Determination of (0,2)-regular sets in graphs

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ABSTRACT

An eigenvalue of a graph is main iff its associated eigenspace is not orthogonal to the all-one vector j. The main characteristic polynomial of a graph G with p main distinct eigenvalues is $m_G(\lambda) = \lambda^p - c_0\lambda^{p-1} - c_1\lambda^{p-2}$-…-$c_{p-2}\lambda - c_{p-1}$ and it has integer coefficients. If G has n vertices, the nxk walk matrix of G is W_k=(j,A_Gj,A_G^2j,…,A_G^{k-1} j) and W, the walk matrix of G, is W_k for which rank(W_k)=k. The number k coincides with the number of distinct main eigenvalues of G. In [2] it was proved that the coefficients of the main characteristic polynomial of G are the solutions of $WX = A_G^p$j. A (κ,τ)-regular set [3] is a subset of the vertices of a graph inducing a κ-regular subgraph such that every vertex not in the subset has τ neighbors in it. In [1], a strategy for the determination of (0,1)-regular sets is described and we generalize it in order to solve the problem of the determination of (0,2)-regular sets in arbitrary graphs. An algorithm for deciding whether or not a given graph has a (0,2)-regular set is described. Its complexity depends on the multiplicity of −2 as an eigenvalue of the adjacency matrix of the graph. When such multiplicity is low, the generalization of the results in [1] assure that the algorithm is polynomial. An example of application of the algorithm to a graph for which this multiplicity is low is also presented.

Cardoso, Sciriha and Zerafa [2] introduced the parametric vector $g_G(\kappa,\tau) = \sum_{j=0}^{p-1} \alpha_j A_G^j \mathbf{j}$ where $\alpha_0, \dots, \alpha_{p-1}$ are the solutions of system (1)

$$\begin{pmatrix} \kappa-\tau & 0 & \dots & 0 & -c_{p-1} \\ -1 & \kappa-\tau & \dots & 0 & -c_{p-2} \\ 0 & -1 & \dots & 0 & -c_{p-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & \kappa-\tau-c_0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{p-2} \\ \alpha_{p-1} \end{pmatrix} = -\tau \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Necessary and sufficient condition for the existence of a (κ,τ)- regular set

Theorem [2]: If G is a graph with p distinct main eigenvalues, then a set of vertices S is (κ,τ)-regular iff $x_S = g_G(\kappa,\tau)$+q,

with $\begin{cases} q = 0 \Leftarrow (\kappa-\tau) \notin \sigma(G) \\ q \in \mathcal{E}(\kappa-\tau) \Leftarrow (\kappa-\tau) \in \sigma(G) \end{cases}$

(0,2)-feasible tuples

Considering a set of vertices $I = \{i_1, \dots, i_m\} \subset V(G)$, vector $x^I = (x_{i_1}, \dots, x_{i_m}) \in \{0,1\}^n$ is (0,2)-feasible if it verifies:

- $(\exists i_r \in I: x_{i_r} = 1) \Rightarrow (\forall i_j \in N_G(i_r) \cap I, x_{i_j} = 0)$.
- $(\exists i_s \in I: N_G(i_s) \subset I) \Rightarrow (\sum_{j \in N_G[i_s]} x_j = 2)$.
- $(\exists i_r \in I: x_{i_r} = 1) \Rightarrow (\forall j \in N_G(i_r), \sum_{k \in (N_G[j] \cap I) \setminus \{i_r\}} x_k = 1)$.

ALGORITHM

Input: Graph G of order n, m=mult(-2) and matrix Q whose columns are the vectors of a basis of $\mathcal{E}$(-2).

Output: A (0,2)-regular set of G or the conclusion that it does not exist.

1. **If** $g_G(0,2) \notin \mathbb{N}$ then **STOP** (there is no solution) **End If;**
2. **If** $m = 0$ **then STOP** ($x_S = g_G(0,2)$) **End If;**
3. **If** $\exists v \in V(G): rank(Q^N) \leq d_G(v) + 1 \quad (N = N_G[v])$ **then STOP** ([1]);
4. Determine $I = \{i_1, \dots, i_m\} \subset V(G): rank(Q^I) = m$ and **set** $g \coloneqq g_G(0,2)$;
5. **Set** $NoSolution \coloneqq TRUE$;
6. **Set** $X \coloneqq \{(x_{i_1}, \dots, x_{i_m})$ that are (0,2)-feasible for G};
7. **While** $NoSolution \wedge X \neq \emptyset$ **do**
   a) $(x_{i_1}, \dots, x_{i_m}) \in X$ and **set** $\boldsymbol{x^I} \coloneqq (x_{i_1}, \dots, x_{i_m})^T$;
   b) **Set** $X \coloneqq X \setminus \{x^I\}$ and determine $\beta: x^I = g^I + Q^I\beta$;
   c) **If** $g + Q\beta \in \{0,1\}^n$ **then** $NoSolution \coloneqq FALSE$ **End If**;
8. **End While;**
9. **If** $NoSolution \coloneqq FALSE$ **then** $x \coloneqq g + Q\beta$ **else** return $NoSolution$;
10. **End.**

EXAMPLE

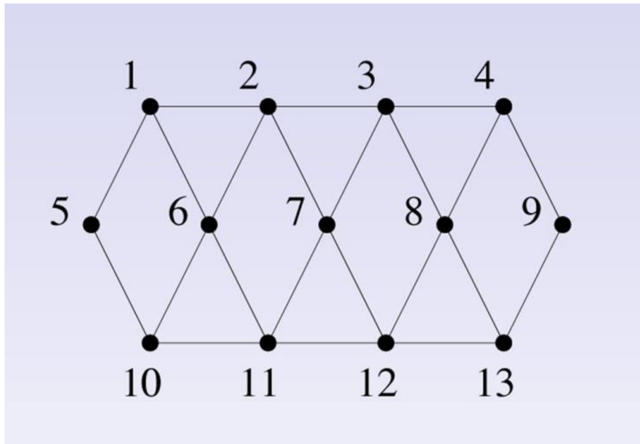

Fig.1 Graph CLP

The walk matrix for CLP is $W = (\mathbf{j}, A_{CLP}\mathbf{j}, A_{CLP}^2\mathbf{j})$.

The solutions of $WX = A_{CLP}^3$ **j** are $\begin{cases} c_2 = 0 \\ c_1 = -5 \\ c_0 = 5 \end{cases}$

The solution of (1) is $\begin{cases} \alpha_0 = 1 \\ \alpha_1 = -\frac{7}{19} \\ \alpha_2 = \frac{1}{19} \end{cases}$ and so the

parametric vector $g_{CLP}(0,2)$ is equal to $\mathbf{j} - \frac{7}{19} A_{CLP}$ **j**+$\frac{1}{19} A_{CLP}^2$ **j.**

Matrix Q =$(q_1, \dots, q_4)$ has the vectors of a basis of the eigenspace associated to -2 as columns.

Looking for a vertex v of degree ≥3 for which the submatrix of Q corresponding to $N_{CLP}[v]$ has full rank, we find that no such vertex exists.

How to proceed?

To the closed neighbourhood of an arbitrarily chosen vertex, another vertex is added. Does the corresponding submatrix of Q have full rank?

It is easily checked that the submatrix of Q corresponding to $N_{CLP}[2] \cup \{4\}$ has full rank so, to proceed, compute the subvector of g corresponding to $I = N_{CLP}[2] \cup \{4\}$={1,2,3,4,6,7}.

Next, supposing that $2 \in S$ and $4 \notin S$ and solving the subsystem x_S^I= $g^I + \sum_{i=1}^4 \beta_i q_i^I$, the values of the βs are obtained and the solution of the complete system $x_S = g + \sum_{i=1}^4 \beta_i q_i$ - a (0,2)-regular set of CLP - is computed:

$x_S = (0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0)^T$.

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