
# Determination of (0,2)-regular sets in graphs

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## ABSTRACT

**An eigenvalue of a graph is main iff its associated eigenspace is not orthogonal to the all-one vector j. The main characteristic polynomial of a graph G with p main distinct eigenvalues is $m_G(\lambda) = \lambda^p - c_0\lambda^{p-1} - c_1\lambda^{p-2}$-...-$c_{p-2}\lambda - c_{p-1}$ and it has integer coefficients. If G has n vertices, the nxk walk matrix of G is $W_k$=(j,$A_G$j,$A_G^2$j,…,$A_G^{k-1}$ j) and W, the walk matrix of G, is $W_k$ for which rank($W_k$)=k. The number k coincides with the number of distinct main eigenvalues of G. In [2] it was proved that the coefficients of the main characteristic polynomial of G are the solutions of $WX = A_G^p$j. A (κ,τ)-regular set [3] is a subset of the vertices of a graph inducing a κ-regular subgraph such that every vertex not in the subset has τ neighbors in it. In [1], a strategy for the determination of (0,1)-regular sets is described and we generalize it in order to solve the problem of the determination of (0,2)-regular sets in arbitrary graphs. An algorithm for deciding whether or not a given graph has a (0,2)-regular set is described. Its complexity depends on the multiplicity of −2 as an eigenvalue of the adjacency matrix of the graph. When such multiplicity is low, the generalization of the results in [1] assure that the algorithm is polynomial. An example of application of the algorithm to a graph for which this multiplicity is low is also presented.**

Cardoso, Sciriha and Zerafa [2] introduced the parametric vector $g_G(\kappa,\tau) = \sum_{j=0}^{p-1} \alpha_j A_G^j$**j** where $\alpha_0, \dots, \alpha_{p-1}$ are the solutions of system (1)

$$\begin{pmatrix} \kappa-\tau & 0 & \dots & 0 & -c_{p-1} \\ -1 & \kappa-\tau & \dots & 0 & -c_{p-2} \\ 0 & -1 & \dots & 0 & -c_{p-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & \kappa-\tau-c_0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{p-2} \\ \alpha_{p-1} \end{pmatrix} = -\tau \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

## Necessary and sufficient condition for the existence of a (κ,τ)- regular set

**Theorem [2]:** If G is a graph with p distinct main eigenvalues, then a set of vertices S is (κ,τ)-regular iff $x_S = g_G(\kappa,\tau)$+q,

with $\begin{cases} q = 0 \Leftarrow (\kappa-\tau) \notin \sigma(G) \\ q \in \mathcal{E}(\kappa-\tau) \Leftarrow (\kappa-\tau) \in \sigma(G) \end{cases}$

## (0,2)-feasible tuples

Considering a set of vertices $I = \{i_1, \dots, i_m\} \subset V(G)$, vector $x^I = (x_{i_1}, \dots, x_{i_m}) \in \{0,1\}^n$ is (0,2)-feasible if it verifies:

- $(\exists i_r \in I: x_{i_r} = 1) \Rightarrow (\forall i_j \in N_G(i_r) \cap I, x_{i_j} = 0)$.
- $(\exists i_s \in I: N_G(i_s) \subset I) \Rightarrow (\sum_{j \in N_G[i_s]} x_j = 2)$.
- $(\exists i_r \in I: x_{i_r} = 1) \Rightarrow (\forall j \in N_G(i_r), \sum_{k \in (N_G[j] \cap I) \setminus \{i_r\}} x_k = 1)$.

## ALGORITHM

**Input:** Graph G of order n, m=mult(-2) and matrix Q whose columns are the vectors of a basis of $\mathcal{E}$(-2).

**Output:** A (0,2)-regular set of G or the conclusion that it does not exist.

1. **If** $g_G(0,2) \notin \mathbb{N}$ then **STOP** (there is no solution) **End If;**
2. **If** $m = 0$ **then STOP** ( $x_S = g_G(0,2)$) **End If;**
3. **If** $\exists \boldsymbol{v} \in V(G): rank(Q^N) \leq d_G(v) + 1 \quad (N = N_G[v])$ **then STOP** ([1]);
4. Determine $I = \{i_1, \dots, i_m\} \subset V(G): rank(Q^I) = m$ and **set** $g \coloneqq g_G(0,2)$;
5. **Set** $NoSolution \coloneqq TRUE$;
6. **Set** $X \coloneqq \{(x_{i_1}, \dots, x_{i_m})$ that are (0,2)-feasible for G};
7. **While** $NoSolution \wedge X \neq \emptyset$ **do**
   a) $(x_{i_1}, \dots, x_{i_m}) \in X$ and **set** $\boldsymbol{x^I} \coloneqq (x_{i_1}, \dots, x_{i_m})^T$;
   b) **Set** $X \coloneqq X \setminus \{x^I\}$ and determine $\beta: x^I = g^I + Q^I\beta$;
   c) **If** $g + Q\beta \in \{0,1\}^n$ **then** $NoSolution \coloneqq FALSE$ **End If**;
8. **End While;**
9. **If** $NoSolution \coloneqq FALSE$ **then** $x \coloneqq g + Q\beta$ **else** return $NoSolution$;
10. **End.**

## EXAMPLE

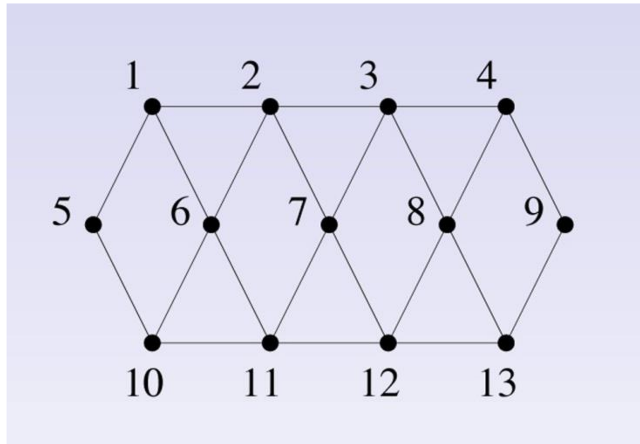

Fig.1 Graph CLP

The walk matrix for CLP is $W = (\mathbf{j}, A_{CLP}\mathbf{j}, A_{CLP}^2\mathbf{j})$.

The solutions of $WX = A_{CLP}^3$ **j** are $\begin{cases} c_2 = 0 \\ c_1 = -5 \\ c_0 = 5 \end{cases}$

The solution of (1) is $\begin{cases} \alpha_0 = 1 \\ \alpha_1 = -\frac{7}{19} \\ \alpha_2 = \frac{1}{19} \end{cases}$ and so the

parametric vector $g_{CLP}(0,2)$ is equal to **j** $- \frac{7}{19} A_{CLP}$ **j**+$\frac{1}{19} A_{CLP}^2$ **j.**

Matrix $Q = (q_1, \dots, q_4)$ has the vectors of a basis of the eigenspace associated to -2 as columns.

Looking for a vertex $v$ of degree ≥3 for which the submatrix of Q corresponding to $N_{CLP}[v]$ has full rank, we find that no such vertex exists.

## How to proceed?

To the closed neighbourhood of an arbitrarily chosen vertex, another vertex is added. Does the corresponding submatrix of Q have full rank?
It is easily checked that the submatrix of Q corresponding to $N_{CLP}[2] \cup \{4\}$ has full rank so, to proceed, compute the subvector of g corresponding to $I = N_{CLP}[2] \cup \{4\}$={1,2,3,4,6,7}.
Next, supposing that $2 \in S$ and $4 \notin S$ and solving the subsystem $x_S^I = g^I + \sum_{i=1}^4 \beta_i q_i^I$, the values of the $\beta$s are obtained and the solution of the complete system $x_S = g + \sum_{i=1}^4 \beta_i q_i$ - a (0,2)-regular set of CLP - is computed:

$x_S = (0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0)^T$.

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[3] D. Cardoso and P. Rama, *Equitable bipartitions of graphs and related results*, Journal of Mathematical Sciences,.120: 869-880, 2004.