COMPUTATION OF CONSERVATION LAWS IN OPTIMAL CONTROL

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Abstract. Making use of a computer algebra system, we define computational tools to identify symmetries and conservation laws in optimal control.

Key words: optimal control, calculus of variations, computer algebra systems, Noether's theorem, symmetries, conservation laws

1. Introduction

The optimal control problems are usually solved with the help of the celebrated Pontryagin maximum principle, that constitutes a generalization of the classic Euler-Lagrange and Weierstrass necessary optimality conditions of the calculus of variations. The method of finding exact optimal solutions via Pontryagin maximum principle is, generally speaking, nontrivial, and very difficult (or even impossible) to implement in practice. One way to address the problem is to obtain conservation laws, i.e., quantities which are preserved along the extremals of the problem. Such conservation laws can be used to simplify the problem [2]. The question is then the following: how to determine conservation laws? It turns out that the classical results of Emmy Noether of the calculus of variations [4], relating the existence of conservation laws with the existence of symmetries, can be generalized to the wider context of optimal control [1, 5], reducing the problem to the one of discovering the invariance-symmetries. The difficulty resides precisely in the determination of the variational symmetries. The theory requires calculations that tend to be tedious even for very simple problems with a linear control system. Therefore, it is of great practical usefulness to have at our disposal computational means for the automatic identification of the symmetries of the optimal control problems. We claim that Computer Algebra Systems are particularly suitable to handle the problem of determining symmetries and conservation laws in optimal control, since they can perform differentiations, simplifications, and solve differential equations, in a reliably way. We illustrate our approach with a concrete optimal control problem borrowed from the literature.

2. Conservation Laws in Optimal Control

The optimal control problem consists to minimize an integral functional,

$$I[\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot)] = \int_a^b L(t, \boldsymbol{x}(t), \boldsymbol{u}(t)) \, \mathrm{d}t \,, \tag{2.1}$$

subject to a control system described by ordinary differential equations,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{\varphi}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)) \,, \tag{2.2}$$

together with appropriate boundary conditions on the values of $\boldsymbol{x}(a)$ and $\boldsymbol{x}(b)$. The Lagrangian $L(\cdot, \cdot, \cdot)$ is a real function, assumed to be continuously differentiable in $[a, b] \times \mathbb{R}^n \times \mathbb{R}^m$; $t \in \mathbb{R}$ the independent variable; $\boldsymbol{x} \colon [a, b] \to \mathbb{R}^n$ the vector of state variables; $\boldsymbol{u} \colon [a, b] \to \Omega \subseteq \mathbb{R}^m$, Ω an open set, the vector of controls, assumed to be piecewise continuous functions; and

$$\boldsymbol{\varphi} \colon [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

the velocity vector, assumed to be a continuously differentiable vector function. We propose a computational method that permits to obtain conservation laws for a given optimal control problem. Our method is based in the version of Noether's theorem established in [1]. Let us consider a one-parameter group of $\mathbb{C}^1$ transformations $\boldsymbol{h}^s : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$,

$$\begin{aligned}\boldsymbol{h}^s(t, \boldsymbol{x}, \boldsymbol{u}, \psi_0, \boldsymbol{\psi}) = \big(h_t^s(t, \boldsymbol{x}, \psi_0, \boldsymbol{\psi}), \boldsymbol{h}_{\boldsymbol{x}}^s(t, \boldsymbol{x}, \psi_0, \boldsymbol{\psi}), \boldsymbol{h}_{\boldsymbol{u}}^s(t, \boldsymbol{x}, \boldsymbol{u}, \psi_0, \boldsymbol{\psi}), \\ \boldsymbol{h}_{\boldsymbol{\psi}}^s(t, \boldsymbol{x}, \boldsymbol{u}, \psi_0, \boldsymbol{\psi})\big) \,,\end{aligned} \tag{2.3}$$

which reduces to the identity transformation when the parameter s vanishes: $h_t^0 = t$, $\boldsymbol{h}_{\boldsymbol{x}}^0 = \boldsymbol{x}$, $\boldsymbol{h}_{\boldsymbol{u}}^0 = \boldsymbol{u}$, $\boldsymbol{h}_{\boldsymbol{\psi}}^0 = \boldsymbol{\psi}$. Associated with a one-parameter group of transformations (2.3), we introduce the infinitesimal *generators*

$$\begin{aligned} T(t, \boldsymbol{x}, \psi_0, \boldsymbol{\psi}) &= \left.\frac{\partial}{\partial s} h_t^s\right|_{s=0} , \qquad \boldsymbol{X}(t, \boldsymbol{x}, \psi_0, \boldsymbol{\psi}) = \left.\frac{\partial}{\partial s} \boldsymbol{h}_{\boldsymbol{x}}^s\right|_{s=0} , \\ \boldsymbol{U}(t, \boldsymbol{x}, \boldsymbol{u}, \psi_0, \boldsymbol{\psi}) &= \left.\frac{\partial}{\partial s} \boldsymbol{h}_{\boldsymbol{u}}^s\right|_{s=0} , \quad \boldsymbol{\Psi}(t, \boldsymbol{x}, \boldsymbol{u}, \psi_0, \boldsymbol{\psi}) = \left.\frac{\partial}{\partial s} \boldsymbol{h}_{\boldsymbol{\psi}}^s\right|_{s=0} . \end{aligned} \tag{2.4}$$

Emmy Noether was the first who established a relation between the existence of invariance transformations of the problem and the existence of conservation laws [4]. Since the work pioneered by Noether, several definitions of invariance have been introduced for the problems of the calculus of variations (see e.g. [4, 6]); and for the problems of optimal control (see e.g. [1, 5, 7]). All these definitions are given with respect to a one-parameter group of transformations (2.3). Although written in a different way one gets, in terms of the generators (2.4), a necessary and sufficient condition of invariance that, essentially, coincide to all those definitions. For this reason, here we define invariance directly in terms of the generators (2.4).

DEFINITION 1. ([1]) We say that an optimal control problem (2.1)-(2.2) is *invariant* under (2.4) or, equivalently, that (2.4) is a *symmetry* of the problem, if, and only if, condition

$$\frac{\partial H}{\partial t}T + \frac{\partial H}{\partial \boldsymbol{x}} \cdot \boldsymbol{X} + \frac{\partial H}{\partial \boldsymbol{u}} \cdot \boldsymbol{U} + \frac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} - \boldsymbol{\Psi}^{\mathsf{T}} \cdot \dot{\boldsymbol{x}} - \boldsymbol{\psi}^{\mathsf{T}} \cdot \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} + H\frac{\mathrm{d}T}{\mathrm{d}t} = 0 \quad (2.5)$$

holds, with H the Hamiltonian:

$$H(t, \boldsymbol{x}, \boldsymbol{u}, \psi_0, \boldsymbol{\psi}) = \psi_0 L(t, \boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{\psi}^{\mathsf{T}} \cdot \boldsymbol{\varphi}(t, \boldsymbol{x}, \boldsymbol{u})\,. \quad (2.6)$$

A symmetry is an intrinsic property of the optimal control problem (2.1)-(2.2) (an intrinsic property of the corresponding Hamiltonian (2.6)), and does not depend on the Pontryagin extremals. If one restricts attention to the quadruples $(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), \psi_0, \boldsymbol{\psi}(\cdot))$ that satisfy the Pontryagin maximum principle, one arrives to Noether's theorem.

Theorem 1. *(Noether's theorem) If* (2.4) *is a* symmetry *of problem* (2.1)-(2.2)*, then*

$$\boldsymbol{\psi}(t)^{\mathsf{T}} \cdot \boldsymbol{X}(t, \boldsymbol{x}(t), \psi_0, \boldsymbol{\psi}(t)) - H\big(t, \boldsymbol{x}(t), \boldsymbol{u}(t), \psi_0, \boldsymbol{\psi}(t)\big) \times T\big(t, \boldsymbol{x}(t), \psi_0, \boldsymbol{\psi}(t)\big) = \mathsf{const} \quad (2.7)$$

is a conservation law.

We remark that Noetherian conservation laws (2.7) only depend on the generators T and $\boldsymbol{X}$ of a symmetry $(T, \boldsymbol{X}, \boldsymbol{U}, \boldsymbol{\Psi})$ (2.4).

3. Computation of Conservation Laws

The conservation laws we are looking for are obtained substituting in (2.7) the components T and $\boldsymbol{X}$ of a symmetry of the problem. We develop the Maple procedure *Noether* to do such calculations for us. The input to this procedure is: the Lagrangian L and the velocity vector $\boldsymbol{\varphi}$, that define the optimal control problem (2.1)-(2.2) and the respective Hamiltonian H; and a symmetry, or a family of symmetries, obtained by means of our procedure *Symmetry*. The output of *Noether* is the correspondent conservation law (2.7). The reader can find the Maple files with the definitions of the *Symmetry* and *Noether* procedures, that constitute our Maple package, at http://www.mat.ua.pt/delfim/maple.htm.

The non-trivial part resides in the determination of the symmetries of the problem. Our algorithm for determining the infinitesimal generators (2.4) is based on the necessary and sufficient condition of invariance (2.5). The key to do the calculations consists in observing that when we substitute the Hamiltonian H and its partial derivatives in the invariance identity (2.5), then the condition become a polynomial in $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{\psi}}$, and one can equal the coefficients to zero. Let us see how it works in more detail.

Substituting H and its partial derivatives into (2.5), and expanding the total derivatives, one can write equation (2.5) as a polynomial in the $2n$ derivatives $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{\psi}}$:

$$\left(\frac{\partial H}{\partial t}T + \frac{\partial H}{\partial \boldsymbol{x}} \cdot \boldsymbol{X} + \frac{\partial H}{\partial \boldsymbol{u}} \cdot \boldsymbol{U} + \frac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} + H\frac{\partial T}{\partial t} - \boldsymbol{\psi}^{\mathsf{T}} \cdot \frac{\partial \boldsymbol{X}}{\partial t}\right) + \left(-\boldsymbol{\Psi}^{\mathsf{T}} + H\frac{\partial T}{\partial \boldsymbol{x}} - \boldsymbol{\psi}^{\mathsf{T}} \cdot \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}\right) \cdot \dot{\boldsymbol{x}} + \left(H\frac{\partial T}{\partial \boldsymbol{\psi}} - \boldsymbol{\psi}^{\mathsf{T}} \cdot \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\psi}}\right) \cdot \dot{\boldsymbol{\psi}} = 0 \,. \quad (3.1)$$

The terms in (3.1) which involve derivatives with respect to vectors are expanded in line-vectors or in matrices, depending, respectively, if the function is a scalar or a vectorial one.

Given an optimal control problem, defined by a Lagrangian L and a control system (2.2), we determine the infinitesimal generators T, $\boldsymbol{X}$, $\boldsymbol{U}$ and $\boldsymbol{\Psi}$, which define a symmetry for the problem, by the following method. Equation (3.1) is a differential equation in the $2n+m+1$ unknown functions T, $X_1, \ldots, X_n$, $U_1, \ldots, U_m$, $\Psi_1, \ldots$, and Ψ_n. This equation must hold for all $\dot{x}_1, \ldots, \dot{x}_n$, $\dot{\psi}_1, \ldots, \dot{\psi}_n$, and therefore all the coefficients of polynomial (3.1) must vanish.

The system of equations obtained from (3.1) is a system of $2n+1$ partial differential equations with $2n+m+1$ unknown functions (so, in general, there exists not a unique symmetry but a whole family of symmetries). Although a system of partial differential equations, its resolution is possible because the system is linear with respect to the unknown functions and their derivatives. However, when dealing with optimal control problems with several state and control variables, the number of calculations is big enough, and the help of the computer is more than welcome. Our Maple procedure *Symmetry* does all the job for us. Since system is homogeneous, we always have, as trivial solution, $(T, \boldsymbol{X}, \boldsymbol{U}, \boldsymbol{\Psi}) = \mathbf{0}$. When the output of *Symmetry* coincides with the trivial solution, that means that the optimal control problem does not admit a symmetry.

Summarizing: given an optimal control problem (2.1)-(2.2) we obtain conservation laws, in an automatic way, through two steps:

(i) with our procedure *Symmetry* we obtain the invariance symmetries of the problem;

(ii) using the obtained symmetries as input to procedure *Noether*, based on Theorem 1, we obtain the correspondent conservation laws.

4. An Illustrative Example

In order to show the functionality and the usefulness of the developed routines, we apply our Maple package to a concrete optimal control problem found in the literature. The calling syntaxes of our procedures are $Symmetry(L, \varphi, t, x, u)$ and $Noether(L, \varphi, t, x, u, S)$, with the respective inputs: Lagrangian L, velocity vector φ, name t of the independent variable, list of names x of the state variables, list of names u of the control variables, and the set S of infinitesimal generators for function $Noether$ (the output of $Symmetry$).

Example. Let us consider the following problem:

$$\int_a^b \left(u_1(t)^2 + u_2(t)^2\right) \mathrm{d}t \longrightarrow \min,$$

$$\begin{cases} \dot{x}_1(t) = u_1(t)\cos x_3(t), \\ \dot{x}_2(t) = u_1(t)\sin x_3(t), \\ \dot{x}_3(t) = u_2(t), \end{cases}$$

where the control system serves as model to the kinematics of a car [3, Example 18, p. 750]. In this case the optimal control problem has three state variables ($n = 3$) and two controls ($m = 2$). With Maple definitions

```
> L:=u[1]^2+u[2]^2; phi:=[u[1]*cos(x[3]),u[1]*sin(x[3]),u[2]];
```

$$L := {u_1}^2 + {u_2}^2$$
$$\varphi := [u_1 \cos(x_3), u_1 \sin(x_3), u_2]$$

our procedure *Symmetry* determines the infinitesimal invariance generators of the optimal control problem under consideration:

```
> Symmetry(L, phi, t, [x[1],x[2],x[3]], [u[1],u[2]]);
```

$$\{T = C_2,\ X_1 = -C_1 x_2 + C_3,\ X_2 = C_1 x_1 + C_4,\ X_3 = C_1,\ U_1 = 0,\ U_2 = 0,\ \Psi_1 = -C_1\psi_2,\ \Psi_2 = C_1\psi_1,\ \Psi_3 = 0\}$$

(C_i, $i = 1, \ldots, 4$, are arbitrary constants). The family of Conservation Laws associated with the generators just obtained is easily obtained through our procedure *Noether* (the sign of percentage (%) is an operator used in Maple to represent the result of the previous command):

```
> Noether(L, phi, t, [x[1],x[2],x[3]], [u[1],u[2]], %);
```

$$(-C_1 x_2(t) + C_3)\,\psi_1(t) + (C_1 x_1(t) + C_4)\,\psi_2(t) + C_1\psi_3(t) - \Big(\psi_0\left(u_1(t)^2 + u_2(t)^2\right) + u_1(t)\cos(x_3(t))\,\psi_1(t) + u_1(t)\sin(x_3(t))\,\psi_2(t) + u_2(t)\psi_3(t)\Big)C_2 = const$$

The obtained conservation law depends on four parameters. With the substitutions $C_1 = 1$ and $C_2 = C_3 = C_4 = 0$ we obtain the conservation law

```
> subs(C[1]=1,C[2]=0,C[3]=0,C[4]=0, %);
```

$$-x_2(t)\psi_1(t) + x_1(t)\psi_2(t) + \psi_3(t) = const$$

which corresponds to the symmetry group of planar (orientation-preserving) isometries given in [3, Example 18, p. 750].

We recall that any problem of the calculus of variations can always be rewritten as an optimal control problem, so we can also apply our Maple procedures *Symmetry* and *Noether* to obtain variational symmetries and conservation laws in the classical context of the calculus of variations.

Acknowledgments

This work was partially supported by the PhD grant PRODEP III/5.3/2003 of Paulo Gouveia, and by FCT via funds from the R&D unit CEOC.

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