh, 1965)

[4] R.T. Cox, Am. J. Phys. 14, 1 (1946)

[5] R.T. Cox, 'The algebra of probable inference' (John Hopkis Press, Baltimore, 1961)

[6] E.T. Jaynes, 'Probability theory. The logic of Science' (Cambridge Univ. Press, Cambridge, 2003)

[7] G. Birkhoff, J. von Neumann, Ann. Math. 37, 823 (1936)

[8] C. Piron, 'Foundations of quantum physics', Benjamin, New York, 1976

[9] G.W. Mackey, 'Mathematical foundations of quantum mechanics' Benjamin, New York, 1963

[10] J.S. Bell, Physics 1, 195 (1964)

[11] S. Kochen, E.P.Specker, J. Math. Mech. 17, 59 (1967)

[12] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt, Phys. Rev. Lett., 23, 880 (1969)

[13] E.P. Wigner, Amer. J. Phys. 38, 1005 (1970)

[14] N.D. Mermin, Rev. Mod. Phys. 65, 803 (1993)

[15] I. Pitowski, Brit. J. Phil. Sci. 45, 95 (1994)

[16] A.A. Klyachko, M.A. Can, S. Binicioglu, A.S. Shumovsky, Phys. Rev. Lett. 101, 020403 (2008)

[17] Y-C Liang, R.W. Spekkens, H.M. Wiseman, Phys. Rep. 506, 1 (2011)

[18] S. Abramsky, A. Brandenburger, New J. Phys. 13, 113036 (2011)

[19] S. Abramsky, L. Hardy, Phys. Rev. A85, 062114 (2012)

[20] A. Cabello, Phys. Rev. Lett. 110, 060402 (2013)