Indirect Parameter Estimation of Continuous-Time Systems using Discrete Time Data

Rui Esteves Araújo, *Member*
Department of Computing and Electrical Engineering
Faculty of Engineering of the University of Porto
Dr. Roberto Frias, s/n
4200-465 Porto, Portugal
raraujo@ fe.up.pt

Américo Vicente Leite
Polytechnic Institute of Bragança
Campus de Sta. Apolónia
Apartado 134

5301-857 Bragança, Portugal
avtl@ipb.pt

Diamantino Silva Freitas
Department of Computing and Electrical Engineering
Faculty of Engineering of the University of Porto
Dr. Roberto Frias, s/n
4200-465 Porto, Portugal
dfreitas@ fe.up.pt

Abstract—This paper addresses the problem of parameter estimation of continuous-time systems using samples of its input-output data. We propose a method based on the bilinear transformation to obtain an equivalent discrete-time model. Introducing a new polynomial pre-filter it is possible to compute the physical parameters via inverse mapping between the discrete-time and the continuous-time models. A simulation example is given to illustrate the noise effects in the parameter estimation results. Using experimental results, we demonstrate the ability of the estimator to handle real measurement problems.

I. INTRODUCTION

In this paper the problem of continuous-time identification is addressed. Two different approaches are currently available to solve this problem. The so-called "indirect method" consists of two parts: firstly the estimation of the discrete-time model parameters using samples of input-output data, and secondly the determination of a continuous-time model, corresponding to the discrete-time model. The other approach is the "direct method", where discrete-time approximations are used for the signals and operators in a continuous-time model. The result is an approximate model in which the parameters of the original model are obtained. The surveys by Young [1], and Unbehauen and Rao [2] describe most of the available techniques.

One limitation of the indirect approach is that the results obtained are strongly dependent on the choice of the sampling interval for the input-output data [3]. Additionally, the necessary evaluation of the natural logarithm of a square matrix presents some difficulties. Remark that the conventional method relies on the calculation of the matrix of a continuous-time model (A) from the estimate of the matrix of a discrete-time model (F), and since $F = e^{AT}$, where the sampling period T is known, the difficulty arises from the calculation of the natural logarithm of F, *i.e.* $AT = ln\ (F)$. More specifically, this calculation can be done by transformation to diagonal or Jordan canonical forms, or by use of Sylvester's interpolation formula. As it has been reported in the literature by Sinha and Lastman [4], these methods do not work efficiently when there are negative or multiple poles. In the first case, a real negative pole in the z-plane is transformed into complex conjugates poles in the s-plane, causing an increase in the order of the system model.

In the case of multiple poles the diagonalization method cannot be used, and an efficient algorithm does not exist to calculate the continuous-time model from the Sylvester interpolation formula. A different approach based on the series expansion of the natural logarithm has also been suggested by Sinha and Lastman in [4]. This technique can sometimes introduce difficulties due to imposition of some restrictions on the poles of the system or on the sampling period to guarantee the convergence of the series.

In order to overcome these problems we propose a novel scheme to compute a discrete-time approximation of a continuous-time model by applying the well-known bilinear transformation. Besides the fact that this transformation gives excellent approximation over a large bandwidth, there is an added advantage, which is the possibility of estimation of the values of the physical parameters of the system. However, the above advantages come with a price. It is necessary to know the a priori structure of the model and the system must be linear.

Many contributions do exist for the estimation of the parameters of continuous-time models from measured input-output data that use the "direct method" [2]. However, only a few are known that use the "indirect method" [5], for which, only an underdeveloped theory is available. One should bear in mind that this approach to identification of continuous-time models has not yet received appropriate coverage in the recent literature.

The present paper is organized as follows: section II gives a general formulation of the parameter estimation problem; section III describes our method to determine the model parameters. To illustrate our approach, we present in section IV a numerical simulation and in section V one practical example is treated. Lastly, we conclude and present some final remarks, in section VI.

II. PROBLEM STATEMENT

A. Model Description

Consider a continuous-time system:

$$G_c(s, \theta_c) = \frac{Y(s)}{U(s)} = \frac{B(s)}{A(s)} \quad (1)$$

with excitation $u(t)$, and response $y(t)$. The polynomials $A(s)$

and $B(s)$ are given by

$$A(s) = s^n + a_{n-1}^c s^{n-1} + \cdots + a_0^c, \ n \geq 1 \qquad (2)$$

$$B(s) = b_m^c s^m + \cdots + b_0^c, \ m \geq 0 \qquad (3)$$

where a_i^c and b_i^c are unknown coefficients and the superscript c denotes continuous parameters. Suppose that the input-output signals of the system are sampled with period T and the measured output $y_m(t)$ is assumed to be corrupted by an additive stationary white noise sequence $e(t)$, with zero mean and variance σ_e^2. Let the measurements consist of samples at time instants, $t = kT, k = 0, 2, \cdots, N$, and the noise effects at those time instants be $e(kT)$. Hence the measured output is

$$y_m(kT) = y(kT) + e(kT) . \qquad (4)$$

The data available for estimation is $\{u(kT), y_m(kT)\}, k \in \{0, 1, 2, \cdots (N-1)\}$ and for convenience they will be denoted as a set $\mathcal{Z} = \{u(k), y_m(k); k = 0, \ldots, N-1\}$. Remark that they denote samples of continuous-time signals with finite time axis.

B. Parameter Estimation Problem

The aim of this sub-section is to formulate the estimation problem. For this purpose, let $\mathcal{M}_c$ be a model structure of a continuous-time system which is a smoothly parametrized set of models

$$\mathcal{M}_c = \{\mathcal{M}_c(\theta_c) : \theta_c \in \mathcal{D}_c\} \qquad (5)$$

where the set $\mathcal{D}_c$ denotes the parameter space to which the parameters vector θ_c is restricted.

We assume that the model system and the measurement data obtained from this system allow an approximate description [6]

$$\mathcal{M}_c(\theta_c) = \{\theta_c \in \mathcal{D}_c : y_m(t) = G_c(p, \theta_c)u(t) + e(t)\} \qquad (6)$$

where p is the differentiation operator and G_c is the transfer function of the system. We further assume that:

A.1 The system G_c under study is linear and asymptotically stable, i.e. $A(s)$ has all zeros in the left-hand side of the plane s.

A.2 The noise $e(t)$ is independent of the input signal $u(t)$.

A.3 The upper bound of the orders n and m is known.

A.4 Only models represented by a finite dimensional parameter vector are considered and all parameters are time-invariant, i.e. $\theta_c \in \mathfrak{R}^h, h = \dim \theta_c, \mathcal{D}_c \subset \mathfrak{R}^h$, and $\dot{\theta}_c \equiv 0$.

Thus, the problem considered in this paper can be formulated in intuitive mathematical terms as follows: Given a set of N measured pairs of input and output data $u(t), y_m(t)$, determine the coefficients $a_{n-1}^c, \cdots, a_0^c$ and $b_m^c \cdots, b_0^c$ of the differential equation model. The parameter vector to be estimated is then:

$$\hat{\theta}_c = \left[a_{n-1}^c, \cdots, a_0^c, b_m^c \cdots, b_0^c\right]^T \qquad (7)$$

where the superscript T denotes the transpose.

III. BASIC THEORY OF INDIRECT METHOD

In this section, we present a solution based on an indirect approach to the parameter estimation problem. This consists of two steps. In the first step a discrete-time model is estimated from the measured input and output signals. Once these parameters are determined, the coefficients of the differential equations can be computed via a mapping function between the discrete-time model and the continuous-time model.

A. From Continuous to Discrete Time

The first step consists in establishing the relation between the continuous-time model and an equivalent discrete-time model. It is assumed that the model and the measurement data obtained from this system, allow a description like the following:

$$y_m(t) = G_c(p, \theta_c)u(t) + e(t) . \qquad (8)$$

In order to establish the model structures for the system, we need to transform the transfer function G_c into a discrete-time model by means of the bilinear transformation:

$$\mathcal{T} : G_d(z, \theta_d) = G_c(s, \theta_c)\big|_{s = \frac{2}{T}\frac{z-1}{z+1}} . \qquad (9)$$

At this point it is worthwhile to interpret the discretization process as a mapping or a transformation, i.e. the discretization may be viewed as a mapping from the $\mathcal{D}_c$ domain to the $\mathcal{D}_d$ domain. Suppose we have two sets, $\mathcal{M}_c$ and $\mathcal{M}_d$. Suppose in addition that we can define a function F that assigns to each element in the set $\mathcal{M}_c$ one element of the set $\mathcal{M}_d$. In other words, F is a function which transforms, or maps, $\mathcal{D}_c$ into $\mathcal{D}_d$ and we will denote this by $F : \mathcal{M}_c \subset \mathfrak{R}^h \to \mathcal{M}_d \subset \mathfrak{R}^l$ such that

$$\theta_d = F(\theta_c) \qquad (10)$$

which can be determined from mathematical modeling, for example with $Maple^{\circledR}$.

After establishing the relation between the continuous-time and discrete-time models, the discrete parameters θ_d will be estimated by a standard identification algorithm $\mathcal{A}$ and a data set $\mathcal{Z}$, i.e. $\hat{\theta}_d = \mathcal{A}(\mathcal{Z})$. For example, an identification algorithm with an output-error method can be used. The orders of the polynomials of the output-error model are defined by the structure of G_d, which may be known in advance based on the continuous-time to discrete-time transformation.

The basis for conversion from the discrete-time domain to the continuous domain is now settled. We consider that, given a continuous-time model for estimation of its parameters, under some reasonable assumptions, it is possible to reformulate the problem in a way that permits the straightforward application of discrete-time black-box identification using, for example, the routines available in the System Identification Toolbox of Matlab [7]. It is quite

obvious that, if the discrete-time model is not accurate, the equivalent continuous-time model will also be inaccurate, so, a number of precautions must be considered in this step.

We turn now our attention to study the possibility of conversion, that is, the calculation the differential equation model coefficients according to the function that performs the mapping between the continuous-time and the discrete-time domains. This is the subject of the next sub-section.

B. The Basic Difficulty

We now come to the topic of inversion of the F function (see (10)). This is a small but extremely important step in our problem. Let us make this idea more precise. Based on the estimated discrete-time parameters $\hat{\theta}_d$, the continuous-time parameters $\hat{\theta}_c$ have to be determined by the inverse function

$$\hat{\theta}_c = F^{-1}(\hat{\theta}_d) \tag{11}$$

when the inverse exists, so giving an unique solution.

The main difficulty with finding this inverse function is associated with the mapping of the zeros of the discrete-time model transfer function to the continuous-time model. It is well known that with a zero-order hold approach the mapping between the discrete-time poles and their continuous-time counterparts is reasonably simple, because these are mapped by means of complex exponentials. However, this is not the general case for the zeros, for which no general closed form equation exists, and only approximate expressions were reported; see for instance in [8] and [9]. Consequently, a poor parameter mapping will usually occur affecting the accuracy of the final continuous-time parameter estimation.

C. Computing the Parameter Vector $\hat{\theta}_c$

The main problem to solve appears with the transfer function zeros. The solution described by Araújo [10] uses a polynomial pre-filter applied to the input signal that guarantees that the continuous-time parameters could be uniquely determined from the discrete-time parameters. As a consequence the measured input is filtered through the polynomial filter. It is shown in [10] that the polynomial filter for input data is given as follows:

$$u^*(k) = \left(1+q^{-1}\right)^r u(k) \tag{12}$$

where q^{-1} is the backward shift operator, r is the relative degree of the model rational function and $u(k)$ it is the input data sample at instant k.

At this stage, we assumed to have the discrete transfer function computed according to (9) and a set of input-output data, $\mathcal{Z}$. The expression (12) implies that each input-output measurement defines a new set denoted by $\mathcal{Z}^*$ containing u^*:

$$\mathcal{Z}^* = \left\{u^*(k), y_m(k); k = 0, \dots, N-1\right\}. \tag{13}$$

So, assuming that the conditions established by the inverse function theorem are all fulfilled, the parameter vector θ_c is calculated by

$$
\overbrace{\hat{\theta}_c = F^{-1}(\hat{\theta}_d)}^{inversion} \quad \Leftarrow \quad
\begin{array}{c}
\overbrace{\hat{\theta}_d = \mathcal{A}(\mathcal{Z}^*)}^{estimation} \\
\text{and} \\
\lim_{T\to 0}\lim_{N\to\infty} \hat{\theta}_d = \theta_d
\end{array}
\tag{14}
$$

where $\hat{\theta}_d$ is estimated by the identification algorithm A using input-output data $\mathcal{Z}^*$. This means that the estimation is close to the true value of the parameter vector for large data sets and small sampling period.

IV. SIMULATION STUDY

In this section, we illustrate the theoretical results discussed in the previous sections by means of a numerical simulation. In section 5 a practical real-data example is also presented. This example is based on the following second-order continuous-time system:

$$G_c(s,\theta_c) = \frac{\dfrac{1}{C}s}{s^2 + \dfrac{s}{RC} + \dfrac{1}{LC}}. \tag{15}$$

The transfer function represents the electric circuit admittance illustrated in figure 1 which is parametrized by the physical parameters R, L, and C, denoted by the parameter vector $\theta_c = \begin{bmatrix} R\,L\,C \end{bmatrix}^T \in \mathfrak{R}^3$.

A. From Continuous-time Model to Discrete-time Model and vice-versa

In order to establish the model structures for this example, we need to use the transfer function (15) which is transformed into a discrete-time model by means of the bilinear transformation:

$$\mathcal{T}: G_d(z,\theta_d) = G_c(s,\theta_c)\big|_{s=\frac{2}{T}\frac{z-1}{z+1}}. \tag{16}$$

After simple manipulations we obtain the following discrete-time model with appropriate reparametrization:

$$G_d^*(z,\theta_d) = \frac{b_0^d}{1 + a_1^d z^{-1} + a_0^d z^{-2}} \tag{17}$$

with
$$\hat{\theta}_d = \begin{bmatrix} b_0^d \\ a_1^d \\ a_0^d \end{bmatrix} = F(\theta_c) = \begin{bmatrix} \dfrac{2RLT}{4RCL+2LT+RT^2} \\ \dfrac{2RT^2-8RLC}{4RCL+2LT+RT^2} \\ \dfrac{RT^2+4RCL-2LT}{4RCL+2LT+RT^2} \end{bmatrix} \tag{18}$$
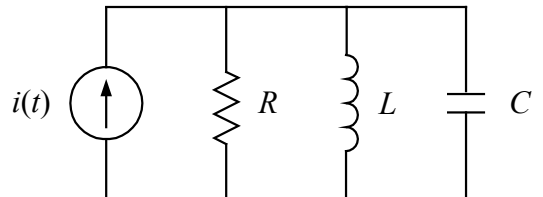

Fig. 1. The circuit under test.

where T is the sampling interval duration. Note that the discrete model is parametrized by three independent parameters that can be estimated with standard prediction error methods. Next, if the discrete-time parameters $\hat{\theta}_d$ are estimated, the continuous-time parameters $\hat{\theta}_c$ can be determined using the inverse function given by

$$\hat{\theta}_c = F^{-1}(\hat{\theta}_d) \Leftrightarrow \begin{bmatrix} R \\ L \\ C \end{bmatrix} = \begin{bmatrix} \dfrac{2\,b_0^d}{1+a_0^d} \\ \dfrac{2\,T\,b_0^d}{1+a_1^d+a_0^d} \\ \dfrac{1}{8}\dfrac{T(a_0^d+1-a_1^d)}{b_0^d} \end{bmatrix}. \qquad (19)$$

An important and interesting aspect of our method is that this algebraic operation can be coded in computer algebraic languages, such as *Maple*®. So, this approach can be used as well in problems of greater dimension.

B. Numerical Illustration

The following illustrative study was simulated using the computer package *Simulink*®. The simulation of a true continuous-time system was implemented and the differential equation solved with constant integration steps using *the Runge-Kutta* method of order 5. The input-output data samples generated by simulation were then considered to be the continuous-time input and output data that were sampled at different sample rates for parameter estimation. Our intention is to analyze the estimation accuracy under different conditions. The true values of parameters of the system under simulation are $\theta_c = [R\ L\ C]^T = [5\,\Omega\ \ 0.02\,\text{H}\ \ 0.5\,\text{F}]^T$. Note that these values are normalized in frequency to preserve the same impedance as the experimental circuit of next section. In order to evaluate the performance of this methodology, the following measure of estimation error was used:

$$\varepsilon = \frac{\|\theta - \theta_0\|}{\|\theta_0\|} \qquad (20)$$

where θ_0 is the true parameter vector and θ is the estimation parameter vector.

The aim of this study is to look at the effect of noise on the accuracy of parameter estimates. In this experiment, the Monte Carlo simulation has 50 runs in each case. The input signal is a square wave with amplitude 2 A e $0.2/\pi$ Hz. The output signal has been disturbed with Gaussian noise sequence. Its variance is adjusted to obtain the desired ratio of noise to signal (N/S) defined by

$$\frac{N}{S} = \frac{\sigma_{e(k)}}{\sigma_{y(k)}} \qquad (21)$$

where σ represents the standard deviation. The sampling period of $T = 0.02\,\text{s}$ has been chosen based on the rule of thumb $(0.003T_n < T < 0.05T_n)$ that utilizes the period of natural system oscillations T_n (Unbehauen and Rao, 1990). Table I summarizes the parameter estimation obtained with the noise to signal ratio that is placed within an assumed range of 10-25%. Note that in this study the θ is a mean value over about 50 Monte Carlo runs. These results allow to obtain some idea on the sensitivity to noise of the identification method presented here. The parameter error norm is indicative of the accuracy of estimation. In general the results show that when the noise increases the accuracy of estimation becomes worse.

The parameter error norm has a large value for small sample size $N = 1000$, but the error norm is improved greatly when the number of samples (N) increases. Indeed, the associated error norm do not exceed 4% even at a considerably high level of the noise (N/S = 25%). In addition, the performance of the identification algorithm becomes more sensitive to the choice of T. A too small T may give erroneous estimates if the number of data points is

TABLE I - RESULTS OF PARAMETER ESTIMATION WITH NOISE

| N	N/S (%)	R (Ω)	L (H)	C (F)	Error Norm
		Mean estimated value $\pm$ standard deviation			
1000	10	4.8376 ± 0.2585	0.0223 ± 0.0039	0.4580 ± 0.0688	0.0334
	15	4.6642 ± 0.6648	0.0224 ± 0.0044	0.4587 ± 0.0743	0.0673
	20	4.4906 ± 0.4486	0.0241 ± 0.0063	0.4378 ± 0.0958	0.1021
	25	3.9253 ± 1.2408	0.0250 ± 0.0068	0.4130 ± 0.1178	0.2146
2000	10	4.9062 ± 0.1641	0.0209 ± 0.002	0.4800 ± 0.0398	0.0191
	15	4.8654 ± 0.2478	0.0212 ± 0.0025	0.4749 ± 0.0470	0.0273
	20	4.8064 ± 0.3019	0.0213 ± 0.0027	0.4719 ± 0.0500	0.0389
	25	4.6461 ± 0.6147	0.0223 ± 0.0037	0.4565 ± 0.0636	0.0710
5000	10	4.9732 ± 0.0536	0.0201 ± 0.0007	0.4940 ± 0.0153	0.0055
	15	4.9710 ± 0.0899	0.0204 ± 0.0011	0.4882 ± 0.0229	0.0062
	20	4.9147 ± 0.1383	0.0210 ± 0.0016	0.4755 ± 0.0339	0.0177
	25	4.8091 ± 0.2574	0.0216 ± 0.0023	0.4645 ± 0.0432	0.0386
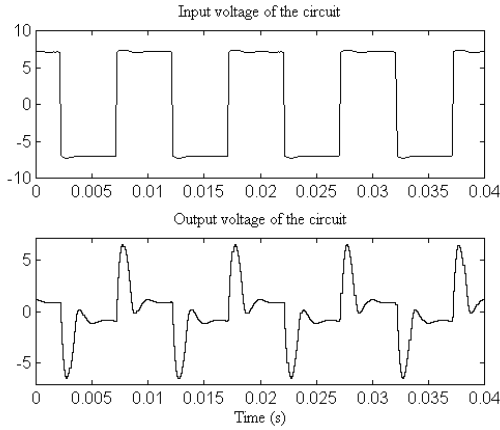
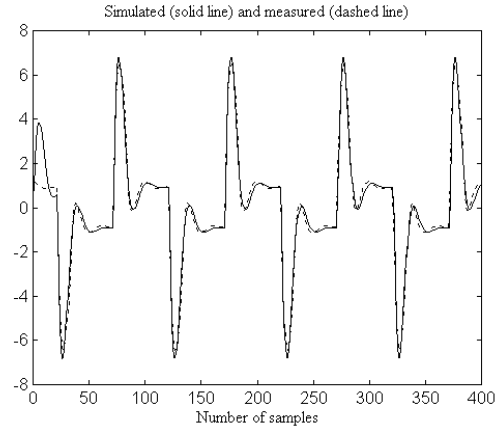Fig. 2. Input and output signals used in experiments.

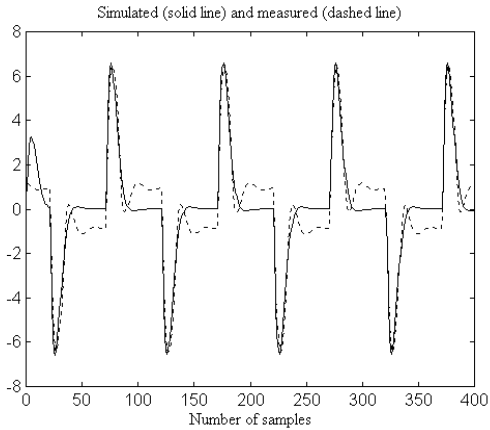Fig. 4. The outputs measured and estimated for experiment 2.

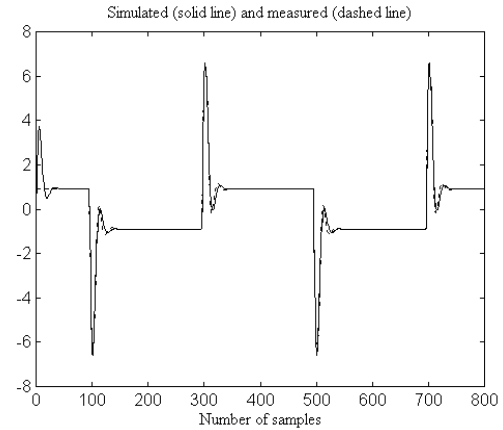Fig. 3. Comparison of output measured with estimated for experiment 1.

Fig. 5. Comparison of output measured with estimated for experiment 3.

not chosen appropriately. As it is known, the increase of N implies that more information about the system is given to the estimator, thus improving the parameter estimates. Another possible observation from table I is that in all case it is possible to obtain reasonable estimates of parameters in the presence of moderate noise.

V. EXPERIMENTAL RESULTS

As a demonstration of the practical application of the method presented here, we estimate the physical parameters of the transfer function of an experimental electrical circuit, which is composed of resistor, inductor and capacitor. The circuit components are measured as R= 987 Ω, L= 0.389 H and C= 0.499 μF. The system inputs for two runs are square wave signals with amplitude of 7.5 V and frequencies 100 and 25 Hz, which are produced by signal generator.

A personal computer with a Keithley DAS-1602/CE board is used for data acquisition. For the A/D converters, the full range of the analog signal is ±10V. The resolution of these converters is 12 bits. Except for the quantization error in the data acquisition, no extra noise is introduced in this system. The sampling frequency is 10 kHz. Table II summarizes the results of the parameter estimation in the three experiments for this electrical circuit. In the first experiment (1) we used a model with three physical parameters. In the other two experiments (2 and 3) we use a model with four parameters in order to account for the resistance of the inductor.

Figure 2 shows the input and output signals measured with the data acquisition system used in experiments 1 and 2. A third experiment was done with a change of the signal frequency. A standard method for model validation is to simulate the system with the actual input and compare the simulated and measured outputs.

Experiment 1- Model with three theoretical components:

TABLE II - ESTIMATED PARAMETERS OF RLC CIRCUIT

Experiment	R (Ω)	L (H)	C (F)	Error Norm
1	668.7	0.436	3.604e-7	0.3225
2	1012	0.364	4.027e-7	0.0255
3	969.5	0.371	4.302e-7	0.0177

In this experiment we consider that real components can be represented by their essential ideal components (*R*, *L* and *C*). From the plots in figure 3 we see that the error in estimation is quite large. Because of the modeling error, i.e. the erroneous theoretical components assumption, the chosen discrete-time model is no longer optimal within the discrete model set. Hence, when the indirect identification method is applied severe errors in the estimated physical parameters are found in the estimation procedure. For this experiment the error norm defined above was 32.3%.

Experiment 2 - Model with four components: The aim of this experiment is to show that when we consider that the inductor also has a series resistance the accuracy of parameter estimation its rather good. In this case the frequency of the input signal is 100Hz. In fact, all real components are not ideal, and their characteristics deviate more or less from the theoretical ones. However, the main difference between real and ideal components in our circuit arises from the inductor. From the plots in figure 4 we conclude that we have a good agreement between the model and the real system. Consequently we have a clear indication that the model has picked up the essential features of the system and is able to reproduce the input-output behavior quite well.

From the discrete-time parameters, the physical parameters can be reconstructed univocally. In this case the error norm is 2.5%.

Experiment 3 - Model with four components: In this case the signal input frequency is 25 Hz with the objective of observing the influence of the input frequency on the identifiability. From the plots in figure 5 we conclude that we have excellent agreement between the model and real system. Note that in this case the error norm is 1.77%, which means that the identifiability of parameters has improved compared to experiment 2.

VI. CONCLUSION

We have studied the problem of parameter estimation in a continuous-time linear system based on an indirect approach. This methodology involves firstly the identification of the parameters of a discrete-time model and secondly a transformation of the corresponding parameters into a continuous-time model. The novel methodology proposed in this paper requires the implementation of a polynomial filter that operates on the input data sequence allowing to establish an inverse function with an unique solution. Numerical test of the identification algorithm exhibits a good accuracy in the presence of Gaussian noise. The experimental results confirm the properties of the proposed approach.

VII. REFERENCES

[1] P Young. "Parameter Estimation for continuous-time models - A survey,". *Automatica*, *17*, (1), 23-39, 1981.

[2] H. Unbehauen, and G. P. Rao, "A review of identification in continuous-time systems," *Annual Reviews in Control*, *22*, 145-171, 1998.

[3] N. K. Sinha and S. Puthenpura, "Choice of the sampling interval for the identification of continuous-time systems from samples of input/output data," *IEE proceedings*, *132*, Pt. D, (6), 263-267, 1985.

[4] N. K. Sinha, and G. J. Lastman, "Identification of continuous-time multivariable systems from sampled data," *Int. J. Control, 35*, (1), 117-126, 1982.

[5] G. Durgaprasad, G. P. Rao, A. Patra and S. Mukhopadhyyay, "Indirect methods of parameter estimation of discrete-time models," *Proceedings of IFAC Identification and System Parameter Estimation*, Budapest, Hungary, 1991.

[6] L. Ljung, *System Identification: Theory for the User*, 2nd edition, Prentice Hall, New Jersey, 1999.

[7] L. Ljung, *The System Identification Toolbox: The Manual*. The MathWorks Inc., 4th edition, Natick, MA, 1995.

[8] K. J.,Åström, P. Hagander and J. Sternby, "Zeros of sampled systems" *Automatica*, *20*, (1), 31-38, 1984.

[9] M. J. Blachuta, ," On zeros of transfer functions". *IEEE Trans. Automatic Control*, *44*, (6), 1229-1234, 1999.

[10] R. E. Araújo, *Modelação, identificação e controlo do motor de indução trifásico*, PhD Thesis (in Portuguese), FEUP, 2001.