RELATIVE ABELIAN KERNELS OF SOME CLASSES OF TRANSFORMATION MONOIDS

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ABSTRACT. We consider members of some well studied classes of finite transformation monoids and give descriptions of their abelian kernels relative to decidable pseudovarieties of abelian groups.

INTRODUCTION

The computability of kernels of finite monoids became a popular problem in Finite Semigroup Theory after a paper of Rhodes and Tilson [22]. Independent solutions were given by Ash [3] and Ribes and Zalesskiĭ [23]. Both solutions contains deep results which led to the development of the theory in various directions. See, for instance, [18, 2]. Computing kernels relative to other pseudovarieties has then interested various researchers. We can refer Ribes and Zalesskiĭ [24], the second author [4] or Steinberg [25]. As computing kernels is directly connected with the Mal'cev product of pseudovarieties of monoids where the second factor is a pseudovariety of groups (see [17]), decidability results can be obtained by this way.

The interest of the second author in using in practice the algorithm to compute the abelian kernel of a finite monoid obtained in [4] led him to work towards a concrete implementation. Having this idea in mind, by detailing some aspects and improving the efficiency of that algorithm (see [5]) a concrete implementation in GAP [26] was then produced.

Fruitful computations have afterwards been achieved by the second and third authors [6, 8] when considering some classes of monoids for which the third author [13, 14, 15] and Gomes, Jesus and the third author [16] gave extremely simple presentations.

We observe that the interest of this kind of computations goes far beyond the computations themselves. For example, they have been at the basis of theoretical results such as those obtained by the second and third authors in [7], which extend the notion of solvability of groups, and later generalized with the collaboration of Margolis and Steinberg [9].

In this work we consider kernels relative to decidable pseudovarieties of abelian groups. An algorithm was obtained by Steinberg [25], having the first and second authors [?] described the necessary details to achieve an implementation using GAP. The implementation used to perform tests that led us to get a better intuition and, ultimately, to the results presented in

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this paper, was achieved by the second author[1] using the GAP language. The GAP package SgpViz [10] has been very useful to visualize the results.

Besides this brief introduction, this paper is divided into four main sections. A fifth section containing a few consequences and conjectures is also included.

The first section contains the preliminaries and is divided into several subsections. Notation and definitions are given. Several results are recalled and some others are new and proved here. These results are to be used in Sections 2 and 4.

In Section 2 we compute the relative abelian kernels of finite cyclic, dihedral and symmetric groups. These are the maximal subgroups of the monoids considered in this paper.

Aiming to keep the paper as self-contained as possible, we present in Section 3 a summary of the results, which are relevant here, on presentations of the transformation monoids $\mathcal{POI}_n$, $\mathcal{PODI}_n$, $\mathcal{POPI}_n$, $\mathcal{PORI}_n$ and $\mathcal{I}_n$.

In Section 4 we give descriptions of the kernels relative to all decidable pseudovarieties of abelian groups of the transformation monoids considered in Section 3.

1. Preliminaries

A pseudovariety of groups is a class of finite groups closed under formation of finite direct products, subgroups and homomorphic images. In this paper we are particularly interested in the class Ab of all finite abelian groups, which is clearly a pseudovariety, and its subpseudovarieties.

It is well known that the computation of the kernel of a finite semigroup relative to a pseudovariety of groups can be reduced to the computation of the kernel of a finite monoid relative to the same pseudovariety of groups, so we will mainly be concerned with monoids, as is usual.

For general background on Green relations and inverse semigroups, we refer the reader to Howie's book [19]. For general notions on profinite topologies and finite semigroups, Almeida's book [1] is our reference.

This section is divided into several subsections. The first one recalls a connection between supernatural numbers and pseudovarieties of abelian groups. Then we introduce some notation. The third subsection recalls an algorithm due to Steinberg [25] to compute the closure of a subgroup of a finitely generated free abelian group relative to the profinite topology of the free abelian group given by a decidable pseudovariety of abelian groups. The definition of relative kernel of a finite monoid is recalled in Subsection 1.4. In Subsection 1.5 we prove two general results that will be applied in Section 2 to the groups considered in this paper. In Subsection 1.6 we prove a combinatorial result that will be used in Subsection 4.4 to describe the relative abelian kernels of the monoids $\mathcal{PORI}_n$.

1.1. Supernatural numbers and pseudovarieties of abelian groups.

A *supernatural number* is a formal product of the form $\pi = \Pi_p p^{n_p}$ where p runs over all natural prime numbers and $0 \leq n_p \leq +\infty$.

We say that a supernatural number Πp^{n_p} has *finite support* if all n_p, except possibly a finite number, are zero. There is an evident notion of division of supernatural numbers. It

[1] The second author wishes to thank José Morais for his help in programming some parts which led to much more efficient implementations.

leads to the also evident notion of greatest common divisor of two supernatural numbers. A supernatural number is said to be *recursive* if the set of all natural numbers which divide it is recursive. In particular, supernatural numbers of finite support are recursive. A supernatural number Πp^{n_p} of finite support such that all the exponents n_p are finite is said to be *finite* and is naturally identified with a positive natural number. All other supernatural numbers are said to be *infinite*. In general we use the Greek letter π for a (possibly infinite) supernatural number, but when the supernatural number is known to be finite, we prefer to use a roman letter, e.g. k. We say that the supernatural number π is *odd* when $gcd(2, \pi) = 1$ and that it is *even* otherwise, that is, when $gcd(2, \pi) = 2$.

To a supernatural number π we associate the pseudovariety of abelian groups H_π of all finite abelian groups whose torsion coefficients divide π (i.e. $\mathsf{H}_\pi = \langle\{\mathbb{Z}/m\mathbb{Z} \mid m \text{ divides } \pi\}\rangle$). See Steinberg's paper [25] for details. Notice that the pseudovariety of abelian groups associated to a natural number k is just $\langle\mathbb{Z}/k\mathbb{Z}\rangle$, the pseudovariety generated by the cyclic group of order k. In particular, the pseudovariety of groups corresponding to the natural number 1 is the trivial pseudovariety. Observe that to the supernatural number Πp^∞, where p runs over all positive prime numbers, is associated the pseudovariety Ab of all finite abelian groups. Observe also that decidable pseudovarieties of abelian groups correspond to recursive supernatural numbers and that the converse is also true. All supernatural numbers considered in this paper are recursive.

For a pseudovariety H of groups and a finite set A, we denote by $F_\mathsf{H}(A)$ the relatively free group on A in the variety of groups (in the Birkhoff sense) generated by H.

Let π be a supernatural number and H_π the corresponding pseudovariety of abelian groups. The following holds:

Proposition 1.1. [25] *Let n be a positive integer and let A be a finite set of cardinality n. Then if π is a natural number, $F_{\mathsf{H}_\pi}(A) = (\mathbb{Z}/\pi\mathbb{Z})^n$. Otherwise, that is, when π is infinite, $F_{\mathsf{H}_\pi}(A) = \mathbb{Z}^n$, the free abelian group on n generators.*

It turns out that the pseudovarieties of abelian groups corresponding to natural numbers are locally finite, while those corresponding to infinite supernatural numbers are non locally finite. The relatively free groups appearing in the last proposition will be turned into topological spaces, the finite ones being discrete.

1.2. **Notation.** Throughout the paper n will denote a positive integer. Without surprise, after Proposition 1.1, the free abelian group $\mathbb{Z}^n$ plays a fundamental role here.

In order to render our notation more understandable, we will use subscripts in some components of the elements of $\mathbb{Z}^n$. For instance, we write $(0, \ldots, 0, 1_{(i)}, 0, \ldots, 0)$ with the meaning of "$(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 is in the position i)". We adopt the usual notation for the neutral element of an abelian group: $(0, \ldots, 0) \in \mathbb{Z}^n$ is simply denoted by 0. The set of non-negative integers, also named natural numbers, is a monoid under addition and is denoted by $\mathbb{N}$.

Let $A = \{a_1, \ldots, a_n\}$ be a finite ordered alphabet. The canonical homomorphism $\gamma : A^* \to \mathbb{Z}^n$, from the free monoid on A into the n-generated free abelian group, defined by $\gamma(a_i) = (0, \ldots, 0, 1_{(i)}, 0, \ldots, 0)$ will be widely used (for the alphabets in the context). The image under γ of a rational language of A^* is a rational subset of $\mathbb{N}^n$ ($\subseteq \mathbb{Z}^n$), thus it is a semilinear set, i.e. a finite union of sets of the form $a + b_1\mathbb{N} + \cdots + b_r\mathbb{N}$. There exists an algorithm (see, for instance, [4, 5]) to compute (a semilinear expression for) the image of a rational language of A^* by γ, when it is given by means of a rational expression.

Suppose that M is an A-generated finite monoid and let $\varphi : A^* \to M$ be an onto homomorphism. We say that an element $x \in M$ *can be represented* by a word $w \in A^*$ (or that w is a representation of x) if $\varphi(w) = x$.

When a monoid M is given through a monoid presentation $\langle A \mid R\rangle$, we always assume that A is an ordered alphabet and φ denotes the homomorphism $\varphi : A^* \to M$ associated to the presentation. Thus, a presentation of a monoid determines two homomorphisms which will be represented throughout the paper by the Greek letters φ and γ.

1.3. **Topologies for the free abelian group.** Let π be an infinite supernatural number and H_π the corresponding pseudovariety of abelian groups.

The pro-H_π topology on $\mathbb{Z}^n$ is the least topology rendering continuous all homomorphisms of $\mathbb{Z}^n$ into groups of H_π. The free abelian group $\mathbb{Z}^n$ endowed with this topology is a topological group. When $\mathsf{H}_\pi = \mathsf{Ab}$, the pseudovariety of all finite abelian groups, the pro-H_π topology is usually referred simply as *profinite* topology.

For a subset X of $\mathbb{Z}^n$, we denote by $\mathsf{Cl}_{\mathsf{H}_\pi}(X)$ the pro-H_π closure of X. The pro-Ab closure will in general be referred as the *profinite* closure. The following holds [4].

Proposition 1.2. *For $a, b_1, \ldots, b_r \in \mathbb{N}^n$, the profinite closure of the subset $a+b_1\mathbb{N}+\cdots+b_r\mathbb{N}$ of $\mathbb{Z}^n$ is $a + b_1\mathbb{Z} + \cdots + b_r\mathbb{Z}$.* □

Next we recall an algorithm to compute the pro-H_π closure of a subgroup G of the free abelian group $\mathbb{Z}^n$. See [**?**, 25] for details.

Let M be a matrix whose rows generate G. We say that M *represents* G. Notice that, by adding rows of zeros when necessary, we may suppose that M is an $n \times n$ matrix.

Algorithm 1.3. INPUT: *a subgroup G of $\mathbb{Z}^n$ given through an integer $n \times n$ matrix M.*
OUTPUT: *a matrix representing the pro-H_π closure of G.*

(1) *Compute invertible integer matrices P and Q such that*

$$PMQ = S = \begin{bmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & a_n \end{bmatrix}$$

is a diagonal matrix. Notice that if, for $1 \le i \le n-1$, $a_i | a_{i+1}$, then the matrix S is in Smith Normal Form.

(2) *For each a_i, compute $b_i = gcd(a_i, \pi)$ (note that we are assuming that π is recursive) and consider the matrix*

$$\overline{S} = \begin{bmatrix} b_1 & 0 & 0 & \ldots & 0 \\ 0 & b_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & b_n \end{bmatrix}.$$

(3) *Return the matrix $\overline{S}Q^{-1}$.*

The matrix returned represents the pro-H_π closure of G.

Next example illustrates the usage of the algorithm. It will be referred in Section 2.2.

Example 1.4. (1) Let $G = n\mathbb{Z}$. Then $\mathsf{Cl}_{\mathsf{H}_\pi}(G) = gcd(n,\pi)\mathbb{Z}$. More generally, let G be a subgroup of $\mathbb{Z}^n$ represented by a diagonal matrix. A matrix representing the pro-H_π closure of G can be obtained from M by replacing each nonzero entry a with $gcd(a,\pi)$.

(2) Let G be the subgroup of $\mathbb{Z}^2$ represented by the matrix $M = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. Then

$$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

In order to use the notation of the algorithm just stated, we write $S = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. We thus have $Q^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

If π is even, then $gcd(2,\pi) = 2$. So, in this case, $\overline{S} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathsf{Cl}_{\mathsf{H}_\pi}(G)$ can be represented by

$$\overline{S}Q^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

and therefore $\mathsf{Cl}_{\mathsf{H}_\pi}(G) = G$.

If π is odd, then $gcd(2,\pi) = 1$ and we get $\overline{S}Q^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Thus $\{(1,1),(0,1)\}$ is a basis of the pro-H_π closure of G and it follows that $\mathsf{Cl}_{\mathsf{H}_\pi}(G) = \mathbb{Z}^2$.

1.4. **Relative kernels of finite monoids.** Let M and N be finite monoids. A *relational morphism of monoids* $\tau : M \longrightarrow\!\!\!\!\!\!\circ\;\; N$ is a function from M into $\mathcal{P}(N)$, the power set of N, such that:

(a) For all $s \in M$, $\tau(s) \neq \emptyset$;
(b) For all $s_1, s_2 \in M$, $\tau(s_1)\tau(s_2) \subseteq \tau(s_1 s_2)$;
(c) $1 \in \tau(1)$.

A relational morphism $\tau : M \longrightarrow\!\!\!\!\!\!\circ\;\; N$ is, in particular, a relation in $M \times N$. Thus, composition of relational morphisms is naturally defined. Homomorphisms, seen as relations, and inverses of onto homomorphisms are examples of relational morphisms.

Given a pseudovariety H of groups, the H-*kernel of a finite monoid* M is the submonoid $\mathsf{K}_{\mathsf{H}}(M) = \bigcap \tau^{-1}(1)$, with the intersection being taken over all groups $G \in \mathsf{H}$ and all relational morphisms of monoids $\tau : M \longrightarrow\!\!\!\!\!\!\circ\;\; G$. Sometimes we refer to the H-kernel simply as *relative kernel*. When H is Ab, we use the terminology *abelian kernel*. Accordingly, when H is a subpseudovariety of Ab, we say *relative abelian kernel*.

As an example we can state the following proposition which was proved in a slightly more general form in [4]. See also [6].

Proposition 1.5. *The abelian kernel of a finite group is precisely its derived subgroup.* □

All statements of the following remark follow directly from the definition or are easy to prove (and well known, in any case).

Remark 1.6. Let H and H' be pseudovarieties of groups and let M be a finite monoid. Then

(1) If $\mathsf{H} \subseteq \mathsf{H}'$, then $\mathsf{K}_{\mathsf{H}'}(M) \subseteq \mathsf{K}_{\mathsf{H}}(M)$.
(2) If N is a subsemigroup of M that is a monoid, then $\mathsf{K}_{\mathsf{H}}(N) \subseteq \mathsf{K}_{\mathsf{H}}(M) \cap N$.
(3) $\mathsf{K}_{\mathsf{H}}(M)$ contains the idempotents of M.

When the subsemigroup of M mentioned in the second statement of previous remark is the group of units, the inclusion can be replaced by an equality, as stated in the following lemma.

Lemma 1.7. [6, Lemma 4.8] *Let G be the group of units of a finite monoid M. Then $\mathsf{K}_{\mathsf{H}}(M) \cap G = \mathsf{K}_{\mathsf{H}}(G)$, for any pseudovariety of groups H. In particular, $\mathsf{K}_{\mathsf{Ab}}(M) \cap G = G'$.* □

As a consequence of Proposition 1.5 and the first step of Remark 1.6 we get the following:

Corollary 1.8. *Any relative abelian kernel of a finite group contains its derived subgroup.* □

Let π be an even supernatural number. It is clear that $\mathsf{H}_2 \subseteq \mathsf{H}_\pi$, thus, for any finite monoid M, $\mathsf{K}_{\mathsf{Ab}}(M) \subseteq \mathsf{K}_{\mathsf{H}_\pi}(M) \subseteq \mathsf{K}_{\mathsf{H}_2}(M)$. Consequently we have the following:

Proposition 1.9. *Let M be a finite monoid such that $\mathsf{K}_{\mathsf{H}_2}(M) = \mathsf{K}_{\mathsf{Ab}}(M)$. Then, for any even supernatural number π, $\mathsf{K}_{\mathsf{H}_\pi}(M) = \mathsf{K}_{\mathsf{Ab}}(M)$.* □

1.5. **Some results concerning relative abelian kernels.** Let π be an infinite supernatural number, let H_π be the corresponding pseudovariety of abelian groups and let M be a finite A-generated monoid. The following result was proved by the second author [4] for the case $\mathsf{H}_\pi = \mathsf{Ab}$ and generalized by Steinberg [25] to cover all other cases.

Proposition 1.10. *Let $x \in M$. Then $x \in \mathsf{K}_{\mathsf{H}_\pi}(M)$ if and only if $0 \in \mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x)))$.*

Next we recall some facts proved by the first and second authors [?]. In order to compute $\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x)))$ we can calculate $\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x)))$ in an intermediate step, as the next result shows.

Proposition 1.11. *Let $x \in M$. Then $\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x))) = \mathsf{Cl}_{\mathsf{H}_\pi}(\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x))))$.*

Now let k be a finite supernatural number and H_k be the corresponding pseudovariety of abelian groups. We consider the projection $c_k : \mathbb{Z}^n \to (\mathbb{Z}/k\mathbb{Z})^n$ (defined by: $c_k(r_1, \dots, r_n) = (r_1 \mod k, \dots, r_n \mod k)$) and the homomorphism $\gamma_k = c_k \circ \gamma : A^* \to (\mathbb{Z}/k\mathbb{Z})^n$. Note that for a word $w \in A^*$, the i^{th} component of $\gamma_k(w)$ is the number of occurrences modulo k of the i^{th} letter of A in w.

Next proposition is similar to Proposition 1.11. It allows us to compute $\gamma_k(\varphi^{-1}(x))$ using $\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x)))$ in an intermediate step.

Proposition 1.12. *Let $x \in M$. Then $\gamma_k(\varphi^{-1}(x)) = c_k(\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x))))$.*

An analogous to Proposition 1.10 was also stated in [?]:

Proposition 1.13. *Let $x \in M$. Then $x \in \mathsf{K}_{\mathsf{H}_\mathsf{k}}(M)$ if and only if $0 \in \gamma_k(\varphi^{-1}(x))$.*

The following is another simple and useful characterization of the H_k-kernel proved in the same paper.

Proposition 1.14. *Let $x \in M$. Then $x \in \mathsf{K}_{\mathsf{H}_\mathsf{k}}(M)$ if and only if x can be represented by a word $w \in A^*$ such that, for any letter $a \in A$, $|w|_a \equiv 0 \mod k$.*

Note that, as a consequence, we get that if $x \in \mathsf{K}_{\mathsf{H_k}}(M)$, then x can be represented by a word whose length is a multiple of k. The following result and its proof is similar to [8, Theorem 3.4].

Theorem 1.15. *Let T be a monoid, let $x_1, x_2, \ldots, x_s, y$ be a set of generators of T such that $y^k = 1$ and let S be the submonoid of T generated by $x_1, x_2, \ldots, x_s$. If for each $i \in \{1, \ldots, s\}$ there exists $u_i \in S$ such that $yx_i = u_i y$, then $\mathsf{K}_{\mathsf{H_k}}(T) \subseteq S$.* □

Proof. Let $x \in \mathsf{K}_{\mathsf{H_k}}(T)$. Then, by Proposition 1.14, we can represent x as a word $w \in \{x_1, x_2, \ldots, x_s, y\}^*$ with a number of occurrences of the generator y that is a multiple of k. Then, applying the relations $yx_i = u_i y$, $i \in \{1, \ldots, s\}$ from the left hand side to the right hand side of w and the relation $y^k = 1$ whenever possible, we can represent x without any occurrence of y, whence $x \in S$. □

Another important result of this subsection is Proposition 1.17 that gives a simple and efficient way to compute the $\mathsf{H_k}$-kernel of a finite group. In order to state it, we need to introduce a subgroup of a group containing the derived subgroup.

Given a finite group G, denote by $G^{[k]}$ the subgroup of G generated by the commutators of G (that is, the elements of the form $xyx^{-1}y^{-1}$, $x, y \in G$) and by the k-powers of G (that is, the elements of the form x^k, $x \in G$). In other words, $G^{[k]}$ is the subgroup of G containing the derived subgroup G' and the k-powers.

Lemma 1.16. *The subgroup $G^{[k]}$ is normal in G.*

Proof. Let G_1 and G_2 be finite groups and let $\varphi : G_1 \to G_2$ be a homomorphism. We have that $\varphi(xyx^{-1}y^{-1}) = \varphi(x)\varphi(y)\varphi(x^{-1})\varphi(y^{-1}) = \varphi(x)\varphi(y)(\varphi(x))^{-1}(\varphi(y))^{-1}$ and $\varphi(x^k) = (\varphi(x))^k$, thus $\varphi({G_1}^{[k]}) \subseteq {G_2}^{[k]}$.

In particular, taking the inner automorphism $\varphi_g : G \to G$ defined by $\varphi_g(x) = gxg^{-1}$, we have that, for any $g \in G$, $gG^{[k]}g^{-1} \subseteq G^{[k]}$, concluding the proof. □

Since $G' \subseteq G^{[k]}$, we have that the factor group $G/G^{[k]}$ is abelian. Furthermore, the order of any element $xG^{[k]}$ of $G/G^{[k]}$ divides k, which implies that $G/G^{[k]} \in \mathsf{H_k}$.

Proposition 1.17. *For a finite group G, we have: $\mathsf{K}_{\mathsf{H_k}}(G) = G^{[k]}$.*

Proof. The projection $p : G \to G/G^{[k]}$ is such that $p^{-1}(1) = G^{[k]}$, thus $\mathsf{K}_{\mathsf{H_k}}(G) \subseteq G^{[k]}$.

Conversely, by Corollary 1.8, $G' \subseteq G^{[k]}$. Since the elements of the form x^k may be written involving each generator of G a multiple of k times, the result follows from Proposition 1.14. □

1.6. **A combinatorial result.** We end the section of preliminaries with a combinatorial result (Proposition 1.22) to be used in Section 4.3.

Let j be an integer. Consider the set

$$B(j) = \{r(n-1) - i + j \mid (i, r) \in \{1, \ldots, n-1\} \times \{0, \ldots, n-1\}\}.$$

Lemma 1.18. *$B(j)$ consists of $n(n-1)$ consecutive integers.*

Proof. Denote by m and M, respectively, the minimum and the maximum of $B(j)$. Clearly $m = 0(n-1) - (n-1) + j$ and $M = (n-1)(n-1) - 1 + j$. As $M - m + 1 = n(n-1)$, it suffices to prove $B(j)$ has cardinality $n(n-1)$ to conclude that all integers between m and M belong to $B(j)$.

On the other hand, as $n(n-1)$ is precisely the cardinality of $\{1,\ldots,n-1\}\times\{0,\ldots,n-1\}$, to prove that $B(j)$ has also cardinality $n(n-1)$ it is enough to observe that, for $(i,r),(i',r')\in\{1,\ldots,n-1\}\times\{0,\ldots,n-1\}$, if $r(n-1)-i+j=r'(n-1)-i'+j$, then $i=i'$ and $r=r'$. Now observe that $-(n-1)<i-i'<n-1$ and $r(n-1)-i+j=r'(n-1)-i'+j \Leftrightarrow (r-r')(n-1)-(i-i')=0$. But this implies that $r=r'$, from what follows that also $i=i'$. □

As an immediate consequence of the preceding lemma we get the following:

Corollary 1.19. *If d is a divisor of $n(n-1)$, then $B(j)$ contains $n(n-1)/d$ multiples of d.*

The proof of Lemma 1.18 has also the following corollary as an immediate consequence.

Corollary 1.20. *The function $g:\{1,\ldots,n-1\}\times\{0,\ldots,n-1\}\to B(j)$ defined by $g(i,r)=r(n-1)-i+j$ is a bijection.*

Let $X=\{1,\ldots,n-1\}\times\{1,\ldots,n\}\times\{0,\ldots,n-1\}$ and suppose now that $1\le j\le n$. Denote by $\mathcal{U}=\cup_{j\in\{1,\ldots,n\}}B(j)\times\{j\}$ the disjoint union of the $B(j)$'s. From the preceding corollary we get immediately the following:

Corollary 1.21. *The function $f:X\to\mathcal{U}$ defined by $f(i,j,r)=(r(n-1)-i+j,j)$ is a bijection.*

Denote by $A(n,d)$ the set of elements of X corresponding, via the bijection of the preceding corollary, to elements of $\mathcal{U}$ whose first component is a multiple of d, that is

$$A(n,d)=\{(i,j,r)\in X : d \mid (r(n-1)+j-i)\}.$$

As a consequence of Corollary 1.19 we get the main result of this section.

Proposition 1.22. *If d is a divisor of $n(n-1)$, then $|A(n,d)|=n^2(n-1)/d$.* □

2. Relative abelian kernels of some finite groups

As already observed, along this paper, cyclic, dihedral and symmetric groups appear frequently. This section provides the computations of relative abelian kernels of finite groups of these kinds. We will consider them as given by the monoid presentations of the following example. In this section n is a positive integer.

Example 2.1. (1) $C_n=\langle g \mid g^n=1\rangle$ (the cyclic group of order n);
(2) $D_{2n}=\langle h,g \mid h^2=g^n=hg^{n-1}hg^{n-1}=1\rangle$ (the dihedral group of order $2n$);
(3) $\mathcal{S}_n=\langle a,g \mid a^2=g^n=(ga)^{n-1}=(ag^{n-1}ag)^3=(ag^{n-j}ag^j)^2=1 \ \ (2\le j\le n-2)\rangle$ (the symmetric group on a base set with n elements).

The presentation given for $\mathcal{S}_n$ requires some words. Consider the transposition $a=(1\,2)$ and the n-cycle $g=(1\,2\,\cdots\,n)$ of $\mathcal{S}_n$. Then, it is well known that $\{a,g\}$ is a set of generators of $\mathcal{S}_n$ and, from a group presentation due to Moore [20], one can easily deduce [15] the monoid presentation for $\mathcal{S}_n$ given in the above example.

As Proposition 1.10 (together with Proposition 1.11 or even Proposition 1.12) makes clear, one possible strategy to compute relative abelian kernels is to start computing profinite closures. This is what we do in this section for the cases of kernels relative to pseudovarieties corresponding to infinite supernatural numbers.

2.1. **Profinite closures.** In this subsection the n-generated free abelian group $\mathbb{Z}^n$ is considered endowed with the profinite topology.

Let $G = \langle A \mid r_1 = r_2 = \cdots = r_s = 1\rangle$ be a monoid presentation of a finite group. As any word in $\varphi^{-1}(1)$ can be obtained from the empty word by inserting or removing some relators, we have that $\gamma(r_1)\mathbb{N} + \cdots + \gamma(r_s)\mathbb{N} \subseteq \gamma(\varphi^{-1}(1)) \subseteq \gamma(r_1)\mathbb{Z} + \cdots + \gamma(r_s)\mathbb{Z}$. Using Proposition 1.2 we have the following lemma.

Lemma 2.2. *With the above notation,* $\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(1))) = \gamma(r_1)\mathbb{Z} + \cdots + \gamma(r_s)\mathbb{Z}$, *that is,* $\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(1)))$ *is the subgroup of* $\mathbb{Z}^n$ *generated by* $\gamma(r_1), \gamma(r_2), \ldots, \gamma(r_s)$.

Next result shows, in particular, that to compute $\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x)))$, where x is an element of the group G, it is not important which representative of x is used.

Lemma 2.3. *Let* $w_x \in A^*$ *be a representative of* $x \in G$. *Then*

$$\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x))) = \gamma(w_x) + \mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(1))).$$

Proof. Observe that $\gamma(w_x) + \mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(1))) = \mathsf{Cl}_{\mathsf{Ab}}(\gamma(w_x) + \gamma(\varphi^{-1}(1)))$, since addition in $\mathbb{Z}^n$ is continuous. As $w_x\varphi^{-1}(1) \subseteq \varphi^{-1}(x)$, we have that $\gamma(w_x) + \gamma(\varphi^{-1}(1)) \subseteq \gamma(\varphi^{-1}(x))$, which implies that $\gamma(w_x) + \mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(1))) \subseteq \mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x)))$.

For the reverse inclusion it suffices to observe that $\gamma(\varphi^{-1}(x)) \subseteq \gamma(w_x) + \mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(1)))$. But this is immediate, since any word w representing x can be obtained from w_x by inserting or removing the relators a finite number of times, thus $\gamma(w) \in \gamma(w_x) + \gamma(r_1)\mathbb{Z} + \cdots + \gamma(r_s)\mathbb{Z} = \gamma(w_x) + \mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(1)))$. $\square$

Next we apply the results just obtained to the groups C_n, D_{2n} and $\mathcal{S}_n$.

The case of the cyclic group C_n.

Let $x \in C_n$ and let r be a non negative integer such that g^r is a word representing x (i.e. $x = \varphi(g^r)$). As by Lemma 2.2 $\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(1))) = n\mathbb{Z}$, by using Lemma 2.3 we have that $\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x))) = \gamma(g^r) + n\mathbb{Z} = r + n\mathbb{Z}$.

The case of the dihedral group D_{2n}.

Let $x \in D_{2n}$ and let $w_x \in A^*$ be a representative of x. By Lemma 2.2, $\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(1))) = (2,0)\mathbb{Z} + (0,n)\mathbb{Z} + (2,2n-2)\mathbb{Z}$. Using now Lemma 2.3 we have that

$$\begin{array}{rcl}\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x))) & = & \gamma(w_x) + (2,0)\mathbb{Z} + (0,n)\mathbb{Z} + (2,2n-2)\mathbb{Z} \\ & = & \gamma(w_x) + (2,0)\mathbb{Z} + (0,n)\mathbb{Z} + (0,-2)\mathbb{Z} \\ & = & \left\{\begin{array}{ll}\gamma(w_x) + (2,0)\mathbb{Z} + (0,1)\mathbb{Z} & \text{if } n \text{ is odd} \\ \gamma(w_x) + (2,0)\mathbb{Z} + (0,2)\mathbb{Z} & \text{if } n \text{ is even.}\end{array}\right.\end{array}$$

The case of the symmetric group $\mathcal{S}_n$.
Using similar notation and arguments we have

$$\begin{aligned}
\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x))) &= \gamma(w_x) + (2,0)\mathbb{Z} + (0,n)\mathbb{Z} + (n-1,n-1)\mathbb{Z} + (6,3n)\mathbb{Z} + (4,2n)\mathbb{Z} \\
&= \gamma(w_x) + (2,0)\mathbb{Z} + (0,n)\mathbb{Z} + (n-1,n-1)\mathbb{Z} \\
&= \begin{cases} \gamma(w_x) + (2,0)\mathbb{Z} + (0,n)\mathbb{Z} + (0,n-1)\mathbb{Z} & \text{if } n \text{ is odd} \\ \gamma(w_x) + (2,0)\mathbb{Z} + (0,n)\mathbb{Z} + (1,1)\mathbb{Z} & \text{if } n \text{ is even} \end{cases} \\
&= \begin{cases} \gamma(w_x) + (2,0)\mathbb{Z} + (0,1)\mathbb{Z} & \text{if } n \text{ is odd} \\ \gamma(w_x) + (2,0)\mathbb{Z} + (1,1)\mathbb{Z} & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

2.2. **Relative abelian kernels.** Next we compute relative abelian kernels of cyclic, dihedral and symmetric groups.

We start by applying to these particular cases the fact that the abelian kernel of a finite group is its derived subgroup (Proposition 1.5). One can easily compute the derived subgroup of a dihedral group (see [11]) and it is well known that the derived subgroup of the symmetric group is the alternating subgroup. We thus have the following:

Lemma 2.4. (1) *The abelian kernel of a finite abelian group is the trivial subgroup;*
(2) *The abelian kernel of the dihedral group D_{2n} of order $2n$ is the subgroup*

$$\langle g^2 \rangle = \begin{cases} \langle g \rangle & \text{if } n \text{ is odd} \\ \{g^{2i} \mid 0 \le i \le \frac{n}{2}\} & \text{if } n \text{ is even} \end{cases}$$

of D_{2n}, where g is the generator of order n;
(3) *The abelian kernel of $\mathcal{S}_n$ is the alternating subgroup $\mathcal{A}_n$.*

For kernels relative to proper subpseudovarieties of Ab we will distinguish the cases of pseudovarieties corresponding to infinite supernatural numbers and those corresponding to natural numbers.

Let π be an infinite supernatural number and let H_π be the corresponding pseudovariety of abelian groups. Recall (Proposition 1.10) that for $x \in G$, $x \in \mathsf{K}_{\mathsf{H}_\pi}(G)$ if and only if $0 \in \mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x)))$. Note also that Proposition 1.11 allows us to use the computations of Subsection 2.1 to calculate $\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x)))$.

The case of the cyclic group C_n.
Let $x = \varphi(g^r) \in C_n$ and let $d = gcd(n,\pi)$. By the Example 1.4 we have $\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x))) = r + d\mathbb{Z}$.

Since there exists $t \in \mathbb{Z}$ such that $0 = r + dt$ if and only if r is a multiple of d, we have

$$\mathsf{K}_{\mathsf{H}_\pi}(C_n) = \langle g^d \rangle,$$

where $d = gcd(n,\pi)$.

The case of the dihedral group D_{2n}.
Let $x \in D_{2n}$. Then

$$\begin{aligned}
\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x))) &= \mathsf{Cl}_{\mathsf{H}_\pi}(\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x)))) \\
&= \begin{cases} \mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(w_x) + (2,0)\mathbb{Z} + (0,1)\mathbb{Z}) & \text{if } n \text{ is odd} \\ \mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(w_x) + (2,0)\mathbb{Z} + (0,2)\mathbb{Z}) & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

We will consider two sub-cases according to whether π is even or π is odd.
Subcase 1: π even. As $gcd(2,\pi)=2$, by making use of Example 1.4, we have that

$$\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x))) = \begin{cases} \gamma(w_x)+(2,0)\mathbb{Z}+(0,1)\mathbb{Z} & \text{if } n \text{ is odd} \\ \gamma(w_x)+(2,0)\mathbb{Z}+(0,2)\mathbb{Z} & \text{if } n \text{ is even} \end{cases} = \mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x))).$$

Therefore

$$\mathsf{K}_{\mathsf{H}_\pi}(D_{2n}) = \mathsf{K}_{\mathsf{Ab}}(D_{2n}) = D'_{2n} = \langle g^2 \rangle.$$

Subcase 2: π odd. As $gcd(2,\pi)=1$, by making again use of Example 1.4, we have $\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x))) = \mathbb{Z}^2$. Thus

$$\mathsf{K}_{\mathsf{H}_\pi}(D_{2n}) = D_{2n}.$$

The case of the symmetric group $\mathcal{S}_n$.
Let $x \in \mathcal{S}_n$. Then

$$\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x))) = \begin{cases} \mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(w_x)+(2,0)\mathbb{Z}+(0,1)\mathbb{Z}) & \text{if } n \text{ is odd} \\ \mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(w_x)+(2,0)\mathbb{Z}+(1,1)\mathbb{Z}) & \text{if } n \text{ is even.} \end{cases}$$

We consider again two subcases and make use of Example 1.4.
Subcase 1: π even. We have

$$\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x))) = \begin{cases} \gamma(w_x)+(2,0)\mathbb{Z}+(0,1)\mathbb{Z} & \text{if } n \text{ is odd} \\ \gamma(w_x)+(0,2)\mathbb{Z}+(1,1)\mathbb{Z} & \text{if } n \text{ is even} \end{cases}$$

thus, also in this case, $\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x))) = \mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x)))$ and therefore,

$$\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{S}_n) = \mathcal{A}_n.$$

Subcase 2: π odd. Again $\mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x))) = \mathbb{Z}^2$ and therefore

$$\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{S}_n) = \mathcal{S}_n.$$

Similar results for finite supernatural numbers could be attained in an entirely analogous way, but here we prefer to observe that these correspond to particular cases of Proposition 1.17 and give alternative proofs. Let k be a natural number and let $\mathsf{H_k}$ be the corresponding pseudovariety of abelian groups.

The case of the cyclic group C_n.
Let $G = C_n$. Since C_n is abelian, there is no need to consider the commutators. Let $d = gcd(n,k)$. As $d \mid k$, we have that $G^{[k]} = \langle g^k \rangle \subseteq \langle g^d \rangle$. Let r and s be integers such that $d = rk + sn$. Thus $g^d = g^{rk}g^{sn} = g^{rk} \in G^{[k]}$. Thus $\langle g^d \rangle \subseteq G^{[k]}$, and therefore $G^{[k]} = \langle g^d \rangle$.

The case of the dihedral group D_{2n}.
Let $G = D_{2n}$. As $\langle g^2 \rangle = G'$, we have that $\langle g^2 \rangle \subseteq G^{[k]}$. Note that the relation $hg^{n-1} = gh$ follows from the defining relations for D_{2n} and therefore the elements of D_{2n} may be written i the form g^i or hg^i, with $i \in \{1,\dots,n\}$. We consider again two subcases, according to whether k is even or odd.
Subcase 1: k even. As $hghg = h^2g^{n-1}g = h^2g^n = 1$, we may conclude that, for $i \in \{1,\dots,n\}$, $(hg^i)^2 = hg^ihg^i = hg^{i-1}hg^{n-1}g^i = hg^{i-1}hg^{i-1} = 1$. It follows that $G^{[k]} = \langle g^2 \rangle = G'$.

Subcase 2: k odd. Note that, as $k = 2r+1$, for some r, $(hg^i)^k = (hg^i)^{2r+1} = hg^i$, for $i \in \{1,\ldots,n\}$. In particular, $h, hg \in G^{[k]}$, thus $g = h \cdot hg \in G^{[k]}$. It follows that $G^{[k]} = G$.

The case of the symmetric group $\mathcal{S}_n$.
Let $\sigma \in \mathcal{S}_n$. Once again, we consider two subcases, according to whether k is even or odd.
Subcase 1: k even. It is clear that $\sigma^k \in \mathcal{A}_n$ and therefore $\mathcal{S}_n^{[k]} = \mathcal{A}_n = \mathcal{S}'_n$.
Subcase 2: k odd. In this case, σ^k has the same parity than σ, thus ${\mathcal{S}_n}^{[k]}$ can not be contained in $\mathcal{A}_n$. Since $\mathcal{A}_n$ has index 2 in $\mathcal{S}_n$, $\mathcal{S}_n^{[k]}$ must be $\mathcal{S}_n$.

Summarizing, we have:

Theorem 2.5. *Let π be a (finite or infinite) supernatural number and let H_π be the corresponding pseudovariety of abelian groups. Then:*

(1) $\mathsf{K}_{\mathsf{H}_\pi}(C_n) = \langle g^d \rangle$, *where $d = gcd(n,\pi)$.*

(2) *If π is even, then $\mathsf{K}_{\mathsf{H}_\pi}(D_{2n}) = \mathsf{K}_{\mathsf{Ab}}(D_{2n}) = D'_{2n} = \langle g^2 \rangle$.*
If π is odd, then $\mathsf{K}_{\mathsf{H}_\pi}(D_{2n}) = D_{2n}$.

(3) *If π is even, $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{S}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{S}_n) = \mathcal{A}_n$.*
If π is odd, $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{S}_n) = \mathcal{S}_n$.

Since any relative kernel of a finite monoid M is a submonoid of M, as a consequence of Remark 1.6 and previous theorem we have:

Corollary 2.6. *Let π be an odd supernatural number and let H_π be the corresponding pseudovariety of abelian groups. Let M be a finite monoid all of whose maximal subgroups are symmetric or dihedral groups. If M is generated by its group-elements, then $\mathsf{K}_{\mathsf{H}_\pi}(M) = M$.*

3. On presentations of some transformation monoids

In this section we give some background on the inverse monoids whose relative abelian kernels will be described.

To avoid ambiguities, from now on we take $n \geq 4$. Notice that, for $n \leq 3$, the relative abelian kernels of the finitely many semigroups under consideration can be easily computed using the already referred implementation in GAP [26] of the algorithm presented in [**?**].

The reader can find more details and the proofs of the facts presented in this section in [12, 13, 14, 15, 16].

3.1. The inverse symmetric monoid $\mathcal{I}_n$.

We begin by recalling some well known facts on the symmetric inverse monoid $\mathcal{I}_n$ on a base set with n elements, i.e. the inverse monoid (under composition) of all injective partial transformations on a set with n elements.

Notice that the symmetric group $\mathcal{S}_n$ is the group of units of the monoid $\mathcal{I}_n$ and that two elements of $\mathcal{I}_n$ are $\mathcal{R}$-related or $\mathcal{L}$-related if they have the same domain or the same image, respectively. Moreover, given $s,t \in \mathcal{I}_n$, we have $s \leq_{\mathcal{J}} t$ if and only if $|\operatorname{Im}(s)| \leq |\operatorname{Im}(t)|$. Hence, for $k \in \{0,1,\ldots,n\}$, being $J_k = \{s \in \mathcal{I}_n \mid |\operatorname{Im}(s)| = k\}$, we have

$$\mathcal{I}_n/\mathcal{J} = \{J_0 <_{\mathcal{J}} J_1 <_{\mathcal{J}} \cdots <_{\mathcal{J}} J_n\}.$$

Since $|J_k| = \binom{n}{k}^2 k!$, for $k \in \{0,1,\ldots,n\}$, it follows that $\mathcal{I}_n$ has $\sum_{k=0}^{n} \binom{n}{k}^2 k!$ elements. Observe that $J_n = \mathcal{S}_n$. Moreover, the maximal subgroups of $\mathcal{I}_n$ contained in J_k are isomorphic to $\mathcal{S}_k$, for $1 \leq k \leq n$. We obtain a generating set of $\mathcal{I}_n$, with three elements, by joining to the permutations $a = (1\,2)$ and $g = (1\,2\,\cdots\,n)$, which generate $\mathcal{S}_n$, any injective partial transformation of rank $n-1$. For instance, if

$$c = \begin{pmatrix} 2 & 3 & \cdots & n \\ 2 & 3 & \cdots & n \end{pmatrix}$$

then the set $\{a, g, c\}$ generates the monoid $\mathcal{I}_n$. In particular, we have the following:

Corollary 3.1. *The inverse symmetric monoid $\mathcal{I}_n$ is generated by its group-elements.*

Combining the monoid presentation for $\mathcal{S}_n$ given in Example 2.1 with the Popova presentation of $\mathcal{I}_n$ [21], one can deduce [15] the following presentation of $\mathcal{I}_n$, in terms of the generators a, g and c:

$$\begin{array}{rl} \langle\, a, g, c \;\mid & a^2 = g^n = (ga)^{n-1} = (ag^{n-1}ag)^3 = (ag^{n-j}ag^j)^2 = 1 \ (2 \le j \le n-2), \\ & g^{n-1}agcg^{n-1}ag = gacag^{n-1} = c = c^2, \ (ca)^2 = cac = (ac)^2 \,\rangle. \end{array}$$

3.2. **Some inverse submonoids of $\mathcal{I}_n$.** A partial transformation σ of a chain X_n with n elements, say $X_n = \{1 < 2 < \cdots < n\}$, is called *order-preserving* [*order-reversing*] if, $x \le y$ implies $x\sigma \le y\sigma$ [$x\sigma \ge y\sigma$], for all $x, y \in \mathrm{Dom}(\sigma)$. We denote by $\mathcal{POI}_n$ the inverse submonoid of $\mathcal{I}_n$ of all order-preserving transformations and by $\mathcal{PODI}_n$ the inverse submonoid of $\mathcal{I}_n$ whose elements are all order-preserving or order-reversing transformations.

Let $c = (c_1, c_2, \ldots, c_t)$ be a sequence of t ($t \ge 0$) elements from the chain X_n. We say that c is *cyclic* [*anti-cyclic*] if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $c_i > c_{i+1}$ [$c_i < c_{i+1}$], where $c_{t+1} = c_1$. Then, given a partial transformation σ on the chain X_n such that $\mathrm{Dom}(\sigma) = \{a_1, \ldots, a_t\}$, with $t \ge 0$ and $a_1 < \cdots < a_t$, we say that σ is *orientation-preserving* [*orientation-reversing*] if the sequence of its images $(a_1\sigma, \ldots, a_t\sigma)$ is cyclic [anti-cyclic]. We denote by $\mathcal{POPI}_n$ the inverse submonoid of $\mathcal{I}_n$ of all orientation-preserving transformations and by $\mathcal{PORI}_n$ the inverse submonoid of $\mathcal{I}_n$ of all orientation-preserving transformations together with all orientation-reversing transformations.

Notice that, $\mathcal{POI}_n \subset \mathcal{PODI}_n \subset \mathcal{PORI}_n$ and $\mathcal{POI}_n \subset \mathcal{POPI}_n \subset \mathcal{PORI}_n$, by definition.

Let M be one of the monoids $\mathcal{POI}_n$, $\mathcal{PODI}_n$, $\mathcal{POPI}_n$ or $\mathcal{PORI}_n$. As for $\mathcal{I}_n$, given two elements $s, t \in M$, we have $s \le_{\mathcal{J}} t$ if and only if $|\mathrm{Im}(s)| \le |\mathrm{Im}(t)|$, whence

$$M/\mathcal{J} = \{J_0 <_{\mathcal{J}} J_1 <_{\mathcal{J}} \cdots <_{\mathcal{J}} J_n\},$$

where $J_k = \{s \in M : |\mathrm{Im}(s)| = k\}$, for $0 \le k \le n$.

Concerning maximal subgroups, the monoid $\mathcal{POI}_n$ is aperiodic, while each $\mathcal{H}$-class of an element $s \in \mathcal{PODI}_n$ has exactly two elements (an order-preserving one and another being order-reversing), unless the rank of s is one or zero, in which case its $\mathcal{H}$-class is trivial. On the other hand, for $1 \le k \le n$, the $\mathcal{H}$-class of an element $s \in \mathcal{POPI}_n$ of rank k has precisely k elements, being a cyclic group of order k if s is a group-element. Finally, given $s \in \mathcal{PORI}_n$, if $|\mathrm{Im}(s)| = k \ge 3$, then the $\mathcal{H}$-class of s has $2k$ elements and, if s is a group-element, it is isomorphic to the dihedral group D_{2k} and if $|\mathrm{Im}(s)| = 2$ then $\mathcal{H}$-class of s has precisely two elements, otherwise it has just one element.

Next, let us consider the elements $x_0, x_1, \ldots, x_{n-1}$ of $\mathcal{POI}_n$ defined by:

(1) $x_0 = \begin{pmatrix} 2 & \cdots & n-1 & n \\ 1 & \cdots & n-2 & n-1 \end{pmatrix}$;

(2) $x_i = \left(\begin{array}{ccc|c|ccc} 1 & \cdots & n-i-1 & n-i & n-i+2 & \cdots & n \\ 1 & \cdots & n-i-1 & n-i+1 & n-i+2 & \cdots & n \end{array}\right)$, for $1 \le i \le n-1$.

Consider also the permutation (that reverts the order)

$$h=\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}$$

of $\mathcal{PODI}_n$ and the n-cycle

$$g=\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix},$$

which is an element of $\mathcal{POPI}_n$. Hence, $A=\{x_0,x_1,\dots,x_{n-1}\}$, $B=A\cup\{h\}$, $C=A\cup\{g\}$ and $D=A\cup\{h,g\}$ are sets of generators of $\mathcal{POI}_n$, $\mathcal{PODI}_n$, $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$, respectively.

Furthermore, consider the following set of monoid relations:

R_1: $x_ix_0=x_0x_{i+1}$, $1\le i\le n-2$;
R_2: $x_jx_i=x_ix_j$, $2\le i+1<j\le n-1$;
R_3: $x_0^2x_1=x_0^2=x_{n-1}x_0^2$;
R_4: $x_{i+1}x_ix_{i+1}=x_{i+1}x_i=x_ix_{i+1}x_i$, $1\le i\le n-2$;
R_5: $x_ix_{i+1}\cdots x_{n-1}x_0x_1\cdots x_{i-1}x_i=x_i$, $0\le i\le n-1$;
R_6: $x_{i+1}\cdots x_{n-1}x_0x_1\cdots x_{i-1}x_i^2=x_i^2$, $1\le i\le n-1$;
R_7: $gx_i=x_{i+1}g$, $1\le i\le n-2$;
R_8: $gx_0x_1=x_1$ and $x_{n-1}x_0g=x_{n-1}$;
R_9: $g^n=1$;
R_{10}: $h^2=1$;
R_{11}: $hx_0=x_1\cdots x_{n-1}h$ and $hx_i=x_{n-i+1}\cdots x_{n-1}x_0x_1\cdots x_{n-i-1}h$, for $1\le i\le n-1$;
R_{12}: $x_0^{n-1}h=x_{n-1}\cdots x_3x_2^2$;
R_{13}: $hg=g^{n-1}h$;
R_{14}: $(g^{n-1}(x_1g)^{n-1})^{n-2}h=(g^{n-1}(x_1g)^{n-1})^{n-2}((x_1g)^{n-2}g^2)^{n-2}g^{n-1}$.

Observe that, we are adopting the following convention: given $i,j\in\{0,\dots,n-1\}$, if $i\le j$ the expression $x_i\cdots x_j$ represents the word of length $j-i+1$ such that the letter in the position $p\in\{1,\dots,j-i+1\}$ is x_{i+p-1} (i.e. the indices of the letters are ordered in the usual way and are consecutive), and if $j<i$ the expression $x_i\cdots x_j$ represents the empty word. For example, the expression $x_3\cdots x_2$ denotes the empty word.

Hence, $\langle A\mid R_1-R_6\rangle$, $\langle B\mid R_1-R_6,R_{10}-R_{12}\rangle$, $\langle C\mid R_1-R_9\rangle$ and $\langle D\mid R_1-R_{14}\rangle$ are presentations of the monoids $\mathcal{POI}_n$, $\mathcal{PODI}_n$, $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$, respectively.

Next, we recall a set of canonical words associated to each of these presentations.

Let $k\in\{1,\dots,n-1\}$, $\ell=n-k$ ($1\le\ell\le n-1$) and $w_j=x_{\ell-j+1}\cdots x_{\ell-j+k}$, for $1\le j\le\ell$. Notice that $|w_j|=k$, for $1\le j\le\ell$. Let A_k $[C_k]$ be the set of all words

$$\Big(\textstyle\prod_{j=1}^{\ell}u_j\Big)x_0^{\ell}\Big(\prod_{j=1}^{\ell}v_j\Big)\quad\Big[g^i\Big(\prod_{j=1}^{\ell}u_j\Big)x_0^{\ell}\Big(\prod_{j=1}^{\ell}v_j\Big)\Big],$$

where $[0\le i\le n-1,]$ u_j is a suffix of w_j and v_j is a prefix of w_j, for $1\le j\le\ell$, $0\le|u_1|\le\cdots\le|u_\ell|\le k$ $[1\le|u_1|\le\cdots\le|u_\ell|\le k]$ and $k\ge|v_1|\ge\cdots\ge|v_\ell|\ge0$. Also, define $A_0=C_0=\{x_0^n\}$, $A_n=\{1\}$ and $C_n=\{g^i\mid 0\le i\le n-1\}$. Then $\bar A=A_0\cup A_1\cup\cdots\cup A_n$ and $\bar C=C_0\cup C_1\cup\cdots\cup C_n$ are sets of canonical words for $\mathcal{POI}_n$ and $\mathcal{POPI}_n$, respectively.

Now, let $B_k=A_k\cup\{wh\mid w\in A_k\}$, for $2\le k\le n$, and $B_k=A_k$, for $k=0,1$. Also, define $D_k=C_k\cup\{wh\mid w\in C_k\}$, for $3\le k\le n$, and $D_k=C_k$, for $0\le k\le 2$. Then $\bar B=B_0\cup B_1\cup\cdots\cup B_n$ and $\bar D=D_0\cup D_1\cup\cdots\cup D_n$ are sets of canonical words for $\mathcal{PODI}_n$ and $\mathcal{PORI}_n$, respectively.

Notice that, for $0 \leq k \leq n$, the sets of words A_k, B_k, C_k and D_k represent the transformations of rank k of $\mathcal{POI}_n$, $\mathcal{PODI}_n$, $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$, respectively.

Of particular interest for us, are the words corresponding to elements of rank $n-1$:

$A_{n-1} = \{x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{j-1} \mid 1 \leq i \leq n, 1 \leq j \leq n\}$,
$B_{n-1} = \{x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{j-1} h^t \mid 1 \leq i \leq n, 1 \leq j \leq n, t = 0, 1\}$,
$C_{n-1} = \{g^r x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{j-1} \mid 1 \leq i \leq n-1, 1 \leq j \leq n, 0 \leq r \leq n-1\}$,
$D_{n-1} = \{g^r x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{j-1} h^t \mid 1 \leq i \leq n-1, 1 \leq j \leq n, 0 \leq r \leq n-1, t = 0, 1\}$.

Again, let M be one of the monoids $\mathcal{POI}_n$, $\mathcal{PODI}_n$, $\mathcal{POPI}_n$ or $\mathcal{PORI}_n$. Denote by X the set of generators of M and by W the set of canonical words of M considered above. Let $\varphi : X^* \longrightarrow M$ be the onto homomorphism extending the map $X \longrightarrow M$, $x \mapsto x$.

Given an element $s \in M$, we denote by w_s the (unique) element of $\varphi^{-1}(s) \cap W$, called the *canonical word associated to s*.

Remark 3.2. Let $M \in \{\mathcal{POI}_n, \mathcal{PODI}_n\}$ and let $s \in M$ be an element of rank $n-1$. Then s is an idempotent if and only if there exists $i \in \{0, \ldots, n-1\}$ such that

$$\mathsf{w}_s = x_i \cdots x_{n-1} x_0 x_1 \cdots x_{i-1}.$$

Notice that, if s is not an idempotent, then $|\mathsf{w}_s|_{x_0} = 1$ and there exists $i \in \{1, \ldots, n-1\}$ such that either $|\mathsf{w}_s|_{x_i} = 0$ or $|\mathsf{w}_s|_{x_i} = 2$.

On the other hand, if $M \in \{\mathcal{POPI}_n, \mathcal{PORI}_n\}$ and $s \in M$ is an element of rank $n-1$, then s is an idempotent if and only if $\mathsf{w}_s = g^{n-1} x_1 \cdots x_{n-1}$ (which corresponds to the same element of M that the word $x_0 x_1 \cdots x_{n-1}$) or there exists $i \in \{1, \ldots, n-1\}$ such that

$$\mathsf{w}_s = x_i \cdots x_{n-1} x_0 x_1 \cdots x_{i-1}.$$

It is known (see [16]) that $\mathcal{PORI}_n$ is generated by the transformations g, h and x_1. Therefore $\{g, h, x_1 g\}$ is also a set of generators of $\mathcal{PORI}_n$. As

$$x_1 g = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & 1 \end{pmatrix}$$

is a partial permutation, we obtain the following:

Corollary 3.3. *The monoid $\mathcal{PORI}_n$ is generated by its group-elements.*

4. Main results

This section is devoted to our main results. We give descriptions of the kernels relative to decidable pseudovarieties of abelian groups of the monoids $\mathcal{POI}_n$, $\mathcal{PODI}_n$, $\mathcal{POPI}_n$, $\mathcal{PORI}_n$ and $\mathcal{I}_n$ for which we already recalled simple presentations. As the kernel of a finite monoid relative to the trivial pseudovariety is the monoid itself, we just need to consider pseudovarieties of abelian groups corresponding to infinite recursive supernatural numbers or to natural numbers greater than 1.

4.1. The case of $\mathcal{POI}_n$.

In this subsection we show that any relative abelian kernel of $\mathcal{POI}_n$ equals the abelian kernel of $\mathcal{POI}_n$.

Theorem 4.1. [6] *The abelian kernel of $\mathcal{POI}_n$ consists of all idempotents and all elements of rank less than $n-1$.* □

The abelian kernel of $\mathcal{POI}_n$ contains all elements of rank less than $n-1$, so it must be the case of any relative abelian kernel of $\mathcal{POI}_n$.

Now, let $x \in \mathcal{POI}_n$ be an element of rank $n-1$. In [6, page 445] it was proved that (considering the letters ordered as follows: $x_0 < x_1 < \cdots < x_{n-1}$)

$$\mathsf{Cl}_{\mathsf{Ab}}(\gamma(\varphi^{-1}(x))) \subseteq \gamma(\mathsf{w}_x) + (1, \ldots, 1)\mathbb{Z},$$

whence, using Proposition 1.12, for any $k > 1$, we have

$$\gamma_k(\varphi^{-1}(x)) \subseteq \gamma_k(\mathsf{w}_x) + (1, \ldots, 1)\mathbb{Z}/k\mathbb{Z}.$$

By Remark 3.2, if x is not idempotent, then $\gamma(\mathsf{w}_x)$ has the first component 1 and the other components are 0 or 2, whence the same happens with $\gamma_k(\mathsf{w}_x)$ (except that if $k = 2$ none of the components is 2) and so $0 \notin \gamma_k(\varphi^{-1}(x))$. It follows that the $\mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{POI}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{POI}_n)$. Moreover, this equality holds even when the supernatural number under consideration is not finite:

Theorem 4.2. *Let $\pi \neq 1$ be a supernatural number. Then $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POI}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{POI}_n)$.*

Proof. Suppose that $k > 1$ is a finite divisor of π. As $\mathsf{H}_k \subseteq \mathsf{H}_\pi$, we have that

$$\mathsf{K}_{\mathsf{Ab}}(\mathcal{POI}_n) \subseteq \mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POI}_n) \subseteq \mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{POI}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{POI}_n).$$

Thus the inclusions must in fact be equalities, concluding the proof. □

4.2. **The case of $\mathcal{PODI}_n$.** In this subsection we compute the relative abelian kernels of the monoid $\mathcal{PODI}_n$.

First we recall:

Theorem 4.3. [8] *If n is an even integer, the abelian kernel of $\mathcal{PODI}_n$ consists precisely of the elements of the abelian kernel of $\mathcal{POI}_n$. If n is an odd integer, then the abelian kernel of $\mathcal{PODI}_n$ consists of the abelian kernel of $\mathcal{POI}_n$ united with the set of elements corresponding to words of the form $x_i \cdots x_{n-1} x_0 x_1 \cdots x_{n-i}$, for $1 \leq i \leq n$.* □

Considering the presentation of $\mathcal{PODI}_n$ recalled in Section 3, as an immediate consequence of Theorem 1.15, we have:

Corollary 4.4. *If k is even, then $\mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{PODI}_n) \subset \mathcal{POI}_n$.* □

Let π be a supernatural number.

Let J_i denote the $\mathcal{J}$-class of $\mathcal{PODI}_n$ of the elements of rank i, for $0 \leq i \leq n$.

As the group of units J_n of $\mathcal{PODI}_n$ is the cyclic group generated by the permutation of order two $h = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}$, by Theorem 2.5 and Lemma 1.7, we have

$$\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n) \cap J_n = \mathsf{K}_{\mathsf{H}_\pi}(J_n) = \langle h^{gcd(2,\pi)} \rangle = \begin{cases} \{1\}, & \text{if 2 divides } \pi \\ \{1, h\}, & \text{otherwise.} \end{cases}$$

Next, we concentrate on the elements of rank less than $n-1$. First, we notice that, by Theorems 4.3 and 4.1, $\mathcal{POI}_n \cap (\cup_{i=0}^{n-2} J_i)$ is contained in the abelian kernel of $\mathcal{PODI}_n$ and so it is contained in $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n)$.

Suppose that π is divisible by 2. Then, as $\mathsf{K}_{\mathsf{H}_2}(\mathcal{PODI}_n) \subset \mathcal{POI}_n$ (by Corollary 4.4), we have

$$\mathcal{POI}_n \cap (\cup_{i=0}^{n-2} J_i) \subseteq \mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n) \cap \left(\cup_{i=0}^{n-2} J_i\right) \subseteq \mathsf{K}_{\mathsf{H}_2}(\mathcal{PODI}_n) \cap \left(\cup_{i=0}^{n-2} J_i\right) \subseteq \mathcal{POI}_n \cap (\cup_{i=0}^{n-2} J_i),$$

whence $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n) \cap \left(\cup_{i=0}^{n-2} J_i\right) = \mathcal{POI}_n \cap (\cup_{i=0}^{n-2} J_i)$.

On the other hand, admit that 2 does not divide π. Then $\left(\mathcal{POI}_n \cap (\cup_{i=0}^{n-2} J_i)\right) \cup \{h\} \subset \mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n)$ and, since any element of $\mathcal{PODI}_n$ can be factorized as a product of a certain element of $\mathcal{POI}_n$ (with the same rank) by h, it follows that $\cup_{i=0}^{n-2} J_i \subset \mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n)$.

Summarizing, we have

$$\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n) \cap \left(\cup_{i=0}^{n-2} J_i\right) = \begin{cases} \mathcal{POI}_n \cap (\cup_{i=0}^{n-2} J_i), & \text{if 2 divides } \pi \\ \cup_{i=0}^{n-2} J_i, & \text{otherwise.} \end{cases}$$

Now, we just have to determine which elements of J_{n-1} belong to $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n)$.

We recall that

$$\{\mathsf{w}_{i,j,t} = x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{j-1} h^t \mid 1 \le i \le n, 1 \le j \le n, 0 \le t \le 1\}$$

is a set of canonical words for J_{n-1}.

It is clear that

$$\gamma(\mathsf{w}_{i,j,t}) = \begin{cases} (1, \ldots, 1, 0_{(j+1)}, \ldots, 0_{(i)}, 1, \ldots, 1, t), & 1 \le j \le i \le n \\ (1, \ldots, 1, 2_{(i+1)}, \ldots, 2_{(j)}, 1, \ldots, 1, t), & 1 \le i < j \le n, \end{cases}$$

for $t = 0, 1$ (considering the letters ordered as follows: $x_0 < x_1 < \cdots < x_{n-1} < h$).

Let $x \in \mathcal{PODI}_n$ be an element of rank $n-1$. The second and third authors showed in [8] that $x \in \mathsf{K}_{\mathsf{Ab}}(\mathcal{PODI}_n)$ if and only if

$$\begin{aligned} \gamma(\mathsf{w}_x) \in\ & (1,1,\ldots,1,0)\mathbb{Z} + (0,0,\ldots,0,2)\mathbb{Z} + (2,0,\ldots,0)\mathbb{Z} \\ & + (0,1,0,\ldots,0,1,0)\mathbb{Z} + \cdots + (0,\ldots,0,1_{\left(\frac{n-1}{2}+1\right)}, 1_{\left(\frac{n+1}{2}+1\right)}, 0, \ldots, 0)\mathbb{Z}, \end{aligned}$$

if n is odd, and

$$\begin{aligned} \gamma(\mathsf{w}_x) \in\ & (1,1,\ldots,1,0)\mathbb{Z} + (0,0,\ldots,0,2)\mathbb{Z} + (2,0,\ldots,0)\mathbb{Z} \\ & + (0,1,0,\ldots,0,1,0)\mathbb{Z} + \cdots + (0,\ldots,0,1_{\left(\frac{n}{2}\right)}, 0, 1_{\left(\frac{n}{2}+2\right)}, 0, \ldots, 0)\mathbb{Z} \\ & + (0,\ldots,0,2_{\left(\frac{n}{2}+1\right)}, 0, \ldots, 0)\mathbb{Z}, \end{aligned}$$

if n is even. Notice that these expressions were deduced from the presentation of $\mathcal{PODI}_n$ [16] recalled in Section 3.

Hence, $x \in \mathsf{K}_{\mathsf{H}_2}(\mathcal{PODI}_n)$ if and only if $\gamma_2(\mathsf{w}_x)$ belongs to

$$\begin{gathered} (1,1,\ldots,1,0)\mathbb{Z}/2\mathbb{Z} + \\ (0,1,0,\ldots,0,1,0)\mathbb{Z}/2\mathbb{Z} + \cdots + (0,\ldots,0,1_{\left(\frac{n-1}{2}+1\right)}, 1_{\left(\frac{n+1}{2}+1\right)}, 0, \ldots, 0)\mathbb{Z}/2\mathbb{Z}, \end{gathered}$$

if n is odd, and to

$$\begin{gathered} (1,1,\ldots,1,0)\mathbb{Z}/2\mathbb{Z} + \\ (0,1,0,\ldots,0,1,0)\mathbb{Z}/2\mathbb{Z} + \cdots + (0,\ldots,0,1_{\left(\frac{n}{2}\right)}, 0, 1_{\left(\frac{n}{2}+2\right)}, 0, \ldots, 0)\mathbb{Z}/2\mathbb{Z}, \end{gathered}$$

if n is even.

Now, suppose that $\mathsf{w}_x = \mathsf{w}_{i,j,t}$, for some $1 \le i \le n, 1 \le j \le n$ and $0 \le t \le 1$. If n is even, then it is clear that $x \in \mathsf{K}_{\mathsf{H}_2}(\mathcal{PODI}_n)$ if and only if $i = j$ and $t = 0$, i.e. $x \in \mathsf{K}_{\mathsf{H}_2}(\mathcal{PODI}_n)$ if and only if x is an idempotent of J_{n-1}, whence $\mathsf{K}_{\mathsf{H}_2}(\mathcal{PODI}_n) \cap J_{n-1} = \mathsf{K}_{\mathsf{Ab}}(\mathcal{PODI}_n) \cap J_{n-1}$. On the other hand, if n is odd, then it is easy to show that $x \in \mathsf{K}_{\mathsf{H}_2}(\mathcal{PODI}_n)$ if and only if $t = 0$ and $i = j$ or $i = n - j + 1$, i.e. $x \in \mathsf{K}_{\mathsf{H}_2}(\mathcal{PODI}_n)$ if and only if x is an idempotent of J_{n-1} or $x = x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{n-i}$, for some $1 \le i \le n$, whence we have, also in this case, $\mathsf{K}_{\mathsf{H}_2}(\mathcal{PODI}_n) \cap J_{n-1} = \mathsf{K}_{\mathsf{Ab}}(\mathcal{PODI}_n) \cap J_{n-1}$.

Hence $\mathsf{K}_{\mathsf{H}_2}(\mathcal{PODI}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{PODI}_n)$, for all $n \in \mathbb{N}$, and so, using Proposition 1.9, we get the following result:

Theorem 4.5. *If π is an even supernatural number, then $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{PODI}_n)$.* □

Next, let k be an odd natural number and consider again an element $x \in \mathcal{PODI}_n$ of rank $n-1$. As $2\mathbb{Z}/k\mathbb{Z} = \mathbb{Z}/k\mathbb{Z}$, we have:

(1) If n is odd, then $x \in \mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{PODI}_n)$ if and only if

$$\begin{aligned}\gamma_k(\mathsf{w}_x) \in\ & (1,1,\dots,1,0)\mathbb{Z}/k\mathbb{Z} + (0,0,\dots,0,2)\mathbb{Z}/k\mathbb{Z} + (2,0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & + (0,1,0,\dots,0,1,0)\mathbb{Z}/k\mathbb{Z} + \cdots + (0,\dots,0,1_{(\frac{n-1}{2}+1)},1_{(\frac{n+1}{2}+1)},0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & = (1,1,\dots,1,0)\mathbb{Z}/k\mathbb{Z} + (0,0,\dots,0,1)\mathbb{Z}/k\mathbb{Z} + (1,0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & + (0,1,0,\dots,0,1,0)\mathbb{Z}/k\mathbb{Z} + \cdots + (0,\dots,0,1_{(\frac{n-1}{2}+1)},1_{(\frac{n+1}{2}+1)},0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & = (1,1,\dots,1,0)\mathbb{Z}/k\mathbb{Z} + (0,0,\dots,0,1)\mathbb{Z}/k\mathbb{Z} \\ & + (0,1,0,\dots,0,1,0)\mathbb{Z}/k\mathbb{Z} + \cdots + (0,\dots,0,1_{(\frac{n-1}{2}+1)},1_{(\frac{n+1}{2}+1)},0,\dots,0)\mathbb{Z}/k\mathbb{Z};\end{aligned}$$

(2) If n is even, then $x \in \mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{PODI}_n)$ if and only if

$$\begin{aligned}\gamma_k(\mathsf{w}_x) \in\ & (1,1,\dots,1,0)\mathbb{Z}/k\mathbb{Z} + (0,0,\dots,0,2)\mathbb{Z}/k\mathbb{Z} + (2,0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & + (0,1,0,\dots,0,1,0)\mathbb{Z}/k\mathbb{Z} + \cdots + (0,\dots,0,1_{(\frac{n}{2})},0,1_{(\frac{n}{2}+2)},0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & + (0,\dots,0,2_{(\frac{n}{2}+1)},0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & = (1,1,\dots,1,0)\mathbb{Z}/k\mathbb{Z} + (0,0,\dots,0,1)\mathbb{Z}/k\mathbb{Z} + (1,0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & + (0,1,0,\dots,0,1,0)\mathbb{Z}/k\mathbb{Z} + \cdots + (0,\dots,0,1_{(\frac{n}{2})},0,1_{(\frac{n}{2}+2)},0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & + (0,\dots,0,1_{(\frac{n}{2}+1)},0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & = (1,1,\dots,1,0)\mathbb{Z}/k\mathbb{Z} + (0,0,\dots,0,1)\mathbb{Z}/k\mathbb{Z} \\ & + (0,1,0,\dots,0,1,0)\mathbb{Z}/k\mathbb{Z} + \cdots + (0,\dots,0,1_{(\frac{n}{2})},0,1_{(\frac{n}{2}+2)},0,\dots,0)\mathbb{Z}/k\mathbb{Z} \\ & + (0,\dots,0,1_{(\frac{n}{2}+1)},0,\dots,0)\mathbb{Z}/k\mathbb{Z}.\end{aligned}$$

Then, supposing that $\mathsf{w}_x = \mathsf{w}_{i,j,t}$, for some $1 \le i \le n, 1 \le j \le n$ and $0 \le t \le 1$, it is easy to show that, for both n odd and even, we have $x \in \mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{PODI}_n)$ if and only if $i = j$ or $i = n-j+1$. Hence, $x \in \mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{PODI}_n)$ if and only if $x = x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{i-1} h^t$ or $x = x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{n-i} h^t$, for some $1 \le i \le n$ and $0 \le t \le 1$.

Now, let π be an odd supernatural. Then, π is divisible by some odd natural number k and we have $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n) \subseteq \mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{PODI}_n)$. In fact, the converse inclusion is also valid, as we will show below.

First, notice that, by relations R_{11}, we have $x_0 h = h x_1 \cdots x_{n-1}$ and

$$x_{n-1} h = h x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{i-1},$$

whence

$$x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{n-i} h = x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{n-i-1} h x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{i-1},$$

for $1 \le i \le n-1$. It follows that, for $1 \le i \le n$, $x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{i-1}$ and $x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{n-i} h$ are $\mathcal{H}$-related elements of $\mathcal{PODI}_n$, whence

$$\{x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{i-1}, x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{n-i} h\}$$

is a cyclic group of order two and so it is contained in $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n)$. Therefore, since $h \in \mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n)$, we have proved:

Theorem 4.6. *If π is an odd supernatural number, then $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PODI}_n)$ consists of all permutations of $\mathcal{PODI}_n$, of all elements of rank $n-1$ of the forms $x_i x_{i+1}\cdots x_{n-1}x_0x_1\cdots x_{i-1}h^t$ and $x_i x_{i+1}\cdots x_{n-1}x_0x_1\cdots x_{n-i}h^t$, with $1 \le i \le n$ and $0 \le t \le 1$, and all transformations with rank less than $n-1$.* □

4.3. **The case of $\mathcal{POPI}_n$.** Let M be a monoid and let U be its group of units. Take $x \in U$ and $s \in M$. Then, it is easy to show that $s\mathcal{L}xs$. Furthermore, as the Green relation $\mathcal{R}$ is compatible with the multiplication on the left, the correspondence

$$\begin{array}{ccc} H_s & \longrightarrow & H_{xs} \\ t & \mapsto & xt \end{array}$$

is a bijection.

Next, let g be the n-cycle permutation and s any element of $\mathcal{I}_n$ with rank $n-1$. Then, $s, gs, g^2s, \ldots, g^{n-1}s$ are n domain-distinct (and so each one lies in a different $\mathcal{R}$-class of $\mathcal{I}_n$) $\mathcal{L}$-related elements of $\mathcal{I}_n$ of rank $n-1$. Hence, $X, gX, g^2X, \ldots, g^{n-1}X$ are n pairwise disjoint subsets of L_s with $|X|$ elements each, for any subset X of H_s.

Now, consider the monoid $\mathcal{POPI}_n$ and let π be a supernatural number. Recall that:

Theorem 4.7. [6] *The abelian kernel of $\mathcal{POPI}_n$ consists of all idempotents and all elements of rank less than $n-1$.* □

Let J_i denote the $\mathcal{J}$-class of $\mathcal{POPI}_n$ of the elements of rank i, for $0 \le i \le n$.

As $J_0, J_1, \ldots, J_{n-2}$ are contained in the abelian kernel of $\mathcal{POPI}_n$, we have also

$$\cup_{i=0}^{n-2} J_i \subset \mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POPI}_n).$$

On the other hand, as the group of units J_n of $\mathcal{POPI}_n$ is the cyclic group generated by the permutation $g = (1\,2\,\cdots\,n)$, by Theorem 2.5 and by Lemma 1.7, we have

$$\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POPI}_n) \cap J_n = \mathsf{K}_{\mathsf{H}_\pi}(J_n) = \langle g^{gcd(n,\pi)} \rangle.$$

Thus, it remains to decide which elements of J_{n-1} belong to $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POPI}_n)$.

Let H be a maximal subgroup of $\mathcal{POPI}_n$ contained in J_{n-1}. Then, H is a cyclic group of order $n-1$ and so, by Theorem 2.5, $\mathsf{K}_{\mathsf{H}_\pi}(H)$ has $\frac{n-1}{gcd(n-1,\pi)}$ elements. As

$$\mathsf{K}_{\mathsf{H}_\pi}(H) \cup g^{gcd(n,\pi)}\mathsf{K}_{\mathsf{H}_\pi}(H) \cup g^{2\,gcd(n,\pi)}\mathsf{K}_{\mathsf{H}_\pi}(H) \cup \cdots \cup g^{(\frac{n}{gcd(n,\pi)}-1)\,gcd(n,\pi)}\mathsf{K}_{\mathsf{H}_\pi}(H)$$

is a subset of $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POPI}_n) \cap J_{n-1}$ (contained in a single $\mathcal{L}$-class of $\mathcal{POPI}_n$) with cardinality $\frac{n}{gcd(n,\pi)} \cdot \frac{n-1}{gcd(n-1,\pi)}$ and J_{n-1} contains n distinct maximal subgroups of $\mathcal{POPI}_n$ (and $\mathcal{POPI}_n$ is an inverse monoid), we have at least $n \cdot \frac{n}{gcd(n,\pi)} \cdot \frac{n-1}{gcd(n-1,\pi)}$ elements in $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POPI}_n) \cap J_{n-1}$.

Now, let k be any natural number.

Recall that

$$\{\mathsf{w}_{i,j,r} = g^r x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{j-1} \mid 1 \le i \le n-1, 1 \le j \le n, 0 \le r \le n-1\}$$

is a set of canonical words for J_{n-1}. Clearly, for all $0 \le r \le n-1$, we have

$$\gamma(\mathsf{w}_{i,j,r}) = \begin{cases} (1, \ldots, 1, 0_{(j+1)}, \ldots, 0_{(i)}, 1, \ldots, 1, r), & 1 \le j \le i \le n-1 \\ (1, \ldots, 1, 2_{(i+1)}, \ldots, 2_{(j)}, 1, \ldots, 1, r), & 1 \le i < j \le n \end{cases}$$

(considering the letters ordered as follows: $x_0 < x_1 < \cdots < x_{n-1} < g$).

Let $x \in J_{n-1}$. In [6, Corollary 4.7], the second and third authors showed that

$$\overline{\gamma(\psi^{-1}(x))} = \gamma(\mathsf{w}_x) + (1,1,\dots,1,0)\mathbb{Z} + \sum_{i=1}^{n-2} f_i\mathbb{Z} + (1,0,\dots,0,1)\mathbb{Z} + (0,0,\dots,0,n)\mathbb{Z},$$

where $f_i = (0_{(1)},\dots,0,-1_{(i+1)},1_{(i+2)},0,\dots,0_{(n+1)})$, for all $1 \le i \le n-2$. Notice that this formula was deduced from the presentation of $\mathcal{POPI}_n$ [13] recalled in the Section 3. It follows that

$$\gamma_k(\psi^{-1}(x)) =$$
$$\gamma_k(\mathsf{w}_x) + (1,1,\dots,1,0)\mathbb{Z}/k\mathbb{Z} + \sum_{i=1}^{n-2} f_i\mathbb{Z}/k\mathbb{Z} + (1,0,\dots,0,1)\mathbb{Z}/k\mathbb{Z} + (0,0,\dots,0,n)\mathbb{Z}/k\mathbb{Z}.$$

Next, admit that $x \in \mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{POPI}_n)$. Hence $0 \in \gamma_k(\psi^{-1}(x))$ and so the system of equations

$$\gamma(\mathsf{w}_x) + (1,1,\dots,1,0)z_0 + \sum_{i=1}^{n-2} f_i z_i + (1,0,\dots,0,1)z_{n-1} + (0,0,\dots,0,n)z_n \equiv 0 \mod k,$$

with integer unknowns $z_0, z_1, \dots, z_n$, has a solution. Let $\gamma(\mathsf{w}_x) = (a_0, a_1, \dots, a_{n-1}, a_n)$. Then, we have

$$\begin{cases} z_0 + z_{n-1} \equiv -a_0 \mod k \\ z_0 - z_1 \equiv -a_1 \mod k \\ z_0 + z_{i-1} - z_i \equiv -a_i \mod k, \;\; 2 \le i \le n-2 \\ z_0 + z_{n-2} \equiv -a_{n-1} \mod k \\ z_{n-1} + nz_n \equiv -a_n \mod k. \end{cases}$$

¿From this system it is easy to deduce the following equation:

$$(n-1)(z_{n-1} + a_0) \equiv (a_1 + \cdots + a_{n-1}) \mod k.$$

Suppose that $\mathsf{w}_x = \mathsf{w}_{i,j,r}$, for some $1 \le i \le n-1$, $1 \le j \le n$ and $0 \le r \le n-1$. Then $a_0 = 1$, $a_1 + \cdots + a_{n-1} = n - i + j - 1$ and $a_n = r$. Hence, the system of equations

$$(1) \qquad \begin{cases} (n-1)(z_{n-1}+1) \equiv (n-i+j-1) \mod k \\ z_{n-1} + nz_n \equiv -r \mod k, \end{cases}$$

with integer unknowns z_{n-1} and z_n, must have a solution. Now, from (1) we have

$$\begin{cases} (n-1)z_{n-1} \equiv (j-i) \mod k \\ (n-1)z_{n-1} + n(n-1)z_n \equiv -r(n-1) \mod k \end{cases}$$

and so the equation

$$n(n-1)z_n \equiv -(r(n-1)+j-i) \mod k$$

has a solution, which implies that $gcd(n(n-1),k)$ divides $r(n-1)+j-i$. It follows, by Proposition 1.22, that the set $\mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{POPI}_n) \cap J_{n-1}$ has at most $\frac{n^2(n-1)}{gcd(n(n-1),k)}$ elements. Since $gcd(n,n-1)=1$, then $gcd(n(n-1),k) = gcd(n,k)\,gcd(n-1,k)$ and so we have precisely

$$|\mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{POPI}_n) \cap J_{n-1}| = \frac{n^2(n-1)}{gcd(n(n-1),k)}.$$

Therefore we have:

Theorem 4.8. *For all $k \in \mathbb{N}$, the relative kernel $\mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{POPI}_n)$ consists of all permutations generated by $g^{gcd(n,k)}$, of all elements of rank $n-1$ of the form $g^r x_i x_{i+1} \cdots x_{n-1} x_0 x_1 \cdots x_{j-1}$, with $1 \le i \le n-1, 1 \le j \le n, 0 \le r \le n-1$ and $gcd(n(n-1),k) \mid r(n-1)+j-i$, and all transformations with rank less than $n-1$.* □

Now, let $k = gcd(n(n-1),\pi)$. Notice that, clearly, $gcd(n,\pi) = gcd(n,k)$, $gcd(n-1,\pi) = gcd(n-1,k)$ and $k = gcd(n(n-1),k) = gcd(n,k)\,gcd(n-1,k) = gcd(n,\pi)\,gcd(n-1,\pi)$, since $gcd(n,n-1) = 1$. Thus, in particular, we have

$$\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POPI}_n) \cap J_n = \langle g^{gcd(n,\pi)} \rangle = \langle g^{gcd(n,k)} \rangle = \mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{POPI}_n) \cap J_n$$

and (as k divides π)

$$\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POPI}_n) \subseteq \mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{POPI}_n).$$

Moreover, since

$$|\mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{POPI}_n) \cap J_{n-1}| = \frac{n^2(n-1)}{gcd(n(n-1),k)} = \frac{n^2(n-1)}{gcd(n,\pi)\,gcd(n-1,\pi)} \le |\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POPI}_n) \cap J_{n-1}|,$$

we have proved:

Theorem 4.9. *Let π be a supernatural number. Then $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{POPI}_n) = \mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{POPI}_n)$, with $k = gcd(n(n-1),\pi)$.* □

4.4. **The case of $\mathcal{PORI}_n$.** First notice that Corollaries 3.3 and 2.6 combined with the fact that the maximal subgroups of $\mathcal{PORI}_n$ are dihedral groups allow us to conclude immediately:

Proposition 4.10. *If π is an odd supernatural number, then $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PORI}_n) = \mathcal{PORI}_n$.* □

Next we will prove that $\mathsf{K}_{\mathsf{H}_2}(\mathcal{PORI}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{PORI}_n)$.

Recall that in [8] it was proved that the abelian kernel of $\mathcal{PORI}_n$ is contained in $\mathcal{POPI}_n$. Denote by J_i the $\mathcal{J}$-class of $\mathcal{PORI}_n$ of all elements of rank i, for $0 \le i \le n$. The following result gives a description of the elements of $\mathcal{PORI}_n \cap J_{n-1}$ that are in the abelian kernel of $\mathcal{PORI}_n$.

Recall that $\{x_0, x_1, \ldots, x_{n-1}, h, g\}$ is a set of generators of $\mathcal{PORI}_n$.

Theorem 4.11. [8] *Let $X = \{x_1, \ldots, x_{n-1}\}$ and let x be the element of $J_{n-1} \cap \mathcal{POPI}_n$ corresponding to the word $w = g^k(x_{n-i} \cdots x_{n-1})x_0(x_1 \cdots x_j)$, with $0 \le k \le n-1$, $1 \le i \le n-1$ and $0 \le j \le n-1$. Then $x \in \mathsf{K}_{\mathsf{Ab}}(\mathcal{PORI}_n)$ if and only if:*

(i) *$|w|_X$ is even, for n odd;*
(ii) *$|w|$ is even, for n even.*

¿From Lemmas 1.7 and 2.4 we have that $\mathsf{K}_{\mathsf{Ab}}(\mathcal{PORI}_n) \cap J_n = \langle g^2 \rangle$ and it was also observed in [8] that, for $k < n-1$, $\mathsf{K}_{\mathsf{Ab}}(\mathcal{PORI}_n) \cap J_k$ consists of the elements of J_k that belong to $\mathcal{POPI}_n$.

Now, using the presentation of $\mathcal{PORI}_n$ recalled in Section 3, as an immediate consequence of Theorem 1.15, we have:

Corollary 4.12. *If k is even, then $\mathsf{K}_{\mathsf{H}_\mathsf{k}}(\mathcal{PORI}_n) \subset \mathcal{POPI}_n$.*

It follows from Proposition 1.14 that if $x \in \mathsf{K}_{\mathsf{H}_2}(\mathcal{PORI}_n)$ then there exists a word w representing x such that $|w|_Y$ is even, for all subset Y of the set of generators, whence, $x \in \mathsf{K}_{\mathsf{Ab}}(\mathcal{PORI}_n)$, by Theorem 4.11. Therefore:

Proposition 4.13. $\mathsf{K}_{\mathsf{H}_2}(\mathcal{PORI}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{PORI}_n)$. $\square$

By Propositions 4.10, 4.13 and 1.9, we get the main result of this subsection:

Theorem 4.14. *Let π be a supernatural number and H_π the associated pseudovariety of abelian groups. Then $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{PORI}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{PORI}_n)$ if π is even and $K_{\mathsf{H}_\pi}(\mathcal{PORI}_n) = \mathcal{PORI}_n$ if π is odd.* $\square$

4.5. **The case of $\mathcal{I}_n$.** We start this subsection, as the previous one, by noticing that Corollaries 3.1 and 2.6 combined with the fact that the maximal subgroups of $\mathcal{I}_n$ are symmetric groups imply the following:

Proposition 4.15. *If π is an odd supernatural number, then $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{I}_n) = \mathcal{I}_n$.* $\square$

Now recall a description of the abelian kernel of $\mathcal{I}_n$ given by the second and third authors [8].

Theorem 4.16. *The abelian kernel of $\mathcal{I}_n$ consists of all even permutations of $\mathcal{S}_n$, of all the $\frac{1}{2}|J_{n-1}|$ elements (with rank $n-1$) of $\mathcal{A}_{n-1}H'\mathcal{A}_{n-1}$, where H is any maximal subgroup of J_{n-1}, and of all transformations with rank less than $n-1$.* $\square$

Next we will prove that $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{I}_n)$. Notice that, by Theorem 2.5, the elements of rank n belong to $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n)$ exactly when they belong to $\mathsf{K}_{\mathsf{Ab}}(\mathcal{I}_n)$. As $\mathsf{K}_{\mathsf{Ab}}(\mathcal{I}_n)$ contains all transformations with rank less than $n-1$ and is contained in $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n)$, it remains to check that $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n) \cap J_{n-1} = \mathsf{K}_{\mathsf{Ab}}(\mathcal{I}_n) \cap J_{n-1}$. Since $\mathsf{K}_{\mathsf{Ab}}(\mathcal{I}_n) \subseteq \mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n)$, can be concluded by showing that both sets have the same number of elements, that is, by showing that $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n) \cap J_{n-1}$ has $\frac{1}{2}|J_{n-1}|$ elements. To achieve this we will follow the strategy used in [8] to prove part of Theorem 4.16. With the same proof as [8, Lemma 6.1], we have the following lemma, which implies that all $\mathcal{H}$-classes of $\mathcal{I}_n \cap J_{n-1}$ have the same number of elements in $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n)$.

First we need some notation. Let $1 \le r, s \le n$. We denote by H_s^r the $\mathcal{H}$-class of J_{n-1} of the elements x such that $\mathrm{Dom}(x) = \{1, 2, \dots, n\} \setminus \{r\}$ and $\mathrm{Im}(x) = \{1, 2, \dots, n\} \setminus \{s\}$. We define $K_s^r = H_s^r \cap \mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n)$.

Lemma 4.17. *For $1 \le r, s, u, t \le n$, there exist $\sigma, \nu \in \mathcal{A}_n$ such that $\sigma K_s^r \nu = K_v^u$ and $\sigma K_v^u \nu = K_s^r$.*

Now we look at the relations of $\mathcal{I}_n$ given in Subsection 3.1. The word cac corresponds to an element of rank $n-2$, whence none of the relations $(ca)^2 = cac = (ac)^2$ can be applied to an element of rank greater than $n-2$. On the other hand, the words involved in the relations $a^2 = g^n = (ga)^{n-1} = (ag^{n-1}ag)^3 = (ag^{n-j}ag^j)^2 = 1$ and $g^{n-1}agcg^{n-1}ag = gacag^{n-1} = c = c^2$ correspond to elements of rank not smaller than $n-1$. Then we have:

Lemma 4.18. *The element of $J_{n-1} \cap \mathcal{I}_n$ represented by the word ac does not belong to $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n)$.*

Proof. By Proposition 1.14, if an element $x \in \mathcal{I}_n$ belongs to $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n)$, then there exists a word u representing x such that

$$\left\{ \begin{array}{ll} |u|_a \equiv 0 & \mod 2 \\ |u|_g \equiv 0 & \mod 2 \end{array} \right. \tag{2}$$

We will conclude that no word representing the same element of $\mathcal{I}_n$ than ac satisfies the condition (2), which proves that the element of $\mathcal{I}_n$ represented by ac does not belong to $\mathsf{K}_{\mathsf{H}_k}(\mathcal{I}_n)$.

We have to take the parity of n into account.

If n is even, then it is easy to check that any word u obtained from ac using the relations above is such that $|u|_a$ and $|u|_g$ have different parities. Thus $|u|_a \mod 2 \neq |u|_g \mod 2$ and so the condition (2) is not verified.

If n is odd, by applying to ac the relations above we only obtain words u such that $|u|_a$ is odd. Thus also in this case condition (2) is not verified, as required. □

Let H be a maximal subgroup of $J_{n-1} \cap \mathcal{I}_n$. Then H is isomorphic to $\mathcal{S}_{n-1}$ and its derived subgroup, H', being isomorphic to $\mathcal{A}_{n-1}$, has index 2 in H. Thus, the only subgroups of H containing H' are H and H' itself. It follows from Lemma 4.18 there exists an element in the $\mathcal{J}$-class J_{n-1} not belonging to $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n)$. As Lemma 4.17 guaranties that all $\mathcal{H}$-classes have the same number of elements in $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n)$, then each $\mathcal{H}$-class has precisely half of the elements in $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n)$.

We have proved the result announced:

Proposition 4.19. $\mathsf{K}_{\mathsf{H}_2}(\mathcal{I}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{I}_n)$. □

Finally,using Propositions 4.15, 4.19 and 1.9 we get the main result of this subsection:

Theorem 4.20. *Let π be a supernatural number and H_π the associated pseudovariety of abelian groups. Then $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{I}_n) = \mathsf{K}_{\mathsf{Ab}}(\mathcal{I}_n)$ if π is even and $\mathsf{K}_{\mathsf{H}_\pi}(\mathcal{I}_n) = \mathcal{I}_n$ if π is odd.* □

5. Consequences

The notion of H-kernel is tightly related to an important operator of pseudovarieties: the *Mal'cev product* (see [17]). Its definition, when the first factor is a pseudovariety V of monoids and the second factor is a pseudovariety H of groups, may be given as follows:

$$\mathsf{V} \,ⓜ\, \mathsf{H} = \{M \in \mathsf{M} : \mathsf{K}_{\mathsf{H}}(M) \in \mathsf{V}\}.$$

Let POI, PODI, POPI and PORI be the pseudovarieties of monoids generated respectively by $\{\mathcal{POI}_n \mid n \in \mathbb{N}\}$, $\{\mathcal{PODI}_n \mid n \in \mathbb{N}\}$, $\{\mathcal{POPI}_n \mid n \in \mathbb{N}\}$ and $\{\mathcal{PORI}_n \mid n \in \mathbb{N}\}$.

In [8, Corollaries 3.7 and 3.8] the second and third authors observed that $\mathsf{PODI} \subset \mathsf{POI} ⓜ \mathsf{Ab}$ and that $\mathsf{PORI} \subseteq \mathsf{POPI} ⓜ \mathsf{Ab}$. From the work done here it follows better upper bounds. In fact, using Corollary 4.4 we obtain the following:

Corollary 5.1. *The inclusion* $\mathsf{PODI} \subseteq \mathsf{POI} ⓜ \mathsf{H}_2$ *holds.* □

Similarly, using Corollary 4.12, we have:

Corollary 5.2. *The inclusion* $\mathsf{PORI} \subseteq \mathsf{POPI} ⓜ \mathsf{H}_2$ *holds.* □

The work presented in this paper was originally motivated by an attempt to compare these pseudovarieties. Although possibly far from obtaining a solution, we leave here the following conjectures:

Conjecture 5.3. *The equality* $\mathsf{PODI} = \mathsf{POI} ⓜ \mathsf{H}_2$ *holds.* □

Conjecture 5.4. *The equality* $\mathsf{PORI} = \mathsf{POPI} ⓜ \mathsf{H}_2$ *holds.* □

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