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IN

INFORMATION TECHNOLOGY AND ELECTRICAL ENGINEERING

CONTRACTION ANALYSIS OF SWITCHED SYSTEMS

WITH APPLICATION TO CONTROL AND OBSERVER DESIGN

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To Claudia,
for her support and love

and the memory of
my beloved dog, Penny

Abstract

In many control problems, such as tracking and regulation, observer design, coordination and synchronization, it is more natural to describe the stability problem in terms of the asymptotic convergence of trajectories with respect to one another, a property known as incremental stability. Contraction analysis exploits the stability properties of the linearized dynamics to infer incremental stability properties of nonlinear systems. However, results available in the literature do not fully encompass the case of switched dynamical systems.

To overcome these limitations, in this thesis we present a novel extension of contraction analysis to such systems based on matrix measures and differential Lyapunov functions. The analysis is conducted first regularizing the system, i.e. approximating it with a smooth dynamical system, and then applying standard contraction results. Based on our new conditions, we present design procedures to synthesize switching control inputs to incrementally stabilize a class of smooth nonlinear systems, and to design state observers for a large class of nonlinear switched systems including those exhibiting sliding motion.

In addition, as further work, we present new conditions for the onset of synchronization and consensus patterns in complex networks. Specifically, we show that if network nodes exhibit some symmetry and if the network topology is properly balanced by an appropriate designed communication protocol, then symmetry of the nodes can be exploited to achieve a synchronization/consensus pattern.

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Contents

CHAPTER 1

Introduction

In a large number of applications the phenomena of interest are captured by an interaction between continuous dynamics and discrete events. For example, in mechanics the motion of a rigid body undergoes velocity jumps and force impulses as a result of friction and impacts [18], in electronic devices switches opening and closing cause discontinuities in currents or voltages, in biological systems some processes are activated or deactivated when concentrations of chemicals reach certain thresholds, and many others.

Discontinuous behaviors are also intentionally designed to control the evolution of dynamical systems. Relay controllers switch the input on and off to regulate a certain state variable around a desired mean value; bang-bang controllers switch between boundary values of admissible input to generate minimum-time trajectories between initial and final states; variable structure systems, such as sliding mode control [140, 139], are used to design control inputs that render the closed-loop system robust to model uncertainties and disturbances acting on the plant, and so on. Also, there exist classes of smooth dynamical systems, e.g. nonholonomic systems [72], that cannot be stabilized by continuous feedback laws.

All these examples fit into the broad class of *switched dynamical systems* [23, 72, 32, 57]. These systems, when modelled as ODEs with discontinuous right-hand sides are also called *Filippov systems* from the name of the mathematician who firstly studied them [44], and characterized by vector fields that are discontinuous on lower dimensional manifolds on which switchings occur. For this class of systems the classical definition of solution for smooth systems is too restrictive and needs to be relaxed. Such systems can also exhibit behaviors that are peculiar to this class, such as *sliding motion*, occurring when a solution is constrained to evolve on the discontinuity manifolds (see Chapter 2 for further details).

As for classical nonlinear systems, a fundamental problem arising in both analysis and control of switched systems is that of determining their stability. The most common approach in the study of stability properties of nonlinear dynamical systems is Lyapunov's direct method [81, 124, 17]. Unfortunately, in many applications, it is not easy to find a suitable Lyapunov function. Moreover, in many control problems, e.g. tracking and regulation, observer design, coordination and synchronization, the steady state solution on which to conduct the Lyapunov analysis is not known a priori. In these contexts it is more natural to describe the stability problem in terms of the asymptotic convergence of trajectories with respect to one another rather than towards some attractor. That is, it is of interest to study the so-called *incremental stability* of the trajectories of the closed-loop system [9, 51].

A particularly interesting and effective approach to obtain sufficient conditions for incremental stability of nonlinear systems comes from contraction analysis [76, 65, 7, 50]. A nonlinear system is said to be *contracting* if initial conditions or temporary state perturbations are forgotten exponentially fast, implying exponential convergence of system trajectories towards each other and consequently towards a steady-state solution which is determined only by the input.

The original definition of contraction requires the system vector field to be *continuously differentiable*, but this is *not* the case for many classes of dynamical systems, such as switched systems. Several results have been presented in the literature to extend contraction analysis to non-differentiable vector fields [77, 117, 36, 136, 95]. An extension of contraction analysis to characterize incremental stability of sliding mode solutions of switched systems was presented in [35, 33] where trajectories are shown to contract only after they have reached a globally attracting sliding manifold. Therefore, contraction is guaranteed only *after a small transient* is extinguished (a *weak* form of contraction [90]). Moreover, the matrix measure is evaluated on the Jacobian matrix of the sliding vector field projected onto the sliding manifold, requiring therefore additional analysis.

Motivated by the previous limitations, in this thesis first we present a new generic approach which indeed does not need the explicit computation of the sliding vector field and guarantees classical (strong) contraction of solutions. The analysis is based on the *regularization* approach [129, 75], consisting in replacing the discontinuous system with a smooth dynamical system that approximates the dynamics of the original system around the points of discontinuity. This allows us to conduct differential analysis and to apply classical conditions for contraction of smooth systems. It is shown that the contraction conditions derived using matrix measures have a simple geometric meaning and can also be applied to nonlinear switched systems.

Then, using the novel theoretical conditions derived, we propose a control design strategy to incrementally stabilize a class of smooth nonlinear systems using switched control actions, and we present conditions for the design of state observers for a large class of nonlinear switched systems including those exhibiting sliding motion.

However, the previous conditions require the switched system to be contracting during *both* flow and switching, excluding from the analysis those systems that contracts only during either one. To overcome this further limitation, we make use of the more general and flexible tools of Finsler-Lyapunov functions [50] and Lyapunov stability theory for hybrid systems [58]. The derived conditions are then applied to two simple examples, clearly illustrating the advantages.

Moreover, we discuss some additional work related to the topic of antagonistic consensus and synchronization in networks of dynamical systems.

1.1 Thesis outline

In Chapter 2 we introduce some basic notions on switched dynamical systems and the definitions of solution for such systems. We recall the derivation of the fundamental solution matrix and how this tool can be extended to Filippov systems using the so-called saltation matrix.

Moreover, the regularization approach to the analysis of discontinuous systems is discussed, on which our contraction analysis of switched systems presented in Chapters 4 and 5 is based. Furthermore, the mathematical concept of matrix measure induced by a norm and its properties are presented.

In Chapter 3, after introducing the concept of incremental stability, we review some of the available literature on contraction analysis. Specifically, we first present the definition of contraction as presented by Slotine and his coauthors and then we discuss those based on matrix measures, convergence and Finsler-Lyapunov functions. We discuss in major details the definition of contraction based on matrix measures since it will be used extensively in Chapter 4. Finally, we review several results that have been presented in the literature to extend contraction analysis to non-differentiable dynamical systems.

In Chapter 4 we address the problem of extending contraction analysis based on matrix measures to switched systems. Instead of directly analyzing the discontinuous system, we first consider its regularization, then we apply standard contraction analysis based on matrix measures. Sufficient conditions for a switched system to be contracting are then obtained as the limit of those for its regularization. Specifically, a bimodal switched system is contracting if both modes of the system are contracting and if the difference of the two vector fields satisfies an additional condition on the switching surface. We then use the results to synthesize switched controllers and state observers.

The results of this chapter have been obtained in collaboration with Prof. John Hogan, Department of Engineering Mathematics, University of Bristol, UK [45].

In Chapter 5, using the same regularization approach proposed in Chapter 4, we derive an extension of the analysis based on Finsler-Lyapunov functions to switched systems. It is shown that the differential dynamics of a switched system has a hybrid nature, combining continuous and discrete dynamics. The continuous dynamics is related to flow and it is described by the Jacobian matrix, while the discrete dynamics is related to switching events and it is captured by the saltation matrix. Then, we formulate sufficient conditions for contraction based on Finsler-Lyapunov functions, and we illustrate them with examples. Finally, we compare these new conditions to those presented in the previous chapter.

The results of this chapter have been obtained in collaboration with Prof. Rodolphe Sepulchre and Dr. Fulvio Forni during a four months visit at the Department of Engineering, University of Cambridge, UK.

In Chapter 6 we present additional work, that has been carried out during the PhD course in collaboration with Dr. Giovanni Russo of the Optimization and Control Group of IBM Research Ireland, on the study of the emergence of antagonistic consensus and synchronization in complex networks of dynamical agents. The problem, known as bipartite consensus, has been recently investigated by Altafini [5] in networks of integrators whose nodes can be divided in two antagonistic groups that converge each one on different and opposite solutions. Exploiting particular symmetries of the nodes' vector fields, we present the definitions of multipartite consensus and synchronization, generalizing the theory to the case of n-dimensional nonlinear agents that can be divided in more than two groups.

In Chapter 7 conclusions are drawn.

The work described in the thesis resulted in the following scientific publications, reported in chronological order:

- Mario di Bernardo and Davide Fiore, "Incremental stability of bimodal Filippov systems in $\mathbb{R}^n$", *Proceedings of the 53rd IEEE Conference on Decision and Control*, pages 4679-4684, 2014

- Mario di Bernardo, Davide Fiore, Giovanni Russo and Francesco Scafuti, "Convergence, Consensus and Synchronization of Complex Networks via Contraction Theory", in *Complex Systems and Networks: Dynamics, Controls and Applications*, edited by Jinhu Lü, Xinghuo Yu, Guanrong Chen and Wenwu Yu, Springer Berlin Heidelberg, pages 313-339, 2016

- Davide Fiore, S. John Hogan and Mario di Bernardo, "Contraction analysis of switched systems via regularization", *Automatica* 73, pages 279-288, 2016

- Mario di Bernardo and Davide Fiore, “Switching control for incremental stabilization of nonlinear systems via contraction theory”, *Proceedings of the European Control Conference*, pages 2054-2059, 2016

- Davide Fiore, Marco Coraggio and Mario di Bernardo, “Observer design for piecewise smooth and switched systems via contraction theory”, accepted to *IFAC 2017 World Congress*

CHAPTER 2

Mathematical preliminaries

In this chapter we introduce some concepts that we are going to use in the rest of this thesis.

2.1 Switched dynamical systems

In what follows we introduce the class of dynamical systems that we are going to study in the rest of this thesis. These systems, also known as *Filippov systems* [44], are a subset of the more general class of *piecewise smooth dynamical systems* (PWS) [32, p.73].

Definition 2.1. *A (state-dependent)* switched system *consists of a finite set of ordinary differential equations*

$$\dot{x} = f(t, x) := f_i(t, x), \quad x \in \mathcal{S}_i, \tag{2-1}$$

where $\mathcal{S}_i$ is an open set with non-empty interior such that $\cup_i \bar{\mathcal{S}}_i = D \subseteq \mathbb{R}^n$. The intersection Σ_{ij} between the closure of the sets $\bar{\mathcal{S}}_i$ and $\bar{\mathcal{S}}_j$ is either an $\mathbb{R}^{(n-1)}$-dimensional manifold included in the boundaries $\partial\mathcal{S}_i$ and $\partial\mathcal{S}_j$, called switching manifold*, or is an empty set. Furthermore, we assume that each vector fields f_i is C^k, $k \geq 1$, in $(t, x) \in [t_0, +\infty) \times \mathcal{S}_i$ and is smoothly extendable to the closure of $\mathcal{S}_i$.*

In Figure 2-1 a graphical representation is shown of the regions of the state space of a switched system. In some context these systems

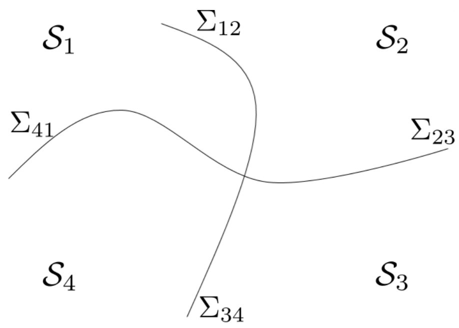

Figure 2-1: Example of regions of state spate of a switched system.

are also called *state-dependent* switched systems, to emphasize that the switchings between vector fields f_i occur when a particular condition is met on the state variable x, and to distinguish them from *time-dependent* switched systems [72], in which the switching depends on the evolution of an auxiliary signal.

Definition 2.2. *A* time-dependent switched system *is a dynamical system of the form*

$$\dot{x} = f(t, x, \sigma(t)),$$

where $\sigma : [t_0, +\infty) \to \{1, \ldots, p\}$ *is a piecewise continuous function, called* switching signal*. Moreover, we suppose that for each fixed* σ *the function* f *is continuously differentiable in both* x *and* t*.*

Note that, according to Definition 2.2, $\sigma(t)$ has a finite number of discontinuities on every bounded time interval and takes constant value on every interval between two consecutive switching times, therefore the accumulation of infinitely many switching events on a finite time interval, a phenomenon also known as *Zeno behavior* or chattering, cannot occur [72].

In the particular case that the vector field f of the switched system (2-1) is continuous but not differentiable on the switching manifolds Σ_{ij}, a system of the form (2-1) is termed as *piecewise smooth continuous* (PWSC).

Definition 2.3. *A piecewise smooth system* (2-1) *is said to be* continuous *if the following conditions hold:*

1. *the function $f(t,x)$ is continuous for all $x \in \mathbb{R}^n$ and for all $t \geq t_0$*
2. *the function $f_i(t,x)$ is continuously differentiable for all $x \in \mathcal{S}_i$ uniformly in t. Furthermore, the Jacobian matrix $\frac{\partial f_i}{\partial x}(t,x)$ can be continuously extended on the boundary ∂S_i.*

Bimodal switched systems In the particular case where the state space of system (2-1) is divided into only two disjoint regions, the switched system is said to be *bimodal*. More importantly, bimodal switched systems can be used to analyze the dynamics of system (2-1) at switching events *locally* to a certain manifold Σ_{ij} and away from points of intersection between manifolds. In this latter case, the system dynamics at these points depends on all the vector fields f_i defined around them and has to be properly defined [44, 32, 39].

A *bimodal switched system* is a dynamical system $\dot{x} = f(x)$ where $f(x)$ is a piecewise continuous vector field having a codimension one submanifold Σ as its discontinuity set and defined as

$$f(x) = \begin{cases} f^+(x) & \text{if } x \in \mathcal{S}^+ \\ f^-(x) & \text{if } x \in \mathcal{S}^- \end{cases} \tag{2-2}$$

where the functions f^+ and f^- are C^k, $k \geq 1$, in $\mathcal{S}^+$ and $\mathcal{S}^-$, respectively. The switching manifold is defined as the zero set of a scalar function $h : D \to \mathbb{R}$, called *indicator function*, that is

$$\Sigma := \{x \in D : h(x) = 0\}, \tag{2-3}$$

and it divides D into two disjoint regions,

$$\mathcal{S}^+ := \{x \in D : h(x) > 0\},$$

and

$$\mathcal{S}^- := \{x \in D : h(x) < 0\},$$

illustrated in Figure 2-2. Furthermore, we assume that h is C^k, $k \geq 2$, and that $0 \in \mathbb{R}$ is a regular value of h, i.e. $\forall x \in \Sigma$

$$\nabla h(x) = \begin{bmatrix} \frac{\partial h(x)}{\partial x_1} & \dots & \frac{\partial h(x)}{\partial x_n} \end{bmatrix} \neq 0.$$

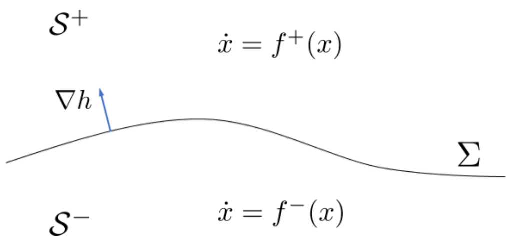

Figure 2-2: Regions of state spate of a *bimodal* switched system.

In the case that the vector fields in (2-2) are affine functions of x and Σ is a hyperplane, the switched system is said to be *piecewise affine* (PWA).

Definition 2.4. *A bimodal PWA system is a system of the form*

$$\dot{x} = \begin{cases} A_1 x + b_1 + Bu & \text{if } h^T x > 0 \\ A_2 x + b_2 + Bu & \text{if } h^T x < 0 \end{cases} \tag{2-4}$$

where x, $h \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $A_i \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $b_i \in \mathbb{R}^n$, $i = 1, 2$, are constant matrices and vectors, respectively.

When $b_i = 0$ the system is called *piecewise linear* (PWL). Moreover, notice that relay feedback systems are a particular class of PWA systems.

2.1.1 Solutions of switched systems

Before presenting some solution concepts for switched systems, let us recall the classical definition of solution to smooth dynamical systems.

Definition 2.5. *Consider a dynamical system of the form*

$$\dot{x} = f(t, x(t)), \quad x(t_0) = x_0, \tag{2-5}$$

where $x \in D \subseteq \mathbb{R}^n$ and $f : [t_0, +\infty) \times D \to \mathbb{R}^n$ is a continuously differentiable vector field. A (classical) *solution to* (2-5) *is a continuously differentiable function $x(t)$ defined on an interval $[t_0, t_1]$ that satisfies* (2-5) *for every $t \in [t_0, t_1]$.*

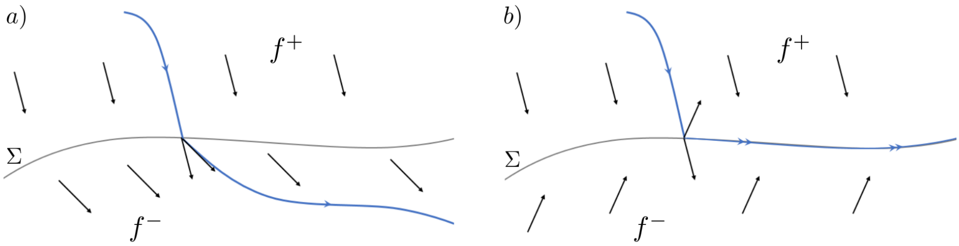

Figure 2-3: Examples of crossing (panel a) and sliding (panel b).

Clearly, this definition cannot be applied to a trajectory of system (2-2) that reaches a point on Σ. In this case two scenarios are possible.

- When the normal components of the vector fields either side of Σ are in the same direction, a trajectory transversally intersects Σ in only one point evolving from one side to the other (Figure 2-3a). The dynamics is described as *crossing* or sewing, and the so-called *Caratheodory* notion of solutions is adopted [44, 60], that relaxes the classical definition allowing solutions not to satisfy the differential equation on a set Σ of zero measure.

- When the normal components of the vector fields on either side of Σ are in the opposite direction, a trajectory is constrained to evolve (or *slide*) on Σ and therefore the previous definition does not apply any longer (Figure 2-3b). When this happens, a new solution concept needs to be used that takes the name of *Filippov solution* [44]. The key idea is to replace the vector field with a set-valued map and the differential equation with a differential inclusion [13]. So that, instead of looking at the value of the vector field at a certain point, we can consider its values in a neighborhood of that point.

In the following these two definitions are discussed in more detail and some conditions for existence and uniqueness are given.

Note that, in general, discontinuous dynamical systems can admit notions of solutions different from those described here. In this thesis we are interested only in *absolutely continuous* solutions, therefore solutions with jumps are not considered here. For further details see [23] and references therein.

2.1.1.1 Caratheodory solutions

Before introducing the definition of Caratheodory solution we recall the following preliminary notions.

Definition 2.6. *A function $g : [t_0, +\infty) \to \mathbb{R}$ is said to be* measurable *if, for any real number a, the set $\{t \in [t_0, +\infty) : g(t) > a\}$ is measurable in the sense of Lebesgue.*

Definition 2.7. *A function $l : [t_0, +\infty) \to \mathbb{R}$ is* summable *if the Lebesgue integral of the absolute value of $l(t)$ exists and is finite.*

Definition 2.8. *A function $z : [a, b] \to \mathbb{R}^n$ is* absolutely continuous *if for all $\varepsilon > 0$ there exists $\delta > 0$ such that, for each finite collection $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ of disjoint open intervals contained in $[a, b]$ with $\sum_i (b_i - a_i) < \delta$, it follows that $\sum_i |z(b_i) - z(a_i)| < \varepsilon$.*

Equivalently, $z(t)$ is absolutely continuous if there exists a Lebesgue integrable function $\kappa : [a, b] \to \mathbb{R}$ such that

$$z(t) = z(a) + \int_a^t \kappa(s)ds, \quad t \in [a, b].$$

Note that every absolutely continuous function is, obviously, continuous and it is differentiable almost everywhere, i.e. except on a set of Lebesgue measure zero.

We can now introduce the Caratheodory conditions of a vector field and the definition of Caratheodory solutions [60, p.28] [44, p.3].

Definition 2.9. *Suppose $D \subseteq \mathbb{R}^n$ is an open set. We say that the vector field $f : [t_0, +\infty) \times D \to \mathbb{R}^n$ satisfies the* Caratheodory conditions *on D if*

1. *the function $f(t, x)$ is continuous in x for each fixed t;*
2. *the function $f(t, x)$ is measurable in t for each fixed x;*
3. *for each compact set of $[t_0, +\infty) \times D$ there is a summable function $m(t)$ such that $|f(t, x)| \leq m(t)$.*

Definition 2.10. *Consider an open set $D \subseteq \mathbb{R}^n$ and a dynamical system of the form*

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2-6}$$

where $f : [t_0, +\infty) \times D \to \mathbb{R}^n$ *satisfies the Caratheodory conditions. A* Caratheodory solution *is an absolutely continuous function* $x(t)$ *defined on an interval* $[t_0, t_1]$ *that satisfies* (2-6) *almost everywhere in* $[t_0, t_1]$ *(in the sense of Lebesgue). Or equivalently, a Caratheodory solution of* (2-6) *is an absolutely continuous function* $x(t)$ *such that*

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau))d\tau, \quad t \in [t_0, t_1].$$

The following theorem provides sufficient conditions for existence and uniqueness of Caratheodory solutions.

Theorem 2.1. *A Caratheodory solution of system* (2-6) *exists if* $f(t,x)$ *satisfies the Caratheodory conditions on its set of definition. Moreover, the solution is unique if for each compact set of* $[t_0, +\infty) \times D$ *the vector field* f *is locally one-sided Lipschitz, that is there exists a summable function* $l(t)$ *such that*

$$(x - y)^T (f(t, x) - f(t, y)) \leq l(t)|x - y|^2.$$

Note that, for differential equations (2-6), the above conditions on uniqueness of solution lead to its continuous dependence on initial conditions [44, p.10]-[8].

2.1.1.2 Filippov solutions

Definition 2.11. *Consider an open set* $D \subseteq \mathbb{R}^n$ *and a dynamical system of the form*

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{2-7}$$

where $f : [t_0, +\infty) \times D \to \mathbb{R}^n$*. A* Filippov solution *of* (2-7) *is an absolutely continuous function* $x(t)$ *defined on an interval* $[t_0, t_1]$ *that satisfies almost everywhere in* $[t_0, t_1]$ *the differential inclusion*

$$\dot{x}(t) \in \mathcal{F}[f](t, x), \tag{2-8}$$

where $\mathcal{F}[f](t, x)$ *is the* Filippov set-valued map $\mathcal{F}[f] : [t_0, +\infty) \times D \to \mathfrak{B}(\mathbb{R}^n)$*, with* $\mathfrak{B}(\mathbb{R}^n)$ *being the collection of all the subsets in* $\mathbb{R}^n$*, defined as*

$$\mathcal{F}[f](t, x) = \bigcap_{\delta > 0} \bigcap_{m(\Sigma) = 0} \overline{\text{co}} \{f(t, \mathcal{B}_\delta(x) \setminus \Sigma)\}, \tag{2-9}$$

where Σ is any set of zero Lebesgue measure $m(\cdot)$, $\mathcal{B}_\delta(x)$ is an open ball centered in x with radius $\delta > 0$, and $\overline{\text{co}}\{\mathcal{I}\}$ denotes the convex closure of a set $\mathcal{I}$.

Equivalently, we can say that a Filippov solution of (2-7) is a Caratheodory solution of the differential inclusion (2-8) [23].

A Filippov solution exists for systems (2-1) under the mild assumption of local essential boundedness of the vector field f, moreover Filippov solutions shows continuous dependence on the initial conditions [23, 44].

Referring for simplicity to the case of bimodal switched systems (2-2), for the consistency property [23], the Filippov set-valued map (2-9) at points of continuity of f, is

$$\mathcal{F}[f](t,x) = \{f(t,x)\}, \quad x \notin \Sigma,$$

that is it takes the same value of f, while on the point of discontinuity it is

$$\mathcal{F}[f](t,x) = \overline{\text{co}}\left\{f^+(t,x), f^-(t,x)\right\}, \quad x \in \Sigma, \tag{2-10}$$

that is the convex combination of f^+ and f^-.

Depending on the directions at which the vectors in (2-10) point we distinguish the following regions on Σ:

1. The *crossing region* is $\Sigma_c := \{x \in \Sigma : f_N^+(x) \cdot f_N^-(x) > 0\}$;
2. The *sliding region* is $\Sigma_s := \{x \in \Sigma : f_N^+(x) < 0,\ f_N^-(x) > 0\}$;
3. The *escaping region* is $\Sigma_e := \{x \in \Sigma : f_N^+(x) > 0,\ f_N^-(x) < 0\}$;

where f_N^+ and f_N^- are the projections of the vectors f^+ and f^- onto the gradient ∇H orthogonal to te switching manifold Σ at the point x.

The sliding region is so called because a solution, that reaches it, is constrained to *slide* on it. The particular choice of the vector field adopted on the sliding region depends on the nature of the problem under consideration. According to the widely used Filippov convention [44, p.50], we can define the *sliding vector field* f^s as the convex combination of f^+ and f^- that is *tangent* to Σ, given for $x \in \Sigma_s$ by

$$f^s(x) = f^-(x) + \lambda\left[f^+(x) - f^-(x)\right], \qquad \lambda \in [0,1] \tag{2-11}$$

with λ such that $\nabla h(x)\, f^s(x) = 0$. Other possible ways to define the sliding vector field were presented in the literature, among them we mention the *equivalent control method* proposed by Utkin [139] and the general definition by Aizerman and Pyatnitskii in [4].

In the following, we assume that for systems (2-2) and (2-11) *right-uniqueness* of Filippov solutions [44, p. 106] holds in D. A sufficient condition for right-uniqueness is given in the following theorem [44, p. 110].

Theorem 2.2. *Consider systems* (2-2) *and* (2-11) *such that* Σ *is a* C^2 *manifold, and* f^+, f^- *and* $f^+ - f^-$ *are* C^1 *functions. If for each* $t \in [t_0, t_1]$ *at each point* $x \in \Sigma$ *at least one of the inequalities* $f_N^- > 0$ *or* $f_N^+ < 0$ *is fulfilled, then right uniqueness of solutions holds in* D.

Therefore, the escaping region is excluded from our analysis, because solutions evolving on it (also called *repulsive sliding mode*) at any instant of time may leave with f^+ or f^-.

Additional results guaranteeing uniqueness of Filippov solutions for specific classes of switched systems (e.g relay feedback systems [102], piecewise affine systems [137], piecewise linear systems [64]) are available in the literature.

2.2 Fundamental solution matrix and saltation matrix

The fundamental solution matrix is a useful tool in the study of stability of perturbations to periodic solutions and bifurcation analysis of nonlinear dynamical systems. The eigenvalues of the fundamental solution matrix evaluated at period T on a solution close to a periodic orbit are called *Floquet multipliers* and provide a measure of the local convergence or divergence to the orbit along specific directions. However, this tool cannot be directly applied to discontinuous systems, in fact in this case such matrix exhibits discontinuities, or *saltations/jumps*, in its time evolution. To cope with this problem, the *saltation matrix* was firstly presented by Aizerman and Gantmakher in [1, 2, 3]. Later it has been used in the context of bifurcation analysis of switched systems, see [32, 70] and references therein, and in the calculation of the Lyapunov exponents [16, 92] for piecewise smooth systems. A review of some properties of this tool and its derivation can be found in [38, 44]. More recently, the saltation matrix has been used to extend the master stability function (MSF) tool for the study of synchronization in networks of coupled non-smooth dynamical systems [22].

In the following we firstly give the definition of the fundamental solution matrix and then we introduce the saltation matrix and some of its properties.

Consider the dynamical system

$$\dot{x} = f(t, x(t)), \quad x(t_0) = x_0, \tag{2-12}$$

where f is a continuously differentiable function. Moreover, denote with $\psi(t, t_0, x_0)$ the value of the solution $x(t)$ at time t with initial condition x_0. The *fundamental solution matrix* Φ is the derivative of the solution with respect to the initial condition, that is

$$\Phi(t, t_0) = \frac{\partial}{\partial x_0} \psi(t, t_0, x_0),$$

so that it satisfies the so-called *first-variation equation* for $t \geq t_0$

$$\dot{\Phi}(t, t_0) = \frac{\partial f}{\partial x}(t, \psi(t, t_0, x_0))\, \Phi(t, t_0), \quad \Phi(t_0, t_0) = I. \tag{2-13}$$

Another way to obtain the fundamental solution matrix is by considering a disturbance $\delta x(t)$ superimposed to a nominal solution $x(t)$, that is

$$\hat{x}(t) = x(t) + \delta x(t).$$

Substituting the previous relation into (2-12) and expanding in a Taylor series around the nominal solution $x(t)$ we obtain

$$\dot{\delta x}(t) = \frac{\partial f}{\partial x}(t, x(t))\, \delta x(t) + O(|\delta x(t)|^2). \tag{2-14}$$

Neglecting all higher order terms in the above equation we get a n-dimensional linear system that has n linearly independent solutions $\delta x_i(t)$ called fundamental set of solutions. The fundamental solution matrix Φ can therefore be defined as the square matrix having this set as columns, that is

$$\Phi(t, t_0) = \begin{bmatrix} \delta x_1(t) & \delta x_2(t) & \dots & \delta x_n(t) \end{bmatrix},$$

and such that

$$\delta x(t) = \Phi(t, t_0)\, \delta x_0 + O(|\delta x_0|^2).$$

The fundamental solution matrix is continuous and non-singular for all $t \geq t_0$, by construction. Furthermore, the following properties hold [38]:

- *Composition property*: if $t > t_1 > t_0$ then

$$\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0).$$

- *Mapping property*: for all $t \geq t_0$

$$f(\psi(t, t_0, x_0)) = \Phi(t, t_0)\, f(x_0).$$

Now consider a bimodal switched systems as in (2-2). In this case the Jacobian matrix $\frac{\partial f}{\partial x}$ in (2-13) is not uniquely defined on Σ, this causes a jump in the evolution of the fundamental solution matrix. As pointed out in the previous section, two type of motions are of interest here, the transversal crossing and attracting sliding. Specifically, consider firstly the case of *transversal crossing* from $\mathcal{S}^-$ to $\mathcal{S}^+$ and denote with $\bar{t}$ the time instant at which a solution $x(t) = \psi(t, t_0, x_0)$ to (2-2) intersects Σ at the point $\bar{x}$. The fundamental solution matrix from t_0 to $\bar{t}^+$, i.e. just after the jump, is defined as

$$\Phi(\bar{t}^+, t_0) = S\, \Phi(\bar{t}^-, t_0), \tag{2-15}$$

where $\Phi(\bar{t}^\pm, t_0) = \lim_{t\to\bar{t}^\pm} \Phi(t, t_0)$. The matrix S is called *saltation matrix*, and it may be thought of as the fundamental solution matrix between $\bar{t}^-$ and $\bar{t}^+$, that is $S = \Phi(\bar{t}^+, \bar{t}^-)$. Specifically, it is defined as

$$S = I + \frac{[f^+(\bar{x}) - f^-(\bar{x})]\, \nabla h(\bar{x})}{\nabla h(\bar{x})\, f^-(\bar{x})} = I + \frac{[f^+(\bar{x}) - f^-(\bar{x})]\, n^T(\bar{x})}{n^T(\bar{x})\, f^-(\bar{x})}, \tag{2-16}$$

where $n(x) = \frac{\nabla h(x)}{|\nabla h(x)|^2}$.

Note that S satisfies the composition property, indeed

$$f^+(\bar{x}) = S\, f^-(\bar{x}),$$

and in the case f is continuous, since $f^+(\bar{x}) = f^-(\bar{x})$ (e.g. PWSC systems), we have that $S = I$.

The following lemma [38, Lemma 2.3] gives information about the spectral properties of the saltation matrix.

Lemma 2.1. *Consider a matrix of the form*

$$S = I + ur^T,$$

where all vectors and matrices have dimension n. Then the eigenvalues of S are given by

$$\{1 + r^T u, 1, \ldots, 1\}.$$

When $r^T u = -1$, the eigenvector associated to the 0 eigenvalue is in the direction of u.

Therefore, saltation matrix (2-16) during crossing is always nonsingular because $\nabla h(\bar{x}) f^-(\bar{x}) \neq 0$. Furthermore, the eigenvalues at 1 are associated to the $n-1$ eigenvectors that span the tangent space to Σ in $\bar{x}$.

In the opposite case of transversal crossing from $\mathcal{S}^+$ to $\mathcal{S}^-$ the same discussion holds with f^+ in place of f^-, and viceversa, that is

$$S^{-1} = I + \frac{\left[f^-(\bar{x}) - f^+(\bar{x})\right] \nabla h(\bar{x})}{\nabla h(\bar{x})\, f^+(\bar{x})}.$$

Finally, in the case in which a solution to (2-2) does not cross Σ but *slides* on it, then the saltation matrix takes the following form [44, p. 119]

$$S = I + \frac{\left[f^s(\bar{x}) - f^-(\bar{x})\right] \nabla h(\bar{x})}{\nabla h(\bar{x})\, f^-(\bar{x})} = I + \frac{\left[f^s(\bar{x}) - f^+(\bar{x})\right] \nabla h(\bar{x})}{\nabla h(\bar{x})\, f^+(\bar{x})}.$$

with f^s as in (2-11). It can be noticed that the saltation matrix takes the same form regardless of whether the solution is coming from $\mathcal{S}^-$ or $\mathcal{S}^+$. Furthermore, during stable sliding the matrix S is singular because the solution is evolving on a lower dimensional manifold and there is no uniqueness of solution backward in time.

2.3 Regularization

A particularly useful approach in the study of a discontinuous system as (2-2) is *regularizing* it, that is replacing it with a smooth dynamical system that approximates the dynamics of the original system around the points of discontinuity [129, 75]. The main advantage of the regularization approach relies in the fact that the regularized vector field is a continuous and, in general, differentiable function. This allows us to conduct differential analysis and, in particular, to compute its Jacobian matrix that plays a main role in the contraction analysis. Then, by taking the limit to 0 of the regularization parameter ε we obtain results that are valid for the original discontinuous system.

There are several ways to regularize system (2-2). We shall adopt the method due to Sotomayor and Teixeira [129], where a smooth approximation of the discontinuous vector field is obtained by means of a monotonic transition function. The regularization can be *symmetrical* or *asymmetrical* depending on how the transition function is defined. The two methodologies are equivalent and their particular choice depends on the application. The regularization approach will be illustrated in detail using the symmetrical version, and then its asymmetrical counterpart will be presented.

2.3.1 Symmetrical regularization

Definition 2.12. *A PWSC function* $\varphi : \mathbb{R} \to \mathbb{R}$ *is a* symmetrical transition function *if*

$$\varphi(s) = \begin{cases} 1 & \textit{if} \quad s \geq 1, \\ \in (-1, 1) & \textit{if} \quad s \in (-1, 1), \\ -1 & \textit{if} \quad s \leq -1, \end{cases} \tag{2-17}$$

and $\varphi'(s) > 0$ *within* $s \in (-1, 1)$.

Definition 2.13. *The* symmetrical φ-regularization *of the bimodal Filippov system* (2-2) *is the one-parameter family of PWSC functions* $f_\varepsilon : D \to \mathbb{R}^n$ *given for* $\varepsilon > 0$ *by*

$$f_\varepsilon(x) = \frac{1}{2}\left[1 + \varphi\left(\tfrac{h(x)}{\varepsilon}\right)\right] f^+(x) + \frac{1}{2}\left[1 - \varphi\left(\tfrac{h(x)}{\varepsilon}\right)\right] f^-(x), \tag{2-18}$$

with φ *defined in* (2-17).

The region of regularization where this process occurs is called *regularization layer* and defined as

$$\mathcal{S}_\varepsilon := \{x \in D : -\varepsilon < h(x) < \varepsilon\}.$$

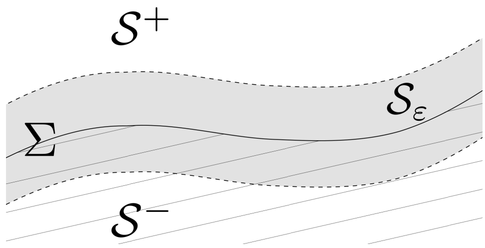

Figure 2-4: Regions of state space of a symmetrical regularized system: the switching manifold $\Sigma := \{x \in D : h(x) = 0\}$, $\mathcal{S}^+ := \{x \in D : h(x) > 0\}$, $\mathcal{S}^- := \{x \in D : h(x) < 0\}$ (hatched zone) and $\mathcal{S}_\varepsilon := \{x \in D : -\varepsilon < h(x) < \varepsilon\}$ (gray zone).

Note that outside $\mathcal{S}_\varepsilon$ the regularized vector field f_ε coincides with the dynamics of the switched system, i.e.

$$f_\varepsilon(x) = \begin{cases} f^+(x) & \text{if } x \in \mathcal{S}^+ \setminus \mathcal{S}_\varepsilon \\ f^-(x) & \text{if } x \in \mathcal{S}^- \setminus \mathcal{S}_\varepsilon \end{cases} \tag{2-19}$$

A graphical representation of the different regions of the state space of the regularized vector field f_ε is depicted in Figure 2-4. Note that, in order that the regularization process have sense, we are assuming that the functions $f^+(x)$ and $f^-(x)$ are defined in $\mathcal{S}^+ \cup \mathcal{S}_\varepsilon$ and $\mathcal{S}^- \cup \mathcal{S}_\varepsilon$, respectively. Furthermore, note that $\mathcal{S}_\varepsilon \to \Sigma$ as $\varepsilon \to 0^+$.

Sotomayor and Teixeira showed that the sliding vector field f^s in (2-11) can be obtained as a limit of the regularized system in the plane. For $\mathbb{R}^n$, a similar result was given in [75, Theorem 1.1]. Here we recover their results directly via the theory of slow-fast systems [68] as follows.

Lemma 2.2. *Consider f in* (2-2) *with $0 \in D$ and its regularization f_ε in* (2-18)*. If for any $x \in \Sigma$ we have that $f_N^+(x) \neq 0$ or $f_N^-(x) \neq 0$ then there exists a singular perturbation problem such that fixed points of the boundary-layer model are critical manifolds, on which the motion of the slow variables is described by the reduced problem, which coincides with the sliding equations* (2-11)*.*
Furthermore, under the hypothesis of right-uniqueness of solutions, de-

noting by $x_\varepsilon(t)$ a solution of the regularized system and by $x(t)$ a solution of the discontinuous system with the same initial conditions x_0, then

$$|x_\varepsilon(t) - x(t)| = O(\varepsilon)$$

uniformly for all $t \geq t_0$ and for all $x_0 \in D$.

Proof. For the sake of clarity, we assume without loss of generality that Σ can be represented, through a local change of coordinates around a point $x \in \Sigma$, by the function $h(x) = x_1$. We use the same notation for both coordinates. Hence our regularized system (2-18) becomes

$$\dot{x} = \frac{1}{2}\left[1 + \varphi\left(\tfrac{x_1}{\varepsilon}\right)\right] f^+(x) + \frac{1}{2}\left[1 - \varphi\left(\tfrac{x_1}{\varepsilon}\right)\right] f^-(x) \tag{2-20}$$

We now write (2-20) as a slow-fast system. Let $\hat{x}_1 = x_1/\varepsilon$, so that the region of regularization becomes $\hat{x}_1 \in (-1, 1)$, and $\hat{x}_i = x_i$ for $i = 2, \ldots, n$. Then (2-20) can be written as

$$\begin{aligned} \varepsilon\dot{\hat{x}}_1 &= \frac{1}{2}\big[1 + \varphi(\hat{x}_1)\big] f_1^+(\hat{x}) + \frac{1}{2}\big[1 - \varphi(\hat{x}_1)\big] f_1^-(\hat{x}), \\ \dot{\hat{x}}_i &= \frac{1}{2}\big[1 + \varphi(\hat{x}_1)\big] f_i^+(\hat{x}) + \frac{1}{2}\big[1 - \varphi(\hat{x}_1)\big] f_i^-(\hat{x}), \end{aligned} \tag{2-21}$$

for $i = 2, \ldots, n$, where $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$. The variable $\hat{x}_1$ is the *fast variable* and the variables $\hat{x}_i$ for $i = 2, \ldots, n$ are the *slow variables*. When $\varepsilon = 0$, we have

$$\begin{aligned} 0 &= \frac{1}{2}\big[1 + \varphi(\hat{x}_1)\big]\, f_1^+(\hat{x}) + \frac{1}{2}\big[1 - \varphi(\hat{x}_1)\big]\, f_1^-(\hat{x}), \\ \dot{\hat{x}}_i &= \frac{1}{2}\big[1 + \varphi(\hat{x}_1)\big]\, f_i^+(\hat{x}) + \frac{1}{2}\big[1 - \varphi(\hat{x}_1)\big]\, f_i^-(\hat{x}), \end{aligned} \tag{2-22}$$

for $i = 2, \ldots, n$, obtaining the so-called *reduced problem*. From the hypotheses we know that $f_1^+(\hat{x}) \neq 0$ or $f_1^-(\hat{x}) \neq 0$, hence we can solve for φ from the first equation

$$\varphi(\hat{x}_1) = -\frac{f_1^+(\hat{x}) + f_1^-(\hat{x})}{f_1^+(\hat{x}) - f_1^-(\hat{x})}, \tag{2-23}$$

that substituted into the second equation in (2-22) gives

$$\dot{\hat{x}}_i = \frac{f_1^+(\hat{x}) f_i^-(\hat{x}) - f_i^+(\hat{x}) f_1^-(\hat{x})}{f_1^+(\hat{x}) - f_1^-(\hat{x})}, \quad i = 2, \ldots, n. \tag{2-24}$$

If we now rescale time $\tau = t/\varepsilon$ and write $()' = d/d\tau$, then (2-21) becomes

$$\begin{aligned} \hat{x}_1' &= \frac{1}{2}\big[1+\varphi(\hat{x}_1)\big]\, f_1^+(\hat{x}) + \frac{1}{2}\big[1-\varphi(\hat{x}_1)\big]\, f_1^-(\hat{x}), \\ \hat{x}_i' &= \frac{\varepsilon}{2}\big[1+\varphi(\hat{x}_1)\big]\, f_i^+(\hat{x}) + \frac{\varepsilon}{2}\big[1-\varphi(\hat{x}_1)\big]\, f_i^-(\hat{x})], \end{aligned} \tag{2-25}$$

for $i = 2, \ldots, n$. The limit $\varepsilon = 0$ of (2-25)

$$\begin{aligned} \hat{x}_1' &= \frac{1}{2}\big[1+\varphi(\hat{x}_1)\big]\, f_1^+(\hat{x}) + \frac{1}{2}\big[1-\varphi(\hat{x}_1)\big]\, f_1^-(\hat{x}), \\ \hat{x}_i' &= 0, \quad i = 2, \ldots, n, \end{aligned}$$

is called the *boundary-layer model*. Its fixed points can be obtained by applying the Implicit Function Theorem to $\hat{x}_1' = 0$, that gives $\hat{x}_1 = g(x_2, \ldots, x_n)$, since $\varphi'(\hat{x}_1) > 0$ for $\hat{x}_1 \in (-1, 1)$ by definition. This in turn implies that $x_1 = \varepsilon\, g(x_2, \ldots, x_n)$.

It now follows directly that the flow of the reduced problem on critical manifolds of the boundary-layer problem coincides with that of the sliding vector field f^s as in (2-11) when the same change of coordinates as in the beginning is considered, i.e. such that $\nabla h = [1\ 0\ \ldots\ 0]$. In fact, after some algebra we get

$$f^s(x) = \left[0,\ \frac{f_1^+ f_2^- - f_2^+ f_1^-}{f_1^+ - f_1^-},\ \ldots\ ,\ \frac{f_1^+ f_n^- - f_n^+ f_1^-}{f_1^+ - f_1^-}\right]^T$$

that coincides with (2-24).

Furthermore, it is a well known fact in singular perturbation problems [67, Theorem 11.1] that, starting from the same initial conditions, the error between solutions $\hat{x}(t)$ of the slow system (2-21) and solutions of its reduced problem (that, as said, coincide with solutions $x_s(t)$ of the sliding vector field) is $O(\varepsilon)$ after some $t_b > t_0$ when the fast variable $\hat{x}_1$ has reached a $O(\varepsilon)$ neighborhood of the slow manifold, i.e. $|\hat{x}(t) - x_s(t)| = O(\varepsilon),\ \ \forall t \geq t_b$. However, in our case the singular perturbation problem is defined only in $\mathcal{S}_\varepsilon$ where any point therein is distant from the slow manifold at most 2ε, therefore the previous estimate is defined uniformly for all $t \geq t_0$ and in any norm due to their equivalence in finite dimensional spaces. On the other hand, from (2-19) outside $\mathcal{S}_\varepsilon$ the regularized vector field is equal to the discontinuous vector field and therefore the error between their solutions is uniformly 0. $\square$

The regularized vector field $f_\varepsilon(x)$ is continuous and differentiable almost everywhere in D, hence it is possible to define its Jacobian matrix function as follows.

Lemma 2.3. *The Jacobian matrix of the regularized vector field* (2-18) *is*

$$\frac{\partial f_\varepsilon}{\partial x}(x) = \alpha(x)\,\frac{\partial f^+}{\partial x}(x) + \beta(x)\,\frac{\partial f^-}{\partial x}(x) + \gamma(x)\left[f^+(x) - f^-(x)\right]\nabla h(x) \tag{2-26}$$

where

$$\begin{aligned} \alpha(x) &:= \frac{1}{2}\left[1 + \varphi\left(\frac{h(x)}{\varepsilon}\right)\right] \\ \beta(x) &:= \frac{1}{2}\left[1 - \varphi\left(\frac{h(x)}{\varepsilon}\right)\right] \\ \gamma(x) &:= \frac{1}{2\varepsilon}\,\varphi'\left(\frac{h(x)}{\varepsilon}\right) \end{aligned}$$

and $\alpha(x) \in [0,1]$, $\beta(x) \in [0,1]$ *and* $\gamma(x) \geq 0$, $\forall x \in D$, $\forall \varepsilon > 0$. *Note that for any transition functions* $\alpha(x) + \beta(x) = 1$, *for all* x.

Proof. The regularized vector field f_ε can be rewritten as

$$f_\varepsilon(x) = \alpha(x) f^+(x) + \beta(x) f^-(x)$$

therefore, taking the derivative with respect to x, we obtain

$$\frac{\partial f_\varepsilon}{\partial x}(x) = \alpha(x)\frac{\partial f^+}{\partial x}(x) + \beta(x)\frac{\partial f^-}{\partial x}(x) + f^+(x)\frac{\partial \alpha}{\partial x}(x) + f^-(x)\frac{\partial \beta}{\partial x}(x). \tag{2-27}$$

Observing that

$$\begin{aligned} \frac{\partial \alpha}{\partial x}(x) =& \frac{1}{2}\,\frac{\partial \varphi}{\partial s}\left(\frac{h(x)}{\varepsilon}\right)\,\frac{\partial}{\partial x}\left[\frac{h(x)}{\varepsilon}\right] \\ =& \frac{1}{2\varepsilon}\,\varphi'\left(\frac{h(x)}{\varepsilon}\right)\nabla h(x) = \gamma(x)\,\nabla h(x) \end{aligned}$$

and

$$\frac{\partial \beta}{\partial x}(x) = -\frac{\partial \alpha}{\partial x}(x),$$

replacing them into (2-27) we finally obtain (2-26). $\square$

The Jacobian matrix is therefore a convex combination of the Jacobian matrices of the two modes, $\frac{\partial f^+}{\partial x}$ and $\frac{\partial f^-}{\partial x}$, plus a rank-1 matrix that depends on the difference of the two vector fields and the gradient ∇h of the indicator function.
Note that if φ is PWSC then the Jacobian matrix (2-26) is a discontinuous function but its restriction to $\mathcal{S}_\varepsilon$ is continuous.

2.3.2 Asymmetrical regularization

Definition 2.14. *A PWSC function $\varphi : \mathbb{R} \to \mathbb{R}$ is an* asymmetrical transition function *if*

$$\varphi(s) = \begin{cases} 1 & \textit{if} \quad s \geq 1, \\ \in (0,1) & \textit{if} \quad s \in (0,1), \\ 0 & \textit{if} \quad s \leq 0, \end{cases} \tag{2-28}$$

and $\varphi'(s) > 0$ within $s \in (0,1)$.

Definition 2.15. *The* asymmetrical φ-regularization *of the bimodal Filippov system* (2-2) *is the one-parameter family of PWSC functions $f_\varepsilon : D \to \mathbb{R}^n$ given for $\varepsilon > 0$ by*

$$f_\varepsilon(x) = f^-(x) + \varphi\left(\tfrac{h(x)}{\varepsilon}\right)\left[f^+(x) - f^-(x)\right], \tag{2-29}$$

with φ defined in (2-28)*.*

Using the definition (2-28) the vector field (2-29) can be further specified as

$$f_\varepsilon(x) = \begin{cases} f^-(x), & h(x) < 0, \\ f^-(x) + \varphi\left(\tfrac{h(x)}{\varepsilon}\right)\left[f^+(x) - f^-(x)\right], & 0 < h(x) < \varepsilon, \\ f^+(x), & h(x) > \varepsilon. \end{cases}$$

In this case the regularization layer is $\mathcal{S}_\varepsilon := \{x \in D : \; 0 \leq h(x) \leq \varepsilon\}$. Moreover, notice that now we are assuming the function $f^-(x)$ to be defined for every point x such that $h(x) \leq \varepsilon$, that is $\mathcal{S}^- \cup \mathcal{S}_\varepsilon$. A graphical representation of these regions of the state space is reported in Figure 2-5. Differentiating the vector field f_ε above for $x \in \mathcal{S}_\varepsilon$ we

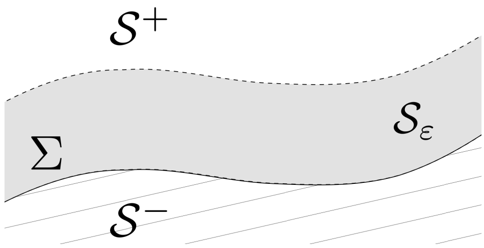

Figure 2-5: Regions of state space of an asymmetrical regularized system: the switching manifold $\Sigma := \{x \in D : h(x) = 0\}$, $\mathcal{S}^+ := \{x \in D : h(x) > 0\}$, $\mathcal{S}^- := \{x \in D : h(x) < 0\}$ (hatched zone) and $\mathcal{S}_\varepsilon := \{x \in D : 0 < h(x) < \varepsilon\}$ (gray zone).

obtain

$$\begin{aligned}\frac{\partial f_\varepsilon}{\partial x}(x) =& \frac{\partial f^-}{\partial x}(x) + \varphi\left(\tfrac{h(x)}{\varepsilon}\right)\left[\frac{\partial f^+}{\partial x}(x) - \frac{\partial f^-}{\partial x}(x)\right] \\ &+ \frac{1}{\varepsilon}\,\varphi'\left(\tfrac{h(x)}{\varepsilon}\right)\left[f^+(x) - f^-(x)\right]\nabla h(x).\end{aligned} \tag{2-30}$$

Linear regularization In the case of *linear* regularization the asymmetrical transition function (2-28) is a picewise linear function such that for $s \in (0,1)$

$$\begin{cases}\varphi(s) = s \\ \varphi'(s) = 1\end{cases}$$

therefore the vector field $f_\varepsilon(x)$ become

$$f_\varepsilon(x) = \begin{cases} f^-(x), & h(x) < 0, \\ f^-(x) + \frac{h(x)}{\varepsilon}\left[f^+(x) - f^-(x)\right], & 0 < h(x) < \varepsilon, \\ f^+(x), & h(x) > \varepsilon. \end{cases} \tag{2-31}$$

The Jacobian matrix of f_ε in this case is

$$\frac{\partial f_\varepsilon}{\partial x}(x) = \begin{cases} \dfrac{\partial f^-}{\partial x}(x), & h(x) < 0, \\ \dfrac{\partial f^-}{\partial x}(x) + \dfrac{h(x)}{\varepsilon}\left[\dfrac{\partial f^+}{\partial x}(x) - \dfrac{\partial f^-}{\partial x}(x)\right] & 0 < h(x) < \varepsilon, \\ \quad +\dfrac{1}{\varepsilon}\left[f^+(x) - f^-(x)\right]\nabla h(x), & \\ \dfrac{\partial f^+}{\partial x}(x), & h(x) > \varepsilon, \end{cases} \tag{2-32}$$

and it is not continuous on the boundaries of the regularization region. Moreover, note that the last term in the second equation above is multiplied by a term that goes to $+\infty$ as $\varepsilon \to 0^+$.

2.4 Matrix measures

In this section we introduce the so-called *matrix measure*, also known as *logarithmic norm*, introduced indipendently by Dahlquist [26] and Lozinskii [78]. More detailed discussions can be found in [126, 133].

Definition 2.16. *Given a real matrix $A \in \mathbb{R}^{n\times n}$ and a norm $|\cdot|$ with its induced matrix norm $\|\cdot\|$, the associated* matrix measure *is the function $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined as*

$$\mu(A) := \lim_{h\to 0^+} \frac{\|I + hA\| - 1}{h}$$

where I denotes the identity matrix.

The limit is know to exist and the convergence is monotonic [126]. The following matrix measures associated to the $p-$norm for $p = 1, 2, \infty$ are often used

$$\mu_1(A) = \max_j \left[a_{jj} + \sum_{i\neq j} |a_{ij}|\right]$$

$$\mu_2(A) = \lambda_{max}\left(\frac{A + A^T}{2}\right)$$

$$\mu_\infty(A) = \max_i \left[a_{ii} + \sum_{j\neq i} |a_{ij}|\right]$$

The matrix measure μ has the following useful properties [144, 31]:

1. $\mu(I) = 1$, $\mu(-I) = -1$.
2. If $A = O_n$, where O_n denotes the zero matrix, then $\mu(A) = 0$.
3. $-\|A\| \leq -\mu(-A) \leq Re\,\lambda_i(A) \leq \mu(A) \leq \|A\|$ for all $i = 1, \ldots, n$, where $Re\,\lambda_i(A)$ denotes the real part of the eigenvalue $\lambda_i(A)$ of A.
4. $\mu(c\,A) = c\,\mu(A)$ for all $c \geq 0$ (positive homogeneity).
5. $\mu(A + B) \leq \mu(A) + \mu(B)$ (subadditivity).
6. Given a constant nonsingular matrix Q, the matrix measure $\mu_{Q,i}$ induced by the weighted vector norm $|x|_{Q,i} = |Qx|_i$ is equal to $\mu_i(QAQ^{-1})$.

The following theorem can be proved [143, 7].

Theorem 2.3. *There exists a positive definite matrix P such that $PA + A^T P < 0$ if and only if $\mu_{Q,2}(A) < 0$, with $Q = P^{1/2}$.*

The condition presented in the following theorem is also known as *Coppel's inequality* [144, p.47].

Theorem 2.4. *Consider the linear time-varying system*

$$\dot{x} = A(t)x, \quad x(t_0) = x_0,$$

where $x \in \mathbb{R}^n$ and $A(t) \in \mathbb{R}^{n \times n}$ is a continuous matrix-value function. Let μ be the matrix measure on $\mathbb{R}^{n \times n}$ induced by a norm $|\cdot|$, then for $t \geq t_0$ we have that

$$|x_0| e^{\int_{t_0}^{t} -\mu(-A(\tau))d\tau} \leq |x(t)| \leq |x_0| e^{\int_{t_0}^{t} \mu(A(\tau))d\tau}.$$

We now present results on the properties of matrix measures of rank-1 matrices, since we will need these in the sequel. We believe that Lemma 2.4 is an original result. For any two vectors $x, y \in \mathbb{R}^n$, $x, y \neq 0$, the matrix $A = xy^T$ has always rank equal to 1. This can be easily proved observing that $xy^T = [y_1 x \; y_2 x \; \ldots \; y_n x]$.

Proposition 2.1. *For any two vectors $x, y \in \mathbb{R}^n$, $x, y \neq 0$ and for any norm we have that $\mu(xy^T) \geq 0$.*

Proof. The proof follows from property 3 of matrix measures as listed above, that is, for any matrix and any norm $\mu(A) \geq Re\,\lambda_i(A)$, $\forall i$, where $Re\,\lambda_i(A)$ denotes the real part of the eigenvalues $\lambda_i(A)$ of A. Therefore, since a rank-1 matrix has $n-1$ zero eigenvalues its measure cannot be less than zero. $\square$

The following result holds for the measure of rank-1 matrices induced by Euclidean norms [45].

Lemma 2.4. *Consider the Euclidean norm $|\cdot|_{Q,2}$, with $Q = P^{1/2}$ and $P = P^T > 0$. For any two vectors $x, y \in \mathbb{R}^n$, $x, y \neq 0$, the following result holds*

$$\mu_{Q,2}(xy^T) = 0 \quad \textit{if and only if} \quad Px = -ay, \; a > 0,$$

otherwise $\mu_{Q,2}(xy^T) > 0$.

Proof. Firstly we prove that $\mu_2(xy^T) = 0$ if and only if x and y are antiparallel, i.e. if $x = -ay$ for some $a > 0$. Indeed, from the definition of Euclidean matrix measure, $\mu_2(xy^T)$ is equal to the maximum eigenvalue of the symmetric part $A_s \equiv (A + A^T)/2$ of the matrix $A = xy^T$. The characteristic polynomial $p_\lambda(A_s)$ of A_s is [15, Fact 4.9.16]

$$\begin{aligned} p_\lambda(A_s) =& \lambda^{n-2}\left\{\lambda^2 - x^T y\lambda - \frac{1}{4}\left[x^T x y^T y - x^T y y^T x\right]\right\} \\ =& \lambda^{n-2}\left\{\lambda^2 - x^T y\lambda - \frac{1}{4}\left[|x|_2^2 |y|_2^2 - (x^T y)^2\right]\right\}. \end{aligned}$$

This polynomial has always $n-2$ zero roots and (in general) two further real roots. It can be easily seen from Descartes' rule that their signs must be opposite. Therefore, the only possibility for them to be nonpositive is that one must be zero while the other is negative. Using again Descartes' rule, this obviously happens if and only if x and y are antiparallel.

Now, assume that $\mu_{Q,2}(xy^T) = 0$ then, using property 6 of matrix measures, we have $\mu_{Q,2}(xy^T) = \mu_2\left(Qxy^T Q^{-1}\right) = \mu_2\left(Qx(Q^{-1}y)^T\right) = 0$, and, from the result proved above, Qx and $Q^{-1}y$ must be antiparallel, i.e. $Qx = -aQ^{-1}y$ for some $a > 0$, or equivalently $Px = -ay$.

To prove sufficiency, suppose that $Px = -ay$, then $Qx = -aQ^{-1}y$ and therefore, using again the result above, we have $\mu_{Q,2}(xy^T) = \mu_2(Qxy^TQ^{-1}) = a^{-1}\mu_2(-Qx(Qx)^T) = 0$. □

Note that when x or y (or both) are equal to 0 then by property 2 of matrix measures $\mu(xy^T) = 0$.

2.5 K-reachable sets

Definition 2.17. *Let $K > 0$ be any positive real number. A subset $\mathcal{C} \subseteq \mathbb{R}^n$ is* K-reachable *if, for any two points x_0 and y_0 in $\mathcal{C}$ there is some continuously differentiable curve $\gamma : [0, 1] \to \mathcal{C}$ such that $\gamma(0) = x_0$, $\gamma(1) = y_0$ and $|\gamma'(r)| \le K|y_0 - x_0|$, $\forall r$.*

For convex sets $\mathcal{C}$, we may pick $\gamma(r) = x_0 + r(y_0 - x_0)$, so $\gamma'(r) = y_0 - x_0$ and we can take $K = 1$. Thus, convex sets are 1-reachable, and it is easy to show that the converse holds.

2.6 Comparison functions

We recall here comparison functions, the class $\mathcal{K}$ and $\mathcal{KL}$ functions, [67, Sec. 4.4], useful in the definitions of stability for nonautonomous systems $\dot{x} = f(t, x)$.

Definition 2.18. *A function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to* class $\mathcal{K}$ *if it is continuous, strictly increasing, and $\alpha(0) = 0$. It is said to belong to* class $\mathcal{K}_\infty$ *if, in addition to this, $\alpha(r) \to +\infty$ for $r \to \infty$.*

Definition 2.19. *A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to belong to* class $\mathcal{KL}$ *if*

- *for each fixed t, the function $\beta(r, t)$ is a* class $\mathcal{K}$*;*
- *for each fixed r, the function $\beta(r, t)$ is nonincreasing with respect to t and $\beta(r, t) \to 0$ for $t \to \infty$.*

CHAPTER 3

Contraction analysis of nonlinear systems: an overview

In this chapter we review some of the available literature on contraction analysis of nonlinear systems. The aim of the chapter is to present the motivations that led to this kind of analysis and to expound some fundamental results. Although different approaches has been presented in the literature, we give particular emphasis to the approach based on matrix measures on which the theory presented in Chapter 4 is derived. Moreover, before concluding the chapter we present some of the available extensions of contraction analysis to non-differentiable systems.

3.1 Introduction

The study of contractions in the context of stability theory dates back at least to the work of Lewis [71, 61]. The basic ideas have been rediscovered independently by Yoshizawa [147, 148] and Demidovich [30], in the latter case with the name of *convergence* [95, 97].

Later the seminal work of Lohmiller and Slotine [76] popularized the concept in the control theory community and the following papers by Slotine and collaborators further extended it, with applications to observer problems, nonlinear regulation, and consensus problems in complex networks [77, 100, 146, 84, 88, 83, 89, 86, 85].

Besides the approach based on Riemannian metrics by Slotine and coworkers, some others methodologies to conduct the same analysis have been used in the last years. In 1958 Dahlquist used matrix measure, also called logarithmic norms, in the study of differential inequalities and to conduct error analysis on numerical integration of differential equations [26]. Later, matrix measures of Jacobian matrices were directly used to study contraction properties of nonlinear systems by Russo, di Bernardo and Sontag [119, 127, 7] extending the analysis to non-Euclidean metrics. The tool was further extended to structured norms [121, 120] and extensively used in coordination problems of complex network [118, 28, 114, 123, 122, 55, 37, 116, 115], and it was proved to be particularly fruitful in biological applications [119, 115, 113]. See [34, 27, 29] for a review on the argument.

Historical reviews on contraction analysis can be found in [65], [95], [51] and the survey [126].

More recently, all methodologies presented above have been unified by mean of the concept of *Finsler–Lyapunov function* proposed by Forni and Sepulchre [50]. The fundamental idea is to view contraction analysis in the context of a *differential* Lyapunov theory, allowing to consider more general Lyapunov functions and to possibly extend all the tools of Lyapunov theory to contraction analysis [49]. The original formulation was presented for dynamical systems defined on Finsler manifolds further generalizing the previous definitions.

However, contraction is a strong property of nonlinear systems and in many applications it is necessary to *weaken* the original formulation. For example, a simple way to relax contraction is by allowing the system to contract after that a small transient is extinguished, or after an overshoot in the state [128, 90, 91]. Another example are the concepts of transverse contraction [87] and its generalization of horizontal contraction [50, 145] that allow a system not to exhibit contraction in some particular directions or subspaces. This is useful for example in limit cycles analysis and synchronization.

Indeed, as a consequence of the original definition, an autonomous contracting system can converge only to an equilibrium point. We mention here that the recent concept of *differential positivity* [52, 48, 47] defining contraction of a cone rather than a ball admits one–dimensional attractors, e.g. limit cycles, as asymptotic behaviors.

To conclude, we highlight that classical contraction analysis requires the system vector field to be *continuously differentiable*, but this is *not* the case for many classes of dynamical systems, e.g. relay feedback systems, switched systems, hybrid systems. Several extensions of contraction analysis has been presented in the literature and briefly reported in the end of this chapter.

3.2 Incremental stability

Let $D \subseteq \mathbb{R}^n$ be an open set. Consider the system of ordinary differential equations, generally time-dependent

$$\dot{x} = f(t, x) \tag{3-1}$$

where f is a *continuously differentiable* vector field defined for $t \in [t_0, \infty)$ and $x \in D$. We denote by $\psi(t, t_0, x_0)$ the value of the solution $x(t)$ at time t of the differential equation (3-1) with initial value $x(t_0) = x_0$. We say that a set $\mathcal{C} \subseteq \mathbb{R}^n$ is *forward invariant* for system (3-1), if $x_0 \in \mathcal{C}$ implies $\psi(t, t_0, x_0) \in \mathcal{C}$ for all $t \geq t_0$.

We can then define notions of *incremental stability* (IS), *incremental asymptotic stability* (IAS) and *incremental exponential stability* (IES) [9, 50].

Definition 3.1. *Consider the differential equation* (3-1) *and its two solutions $x(t) = \psi(t, t_0, x_0)$ and $y(t) = \psi(t, t_0, y_0)$. Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a forward invariant set and $|\cdot|$ some norm on $\mathbb{R}^n$. The system is said to be*

- incrementally stable *(IS) in $\mathcal{C}$ if there exists a function α of class $\mathcal{K}$ such that $\forall t \geq t_0$, $\forall x_0, y_0 \in \mathcal{C}$*

$$|x(t) - y(t)| \leq \alpha\left(|x_0 - y_0|\right)$$

- incrementally asymptotically stable *(IAS) in $\mathcal{C}$ if it is incrementally stable and $\forall t \geq t_0$, $\forall x_0, y_0 \in \mathcal{C}$*

$$\lim_{t \to \infty} |x(t) - y(t)| = 0$$

or, equivalently, there exists a function β *of class* $\mathcal{KL}$ *such that* $\forall t \geq t_0$, $\forall x_0, y_0 \in \mathcal{C}$

$$|x(t) - y(t)| \leq \beta\left(|x_0 - y_0|, t\right)$$

- incrementally exponentially stable *(IES) in* $\mathcal{C}$ *if there exist constants* $K \geq 1$ *and* $\lambda > 0$ *such that* $\forall t \geq t_0$, $\forall x_0, y_0 \in \mathcal{C}$

$$|x(t) - y(t)| \leq K\, e^{-\lambda(t-t_0)}\, |x_0 - y_0| \tag{3-2}$$

These definitions are the incremental versions of the classical notions of stability, asymptotic stability and exponential stability. *Global, semiglobal*, and *local* notions of stability are specified through the definition of the set $\mathcal{C}$. For example, in the case of $\mathcal{C} \equiv \mathbb{R}^n$ incremental stability holds globally (IGS, IGAS, or IGES).

3.3 Contraction analysis using Riemannian metrics

In this section we present the original definition of contraction property presented to the control community in the seminal paper of Lohmiller and Slotine [76]. We report here the derivation of the main results in Euclidean norm and its generalization to Riemannian metrics. Further details can be found in the cited papers.

The dynamical system in (3-1) can be thought of as an n–dimensional fluid flow, where $\dot{x}$ is the n dimensional *velocity* vector at the n–dimensional *position* x and time t. Assuming $f(t, x)$ being continuously differentiable, we can obtain the following *differential dynamics*

$$\dot{\delta x} = \frac{\partial f}{\partial x}(t, x)\, \delta x, \tag{3-3}$$

where δx is the *differential displacement*, i.e. an infinitesimal displacement at fixed time. Formally, δx defines a tangent differential form [11, 24]. Consider now two neighboring trajectories of the flow defined by (3-1) and the differential displacement δx between them. The evolution in time of squared Euclidean distance $\delta x^T \delta x$ between these trajectories is given as

$$\frac{d}{dt}\left(\delta x^T \delta x\right) = 2\,\delta x^T \dot{\delta x} = 2\,\delta x^T \frac{\partial f}{\partial x}(t,x)\delta x,$$

Denoting by $\bar{\lambda}_{\max}(t,x)$ the largest eigenvalue of the symmetric part of $\frac{\partial f}{\partial x}(t,x)$ we have

$$\frac{d}{dt}\left(\delta x^T \delta x\right) \leq 2\,\bar{\lambda}_{\max}(t,x)\,\delta x^T \delta x, \tag{3-4}$$

and therefore, by integration

$$|\delta x(t)|^2 \leq |\delta x(t_0)|^2\, e^{\int_{t_0}^{t} \bar{\lambda}_{\max}(\tau,x)d\tau}. \tag{3-5}$$

Assuming that $\bar{\lambda}_{\max}(t,x)$ is uniformly negative definite in a forward invariant set $\mathcal{C}$, i.e.

$$\exists\, c > 0 \;:\; \forall x \in \mathcal{C},\, \forall t \geq t_0, \quad \bar{\lambda}_{\max}(t,x) \leq -c, \tag{3-6}$$

then (3-5) implies that any infinitesimal length converges exponentially to zero with convergence rate c. In other words, if for dynamical system (3-1) condition (3-6) holds, then a ball of radius δx centered about any given trajectory $x(t)$ *contracts* exponentially fast and the neighboring trajectories inside the ball converge towards each other. This in turns implies that by path integration the length of any finite curve between any two trajectories $x(t)$ and $y(t)$ in $\mathcal{C}$ converges exponentially to zero. Therefore, the nonlinear dynamical system is incrementally exponentially stable in $\mathcal{C}$ and (3-2) holds.

A more general result can be obtained using the differential change of coordinates

$$\delta z = \Theta(t,x)\delta x, \tag{3-7}$$

where $\Theta(t,x)$ is a nonsingular square matrix with bounded inverse. This leads to the generalization of the squared distance

$$\delta z^T \delta z = \delta x^T M(t,x)\delta x, \tag{3-8}$$

where $M(t,x) = \Theta^T\Theta$ is continuously differentiable and defines a *Riemannian metric*. The matrix M is assumed to be uniformly positive definite so that convergence to zero of δz also implies convergence to zero of δx. The time derivative of $\delta z = \Theta(t,x)\delta x$ is

$$\dot{\delta z} = \dot{\Theta}\delta x + \Theta\dot{\delta x} = \left(\dot{\Theta} + \Theta\frac{\partial f}{\partial x}\right)\Theta^{-1}\delta z = F(t,x)\delta z \tag{3-9}$$

where the matrix

$$F(t,x) := \left(\dot{\Theta} + \Theta\frac{\partial f}{\partial x}\right)\Theta^{-1} \tag{3-10}$$

is the so-called *generalized Jacobian matrix*. The rate of change of the squared Euclidean distance $\delta z^T \delta z$ is therefore

$$\frac{d}{dt}\left(\delta z^T \delta z\right) = 2\,\delta z^T \dot{\delta z} = 2\,\delta z^T F \delta z. \tag{3-11}$$

Therefore, following a similar argument as before, if the generalized Jacobian matrix F is uniformly negative definite in $\mathcal{C}$ then δz exponentially converges to zero (and thus also δx).

We can summarize the previous analysis as follows [76].

Definition 3.2. *A dynamical system* (3-1) *is said to be* contracting *if there exists a metric $M(t,x)$, that is uniformly positive definite matrix function, such that for some positive constant α*

$$|\delta z(t)| \leq |\delta z(t_0)|\, e^{-\alpha(t-t_0)}, \quad \forall t \geq t_0.$$

We now restate the main result of contraction analysis [76].

Theorem 3.1. *Let $\mathcal{C} \subseteq D$ be a forward invariant set. A dynamical system* (3-1) *is contracting if there exists a metric $M(t,x)$ such that for some positive constant α*

$$\dot{M} + \frac{\partial f}{\partial x}^T M + M\frac{\partial f}{\partial x} \leq -2\alpha M, \tag{3-12}$$

or, equivalently, the generalized Jacobian matrix F as (3-10) *is uniformly negative definite for all $x \in \mathcal{C}$ and for $t \geq t_0$. Then, every trajectories starting in $\mathcal{C}$ converge exponentially towards each other with rate c.*

More importantly, from the previous theorem, we can conclude that if an autonomous nonlinear dynamical system is *contracting* in a forward invariant set then there exists a *unique equilibrium point* in $\mathcal{C}$ to whom all trajectories converge exponentially.

Discrete-time systems Analogous results hold for discrete-time systems given by

$$x_{k+1} = f_k(k, x_k), \tag{3-13}$$

where $x \in \mathbb{R}^n$, $k \in \mathbb{N}$ and f is a continuously differentiable vector field. Similarly, defining the differential displacement δx_k and using the change of coordinates $\delta z_k = \Theta_k(k, x_k)\delta x_k$, where $\Theta_k(k, x_k)$ is again a nonsingular square matrix with bounded inverse, we obtain the differential dynamics

$$\delta z_{k+1} = F_k \, \delta z_k,$$

where, in this case, the generalized Jacobian matrix is defined as

$$F_k(k, x) := \Theta_{k+1} \frac{\partial f_k}{\partial x_k} \Theta_k^{-1}. \tag{3-14}$$

Here we give the definition of contracting discrete-time system [76].

Definition 3.3. *A discrete-time system* (3-13) *is said to be* contracting *if there exists a metric $M_k(k, x_k) = \Theta_k^T \Theta_k$ such that for some constant β with modulus lesser than 1 (i.e. $|\beta| < 1$)*

$$|\delta z_k| \leq |\delta z_{k_0}| \, \beta^{(k-k_0)}, \quad \forall k \geq k_0.$$

The analogous theorem for dicrete-time systems is as follows [76].

Theorem 3.2. *Let $\mathcal{C} \subseteq D$ be a forward invariant set. A discrete-time system* (3-13) *is contracting if there exists a metric $M_k(k, x_k)$ such that for some positive constant β*

$$F_k^T F_k - I \leq -\beta I, \quad \forall x \in \mathcal{C}, \, \forall k \geq k_0.$$

Then, every trajectories starting in $\mathcal{C}$ converge exponentially towards each others with rate $\ln \beta$.

The previous result may be viewed as an extension of the Banach-Caccioppoli contraction mapping theorem to non-autonomous systems [76].

3.4 Contraction analysis using matrix measures

The definitions and results on contracting systems discussed in the previous section are based on Riemannian metrics. Note that, although Euclidean norms are particular *constant* Riemannian metrics, in this thesis we are interested in contraction with respect to general norms, not necessarily quadratic norms.

The mathematical tool that we are going to use for this purpose here and in the next chapter is the *matrix measure* [26, 78, 31, 144, 126] (See Section 2.4 for defintion and some of important properties). The matrix measure, also called logarithmic norm, was independently introduced by Dahlquist [26] and Lozinskii [78] to derive error bounds in initial value problems of linear dynamical systems, using differential inequalities that distinguished between forward and reverse time integration. More specifically, the original idea was to derive a topological condition on a matrix $A \in \mathbb{R}^{n\times n}$ that would guarantee that solutions to a linear dynamical system

$$\dot{x} = Ax + r$$

remain bounded whenever r is a bounded function of t. Indeed, for $t \geq 0$ the norm of x satisfies the differential inequality

$$\begin{aligned}
D_t^+|x| &= \lim_{h\to 0^+} \sup \frac{|x(t+h)| - |x(t)|}{h} \\
&= \lim_{h\to 0^+} \frac{|x(t) + h\dot{x}(t)| - |x(t)|}{h} \\
&\leq \lim_{h\to 0^+} \frac{|(I+hA)x(t)| - |x(t)|}{h} + |r(t)| \\
&\leq \lim_{h\to 0^+} \frac{|I+hA| - 1}{h}|x(t)| + |r(t)| \\
&= \mu(A)\cdot|x(t)| + |r(t)|,
\end{aligned}$$

where D_t^+ is the upper right Dini derivative with respect to time t. Integrating the previous relation we obtain

$$|x(t)| \leq e^{t\mu(A)}|x(0)| + \int_0^t e^{(t-\tau)\mu(A)}|r(t)|d\tau \tag{3-15}$$

If $r = 0$, an initial condition $x(0)$ gives the solution $x(t) = e^{At}x(0)$. Therefore, the previous formula gives $|e^{tA}x(0)| \leq e^{t\mu(A)}|x(0)|$ and we can conclude that the matrix exponential is bounded by

$$|e^{tA}| \leq e^{t\mu(A)}, \quad t \geq 0 \tag{3-16}$$

Thus, if $\mu(A) \leq 0$ the zero solution is stable, with exponential stability when $\mu(A) < 0$ [126]. Therefore, the matrix measure can be viewed as an extension of the notion of *real part*, indeed recalling property 3 in Section 2.4, we know that

$$-\mu(-A) \leq Re\,\lambda_i(A) \leq \mu(A)$$

for all $i = 1, 2, \ldots, n$, where $Re\,\lambda_i(A)$ denotes the real part of the eigenvalue $\lambda_i(A)$ of A, therefore $\mu(A)$ is an upper bound of the real part of the eigenvalues of A.

Turning back the attention on nonlinear systems as (3-1), assuming $f(t, x)$ being continuously differentiable, we can obtain the following differential dynamics

$$\dot{\delta x} = \frac{\partial f}{\partial x}(t, x)\,\delta x,$$

where δx is again the differential displacement. The previous differential system can be thought of as a linear time-varying system, hence an upper bound of its solutions can be obtained by means of the Coppel's inequality (see Theorem 2.4), yielding

$$|\delta x(t)| \leq |\delta x_0| e^{\int_{t_0}^{t} \mu\left(\frac{\partial f}{\partial x}(\tau)\right) d\tau}.$$

Therefore, assuming that the matrix measure $\mu\left(\frac{\partial f}{\partial x}\right)$ is uniformly negative definite in the set of interest, we have that there exist some $b > 0$ and $c > 0$ such that

$$|\delta x(t)| \leq b\, e^{-c(t-t_0)}.$$

Thus, trajectories starting inside a ball of radius δx converge exponentially towards each other.

More formally, we report here the definition of contracting system [119].

Definition 3.4. *A dynamical system* (3-1)*, or the vector field f, is said to be (infinitesimally)* contracting *on a set $\mathcal{C} \subseteq U$ if there exists* *some norm in $\mathcal{C}$, with associated matrix measure μ, such that, for some constant $c > 0$ (the* contraction rate*)*

$$\mu\left(\frac{\partial f}{\partial x}(t,x)\right) \le -c \qquad \forall x \in \mathcal{C},\ \forall t \ge t_0. \tag{3-17}$$

The main result of contraction theory for continuosly differentiable systems based on matrix measures is as follows [119].

Theorem 3.3. *Let $\mathcal{C} \subseteq U$ be a forward invariant K-reachable set. If the dynamical system (3-1) is contracting in $\mathcal{C}$, then the system is incrementally exponentially stable in $\mathcal{C}$ with convergence rate c, that is*

$$|x(t) - y(t)| \le K\, e^{-c(t-t_0)}\, |x_0 - y_0|, \quad \forall t \ge t_0.$$

For a self-contained discussion we report next the proof of the previous theorem from [119].

Proof. Given any two points $x(0) = x_0$ and $y(0) = y_0$ in $\mathcal{C}$, take a smooth curve $\gamma : [0,1] \to \mathcal{C}$, such that $\gamma(0) = x_0$ and $\gamma(1) = y_0$. Let $\psi(t, t_0, \gamma(s))$ be the solution of system (3-1) from the initial condition $\gamma(s)$, $s \in [0,1]$, such that $x(t) = \psi(t, t_0, \gamma(0))$ and $y(t) = \psi(t, t_0, \gamma(1))$. Note that this function is continuously differentiable in both arguments since $f(t,x)$ is C^1 and $\gamma(s)$ is a smooth curve, and moreover

$$\frac{\partial}{\partial t}\psi(t, t_0, \gamma(s)) = f\big(t, \psi(t, t_0, \gamma(s))\big).$$

Define

$$w(t,s) := \frac{\partial}{\partial s}\psi(t, t_0, \gamma(s)),$$

then it follows that

$$\begin{aligned}
\dot{w}(t,s) =& \frac{\partial}{\partial t}\frac{\partial}{\partial s}\psi(t, t_0, \gamma(s)) \\
=& \frac{\partial}{\partial s}\frac{\partial}{\partial t}\psi(t, t_0, \gamma(s)) \\
=& \frac{\partial}{\partial s} f\big(t, \psi(t, t_0, \gamma(s))\big) \\
=& \frac{\partial f}{\partial x}\big(t, \psi(t, t_0, \gamma(s))\big)\frac{\partial}{\partial s}\psi(t, t_0, \gamma(s)) \\
=& \frac{\partial f}{\partial x}\big(t, \psi(t, t_0, \gamma(s))\big)\, w(t,s)
\end{aligned}$$

Using Coppel's inequality (see Theorem 2.4), we have

$$|w(t,s)| \leq |w(t_0,s)|\, e^{\int_{t_0}^{t} \mu\left(\frac{\partial f}{\partial x}(\tau)\right) d\tau}, \tag{3-18}$$

for all $x \in \mathcal{C}$, $t \geq t_0$, and $s \in [0,1]$. By the Fundamental Theorem of Calculus, we can write $\int_0^1 w(t,s)ds = \psi(t,t_0,\gamma(1)) - \psi(t,t_0,\gamma(0))$, hence we obtain $|x(t) - y(t)| \leq \int_0^1 |w(t,s)|ds$. Now, using (3-18) and the fact that $\mathcal{C}$ is a K-reachable set, the above inequality becomes

$$|x(t) - y(t)| \leq \int_0^1 \left[|w(t_0,s)|\, e^{\int_{t_0}^{t} \mu\left(\frac{\partial f}{\partial x}(\tau)\right) d\tau} \right] ds \leq K|x_0 - y_0| e^{-c(t-t_0)}.$$

□

As a result, if an autonomous nonlinear system is contracting in a (bounded) forward invariant subset then it converges towards an equilibrium point therein. Moreover, such equilibrium is also unique [119, 76].

It can be easily proved that condition (3-17) with $\mu_{Q,2}$ induced by Euclidean norms and condition (3-6) are equivalent for $\Theta = Q$. Indeed, by definition

$$\begin{aligned} \mu_{Q,2}\left(\frac{\partial f}{\partial x}\right) &= \mu_2\left(Q \frac{\partial f}{\partial x} Q^{-1}\right) = \mu_2\left(\Theta \frac{\partial f}{\partial x} \Theta^{-1}\right) \\ &= \mu_2(F) = \lambda_{\max}\left(\frac{F + F^T}{2}\right), \end{aligned}$$

where F is the generalized Jacobian matrix. Note that $\dot{\Theta} = \dot{Q} = 0$ since Q is constant.

Properties of contracting systems Contracting systems have been shown to possess several useful properties [76]. Here we review some of them in terms of matrix measures (See [119, 127] for proofs and further details).

Cascades of contracting systems are again contracting. Consider a system of the following form:

$$\begin{aligned} \dot{x} &= f(x,t) \\ \dot{y} &= g(x,y,t) \end{aligned}$$

where $x(t) \in \mathcal{C}_1 \subseteq \mathbb{R}^{n_1}$ and $y(t) \in \mathcal{C}_2 \subseteq \mathbb{R}^{n_2}$ for all t.
The Jacobian matrix of this system is

$$J = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \tag{3-19}$$

where we have written the Jacobian matrix of f with respect to x as $A(x,t) = \frac{\partial f}{\partial x}(x,t)$, the Jacobian matrix of g with respect to x as $B(x,y,t) = \frac{\partial g}{\partial x}(x,y,t)$, and the Jacobian matrix of g with respect to y as $C(x,y,t) = \frac{\partial g}{\partial y}(x,y,t)$.

Theorem 3.4. *Suppose that*

- *the system $\dot{x} = f(x,t)$ is infinitesimally contracting with contraction rate c_1*
- *the system $\dot{y} = g(x,y,t)$ is infinitesimally contracting with contraction rate c_2 when x is viewed as a parameter*
- *the mixed Jacobian matrix $B(x,y,t)$ is bounded, that is $\|B(x,y,t)\| \leq k$, $k > 0$*

then the cascade system is infinitesimally contracting. More precisely, pick any two positive numbers p_1 and p_2 such that $c_1 - \frac{p_2}{p_1}k > 0$ and let $c := \min\left\{c_1 - \frac{p_2}{p_1}k,\ c_2\right\}$ then $\mu(J) \leq -c$.

Another useful property, often exploited in applications such as synchronization and tracking, refers to the case where a contracting system is forced by an external periodic signal, also called *entrainment*. In particular, given a number $T > 0$, we will say that system (3-1) is *T-periodic* if it holds that

$$f(x, t+T) = f(x,t) \qquad \forall t \geq 0$$

Notice that a system $\dot{x} = f(x, u(t))$ with input $u(t)$ is T-periodic if $u(t)$ is itself a periodic function of period T. We can then state the following basic result about existence and stability of periodic orbits.

Theorem 3.5. *Suppose that*

- *$\mathcal{C}$ is a closed convex subset of $\mathbb{R}^n$;*

- *f is infinitesimally contracting with contraction rate c;*
- *f is T-periodic.*

Then there is a unique periodic orbit $\widehat{\omega}$ in $\mathcal{C}$ of (3-1) *of period T and, for every solution $x(t)$ starting in $\mathcal{C}$, it holds that* $\operatorname{dist}(x(t), \widehat{\omega}) \to 0$ *as $t \to \infty$.*

This property was used in [119] to prove global entrainment of transcriptional biological networks and can be effectively used whenever the goal is to prove entrainability of a system or network of interest.

The next result provides a *robustness margin* that says that any solution of the original system and any solution of the *perturbed* system $\dot{x} = f(t, x) + d(t)$ also exponentially converge toward each other, provided that $d(t)$ goes to 0 exponentially.

Theorem 3.6. *Assume that the system $\dot{x} = f(t, x)$ is infinitesimally contracting. Let $d(t)$ be a vector function satisfying $|d(t)| \leq Le^{-kt}$, for all $t \geq 0$ for some $k > 0$ and $L \geq 0$. Then, there exists constants $\ell > 0$ and κ such that for any solution $x(t)$ of the original system and any solution $y(t)$ of the perturbed system*

$$|x(t) - y(t)| \leq e^{-\ell(t-t_0)} \left(\kappa + |x_0 - y_0|\right), \quad \forall t \geq t_0.$$

3.5 Convergent systems

In this section we briefly present the notion of *convergence* originally presented by the Russian mathematician Demidovich in the study of dissipativity and convergence properties of nonlinear systems [30]. Convergent systems are dynamical systems that have a uniquely defined globally asymptotically stable steady-state solution. This in turn implies that all solutions converges to the steady-state solution forgetting their initial condition. The original formulation was presented for closed dynamical systems, successively it has been extended to systems with inputs [97]. A detailed review can be found in [95, 96].

Definition 3.5. *A dynamical system* (3-13) *is said to be* convergent *if there exists a unique solution $\bar{x}(t)$ such that*

1. *it is defined and bounded for all t,*
2. *it is globally asymptotically stable.*

If $\bar{x}(t)$ is uniformly (exponentially) asymptotically stable, then system (3-1) *is said to be* uniformly (exponentially) convergent.

Therefore, all solutions of a convergent system forget their initial condition and converge one to another and towards some nominal motion $\bar{x}(t)$.

In the case that the time dependency of the right-hand side of (3-1) is due to some input, we can consider the system defined as

$$\dot{x} = f(x, w(t)), \tag{3-20}$$

where f is a continuously differentiable function and $w : [t_0, +\infty) \to \mathbb{R}^m$ is an exogenous signal bounded for all t, that is $|w(t)| \leq r$, $\forall t$ for some $r > 0$. Then, the following definition follows.

Definition 3.6. *A dynamical system* (3-20) *is said to be* (uniformly, exponentially) convergent *for a class of input $\mathcal{I}$ if it is (uniformly, exponentially) convergent for every input $w(t) \in \mathcal{I}$ from that class.*

Convergent systems have similar properties to those of contracting systems. Specifically, if the input w is constant every solutions converge to a unique equilibrium point, while if it is periodic of period T they converge to a unique periodic solution $\bar{x}_w(t)$ with same period T (entrainment property, Section 3.4). Moreover, in [112] it was shown that the notions of convergence and incremental stability are equivalent on compact sets.

The following theorem gives a sufficient condition for a dynamical system to be convergent [30, 95].

Theorem 3.7. *The dynamical system* (3-20) *is globally exponentially convergent if there exist two positive definite matrices P and Q such that*

$$P\frac{\partial f}{\partial x}(x, w) + \frac{\partial f}{\partial x}^T(x, w)P \leq -Q, \quad \forall x \in \mathbb{R}^n,\ w \in \mathbb{R}^m. \tag{3-21}$$

As discussed in [50], condition (3-21) can be interpreted as a particular case of contraction with respect to a constant Riemannian metric or, equivalently, as contraction based on matrix measure induced by Euclidean norms, that is $\mu_{Q,2}$ with $Q = P^{1/2}$. The latter case can be easily proved by mean of Theorem 2.3.

3.6 Contraction analysis using Finsler–Lyapunov functions

Recently, all definitions of contraction presented in the previous sections have been unified by mean of the concept of *Finsler–Lyapunov function* [50]. The proposed idea is to view contraction analysis as a *differential* Lyapunov theory, allowing to consider more general Lyapunov functions and to possibly extend all the tools of Lyapunov theory to contraction analysis. The original formulation was presented for dynamical systems defined on more general Finsler manifolds (see the original paper [50] for a more detailed discussion), here we report a simplified version in Euclidean space as reported in [48].

Consider the so-called *prolonged system* [24] represented by the system dynamics aggregated with the linearized dynamics

$$\begin{cases} \dot{x} = f(t,x) \\ \dot{\delta x} = \dfrac{\partial f(t,x)}{\partial x}\delta x \end{cases} \qquad (x,\delta x) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{3-22}$$

then we have the following result.

Theorem 3.8. *A dynamical system* (3-1) *is contracting in a connected and forward invariant set* $\mathcal{C}$ *if there exists a* C^1 *function* $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that*

1. *there exist positive constants* $0 < \alpha_1 < \alpha_2$*, a positive integer* p *and a Finsler metric* $|\cdot|_x$ *such that*

$$\alpha_1 |\delta x|_x^p \leq V(x,\delta x) \leq \alpha_2 |\delta x|_x^p, \tag{3-23}$$

for all $(x,\delta x) \in \mathcal{C} \times \mathbb{R}^n$*;*

2. *there exists a positive constant* c *(the contraction rate) such that*

$$\dot{V}(x,\delta x) = \frac{\partial V(x,\delta x)}{\partial x} f(t,x) + \frac{\partial V(x,\delta x)}{\partial \delta x}\frac{\partial f(t,x)}{\partial x}\delta x \leq -cV(x,\delta x) \tag{3-24}$$

for all $t \geq t_0$ *and for all* $(x,\delta x) \in \mathcal{C} \times \mathbb{R}^n$*.*

Condition (3-23) says that the Finsler–Lyapunov function V is a measure of the length of the tangent vector δx, i.e. the distance between two infinitesimal neighboring trajectories, while condition (3-24) assures that for every x this length converges exponentially to zero. Therefore, since the Finsler metric $|\cdot|_x$ induce via integration a well-defined distance $d(x(t), y(t))$ between any two trajectories, we have that there exists $K \geq 1$ such that

$$d(x(t), y(t)) \leq K e^{-c(t-t_0)} d(x_0, y_0), \quad \forall t \geq t_0.$$

Note that continuously differentiability of V can be relaxed as in classical Lyapunov theory [50, Remark 2].

As said above, this approach extends the definitions of contraction presented in the previous sections. Specifically,

- contraction with respect to Riemannian metrics (Section 3.3) corresponds to the use of the function $V(x, \delta x) = \delta x^T M(t, x) \delta x$;
- contraction with respect to matrix measure (Section 3.4) corresponds to $V(x, \delta x) = |\delta x|$, where $|\cdot|$ denote any vector norm;
- convergence (Section 3.5) corresponds to $V(x, \delta x) = \delta x^T P \delta x$, with $P = P^T > 0$, or equivalently $V(x, \delta x) = |\delta x|_{Q,2}$, with $Q = P^{1/2}$.

See [50] for further details, next for completeness we report the proof of the equivalence of condition (3-17) in Definition 3.4 and condition (3-24) with $V(x, \delta x) = |\delta x|$. Denoting for brevity $J(t, x) = \frac{\partial f(t,x)}{\partial x}$, the equivalence follows from

$$\begin{aligned}
\dot{V}(x, \delta x) =& \frac{\partial V(x, \delta x)}{\partial \delta x} J(t, x) \delta x \\
=& \lim_{h \to 0^+} \frac{V(x, \delta x + hJ(t, x)\delta x) - V(x, \delta x)}{h} \\
\leq& \lim_{h \to 0^+} \frac{|I + hJ(t, x)| \, |\delta x| - |\delta x|}{h} \\
=& \lim_{h \to 0^+} \frac{|I + hJ(t, x)| - 1}{h} |\delta x| \\
=& \mu(J(t, x)) V(x, \delta x) \\
=& -cV(x, \delta x)
\end{aligned}$$

for each $t \geq t_0$, $x \in \mathcal{C}$, $\delta x \in \mathbb{R}^n$.

3.7 Contracting non–differentiable systems

Here we review several results that have been presented in the literature to extend contraction analysis to non–differentiable dynamical systems.

PWSC and time–dependent switched systems An extension to piecewise smooth continuous systems was outlined in [77, 117] and formalized in [36].

Theorem 3.9. *Let $\mathcal{C} \subseteq D$ be a K-reachable set. Consider a PWSC system as in Definition 2.3 such that it fulfills conditions for the existence and uniqueness of a Carathéodory solution. If there exists a unique matrix measure such that for some positive constants c_i*

$$\mu\left(\frac{\partial f_i}{\partial x}(t,x)\right) \leq -c_i,$$

for all $x \in \bar{\mathcal{S}}_i$, for all $t \geq t_0$ and for all i, then the system is incrementally exponentially stable in $\mathcal{C}$ with convergence rate $c := \min_i c_i$.

Basically, a PWSC system is contracting if so they are all its modes f_i with respect to the same norm. A similar result using Euclidean norms was previously presented in [97, Theorem 2.33] in terms of convergent systems.

An analogous result for time-dependent switched systems was also presented in [36].

Theorem 3.10. *Let $\mathcal{C} \subseteq D$ be a K-reachable set. Consider a time-dependent switched system as in Definition 2.2 such that it fulfills conditions for the existence and uniqueness of a Carathéodory solution. If there exists a unique matrix measure such that for some positive constants c_σ*

$$\mu\left(\frac{\partial f}{\partial x}(t,x,\sigma)\right) \leq -c_\sigma,$$

for all $x \in \mathcal{C}$, for all $t \geq t_0$ and for all $\sigma \in \{1,\ldots,p\}$, then the system is incrementally exponentially stable in $\mathcal{C}$ with convergence rate $c := \min_\sigma c_\sigma$.

Note that the previous theorem requires the existence of a *unique* matrix measure for all Jacobian matrices. A relaxation to this assumption was presented in [79, 80] using a *switched matrix measure* and *transaction coefficients* between norms.

Moreover, as proved in [117, 80], incremental stability of a time-switched system is still guaranteed even in the case that some of its modes are unstable (or not contracting) over some time intervals, extending a previous result presented in [104].

PWA systems Incremental stability properties of bimodal piecewise affine systems were studied in [95, 97, 142] in terms of *convergence* (see Section 3.5 above for further details).

Theorem 3.11. *A bimodal PWA system* (2-4) *is incrementally exponentially stable if there exist a positive definite matrix* $P = P^T > 0$*, a number* $\gamma \in \{0, 1\}$ *and a vector* $g \in \mathbb{R}^n$ *such that*

1. $PA_i + A_i^T P < 0, \quad i = 1, 2,$
2. $\Delta A = gh^T,$
3. $P\Delta b = -\gamma h,$

where $\Delta A = A_1 - A_2$ *and* $\Delta b = b_1 - b_2$*.*

The first condition requires the existence of a common Lyapunov function $V(x) = x^T Px$ for the two modes. The second condition assumes that the linear part of the two modes is continuous on the switching plane. There are two cases in the third condition [95, Remark 4]. For $\gamma = 0$, the PWA system (2-4) is continuous. For $\gamma = 1$, the discontinuity is due only to the b_i and, together with the first condition, implies that the two modes of the PWA system (2-4) are simultaneously strictly passive.

Moreover, in [142] based on the above conditions and further generalizations, the tracking problem for PWA systems is discussed and algorithms based on LMI formulations are presented for the design of state feedback and observer-based output feedback controllers.

As discussed in Section 3.5, convergence conditions can be viewed as a particular case of conditions based on matrix measure induced by Euclidean norms. In the same way, in Chapter 4 it will be shown that conditions of Theorem 3.11 can be derived as a particular case of a more general result for nonlinear switched systems (2-1).

Sliding mode solutions An extension of contraction theory, related to the concept of *weak* contraction after short transient [128], to characterize incremental stability of sliding mode solutions was first presented in [35] for planar systems and later extended to n–dimensional switched systems in [33].

In these papers, the sliding region $\Sigma_s \subseteq \Sigma$ is assumed to be globally attractive for solutions with initial conditions outside of it, then sufficient conditions for contraction of sliding mode solutions are derived based on the matrix measure of the *projected* Jacobian matrix onto the switching manifold Σ. This is necessary since $f^s(t,x)$ is not contracting in the direction orthogonal to Σ. Specifically, the n–dimensional sliding vector field $f^s(t,x)$ is replaced by the lower dimensional vector field $\hat{f}^s(t,z)$ defined as

$$\hat{f}^s(t,z) = P^T(\gamma(z))\, f^s(t,\gamma(z)),$$

where P is the projection matrix onto Σ and $z \in \mathbb{R}^{n-1}$ is a set of constrained coordinates on Σ, i.e. such that $x = \gamma(z) \in \Sigma$, $\forall z$. In this way the direction orthogonal to Σ where the sliding vector field $f^s(t,x)$ does not contract is removed from the analysis.

Theorem 3.12. *Let $\mathcal{C} \subseteq D$ be a K-reachable set. Consider a bimodal switched system as (2-2), with indicator function $h(x) = h^T(x - x_h)$, h, $x_h \in \mathbb{R}^n$ and such that $\Sigma_s \cap \mathcal{C} \neq \varnothing$. If the following conditions hold*

1. *every trajectory with initial condition outside Σ reaches Σ_s in finite time, that is Σ_s is a globally attractive set,*

2. *there exists a matrix measure such that for some positive constants c*

$$\mu\left(\frac{\partial \hat{f}^s}{\partial z}(t,z)\right) \leq -c, \quad \forall z : \gamma(z) \in \Sigma_s, \forall t \geq t_0,$$

then after a short time all trajectories exponentially converge towards each others, that is there exists a time instant $\tau > t_0$ such that

$$|x(t) - y(t)| \leq K e^{-c(t-\tau)} |x(\tau) - y(\tau)|, \quad \forall t \geq \tau.$$

Since no hypotheses on incremental stability outside Σ are made, the solutions are guaranteed to contract only after they reached Σ_s.

Another result has been recently presented in [109] for the tracking problem in fully-actuated mechanical port-Hamiltonian systems. The proposed controller renders a desired sliding manifold (where the reference trajectory lies) attractive by making the corresponding error system partially contracting.

Resetting hybrid systems Contraction analysis of resetting hybrid systems was firstly presented in [77] and then extended in [111]. Resetting hybrid systems, also known as impulsive systems, are dynamical systems that combine both continuous and discrete dynamics. A continuous-time dynamics evolves according to the vector field f until a discrete event generated by the law g occurs at time t_j. At this point the state is reset to the new value x^+ that is used as initial condition for the continuous dynamics, and so on. Specifically, this class of dynamical system can be expressed by the equations

$$\begin{aligned} \dot{x} &= f(t, x(t)), \qquad && t \neq t_j \\ x^+ &= g(t, x(t)), && t = t_j \\ j^+ &= h(t, x(t), j(t)), \end{aligned}$$

where $j \in \mathbb{N}$ is a piecewise constant signal called resetting index, and t_j is the j-th resetting time.

The associated differential dynamics of this system is

$$\begin{aligned} \dot{\delta x} &= \frac{\partial f}{\partial x}(t, x)\, \delta x, \quad && t \neq t_j, \\ \delta x^+ &= \frac{\partial g}{\partial x}(t, x)\, \delta x, && t = t_j. \end{aligned}$$

Denote by $F(t, x)$ the generalized Jacobian matrix associated to the continuous dynamics $f(t, x)$ as in (3-10), that is

$$F(t, x) = \left(\dot{\Theta}_c + \Theta_c \frac{\partial f}{\partial x} \right) \Theta_c^{-1},$$

and by $G(k, x)$ the generalized Jacobian matrix associated to the resetting dynamics $g(t, x)$ as in (3-14), that is

$$G(k, x) = \Theta_{d,k+1} \frac{\partial g}{\partial x_k} \Theta_{d,k}^{-1}.$$

Also denote with Δt_{rj} the period between two resets following the j-th reset. Assume for the sake of simplicity that the continuous metric $M_c(t,x) = \Theta_c^T\Theta_c$ and the discrete metric $M_d(k,x) = \Theta_d^T\Theta_d$ are the same, then the following result holds [111].

Theorem 3.13. *A resetting hybrid systems is contracting if there exists a positive constant η such that*

$$\alpha + \frac{\beta}{\Delta t_{rj}} \leq -\eta, \quad \forall\, j,$$

where α is the largest eigenvalue of the symmetric part of F and β is the largest singular value of G.

Note that the system can be contracting even if one dynamics is unstable, in this case the theorem can be generalized using the concept of average dwell-time. Moreover, the two metrics M_c and M_d can be different and time-varying. These further generalizations are also presented in [111].

Finally, note that the class of resetting hybrid systems is only a particular one of the more general class of hybrid dynamical systems [57, 58]. A systematic extension of contraction analysis to the latter is ongoing research.

Transverse contraction of hybrid limit cycles In the paper [136] a transverse contraction framework for analysis of hybrid limit cycles was proposed, based on the work of transversal surface construction in [82], and continuous transverse contraction in [87]. The class of hybrid systems under consideration has a continuous dynamics $f(x)$ and a reset map $g(x)$ with resets occurring when the continuous flow intersects a certain switching surface Σ (here we report only the case of one surface). The dynamics can hence be described as

$$\begin{aligned} \dot{x} =& f(x), \quad x \notin \Sigma \\ x^+ =& g(x), \quad x \in \Sigma \end{aligned} \tag{3-25}$$

with associated differential dynamics

$$\begin{aligned} \dot{\delta x} &= \frac{\partial f(x)}{\partial x}\delta x \\ \delta x^+ &= \frac{\partial g(x)}{\partial x}\delta x \end{aligned}$$

The stability of a unique hybrid limit cycle is guaranteed by *transverse* contraction of every solution by mean of the following theorem.

Theorem 3.14. *If there exists a Riemannian metric* $V(x, \delta x) = \sqrt{\delta x^T M(x) \delta x}$ *such that*

1. *the continuous dynamics is transverse contracting, that is there exists a positive constant* λ *such that*

$$\frac{\partial V(x, \delta x)}{\partial x} f(x) + \frac{\partial V(x, \delta x)}{\partial \delta x} \frac{\partial f(x)}{\partial x} \delta x \leq -\lambda V(x, \delta x)$$

   for all $\delta x \neq 0$ *such that* $\frac{\partial V}{\partial \delta x} f(x) = 0$*;*

2. *every trajectory approaches the switching surface* Σ *orthogonally with respect to the metric* $M(x)$*, that is*

$$\delta x^T M(x) f(x) = 0$$

   for all $x \in \Sigma$ *and for all* δx *in the tangent space to* Σ *at point* x*;*

3. *the metric* $V(x, \delta x)$ *is not increasing at resetting events, that is*

$$\frac{\partial g(x)}{\partial x}^T M(x) \frac{\partial g(x)}{\partial x} - M(x) \leq 0,$$

   for all $\delta x^T M(x) f(x) = 0$*,*

then there exists a unique and stable limit cycle to whom every solution to hybrid system (3-25) *exponentially converges.*

The original theorem in [136] was in terms of Zhukovski stability, that is stability under time reparametrization, implying that every trajectory converges on the same periodic orbit with different phases. Moreover, the above conditions was also formulated as a convex optimization problem in terms of pointwise LMIs.

An interesting problem left for future work is to study in greater detail the differential dynamics of the reset maps by using the formulation presented in [32].

CHAPTER 4

Contraction analysis of switched systems via regularization

In this chapter, we take a different approach to the study of contraction in n–dimensional Filippov systems than the one taken in [35, 33]. In those papers, the sliding manifold was assumed to be attractive for every solution with initial condition outside of it and then the contraction properties of the projection of the sliding vector field f^s onto the switching manifold was considered (together with a suitable change of coordinates). Here we adopt a new generic approach which directly uses the vector fields $f^{\pm}$ and does not need the explicit computation of the sliding vector field f^s. The presented method has a simple geometric meaning and, unlike other methods, can also be applied to nonlinear switched systems.

Instead of directly analysing the Filippov system, we first consider a *regularized* version; one where the switching manifold Σ has been replaced by a boundary layer of width 2ε. We choose the symmetric regularization method of Sotomayor and Teixeira [129]. We then apply standard contraction theory results based on matrix measures to this new system, before taking the limit $\varepsilon \to 0$ in order to recover results that are valid for our Filippov system.

Then, in the second part of the chapter, we illustrate two applications to design problems of the theoretical conditions presented in the first part. Firstly, we present a control strategy to incrementally stabilize a class of smooth nonlinear systems using switched control actions. Finally, we present new conditions for the design of state observers for a large class of nonlinear switched systems including those exhibiting sliding motion. The theoretical results are then illustrated through simple but representative examples.

4.1 Contracting switched systems

In this section we present our two main results, Theorems 4.1 and 4.2, for switched Filippov systems. Theorem 4.1, using Lemma 2.2, shows that if the regularized system $\dot{x} = f_\varepsilon(x)$ is incrementally exponentially stable so it is the Filippov system from which it is derived. Theorem 4.2 then gives sufficient conditions for the discontinuous vector field to be incrementally exponentially stable.

Theorem 4.1. *Let $\mathcal{C} \subseteq D$ be a K-reachable set. If there exists a positive constant $\bar{\varepsilon} < 1$ such that for all $\varepsilon < \bar{\varepsilon}$ the regularized vector field f_ε* (2-18) *is incrementally exponentially stable in $\mathcal{C}$ with convergence rate c, then in the limit for $\varepsilon \to 0^+$ any two solutions $x(t) = \psi(t, t_0, x_0)$ and $y(t) = \psi(t, t_0, y_0)$, with $x_0, y_0 \in \mathcal{C}$, of the bimodal Filippov system* (2-2) *converge towards each other in $\mathcal{C}$, i.e.*

$$|x(t) - y(t)| \leq K\, e^{-c(t-t_0)}|x_0 - y_0|, \quad \forall t \geq t_0. \tag{4-1}$$

Proof. From Lemma 2.2 we know that the error between any two solutions $x_\varepsilon(t)$ and $y_\varepsilon(t)$ of the regularized vector field f_ε and their respective limit solutions $x(t)$ and $y(t)$ of the discontinuous system is $O(\varepsilon)$, i.e. $|x_\varepsilon(t) - x(t)| = O(\varepsilon)$ and $|y_\varepsilon(t) - y(t)| = O(\varepsilon)$, $\forall t \geq t_0$. Therefore, from the hypothesis of f_ε being incrementally exponentially stable (3-2) holds and applying the triangular inequality of norms we have

$$\begin{aligned}
|x(t) - y(t)| &\leq |x(t) - x_\varepsilon(t)| + |x_\varepsilon(t) - y(t)| \\
&\leq |x(t) - x_\varepsilon(t)| + |x_\varepsilon(t) - y_\varepsilon(t)| + |y_\varepsilon(t) - y(t)| \\
&\leq K\, e^{-c(t-t_0)}|x_\varepsilon(t_0) - y_\varepsilon(t_0)| + 2\, O(\varepsilon)
\end{aligned}$$

for $x_\varepsilon(t_0), y_\varepsilon(t_0) \in \mathcal{C}$ and for every $t \geq t_0$. The theorem is then proved by taking the limit for $\varepsilon \to 0^+$. □

If the chosen transition function φ is a $C^1(\mathbb{R})$ function, then the regularized vector field f_ε is $C^1(D, \mathbb{R}^n)$ and Theorem 3.3 can be directly applied to study its incremental stability. On the other hand, if the transition function is not C^1 but it is at least a PWSC function as in Definition 2.3, with $\mathcal{S}_1 = (-\infty, -1)$, $\mathcal{S}_2 = (-1, 1)$ and $\mathcal{S}_3 = (1, +\infty)$, then the regularized vector field f_ε is itself a PWSC vector field and Theorem 3.9 applies. This is the case for $\varphi(s) = \mathrm{sat}(s)$. This function is $C^0(\mathbb{R})$ but its restrictions to each subsets $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are smooth functions. Theorem 4.1 and Lemma 2.3 then allow us to derive our second main result, as follows.

Theorem 4.2. *Let $\mathcal{C} \subseteq D$ be a K-reachable set. A bimodal Filippov system* (2-2) *is incrementally exponentially stable in $\mathcal{C}$ with convergence rate $c := \min\{c_1, c_2\}$ if there exists some norm in $\mathcal{C}$, with associated matrix measure μ, such that for some positive constants c_1, c_2*

$$\mu\left(\frac{\partial f^+}{\partial x}(x)\right) \leq -c_1, \quad \forall x \in \bar{\mathcal{S}}^+ \tag{4-2}$$

$$\mu\left(\frac{\partial f^-}{\partial x}(x)\right) \leq -c_2, \quad \forall x \in \bar{\mathcal{S}}^- \tag{4-3}$$

$$\mu\left(\left[f^+(x) - f^-(x)\right] \nabla h(x)\right) = 0, \quad \forall x \in \Sigma. \tag{4-4}$$

Proof. The transition function φ is a PWSC function hence the resulting regularized vector field f_ε is also PWSC, i.e. it is continuous in all D and such that its restrictions to the subsets $\bar{\mathcal{S}}^+ \setminus \mathcal{S}_\varepsilon$, $\bar{\mathcal{S}}^- \setminus \mathcal{S}_\varepsilon$ and $\bar{\mathcal{S}}_\varepsilon$ are continuously differentiable. Therefore Theorem 3.9 can be directly applied and we have that f_ε is contracting in $\mathcal{C}$ if there exist positive constants c_1, c_2, c_3 such that

$$\mu\left(\frac{\partial f^+}{\partial x}(x)\right) \leq -c_1, \quad \forall x \in \bar{\mathcal{S}}^+ \setminus \mathcal{S}_\varepsilon \tag{4-5}$$

$$\mu\left(\frac{\partial f^-}{\partial x}(x)\right) \leq -c_2, \quad \forall x \in \bar{\mathcal{S}}^- \setminus \mathcal{S}_\varepsilon \tag{4-6}$$

$$\mu\left(\frac{\partial f_\varepsilon}{\partial x}(x)\right) \leq -c_3, \quad \forall x \in \bar{\mathcal{S}}_\varepsilon. \tag{4-7}$$

Thus, by Lemma 2.3, substituting (2-26) into (4-7) and using the subadditivity and positive homogeneity properties of the matrix measures, we obtain

$$\begin{aligned}\mu\left(\frac{\partial f_\varepsilon}{\partial x}(x)\right) \leq \alpha(x)\,\mu\left(\frac{\partial f^+}{\partial x}(x)\right) + \beta(x)\,\mu\left(\frac{\partial f^-}{\partial x}(x)\right)\\ + \gamma(x)\,\mu\left(\left[f^+(x) - f^-(x)\right]\nabla h(x)\right)\end{aligned} \tag{4-8}$$

Therefore, conditions (4-5)-(4-7) are satisfied if

$$\mu\left(\frac{\partial f^+}{\partial x}(x)\right) \leq -c_1, \quad \forall x \in \bar{\mathcal{S}}^+ \cup \bar{\mathcal{S}}_\varepsilon \tag{4-9}$$

$$\mu\left(\frac{\partial f^-}{\partial x}(x)\right) \leq -c_2, \quad \forall x \in \bar{\mathcal{S}}^- \cup \bar{\mathcal{S}}_\varepsilon \tag{4-10}$$

$$\mu\left(\left[f^+(x) - f^-(x)\right]\nabla h(x)\right) = 0, \qquad \forall x \in \bar{\mathcal{S}}_\varepsilon \tag{4-11}$$

and $c_3 \geq \min\{c_1, c_2\}$. Finally, considering that $\bar{\mathcal{S}}_\varepsilon \to \Sigma$ in the limit for $\varepsilon \to 0^+$ we obtain conditions (4-2)-(4-4) stated in the theorem. Therefore, by virtue of Theorem 4.1, these conditions are sufficient for the bimodal Filippov system (2-2) to be incrementally exponentially stable. □

Remark 4.1. *If φ is $C^1(\mathbb{R})$ it can be easily proved (by using Lemma 2.3 and the subadditivity property of matrix measures) that conditions* (4-9)-(4-11) *are sufficient for the measure of the Jacobian of $f_\varepsilon(x)$ to be negative definite over the entire region of interest.*

The first two conditions (4-2) and (4-3) in Theorem 4.2 guarantee that the regularized vector field f_ε is contracting outside the region $\mathcal{S}_\varepsilon$, and therefore imply that any two trajectories in $\mathcal{C} \setminus \mathcal{S}_\varepsilon$ converge towards each other exponentially. Condition (4-4) assures that the third term in (4-8) does not diverge as $\varepsilon \to 0^+$ and therefore that negative definiteness of the measures of the Jacobian matrices of two modes, f^+ and f^-, is enough to guarantee incremental exponential stability of f_ε inside $\mathcal{S}_\varepsilon$.

Notice that Theorem 4.2 gives conditions in terms of a generic norm. When a specific norm is chosen, it is possible to further specify the conditions of Theorem 4.2, as we now show.

Proposition 4.1. *Assume that through a local change of coordinates around a point $x \in \Sigma$ the switching manifold Σ is represented by the function $h(x) = x_1$ and let $\Delta f(x) = f^+(x) - f^-(x)$. Let $D = \text{diag}\{d_1, \ldots, d_n\}$, with $d_i > 0\ \forall i$, be a diagonal matrix and $P = Q^2$ be a positive definite matrix. Assuming that $\Delta f(x) \neq 0\ \ \forall x \in \Sigma$, then*

1. *$\mu_{D,1}(\Delta f(x)\,\nabla h) = 0$ if and only if*

$$\begin{cases} \Delta f_1(x) < 0 \\ |\Delta f_1(x)| \geq |d_2 \Delta f_2(x) d_1^{-1}| + \cdots + |d_n \Delta f_n(x) d_1^{-1}| \end{cases}$$

2. *$\mu_{Q,2}(\Delta f(x)\,\nabla h) = 0$ if and only if $P\Delta f(x) = -a\nabla h^T$, $a > 0$.*

3. *$\mu_{D,\infty}(\Delta f(x)\,\nabla h) = 0$ if and only if $\Delta f(x)$ and ∇h^T are antiparallel.*

Proof. The matrix $(\Delta f(x)\,\nabla h)$ has rank equal to 1 and, since $\nabla h = [1\ \ 0\ \ \ldots\ \ 0]$, it can be written as

$$\Delta f(x)\,\nabla h = \begin{bmatrix} \Delta f_1(x) & 0 & \ldots & 0 \\ \Delta f_2(x) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Delta f_n(x) & 0 & \ldots & 0 \end{bmatrix}$$

1. From [143, Lemma 4] we have

$$\mu_{D,1}(\Delta f(x)\,\nabla h) = \max\{\Delta f_1(x) + |d_2 \Delta f_2(x) d_1^{-1}| \\ + |d_n \Delta f_n(x) d_1^{-1}|; 0;\ \ldots;\ 0\}.$$

   This measure is equal to zero if and only if

$$\Delta f_1(x) + |d_2 \Delta f_2(x) d_1^{-1}| + \cdots + |d_n \Delta f_n(x) d_1^{-1}| \leq 0.$$

2. The proof for $\mu_{Q,2}$ comes from Lemma 2.4.

3. Again, from [143, Lemma 4] we have

$$\mu_{D,\infty}(\Delta f(x)\,\nabla h) = \max\{\Delta f_1(x);\ |d_2 \Delta f_2(x) d_1^{-1}|; \\ \ldots;|d_n \Delta f_n(x) d_1^{-1}|\}.$$

   The above measure is equal to zero if and only if $\Delta f_1(x) < 0$ and $\Delta f_i(x) = 0$, $\forall i \neq 1$, that is if $\Delta f(x)$ is antiparallel to ∇h^T.

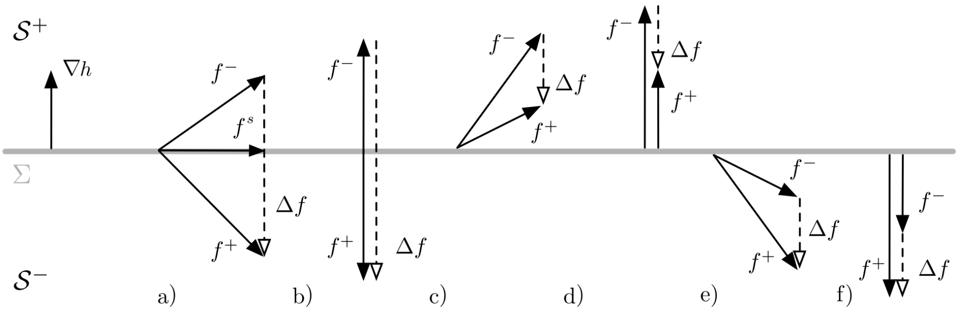

Figure 4-1: Geometrical interpretation of condition (4-4) using Euclidean norm (with $Q = I$) and ∞-norm in $\mathbb{R}^2$. The horizontal line is Σ. Sliding is represented in a) and b), while crossing occurs in c), d), e), f). In all cases the difference vector field Δf is antiparallel to ∇h.

$\square$

Hence, using the ℓ_1-norm there always exist a matrix D and a change of coordinates such that the condition holds assuming that the scalar product between ∇h and Δf is negative, that is $\nabla h(x)\,\Delta f(x) < 0$, $\forall x \in \Sigma$. Moreover, using the Euclidean norm a matrix P such that the condition holds exists only if $\nabla h(x)\,\Delta f(x) < 0$, $\forall x \in \Sigma$, as proved next.

Proposition 4.2. *Assume that $\Delta f(\bar{x}) \neq 0$ with $\bar{x} \in \Sigma$, then a Euclidean norm $|\cdot|_{Q,2}$, with $Q > 0$, such that $\mu_{Q,2}(\Delta f(\bar{x})\,\nabla h(\bar{x})) = 0$ exists if and only if $\nabla h(\bar{x})\,\Delta f(\bar{x}) < 0$.*

Proof. Firstly, note that from Proposition 4.1 and from Lemma 2.4 we know that
$\mu_{Q,2}(\Delta f(\bar{x})\,\nabla h(\bar{x})) = 0$ if and only if a matrix $P = Q^2$ exists such that $P\Delta f(\bar{x}) = -a\nabla h(\bar{x})$, $a > 0$. Now, from the definition of positive definite matrices it follows that given the two nonzero vectors $\Delta f(\bar{x})$ and $\nabla h(\bar{x})$ such a positive definite matrix P exists if and only if $-\nabla h(\bar{x})\,\Delta f(\bar{x}) > 0$, that is $\nabla h(\bar{x})\,\Delta f(\bar{x}) < 0$. $\square$

Furthermore, note that when $\Delta f(x) = 0$, $\forall x \in \Sigma$, that is when the system is continuous on Σ as in the case of PWSC systems, we have that $\mu(\Delta f(x)\,\nabla h(x)) = \mu(O_n) = 0$. Therefore condition (4-4) is always satisfied and Theorem 4.2 coincides with Theorem 3.9.

In Figure 4-1 the geometrical interpretation of condition (4-4) in $\mathbb{R}^2$ is shown schematically when either the Euclidean norm (with $Q = I$) or the ∞-norm are used.

One significant advantage of our method is that it can deal with nonlinear switched systems, as we shall now demonstrate next. In our first example, we are able to show that our system is globally incrementally exponentially stable. In the numerical examples the ℓ_1-norm will be used to highlight that non-Euclidean norms can be used as an alternative to Euclidean norms and that the analysis can be much easier if they are used. All simulations presented here were computed using the numerical solver in [101].

Example 1 Consider the switched system (2-2) with

$$f^+(x) = \begin{bmatrix} -4x_1 \\ -9x_2 - x_2^2 - 18 \end{bmatrix}, \quad f^-(x) = \begin{bmatrix} -4x_1 \\ -9x_2 + x_2^2 + 18 \end{bmatrix}$$

and $h(x) = x_2$. We can easily check that all three conditions of Theorem 4.2 are satisfied in the ℓ_1-norm. Indeed, for the first condition we have

$$\mu_1\left(\frac{\partial f^+}{\partial x}(x)\right) = \max\{-4;\ -2x_2 - 9\} = -4$$

because $-2x_2 - 9 < -9, \ \ \forall x \in \mathcal{S}^+$. Similarly for the second condition we have

$$\mu_1\left(\frac{\partial f^-}{\partial x}(x)\right) = \max\{-4;\ 2x_2 - 9\} = -4$$

because $2x_2 - 9 < -9, \ \ \forall x \in \mathcal{S}^-$. Finally, for the third condition we have that for all $x \in \Sigma$

$$\begin{aligned} \mu_1\left(\left[f^+(x) - f^-(x)\right]\nabla h(x)\right) &= \mu_1\left(\begin{bmatrix} 0 & 0 \\ 0 & -2x_2^2 - 36 \end{bmatrix}\right) \\ &= \max\{0;\ -2x_2^2 - 36\} = 0. \end{aligned}$$

Therefore the switched system considered here is incrementally exponentially stable in all $\mathbb{R}^2$ with convergence rate $c = 4$. In Figure 4-2a we show numerical simulations which confirm the analytical estimation (4-1). In our second example, we show how our method can find a subset $\mathcal{C} \subset \mathbb{R}^2$ in which a nonlinear switched system is incrementally exponentially stable with respect to a given norm.

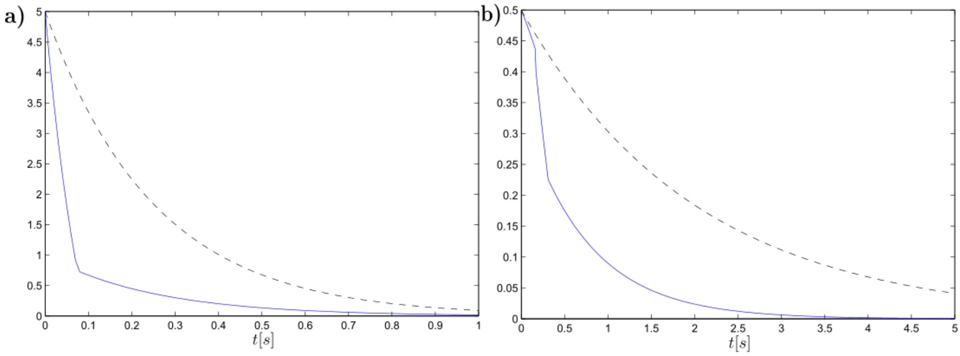

Figure 4-2: Norm of the difference between two trajectories for (a) Example 1 and (b) Example 2. Initial conditions are respectively $x_0 = [2 \quad 2]^T \in \mathcal{S}^+$, $y_0 = [3 \quad -2]^T \in \mathcal{S}^-$ and $x_0 = [0 \quad -1]^T \in \mathcal{S}^-$, $y_0 = [0 \quad -0.5]^T \in \mathcal{S}^-$. The dashed lines represent the analytical estimates (4-1) with (a) $c = 4$ and (b) $c = 1/2$, and $K = 1$.

Example 2 Consider the switched system (2-2) with

$$f^+(x) = \begin{bmatrix} -2x_1 - \frac{2}{9}x_2^2 + 2 \\ x_1 - x_2 - 3 \end{bmatrix}, \quad f^-(x) = \begin{bmatrix} -2x_1 + \frac{2}{9}x_2^2 - 2 \\ x_1 - x_2 + 3 \end{bmatrix}$$

and $h(x) = x_2$. For the first condition of Theorem 4.2 we have

$$\begin{aligned} \mu_1\left(\frac{\partial f^+}{\partial x}(x)\right) &= \max\left\{-1;\ -1 + \frac{4}{9}\,|x_2|\right\} = \\ &= -1 + \frac{4}{9}\,|x_2| \end{aligned}$$

Therefore f^+ is contracting in the ℓ_1-norm for $|x_2| < 9/4$. If we want to guarantee a certain contraction rate c we need to consider the subset $|x_2| < 9/4(1-c)$ instead. An identical result holds for f^-. Finally, for the third condition of Theorem 4.2 we have

$$\begin{aligned} \mu_1\left(\left[f^+(x) - f^-(x)\right]\nabla h(x)\right) =& \mu_1\left(\begin{bmatrix} 0 & -\frac{4}{9}x_2^2 + 4 \\ 0 & -6 \end{bmatrix}\right) \\ =& \max\left\{0;\ -2 + \frac{4}{9}x_2^2\right\} = 0 \end{aligned}$$

for all $x \in \Sigma$, that is $x_2 = 0$. We can conclude that the switched system taken into example satisfies Theorem 4.2 in the subset $\mathcal{C} = \{x \in \mathbb{R}^2 : |x_2| < 9/8\}$ and therefore it is incrementally exponentially stable with convergence rate $c = 1/2$ therein. The previous results are confirmed by numerical simulations shown in Figure 4-2b.

4.1.1 Application to PWA systems

We now present the application of the theoretical results of the previous section to the class of PWA systems (2-4).

Proposition 4.3. *The PWA system* (2-4) *is incrementally exponentially stable in a K-reachable set $\mathcal{C} \subseteq D$ with convergence rate $c := \min\{c_1, c_2\}$ if there exists some norm in $\mathcal{C}$, with associated matrix measure μ, such that for some positive constants c_1, c_2 and for all $x \in \Sigma$*

$$\mu(A_1) \leq -c_1 \tag{4-12}$$
$$\mu(A_2) \leq -c_2 \tag{4-13}$$
$$\mu\left(\Delta A x h^T\right) = 0 \tag{4-14}$$
$$\mu\left(\Delta b h^T\right) = 0 \tag{4-15}$$

Proof. The proof follows directly from Theorem 4.2 noting that $\frac{\partial f^+}{\partial x} = A_1$, $\frac{\partial f^-}{\partial x} = A_2$, $f^+(x) - f^-(x) = \Delta A x + \Delta b$, and $\nabla h(x) = h^T$. Indeed

$$\begin{aligned}\mu\left(\left[f^+(x) - f^-(x)\right]\nabla h(x)\right) =& \mu\left(\left[\Delta A x + \Delta b\right]h^T\right)\\ \leq& \mu\left(\Delta A x h^T\right) + \mu\left(\Delta b h^T\right).\end{aligned}$$

□

Remark 4.2. *When Euclidean norms $|\cdot|_{Q,2}$ are used, with $Q = P^{1/2}$, the conditions of Proposition 4.3 become the same as those in Theorem 3.11. It is easy to show that the conditions of Theorem 3.11 are sufficient for those of our Proposition to hold. In fact, from Theorem 2.3, condition 1 of Theorem 3.11 on the matrices A_1 and A_2 implies that their measures $\mu_{Q,2}(A_1)$ and $\mu_{Q,2}(A_2)$ are negative definite. Condition 2 of Theorem 3.11 implies that in any norm $\mu\left(\Delta A x h^T\right) = \mu\left(g\left(h^T x\right)h^T\right) = 0$ since $h^T x = 0$, $\forall x \in \Sigma$. Condition 3 of Theorem 3.11 can be rewritten as $Q\Delta b = -Q^{-1}h$, therefore $\mu_{Q,2}\left(\Delta b h^T\right) = \mu_2\left(Q\Delta b h^T Q^{-1}\right) = \mu_2\left(-Q^{-1}h\left(Q^{-1}h\right)^T\right) = 0$ for Lemma 2.4, since vectors $Q^{-1}h$ and $-Q^{-1}h$ are antiparallel.*

Example Consider a PWA system of the form (2-4) with

$$A_1 = \begin{bmatrix} -2 & -1 \\ 1 & -3 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1 \\ -3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$

and $B = [0 \ \ 1]^T$, $h = [0 \ \ 1]^T$. Using the ℓ_1-norm the first two conditions of Proposition 4.3 are satisfied, in fact $\mu_1(A_1) = -1$ and $\mu_1(A_2) = -1$. The third condition is also satisfied since we have that

$$\mu_1(\Delta A x h^T) = \mu_1 \left(\begin{bmatrix} 0 & 0 \\ 0 & x_2 \end{bmatrix} \right) = x_2 = 0, \qquad \forall x \in \Sigma.$$

Finally, the fourth condition is satisfied as it can be easily proved that $\mu_1(\Delta b h^T) = 0$. Therefore, from Proposition 4.3, the PWA system considered here is incrementally exponentially stable. In Figure 4-3a we show numerical simulations of the norm of the difference between two trajectories for this PWA system. Similar qualitative behavior was observed for different choices of the initial conditions. The dashed line is the estimated exponential decay from (4-1) with $c = 1$ and $K = 1$. It can be seen that as expected from the theoretical analysis $|x(t) - y(t)|_1 \leq e^{-t}|x_0 - y_0|_1$, $\forall t \geq 0$.

The evolution of the system state $x_2(t)$ is reported in Figure 4-3b when the periodic signal $u(t) = 6 \sin(2\pi\, t)$ is chosen as a forcing input. As expected for contracting systems, all trajectories converge towards a unique periodic (non-smooth) solution with the same period of the excitation $u(t)$ (confirming the entrainment property of contracting systems, see Theorem 3.5).

Relay feedback systems We present here a similar result for relay feedback systems.

Proposition 4.4. *A relay feedback system of the form*

$$\begin{aligned} \dot{x} &= Ax - b\, \mathrm{sgn}(y) \\ y &= c^T x \end{aligned} \tag{4-16}$$

where $A \in \mathbb{R}^{n \times n}$, b, $c \in \mathbb{R}^n$, is incrementally exponentially stable in a K-reachable set $\mathcal{C} \subseteq D$ with convergence rate $\bar{c}$ if there exists some

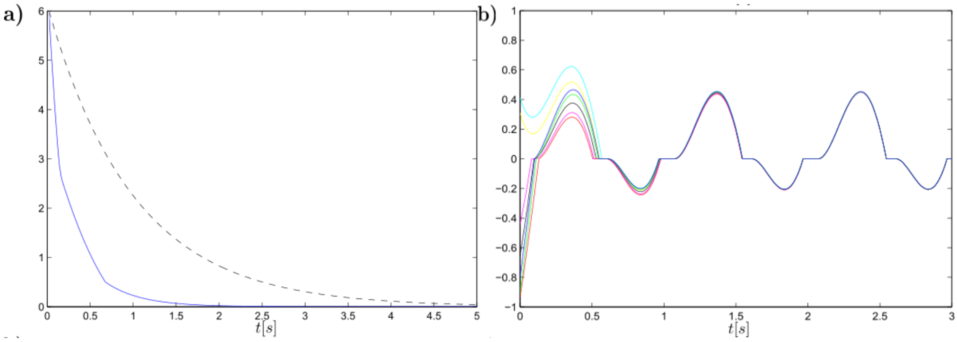

Figure 4-3: Norm of the difference between two trajectories for Example 3, panel (a). Initial conditions are $x_0 = [4 \quad 4]^T \in \mathcal{S}^+$, $y_0 = [3 \quad -1]^T \in \mathcal{S}^-$. The dashed lines represent the analytical estimates (4-1) with $K = 1$ and $c = 1$. Panel (b) depicts the time evolution of the state $x_2(t)$ of Example 3 from different initial conditions and with $u(t) = 6\sin(2\pi t)$ set as a periodic input signal.

norm in $\mathcal{C}$, with associated matrix measure μ, such that for some positive constant $\bar{c}$

$$\mu(A) \leq -\bar{c} \tag{4-17}$$
$$\mu\left(-bc^T\right) = 0. \tag{4-18}$$

Proof. The proof follows observing that the relay feedback system is a PWA system of the form

$$\dot{x} = \begin{cases} Ax - b & \text{if } c^T x > 0 \\ Ax + b & \text{if } c^T x < 0 \end{cases}$$

with $A_1 = A_2 = A$, $\Delta A = O_n$, $\Delta b = -2b$ and $h = c$. Therefore applying Proposition 4.3 to it we get $\mu\left(\Delta A x h^T\right) + \mu\left(\Delta b h^T\right) = 0 + \mu\left(-2bc^T\right) = 2\,\mu\left(-bc^T\right)$, and the assertion is proved. □

It is well-known in literature [12, 138] that under certain conditions a relay feedback system can exhibit self-sustained oscillations, i.e. limit cycles. According to the theory presented in Chapter 3, we know that a contracting autonomous system in a forward invariant set must converge towards an equilibrium point, therefore so we expect a contracting relay feedback system should behave. Indeed, if conditions (4-17) and (4-18) hold then a *planar* relay feedback system (4-16) cannot converge to a limit cycle. In Euclidean norms condition (4-17) implies from Theorem 2.3 that A is Hurwitz, this in turn implies that its trace is negative, i.e. $\mathrm{tr}(A) < 0$. Condition (4-18) implies from Lemma 2.4 that $Pb = c$ where P is a positive definite matrix, this means that $c^T b = (Pb)^T b = b^T P b > 0$ for any $b \neq 0$. The regularized vector field of (4-16) is

$$f_\varepsilon(x) = Ax - b\,\varphi\left(\frac{c^T x}{\varepsilon}\right)$$

If $\varphi \in C^1$ so it is also f_ε and its divergence is

$$\mathrm{div}(f_\varepsilon(x)) = \begin{cases} \mathrm{tr}(A) - \frac{1}{\varepsilon}\varphi'\left(\frac{c^T x}{\varepsilon}\right) c^T b, & \text{if } x \in \mathcal{S}_\varepsilon \\ \mathrm{tr}(A), & \text{if } x \notin \mathcal{S}_\varepsilon \end{cases}$$

Since we know that $\varphi'(s) \geq 0$ for all s and $\varepsilon > 0$, we can conclude that conditions (4-17) and (4-18) imply that $\mathrm{div}(f_\varepsilon(x)) < 0$ for all $x \in \mathbb{R}^2$ and, from Bendixon-Dulac theorem [67, Lemma 2.2], $\dot{x} = f_\varepsilon(x)$ cannot have limit cycles. Hence, from Theorem 4.1 the relay feedback system from which f_ε was derived cannot exhibit limit cycles.

As illustration of Proposition 4.4, consider the following example.

Example Consider a relay feedback system (4-16) with

$$A = \begin{bmatrix} -2 & -1 \\ 1 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad c^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Using the linear transition function $\varphi(s) = \mathrm{sat}(s)$ the corresponding regularized vector field (2-18) becomes

$$f_\varepsilon(x) = \begin{cases} Ax - b & \text{if } c^T x > \varepsilon \\ \left(A - \dfrac{1}{\varepsilon} b c^T\right) x & \text{if } -\varepsilon < c^T x < \varepsilon \\ Ax + b & \text{if } c^T x < -\varepsilon \end{cases}$$

Outside $\mathcal{S}_\varepsilon$ the Jacobian of f_ε is equal to A, and hence its measure does not depend on ε. On the other hand, since using the ℓ_1-norm we

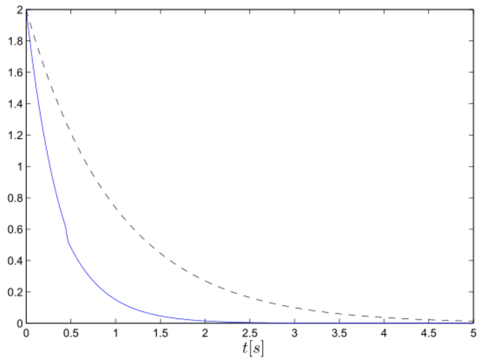

Figure 4-4: Norm of the difference between two trajectories for Example 4. Initial conditions are $x_0 = [2 \quad 2]^T \in \mathcal{S}^+$, $y_0 = [2 \quad -2]^T \in \mathcal{S}^-$. The dashed lines represent the analytical estimates (4-1) with $K = 1$ and $c = 1$.

have that $\mu_1(A) = \max\{-2 + |1|;\ -3 + |-1|\} = -1$, and $\mu_1(-bc^T) = \max\{0;\ -3 + |-1|\} = 0$, then when $x \in \mathcal{S}_\varepsilon$

$$\mu\left(\frac{\partial f_\varepsilon}{\partial x}\right) \leq \mu(A) + \frac{1}{\varepsilon}\,\mu(-bc^T) = -1.$$

Therefore the regularized vector field f_ε remains contracting in the ℓ_1-norm for any value of ε, as should be expected since conditions of Proposition 4.4 are satisfied in this norm. Hence, from Theorem 4.1 we can conclude that the relay feedback system taken into example is incrementally exponentially stable in the ℓ_1-norm. In Figure 4-4, we show numerical simulations of the evolution of the difference between two trajectories for this system. The dashed line is the estimated exponential decay from (4-1) with $\bar{c} = 1$ and $K = 1$.

4.2 Application to the design of switching controls and observers

Next, we illustrate some simple yet effective applications of the theoretical results derived so far using regularization to the synthesis of switched controllers and observers.

4.2.1 Incrementally stabilizing switching control

We start by presenting the synthesis of a switching control strategy to incrementally stabilize a class of nonlinear dynamical systems over some set of interest. The proposed approach is based on the analytical results on contraction and incremental stability of bimodal switched systems presented in the previous section. In particular, the switching control action resulting from our design procedure is active only where the open-loop system is not sufficiently incrementally stable. Such property can be usefully exploited to reduce the required control input energy.

Problem formulation The class of dynamical systems considered is defined by

$$\dot{x} = f(x) + B(x)\, u(x) \tag{4-19}$$

where $x \in \mathbb{R}^n$, $u(x) \in \mathbb{R}^m$ are state and feedback control input, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuously differentiable.

We want to find a discontinuous feedback control input u for system (4-19) such that the resulting closed-loop system is incrementally stabilized, either locally or globally. The control input $u(x)$ we are looking for has the following form

$$u(x) = \begin{cases} u^+(x) & \text{if } h(x) > 0 \\ u^-(x) & \text{if } h(x) < 0 \end{cases} \tag{4-20}$$

where $u^{\pm}(x)$ are continuously differentiable vector fields, and $h(x)$ is a scalar function as in (2-3).

In particular, to minimize the control effort we want to exploit possible contracting properties of the open-loop vector field $f(x)$ to design a control input that is not active in the regions where $f(x)$ is already sufficiently incrementally stable.

Main theorem The main result of this section follows directly from Theorem 4.2.

Theorem 4.3. *The dynamical systems* (4-19) *with the switching control input* (4-20) *is incrementally exponentially stable in a K-reachable set* $\mathcal{C} \subseteq D$ *with convergence rate* $c := \min\{c_1, c_2\}$ *if there exist some* *norm in $\mathcal{C}$, with associated matrix measure μ such that for some positive constants c_1, c_2*

$$\mu\left(\frac{\partial f}{\partial x}(x) + \frac{\partial}{\partial x}\left[B(x)\, u^+(x)\right]\right) \leq -c_1, \quad \forall x \in \bar{\mathcal{S}}^+ \tag{4-21}$$

$$\mu\left(\frac{\partial f}{\partial x}(x) + \frac{\partial}{\partial x}\left[B(x)\, u^-(x)\right]\right) \leq -c_2, \quad \forall x \in \bar{\mathcal{S}}^- \tag{4-22}$$

$$\mu\Big(B(x)\,\left[u^+(x) - u^-(x)\right] \cdot \nabla h(x)\Big) = 0, \quad \forall x \in \Sigma \tag{4-23}$$

Proof. The closed-loop system with switching control (4-20) is a Filippov system as (2-2) of the form

$$\dot{x} = \begin{cases} f^+(x) := f(x) + B(x)\, u^+(x) & \text{if } h(x) > 0 \\ f^-(x) := f(x) + B(x)\, u^-(x) & \text{if } h(x) < 0 \end{cases} \tag{4-24}$$

therefore Theorem 4.2 can be directly applied giving the previous three conditions. And thus if these conditions hold then the switching control (4-20) incrementally stabilizes system (4-19) with convergence rate c. $\square$

Note that

$$\frac{\partial}{\partial x}\left[B(x)\, u^{\pm}(x)\right] = \sum_{i=1}^{m}\left(\frac{\partial b_i}{\partial x}(x)\, u_i^{\pm}(x) + b_i(x)\,\frac{\partial u_i^{\pm}}{\partial x}(x)\right)$$

where we denoted with b_i and $u_i^{\pm}$ the i-th column of $B(x)$ and the i-th component of $u^{\pm}(x)$, respectively.

Design procedure In the following we present a possible approach to design a switching controller (4-20) that incrementally stabilize system (4-19) in a desired set using conditions of Theorem 4.3. Indeed if the designed $u(x)$ is such that conditions (4-21)-(4-23) are satisfied for a desired c then the discontinuous closed-loop system (4-24) is incrementally exponentially stable as required.

Specifically, suppose that the closed-loop system (4-24) is required to be incrementally stable with convergence rate $\bar{c}$ in a certain set $\mathcal{C}_d$ (where the open-loop system (4-19) is not sufficiently contracting).

Suppose that in $\mathcal{C}_d$ there can be identified two disjoint subregions, one where

$$\mu\left(\frac{\partial f}{\partial x}(x)\right) \leq -\bar{c} \tag{4-25}$$

is not satisfied and the other one where it is satisfied (without the equality sign). Specifically, the two subregions are

$$\mathcal{S}^+ := \left\{x \in \mathcal{C}_d : \mu\left(\frac{\partial f}{\partial x}(x)\right) > -\bar{c}\right\},$$
$$\mathcal{S}^- := \left\{x \in \mathcal{C}_d : \mu\left(\frac{\partial f}{\partial x}(x)\right) < -\bar{c}\right\}.$$

The key design idea is to choose the scalar function h in (4-20) as

$$h(x) = \mu\left(\frac{\partial f}{\partial x}(x)\right) + \bar{c}, \tag{4-26}$$

in this way the switching manifold Σ is defined as

$$\Sigma := \left\{x \in \mathcal{C}_d : \mu\left(\frac{\partial f}{\partial x}(x)\right) = -\bar{c}\right\}. \tag{4-27}$$

The final step is to find u^+ and u^- such that conditions (4-21)-(4-23) are satisfied. Obviously with the selection of $h(x)$ made in (4-26) the open-loop vector field f already satisfies the design requirements in $\mathcal{S}^-$, therefore in this case the simplest choice is

$$u^-(x) = 0, \tag{4-28}$$

and the control problem is reduced to find a u^+ that satisfies (4-21) and (4-23). In other terms, by selecting (4-27) as switching manifold the resulting switching control input can be active only in the region where the controlled system is not sufficiently contracting.

This property can be exploited to reduce the average control energy compared to the one required by a continuous control input defined in the whole set $\mathcal{C}_d$ (eventually globally), as we will show in the next section through a simple example.

Representative examples Here we present examples to illustrate the design procedure described in the previous section. The unweighted 1-norm will be used to highlight that non-Euclidean norms can be chosen in some cases as an alternative to Euclidean norms and that not only the analysis but the control synthesis too can be easier.

The nonlinear system (4-19) that we want to incrementally stabilize in a certain set is

$$\dot{x} = \begin{bmatrix} -4x_1 \\ x_2^2 - 6x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(x) \tag{4-29}$$

The desired convergence rate $\bar{c}$ in this examples is set to 2, i.e. $\bar{c} = 2$. It can be easily seen that

$$\begin{aligned} \mu_1\left(\frac{\partial f}{\partial x}(x)\right) &= \mu_1\left(\begin{bmatrix} -4 & 0 \\ 0 & 2x_2 - 6 \end{bmatrix}\right) \\ &= \max\{-4;\ 2x_2 - 6\} \\ &= \begin{cases} -4 & \text{if } x_2 \leq 1 \\ 2x_2 - 6 & \text{if } x_2 > 1 \end{cases} \end{aligned}$$

Therefore the set $\mathcal{C}$ where system (4-29) is contracting with contraction rate $\bar{c}$, that is where it satisfies condition (4-25), is

$$\mathcal{C} = \{x \in \mathbb{R}^2 :\ x_2 < 2\}.$$

In the following two design examples will be presented and discussed. In the first one we want to extend the region $\mathcal{C}$ where the system is incrementally stable to the set $\mathcal{C}_d \supset \mathcal{C}$, and in the second one we want to make the system globally incrementally stable, that is $\mathcal{C}_d \equiv \mathbb{R}^2$.

In both cases, following the design procedure presented above, the scalar function h of the switching controller is set as

$$h(x) = \mu_1\left(\frac{\partial f}{\partial x}(x)\right) + 2$$

and the switching manifold Σ as its zero set, that is as

$$\Sigma = \{x \in \mathcal{C}_d :\ x_2 = 2\}$$

Furthermore, as expected the control requirements are already satisfied in $\mathcal{S}^-$, and thus $u^-(x) = 0$. The problem is now reduced to find a function $u^+(x)$ such that conditions (4-21) and (4-23) hold. Specifically, condition (4-21) is satisfied if the following quantity is made less than $-\bar{c}$

$$\begin{aligned} \mu_1\left(\frac{\partial}{\partial x}\left[f(x)+B(x)\,u^+(x)\right]\right) =&\mu_1\left(\begin{bmatrix} -4+u_{x1} & u_{x2} \\ 2u_{x1} & 2x_2-6+2u_{x2} \end{bmatrix}\right) \\ =&\max\{-4+u_{x1}+|2u_{x1}|; \\ &\qquad 2x_2-6+2u_{x2}+|u_{x2}|\} \end{aligned} \tag{4-30}$$

with $\frac{\partial u^+}{\partial x} = [u_{x1} \;\; u_{x2}]$.
In this simple example the first term in (4-30) does not depend on x so it can be made less than $-\bar{c}$ by simply setting $u_{x1} = 0$. Therefore, in conclusion we need to find u_{x2} such that

$$2x_2 - 6 + 2u_{x2} + |u_{x2}| \leq -2, \qquad \forall x \in \bar{\mathcal{S}}^+ \tag{4-31}$$

and then check if the resulting $u(x)$ satisfies (4-23) where $\nabla h = [0 \;\; 1]$.

Example 1 In this first example we want to extend the region where system (4-29) is contracting to a new set $\mathcal{C}_d$, in particular we choose $\mathcal{C}_d = \{x \in \mathbb{R}^2 : \, x_2 < 7\}$. Therefore $\mathcal{S}^+ = \{x \in \mathcal{C}_d : \; 2 < x_2 < 7\}$, and it can be easily proved that (4-31) is satisfied for $u_{x2} \leq -10$, and thus, by integration, we have

$$u^+(x) = -10x_2$$

Condition (4-23) is also satisfied, since we have that for all $x \in \Sigma$

$$\begin{aligned} \mu_1\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot (-10x_2) \cdot \begin{bmatrix} 0 & 1 \end{bmatrix}\right) =&10\,\mu_1\left(\begin{bmatrix} 0 & -x_2 \\ 0 & -2x_2 \end{bmatrix}\right) \\ =&10\,\max\{0;\; -2x_2+|x_2|\} \\ =&10\,\max\{0;\; -2\} = 0. \end{aligned}$$

In conclusion a switching control input that incrementally stabilize (4-29) in $\mathcal{C}_d$ is

$$u(x) = \begin{cases} -10x_2 & \text{if } x_2 > 2 \\ 0 & \text{if } x_2 < 2 \end{cases} \tag{4-32}$$

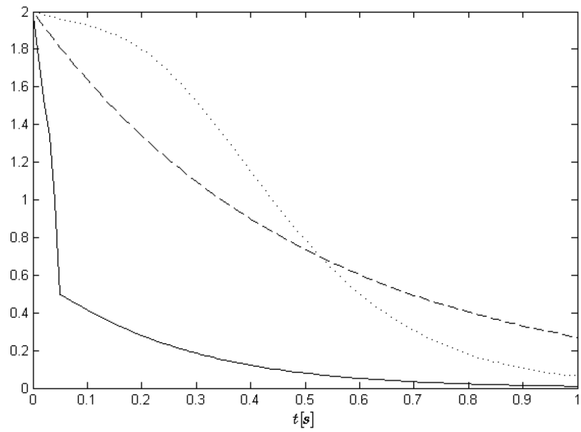

Figure 4-5: System (4-29) in open-loop (dotted line) and with control (4-32) (solid line). Initial conditions in $x_0 = [1 \quad 4]^T$ and $y_0 = [2 \quad 5]^T$. The dashed line is the estimated exponential upper bound with $\lambda = \bar{c} = 2$ and $K = 1$.

In Figure 4-5, we report numerical simulations of the evolution of the difference between two trajectories. The dashed line is the estimated exponential upper bound with $c = 2$ and $K = 1$, that is

$$|x(t) - y(t)|_1 \leq e^{-2t} \, |x_0 - y_0|_1, \qquad \forall t > 0.$$

Example 2 Next we consider the problem of making system (4-29) to be globally incrementally stable (that is $\mathcal{C}_d \equiv \mathbb{R}^2$) condition (4-31) has to be verified with $\mathcal{S}^+ = \{x \in \mathbb{R}^2 : \; x_2 > 2\}$. It can be proved that such condition is satisfied choosing for example $u_{x2} = -2x_2$, and therefore by integration the control input defined in $\mathcal{S}^+$ is

$$u^+(x) = -x_2^2.$$

Again, condition (4-23) is satisfied since

$$\begin{aligned} \mu_1 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot (-x_2^2) \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} \right) &= \max \{0; \; -2x_2^2 + |-x_2^2|\} \\ &= \max \{0; \; -4\} = 0 \end{aligned}$$

for all $x \in \Sigma$.

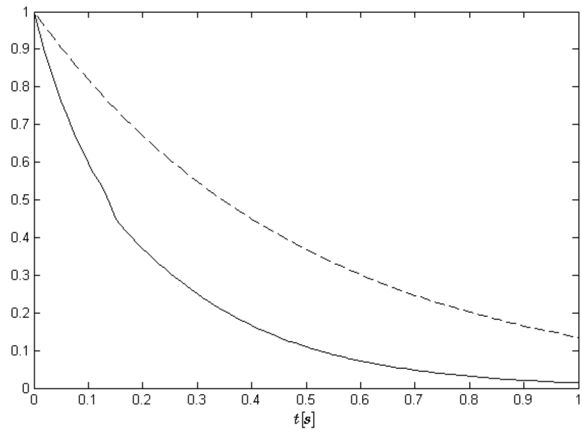

Figure 4-6: Closed-loop system with control (4-33). Initial conditions in $x_0 = [1 \;\; 8]^T$ and $y_0 = [1 \;\; 9]^T$. The dashed line is the estimated exponential upper bound with $\lambda = \bar{c} = 2$ and $K = 1$.

To conclude, system (4-29) is globally incrementally stabilized by the switching controller

$$u(x) = \begin{cases} -x_2^2 & \text{if } x_2 > 2 \\ 0 & \text{if } x_2 < 2 \end{cases} \tag{4-33}$$

In Figure 4-6, we show numerical simulations of the evolution of the difference between two trajectories that confirm the theoretical results. Open-loop simulations are not reported in this case since the system is unstable for chosen initial conditions.

Discussion As highlighted previously, the control input presented here is active only in the region $\mathcal{S}^+$ of the state space where the open-loop system is not sufficiently incrementally stable, otherwise it is turned off, reducing the required control energy. On the other hand, to satisfy the same stability requirements a continuous control feedback $\widehat{u}(x)$ has to be design such that

$$\mu\left(\frac{\partial}{\partial x}\left[f(x) + B(x)\,\widehat{u}(x)\right]\right) \le -\bar{c} \qquad \forall x \in \mathcal{C}_d,$$

and thus it has to take non-zero values on the whole $\mathcal{C}_d$.

For example, a continuous feedback control that satisfies control requirements as in Example 1 is

$$\widehat{u}(x) = -10x_2 \qquad \forall x \in \mathcal{C}_d, \tag{4-34}$$

that is the same input $u^+(x)$ in (4-32) extended to the whole state space. Hence, trivially, it is clear that control input (4-32) uses less energy than (4-34).

Instead, for what concerns Example 2, a continuous function $\widehat{u}(x)$ such that (4-31) holds on all $\mathbb{R}^2$ has to be at least cubic (while (4-33) is quadratic). Since their derivatives have to satisfy the same linear constraint (4-31) in $\mathcal{S}^+$, it follows that the L_2-norm of the continuous control input will in general greater than the one of the discontinuous input.

4.2.2 Observer design for switched systems

The problem of designing state observers for nondifferentiable systems is the subject of current research. For example, the design of observers for Lipschitz continuous nonlinear systems was investigated in [107, 149], while in [10, 19] design approaches based on passivity theory were proposed for Lur'e-type systems. Also, in [66, 42] sufficient conditions were presented to ensure stability of the estimation error for state observers of bimodal piecewise linear (PWL) systems (both continuous and discontinuous on the switching surface). The analysis was conducted analyzing the quadri-modal estimation error dynamics based on quadratic Lyapunov functions and LMIs. Related results were presented in [142] for the case of piecewise affine (PWA) systems. Therein, using theoretical results developed in [95], sufficient conditions guaranteeing exponential stability of the estimation error were given in terms of a set of appropriate LMIs. More recently, the state estimation problem was investigated in [62] for linear complementarity systems and in [53] for hybrid systems with impacts.

In this section we propose a methodology to design state observers for nondifferentiable bimodal vector fields, which stems from the results presented in first part of this chapter. Specifically, we derive conditions in terms of matrix measures of the Jacobian of the observer dynamics and of an additional condition on the vector fields on the discontinuity set such that the estimation error converges exponentially to zero.

These conditions, when particularized to the case of PWA systems, generalize those presented in [142] to the case of non-Euclidean norms.

Problem formulation Consider the bimodal switched system

$$\dot{x} = \begin{cases} f^+(x) + u(t), & h(x) > 0 \\ f^-(x) + u(t), & h(x) < 0 \end{cases}, \tag{4-35}$$

$$y = g(x), \tag{4-36}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^n$ are the state, output and the input of the system, respectively, and f^+, f^-, g are continuously differentiable vector fields.

As an observer for the system (4-35)-(4-36), we propose a bimodal switched observer with the Luenberger-type structure

$$\dot{\hat{x}} = \begin{cases} f^+(\hat{x}) + L^+(y - \hat{y}) + u(t), & h(\hat{x}) > 0 \\ f^-(\hat{x}) + L^-(y - \hat{y}) + u(t), & h(\hat{x}) < 0 \end{cases}, \tag{4-37}$$

$$\hat{y} = g(\hat{x}), \tag{4-38}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimated state and $L^+, L^- \in \mathbb{R}^{n\times p}$ are observer gain matrices to be selected appropriately.

We are interested to derive conditions on the observer gain matrices L^+ and L^- that guarantee exponential convergence to 0 of the estimation error $e(t) := x(t) - \hat{x}(t)$ for all $x(t) : \mathbb{R}^+ \to \mathbb{R}^n$ satisfying (4-35)-(4-36) for any given continuous function $u(t) : \mathbb{R}^+ \to \mathbb{R}^n$. Note that in what follows it is not required for system (4-35)-(4-36) to be contracting, i.e. Theorem 4.2 must not necessarily hold for this system. Instead, contraction will be used to analyze convergence of the system describing the dynamics of the estimation error.

Theorem 4.4. *The state estimation error $e(t)$ converges exponentially to zero with convergence rate $c := \min\{c_1, c_2\}$, that is*

$$|e(t)| \le K\, e^{-c(t-t_0)}\, |x(t_0)|\,, \qquad \forall t \ge t_0, \tag{4-39}$$

if there exists some norm, with associated matrix measure μ, such that, *for some positive constants c_1, c_2,*

$$\mu\left(\frac{\partial f^+}{\partial x}(\hat{x}) - L^+\frac{\partial g}{\partial x}(\hat{x})\right) \le -c_1, \quad \forall \hat{x} : h(\hat{x}) > 0, \qquad (4\text{-}40)$$

$$\mu\left(\frac{\partial f^-}{\partial x}(\hat{x}) - L^-\frac{\partial g}{\partial x}(\hat{x})\right) \le -c_2, \quad \forall \hat{x} : h(\hat{x}) < 0, \qquad (4\text{-}41)$$

$$\mu\left(\left[\Delta f(\hat{x}) + \Delta L(y-\hat{y})\right]\nabla h(\hat{x})\right) = 0, \quad \forall \hat{x} : h(\hat{x}) = 0, \qquad (4\text{-}42)$$

where $\Delta f(\hat{x}) = f^+(\hat{x}) - f^-(\hat{x})$ *and* $\Delta L = L^+ - L^-$.

Proof. Conditions (4-2)-(4-4) come from the application of Theorem 4.2 to the dynamics of state observer (4-37)-(4-38) by rewriting them as

$$\dot{\hat{x}} = \begin{cases} \bar{f}^+(\hat{x}) + \eta^+(t), & h(\hat{x}) > 0 \\ \bar{f}^-(\hat{x}) + \eta^-(t), & h(\hat{x}) < 0 \end{cases},$$

where $\bar{f}^\pm(\hat{x}) = f^\pm(\hat{x}) - L^\pm g(\hat{x})$ depends only on $\hat{x}$, and $\eta^\pm(t) = L^\pm g(x(t)) + u(t)$ is a function of t. Hence, if such conditions are satisfied, then the state observer is contracting; this in turn implies that, for two generic solutions $\hat{x}_1(t)$ and $\hat{x}_2(t)$, (4-1) holds, i.e.

$$|\hat{x}_1(t) - \hat{x}_2(t)| \le K\, e^{-c(t-t_0)}|\hat{x}_1(t_0) - \hat{x}_2(t_0)|, \ \forall t \ge t_0.$$

Now, noticing that a solution $x(t)$ of system (4-35) is a particular solution of the observer (4-37) — because (4-35) and (4-37) have the same structure, except for the correction term $g(x)-g(\hat{x})$, which is null when considering $x(t)$ as a solution of the observer — we can replace $\hat{x}_2(t)$ with $x(t)$, rename $\hat{x}_1(t)$ as the general solution $\hat{x}(t)$, and write

$$|e(t)| = |x(t) - \hat{x}(t)| \le K\, e^{-c(t-t_0)}|x(t_0)|,$$

for all $t \ge t_0$, where $\hat{x}(t_0) = 0$ as usual in observer design. Hence, the exponential convergence to zero of the estimation error is proved. □

Alternatively, the theorem can be proved considering the regularized dynamics of both system (4-35) and observer (4-37). Denoting by $x_\varepsilon(t)$ a solution to the regularized switched system (4-35), and by $\hat{x}_\varepsilon(t)$ a solution to the regularized observer (4-37), we have

$$\begin{aligned}|e(t)| =& |x(t) - \hat{x}(t)| \\ \le& |x(t) - x_\varepsilon(t)| + |x_\varepsilon(t) - \hat{x}_\varepsilon(t)| + |\hat{x}_\varepsilon(t) - \hat{x}(t)|.\end{aligned}$$

The first and the third terms are the error between a solution to the discontinuous system and a solution to its regularized counterpart; hence, from Lemma 2.2 we know that

$$\begin{aligned} |x(t) - x_\varepsilon(t)| &= O(\varepsilon), \\ |\hat{x}(t) - \hat{x}_\varepsilon(t)| &= O(\varepsilon). \end{aligned}$$

Furthermore, similarly to what done in Section 4.1, it can be shown that conditions (4-40)-(4-42) imply incremental stability of the trajectories of the regularized observer, thus

$$|\hat{x}_{\varepsilon,1}(t) - \hat{x}_{\varepsilon,2}(t)| \leq K\, e^{-c(t-t_0)}|\hat{x}_{\varepsilon,1}(t_0) - \hat{x}_{\varepsilon,2}(t_0)|, \forall t \geq t_0.$$

The theorem is finally proved by taking the limit for $\varepsilon \to 0^+$ and following the last step as in the above proof.

Remark 4.3. *In the case that one of the two modes, f^+ or f^-, of the observed system* (4-35) *is already contracting, the respective observer gain in* (4-37) *can be set to zero to simplify the design problem. The drawback is a convergence rate of the estimation error that depends on the open loop contraction rate of the respective mode.*

Remark 4.4. *In presence of bounded disturbances or uncertainties on the models, contraction properties of the vector fields guarantee boundedness of the estimation error (a more detailed analysis is not the aim of the current discussion; the interested reader can refer to [76]).*

Representative examples Here we present examples to illustrate how state observers for different classes of piecewise smooth systems may be designed using Theorem 4.4.

Example 1 Consider a nonlinear bimodal switched system as in (4-35)-(4-36) with

$$f^+(x) = \begin{bmatrix} -9x_1 - 3x_1^2 - 18 \\ -4x_2 \end{bmatrix}, \; f^-(x) = \begin{bmatrix} -9x_1 + 3x_1^2 + 18 \\ -4x_2 \end{bmatrix},$$

and $h(x) = x_1$, $y = g(x) = x_1^2$.

According to Theorem 4.4, a state observer as in (4-37)-(4-38) with $L^+ = \begin{bmatrix} \ell_1^+ & \ell_2^+ \end{bmatrix}^{\mathrm{T}}$ and $L^- = \begin{bmatrix} \ell_1^- & \ell_2^- \end{bmatrix}^{\mathrm{T}}$ for this system has the property that its estimation error converges exponentially to zero if there exist choices of the gain matrices L^+ and L^- so that all three conditions (4-2)-(4-4) are satisfied.

To find L^+ and L^-, it is first necessary to select a specific matrix measure; here we use the measure μ_1, associated to the so-called ℓ^1-norm. Therefore, conditions (4-40) and (4-41) translate respectively to

$$\mu_1\left(\begin{bmatrix} -9-6\hat{x}_1-2\ell_1^+\hat{x}_1 & 0 \\ -2\ell_2^+\hat{x}_1 & -4 \end{bmatrix}\right) < 0, \quad \text{with } \hat{x}_1 > 0,$$
$$\mu_1\left(\begin{bmatrix} -9+6\hat{x}_1-2\ell_1^-\hat{x}_1 & 0 \\ -2\ell_2^-\hat{x}_1 & -4 \end{bmatrix}\right) < 0, \quad \text{with } \hat{x}_1 < 0.$$

Selecting for simplicity $\ell_2^+ = \ell_2^- = 0$, the above inequalities are satisfied if

$$\max\{-9-6\hat{x}_1-2\ell_1^+\hat{x}_1;\ -4\} < 0, \quad \text{with } \hat{x}_1 > 0,$$
$$\max\{-9+6\hat{x}_1-2\ell_1^-\hat{x}_1;\ -4\} < 0, \quad \text{with } \hat{x}_1 < 0.$$

This is true if $\ell_1^+ > -3$ and $\ell_1^- < 3$.

Next, from the the third condition (4-42), we have

$$\mu_1\left(\begin{bmatrix} -6\hat{x}_1^2-36+(\ell_1^+-\ell_1^-)(x_1^2-\hat{x}_1^2) \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\right) = 0,$$

with $\hat{x}_1 = 0$, which is verified if

$$\max\{-36+(\ell_1^+-\ell_1^-)x_1^2;\ 0\} = 0,$$

i.e. if

$$-36+(\ell_1^+-\ell_1^-)x_1^2 < 0,$$

which holds for all x_1 if $\ell_1^+ < \ell_1^-$. Therefore, to satisfy all three conditions of Theorem 4.4 it is possible for example to select $L^+ = [-2\ \ 0]^{\mathrm{T}}$ and $L^- = [2\ \ 0]^{\mathrm{T}}$. The resulting state observer is contracting and its estimation error satisfies (4-39) with convergence rate $c = 4$. In Fig. 4-7(a) we show numerical simulations of the evolution of the states x_1 and $\hat{x}_1$ when an input $u(t) = [1\ \ 1]^{\mathrm{T}} \sin(2\pi t)$ of period $T = 1$ is applied to the system. In Fig. 4-7(b) the evolution of the ℓ^1-norm of the state estimation error $e(t)$ is reported, confirming the analytical estimate (4-39).

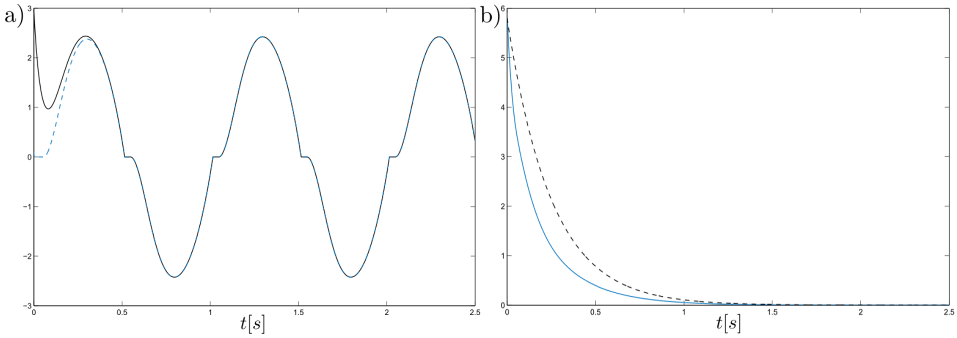

Figure 4-7: Panel a: Time evolution of the states $x_1(t)$ (solid line) and $\hat{x}_1(t)$ (dashed line) of Example 1. Initial conditions are respectively $x_0 = [3 \quad 3]^{\mathrm{T}}$, $\hat{x}_0 = [0 \quad 0]^{\mathrm{T}}$. Panel b: Norm of the corresponding estimation error $|e(t)|_1$. The dashed line represents the analytical estimate (4-39) with $c = 4$ and $K = 1$. Parameters: $L^+ = [-2 \quad 0]^{\mathrm{T}}$ and $L^- = [2 \ \ 0]^{\mathrm{T}}$.

Example 2 Consider a piecewise affine (PWA) system of the form

$$\dot{x} = \begin{cases} A_1 x + b_1 + Bu, & \text{if } h^{\mathrm{T}} x > 0 \\ A_2 x + b_2 + Bu, & \text{if } h^{\mathrm{T}} x < 0 \end{cases}, \tag{4-43}$$

$$y = c^{\mathrm{T}} x, \tag{4-44}$$

where

$$A_1 = \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1 \\ -3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix},$$

and $B = [0 \ \ 1]^{\mathrm{T}}$, $h = [0 \ \ 1]^{\mathrm{T}}$, $c = [1 \ \ 1]^{\mathrm{T}}$.

A state observer as in (4-37)-(4-38) for this system has the structure

$$\dot{\hat{x}} = \begin{cases} A_1 \hat{x} + b_1 + L^+(y - \hat{y}) + Bu, & \text{if } h^{\mathrm{T}} \hat{x} > 0 \\ A_2 \hat{x} + b_2 + L^-(y - \hat{y}) + Bu, & \text{if } h^{\mathrm{T}} \hat{x} < 0 \end{cases}, \tag{4-45}$$

$$\hat{y} = c^{\mathrm{T}} \hat{x}, \tag{4-46}$$

where, for the sake of simplicity, we choose $L^+ = L^- = L$. Again we decide to proceed using the matrix measure induced by the ℓ^1-norm. In this case, conditions (4-40) and (4-41) yield respectively

$$\begin{aligned}\mu_1\left(A_1 - Lc^{\mathrm{T}}\right) =& \mu_1\left(\begin{bmatrix}-1-\ell_1 & -\ell_1\\ 2-\ell_2 & -2-\ell_2\end{bmatrix}\right)\\ &= \max\{-1-\ell_1+|2-\ell_2|; -2-\ell_2+|\ell_1|\}\end{aligned}$$

and

$$\begin{aligned}\mu_1\left(A_2 - Lc^{\mathrm{T}}\right) =& \mu_1\left(\begin{bmatrix}-1-\ell_1 & -\ell_1\\ 2-\ell_2 & -3-\ell_2\end{bmatrix}\right)\\ &= \max\{-1-\ell_1+|2-\ell_2|; -3-\ell_2+|\ell_1|\}.\end{aligned}$$

It is easy to verify that choosing $\ell_1 = \ell_2 = 1$ both measures are equal to -1. Condition (4-4) translates into

$$\mu_1\left(\begin{bmatrix}0 & -3\\ 0 & \hat{x}_2 - 7\end{bmatrix}\right) = 0, \quad \text{with } \hat{x}_2 = 0,$$

which is identically verified, independently of L.

Hence, the designed observer (4-45) is contracting and the estimation error converges exponentially to zero with rate $c = 1$. In Fig. 4-8(a) we show numerical simulations of the evolution of the states x_2 and $\hat{x}_2$ when an input $u(t) = 4\,\sin(2\pi t)$ of period $T = 1$ is applied to the system. In Fig. 4-8(b) the evolution is reported of the ℓ^1-norm of the state estimation error $e(t)$.

Note that faster convergence can be obtained by choosing higher values of ℓ_1 and ℓ_2 fulfilling conditions (25)-(26). For example choosing $L = [1.5\ \ 2]^T$ we obtain a convergence rate $c = 2.5$, as shown in Fig. 4-8(c).

Example 3 Consider now a harmonic oscillator affected by Coulomb friction, described by the equations

$$\begin{cases}\dot{x}_1 = x_2,\\ \dot{x}_2 = -\omega_{\mathrm{n}} x_1 - \dfrac{\omega_{\mathrm{n}}}{Q} x_2 - \dfrac{F_{\mathrm{f}}}{m}\mathrm{sgn}(x_2) + \dfrac{F_{\mathrm{d}}}{m}\sin(\omega_{\mathrm{d}} t),\end{cases} \tag{4-47}$$

$$y = x_1, \tag{4-48}$$

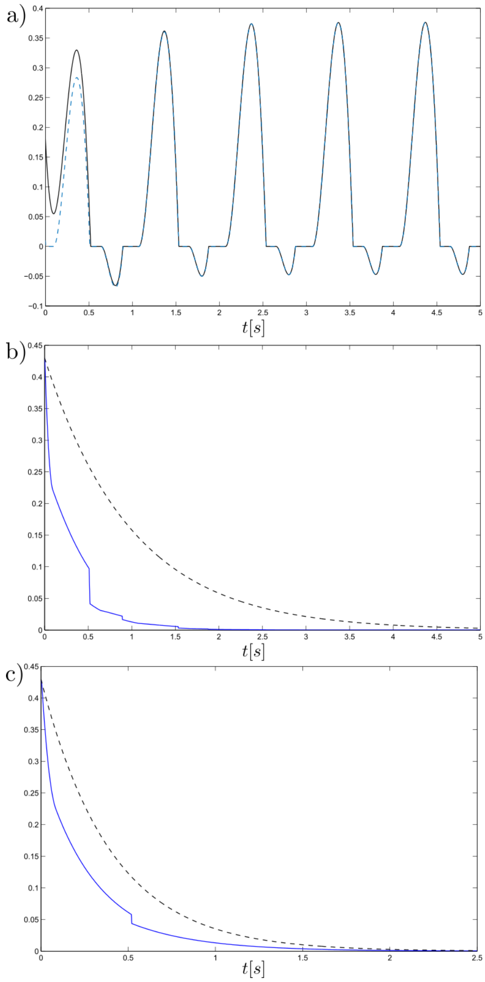

Figure 4-8: Panel a: Time evolution of the states $x_2(t)$ (solid line) and $\hat{x}_2(t)$ (dashed line) of Example 2. Initial conditions are respectively $x_0 = [0.3 \quad 0.3]^{\mathrm{T}}$, $\hat{x}_0 = [0 \quad 0]^{\mathrm{T}}$. Panel b: Norm of the corresponding estimation error $|e(t)|_1$. The dashed line represents the analytical estimate (4-39) with $c = 1$ and $K = 1$. Parameters: $L^+ = L^- = [1 \ \ 1]^{\mathrm{T}}$. Panel c: Norm of the estimation error using observer gain $L = [1.5 \ \ 2]^{\mathrm{T}}$.

where $x_1 \in \mathbb{R}$ is the position of the oscillator, $x_2 \in \mathbb{R}$ is its velocity, ω_{n} is its natural frequency, Q is said *Q factor* and is inversely proportional to the damping, m is the mass of the oscillator, F_d is the amplitude of the driving force, ω_d is the driving frequency and F_f is the amplitude of the dry friction force which is modeled through the sign function as in [25]. The proposed observer for system (4-47)-(4-48) has the form

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \ell_1(x_1 - \hat{x}_1) \\ \dot{\hat{x}}_2 = -\omega_\mathrm{n}\hat{x}_1 - \dfrac{\omega_\mathrm{n}}{Q}\hat{x}_2 - \dfrac{F_\mathrm{f}}{m}\mathrm{sgn}(\hat{x}_2) \\ \qquad +\ell_2(x_1 - \hat{x}_1) + \dfrac{F_\mathrm{d}}{m}\sin(\omega_\mathrm{d} t) \end{cases}, \quad \hat{y} = \hat{x}_1.$$

Note that system (4-47) may be viewed as a PWA system (4-43) where

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -\omega_\mathrm{d} & -\omega_\mathrm{d}/Q \end{bmatrix},$$

$B = [0 \;\; 1/m]^\mathrm{T}$, $b_1 = [0 \;\; -F_\mathrm{f}/m]^\mathrm{T}$, $b_2 = [0 \;\; F_\mathrm{f}/m]^\mathrm{T}$, $h = [0 \;\; 1]^\mathrm{T}$, and excited by an input $u(t) = F_\mathrm{d}\sin(\omega_\mathrm{d} t)$.

Using the measure μ_∞, induced by the uniform norm, conditions (4-40) and (4-41) of Theorem 4.4, combined, translate to

$$\mu_\infty\left(\begin{bmatrix} -\ell_1 & 1 \\ -\omega_\mathrm{n} - \ell_2 & -\omega_\mathrm{n}/Q \end{bmatrix}\right) < 0, \quad \text{with } \hat{x}_2 \neq 0,$$

which in turn is equivalent to

$$\max\{-\ell_1 + 1;\; -\omega_\mathrm{n}/Q + |-\omega_\mathrm{n} - \ell_2|\} < 0, \quad \text{with } \hat{x}_2 \neq 0.$$

Therefore ℓ_1 and ℓ_2 must be chosen so that

$$\begin{aligned} & \ell_1 > 1, \\ -\omega_\mathrm{n}\left(1 + \frac{1}{Q}\right) < \; & \ell_2 < -\omega_\mathrm{n}\left(1 - \frac{1}{Q}\right). \end{aligned}$$

Furthermore, condition (4-4) is verified if

$$\mu_\infty\left(\begin{bmatrix} 0 \\ -2F_\mathrm{f}/m \end{bmatrix}[0 \;\; 1]\right) = 0, \quad \text{with } \hat{x}_2 = 0,$$

i.e. if

$$\max\{0;\; -F_\mathrm{f}/m\} = 0, \quad \text{with } \hat{x}_2 = 0,$$

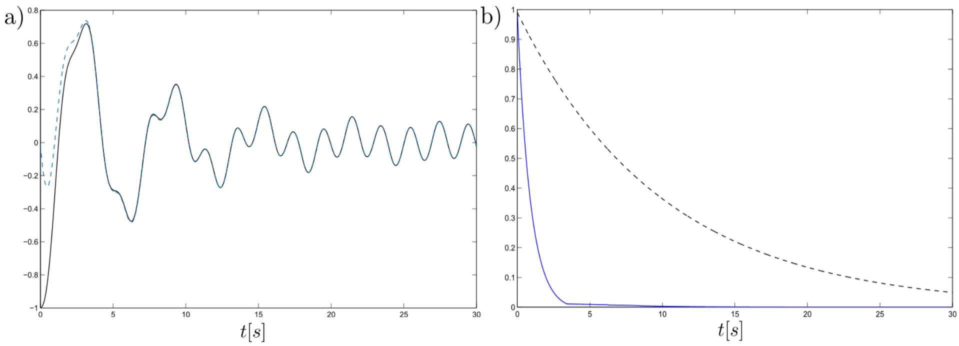

Figure 4-9: Panel a: time evolution of the states $x_1(t)$ (solid line) and $\hat{x}_1(t)$ (dashed line) of Example 3. Initial conditions are respectively $x_0 = [-1 \quad 0]^{\mathrm{T}}$, $\hat{x}_0 = [0 \quad 0]^{\mathrm{T}}$. Panel b: Norm of the corresponding estimation error $|e(t)|_\infty$. The dashed line represents the analytical estimate (4-39) with $c = 0.1$ and $K = 1$. Parameters: $\omega_{\mathrm{n}} = 1$ rad/s, $Q = 10$, $m = 1$ kg, $F_{\mathrm{d}} = 1$ N, $\omega_{\mathrm{d}} = \pi$ rad/s, $F_{\mathrm{f}} = 0.1$ N, $\ell_1 = 1.1$, $\ell_2 = -1$.

which always holds because $F_{\mathrm{f}}, m > 0$.

Numerical simulations reported in Fig. 4-9(a)-(b) confirm the theoretical predictions, showing that the estimation error converges to zero as expected. In practice, the exact value of the parameter F_f is not known. This implies bounded convergence of the estimation error, as stated in Remark 4.4.

CHAPTER 5

Finsler-Lyapunov analysis of switched systems via regularization

In this chapter we investigate an extension of the differential stability theory based on Finsler-Lyapunov functions [50] (see also Section 3.6) to switched dynamical systems. Due to non-differentiability of such systems the original formulation of the theory cannot be directly applied to them. To overcome this limitation we present here a new approach based on regularization. The idea of studying differential stability of switched systems via regularization was formerly introduced in [45], where incremental stability was investigated using matrix measures (see Chapter 4 for further details).

We derive the *prolonged system* (3-22) of a bimodal switched system as the limit of the one of its regularization. We show that the prolonged system has a hybrid nature, combining continuous and discrete dynamics. The continuous dynamics is related to flow and it is described by the Jacobian matrix, while the discrete dynamics is related to switching events and it is captured by the *saltation matrix*. We find that such discrete map can at most contract a ball in a direction transversal to the switching surface, leaving the others unchanged. Therefore, in the study of incremental stability it is necessary to take into account this hybrid nature and to look at variations of an appropriate Finsler-Lyapunov function during both flow and switchings with analysis tools typical of hybrid systems [58].

The idea presented here is then applied to three simple switched linear systems. Further details on the saltation matrix are reported in Section 2.2 and a comparison with the conditions presented in [45] are reported in Section 5.3. Furthermore, note that in this chapter we use ∂f as short notation for the Jacobian matrix $\frac{\partial f}{\partial x}$.

5.1 Differential stability of switched systems

To extend contraction analysis based on Finsler-Lyapunov functions to switched systems as (2-1) is first necessary to extend the definition of prolonged system. We make the following assumption.

Assumption 5.1. *The switched system* (2-1) *fulfills the conditions for existence and uniqueness of Carathéodory solutions of Theorem 2.1. Moreover, every intersection of a solution with a switching manifold* Σ_{ij} *occurs transversally and at most with one manifold at time.*

From this assumption, the dynamics of system (2-1) in the neighborhood of crossing events can be described by a bimodal switched system (2-2) for whom the following result holds.

Proposition 5.1. *The* prolonged system *of a bimodal switched system* (2-2) *along a solution that starts in* $\mathcal{S}^-$*, transversally crosses* Σ *and ends in* $\mathcal{S}^+$*, is the composition of three systems: two continuous-time systems that describe the dynamics before and after the switching event,*

$$\begin{cases} \dot{x} = f^-(x) \\ \dot{\delta x} = \partial f^-(x)\delta x \end{cases}, \quad (x, \delta x) \in \mathcal{S}^- \times \mathbb{R}^n \tag{5-1}$$

and

$$\begin{cases} \dot{x} = f^+(x) \\ \dot{\delta x} = \partial f^+(x)\delta x \end{cases}, \quad (x, \delta x) \in \mathcal{S}^+ \times \mathbb{R}^n \tag{5-2}$$

respectively, and a third discrete-time system

$$\begin{cases} x^+ = x \\ \delta x^+ = S(x)\delta x \end{cases}, \quad (x, \delta x) \in \Sigma \times \mathbb{R}^n, \tag{5-3}$$

that takes into account the switch, where the linear discrete map

$$S(x) = I + \frac{[f^+(x) - f^-(x)]\nabla h(x)}{\nabla h(x) f^-(x)}$$

is the saltation matrix *and describes how the vector δx is mapped into δx^+ at the switching time instant.*

Further details about the saltation matrix are reported in Section 2.2. Our derivation of the prolonged system is based on the regularization approach and it follows steps similar to those presented in [70].

Proof. Instead of directly analyzing the discontinuous system (2-2) we consider its *linear regularization* (2-31), then we will show that as $\varepsilon \to 0$ the prolonged system of the regularized system evaluated along a solution moving from $\mathcal{S}^-$ to $\mathcal{S}^+$ tends to the three subsystems (5-1)-(5-3).

The *prolonged system* of the regularized system (2-29) is

$$\begin{cases} \dot{x}_\varepsilon = f_\varepsilon(x_\varepsilon) \\ \dot{\delta x}_\varepsilon = \partial f_\varepsilon(x_\varepsilon)\, \delta x_\varepsilon \end{cases} \tag{5-4}$$

where f_ε and ∂f_ε are those in (2-31) and (2-32), respectively. The subscript ε in x_ε and δx_ε emphasizes that the solutions to (2-31) depends on the value of $\varepsilon > 0$.

A solution to (5-4) starting in $\mathcal{S}^-$ that *transversally* crosses Σ , that is

$$\nabla h(x) f^-(x) \neq 0 \quad \forall x \in \Sigma, \tag{5-5}$$

evolves in the regularization layer Σ_ε and then exits it when $h(x_\varepsilon) = \varepsilon$, continuing its evolution in $\mathcal{S}^+$. Since no approximation was made in $\mathcal{S}^-$ we have that $x_\varepsilon \equiv x$ therein, and therefore for all x_ε such that $h(x_\varepsilon) < 0$ (i.e. $x_\varepsilon \in \mathcal{S}^-$) system (5-4) coincides with (5-1). Hence, in the following analysis we can consider as initial condition a point on Σ. More precisely, denote with $\bar{x} = x_\varepsilon(0)$ the state at time $t = 0$ when the solution to (5-4) crosses Σ, that is $h(x_\varepsilon(0)) = h(\bar{x}) = 0$, and with $x_\varepsilon(t_\varepsilon) = \bar{x} + \Delta x_\varepsilon(t_\varepsilon)$ the state at time $t = t_\varepsilon$ when the solution exits the regularization layer, that is $h(x_\varepsilon(t_\varepsilon)) = \varepsilon$. (See Figure 5-1).

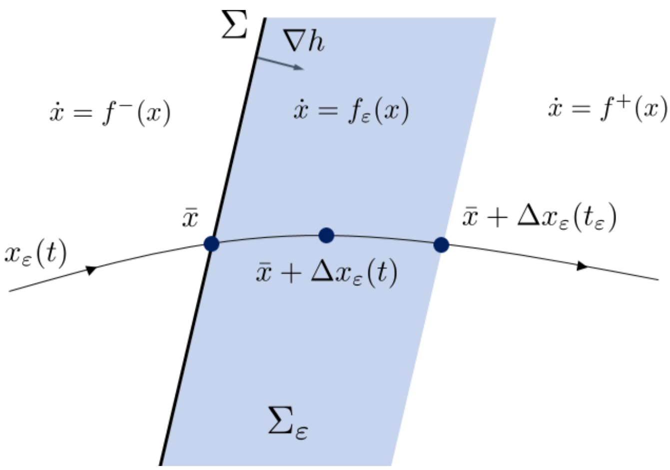

Figure 5-1: Evolution of the solution $x_\varepsilon(t)$ in the regularization layer.

The solution $x_\varepsilon(t)$ to $\dot{x}_\varepsilon = f_\varepsilon(x_\varepsilon)$ will evolve in the regularization layer Σ_ε for an interval of time $\Delta t = t_\varepsilon$, and its value after this interval is

$$x_\varepsilon(t_\varepsilon) = \bar{x} + \Delta x_\varepsilon(t_\varepsilon) = \bar{x} + \int_0^{t_\varepsilon} f_\varepsilon(x_\varepsilon(\tau))\, d\tau. \tag{5-6}$$

From the previous relation it follows that

$$\Delta x_\varepsilon(t_\varepsilon) = f_\varepsilon(x_\varepsilon(0))\, t_\varepsilon + O(t_\varepsilon^2) = f^-(\bar{x})\, t_\varepsilon + O(t_\varepsilon^2), \tag{5-7}$$

since, from (2-31), $f_\varepsilon(x) = f^-(x)$, $\forall x \in \Sigma$.

On the other hand, the evolution of the vector δx_ε in the same interval of time is given as

$$\delta x_\varepsilon(t_\varepsilon) = \delta x_\varepsilon(0) + \int_0^{t_\varepsilon} \partial f_\varepsilon(x_\varepsilon(\tau))\, \delta x_\varepsilon(\tau) d\tau. \tag{5-8}$$

To analyze the limit for $\varepsilon \to 0^+$ of equation (5-8), note that it can be rewritten as

$$\delta x_\varepsilon(t_\varepsilon) = \delta x_\varepsilon(0) + \partial f_\varepsilon(x_\varepsilon(0^+))\, \delta x_\varepsilon(0)\, t_\varepsilon + O(t_\varepsilon^2), \tag{5-9}$$

where

$$\partial f_\varepsilon(x_\varepsilon(0^+)) = \lim_{t \to 0^+} \partial f_\varepsilon(x_\varepsilon(t)),$$

because in the linear regularization the Jacobian matrix ∂f_ε is not continuous on the boundary $\partial\Sigma_\varepsilon$.
Substituting the expression of ∂f_ε from (2-32) in (5-9) we get

$$\begin{aligned}
\delta x_\varepsilon(t_\varepsilon) = \delta x_\varepsilon(0) + \Big[&\partial f^-(x_\varepsilon(0)) + \frac{h(x_\varepsilon(0))}{\varepsilon}\partial\Delta f(x_\varepsilon(0)) \\
&+ \frac{1}{\varepsilon}\Delta f(x_\varepsilon(0))\nabla h(x_\varepsilon(0))\Big]\,\delta x_\varepsilon(0)\,t_\varepsilon + O(t_\varepsilon^2) \\
= \delta x_\varepsilon(0) &+ \partial f^-(x_\varepsilon(0))\,\delta x_\varepsilon(0)\,t_\varepsilon \\
&+ \frac{h(x_\varepsilon(0))}{\varepsilon}\partial\Delta f(x_\varepsilon(0))\,\delta x_\varepsilon(0)\,t_\varepsilon \\
&+ \frac{1}{\varepsilon}\Delta f(x_\varepsilon(0))\nabla h(x_\varepsilon(0))\,\delta x_\varepsilon(0)\,t_\varepsilon + O(t_\varepsilon^2) \\
= \delta x_\varepsilon(0) &+ \partial f^-(\bar{x})\,\delta x_\varepsilon(0)\,t_\varepsilon + \frac{h(\bar{x})}{\varepsilon}\partial\Delta f(\bar{x})\,\delta x_\varepsilon(0)\,t_\varepsilon \\
&+ \frac{1}{\varepsilon}\Delta f(\bar{x})\nabla h(\bar{x})\,\delta x_\varepsilon(0)\,t_\varepsilon + O(t_\varepsilon^2),
\end{aligned} \tag{5-10}$$

where we denoted $\partial\Delta f(x) = \partial f^+(x) - \partial f^-(x)$ and $\Delta f(x) = f^+(x) - f^-(x)$.

We can easily note that

$$\lim_{t_\varepsilon\to 0^+} \partial f^-(\bar{x})\,\delta x_\varepsilon(0)\,t_\varepsilon = 0,$$

and that, since $h(\bar{x}) = 0$, for all $\varepsilon > 0$

$$\frac{h(\bar{x})}{\varepsilon}\partial\Delta f(\bar{x})\,\delta x_\varepsilon(0)\,t_\varepsilon = 0.$$

For what concerns the last term, we need to analyze the relation between t_ε and ε. Recalling that $h(x_\varepsilon(t_\varepsilon)) = \varepsilon$, we have

$$\begin{aligned}
h(x_\varepsilon(t_\varepsilon)) &= h(\bar{x}) + \nabla h(\bar{x})\Delta x_\varepsilon(t_\varepsilon) + O\left(|\Delta x_\varepsilon(t_\varepsilon)|^2\right) \\
&= 0 + \nabla h(\bar{x})\, f^-(\bar{x})\,t_\varepsilon + O(t_\varepsilon^2) = \varepsilon
\end{aligned} \tag{5-11}$$

where we have used (5-7). Therefore, in the hypothesis of transversal crossing (5-5), taking the limit of the last term in (5-10) and using (5-11) we have

$$\begin{aligned}
\lim_{t_\varepsilon\to 0^+} \frac{1}{\varepsilon}\Delta f(\bar{x})\nabla h(\bar{x})\,\delta x_\varepsilon(0)\,t_\varepsilon &= \lim_{t_\varepsilon\to 0^+} \frac{\Delta f(\bar{x})\nabla h(\bar{x})}{\nabla h(\bar{x})\, f^-(\bar{x})t_\varepsilon + O(t_\varepsilon^2)}\,\delta x_\varepsilon(0)\,t_\varepsilon \\
&= \frac{\Delta f(\bar{x})\nabla h(\bar{x})}{\nabla h(\bar{x})\, f^-(\bar{x})}\,\delta x_0(0),
\end{aligned}$$

where we denoted
$$\lim_{t_\varepsilon \to 0^+} \delta x_\varepsilon(0) = \delta x_0(0).$$
Therefore, taking the limit of (5-10) we get
$$\begin{aligned}\delta x_0(0^+) &= \delta x_0(0) + \frac{\Delta f(\bar{x})\nabla h(\bar{x})}{\nabla h(\bar{x})\, f^-(\bar{x})}\,\delta x_0(0)\\ &= \left[I + \frac{\Delta f(\bar{x})\nabla h(\bar{x})}{\nabla h(\bar{x})\, f^-(\bar{x})}\right]\,\delta x_0(0)\end{aligned}$$
where we denoted
$$\lim_{t_\varepsilon \to 0^+} \delta x_\varepsilon(t_\varepsilon) = \delta x_0(0^+).$$
Since $\delta x_0(0)$ and $\delta x_0(0^+)$ are respectively the vector δx before and after the crossing of discontinuous system (2-2), we can conclude that
$$\delta x^+ = S(\bar{x})\,\delta x$$
where $S(\bar{x})$ is the saltation matrix.

Furthermore, taking the limit of (5-6) and (5-7) we obtain $x_0(0^+) = x_0(0)$, as expected since Caratheodory's solutions are absolute continuous functions. Therefore, in conclusion, the limit for $\varepsilon \to 0^+$ of the prolonged system (5-4) for $x_\varepsilon \in \Sigma_\varepsilon$ is the discrete-time system
$$\begin{cases} x^+ = \bar{x} \\ \delta x^+ = S(\bar{x})\,\delta x \end{cases}$$
that coincides with (5-3).

Finally, note that for all x_ε such that $h(x_\varepsilon) > \varepsilon$ (i.e. $x_\varepsilon \in \mathcal{S}^+ \setminus \Sigma_\varepsilon$) prolonged system (5-4) coincides with (5-2), therefore since $\mathcal{S}^+ \setminus \Sigma_\varepsilon \to \mathcal{S}^+$ as $\varepsilon \to 0^+$ the proof in concluded. □

Incremental stability of switched systems (2-1) can therefore be studied taking into account the *hybrid* nature of the prolonged system, combining both *continuous* and *discrete* dynamics. Denoting by Σ the union of all switching surfaces Σ_{ij}, that is $\Sigma := \bigcup_{i,j} \Sigma_{ij}$, we have the following result.

Theorem 5.1. *Under Assumption 5.1, a switched system* (2-1) *is contracting in a connected and forward invariant set $\mathcal{C}$ if there exists a C^1 function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that*

1. *there exist positive constants* $0 < \alpha_1 < \alpha_2$, *a positive integer* p *and a Finsler metric* $|\cdot|_x$ *such that*

$$\alpha_1 |\delta x|_x^p \leq V(x, \delta x) \leq \alpha_2 |\delta x|_x^p, \tag{5-12}$$

for all $(x, \delta x) \in \mathcal{C} \times \mathbb{R}^n$;

2. *there exist constants* $\lambda_c > 0$ *and* $0 < \lambda_d < 1$ *such that for all* $t \geq t_0$

$$\dot{V}(x, \delta x) \leq -\lambda_c V(x, \delta x), \qquad \forall x \in \mathcal{C} \setminus \Sigma,\ \forall \delta x \in \mathbb{R}^n, \tag{5-13}$$
$$V(x^+, \delta x^+) \leq \lambda_d V(x, \delta x), \qquad \forall x \in \mathcal{C} \cap \Sigma,\ \forall \delta x \in \mathbb{R}^n. \tag{5-14}$$

The complete proof of this theorem will be presented elsewhere. For the sake of completeness, a sketch proof is given below.

Sketch Proof. By Assumption 5.1 every solution to system (2-1) in a neighborhood of a crossing event can be described by a bimodal switched system (2-2) whose prolonged system is derived in Proposition 5.1. Therefore, the proof directly follows by combining Theorem 3.8 from Section 3.6 and Theorem 3.18 from [58, p.52]. □

Note that, as in Theorem 3.8, the previous theorem can be extended to piecewise continuously differentiable and locally Lipschitz candidate Finsler-Lyapunov functions V [50, Remark 2]. Moreover, the switched system is still contracting even if one of two conditions (5-13)-(5-14) is not satisfied, as long as V is *non-increasing* along any solution (see Propositions 3.24 and 3.25 in [58, p.60]). This will be illustrated in the next section.

5.2 Examples

In the following we present two illustrative examples of the application of the differential Lyapunov stability analysis to planar switched linear systems. Specifically, in the first example a Finsler-Lyapunov function is shown to be decreasing along any solution, while in the second example it is only non-increasing so further analysis is necessary. Note that Theorem 4.2 presented in Chapter 4 cannot be applied to these systems because both flows are not contracting in any norm.

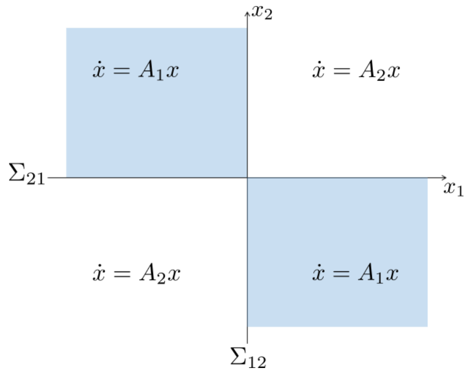

Figure 5-2: Regions of the state space of system (5-15) in Example 1.

5.2.1 Example 1: PWL system with decreasing V at switching instants

We consider a switched linear system such that each subsystem is a harmonic oscillator. Although no Finsler Lyapunov function can be found that verifies (5-13) during the flow, it will be shown that condition (5-14) at switching events is enough to guarantee that the switched system is contracting. Indeed, the saltation matrices S_{ij} are such that the system is contracting in the direction normal to the switching manifolds.

Consider a piecewise linear (PWL) system

$$\dot{x} = \begin{cases} A_1\, x, & x_1 x_2 < 0 \\ A_2\, x, & x_1 x_2 > 0 \end{cases} \tag{5-15}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1/2 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1/2 \\ -1 & 0 \end{bmatrix}$$

In Figure 5-2 it is reported the illustration of the regions where A_1 and A_2 are active.

This system has been previously used as an example in [72, p.68]. It has been shown therein that the origin of this system is globally asymptotically stable, indeed the function $V(x) = x^T x$ is decreasing for every trajectories of the system apart the points on the axes, which are not invariant sets for the system trajectories. Note that the function $V(x, \delta x) = \delta x^T \delta x$ cannot be used to assess differential stability of this system because it is not a *common* Lyapunov function for A_1 and A_2.

The saltation matrices are constant along the switching surfaces and defined as

$$S_{12} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

when the trajectories switch from A_1 to A_2, and

$$S_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

when they switch from A_2 to A_1.

The prolonged system during flow is

$$\begin{cases} \dot{x} = A_i x \\ \dot{\delta x} = A_i \delta x \end{cases} \qquad i = 1, 2 \tag{5-16}$$

while at switching instants is

$$\begin{cases} x_+ = x \\ \delta x_+ = S_{ij} \delta x \end{cases} \qquad i \neq j \tag{5-17}$$

where S_{ij} is the saltation matrix from A_i to A_j.

To study the incremental stability of the system, consider as candidate Finsler-Lyapunov function $V(x, \delta x) = \delta x^T P_i \delta x$, where

$$P_i = \begin{cases} P_1, & x_1 x_2 < 0 \\ P_2, & x_1 x_2 > 0 \end{cases}$$

with

$$P_1 = \rho_1 \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}, \ \rho_1 > 0; \qquad P_2 = \rho_2 \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \ \rho_2 > 0.$$

During flow $V(\cdot)$ is constant, indeed

$$\dot{V}(x, \delta x) = \delta x^T (A_i^T P_i + P_i A_i) \delta x = 0, \quad \forall \delta x, \ i = 1, 2,$$

therefore the PWL system is not contracting during flow.

On the other hand, during a switch from A_i to A_j the variation of ΔV of the function V is

$$\begin{aligned} \Delta V_{ij} &= V_j^+ - V_i \\ &= \delta x_+^T P_j \delta x_+ - \delta x^T P_i \delta x \\ &= \delta x^T (S_{ij}^T P_j S_{ij} - P_i) \delta x, \end{aligned} \tag{5-18}$$

that is negative for every δx if $S_{ij}^T P_j S_{ij} - P_i < 0$.

Specifically, in the example of interest, when a solution $x(t)$ switches from A_1 to A_2, we have

$$S_{12}^T P_2 S_{12} - P_1 = \begin{bmatrix} \frac{1}{2}\left(\frac{\rho_2}{2} - \rho_1\right) & 0 \\ 0 & \frac{\rho_2}{2} - \rho_1 \end{bmatrix}$$

and such matrix is definite negative if and only if $\rho_2 < 2\rho_1$. Likewise, when a solution $x(t)$ switches from A_2 to A_1 we have

$$S_{21}^T P_1 S_{21} - P_2 = \begin{bmatrix} \frac{\rho_1}{2} - \rho_2 & 0 \\ 0 & \frac{1}{2}\left(\frac{\rho_1}{2} - \rho_2\right) \end{bmatrix}$$

and such matrix is definite negative if and only if $\frac{\rho_1}{2} < \rho_2$. Therefore, the variation of V is negative for any switch if and only if

$$\frac{\rho_1}{2} < \rho_2 < 2\rho_1.$$

For example, for $\rho_1 = \rho_2 = 4$ we have

$$\begin{aligned} \Delta V_{12} &= -\delta x_1^2 - 2\delta x_2^2 < 0 \\ \Delta V_{21} &= -2\delta x_1^2 - \delta x_2^2 < 0 \end{aligned}$$

Therefore, even though the Finsler-Lyapunov function $V(x, \delta x) = \delta x^T P_i \delta x$ does not decrease during flow, it does during the switches. And thus this implies that system (5-15) is contracting. In Figure 5-3 the evolution of a ball of initial conditions is schematically shown along a solution of system (5-15). It can be clearly seen that the ball contracts at every switch while it is only deformed during flow.

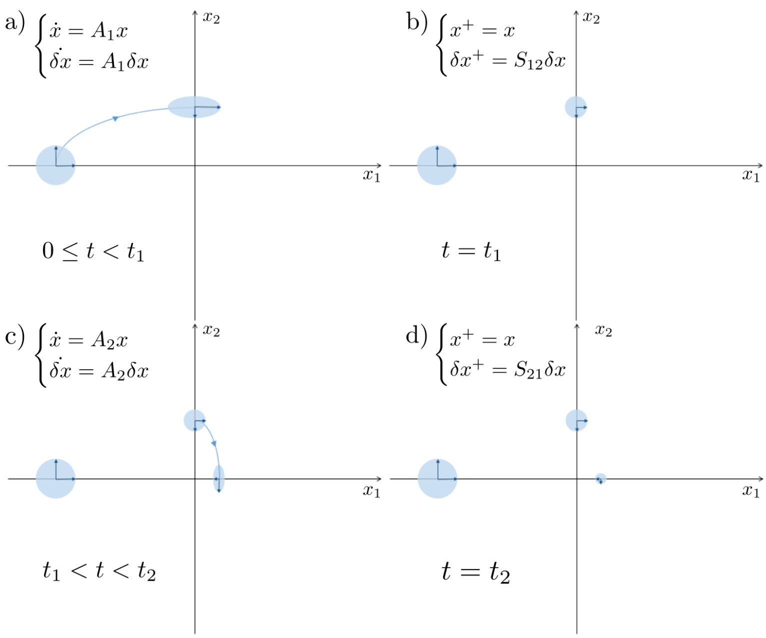

Figure 5-3: A ball with initial radius δx_0 contracts along system trajectories. Panel a: the ball is deformed and rotated but not shrunk by the continuous flow $\dot{\delta x} = A_1 \delta x$; panel b: the ball contracts in the normal direction to the switching surface Σ_{12}, according to the map S_{12}; panel c: the ball is deformed and rotated but not shrunk by the continuous flow $\dot{\delta x} = A_2 \delta x$; panel d: the ball contracts in the normal direction to the switching surface Σ_{21}, according to the map S_{21}.

5.2.2 Example 2: PWL system with nonincreasing V at switching instants

In this section we present an example that is a slight modification of the previous one. Specifically, in this case the vector field is continuous on one of the two switching surfaces, therefore the corresponding saltation matrix S_{12} is the identity matrix. Hence, the sole contribution to the decrease of a certain Finsler-Lyapunov function V comes from the saltation matrix S_{21}.

Moreover, since the Finsler-Lyapunov function V chosen in the analysis

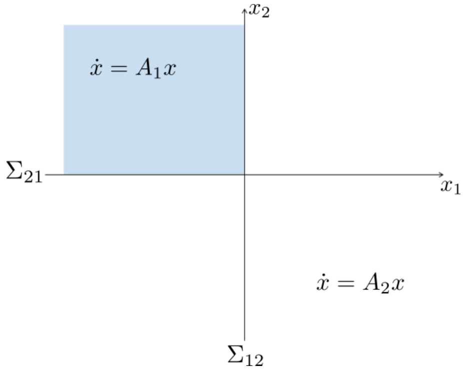

Figure 5-4: Regions of the state space of system (5-19) in Example 2.

can be proved to be non-increasing only (ΔV does not depend on δx_1), by further investigation on the evolution of the system trajectories it is proved that $\Delta V < 0$ for every trajectory such that $\delta x_0 \neq 0$. Therefore, the system is incrementally asymptotically stable.

Consider a PWL system defined as

$$\dot{x} = \begin{cases} A_1\, x, & x_1 < 0,\ x_2 > 0 \\ A_2\, x, & \text{otherwise} \end{cases} \tag{5-19}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1/2 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This system can possibly describe an *undamped* mass-spring system that can be incrementally stabilized by switching between two different values of the spring stiffness. In Figure 5-4 it is reported the illustration of the regions where A_1 and A_2 are active.

Note that since for some vector $e \in \mathbb{R}^n$

$$\Delta A = A_2 - A_1 = \begin{bmatrix} 0 & 0 \\ -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = e\, c_{12}^T$$

system (5-19) is continuous on the switching surface Σ_{12}, where $x_1 = 0$, therefore the saltation matrix from A_1 to A_2 is

$$S_{12} = I.$$

On the other hand, the saltation matrix from A_2 to A_1 on Σ_{21} is

$$S_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Moreover, the prolonged system has the form (5-16)-(5-17) as in the previous example.

To analyze the incremental stability of this system we consider as candidate Finsler-Lyapunov function $V(x, \delta x) = \delta x^T P_i \delta x$, where

$$P_i = \begin{cases} P_1, & x_1 < 0,\, x_2 > 0 \\ P_2, & \text{otherwise} \end{cases}$$

with

$$P_1 = \rho_1 \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}, \ \rho_1 > 0; \qquad P_2 = \rho_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \rho_2 > 0.$$

During flow $V(x, \delta x)$ is constant, indeed

$$\dot{V}(x, \delta x) = \delta x^T (A_i^T P_i + P_i A_i) \delta x = 0, \quad \forall \delta x,\ i = 1, 2,$$

therefore the PWL system in no contracting during flow.

On the other hand, when a solution $x(t)$ switches from A_1 to A_2, from (5-18), the value of the variation ΔV_{12} is

$$\Delta V_{12} = \delta x^T (S_{12}^T P_2 S_{12} - P_1) \delta x,$$

where

$$S_{12}^T P_2 S_{12} - P_1 = \begin{bmatrix} \rho_2 - \frac{\rho_1}{2} & 0 \\ 0 & \rho_2 - \rho_1 \end{bmatrix}$$

therefore, ΔV_{12} is non-positive if and only if $\rho_2 \leq \frac{\rho_1}{2}$ (since the above matrix can at most be semi-definite).
Likewise, when a solution $x(t)$ switches from A_2 to A_1 we have

$$S_{21}^T P_1 S_{21} - P_2 = \begin{bmatrix} \frac{\rho_1}{2} - \rho_2 & 0 \\ 0 & \frac{\rho_1}{4} - \rho_2 \end{bmatrix}$$

and such matrix is semi-definite negative if and only if $\rho_2 \geq \frac{\rho_1}{2}$. Therefore, the variation of V is non-positive for any switch if and only if

$$\rho_2 = \frac{\rho_1}{2},$$

otherwise its sign remains undefined. For example, for $\rho_1 = 4$, $\rho_2 = 2$ we have

$$\Delta V_{12} = -2\delta x_2^2 \leq 0$$
$$\Delta V_{21} = -\delta x_2^2 \leq 0$$

Note that the variation of V does not depend on δx_1, therefore it may be possible for trajectories to exist such that V never decreases. In the following it will be shown that this can occur only for solutions such that $\delta x_0 = 0$. Therefore, we can conclude that the PWL system is contacting.

Consider a solution $x(t)$ to system (5-19) with initial condition $x(0) = x_0$ on Σ_{21}. For $t \in [0, t_1)$ its evolution is described by $x(t) = e^{A_1 t} x_0$, where

$$e^{A_1 t} = \begin{bmatrix} \cos\left(\frac{t}{\sqrt{2}}\right) & \sqrt{2}\sin\left(\frac{t}{\sqrt{2}}\right) \\ -\frac{1}{\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right) & \cos\left(\frac{t}{\sqrt{2}}\right) \end{bmatrix}$$

and t_1 is the time instant when the solution crosses the switching surface Σ_{12}, while for $t \in [t_1, t_2)$ it is described by $x(t) = e^{A_2(t-t_1)} x(t_1)$, where

$$e^{A_2 t} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

and t_2 is the time instant when the solution crosses Σ_{21}. Specifically, the two switching time instants are $t_1 = \sqrt{2}\frac{\pi}{2}$ and $t_2 = t_1 + \frac{3}{2}\pi$. In this way it is possible to describe the evolution in time of the vector $\delta x(t)$ along the solution $x(t)$ with initial condition $\delta x(0) = \delta x_0$.

For $t \in [0, t_1)$ we have that $\delta x(t) = e^{A_1 t}\delta x_0$, therefore its value just before the first switch is

$$\delta x(t_1) = e^{A_1 t_1}\delta x_0 = \begin{bmatrix} 0 & \sqrt{2} \\ -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \delta x_{0,1} \\ \delta x_{0,2} \end{bmatrix} = \begin{bmatrix} \sqrt{2}\,\delta x_{0,2} \\ -\frac{1}{\sqrt{2}}\delta x_{0,1} \end{bmatrix}$$

Likewise, for $t \in [t_1, t_2)$ we have that

$$\delta x(t) = e^{A_2(t-t_1)}\delta x(t_1^+) = e^{A_2(t-t_1)} S_{12}\delta x(t_1) = e^{A_2(t-t_1)}\delta x(t_1),$$

since $S_{12} = I$, therefore its value just before the second switch is

$$\delta x(t_2) = e^{A_2(t_2-t_1)}\delta x(t_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}\,\delta x_{0,2} \\ -\frac{1}{\sqrt{2}}\delta x_{0,1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}\delta x_{0,1} \\ \sqrt{2}\,\delta x_{0,2} \end{bmatrix}$$

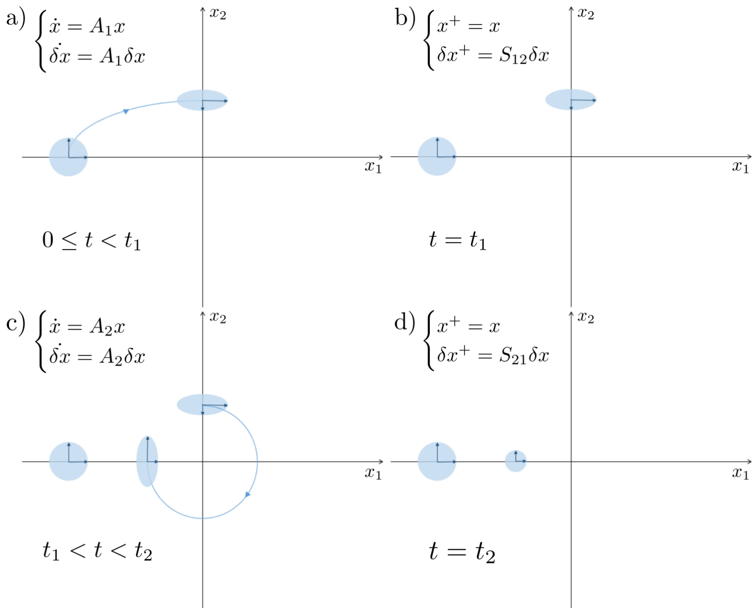

Figure 5-5: A ball with initial radius δx_0 contracts along system trajectories. Panel a: the ball is deformed and rotated but not shrunk; panel b: since on the switching surface Σ_{12} the system is continuous the ball is left unchanged; panel c: the ball is only rotated by the continuous flow $\dot{\delta x} = A_2 \delta x$; panel d: the ball contracts in the normal direction to the switching surface Σ_{21}, according to the map S_{21}.

Finally, after the second switch we have

$$\begin{aligned}\delta x(t_2^+) =& S_{21}\delta x(t_2) = S_{21} e^{A_2(t_2-t_1)} S_{12} e^{A_1 t_1} \delta x_0\\ =& \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \delta x_0 = \frac{1}{\sqrt{2}} \delta x_0.\end{aligned}$$

Hence, after a whole period of $T = t_2 = \frac{3+\sqrt{2}}{2}\pi \approx 6.93\,s$ a ball of initial radius δx_0 has contracted by a factor of $1/\sqrt{2} < 1$ in every directions, as shown graphically in Figure 5-5.

Moreover, the value of ΔV_{12} at t_1 for this solution is

$$\Delta V_{12}(t_1) = -2\delta x_2^2(t_1) = -2\left(-\frac{1}{\sqrt{2}}\delta x_{0,1}\right)^2 = -\delta x_{0,1}^2,$$

while the value of ΔV_{21} at t_2 is

$$\Delta V_{21}(t_2) = -\delta x_2^2(t_2) = -\left(-\sqrt{2}\delta x_{0,2}\right)^2 = -2\delta x_{0,2}^2.$$

Since the only possible initial condition such that variations ΔV_{ij} are both zero is $\delta x_0 = [0 \quad 0]^T$ and the previous analysis holds for any non-zero solution to (5-19), we can conclude that the Finsler-Lyapunov function $V(x, \delta x)$ is decreasing along every trajectories such that $\delta x_0 \neq 0$, hence system (5-19) is contracting.

5.3 Comparison with conditions based on matrix measures

The saltation matrix from f^+ to f^- at the point $x \in \Sigma$ is defined as

$$S(x) = I + \frac{[f^+(x) - f^-(x)]\nabla h(x)}{\nabla h(x) f^-(x)} \tag{5-20}$$

where f^+ and f^- denote the vector field after and before the crossing, respectively. Notice that, under the hypothesis of transversal intersection of solutions on Σ, the denominator $\nabla h(x) f^-(x)$ is nonzero. Furthermore, notice that the matrix $S(x)$ is a function of the point $x \in \Sigma$ where the solution $x(t)$ intersects the switching surface Σ. From Lemma 2.1 we have that the eigenvalues of the saltation matrix S are

$$\left\{1 + \frac{\nabla h(x)[f^+(x) - f^-(x)]}{\nabla h(x) f^-(x)},\ 1, \ldots, 1\right\}. \tag{5-21}$$

Furthermore, the $n-1$ eigenvectors associated to unitary eigenvalues are all tangent to Σ (i.e. orthogonal to ∇h), while the other one is in the direction of $\Delta f = f^+ - f^-$. The following result holds.

Proposition 5.2. *The differential map $\delta x^+ = S(x)\delta x$ is* nonexpanding *(i.e. every eigenvalue is in the unitary circle) if and only if $f_N^+ \leq f_N^-$, that is the normal component of the vector field after the crossing is lesser or equal than the one before it.*

Proof. The map S is non-expanding if all its eigenvalues are in modulus lesser or equal than 1, that is, from (5-21), if

$$\left|1 + \frac{\nabla h(x)[f^+(x) - f^-(x)]}{\nabla h(x) f^-(x)}\right| \leq 1.$$

In the case of transversal crossing from f^- to f^+ we have that

$$\begin{cases} f_N^- = \nabla h\, f^- > 0 \\ f_N^+ = \nabla h\, f^+ > 0 \end{cases},$$

where we omitted the dependency on x for the sake of brevity, therefore

$$\left|1 + \frac{\nabla h(f^+ - f^-)}{\nabla h f^-}\right| = \left|1 + \frac{f_N^+ - f_N^-}{f_N^-}\right| = \left|1 + \frac{f_N^+}{f_N^-} - 1\right| = \frac{|f_N^+|}{|f_N^-|} = \frac{f_N^+}{f_N^-},$$

that substituted in the relation above gives $f_N^+ \leq f_N^-$.

The opposite case, that is crossing from f^+ to f^-, gives the opposite result but with the same meaning, since in this other case we have that

$$\begin{cases} f_N^- = \nabla h\, f^- < 0 \\ f_N^+ = \nabla h\, f^+ < 0 \end{cases}$$

Finally, when a trajectory moves from f^- (or f^+) to stable sliding mode (remember that sliding mode is stable if $f_N^+ \leq f_N^-$, see Section 2.1) with vector field f^s in (2-11) the saltation matrix is singular since the first eigenvalue is 0, and therefore it is still nonexpanding. □

In Theorem 4.2 it was proved that a bimodal Filippov system (2-2) is contracting if there exists a norm such that the measure μ of the Jacobian matrices of the two modes are definite negative and for any point on Σ it holds that $\mu(\Delta f(x) \nabla h(x)) = 0$. The relation between the latter condition and the Lyapunov analysis of the prolonged system presented before is stated next.

Proposition 5.3. *The differential map* $\delta x^+ = S(x)\delta x$ *is* nonexpanding *if there exists some norm with associated matrix measure* μ *such that for all* $x \in \Sigma$

$$\mu(\Delta f(x) \nabla h(x)) = 0. \tag{5-22}$$

Proof. Since the matrix measure of a matrix A gives an upper bound of the real part of the eigenvalues of A, that is

$$\mathrm{Re}(\lambda_i(A)) \leq \mu(A), \qquad \forall i,$$

and since the matrix $\Delta f \, \nabla h$ is a rank-1 matrix with eigenvalues

$$\{\nabla h \, \Delta f, \, 0, \ldots, 0\},$$

condition (5-22) implies that $\nabla h \, \Delta f \leq 0$. In the case of crossing from f^- to f^+, this means that

$$\nabla h \, (f^+ - f^-) = \nabla \, h f^+ - \nabla h \, f^- = f_N^+ - f_N^- \leq 0 \quad \Longrightarrow \quad f_N^+ \leq f_N^-,$$

and therefore the saltation matrix is non-expanding. The same holds for crossing from f^+ to f^-. $\square$

In the following we revisit the example of the relay feedback system presented in Section 4.1.1 where the system was studied using the matrix measure induced by the 1-norm, $|x|_1 = |x_1| + \cdots + |x_n|$. It was shown that the two continuous flows are contracting in this norm (i.e. $\mu_1(A) < 0$) and that the condition on Σ is satisfied (that is, $\mu_1(\Delta f \nabla h) = \mu_1(-2bc^T) = 0$). We will now repeat the analysis using the Finsler-Lyapunov function $V(x, \delta x) = |\delta x|_1$.

Example Consider the relay feedback system defined as

$$\dot{x} = \begin{cases} f^+(x) := Ax - b, & c^T x > 0, \\ f^-(x) := Ax + b, & c^T x < 0 \end{cases}$$

with

$$A = \begin{bmatrix} -2 & -1 \\ 1 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This system has a switching manifold Σ defined by $x_2 = 0$ that presents three different regions:

1. for $x_1 < -3$ there is a crossing region from f^+ to f^-;
2. for $-3 < x_1 < 3$ there is a sliding region;
3. for $x_1 > 3$ there is a crossing region from f^- to f^+.

Here we will study only the last case, specifically the saltation matrix for this region is

$$S^{\mp}(x) = I + \frac{(f^+ - f^-)\nabla h}{\nabla h f^-} = I + \frac{-2bc^T}{c^T(Ax+b)} = \begin{bmatrix} 1 & -\frac{2}{x_1+3} \\ 0 & 1-\frac{6}{x_1+3} \end{bmatrix}$$

Its non-trivial eigenvalue is

$$\lambda = \frac{x_1 - 3}{x_1 + 3}$$

and is always less than 1 for $x_1 > 3$ (it is equal to 1 in the limit for $x_1 \to +\infty$). Therefore the saltation matrix is always non-expanding.

For what concerns the continuous flows, the prolonged system is

$$\begin{cases} \dot{x} = A\,x \pm b \\ \dot{\delta x} = A\,\delta x \end{cases}$$

therefore, assuming $\delta x_i \neq 0, \forall i$ for the sake of simplicity, we have

$$\begin{aligned}
\dot{V}(x,\delta x) &= \partial|\delta x|_1^T\,\dot{\delta x} = \begin{bmatrix} \partial|\delta x_1| & \partial|\delta x_2| \end{bmatrix} A\delta x \\
&= \begin{bmatrix} \frac{\delta x_1}{|\delta x_1|} & \frac{\delta x_2}{|\delta x_2|} \end{bmatrix} \begin{bmatrix} -2\delta x_1 - \delta x_2 \\ \delta x_1 - 3\delta x_2 \end{bmatrix} \\
&= \frac{\delta x_1}{|\delta x_1|}(-2\delta x_1 - \delta x_2) + \frac{\delta x_2}{|\delta x_2|}(\delta x_1 - 3\delta x_2) \\
&= -2|\delta x_1| - 3|\delta x_2| + (-|\delta x_2| + |\delta x_1|)\mathrm{sgn}(\delta x_1)\mathrm{sgn}(\delta x_2) \\
&\leq -|\delta x_1| - 2|\delta x_2| \\
&\leq -|\delta x_1| - |\delta x_2| = \; -1 \cdot V(x,\delta x)
\end{aligned}$$

Thus, the continuous flow contracts with convergence rate $c = -1$.

On the discontinuity manifold the prolonged system is

$$\begin{cases} x^+ = x, \\ \delta x^+ = S^{\mp}(x)\,\delta x \end{cases}$$

where

$$S^{\mp}(x) = \begin{bmatrix} 1 & -\frac{2}{x_1+3} \\ 0 & 1-\frac{6}{x_1+3} \end{bmatrix}$$

and we have

$$
\begin{aligned}
V(x^+, \delta x^+) &= |\delta x_1^+| + |\delta x_2^+| \\
&= \left| \delta x_1 - \frac{2}{x_1+3}\delta x_2 \right| + \left| \frac{x_1-3}{x_1+3}\delta x_2 \right| \\
&\leq |\delta x_1| + \left| \frac{2}{x_1+3} \right| |\delta x_2| + \left| \frac{x_1-3}{x_1+3} \right| |\delta x_2| \\
&= |\delta x_1| + \frac{2+|x_1-3|}{|x_1+3|}|\delta x_2| \\
&< |\delta x_1| + |\delta x_2| = V(x, \delta x)
\end{aligned}
$$

The Finsler-Lyapunov function is decreasing at every switching instant for $x_1 > 3$, and therefore the system is contracting. We can conclude that the two methods, the one with matrix measure μ_1 and the one with differential Lyapunov function $V(x, \delta x) = |\delta x|_1$, give the same result for this system.

CHAPTER 6

Other work

In this chapter we present some other work that was carried out during the PhD related to the study of convergence in networks of dynamical agents. In particular, we study the bipartite consensus problem [5] and its generalization to multipartite consensus in a network of nonlinear agents. In bipartite consensus the network nodes, represented by simple integrators, are divided in two *antagonistic* groups that converge each one on a different solution. It will be shown that this idea can be further extended to nonlinear multidimensional agents and generalized to more than two groups of agents by exploiting some symmetries of the nodes' vector fields.

6.1 Introduction

Network control is of utmost importance in many application areas from computer science to power engineering, the Internet of Things and systems biology [74]. Over the past few years there has been considerable interest in the problem of steering the dynamics of network agents towards some coordinated collective behavior, see e.g. [20] and references therein. Synchronization and consensus are two examples where all the agents of the network *cooperate* in order for a common asymptotic behavior to emerge [94], [134], [41].

Often, in applications, interactions between neighboring network nodes are not all collaborative as there might be certain nodes that have *antagonistic* relationships with neighbors. This is the case, for example, of social networks, where network agents might have different opinions [54], or biochemical and gene regulatory networks, where interactions between nodes are either activations or inhibitions [135]. Similar antagonistic interactions also arise in technological systems, as for example in the so-called Internet of Things where a number of objects is required to collect field information and maximize different (often conflicting) utility functions. A remarkable example of this instance is given in [131], where a distributed system for charging electric vehicles is developed with the goal of balancing multiple utility functions.

A convenient way for modeling the presence of collaborative and antagonistic relationships among nodes in a network is to use *signed graphs*, [63]. Motivated by the many applications, an increasing number of papers in the literature is focusing on the study of the collective dynamics emerging in this type of networks. For example, in [153], partial synchronization of Rössler oscillators over a ring is studied via the Master Stability Function (MSF), while in [103] the same phenomenon is studied within the broader framework of symmetries intrinsic to the network structure (see also [123] for a discussion between the interplay between symmetries and synchronization). Symmetries of the network topology are also been exploited in [99], where the MSF is used to study local stability of synchronized clusters of nodes. The study of consensus dynamics over signed graphs has also attracted an increasing amount of interest. A particularly interesting problem is the one considered in [5], where sufficient conditions are given for a signed network to achieve a form of *"agreed upon" dissensus*. The model proposed in [5] has been also used in a number of applications, like flocking [43] and extended to the case of Linear Time Invariant systems, discrete-time integrator dynamics and time-varying topologies, see e.g. [151], [152], [141], [73], [105], [125]. More recently, bipartite synchronization in a network of scalar nonlinear systems whose vector fields are odd functions has been studied in [150].

In this chapter, we follow a different approach to ensure that a given network exhibits a synchronization/consensus pattern. We start by presenting a sufficient condition ensuring that a network of nonlinear nodes exhibits a synchronization pattern consisting of two separate clusters of synchronized nodes. Specifically, we will show that symmetries of the vector fields of the network nodes, rather than those of the network topology, can be exploited to induce the formation of such synchronized groups. Then, we focus on consensus problems in networks of linear systems. For this case, we introduce a sufficient condition ensuring the emergence of a bipartite consensus pattern. With respect to this, we remark here that the symmetries and synchronization patterns we consider represent a wider class than those studied in [5, 152, 150], where only the odd symmetry and anti-phase synchronization pattern were studied.

We also show how the conditions we derive can be turned into design guidelines to synthesize communication protocols between the nodes able to *drive* the network evolution towards the desired synchronization/consensus pattern. Specifically, we present a systematic methodology that, given some bipartite synchronization/consensus pattern, allows to design the network so that this desired pattern is effectively attained.

Finally, another contribution of this chapter is that it introduces sufficient conditions for a network to exhibit a multipartite synchronization/consensus pattern. This has been done by generalizing our approach to the case of ODEs having multiple symmetries.

We demonstrate the effectiveness of our results by using some representative examples. Namely, we first consider the problem of analyzing the onset of anti-synchronization in a network of FitzHugh-Nagumo oscillators, then we show how our results can be used to design a desired bipartite consensus pattern in a network of generic nonlinear dynamical systems. Finally, we use our results to prove the onset of a multipartite synchronization pattern in a network of coupled oscillators.

6.2 Mathematical preliminaries

Throughout this chapter, we will denote by I_n the $n \times n$ identity matrix and by O_n the $n \times n$ matrix with all zero elements. The orthogonal symmetry group will be denoted by $\mathcal{O}(n)$ (see [59] for more details on symmetry groups and their definitions).

Networks of interest

We will consider a connected undirected network of $N > 1$ smooth n-dimensional dynamical systems described by the following equation:

$$\dot{x}_i = f(t, x_i) + k \sum_{j=1}^{N} a_{ij} \left(g_{ij}(x_j) - x_i \right), \tag{6-1}$$

where $x_i \in \mathbb{R}^n$, $i = 1, \ldots, N$, is the state vector of node i, $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ describes the intrinsic dynamics all nodes share, $k > 0$ is the coupling strength, $a_{ij} \in \{0, 1\}$ are the elements of the adjacency matrix, the functions $g_{ij}(\cdot)$ are the coupling functions that will be designed in this chapter to obtain a specific synchronization pattern (as defined in Section 6.3).

Note that, if in (6-1) we set $g_{ij}(x) = x$, $\forall i, j = 1, \ldots, N$, then we obtain the well-known dynamics for a set of diffusively coupled nodes. Such a dynamics can be written in compact form as:

$$\dot{X} = F(t, X) - k\,(L \otimes I_n)\,X, \tag{6-2}$$

with $X = \left[x_1^T, \ldots, x_N^T\right]^T \in \mathbb{R}^{nN}$, $F(t, X) = \left[f(t, x_1)^T, \ldots, f(t, x_N)^T\right]^T$, and L being the $N \times N$ Laplacian matrix, [56]. In the rest of the chapter we will refer to networks of the form (6-2) as *auxiliary networks* associated to (6-1). Specifically, we will provide conditions for the onset of synchronization patterns for network (6-1) which are based on the onset of synchronization for network (6-2), defined as follows.

Definition 6.1. *Let $\dot{s} = f(t, s)$. We will say that (6-2) achieves synchronization if* $\lim_{t \to +\infty} |x_i(t) - s(t)| = 0$, $\forall i = 1, \ldots, N$.

Note that in the case where nodes' dynamics are integrators, then the definition of synchronization above simply becomes a definition for consensus.

Symmetries of ODEs

The symmetries of a system of ODEs are described in terms of a group of transformations of the variables that preserves the structure of the equation and its solutions (see [59, 40] for a detailed discussion and proofs of the material reported in this Section). Here, we will consider symmetries of ODEs specified in terms of compact Lie groups acting on $\mathbb{R}^n$ (see Section 6.3). These groups can be identified as a subgroup of $\mathcal{O}(n)$, such that $\forall\gamma \in \mathcal{O}(n)$, it holds $\gamma^{-1} = \gamma^T$.

Let us consider a dynamical system of the form

$$\dot{x} = f(t, x), \quad x \in \mathbb{R}^n. \tag{6-3}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field. We will use the following standard definitions [59].

Definition 6.2. *The group element $\gamma \in \mathcal{O}(n)$ is a* symmetry *of* (6-3) *if for every solution $x(t)$ of* (6-3)*, $\gamma x(t)$ is also a solution.*

Definition 6.3. *Let Γ be a compact Lie group acting on $\mathbb{R}^n$. Then, f is* Γ-equivariant *if $f(t, \gamma x) = \gamma f(t, x)$ for all $\gamma \in \Gamma$, $x \in \mathbb{R}^n$.*

Essentially, Γ-equivariance means that γ *commutes* with f and it implies that γ is a symmetry of (6-3). In fact, let $y(t) = \gamma x(t)$, we have that $\dot{y} = \gamma\dot{x} = \gamma f(t, x) = f(t, \gamma x) = f(t, y)$.

We now introduce the following Lemma which will be used in the rest of this chapter. The proof of this result can be immediately obtained from [40] and is reported here for the sake of completeness.

Lemma 6.1. *Assume that, for system* (6-3)*, $f(t, x)$ is Γ-equivariant. Let*

$$D := \operatorname{diag}\{\sigma_1, \dots, \sigma_N\}, \tag{6-4}$$

with $\sigma_i \in \Gamma$, $i = 1, \dots, N$. Then, for all X, $DF(t, X) = F(t, DX)$.

Proof. The proof immediately follows from the application of the definition of Γ-equivariance for f. In particular, as $\sigma_i f(t, x) = f(t, \sigma_i x)$, $\forall\sigma_i \in \Gamma$, we have:

$$DFf(t, X) = \begin{bmatrix} \sigma_1 & O_n & \dots & O_n \\ O_n & \sigma_2 & \dots & O_n \\ \vdots & \vdots & \ddots & \vdots \\ O_n & O_n & \dots & \sigma_N \end{bmatrix} \begin{bmatrix} f(t, x_1) \\ \vdots \\ f(t, x_N) \end{bmatrix} =$$

$$= \begin{bmatrix} \sigma_1 f(t, x_1) \\ \vdots \\ \sigma_N f(t, x_N) \end{bmatrix} = \begin{bmatrix} f(t, \sigma_1 x_1) \\ \vdots \\ f(t, \sigma_N x_N) \end{bmatrix} = F(t, DX).$$

□

Also, note that by construction $D \in \mathcal{O}(nN)$ and therefore $D^{-1} = D^T$. Essentially, Lemma 6.1 implies that whenever a given function of interest $f(t,x)$ is Γ-equivariant, i.e. f commutes with some $\sigma_i \in \Gamma$, then the stack F commutes with the matrix D as defined above.

6.3 Bipartite synchronization

Problem Statement

Let $\mathcal{G}_N := \{1, \dots, N\}$ be the set of all network nodes and let $\mathcal{G}$ and $\mathcal{G}^*$ be two subsets (or groups) such that: $\mathcal{G} \cap \mathcal{G}^* = \{\emptyset\}$, $\mathcal{G} \cup \mathcal{G}^* = \mathcal{G}_N$, with the cardinality of $\mathcal{G}$ being equal to h and the cardinality of $\mathcal{G}^*$ being $N-h$. Clearly, the two sets above generate a partition of the network nodes. Throughout this chapter, no hypotheses will be made on the network partition, i.e. nodes can be partitioned arbitrarily, furthermore nodes belonging to the same group do not necessarily need to be directly interconnected.

Definition 6.4. *Consider network* (6-1) *and let f be γ-equivariant. We say that (6-1) achieves a γ-bipartite synchronization pattern if*

$$\lim_{t \to +\infty} |x_i(t) - s(t)| = 0, \quad \forall i \in \mathcal{G};$$
$$\lim_{t \to +\infty} |x_i(t) - s^*(t)| = 0, \quad \forall i \in \mathcal{G}^*,$$

where $s(t) = \gamma\, s^*(t)$.

Definition 6.4 implies that the collective behavior emerging from the network dynamics will encompass two groups of nodes synchronized onto two different common solutions related via the symmetry γ. Note that this is a more general definition than that presented in [5] where the scalar asymptotic solutions considered therein agree in modulus but differ in sign. In our case the two solutions s and s^* still share the same norm[1] but are related by the more generic symmetry transformation γ.

[1]This can be immediately proved by using the fundamental properties of orthogonal matrices

Main Result

The following result provides a sufficient condition for network (6-1) to achieve a γ-bipartite synchronization pattern.

Theorem 6.1. *A γ-bipartite synchronization pattern arises for (6-1) if:*

H1 *the intrinsic node dynamics f is γ-equivariant, with $\gamma \in \mathcal{O}(n)$;*

H2 *g_{ij} is defined as follows:*

$$g_{ij}(x_j) := \begin{cases} x_j, & x_i, x_j \in \mathcal{G} \ \ or \ \ x_i, x_j \in \mathcal{G}^* \\ \gamma\, x_j, & x_i \in \mathcal{G} \ \ and \ \ x_j \in \mathcal{G}^* \\ \gamma^T x_j, & x_i \in \mathcal{G}^* \ \ and \ \ x_j \in \mathcal{G} \end{cases}$$

H3 *the associated auxiliary network (6-2) synchronizes.*

Proof. Without loss of generality, let us consider the first h nodes belonging to the subset $\mathcal{G}$, that is $\mathcal{G} = \{1, \dots, h\}$, and the remaining nodes to $\mathcal{G}^*$, that is $\mathcal{G}^* = \{h+1, \dots, N\}$. Hypothesis **H2** implies that the dynamics of network(6-1) can be written as follows.

$$\dot{x}_i = f(t, x_i) - k\left[l_{ii}x_i + \sum_{j=1}^{h} l_{ij}x_j + \sum_{j=h+1}^{N} l_{ij}\gamma\, x_j\right], \quad \text{if } i \in \mathcal{G};$$
$$\dot{x}_i = f(t, x_i) - k\left[l_{ii}x_i + \sum_{j=1}^{h} l_{ij}\gamma^T x_j + \sum_{j=h+1}^{N} l_{ij}x_j\right], \quad \text{if } i \in \mathcal{G}^*,$$

where l_{ij} are the elements of the Laplacian matrix. Now, let D be the $nN \times nN$ block-diagonal matrix having on its main block-diagonal

$$\sigma_i = \begin{cases} I_n & \text{if node } i \text{ belongs to } \mathcal{G} \\ \gamma & \text{if node } i \text{ belongs to } \mathcal{G}^* \end{cases} \tag{6-5}$$

Then the above dynamics can be rewritten in compact form as (recall that $D^T = D^{-1}$):

$$\dot{X} = F(t, X) - k\, D^T(L \otimes I_n)DX, \tag{6-6}$$

Let $Z = DX$. From (6-6) we have:

$$\begin{aligned}\dot{Z} =& D\dot{X} = \\ =& DF(t,X) - k\, DD^T(L \otimes I_n)DX = \\ =& F(t,DX) - k\,(L \otimes I_n)DX = \\ =& F(t,Z) - k\,(L \otimes I_n)Z,\end{aligned}$$

where we used **H1** and Lemma 6.1. Now, note that in the new state variables, the network dynamics can be recast as

$$\dot{Z} = F(t,Z) - k(L \otimes I_n)Z, \tag{6-7}$$

that has the same form as the auxiliary network (6-2).

Now, from hypothesis **H3**, since the auxiliary network synchronizes, then so does network (6-7) which shares the same network dynamics. Therefore, there exists some $\dot{s} = f(t,s)$ such that, $\forall i = 1,\ldots,N$:

$$\lim_{t\to+\infty} |z_i(t) - s(t)| = 0, \quad \forall i.$$

Finally, $X = D^T Z$ yields

$$\lim_{t\to+\infty} x_i(t) = \begin{cases} I_n\, z_i(t) = s(t), & \text{if } i \in \mathcal{G}; \\ \gamma^T z_i(t) = \gamma^T s(t) = s^*(t), & \text{if } i \in \mathcal{G}^* \end{cases}$$

and the theorem is proved. □

Remark 6.1. *Essentially, the spirit of Theorem 6.1 is that of providing a condition for the onset of a γ-bipartite synchronization pattern for a given network of interest, based on a synchronization condition for the associated auxiliary network. This approach is motivated by the fact that proving synchronization of a diffusively coupled network is easier than proving the emergence of synchronization patterns. To this end, many results are currently available which can be used to prove network synchronization under different technical assumptions. Rather than choosing a specific approach, we leave the reader to select the most appropriate for the application of interest. Examples of the available results are those obtained via the use of Lyapunov functions, passivity theory, contraction theory, monotone systems (see e.g. [93], [130], [6], [21], [115], [28] and references therein) and, within the Physics Community, the Master Stability Function, see e.g. [98].*

Remark 6.2. *In the proof of Theorem 6.1, we use a transformation matrix D which is a generalization of the one used in [5]. In such a paper, the consensus dynamics is studied of networks of integrators with antagonistic interactions and therefore only the set of* gauge transformations *was considered. This set of transformations is defined as* $\mathcal{D} = \mathrm{diag}\{\sigma_1, \ldots, \sigma_N\}$*, but with σ_i being scalars either equal to* 1 *or* -1*. Indeed, when $n = 1$ the orthogonal group we consider degenerates into* $\{1, -1\}$.

Remark 6.3. *Theorem 6.1 is also related in the spirit to the results of [103]. In such a paper, the authors analyzed the stability of synchronization patterns arising from symmetries of the network structure. Such symmetries were described by means of permutation matrices. Here, we use instead symmetries in the nodes' dynamics to induce synchronization patterns in the network.*

Remark 6.4. *Theorem 6.1 can be extended to study γ-bipartite synchronization for discrete time networks. Specifically, consider a discrete time network of the form*

$$x_i(t+1) = f\left(t, x_i(t)\right) + k \sum_{j=1}^{N} a_{ij} \left(g_{ij}\left(x_j(t)\right) - x_i(t)\right), \qquad \text{(6-8)}$$

*where t belongs to the set of positive integers, i.e. $t \in \mathbb{N}^+$. Then, a γ-bipartite synchronization pattern arises for network (6-8) under hypotheses **H1** - **H3**, where in this case the auxiliary network* (6-2) *is replaced with*

$$X(t+1) = F(t, X(t)) - k(L \otimes I_n)X(t).$$

The proof of this result follows similar steps to those used to prove Theorem 6.1 and therefore it is omitted here for the sake of brevity.

Example - Anti-synchronization of FitzHugh-Nagumo oscillators

As an example of application of Theorem 6.1 we now address the problem of generating an anti-synchronization pattern for a network of FitzHugh-Nagumo (FN, see [46]) oscillators where two groups of oscillators emerge each synchronized onto a phase and anti-phase solution, respectively (for further details on anti-synchronization see [14]). The oscillators are described by the following set of differential equations

$$\begin{aligned} \dot{v}_i &= c\left(v_i + w_i - \tfrac{1}{3}v_i^3\right) + k\sum_{j=1}^{N} a_{ij}\left(v_j - v_i\right), \\ \dot{w}_i &= -\tfrac{1}{c}\left(v_i + bw_i\right) + k\sum_{j=1}^{N} a_{ij}\left(w_j - w_i\right), \end{aligned} \tag{6-9}$$

where v_i and w_i are the membrane potential and the recovery variable for the i-th FN oscillator ($i = 1, \ldots, N$). In terms of the formalism introduced in Definition 6.4, anti-synchronization will correspond to the case where $s(t) = -s^*(t)$ so that

$$\gamma = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Now, let $x = [v, w]^T$, then $f(t, x) = \left[c\left(v + w - \frac{1}{3}v^3\right), -\frac{1}{c}(v + bw)\right]^T$ is the intrinsic dynamics of the FN oscillator. It is straightforward to check that hypothesis **H1** of Theorem 6.1 is fulfilled for the FNs. Indeed:

$$\begin{aligned} \gamma f(x) &= \begin{bmatrix} -c\left(v + w - \frac{1}{3}v^3\right) \\ \frac{1}{c}(v + bw) \end{bmatrix} = \\ &= \begin{bmatrix} c\left(-v - w + \frac{1}{3}v^3\right) \\ -\frac{1}{c}(-v - bw) \end{bmatrix} = f(\gamma x) \end{aligned}$$

Consider now the network structure shown in Figure 6-1 (left panel) and its partition illustrated in Figure 6-1 (right panel), obtained by dividing nodes into the two groups $\mathcal{G} = \{1, 3\}$ and $\mathcal{G}^* = \{2, 4, 5\}$. The nodes' dynamics can then be written according to (6-1) as

$$\begin{aligned} \dot{x}_1 &= f\left(x_1\right) + k\left(g_{12}\left(x_2\right) + g_{13}\left(x_3\right) - 2x_1\right) \\ \dot{x}_2 &= f\left(x_2\right) + k\left(g_{21}\left(x_1\right) - x_2\right) \\ \dot{x}_3 &= f\left(x_3\right) + k\left(g_{31}\left(x_1\right) + g_{34}\left(x_4\right) + g_{35}\left(x_5\right) - 3x_3\right) \\ \dot{x}_4 &= f\left(x_4\right) + k\left(g_{43}\left(x_3\right) - x_4\right) \\ \dot{x}_5 &= f\left(x_5\right) + k\left(g_{53}\left(x_3\right) - x_5\right) \end{aligned} \tag{6-10}$$

It is well know from the literature that the auxiliary network (6-2) associated to the above dynamics synchronizes if k is sufficiently large [132]. Therefore, by choosing the coupling gain k sufficiently high, hypothesis **H3** of Theorem 6.1 will also be fulfilled.

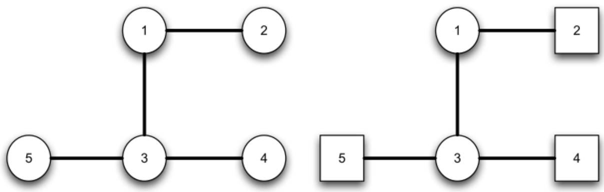

Figure 6-1: Left panel: a network of diffusively coupled identical oscillators. Right panel: an arbitrary partition of the set $\mathcal{G}_N$. Nodes belonging to the same group are denoted with the same shape. Note that the two groups does not need to be connected. Nodes 1 and 3 belong to group $\mathcal{G}$, nodes 2, 4 and 5 belong to group $\mathcal{G}^*$.

Finally, by choosing the coupling functions g_{ij} as

$$g_{ij}(x_j) := \begin{cases} x_j, & x_i, x_j \in \mathcal{G} \text{ or } x_i, x_j \in \mathcal{G}^*; \\ \gamma x_j = -x_j, & x_i \in \mathcal{G} \text{ and } x_j \in \mathcal{G}^*; \\ \gamma^T x_j = -x_j, & x_i \in \mathcal{G}^* \text{ and } x_j \in \mathcal{G}, \end{cases}$$

hypothesis **H2** of Theorem 6.1 is also fulfilled implying that anti-synchronization will be attained with nodes 1 and 3 converging onto the same trajectory, $s(t)$ while nodes 2, 4 and 5 onto $s^* = -s(t)$. Figure 6-2 clearly confirms this theoretical prediction.

6.4 Bipartite synchronization and consensus of linear systems

We now turn our attention to the consensus problem, where a network of linear agents needs to agree upon a given quantity of interest. Consensus problems have been widely investigated within the control theoretic community for its application, see e.g. the pioneering work in [108] and [94]. In this Section we consider the case where each agent is modeled by an n-dimensional LTI system and we investigate under what conditions a bipartite consensus pattern arises in the network.

Consider a set of $N > 1$ LTI agents described by

$$\dot{x}_i = Ax_i + Bu_i \tag{6-11}$$

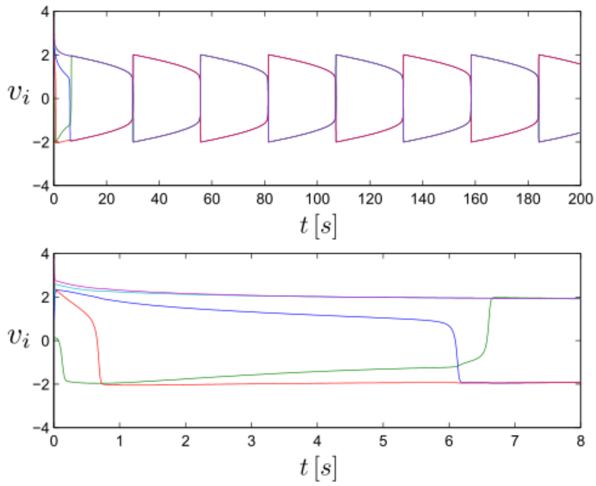

Figure 6-2: Top panel: time behavior of (6-9), with $k = 1$. Note that two groups of anti-synchronized nodes arise. Bottom panel: transient behavior of network nodes illustrating how the two groups arise. Initial conditions are randomly taken from the standard distribution.

where $i = 1, \ldots, N$, $x_i \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and assume they are networked through the interconnection protocol $u_i \in \mathbb{R}^m$ given by

$$u_i = K \sum_{j=1}^{N} a_{ij}(g_{ij}(x_j) - x_i) \tag{6-12}$$

where $K \in \mathbb{R}^{m\times n}$ is the control gain matrix, $a_{ij} \in \{0, 1\}$ and g_{ij} is the coupling function defined as before. Substituting (6-12) into (6-11), we obtain

$$\dot{x}_i = Ax_i + BK \sum_{j=1}^{N} a_{ij}(g_{ij}(x_j) - x_i) \tag{6-13}$$

for $i = 1, \ldots, N$. As noted in Section 6.3, if we select the coupling functions as $g_{ij}(x) = x$, then we obtain a diffusively coupled network that can be written in compact form as [152]

$$\dot{X} = (I_N \otimes A)X - (L \otimes BK)X \tag{6-14}$$

where L is the Laplacian matrix. Again, we will refer to network (6-14) as the *auxiliary network* associated to (6-13).

Corollary 6.1. *A γ-bipartite consensus pattern arises for* (6-13) *if:*

H1 *there exists some $\gamma \in \mathcal{O}(n)$, $\gamma^T = \gamma^{-1}$, such that $A\gamma = \gamma A$;*

H2 *g_{ij} is defined as follows:*

$$g_{ij}(x_j) := \begin{cases} x_j, & x_i, x_j \in \mathcal{G} \;\; or \;\; x_i, x_j \in \mathcal{G}^* \\ \gamma\, x_j, & x_i \in \mathcal{G} \;\; and \;\; x_j \in \mathcal{G}^* \\ \gamma^T x_j, & x_i \in \mathcal{G}^* \;\; and \;\; x_j \in \mathcal{G} \end{cases}$$

H3 *the associated auxiliary network* (6-14) *reaches consensus.*

Proof. The proof follows the same steps of that of Theorem 6.1. In particular, using hypothesis **H2**, we can rewrite (6-13) as

$$\dot{X} = (I_N \otimes A)X - D^T(L \otimes BK)DX$$

where D is defined as in (6-4) and (6-5).

Now note that when $f(x) = Ax$, f being γ-equivariant simply means that the matrices A and γ commute (**H1**). In fact,

$$f(\gamma\, x) = A\,(\gamma\, x) = \gamma\,(A\, x) = \gamma\, f(x).$$

Furthermore, it is easy to prove that

$$D(I_N \otimes A) = (I_N \otimes A)D$$

since D and $(I_N \otimes A)$ are block diagonal matrices whose respective diagonal blocks commute with each other.

Therefore, taking $Z = DX$ we obtain

$$\begin{aligned} \dot{Z} =& D\dot{X} = \\ =& D\,(I_N \otimes A)X - DD^T(L \otimes BK)DX = \\ =& (I_N \otimes A)DX - (L \otimes BK)DX = \\ =& (I_N \otimes A)Z - (L \otimes BK)Z \end{aligned}$$

that has the same form of the auxiliary network (6-14). From **H3**, this latter network achievess consensus and therefore following the same line of reasoning in the proof of Theorem 6.1, we can conclude that network (6-13) reaches γ-bipartite consensus. □

Remark 6.5. *In [152] the authors studied bipartite consensus of multi-agent system over directed signed graph showing the equivalence between bipartite consensus problems [5] and classical consensus problems in diffusive networks. However, the authors considered only the anti-symmetric case, that is $\gamma = -I_n$. Note that the matrix $-I_n$ commutes with every square matrix and therefore this is a special case of the more general results presented here.*

Networks of integrators

As a specific example we move next to networks of simple and higher-order integrators, which have been shown to be of relevance in a wide range of applications, see e.g. [106], [110] and references therein.

Higher-order integrators

We now consider a connected undirected network of $N > 1$ n dimensional integrators, which can be written in compact form as (6-13), with

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{6-15}$$

and $K \in \mathbb{R}^{1\times n}$.

Proposition 6.1. *Assume that for network (6-13), with A and B defined as in (6-15), the assumptions of Corollary 6.1 are fulfilled. Then, a γ-bipartite consensus pattern arises and $\gamma = -I_n$.*

Proof. Recalling that for LTI systems hypothesis **H1** of Corollary 6.1 requires that $A\gamma = \gamma A$, we have from (6-15) and simple matrix multiplications that

$$A\gamma = \begin{bmatrix} \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \cdots & \gamma_{2,n-1} & \gamma_{2,n} \\ \gamma_{3,1} & \gamma_{3,2} & \gamma_{3,3} & \cdots & \gamma_{3,n-1} & \gamma_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-1,3} & \cdots & \gamma_{n-1,n-1} & \gamma_{n-1,n} \\ \gamma_{n,1} & \gamma_{n,2} & \gamma_{n,3} & \cdots & \gamma_{n,n-1} & \gamma_{n,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$\gamma A = \begin{bmatrix} 0 & \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,n-2} & \gamma_{1,n-1} \\ 0 & \gamma_{2,1} & \gamma_{2,2} & \cdots & \gamma_{2,n-2} & \gamma_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \gamma_{n-2,1} & \gamma_{n-2,2} & \cdots & \gamma_{n-2,n-2} & \gamma_{n-2,n-1} \\ 0 & \gamma_{n-1,1} & \gamma_{n-1,2} & \cdots & \gamma_{n-1,n-2} & \gamma_{n-1,n-1} \\ 0 & \gamma_{n,1} & \gamma_{n,2} & \cdots & \gamma_{n,n-2} & \gamma_{n,n-1} \end{bmatrix}.$$

Hence, by means of the equivalence principle, we get

$$\gamma_{2,1} = \gamma_{3,1} = \cdots = \gamma_{n-1,2} = \gamma_{n,1} = 0, \tag{6-16}$$

and

$$\gamma_{1,1} = \gamma_{2,2} = \cdots = \gamma_{n-1,n-1} = \gamma_{n,n}. \tag{6-17}$$

Since $\gamma \in \mathcal{O}(n)$ we know that every column vector of γ has norm equal 1, thus from (6-16) we have that $|\gamma_{1,1}| = 1$ and therefore, from (6-17), we can conclude that $\gamma = \{I_n, -I_n\}$. □

Simple Integrators

In the case the nodes are simple integrators the network dynamics is described by

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij}(g_{ij}(x_j) - x_i) \tag{6-18}$$

for $i = 1, \ldots, N$ and with $k = 1$. In this simpler case $n = 1$ and we have $\mathcal{O}(1) = \{1, -1\}$, therefore the only possible non-trivial symmetry is the odd symmetry (i.e., $\gamma = -1$). This is the case investigated in [5] by means of the signed graph model:

$$\dot{x}_i = \sum_{j=1}^{N} |a_{ij}|(\mathrm{sgn}(a_{ij})x_j - x_i),$$

where $a_{ij} = \{0, \pm 1\}$ to take into account *negative* (antagonistic) couplings. In our framework it is not necessary to consider negative elements of the adjacency matrix a_{ij} since *antagonistic interactions* can described by setting the coupling functions $g_{ij}(x_j)$ equal to $\pm x_j$ in (6-18).

6.5 Distributed control design

Next, we present a procedure based on the results presented so far that can be used to design control strategies ensuring that a generic network of interest attains a desired γ-bipartite synchronization pattern.

The key idea behind the methodology described below is to design a *local nonlinear controller*, $v(x_i)$, at the node level inducing the desired symmetry of its closed-loop vector field and a *communication protocol* that exploiting the symmetry induced by the local node controller allows the desired synchronization pattern to emerge. The resulting network dynamics will then have the form

$$\dot{x}_i = f(t, x_i) + v(x_i) + k \sum_{j=1}^{N} a_{ij} \left(g_{ij}(x_j) - x_i \right), \tag{6-19}$$

where $i = 1, \ldots, N$ and where $v(x_i)$ is the local controller to be designed while g_{ij} is the functions determining the type of communication between node j and node i. The control task in this case is to ensure that a desired γ-bipartite pattern is achieved by the network.

To achieve this goal, our procedure consists of the following steps:

1. Determine the desired symmetry, γ (recall that $\gamma \in \mathcal{O}(n)$, $\gamma^{-1} = \gamma^T$);

2. Check whether $f(t, x)$ is γ-equivariant. If this condition is verified, then set $v(x) = 0$ and go to step 3. Otherwise, design local nonlinear control input such that the closed-loop vector field

$$\widehat{f}(t, x) := f(t, x) + v(x)$$

   is γ-equivariant. Then, go to step 3;

3. Choose a partition of the network nodes so that a group belongs to $\mathcal{G}$ and the other to $\mathcal{G}^*$;

4. Design the communication protocols g_{ij} in accordance to Theorem 6.1, i.e.:

$$g_{ij}\left(x_j\right) := \begin{cases} x_j, & x_i, x_j \in \mathcal{G} \text{ or } x_i, x_j \in \mathcal{G}^*; \\ \gamma\, x_j, & x_i \in \mathcal{G} \text{ and } x_j \in \mathcal{G}^*; \\ \gamma^T x_j, & x_i \in \mathcal{G}^* \text{ and } x_j \in \mathcal{G}. \end{cases}$$

Application example

As an illustration of the methodology, we suppose that the goal is for a γ-bipartite synchronization to arise in a network of continuous-time integrators, whose intrinsic dynamics is described by the vector field $f(t, x_i) = 0$. Therefore, from (6-19), the dynamics of each node in the network can be written as

$$\dot{x}_i = v(x_i) + k \sum_{j=1}^{N} a_{ij}\left(g_{ij}(x_j) - x_i\right)$$

For the sake of simplicity we again consider the network topology in Figure 6-1 (left panel). Next we revisit the steps of the procedure in Section 6.5.

1. We choose as desired symmetry the odd symmetry, that is $\gamma = -1$.

2. The open-loop node dynamics $f(t, x_i) = 0$ do not exhibit this symmetry, therefore local controllers $v(x_i)$ in (6-19) need to be designed. Without loss of generality, we choose $v(x_i) = 5x_i - x_i^3$, that gives the closed loop dynamics the required symmetry γ.

   As in the previous example, the desired node partition is represented in Figure 6-1 (right panel). Therefore we set $\mathcal{G} = \{1, 3\}$ and $\mathcal{G}^* = \{2, 4, 5\}$.

3. Finally, the communication protocols g_{ij} are set as

$$g_{ij}\left(x_j\right) := \begin{cases} x_j, & x_i, x_j \in \mathcal{G} \text{ or } x_i, x_j \in \mathcal{G}^*; \\ -x_j, & x_i \in \mathcal{G} \text{ and } x_j \in \mathcal{G}^*; \\ -x_j, & x_i \in \mathcal{G}^* \text{ and } x_j \in \mathcal{G}, \end{cases}$$

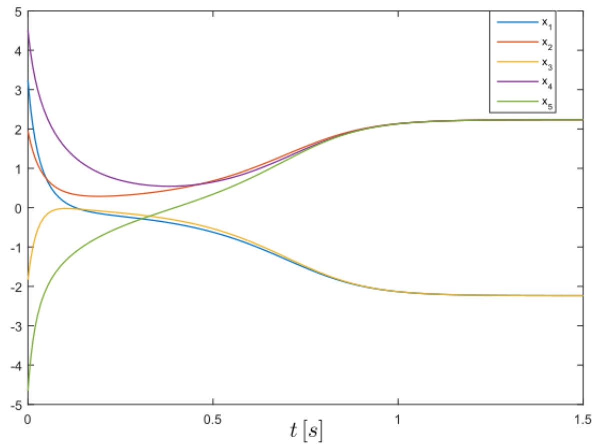

Figure 6-3: Time behavior of (6-20), with $k = 10$. Initial conditions are randomly taken from the uniform distribution.

Therefore the resulting nodes' dynamics (6-19) becomes

$$\begin{aligned}
\dot{x}_1 &= 5x_1 - x_1^3 + k\left(-x_2 + x_3 - 2x_1\right) \\
\dot{x}_2 &= 5x_2 - x_2^3 + k\left(-x_1 - x_2\right) \\
\dot{x}_3 &= 5x_3 - x_3^3 + k\left(x_1 - x_4 - x_5 - 3x_3\right) \\
\dot{x}_4 &= 5x_4 - x_4^3 + k\left(-x_3 - x_4\right) \\
\dot{x}_5 &= 5x_5 - x_5^3 + k\left(-x_3 - x_5\right)
\end{aligned} \tag{6-20}$$

Figure 6-3 shows the evolution of the nodes dynamics starting from random initial conditions. As predicted the desired synchronization pattern emerges with nodes in group $\mathcal{G}$ converging to a solution which is antithetic to that of nodes in group $\mathcal{G}^*$.

6.6 Multipartite synchronization

We end the chapter by presenting a generalization of the results presented in Section 6.3 to the case of ODEs having more than one symmetry.

Let $\mathcal{G}_N := \{1, \ldots, N\}$ be the set of the network nodes and let $\mathcal{G}_1, \ldots, \mathcal{G}_r$ be $r \geq 2$ non-empty subsets forming a partition for $\mathcal{G}_N$, that is $\mathcal{G}_i \cap \mathcal{G}_j = \{\emptyset\}$, for all i, j, with $i \neq j$, and $\bigcup_{i=1}^{r} \mathcal{G}_i = \mathcal{G}_N$.

Definition 6.5. *Consider network* (6-1) *and let f be Γ-equivariant. We say that (6-1) achieves a Γ-multipartite synchronization pattern if*

$$\begin{aligned}\lim_{t\to+\infty} |x_i(t) - s_1(t)| = 0, \quad \forall i \in \mathcal{G}_1;\\ \vdots \\ \lim_{t\to+\infty} |x_i(t) - s_r(t)| = 0, \quad \forall i \in \mathcal{G}_r,\end{aligned}$$

where

$$\begin{aligned}s_1(t) &= \gamma_1\, s_1(t) = I_n\, s_1(t)\\ s_1(t) &= \gamma_2\, s_2(t)\\ &\vdots\\ s_1(t) &= \gamma_r\, s_r(t).\end{aligned}$$

and $\{\gamma_1, \ldots, \gamma_r\} \in \Gamma$, $\gamma_i \in \mathcal{O}(n)$, $\forall i$.

Theorem 6.2. *Network (6-1) achieves a Γ-multipartite synchronization pattern if:*

- **H1** *the intrinsic node dynamics f is Γ-equivariant, and there exist r symmetries $\{\gamma_1, \ldots, \gamma_r\} \in \Gamma$;*
- **H2** *g_{ij} is defined as follows:*

$$g_{ij}\,(x_j) := \gamma_h^T\, \gamma_k\, x_j, \quad x_i \in \mathcal{G}_h \;\; \text{and} \;\; x_j \in \mathcal{G}_k$$

- **H3** *the associated auxiliary network (6-2) synchronizes.*

Proof. Without loss of generality, let us relabel the network nodes such that the first ℓ_1 nodes belong to $\mathcal{G}_1$, i.e. $\mathcal{G}_1 = \{1, \ldots, \ell_1\}$, then the other $\ell_2 - \ell_1$ nodes belong to $\mathcal{G}_2$, i.e. $\mathcal{G}_2 = \{\ell_1 + 1, \ldots, \ell_2\}$, and so on until $\mathcal{G}_r = \{\ell_{r-1}{+}1, \ldots, \ell_r\}$, with $\ell_r = N$. From hypothesis **H2** the network dynamics (6-1) can then be written as

$$\begin{aligned}\dot{x}_i = f(t, x_i) - k \Bigg[l_{ii} x_i &+ \sum_{j=1}^{\ell_1} l_{ij}\, \gamma_h^T \gamma_1\, x_j \\ &+ \sum_{j=\ell_1+1}^{\ell_2} l_{ij}\, \gamma_h^T \gamma_2\, x_j + \ldots \\ &+ \sum_{j=\ell_{r-1}+1}^{\ell_r} l_{ij}\, \gamma_h^T \gamma_r\, x_j \Bigg],\end{aligned}$$

for any $i \in \mathcal{G}_h$ and $h \in \{1, \ldots, r\}$, where l_{ij} are the elements of the Laplacian matrix. Now, let D be the $nN \times nN$ block-diagonal matrix defined in (6-4) having on its main diagonal the blocks

$$\sigma_i = \gamma_h \qquad \text{if node } i \text{ belongs to } \mathcal{G}_h \tag{6-21}$$

with $h \in \{1, \ldots, r\}$. Then the above dynamics can be rewritten in compact form as (recall that $D^T = D^{-1}$):

$$\dot{X} = F(t, X) - k\, D^T(L \otimes I_n)DX. \tag{6-22}$$

Let $Z = DX$. From (6-22) we have:

$$\begin{aligned}\dot{Z} =& D\dot{X} = \\ =& DF(t, X) - k\, DD^T(L \otimes I_n)DX = \\ =& F(t, DX) - k\,(L \otimes I_n)DX = \\ =& F(t, Z) - k\,(L \otimes I_n)Z,\end{aligned}$$

where we used **H1** and Lemma 6.1. Now, note that as before in the new state variables, the network dynamics can recast as

$$\dot{Z} = F(t, Z) - k\,(L \otimes I_n)Z, \tag{6-23}$$

that has the same form as the auxiliary network (6-2).

Since, by hypothesis **H3**, the auxiliary network synchronizes, then also does network (6-23). Therefore, there exists some $\dot{s}_1 = f(t, s_1)$ such that, $\forall i = 1, \ldots, N$:

$$\lim_{t \to +\infty} |z_i(t) - s_1(t)| = 0, \quad \forall i.$$

Since $X = D^T Z$, we finally have that

$$\begin{aligned}\lim_{t \to +\infty} x_i(t) =& \lim_{t \to +\infty} \sigma_i^T\, z_i(t) = \\ =& \begin{cases} \gamma_1^T\, s_1(t) = I_n\, s_1(t) = s_1(t), & \text{if } i \in \mathcal{G}_1; \\ \gamma_2^T\, s_1(t) = s_2(t), & \text{if } i \in \mathcal{G}_2; \\ \vdots & \\ \gamma_r^T\, s_1(t) = s_r(t), & \text{if } i \in \mathcal{G}_r. \end{cases}\end{aligned}$$

□

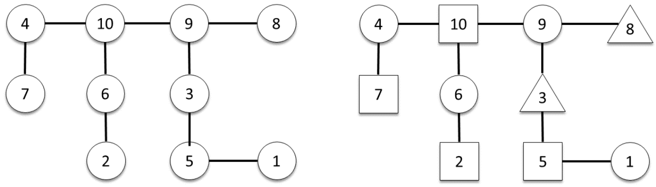

Figure 6-4: Left panel: a network of diffusively coupled harmonic oscillators (6-24). Right panel: An arbitrary partition of the set $\mathcal{G}_N$. Nodes belonging to the same group are denoted with the same shape: $\mathcal{G}_1$ (squares), $\mathcal{G}_2$ (circles), $\mathcal{G}_3$ (triangles).

An example

In order to illustrate the application of Theorem 6.2, we consider as a representative example the problem of finding sufficient conditions for the onset of Γ-multipartite synchronization patterns in networks of identical harmonic oscillators. Specifically, we consider a network of $N = 10$ identical harmonic oscillators with topology as in Figure 6-4 (left panel). The harmonic oscillator dynamics is described by

$$\dot{x} = Ax = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} x. \tag{6-24}$$

The symmetries of (6-24) are those described by rotations by an angle $\phi \in [0, 2\pi)$. That is, γ belongs to the special orthonormal group $\mathcal{SO}(2)$ or, in matrix form,

$$\gamma = \begin{bmatrix} \cos\,\phi & -\sin\,\phi \\ \sin\,\phi & \cos\,\phi \end{bmatrix}$$

It is important to note that a set of weakly coupled nonlinear oscillators can be transformed via the so-called *phase reduction* [69] into a new set of ODEs that is equivariant with respect to the circle group S^1, which is isomorphic to $\mathcal{SO}(2)$.

To satisfy hypothesis **H1** of Theorem 6.2, consider, for example, three symmetries γ_1, γ_2 and γ_3 associated to rotations by $\phi_1 = 0°$, $\phi_2 = 120°$ and $\phi_3 = 240°$, respectively, and consider the network nodes partitioned into $\mathcal{G}_1 = \{2, 5, 7, 10\}$, $\mathcal{G}_2 = \{1, 4, 6, 9\}$ and $\mathcal{G}_3 = \{3, 8\}$ associated to the respective symmetries, as reported in Figure 6-4 (right

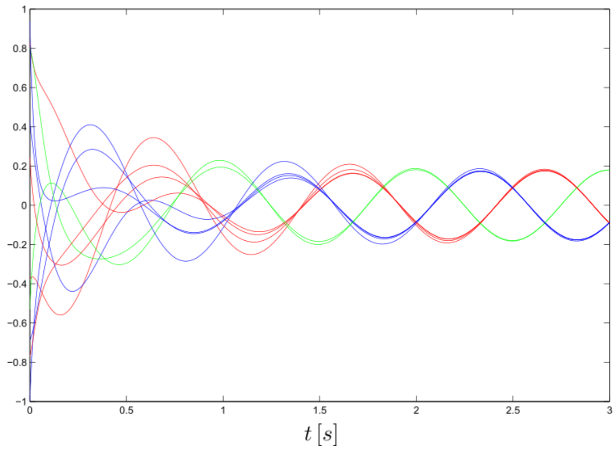

Figure 6-5: Time behavior of the state component x_1 of every nodes in groups $\mathcal{G}_1$ (blue), $\mathcal{G}_2$ (red) and $\mathcal{G}_3$ (green) in Example 6.6, with $\omega = 1$ and $k = 10$. Initial conditions are randomly taken from the uniform distribution on the unit circle.

panel) where nodes belonging to the same group are depicted with the same shape. Note that, as mentioned before, the nodes belonging to the same group do not need to be directly connected between each others. Applying the coupling functions in accordance to **H2** of Theorem 6.2, the network dynamics is described by (6-22) where $F(X) = (I_N \otimes A)X$ and the matrix D is the diagonal matrix

$$D = \text{diag}\left\{\gamma_2, \gamma_1, \gamma_3, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_3, \gamma_2, \gamma_1\right\}.$$

Furthermore, following Theorem 5.1 in [36], it can be shown that the auxiliary network (6-23) synchronizes for any $k > 0$, and therefore all hypotheses of Theorem 6.2 are verified. In Figure 6-5 the time evolution is reported for the first state component (i.e. x_1) of each oscillator of the network. As expected, the nodes belonging to the same group synchronize between each others and there is a phase delay of 120° between the three groups as required.

CHAPTER 7

Conclusions

In many control problems, such as tracking and regulation, observer design, coordination and synchronization, it is more natural to describe the stability problem in terms of the asymptotic convergence of trajectories with respect to one another rather than towards some attractor. That is, instead of studying the Lyapunov stability of some nominal solution, we are more interested to analyze the incremental stability among solutions. Contraction analysis exploits indeed the stability properties of the linearized dynamics to infer incremental stability properties of nonlinear systems. However, results available in the literature do not fully encompass the case of dynamical systems with discontinuous right-hand side (i.e. Filippov systems).

To overcome these limitations, in this thesis we presented a novel extension of contraction analysis to switched systems based on matrix measures. The analysis was conducted first regularizing the system, and then applying standard contraction results. The conditions we developed guarantee contraction of solutions for every initial conditions and do not require explicit evaluation of the sliding vector field as in previous results available in the literature.

Moreover, based on these new conditions, we firstly presented a design procedure to synthesize switching control inputs to incrementally stabilize a class of smooth nonlinear systems, and then we developed results for the design of state observers for a large class of nonlinear switched systems including those exhibiting sliding motion.

Nevertheless, our previous conditions require the system to contract during both flow and switching, a condition that can be too restrictive in certain applications. To relax this requirement, a more detailed analysis of the differential dynamics of switched systems was conducted and revealed that the differential dynamics has a hybrid nature, combining continuous and discrete dynamics. This allowed us to analyze flow and switching separately, and to finally formulate a more general contraction analysis based on Finsler-Lyapunov functions. The theory was then illustrated through a set of representative examples.

Moreover, as a further work, we presented new conditions for the onset of synchronization and consensus patterns in complex networks. Specifically, we showed that if network nodes exhibit some symmetry and if the network topology is properly balanced by an appropriate choice of the communication protocol, then symmetry of the nodes can be exploited to achieve a synchronization/consensus pattern. The symmetries we considered for synchronization were those belonging to the orthogonal group. After presenting some analysis, we showed that our approach can be turned into a design methodology and demonstrated the effectiveness of our results via a set of representative examples including networks of linear and nonlinear systems.

7.1 Future work

Several open problems are left for further developments.

- Contraction analysis based on Finsler-Lyapunov functions allowed to derive Theorem 5.1 from well-known Lyapunov stability analysis of hybrid dynamical systems [58]. In analogy to hybrid systems, the conditions in Theorem 5.1 can be further relaxed, e.g. using dwell-time.

- Sliding mode solutions were excluded from the analysis in Chapter 5. Such solutions are constrained to evolve onto a lower dimensional manifold, and therefore, contraction has to be verified only in this subspace (as in *horizontal contraction* [50] and related to what done in [35, 33]). Moreover, as described in [38, 44], when a solution dives into the sliding manifold, the saltation matrix has an eigenvalue equal to zero, that corresponds to instantaneous contraction to zero of one (transversal to the sliding manifold) dimension of a ball.

- Switched systems are also known to exhibit self-sustained oscillations under certain conditions [12]. Whereas a contracting system in a forward invariant set can converge only to an equilibrium point, a differentially positive system can instead converge to a limit cycle [52]. It is worth to investigate how differential positivity can also be extended to switched dynamical systems.

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