Springer Proceedings in Mathematics & Statistics

Clara Grácio
Daniele Fournier-Prunaret
Tetsushi Ueta
Yoshifumi Nishio *Editors*

Nonlinear Maps and their Applications

Selected Contributions from the NOMA 2011 International Workshop

Springer

Chapter 1
Renormalization of circle diffeomorphism sequences and Markov sequences

João P. Almeida, Alberto A. Pinto and David A. Rand

Abstract We show a one-to-one correspondence between circle diffeomorphism sequences that are C^{1+} n-periodic points of renormalization and smooth Markov sequences.

1.1 Introduction

Following [2–9, 20–23], we present the concept of renormalization applied to circle diffeomorphism sequences. These concepts are essential to extend the results presented in [8, 9] to all Anosov diffeomorphisms on surfaces, i.e. to prove a one-to-one correspondence between C^{1+} conjugacy classes of Anosov diffeomorphisms and pairs of C^{1+} circle diffeomorphism sequences that are C^{1+} n-periodic points of renormalization (see also [1, 8, 9, 19]). The main point in this paper is to show the existence of a one-to-one correspondence between C^{1+} circle diffeomorphism sequences that are C^{1+} n-periodic points of renormalization and smooth Markov sequences. This correspondence is a key step in passing from circle diffeomorphisms to Anosov diffeomorphisms because the Markov sequences encode the smooth in-

João P. Almeida
LIAAD-INESC TEC and Department of Mathematics, School of Technology and Management, Polytechnic Institute of Bragança. Campus de Santa Apolónia, Ap. 1134, 5301-857 Bragança, Portugal. e-mail: jpa@ipb.pt

Alberto A. Pinto
LIAAD-INESC TEC and Department of Mathematics, Faculty of Sciences, University of Porto. Rua do Campo Alegre, 4169-007 Porto, Portugal. e-mail: aapinto@fc.up.pt

David A. Rand
Warwick Systems Biology & Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom.
e-mail: dar@maths.warwick.ac.uk

formation of the expanding and contracting laminations of the Anosov diffeomorphisms [10–18]).

1.2 Circle difeomorphisms

Let $\mathbf{a} = (a_i)_{i=0}^{\infty}$ be a sequence of positive integers and let $\gamma(\mathbf{a}) = 1/(a_0 + 1/(a_1 + 1/\cdots))$. For every $i \in \mathbb{N}_0$, let $\gamma_i = \gamma_i(\mathbf{a}) = 1/(a_i + 1/(a_{i+1} + 1/\cdots))$ and let $\mathbb{S}_i$ be a *counterclockwise oriented circle* homeomorphic to the circle $\underline{\mathbb{S}}_i = \mathbb{R}/(1+\gamma_i)\mathbb{Z}$.

An *arc* in $\mathbb{S}_i$ is the image of a non trivial interval I in $\mathbb{R}$ by a homeomorphism $\alpha : I \to \mathbb{S}_i$. If I is closed (resp. open) we say that $\alpha(I)$ is a *closed* (resp. *open*) *arc* in $\mathbb{S}_i$. We denote by (a,b) (resp. $[a,b]$) the positively oriented open (resp. closed) arc on $\mathbb{S}_i$ starting at the point $a \in \mathbb{S}_i$ and ending at the point $b \in \mathbb{S}_i$. A C^{1+} atlas $\mathscr{A}_i$ in $\mathbb{S}_i$ is a set of charts such that (i) every small arc of $\mathbb{S}_i$ is contained in the domain of some chart in $\mathscr{A}_i$, and (ii) the overlap maps are $C^{1+\alpha}$ compatible, for some $\alpha > 0$.

Let $\underline{\mathscr{A}}_i$ denote the affine atlas whose charts are isometries with respect to the usual norm in $\underline{\mathbb{S}}_i$. Let the *rigid rotation* $\underline{g}_i : \underline{\mathbb{S}}_i \to \underline{\mathbb{S}}_i$ be the affine homeomorphism, with respect to the atlas $\underline{\mathscr{A}}_i$, with rotation number $\gamma_i/(1+\gamma_i)$.

A homeomorphism $h : \underline{\mathbb{S}}_i \to \mathbb{S}_i$ is *quasisymmetric* if there exists a constant $C > 1$ such that for each two arcs I_1 and I_2 of $\underline{\mathbb{S}}_i$ with a common endpoint and such that $|I_1|_i = |I_2|_i$, we have $|h(I_1)|_i/|h(I_2)|_i < C$, where the lengths are measured in the charts of $\underline{\mathscr{A}_i}$ and $\mathscr{A}_i$.

A C^{1+} *circle diffeomorphism sequence* $(g_i, \mathbb{S}_i, \mathscr{A}_i)_{i=0}^{\infty}$ is a sequence of triples $(g_i, \mathbb{S}_i, \mathscr{A}_i)$ with the following properties: (i) $g_i : \mathbb{S}_i \to \mathbb{S}_i$ is a $C^{1+\alpha}$ diffeomorphism, with respect to the $C^{1+\alpha}$ atlas $\mathscr{A}_i$, for some $\alpha > 0$; and (ii) g_i is quasi-symmetric conjugate to the rigid rotation $\underline{g}_i$ with respect to the atlas $\underline{\mathscr{A}}_i$.

We denote the C^{1+} circle diffeomorphism $(g_i, \mathbb{S}_i, \mathscr{A}_i)$ by g_i. In particular, we denote the rigid rotation $(\underline{g}_i, \underline{\mathbb{S}}_i, \underline{\mathscr{A}}_i)$ by $\underline{g}_i$.

1.2.1 Horocycles

Let us mark a point in $\mathbb{S}_i$ that we will denote by $0_i \in \mathbb{S}_i$. Let $S_i^0 = [0_i, g_i(0_i)]$ be the oriented closed arc in $\mathbb{S}_i$, with endpoints 0_i and $g_i(0_i)$. For every $k \in \{0, \ldots, a_i\}$, let $S_i^k = \left[g_i^k(0_i), g_i^{k+1}(0_i)\right]$ be the oriented closed arc in $\mathbb{S}_i$ with endpoints $g_i^k(0_i)$ and $g_i^{k+1}(0_i)$ and such that $S_i^k \cap S_i^{k-1} = \{g_i^k(0)\}$. Let $S_i^{a_i+1} = [g_i^{a_i+1}(0_i), 0_i]$ be the oriented closed arc in $\mathbb{S}_i$, with endpoints $g_i^{a_i+1}(0_i)$ and 0_i.

We introduce an *equivalence relation* $\sim$ in $\mathbb{S}_i$ by identifying the $a_i + 1$ points $g_i(0), \ldots, g_i^{a_i+1}(0)$ and form the topological space $H_i(\mathbb{S}_i, g_i) = \mathbb{S}_i/\sim$. We take the orientation in H_i as the reverse orientation of the one induced by $\mathbb{S}_i$. We call this oriented topological space the *horocycle* and we denote it by $H_i = H_i(\mathbb{S}_i, g_i)$. We consider the quotient topology in H_i. Let $\pi_{g_i} : \mathbb{S}_i \to H_i$ be the natural projection. The

point
$$\xi_i = \pi_{g_i}(g_i(0_i)) = \cdots = \pi_{g_i}(g_i^{a_i+1}(0_i)) \in H_i$$
is called the *junction* of the horocycle H_i. For every $k \in \{0,\ldots,a_i\}$, let $S^k_{i,H} = S^k_{i,H}(\mathbb{S}_i, g_i) \subset H_i$ be the projection by π_{g_i} of the closed arc S^k_i. Let $R_i\mathbb{S}_i = S^0_{i,H} \cup S^{a+1}_{i,H}$ be the *renormalized circle* in H_i. The horocycle H_i is the union of the renormalized circle $R_i\mathbb{S}_i$ with the circles $S^k_{i,H}$ for every $k \in \{1,\ldots,a_i\}$.

A *parametrization* in H_i is the image of a non trivial interval I in $\mathbb{R}$ by a homeomorphism $\alpha : I \to H_i$. If I is closed (resp. open) we say that $\alpha(I)$ is a *closed* (resp. *open*) *arc* in H_i. A *chart* in H_i is the inverse of a parametrization. A *topological atlas* $\mathscr{B}$ on the horocycle H_i is a set of charts $\{(j,J)\}$, on H_i, with the property that every small arc is contained in the domain of a chart in $\mathscr{B}$, i.e. for any open arc K in H_i and any $x \in K$ there exists a chart $\{(j,J)\} \in \mathscr{B}$ such that $J \cap K$ is a non trivial open arc in H_i and $x \in J \cap K$. A C^{1+} *atlas* $\mathscr{B}$ *in* H_i is a topological atlas $\mathscr{B}$ such that the overlap maps are $C^{1+\alpha}$ and have $C^{1+\alpha}$ uniformly bounded norms, for some $\alpha > 0$.

Let $\mathscr{A}_i$ be a C^{1+} atlas on $\mathbb{S}_i$ in which $g_i : \mathbb{S}_i \to \mathbb{S}_i$ is a C^{1+} circle diffeomorphism. We are going to construct a C^{1+} atlas $\mathscr{A}^H_i$ on H_i that we call the *extended pushforward* $\mathscr{A}^H_i = (\pi_{g_i})_* \mathscr{A}_i$ of the atlas $\mathscr{A}_i$ on $\mathbb{S}_i$. If $x \in H_i \backslash \{\xi_i\}$ then there exists a sufficiently small open arc $J \subset H_i$ containing x and such that $\pi^{-1}_{g_i}(J)$ is contained in the domain of some chart $(I, \hat{\imath})$ of $\mathscr{A}_i$. In this case, we define $(J, \hat{\imath} \circ \pi^{-1}_{g_i})$ as a chart in $\mathscr{A}^H_i$. If $x = \xi_i$ and J is a small arc containing ξ_i, then either (i) $\pi^{-1}_{g_i}(J)$ is an arc in $\mathbb{S}_i$ or (ii) $\pi^{-1}_{g_i}(J)$ is a disconnected set that consists of a union of two connected components.

In case (i), $\pi^{-1}_{g_i}(J)$ is connected and it is contained in the domain of some chart $(I, \hat{\imath}) \in \mathscr{A}_i$. Therefore we define $\left(J, \hat{\imath} \circ \pi^{-1}_{g_i}\right)$ as a chart in $\mathscr{A}^H_i$.

In case (ii), $\pi^{-1}_{g_i}(J)$ is a disconnected set that is the union of two connected arcs I^L_l and I^R_r of the form $I^L_l = (c^L_l, g^l_i(0)]$ and $I^R_r = [g^r_i(0), c^R_r)$, respectively, for all $l,r \in \{1,\ldots,a_i+1\}$. Let J^L_l and J^R_r be the arcs in H_i defined by $J^L_l = \pi_{g_i}(I^L_l)$ and $\pi_{g_i}(I^R_r)$ respectively. Then $J = J^L_l \cup J^R_r$ is an arc in H_i with the property that $J^L_l \cap J^R_r = \{\xi_i\}$, for every $l,r \in \{1,\ldots,a_i+1\}$. We call such arc J a (l,r)-*arc* and we denote it by $J_{l,r}$. Let $j_{l,r} : J_{l,r} \to \mathbb{R}$ be defined by,

$$j_{l,r}(x) = \begin{cases} \hat{\imath} \circ \pi^{-1}_{g_i}(x) & \text{if } x \in J^R_r \\ \hat{\imath} \circ g_i^{r-l} \circ \pi^{-1}_{g_i}(x) & \text{if } x \in J^L_l \end{cases}.$$

Let $(I, \hat{\imath}) \in \mathscr{A}_i$ be a chart such that $\pi_{g_i}(I) \supset J_{l,r}$. Then we define $(J_{l,r}, j_{l,r})$ as a chart in $\mathscr{A}^H_i$ (see Figure 1.1). We call the atlas determined by these charts the *extended pushforward atlas of* $\mathscr{A}_i$ and, by abuse of notation, we will denote it by $\mathscr{A}^H_i = (\pi_{g_i})_* \mathscr{A}_i$.

Let the marked point $\underline{0}_i$ in $\underline{\mathbb{S}}_i$ be the natural projection of $0 \in \mathbb{R}$ onto $\underline{0}_i \in \underline{\mathbb{S}}_i = \mathbb{R}/(1+\gamma_i)\mathbb{Z}$. Let $\underline{S}^0_i = [0_i, \underline{g}_i(0_i)]$ and $\underline{S}^k_i = [\underline{g}^k_i(0_i), \underline{g}^{k+1}_i(0_i)]$. Furthermore, let

$$\underline{H}_i = H_i(\underline{\mathbb{S}}_i, \underline{g}_i)\ ,\ \underline{S}^k_{i,H} = S^k_{i,H}(\underline{\mathbb{S}}_i, \underline{g}_i)\ ,\ R_i\underline{\mathbb{S}}_i = \underline{\mathbb{S}}^0_{i,H} \cup \underline{\mathbb{S}}^{a+1}_{i,H} \text{ and } \underline{\mathscr{A}}^{\mathrm{H_i}} = \left(\pi_{\underline{\mathrm{g}}_\mathrm{i}}\right)_* \underline{\mathscr{A}}_\mathrm{i}.$$

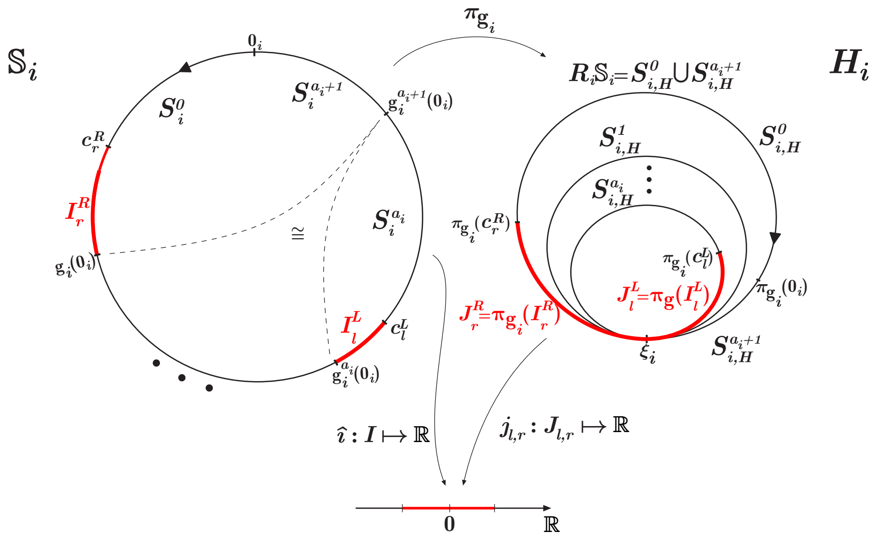

Fig. 1.1: The horocycle H_i and the chart $j_{l,r} : J_{l,r} \to \mathbb{R}$ in case (ii). The junction ξ_i of the horocycle is equal to $\xi_i = \pi_{g_i}(g_i(0_i)) = \cdots \pi_{g_i}(g_i^{a_i}(0_i)) = \pi_{g_i}(g_i^{a_i+1}(0_i))$.

1.3 Renormalization

The *renormalization of a* C^{1+} *circle diffeomorphism* g_i is the triple $(R_i g_i, R_i\mathbb{S}_i, R_i\mathscr{A}_i)$ where (i) $R_i\mathbb{S}_i$ is the renormalized circle with the orientation of the horocycle H_i, i.e. the reversed orientation of the orientation induced by $\mathbb{S}_i$; (ii) the *renormalized atlas* $R_i\mathscr{A}_i = \mathscr{A}_i^H|_{R_i\mathbb{S}_i}$ is the set of all charts in $\mathscr{A}_i^H$ with domains contained in $R_i\mathbb{S}_i$; and (iii) $R_i g_i : R_i\mathbb{S}_i \to R_i\mathbb{S}_i$ is the continuous map given by

$$R_i g_i(x) = \begin{cases} \pi_{g_i} \circ g_i^{a_i+1} \circ \left(\pi_{g_i}|_{S^0_{i,H}}\right)^{-1}(x) \;\; if\;\; x \in S^0_{i,H} \\ \pi_{g_i} \circ g_i \circ \left(\pi_{g_i}|_{S^{a_i+1}_{i,H}}\right)^{-1}(x) \quad if\;\; x \in S^{a_i+1}_{i,H} \end{cases} .$$

We denote the C^{1+} renormalization $(R_i g_i, R_i\mathbb{S}_i, R_i\mathscr{A}_i)$ of g_i by $R_i g_i$.

By construction, the renormalization $R_i\underline{g}_i$ of the rigid rotation $\underline{g}_i$ is affine conjugate to the rigid rotation $\underline{g}_{i+1}$. Hence, from now on, we identify $(R_i\underline{g}_i, R_i\underline{\mathbb{S}}_i, R_i\underline{\mathscr{A}}_i)$ with $(\underline{g}_{i+1}, \underline{\mathbb{S}}_{i+1}, \underline{\mathscr{A}}_{i+1})$.

Recall that a C^{1+} *circle diffeomorphism* $g : \mathbb{S}_i \to \mathbb{S}_i$ is a $C^{1+\alpha}$ diffeomorphism with respect to a $C^{1+\alpha}$ atlas $\mathscr{A}$ on $\mathbb{S}_i$, for some $\alpha > 0$, that is quasi-symmetric conjugate to a rigid rotation $\underline{g} : \mathbb{S}_i \to \mathbb{S}_i$ with respect to an affine atlas $\underline{\mathscr{A}}$ on $\underline{\mathbb{S}}_i$.

The renormalization $R_i g_i$ is a C^{1+} circle diffeomorphism quasi-symmetric conjugate to the rigid rotation $\underline{g}_{i+1}$. Hence, $R_i g_i$ is quasi-symmetric conjugate to the C^{1+}

circle diffeomorphism g_{i+1}. The marked point $0_i \in \mathbb{S}_i$ determines the marked point $0_{R_i\mathbb{S}_i} = \pi_{g_i}(0_i)$ in the circle $R_i\mathbb{S}_i$. Thus, there is a unique topological conjugacy h_i between $R_i g_i$ and g_{i+1} such that $h_i(0_{R_i\mathbb{S}_i}) = 0_{i+1}$.

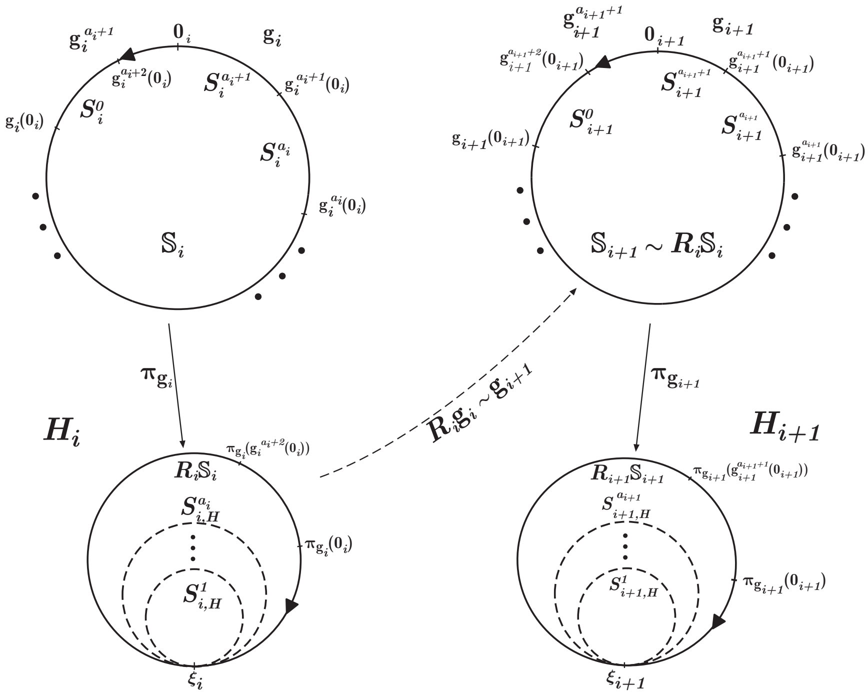

Fig. 1.2: The horocycles H_i and H_{i+1}, and the renormalized map $R_i g_i : R_i\mathbb{S}_i \to R_i\mathbb{S}_i$. Here $\xi_i = g_i(0_i) = \ldots = g_i^{a_i+1}(0_i)$, $\xi_{i+1} = g_{i+1}(0_{i+1}) = \ldots = g_{i+1}^{a_{i+1}+1}(0_{i+1}))$ and the map $R_i g_i$ is identified with g_{i+1}.

A C^{1+} circle diffeomorphism g_0 determines a unique C^{1+} *renormalization circle diffeomorphism sequence* $\mathbf{R}(g_0) = (g_i, \mathbb{S}_i, \mathscr{A}_i)_{i=0}^{\infty}$ given by

$$(g_i, \mathbb{S}_i, \mathscr{A}_i) = (R_i \circ \ldots \circ R_0 g_0, R_i \circ \ldots \circ R_0 \mathbb{S}_0, R_i \circ \ldots \circ R_0 \mathscr{A}_0).$$

We note that the C^{1+} renormalization circle diffeomorphism sequence $\mathbf{R}(g_0)$ is a C^{1+} circle diffeomorphism sequence.

We say that $\mathbf{a} = (a_i)_{i=0}^{\infty}$ a sequence is *n-periodic* if n is the least integer such that $a_{i+n} = a_i$, for every $i \in \mathbb{N}_0$. We observe that given a n-periodic sequence of positive integers $\mathbf{a} = (a_i)_{i=0}^{\infty}$, $\gamma_i = \gamma_i(\mathbf{a}) = 1/(a_i + 1/(a_{i+1} + 1/\cdots))$ is equal to γ_{i+n} for every $i \in \mathbb{N}_0$. Hence, there exists a topological conjugacy $\phi_i : \mathbb{S}_i \to \mathbb{S}_{i+n}$ such that

$$\phi_i \circ g_i = g_{i+n} \circ \phi_i,$$

because g_i and $g_{i+n} = R_{i+n} \circ \ldots \circ R_i g_i$ are C^{1+} circle diffeomorphisms with the same rotation number $\gamma_i = \gamma_{i+n}$.

We say that a sequence $\mathbf{R}(g_0)$ is a C^{1+} *n-periodic point of renormalization*, if ϕ_i is C^{1+} for every $i \in \mathbb{N}$.

1.4 Markov maps

Let $\mathbf{R}(g_0)$ be the renormalization circle diffeomorphism sequence associated to the C^{1+} circle diffeomorphism g_0. The *Markov map* $M_i : H_i \to H_{i+1}$ is given by

$$M_i(x) = \begin{cases} \pi_{g_{i+1}}(x) & if\ x \in R_i \mathbb{S}_i \\ \pi_{g_{i+1}} \circ \pi_{g_i} \circ g_i^{-k} \circ \pi_{g_i}^{-1}(x) & if\ x \in S^k_{i,H_i},\ \text{for}\ \ k = 1, \ldots, a_i \end{cases} .$$

The *Markov sequence* $(M_i)_{i=0}^{\infty}(g_0)$ associated to a C^{1+} circle diffeomorphism g_0 is the sequence of Markov maps $M_i : H_i \to H_{i+1}$, for $i \in \mathbb{N}_0$. Two Markov sequences $(M_i)_{i=0}^{\infty}(g_0)$ and $(N_i)_{i=0}^{\infty}(g_0)$ are *quasi-symmetric conjugate* if there is a sequence $(h_i)_{i=0}^{\infty}$ of quasi-symmetric maps h_i such that $M_{i+1} \circ h_i = h_{i+1} \circ M_i$, for each $i \in \mathbb{N}_0$.

The *rigid Markov sequence* $(\underline{M}_i)_{i=0}^{\infty} = (M_i)_{i=0}^{\infty}(\underline{g}_0)$ is the Markov sequence associated to the rigid rotation $\underline{g}_0$. The rigid Markov maps $\underline{M}_i : \underline{H}_i \to \underline{H}_{i+1}$ are affine with respect to the atlases $\underline{\mathscr{A}}_i^H$ and $\underline{\mathscr{A}}_{i+1}^H$ (see Figure 1.3).

The Markov sequence $(M_i)_{i=0}^{\infty}(g_0)$ has the following properties: (i) the Markov maps M_i are local $C^{1+\alpha}$ diffeomorphisms, for some $\alpha > 0$, and (ii) the Markov sequence $(M_i)_{i=0}^{\infty}(g_0)$ is quasi-symmetric conjugate to the rigid Markov sequence $(\underline{M}_i)_{i=0}^{\infty}(\underline{g}_0)$ because g_i is quasi-symmetric conjugate to $\underline{g}_i$.

The *n-extended Markov sequence* $(\mathbf{M}_i)_{i=0}^{n-1}(g_0)$ is the sequence of the *n-extended Markov maps* $\mathbf{M}_i(g_0) : H_i \to H_i$ defined by

$$\mathbf{M}_i(g_0) = \phi_i^{-1} \circ M_{i+n} \circ \cdots \circ M_i .$$

We observe that a sequence $\mathbf{R}(g_0)$ is a C^{1+} *n-periodic point of renormalization* if, and only if, the n-extended Markov maps $\mathbf{M}_i : H_i \to H_i$ are C^{1+} for every $i \in \mathbb{N}$.

The *rigid n-extended Markov sequence* $(\underline{\mathbf{M}}_i)_{i=0}^{n-1} = (\mathbf{M}_i)_{i=0}^{n-1}(\underline{g}_0)$ is the n-extended Markov sequence associated to the rigid rotation $\underline{g}_0$. The rigid n-extended Markov maps $\mathbf{M}_0, \ldots, \mathbf{M}_{n-1}$ are affine with respect to the atlases $\underline{\mathscr{A}}_0^H, \ldots, \underline{\mathscr{A}}_{n-1}^H$, respectively, because the conjugacy maps $\underline{\phi}_i : \underline{\mathbb{S}}_i \to \underline{\mathbb{S}}_{i+n}$ are affine.

If $\phi_i : \mathbb{S}_i \to \mathbb{S}_{i+n}$ is $C^{1+\alpha}$ then the n-extended Markov sequence $(\mathbf{M}_i)_{i=0}^{n-1}(g_0)$ has the following properties: (i) the n-extended Markov maps $\mathbf{M}_i$ are local $C^{1+\alpha}$ diffeomorphisms, for some $\alpha > 0$, because the Markov maps $M_0, \ldots M_{n-1}$ of the sequence $(M_i)_{i=0}^{n-1}(g_0)$ are local $C^{1+\alpha}$ diffeomorphisms, and (ii) the n-extended Markov maps $\mathbf{M}_i$ are quasi-symmetric conjugate to the rigid n-extended Markov maps $\underline{\mathbf{M}}_i$ because the Markov maps $M_0, \ldots M_{n-1}$ are quasi-symmetric conjugate to $\underline{M}_0, \ldots, \underline{M}_{n-1}$.

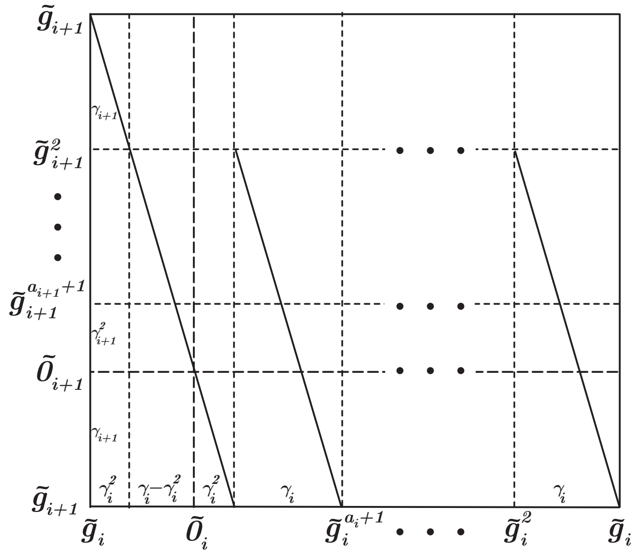

Fig. 1.3: A representation of the rigid Markov map $\underline{M}_i : \underline{H}_i \to \underline{H}_{i+1}$, with respect to the atlases $\underline{\mathscr{A}}_i^H$ and $\underline{\mathscr{A}}_{i+1}^H$, respectively. Here we represent by $\tilde{0}_i$ and $\tilde{0}_{i+1}$ the points $\pi_{\underline{g}_i}(\underline{0}_i)$ and $\pi_{\underline{g}_{i+1}}(\underline{0}_{i+1})$, respectively, and by $\tilde{g}_k^l$ the points $\pi_{\underline{g}_k} \circ \underline{g}_k^l(\underline{0}_k)$, for $k \in \{i, i+1\}$ and $l \in \{1, \ldots, a_k + 1\}$.

Acknowledgements We thank the financial support of LIAAD–INESC TEC through program PEst, USP-UP project, Faculty of Sciences, University of Porto, Calouste Gulbenkian Foundation, FEDER and COMPETE Programmes, PTDC/MAT/121107/2010 and Fundação para a Ciência e a Tecnologia (FCT). J. P. Almeida acknowledges the FCT support given through grant SFRH/PROTEC/49754/2009.

References

1. Almeida, J. P., Fisher, A. M., Pinto, A. A. and Rand, D. A.: Anosov and circle diffeomorphisms, Dynamics Games and Science I, Springer Proceedings in Mathematics (eds. M. Peixoto, A. Pinto and D. Rand), Springer Verlag, 11-23 (2011).
2. Arnol'd, V. I.: Small denominators I: on the mapping of a circle into itself. *Investijia Akad. Nauk. Math.*, 25, 1, (1961), 21-96. Transl. A.M.S., 2nd series, 46, 213-284.
3. Coullet, P. and Tresser, C.: Itération d'endomorphismes et groupe de renormalisation. J. de Physique, C5 25 (1978).
4. Herman, M. R.: Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. Publ. IHES **49**, 1979, 5-233.
5. Lanford, O.: Renormalization group methods for critical circle mappings with general rotation number. VIIIth International Congress on Mathematical Physics. World Sci. Publishing, Singapore, 532–536 (1987).

6. Masur, H.: Interval exchange transformations and measured foliations. The Annals of Mathematics. 2nd Ser., **115** (1), 169–200 (1982).
7. Penner, R. C. and Harer, J. L.: *Combinatorics of Train-Tracks*. Princeton University Press, Princeton, New Jersey (1992).
8. Pinto, A. A., Almeida, J. P. and Portela, A.: Golden tilings. Transactions of the American Mathematical Society, 364 (2012), 2261-2280.
9. Pinto, A. A., Almeida, J. P. and Rand, D. A.: Anosov and renormalized circle diffeomorphisms. pp. 1–33 (2012) (*Submitted*).
10. Pinto, A. A. and Rand, D. A.: Train-tracks with C^{1+} self-renormalisable structures. Journal of Difference Equations and Applications, 16(8), 945-962 (2010).
11. Pinto, A. A. and Rand, D. A.: Solenoid functions for hyperbolic sets on surfaces. Dynamics, Ergodic Theory and Geometry, Boris Hasselblat (ed.), MSRI Publications, 54 145-178 (2007).
12. Pinto, A. A. and Rand, D. A.: Rigidity of hyperbolic sets on surfaces. J. London Math. Soc., 71 2 481-502 (2004).
13. Pinto, A. A. and Rand, D. A.: Smoothness of holonomies for codimension 1 hyperbolic dynamics. Bull. London Math. Soc., 34 341-352 (2002).
14. Pinto, A. A. and Rand, D. A.: Teichmüller spaces and HR structures for hyperbolic surface dynamics. Ergod. Th. Dynam. Sys, 22 1905-1931 (2002).
15. Pinto, A. A. and Rand, D. A.: Existence, uniqueness and ratio decomposition for Gibbs states via duality. Ergod. Th. & Dynam. Sys., 21 533-543 (2001).
16. Pinto, A. A. and Rand, D. A.: Characterising rigidity and flexibility of pseudo-Anosov and other transversally laminated dynamical systems on surfaces. Warwick Preprint 1995.
17. Pinto, A. A., Rand, D. A. and Ferreira, F.: Arc exchange systems and renormalization. Journal of Difference Equations and Applications 16 (2010), no. 4, 347-371.
18. Pinto, A. A., Rand, D. A. and Ferreira, F.: Cantor exchange systems and renormalization. Journal of Differential Equations, 243 593-616 (2007).
19. Pinto, A. A. Rand, D. A., Ferreira, F.: *Fine structures of hyperbolic diffeomorphisms*. Springer Monographs in Mathematics, Springer (2010).
20. Pinto, A. A. and Sullivan, D.: The circle and the solenoid. Dedicated to Anatole Katok On the Occasion of his 60th Birthday. *DCDS A*, 16 2 463-504 (2006).
21. Rand, D. A.: Existence, nonexistence and universal breakdown of dissipative golden invariant tori. I. Golden critical circle maps. Nonlinearity, **5** 3 639-662 (1992).
22. Veech, W.: Gauss measures for transformations on the space of interval exchange maps. The Annals of Mathematics, 2nd Ser. 115 2 201-242 (1982).
23. Yoccoz J. C.: Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne. Ann. Scient. Éc. Norm. Sup., 4 série, t, **17** (1984) 333-359.