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Countable and Uncountable in Group Theory

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Our *chit-chats* made me the student I am and gave birth to the hereunder work. I hope we'll always have time for these fruitful conversations.
Thanks *Prof*

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Introduction

letter to a Young Mathematician

Whether one wants it or not, infinite is out there. No, not in the physical world (perhaps!) but in the rather more comfortable and insidious world of ideas. It is probably the first unintuitive and unnatural concept, apart from God (even if, for someone, the two coincide), we meet in our lives.

It took eons and a bunch of brilliant minds for a satisfying definition of infinite to appear and now, now that we have one, it's on us all to exploit its logical (and beautifully "illogical") consequences in every area of mathematics. Most of these areas implicitly and informally dealt with the infinite since the dawn of time. Therefore, in those, one could satisfyingly test if the infinite is correctly represented by our modern definition. Others were born from the finite and spread their wings towards the infinite under the influence of this contemporary revolution. Group Theory is one of these.

The majority of group theorists have always worked with finite, and there is nothing wrong in that: group theory deserves to be studied in whole and finite group theory has such results of incomparable beauty. But that is not the end of the story. Many decades ago, people started to study infinite groups with an eye on the finite, they began the study of groups with finiteness conditions (i.e., conditions automatically satisfied by all finite groups), and their efforts produced results that can stand the comparison with the finite ones. Some of those pioneers also detached themselves from the finiteness restrictions and began their path outside the Cave.

They saw inconceivable and wonderful groups. But what matters, they saw light, they saw that there is something worth to be studied. Their fire today still burn in someone.

However, that is not enough for most people. Infinite group theory is largely not applicable, at the moment, and that seems to be the main reason for the full disregard people offer to it.

There is perhaps something better than study a completely useless (but intriguing) object? Think about it: working, for instance, on such unnatural groups, means working on a completely pure world where the potential became actual, where almost anything, not corrupted from the real world, is beautiful on its own sake (one does not need to find the beauty in some other place).

It is understandable that the human kind needs to make tangible progresses and in order to do so needs to study useful things, but, as the history proves, often the most groundbreaking results follows from using old useless things.

I truly think that infinite group theory is beauty and still rich in fascinating things waiting only to be discovered. So, dear peer, consider for yourself a bath in those objects and see with your eyes if those are worth your time (do not be caught in the depraved mechanism society is imposing now onto us: the continuous search for applicability), see if you can keep alive the sparkle moving me and others.

an introduction

Let's cut the chase and start to describe the mixture of infinite and groups that grabbed my attention in these years.

Recall that a group G is said to have *finite rank* r if every finitely generated subgroup of G can be generated by r elements, and r is the least positive integer with such a property. If such an r does not exists, we will say that G has *infinite rank*.

In a long series of papers, it has been shown that the structure of a (generalized) soluble group of infinite rank is strongly influenced by that of its proper subgroups of infinite rank (see for instance [17], where a full reference list on this subject can be found). The results in these papers suggest that the behavior of *large* subgroups strongly influence the structure of the group itself, at least for a right choice of the definition of largeness and within a suitable universe.

Moving from this, I started to study how subgroups of uncountable cardinality affect an uncountable group. Let $\mathfrak{X}$ be a group theoretical property, let G be a group of uncountable cardinality and suppose that all its proper uncountable subgroups satisfy $\mathfrak{X}$. Is it true that all (proper) subgroups of G satisfy $\mathfrak{X}$? In the affirmative case we will generically say that the class $\mathfrak{X}$ is *uncountably recognizable*.

The thesis exploits this question (see Chapter 2), showing that, under some (generalized) soluble conditions, the answer is often positive, whenever $\mathfrak{X}$ is an absolute, countably recognizable property. Remember that a group class $\mathfrak{X}$ is said to be *countably recognizable* if a group G is an $\mathfrak{X}$-group whenever all its countable subgroups belong to $\mathfrak{X}$.

Conjecture — In a suitable universe of groups, is every countably recognizable class also uncountably recognizable?

Undertaking the study of this conjecture I was lead to the study of countably recognizable classes of groups themselves. It turns out, as will be shown here (see Chapter 1), that almost all reasonable group theoretical classes you can think of are countably recognizable.

Most of the notation is standard and can be found in [61]. However, in what follows, we have tried to catch up all the notions we needed in the former (and main) sections of both chapters.

groups with restricted conjugacy classes

Both chapters first deal with groups with restricted conjugacy classes, and in particular with groups with finite conjugacy classes. It will be proved that most of the classes of groups defined by restrictions on the conjugacy classes are countably recognizable and that being FC is uncountably recognizable, at least in a suitable universe of groups. Here we recall some notions about these groups.

If G is a group, the elements of G admitting only finitely many conjugates form a subgroup $FC(G)$, called the FC-*centre* of G, and G is an FC-*group* if it coincides with the FC-centre, i.e. if all conjugacy classes of elements of G are finite. Thus a group G has the FC-property if and only if the index $|G : C_G(x)|$ is finite for each element x of G. Clearly, all abelian groups and all finite groups have the FC-property, and the study of FC-groups was initially developed with the aim of finding properties common to these two relevant group classes. We refer to the monographs [75] and [12] for a detailed description of results and properties concerning this important chapter of the theory of infinite groups. It is obvious that finitely generated FC-groups are finite over the centre. Among the basic results, it should be also mentioned that if G is any FC-group, then the factor group $G/Z(G)$ is periodic and residually finite; it follows that the commutator subgroup of any FC-group is periodic, so that in particular torsion-free FC-groups are abelian.

Moreover, it is easy to show that a periodic group has the FC-property if and only if it is covered by finite normal subgroups (this result is usually known as Dietzmann's Lemma). If $\mathfrak{X}$ is any group class, we shall denote by $M\mathfrak{X}$ the class of all groups in which every finite subset lies in a normal $\mathfrak{X}$-subgroup. Thus Dietzmann's Lemma just says that $\boldsymbol{M}\mathfrak{F}$ is the class of all periodic FC-groups.

Recall also that a group G is said to be a BFC-*group* if it has boundedly finite conjugacy classes, i.e. if there is a positive integer k such that $|G : C_G(x)| \leqslant k$ for all elements x of G. It was proved by B.H. Neumann [54] that a group has the BFC-property if and only if its commutator subgroup is finite.

If G is a group, the *upper* FC-*central series* of G is the ascending characteristic series $\{FC_\alpha(G)\}_\alpha$ defined by setting $FC_0(G) = \{1\}$,

$$FC_{\alpha+1}(G)/FC_\alpha(G) = FC\big(G/FC_\alpha(G)\big)$$

for each ordinal α and

$$FC_\lambda(G) = \bigcup_{\alpha<\lambda} FC_\alpha(G)$$

if λ is a limit ordinal. The last term of the upper FC-central series of G is called the FC-*hypercentre* of G, and G is said to be FC-*hypercentral* if it coincides with the FC-hypercentre. Moreover, G is called FC-*nilpotent* if $FC_k(G) = G$ for some non-negative integer k, and in this case the smallest such k is the FC-*nilpotency class* of G; then a group has the FC-property if and only if it is FC-nilpotent of class $\leqslant 1$.

Obviously, all nilpotent-by-finite groups, i.e., groups with a normal nilpotent subgroup of finite index, are FC-nilpotent. On the other hand, if p is any prime number and G is the semidirect product of a group P of type p^∞ by the cyclic group generated by an automorphism of P of infinite order, then G is FC-nilpotent, but it is not nilpotent-by-finite, since P is the Fitting subgroup of G. This situation cannot occur in the case of finitely generated groups. In fact, the following result, due to D.H. McLain [50], shows that for finitely generated groups the properties of being FC-hypercentral, FC-nilpotent and nilpotent-by-finite are equivalent. In particular, finitely generated FC-hypercentral groups satisfy the maximal condition on subgroups.

Lemma *Let G be a finitely generated FC-hypercentral group. Then G is nilpotent-by-finite.*

PROOF — Let μ be the smallest ordinal with the property that the factor group $G/FC_\mu(G)$ satisfies the maximal condition on subgroups, and assume that $\mu > 0$. Then $G/FC_\mu(G)$ is FC-nilpotent and all factors of its upper FC-central series are central-by-finite, so that in particular $G/FC_\mu(G)$ is polycyclic-by-finite. It follows that $FC_\mu(G)$ is the normal closure of a finite subset of G, and so μ cannot be a limit ordinal. Thus the FC-centre $FC_\mu(G)/FC_{\mu-1}(G)$ of $G/FC_{\mu-1}(G)$ is finitely generated, so that it satisfies the maximal condition and hence also $G/FC_{\mu-1}(G)$ satisfies the maximal condition. This contradiction shows that $\mu = 0$, and so G satisfies the maximal condition on subgroups; in particular, G is FC-nilpotent, and so $G = FC_k(G)$ for some non-negative integer k.

For each positive integer $i \leqslant k$, the FC-centre $FC_i(G)/FC_{i-1}(G)$ of $G/FC_{i-1}(G)$ is finitely generated. This implies that also that the index

$$|G : C_G\big(FC_i(G)/FC_{i-1}(G)\big)|$$

is finite. Therefore

$$C = \bigcap_{i=1}^{k} C_G\big(FC_i(G)/FC_{i-1}(G)\big)$$

is a nilpotent subgroup of finite index of G, and therefore G is nilpotent-by-finite. □

Let FC^0 be the class of all finite groups, and for each non-negative integer n define by induction FC^{n+1} as the class consisting of all groups G such that $G/C_G(\langle x\rangle^G)$ belongs to FC^n for every element x of G. Notice that FC^1 is precisely the class of all FC-groups, and that the class FC^n is closed with respect to subgroups and homomorphic images for all n. It is also clear that FC^n contains all nilpotent groups of class at most n.

Groups with the FC^n-property have been introduced in [29], where it was proved in particular that if G is an FC^n-group for some positive integer n, then the subgroup $\gamma_n(G)$ is contained in the FC-centre of G, and so G is FC-nilpotent of class at most n. It follows easily that $\gamma_{n+1}(G)$ is periodic for every FC^n-group G, and in particular torsion-free groups with the FC^n-property are nilpotent of class at most n. The consideration of the infinite dihedral group shows that

$$FC^* = \bigcup_{n\in\mathbb{N}} FC^n$$

is properly contained in the class of FC-nilpotent groups.

In [29] it was also studied the class FC^∞ consisting of all groups G such that for each element x the factor group $G/C_G(\langle x\rangle^G)$ belongs to FC^n for some non-negative integer n depending on x. For each positive integer n, let G_n be a finitely generated nilpotent group of class n such that $G_n/Z_{n-1}(G_n)$ is infinite; the direct product

$$G = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, G_n$$

is an FC^∞-group which does not have the FC^n-property for any n. Therefore FC^∞ properly contains the class FC^*.

In a celebrated paper of 1955, B.H. Neumann [55] started the investigation of groups in which all subgroups are normal up to the obstruction of a finite section, and proved that such groups are close to be abelian. In fact, he proved that a group G has finite conjugacy classes of subgroups (or equivalently each subgroup of G is normal in a subgroup of finite index) if and only if the centre $Z(G)$ has finite index in G, while in a group G every subgroup has finite index in its normal closure if and only if the commutator subgroup G' is finite, and so if and only if G is a BFC-group. A third natural normality condition was considered forty years later and it is in some sense much more difficult to handle. A group G is called a CF-*group* if the index $|X : X_G|$ is finite for each subgroup X of G. The consideration of the locally dihedral 2-group shows that locally finite groups with the CF-property need not be FC-groups. The CF-property has been introduced in [9], where it was proved that any locally finite CF-group contains an abelian subgroup of finite index; this result was later extended to locally (soluble-by-finite) CF-groups (see [73]), but it cannot be proved in the general case, as Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) have obviously the CF-property.

A group G is said to have the BCF-*property* if there exists a positive integer k such that $|X : X_G| \leqslant k$ for all subgroups X of G. It can be proved that locally finite CF-groups have the BCF-property (see [9]), and that locally graded BCF-groups are abelian-by-finite (see [73]). Recall here that a group G is *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index; in particular, all locally (soluble-by-finite) groups are locally graded.

Let G be a group, and let X be a subgroup of G. The *normal oscilla-*

tion of X in G is the cardinal number

$$\min\{|X : X_G|, |X^G : X|\}.$$

Clearly, X is normal in G if and only if it has normal oscillation 1. Moreover, X has finite normal oscillation in G if and only if either X has finite index in its normal closure X^G or it is finite over its core X_G; in particular, finite subgroups and subgroups of finite index have finite normal oscillation. We shall say that a group G is an FNO-*group* if all its subgroups have finite normal oscillation. Then all groups with finite commutator subgroup and all CF-groups have the FNO-property. It has recently been proved that any locally finite FNO-group is nilpotent-by-finite (see [28]).

Let G be a group and $\mathfrak{X}$ a class of groups. Remember that G is said to be an $\mathfrak{X}C$-*group* (or to have $\mathfrak{X}$-*conjugacy classes*) if the factor group $G/C_G(\langle g\rangle^G)$ belongs to $\mathfrak{X}$ for each element g of G. Thus $\mathfrak{F}C$ is precisely the class of FC-groups. If $\mathfrak{X}$ is chosen to be either the class $\mathfrak{C}$ of Černikov groups or the class $\mathfrak{P}$ of polycyclic-by-finite groups, we obtain the relevant classes of CC-*groups* and PC-*groups*, introduced in [60] and [25], respectively. We mention here that a periodic group G has the CC-property if and only if $\langle x\rangle^G$ is a Černikov group for each element x, while PC-groups can be characterized as those groups which can be covered by their polycyclic-by-finite normal subgroups. This means that $\mathit{M}\mathfrak{C}$ is the class of periodic CC-groups and $\mathit{M}\mathfrak{P}$ is the class of all PC-groups.

Further classes of generalized FC-groups have been considered by several authors. Among the most interesting ones, we mention the class of FCI-groups and that of FNI-groups. A group G is said to be an FCI-*group* if every cyclic non-normal subgroup has finite index in its centralizer, while G is an FNI-*group* if each non-normal subgroup of G has finite index in its normalizer. These group classes have been completely described in the locally (soluble-by-finite) case by D.J.S. Robinson [64].

generalized nilpotency properties

The second sections of both chapters concern groups with generalized nilpotency properties. Here, the main result is that almost all (natural) generalized nilpotency properties we will describe below are countably recognizable and that both being nilpotent and being locally nilpotent are uncountably recognizable properties in a suitable universe.

Let $\mathfrak{X}$ be a group class. We say that $\mathfrak{X}$ is a class of *generalized nilpotent groups* if every nilpotent group belongs to $\mathfrak{X}$ and all finite groups in $\mathfrak{X}$ are nilpotent. The most relevant classes of generalized nilpotent groups are described in two diagrams represented at pages 3 and 13 of [61], Part 2.

Obviously, the local class $\boldsymbol{L}\mathfrak{N}$ of all locally nilpotent groups is a class of generalized nilpotent groups, and a subclass $\mathfrak{X}$ of $\boldsymbol{L}\mathfrak{N}$ is a class of generalized nilpotent groups if and only if it contains all nilpotent groups. The most relevant classes of generalized nilpotent groups contained in $\boldsymbol{L}\mathfrak{N}$ are the following.

- The class of Gruenberg groups: a group G is a *Gruenberg group* if it is generated by its abelian ascendant subgroups, or equivalently if all finitely generated subgroups of G are ascendant. It follows from the Hirsch-Plotkin theorem that these groups are locally nilpotent.

- The class of Baer groups: a group G is a *Baer group* if it is generated by its abelian subnormal subgroups, or equivalently if all finitely generated subgroups of G are subnormal. Of course, every Baer group is a Gruenberg group.

- The class of Fitting groups: a group G is a *Fitting group* if it is generated by its nilpotent normal subgroups, or equivalently if G is covered by its nilpotent normal subgroups. Obviously, all Fitting groups are Baer groups.

- The class of $\mathcal{N}$-groups: a group G is an $\mathcal{N}$*-group* if all its subgroups are ascendant, or equivalently if every proper subgroup of G is properly contained in its normalizer. Thus every $\mathcal{N}$-group is a Gruenberg group.

- The class of hypercentral groups: a group G is *hypercentral* if its upper central series terminates with G. It can be easily proved that any hypercentral group is an $\mathcal{N}$-group.

- The class of $\mathcal{N}_1$-groups; a group G is an $\mathcal{N}_1$*-group* if all its subgroups are subnormal. It is known that if all subgroups of a group are subnormal of bounded defect, then the group is nilpotent (see [61] Part 2, p.71), but H. Heineken and I.J. Mohamed [37] gave an example of a periodic metabelian group with trivial centre and such that all proper subgroups are subnormal and nilpotent. On the other hand, a relevant result of W. Möhres [52] shows that any $\mathfrak{N}_1$-group is at least soluble.

The main classes of generalized nilpotent groups which are not contained in $\boldsymbol{L}\mathfrak{N}$ are reported in the second diagram quoted from [61]. Most of them are local classes, and so also countably recognizable. We mention in particular that it follows from Mal'cev local theorem that Z-groups, $\overline{Z}$-groups and $\overline{\overline{Z}}$-groups form local classes (see [61] Part 2, p.99). Recall here that a group G is called a *Z-group* if it has a central series (of arbitrary order type), while G is said to be a *$\overline{Z}$-group* if all its homomorphic images are Z-groups, or equivalently if every chief factor of G is central; finally, G is a *$\overline{\overline{Z}}$-group* if all its subgroups are $\overline{Z}$-groups. Note that also the class of *$\widetilde{\mathfrak{N}}$-groups* (i.e. groups in which every subgroup is serial) has been proved to be local (see [3]). However, there are two relevant classes in this diagram which are not local.

- The class of hypocentral groups: a group G is *hypocentral* if its lower central series terminates with $\{1\}$.
- The class of residually nilpotent groups: if $\mathfrak{X}$ is any class of groups, the *$\mathfrak{X}$-residual* $\rho^*_{\mathfrak{X}}(G)$ of a group G is the intersection of all normal subgroups N of G such that G/N belongs to $\mathfrak{X}$, and G is *residually* $\mathfrak{X}$ if its $\mathfrak{X}$-residual is trivial (the class of residually $\mathfrak{X}$-groups will be denoted by $\boldsymbol{R}\mathfrak{X}$). In particular, a group G is residually nilpotent if and only if the ω-th term

$$\gamma_\omega(G) = \bigcap_{n\in\mathbb{N}} \gamma_n(G)$$

of the lower central series of G is trivial. Thus all residually nilpotent groups are hypocentral.

Chapter 1

Countable in Group Theory

A group class $\mathfrak{X}$ is said to be *countably recognizable* if a group G is an $\mathfrak{X}$-group whenever all its countable subgroups belong to $\mathfrak{X}$. Countably recognizable classes of groups were introduced and studied by R. Baer [2] in 1962, but already in the fifties the property of being hyperabelian and that of being hypercentral were proved to be detectable from the behaviour of countable subgroups by Baer and S.N. Černikov, respectively (see for instance [61] Part 1, Theorem 2.15 and Theorem 2.19).

Among the countably recognizable group classes there are the so-called local class: a group class $\mathfrak{X}$ is said to be *local* if it contains all groups in which every finite subset lies in an $\mathfrak{X}$-subgroup. Clearly nilpotent groups of class at most n form a local class $\mathfrak{N}_n$ for each positive integer n, and similarly the class $\mathfrak{S}_n$, consisting of all soluble groups of derived length at most n, is local. Although the class $\mathfrak{N}$ of nilpotent groups and the class $\mathfrak{S}$ of soluble groups are not local, the following easy elementary lemma due to Baer shows in particular that they are at least countably recognizable.

Lemma 1.1 *Let $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ be a countable collection of subgroup closed and countably recognizable group classes. Then also the class*

$$\mathfrak{X} = \bigcup_{n\in\mathbb{N}} \mathfrak{X}_n$$

is countably recognizable.

Proof — Let G be a group whose countable subgroups belong to $\mathfrak{X}$, and assume for a contradiction that G is not an $\mathfrak{X}$-group. Then for each positive integer n there exists a countable subgroup H_n of G which is not in $\mathfrak{X}_n$. As all classes $\mathfrak{X}_n$ are subgroup closed, it follows that the countable subgroup

$$\langle H_n \mid n \in \mathbb{N}\rangle$$

cannot be in $\mathfrak{X}$, and this contradiction proves the statement. □

In his paper, Baer produced many interesting examples of countably recognizable group classes which are not local; it follows for instance from Baer's methods that if $\mathfrak{X}$ is a countably recognizable group class, closed with respect to subgroups and homomorphic images, then the class of all groups admitting an ascending normal series with $\mathfrak{X}$-factors is likewise countably recognizable, and this class is not local for many natural choices of $\mathfrak{X}$.

Later many other relevant countably recognizable group classes were discovered. In particular, B.H. Neumann [56] proved that residually finite groups (i.e. groups in which the intersection of all normal subgroups of finite index is trivial) form a countably recognizable class. Moreover, R.E. Phillips ([58], [59]) proved that the class of groups in which every subgroup has all its maximal subgroups of finite index is countably recognizable, and that the same conclusion holds for the class of groups whose simple sections belong to a subgroup closed and countably recognizable group class.

Further interesting examples of countably recognizable group classes can be found in [23] and [72], where it is proved for instance that groups for which some term (with finite ordinal type) of the derived series has finite rank form a countably recognizable class, and that a corresponding result holds if the derived series is replaced by the lower central series.

Examples of important group classes which are not countably recognizable are known in the literature. It was proved by G. Higman [40] that there exists a group of cardinality $\aleph_1$ which is not free but whose countable subgroups are free. Therefore the class of free groups is not countably recognizable. Note that also the property of being free abelian cannot be detected from the behaviour of countable subgroups; in fact, the cartesian product of any infinite collection of infinite cyclic groups cannot be decomposed into a direct product of infinite cyclic groups, but all its countable subgroups are free abelian (see for instance [26], Theorem 19.2). Moreover, M.I. Kargapolov [45] constructed a locally nilpotent group with no abelian non-trivial ascendant subgroups, and this example shows that the class SN^* of all groups admitting an ascending series with abelian factors is not countably recognizable.

It is straightforward to show that the class of groups with finite conjugacy classes is countably recognizable, and the aim of the first section of this chapter is to prove that also many other relevant group classes defined by restrictions on the conjugacy classes are countably recognizable (see also [33]). We then move on to prove that most of the generalized nilpotency properties we stated in the introduction have countable character (see also [31]). Finally we obtain the countably recognizability of minimax groups (namely, groups with a finite series whose factors satisfy either the minimal or the maximal condition on subgroups) and informations about the influence of closed countable subgroups in topological groups (see also [36] and [32]).

1.1 Groups with Restricted Conjugacy Classes

If x is an element of a group G admitting infinitely many conjugates, it is obvious that x belongs to some countable subgroup of G where it has again infinitely many conjugates. It follows that the class of FC-groups is countably recognizable. It is also clear that groups with finite commutator subgroup (which are precisely the groups with the BFC-property) form a countably recognizable group class. In this section it will be proved that all of the relevant classes of generalized FC-groups we saw in the introduction are countably recognizable. It was proved by Baer [2] that the class of FC-hypercentral groups is countably recognizable, and it is also known that the property of being nilpotent-by-finite can be detected from the behaviour of countable subgroups (see for instance [28]). Our first aim is to prove that also the intermediate class of FC-nilpotent groups is countably recognizable.

Note first that the class FC_n, consisting of all FC-nilpotent groups of class at most n, is not local for any positive integer n. To see this, it is enough to consider any locally finite group with trivial FC-centre, like for instance an infinite simple locally finite group or one of the periodic metabelian groups constructed by V.S. Čarin (see [61] Part 1, p.152).

Lemma 1.1.2 *Let G be a group, and let X be a countable subgroup of G. Then for each non-negative integer n there exists a countable subgroup H_n of G containing X such that $H_n \cap FC_n(G) = FC_n(H_n)$.*

Proof — The proof is by induction on n, the statement being obvious for $n = 0$. If the subgroup X is contained in $FC_{n+1}(G)$, it is enough to put $H_{n+1} = X$. Assume now that X is not contained in $FC_{n+1}(G)$, and let x be any element of $X \setminus FC_{n+1}(G)$. Then the coset $xFC_n(G)$ has infinitely many conjugates in $G/FC_n(G)$, and so there exists a countably infinite subset Y_x of G such that

$$x^{y_1}FC_n(G) \neq x^{y_2}FC_n(G)$$

for all elements y_1 and y_2 of Y_x such that $y_1 \neq y_2$. Clearly, the subgroup

$$K = \langle X, Y_x \mid x \in X \setminus FC_{n+1}(G)\rangle$$

is countable, so by induction we can find a countable subgroup U_1 of G containing K such that $U_1 \cap FC_n(G) = FC_n(U_1)$. Apply now the same argument to U_1 in order to obtain a new countable subgroup U_2 containing U_1, with $U_2 \cap FC_n(G) = FC_n(U_2)$ and such that for every $a \in U_1 \setminus FC_{n+1}(G)$ there is a countably infinite subset Z_a of U_2 for which $a^{z_1}FC_n(G) \neq a^{z_2}FC_n(G)$ whenever z_1 and z_2 are different elements of Z_a. In this way we can construct an increasing sequence $(U_k)_{k\in\mathbb{N}}$ of countable subgroups of G such that

$$U_k \cap FC_n(G) = FC_n(U_k).$$

Consider the countable subgroup

$$U = \bigcup_{k\in\mathbb{N}} U_k.$$

If u is any element of $FC_n(U)$ and if k is a positive integer such that $u \in U_k$, then u belongs to $FC_n(U_k)$ and so also to $FC_n(G)$. Therefore $U \cap FC_n(G) = FC_n(U)$. Let v be any element of $FC_{n+1}(U)$, and assume for a contradiction that u does not belong to $FC_{n+1}(G)$. Fix a positive integer k such that $v \in U_k$. It follows from our construction that U_k contains a countably infinite subset $W = \{w_i \mid i \in \mathbb{N}\}$ such that $v^{w_i}FC_n(G) \neq v^{w_j}FC_n(G)$ if $i \neq j$. As $FC_n(U)$ is contained in $FC_n(G)$, we have also that $v^{w_i}FC_n(U) \neq v^{w_j}FC_n(U)$ if $i \neq j$, contradicting the assumption that v belongs to $FC_{n+1}(U)$. Therefore $FC_{n+1}(U)$ is contained in $FC_{n+1}(G)$, and the proof of the statement can be completed by choosing $H_{n+1} = U$. □

Theorem 1.1.3 *For each positive integer* n*, the class of* FC*-nilpotent groups of class at most* n *is countably recognizable.*

Proof — Let G be a group whose countable subgroups are FC-nilpotent with class at most n. It follows from Lemma 1.1.2 that every countable subgroup of G is contained in $FC_n(G)$, so that $FC_n(G) = G$ and G is FC-nilpotent with class at most n. □

Corollary 1.1.4 *The class of* FC*-nilpotent groups is countably recognizable.*

Proof — Let FC_∞ be the class of FC-nilpotent groups. Then

$$FC_\infty = \bigcup_{n\in\mathbb{N}} FC_n,$$

and so the statement of the corollary is a direct consequence of Theorem 1.1.3 and Lemma 1.1. □

It was claimed in [68], Lemma 5, that the class FC^n is countably recognizable for each non-negative integer n, but unfortunately the proof of this result contains a mistake. Here we prove in a different way that this statement holds; as a consequence, it can be deduced that also Theorem 2 of [68] remains true.

Lemma 1.1.5 *Let G be a group, and let X be a countable subgroup of G. Then for every element g of G, there exists a countable subgroup Y of G such that $\langle g, X\rangle \leqslant Y$ and $C_X(\langle g\rangle^G) = C_X(\langle g\rangle^Y)$.*

Proof — It can obviously be assumed that $C_X(\langle g\rangle^G) \neq X$, so that in particular $X = \langle X \setminus C_X(\langle g\rangle^G)\rangle$. For each element x of $X \setminus C_X(\langle g\rangle^G)$, choose a finite subset E_x of G such that $x \notin C_X(\langle g\rangle^{E_x})$, and put

$$Y = \langle g, x, E_x \mid x \in X \setminus C_X(\langle x\rangle^G)\rangle.$$

Then Y is a countable subgroup of G such that $\langle g, X\rangle \leqslant Y$ and

$$C_X(\langle g\rangle^G) = C_X(\langle g\rangle^Y).$$

The statement is proved. □

Theorem 1.1.6 *The class FC^n is countably recognizable for each non-negative integer n.*

Proof — The classes $FC^0 = \mathfrak{F}$ and $FC^1 = FC$ are obviously countably recognizable. Suppose by induction that FC^n is countably recognizable, and assume for a contradiction that there exists a group G which is not an FC^{n+1}-group but whose countable subgroups have all the FC^{n+1}-property. Then there exists an element g of G such that $G/C_G(\langle g\rangle^G)$ is not an FC^n-group, and so there is a countable subgroup X of G such that $X/C_X(\langle g\rangle^G)$ does not have the FC^n-property. It follows from Lemma 1.1.5 that there is a countable subgroup Y of G such that $\langle g, X\rangle \leqslant Y$ and $C_X(\langle g\rangle^G) = C_X(\langle g\rangle^Y)$. By hypothesis, Y is an FC^{n+1}-group, and so $Y/C_Y(\langle g\rangle^Y)$ belongs to FC^n. Then also

$$X/C_X(\langle g\rangle^G) = X/C_X(\langle g\rangle^Y) \simeq XC_Y(\langle g\rangle^Y)/C_Y(\langle g\rangle^Y)$$

is an FC^n-group, and this contradiction completes the proof. □

Of course, it follows from Lemma 1.1 and Theorem 1.1.6 that the class

$$\mathrm{FC}^* = \bigcup_{n \in \mathbb{N}_0} \mathrm{FC}^n$$

is countably recognizable. Furthermore the FC^∞-property can be detected from the behaviour of countable subgroups, as our next result shows.

Theorem 1.1.7 *The class* FC^∞ *is countably recognizable.*

Proof — Let G be a group. Suppose that all countable subgroups of G are FC^∞-groups, and assume for a contradiction that there exists an element g of G such that the factor group $G/C_G(\langle x\rangle^G)$ does not belong to the class FC^*. If n is any non-negative integer n, the class FC^n is countably recognizable class, and so there exists a countable subgroup X_n of G such that $X_n C_G(\langle g\rangle^G)/C_G(\langle g\rangle^G)$ is not in the class FC^n. It follows from Lemma 1.1.5 that for each n there is a countable subgroup Y_n of G such that

$$\langle g, X_n\rangle \leqslant Y_n \text{ and } C_{X_n}(\langle g\rangle^G) = C_{X_n}(\langle g\rangle^{Y_n}).$$

The subgroup

$$Y = \langle Y_n \mid n \in \mathbb{N}_0\rangle$$

is countable, so that $Y/C_Y(\langle g\rangle^Y)$ is an FC^k-group for some non-negative integer k, and hence also $X_k C_Y(\langle g\rangle^Y)/C_Y(\langle g\rangle^Y)$ belongs to FC^k. On the other hand,

$$C_{X_k}(\langle g\rangle^Y) \leqslant C_{X_k}(\langle g\rangle^{Y_k}) = C_{X_k}(\langle g\rangle^G),$$

and so $C_{X_k}(\langle g\rangle^Y) = C_{X_k}(\langle g\rangle^G)$. Therefore $X_k C_G(\langle g\rangle^G)/C_G(\langle y\rangle^G)$ is an FC^k-group, and this contradiction proves the statement. □

It is easy to show that, like the class of finite-by-abelian groups, also the class of groups which are finite over the centre is countably recognizable. Our next result proves that the third class of groups considered by Neumann has the same property.

Theorem 1.1.8 *The class of* CF*-groups is countably recognizable.*

Proof — Let G be a group such that all countable subgroups have the CF-property. Assume for a contradiction that G is not a CF-group, and let X be a subgroup of G such that the index $|X : X_G|$ is infinite. Then X/X_G contains a countably infinite subgroup Y/X_G. Let W be a transversal to X_G in Y. If y and z are distinct elements of W, the product $y^{-1}z$ does not belong to $X_G = Y_G$, and so there exists an element $g(y,z)$ of G such that $y^{-1}z$ is not in $Y^{g(y,z)}$. Put $K = \langle W \rangle$, and consider the countable subgroup

$$H = \langle K, g(y,z) \mid y, z \in W, y \neq z \rangle.$$

Then H is a CF-group, so that the index $|K : K_H|$ must be finite, and hence there exist distinct elements y, z of W such that $y^{-1}z$ lies in K_H, a contradiction because

$$K_H \leqslant K^{g(y,z)} \leqslant Y^{g(y,z)}.$$

Therefore G has the CF-property, and the class CF is countably recognizable. □

If n is a positive integer, we say that a group G has the CF_n-*property* if $|X : X_G| \leqslant n$ for all subgroups X of G. The same argument used in the proof of Theorem 1.1.8 shows that the class of CF_n-groups is countably recognizable for each n. Therefore the class

$$BCF = \bigcup_{n \in \mathbb{N}} CF_n$$

is likewise countably recognizable by Lemma 1.1.

Groups with the FNO-property are of course related to CF-groups, and they form another countably recognizable group class.

Theorem 1.1.9 *The class of* FNO*-groups is countably recognizable.*

Proof — Let G be a group such that all countable subgroups have the FNO-property, and assume for a contradiction that G contains a subgroup X such that both indices $|X^G : X|$ and $|X : X_G|$ are infinite. Let $(x_n)_{n \in \mathbb{N}}$ be a countably infinite collection of elements of X such that $x_iX_G \neq x_jX_G$ if $i \neq j$, and put $Y = \langle x_n \mid n \in \mathbb{N} \rangle$. For all positive integers i and j such that $i \neq j$ there exists an element $g(i,j)$ of G such that $x_i^{-1}x_j$ does not belong to $X^{g(i,j)}$. On the other hand, as the index $|X^G : X|$ is infinite, there exist countable subgroups Z of X and U of G such that $Y \leqslant Z$ and the normal closure Z^U contains an infinite subset W for which $w_1X \neq w_2X$, whenever w_1, w_2 are elements of W and $w_1 \neq w_2$. Then

$$H = \langle Z, U, g(i,j) \mid i \neq j\rangle$$

is a countable subgroup of G, and $x_i^{-1}x_j$ is not in $Z^{g(i,j)}$ if $i \neq j$. It follows that $x_iZ_H \neq x_jZ_H$ for all $i \neq j$, and so the index $|Z : Z_H|$ is infinite. Moreover, $Z^H \geqslant Z^U \geqslant W$ and hence also the index $|Z^H : Z|$ is infinite, a contradiction because H is an FNO-group. Therefore G is an FNO-group, and the class FNO is countably recognizable. □

We will consider now the classes of groups with restricted conjugacy classes studied by Robinson.

Theorem 1.1.10 *The classes* FCI *and* FNI *are countably recognizable.*

Proof — Suppose first that G is a group such that all countable subgroups have the FCI-property, and assume for a contradiction that G contains a cyclic non-normal subgroup $\langle x\rangle$ such that the index $|C_G(x) : \langle x\rangle|$ is infinite. Let g be an element of G with the property that $\langle x\rangle^g \neq \langle x\rangle$, and consider a countably infinite subgroup $U/\langle x\rangle$ of $C_G(x)/\langle x\rangle$. Then $H = \langle U, g\rangle$ is a countable subgroup of G, and $\langle x\rangle$ is a non-normal subgroup of H which has infinite index in the centralizer $C_H(x)$. This contradiction shows that G is an FCI-group, and so the class FCI is countably recognizable.

Suppose now that G is a group whose countable subgroups belong to FNI, and assume that G contains a non-normal subgroup X such that the index $|N_G(X) : X|$ is infinite. Let $x \in X$ and $g \in G$ be elements such that x^g is not in X, and consider a countable subgroup U of $N_G(X)$ such that x belongs to U and UX/X is infinite. Then $H = \langle U, g\rangle$ is a countable subgroup of G, and $X \cap U$ is a non-normal subgroup of H. Moreover, U is contained in the normalizer $N_H(X \cap U)$, and hence $X \cap U$ has infinite index in $N_H(X \cap U)$. This contradiction proves that G is an FNI-group, and so FNI is a countably recognizable class. □

As we mentioned, the class of periodic FC-groups is precisely the class $\boldsymbol{M}\mathfrak{F}$. Our next result shows that many classes of the form $\boldsymbol{M}\mathfrak{X}$ are countably recognizable.

Theorem 1.1.11 *Let $\mathfrak{X}$ be a subgroup closed and countably recognizable group class. Then the class $\boldsymbol{M}\mathfrak{X}$ is countably recognizable.*

Proof — Let G be a group such that all countable subgroups belong to $\boldsymbol{M}\mathfrak{X}$, and assume for a contradiction that G is not in $\boldsymbol{M}\mathfrak{X}$. As $\mathfrak{X}$ is subgroup closed, it follows that there exists a finitely generated subgroup E of G such that the normal closure E^G does not belong to $\mathfrak{X}$. But $\mathfrak{X}$ is countably recognizable, and so E^G contains a countable subgroup U which is not in $\mathfrak{X}$. Let X be a countable subgroup of G such that $U \leqslant E^X$. Then $H = \langle E, X\rangle$ is a countable subgroup of G, and the normal closure E^H is not in $\mathfrak{X}$, because $U \leqslant E^X \leqslant E^H$. This contradiction shows that G lies in $\boldsymbol{M}\mathfrak{X}$, and hence $\boldsymbol{M}\mathfrak{X}$ is countably recognizable. □

Since it is known that both the class $\mathfrak{P}$ of all polycyclic-by-finite groups and the class $\mathfrak{C}$ of all Černikov groups are countably recognizable, it follows from the above theorem that the class $\boldsymbol{M}\mathfrak{P}$ (which coincides with the class of all PC-groups) and the class $\boldsymbol{M}\mathfrak{C}$ are countably recognizable. Our next result shows that many other similar classes, and in particular that of arbitrary CC-groups, have countable character.

Theorem 1.1.12 *Let $\mathfrak{X}$ be a subgroup closed and countably recognizable class of groups. Then the class $\mathfrak{X}C$, consisting of all groups with $\mathfrak{X}$-conjugacy classes, is countably recognizable.*

Proof — Let G be a group whose countable subgroups belong to $\mathfrak{X}C$, and assume for a contradiction that G contains an element g such that $G/C_G(\langle g\rangle^G)$ is not an $\mathfrak{X}$-group. As the class $\mathfrak{X}$ is countably recognizable, there exists a subgroup $H/C_G(\langle g\rangle^G)$ of $G/C_G(\langle g\rangle^G)$ which is countable but not in $\mathfrak{X}$. Clearly $H = XC_G(\langle g\rangle^G)$, where X is a suitable countable subgroup, and $X/C_X(\langle g\rangle^G) \simeq H/C_G(\langle g\rangle^G)$ is not in $\mathfrak{X}$. By Lemma 1.1.5 there exists a countable subgroup Y of G containing $\langle g, X\rangle$ and such that $C_X(\langle g\rangle^G) = C_X(\langle g\rangle^Y)$. Then

$$X/C_X(\langle g\rangle^G) = X/C_X(\langle g\rangle^Y) \simeq XC_Y(\langle g\rangle^Y)/C_Y(\langle g\rangle^Y) \leqslant Y/C_Y(\langle g\rangle^Y),$$

a contradiction, because $\mathfrak{X}$ is $\boldsymbol{S}$-closed and $Y/C_Y(\langle g\rangle^Y)$ belongs to $\mathfrak{X}$. Therefore G is an $\mathfrak{X}C$-group and the class $\mathfrak{X}C$ is countably recognizable. □

A relevant problem in the theory of groups with finite conjugacy classes is to establish conditions under which a periodic residually finite FC-group can be embedded into the direct product of a collection of finite groups. Therefore groups which are isomorphic to subgroups of direct products of finite groups form an important class of FC-groups, which is denoted by $\boldsymbol{SD}\mathfrak{F}$. We point out here that the class $\boldsymbol{SD}\mathfrak{F}$ is not countably recognizable. As P. Hall proved that any countable periodic residually finite FC-group belongs to $\boldsymbol{SD}\mathfrak{F}$ (see for instance [12], Theorem 1.5.1), it is enough to show that there exists an uncountable periodic residually finite FC-group which is not isomorphic to a subgroup of a direct product of finite groups. To prove this, fix a prime number p, and for each positive integer n let C_n be a cyclic group of order p^n. Consider the cartesian product C of the collection $(C_n)_{n\in\mathbb{N}}$, and let G be the subgroup of all elements of finite order of C. Then G is residually finite, but it cannot be embedded into a direct product of finite groups. Therefore the class $\boldsymbol{SD}\mathfrak{F}$ is not countably recognizable.

1.2 Generalized Nilpotency Properties

The main purpose of this section is to prove that all the classes of generalized nilpotent groups (apart from the local ones) we described in the introduction (with a single exception) are countably recognizable. We will begin with the classes of groups which are locally nilpotent.

Among these, the class of Gruenberg groups is the only one which is not countably recognizable. In fact, it is easy to see that any countable locally nilpotent group is a Gruenberg group, and Kargapolov [45], as we saw in the introduction to this chapter, constructed a locally nilpotent group with no abelian non-trivial ascendant subgroups.

Before proving that the class of Baer groups is countably recognizable, we will make some general remarks on subnormal subgroups of uncountable groups. Actually, it was shown by Baer [2] that if X is a subgroup of a group G which is subnormal of defect at most k in $\langle X, U\rangle$ for each countable subgroup U of G, then X is subnormal in G with defect at most $k+1$. Moreover, Baer mentioned that E. Wirsing was able to prove, under the same assumptions, that the defect of the subnormal subgroup X is bounded by k. However, this latter result was not published, and an easy proof of it is presented here.

Let G be a group, and let X be a subgroup of G. Recall that the *series of normal closures* $\{X^{G,n}\}_{n\in\mathbb{N}_0}$ of X in G is defined by putting $X^{G,0} = G$ and

$$X^{G,n+1} = X^{X^{G,n}}$$

for each non-negative integer n. In particular, $X \leqslant X^{G,n}$ for all n, and $X^{G,1} = X^G$, the normal closure of X in G. Note that X is subnormal in G of defect at most k if and only if $X^{G,k} = X$.

Lemma 1.2.13 *Let G be a group, and let X be a subgroup of G. If Y is a countable subgroup of G and $Y \leqslant X^{G,n}$ for some positive integer n, then there exists a countable subgroup U of G such that $Y \leqslant X^{U,n}$.*

Proof — The statement is obvious if $n = 1$, since Y is countable and $X^{G,1} = X^G$. Assume by induction that it holds for all countable subgroups of $X^{G,n}$, for some positive integer n, and let Y be a countable subgroup of $X^{G,n+1}$. As $X^{G,n+1}$ is the normal closure of X in $X^{G,n}$, there exists a countable subgroup V of $X^{G,n}$ such that $Y \leqslant X^V$. By assumption, there is a countable subgroup W of G such that $V \leqslant X^{W,n}$, and hence

$$Y \leqslant X^V \leqslant X^{X^{W,n}} = X^{W,n+1}.$$

The lemma is proved. □

Lemma 1.2.14 *Let G be a group, and let X be a subgroup of G which is properly contained in $X^{G,n}$ for some positive integer n. Then there exists a countable subgroup U of G such that X is a proper subgroup of $X^{U,n}$.*

Proof — The statement is obvious if $n = 1$. Suppose $n > 1$. As $X^{G,n}$ is the normal closure of X in $X^{G,n-1}$, there exists a countable subgroup V of $X^{G,n-1}$ such that $X \neq X^V$. It follows from Lemma 1.2.13 that there is a countable subgroup U of G such that V is contained in $X^{U,n-1}$, and hence

$$X < X^V \leqslant X^{X^{U,n-1}} = X^{U,n},$$

which proves the lemma. □

Theorem 1.2.15 *Let G be a group, and let X be a subgroup of G.*

(a) *If X is subnormal in $\langle X, U\rangle$ for each countable subgroup U of G, then X is subnormal in G.*

(b) *If k is a positive integer and X is subnormal in $\langle X, U\rangle$ of defect at most k for each countable subgroup U of G, then X is subnormal in G of defect at most k.*

Proof — Assume that the subgroup X is not subnormal with defect at most k for some positive integer k. Then $X < X^{G,k}$, and so it follows from Lemma 1.2.14 that G contains a countable subgroup U_k such that $X < X^{U_k,k}$. Then X is not subnormal with defect at most k in $\langle X, U_k\rangle$, and hence part (b) of the statement is proved. Moreover, if X is not subnormal in G, we have $X < X^{G,k}$ for all k, and so the countable subgroup U_k can be chosen for each positive integer k. Then

$$U = \langle U_k \mid k \in \mathbb{N}\rangle$$

is a countable subgroup of G, and $X < X^{U,k}$ for each k, so that X is not subnormal in $\langle X, U\rangle$. The proof of the theorem is complete. □

As a direct consequence of the first part of Theorem 1.2.15, we have:

Corollary 1.2.16 *The class of Baer groups is countably recognizable.*

Our next result shows that also the property of being a Fitting group can be detected from the behaviour of countable subgroups.

Theorem 1.2.17 *The class of Fitting groups is countably recognizable.*

Proof — Let G be a group whose countable subgroups are Fitting groups, and assume for a contradiction that G contains an element x such that the normal closure $\langle x\rangle^G$ is not nilpotent. As the class of nilpotent groups is countably recognizable, there exists a countable non-nilpotent subgroup H of $\langle x\rangle^G$ such that $x \in H$. Moreover, for each element h of H, there exists a finitely generated subgroup $U(h)$ of G such that h belongs to the normal closure $\langle x\rangle^{U(h)}$. Then

$$U = \langle x, U(h) \mid h \in H\rangle$$

is a countable subgroup of G, and H is contained in $\langle x\rangle^U$, so that $\langle x\rangle^U$ is not nilpotent and hence U is not a Fitting group. This contradiction proves the statement. □

The first non-local class of generalized nilpotent groups which has been proved to be countably recognizable is probably that of hypercentral groups. In fact, S.N. Černikov [13] proved in 1950 that a group G is hypercentral if and only if given two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements of G such that $[x_n, y_n] = x_{n+1}$ for all n, there exists a positive integer m such that $x_m = 1$. The countable recognizability of the class of hypercentral groups can also be obtained as a special case of a result on ascending normal series that will be proved in the last section. Note also that Baer [2] showed that also the wider class of $\mathfrak{N}$-groups is countably recognizable.

To complete our analysis of the main classes of generalized nilpotent groups contained in $\boldsymbol{L}\mathfrak{N}$, we consider now the case of $\mathfrak{N}_1$-groups.

Theorem 1.2.18 *The class of $\mathfrak{N}_1$-groups is countably recognizable.*

Proof — Let G be a group whose countable subgroups belong to the class $\mathfrak{N}_1$, and assume for a contradiction that G contains a subgroup X which is not subnormal. Then for each positive integer n, we have that X is properly contained in $X^{G,n}$, and so Lemma 1.2.14 yields that there exists a countable subgroup U_n of the group G such that $X < X^{U_n,n}$. It follows that for every n we can choose a countable subgroup V_n of X such that $V_n^{U_n,n}$ is not contained in X. Put

$$U = \langle U_n \mid n \in \mathbb{N}\rangle \quad \text{and} \quad V = \langle V_n \mid n \in \mathbb{N}\rangle,$$

and consider the countable subgroup $H = \langle U, V\rangle$ of G. Then H is an $\mathcal{N}_1$-group, so that V is subnormal in H, with defect k, say, and hence $V^{H,k} = V \leqslant X$. But $V^{H,k}$ contains the subgroup $V_k^{U_k,k}$, and so it cannot be contained in X. This contradiction shows that G is an $\mathcal{N}_1$-group. □

We now move on the main classes of generalized nilpotent groups which are not contained in $\boldsymbol{L}\mathfrak{N}$. We begin by proving that the class of hypocentral groups is countably recognizable and, in order to do so, we need the following lemma.

Lemma 1.2.19 *Let G be a group, and let N be a non-trivial normal subgroup of G such that $[N, G] = N$. Then there exists a countable subgroup H of G such that $N \cap H \neq \{1\}$ and $[N \cap H, H] = N \cap H$.*

Proof — Let x be any non-trivial element of N. Then there exist countable subgroups X of N and Y of G such that x belongs to $[X, Y]$. Then $H_1 = \langle X, Y\rangle$ is countable. Assume that a countable subgroup H_n has been chosen. For each element h of $N \cap H_n$, let $X_n(h)$ and $Y_n(h)$ be countable subgroups of N and G, respectively, such that h lies in $[X_n(h), Y_n(h)]$, and put

$$H_{n+1} = \langle X_n(h), Y_n(h) \mid h \in H_n\rangle.$$

Then $(H_n)_{n\in\mathbb{N}}$ is an ascending sequence of countable subgroups, and so

$$H = \bigcup_{n\in\mathbb{N}} H_n$$

is likewise a countable subgroup of G with $N \cap H \neq \{1\}$.

If a is any element of $H \cap N$, there is a positive integer m such that $a \in H_m$, and hence a belongs to $[X_m(a), Y_m(a)] \leqslant [N \cap H, H]$. Therefore $[N \cap H, H] = N \cap H$, and the statement is proved. □

Theorem 1.2.20 *The class of hypocentral groups is countably recognizable.*

Proof — Let G be a group such that all its countable subgroups are hypocentral, and assume for a contradiction that G is not hypocentral. Then $\gamma_\tau(G) = \gamma_{\tau+1}(G) \neq \{1\}$ for a suitable ordinal τ. It follows from Lemma 1.2.19 that there exists a countable subgroup H of G such that $\gamma_\tau(G) \cap H \neq \{1\}$ and $[\gamma_\tau(G) \cap H, H] = \gamma_\tau(G) \cap H$, a contradiction since H is hypocentral. □

The class of residually nilpotent groups was discovered to be countably recognizable by B.H. Neumann [56]. Here we obtain this information as a special case of a result on sequences of varieties.

Let W be a set of words in countably many variables. If G is any group, the *verbal subgroup* of G determined by W is defined as the subgroup $W(G)$ generated by all values of words in W on elements of G. Recall also that the *variety* determined by W is the class $\mathfrak{B}(W)$ consisting of all groups G such that each word in W reduces to the identity when the variables are replaced by arbitrary elements of G; thus a group G belongs to $\mathfrak{B}(G)$ if and only if $W(G) = \{1\}$. Clearly, every variety is ***S***, ***H***, ***L*** and ***R***-closed; on the other hand, it is well-known that a group class is a variety if and only if it is ***H*** and ***R***-closed (see for instance [61] Part 1, Theorem 1.13).

Lemma 1.2.21 *Let $(\mathfrak{B}_n)_{n\in\mathbb{N}}$ be a countable collection of varieties made by groups, and let*

$$\mathfrak{B} = \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n.$$

Then the class $\boldsymbol{R}\mathfrak{B}$ of residually $\mathfrak{B}$-groups is countably recognizable.

Proof — For each positive integer n, let W_n be a set of words determining the variety $\mathfrak{B}_n$. Consider a group G whose countable subgroups are residually $\mathfrak{B}$, and let x be any element of the $\mathfrak{B}$-residual $\rho^*_{\mathfrak{B}}(G)$ of G. Clearly,

$$\rho^*_{\mathfrak{B}}(G) = \bigcap_{n\in\mathbb{N}} W_n(G),$$

where $W_n(G)$ is the verbal subgroup of G determined by W_n. Then x belongs to $W_n(G)$ for each positive integer n, and so there exists a countable subgroup H_n of G such that x lies also in the verbal subgroup $W_n(H_n)$. Clearly

$$H = \langle H_n \mid n \in \mathbb{N}\rangle$$

is a countable subgroup of G, and $x \in W_n(H)$ for all n, so that x is an element of the $\mathfrak{B}$-residual of H. As H is a residually $\mathfrak{B}$-group, it follows that $x = 1$, and hence G is residually $\mathfrak{B}$. □

Theorem 1.2.22 *The class $\boldsymbol{R}\mathfrak{N}$ of residually nilpotent groups is countably recognizable.*

Proof — For each positive integer n, the class $\mathfrak{N}_n$ of all nilpotent groups of class at most n is a variety, and hence the statement is a special case of Lemma 1.2.21. □

More Nilpotency Properties

The aim of this paragraph is to study the countable recognizability of classes of groups which are close to be nilpotent (the obstruction being, for instance, a finite section; on this subject see also [39]). The first of these classes is that consisting of all groups containing a nilpotent subgroup of finite index.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be group classes. We shall denote by $\mathfrak{X}\mathfrak{Y}$ the class consisting of all ($\mathfrak{X}$-by-$\mathfrak{Y}$)-groups, i.e. the class of all groups G containing a normal $\mathfrak{X}$-subgroup N such that the factor group G/N belongs to $\mathfrak{Y}$. In particular, if $\mathfrak{X}$ is any group class and $\mathfrak{F}$ is the class of finite groups, $\mathfrak{X}\mathfrak{F}$ is the class of all groups containing a normal $\mathfrak{X}$-subgroup of finite index. Thus $\mathfrak{S}\mathfrak{F}$, $\mathfrak{N}\mathfrak{F}$ and $\mathfrak{A}\mathfrak{F}$ are the classes of soluble-by-finite, nilpotent-by-finite and abelian-by-finite groups, respectively.

Our next theorem provides a number of countably recognizable group classes. It shows in particular that if $\mathfrak{X}$ is a class of groups such that $\boldsymbol{S}\mathfrak{X} = \boldsymbol{L}\mathfrak{X} = \mathfrak{X}$, then the class $\mathfrak{X}\mathfrak{F}$ is countably recognizable. In order to prove this result, we need the following result due to Baer, for a proof of which we refer to [46], Proposition 1.K.2.

Lemma 1.2.23 *Let $\mathfrak{X}$ be an $\boldsymbol{S}$-closed class of groups, and let G be a group in which every finitely generated subgroup contains an $\mathfrak{X}$-subgroup of index at most k, where k is a fixed positive integer. Then G contains a subgroup of index at most k which is locally $\mathfrak{X}$.*

Theorem 1.2.24 *Let $(\mathfrak{X}_n)_{n\in\mathbb{N}}$ be a collection of group classes which are $\boldsymbol{S}$ and $\boldsymbol{L}$-closed, and let*

$$\mathfrak{X} = \bigcup_{n\in\mathbb{N}} \mathfrak{X}_n.$$

Then the class $\mathfrak{X}\mathfrak{F}$ is countably recognizable.

Proof — Let G be a group in which every countable subgroup contains an $\mathfrak{X}$-subgroup of finite index, and assume for a contradiction that G does not belong to the class $\mathfrak{X}\mathfrak{F}$. In particular, G is not in $\mathfrak{X}_n\mathfrak{F}$ for any n, and so it follows from Lemma 1.2.23 that for all positive integers n and k there exists a finitely generated subgroup $E_{n,k}$ of G which has no $\mathfrak{X}_n$-subgroups of index at most k. Clearly,

$$E = \langle E_{n,k} \mid n, k \in \mathbb{N} \rangle$$

is a countable subgroup of G, and hence it contains an $\mathfrak{X}$-subgroup X of finite index h. Let m be a positive integer such that X lies in $\mathfrak{X}_m$. Then $X \cap E_{m,h}$ is an $\mathfrak{X}_m$-subgroup of $E_{m,h}$ and

$$|E_{m,h} : X \cap E_{m,h}| \leqslant h.$$

This contradiction proves the statement. □

Corollary 1.2.25 *The group classes $\mathfrak{A}\mathfrak{F}$ and $\mathfrak{N}\mathfrak{F}$ are countably recognizable.*

Proof — For each positive integer n, the class $\mathfrak{N}_n$ is obviously $\boldsymbol{S}$ and $\boldsymbol{L}$-closed, and so also the class $\mathfrak{N}_n\mathfrak{F}$ is countably recognizable by Theorem 1.2.24. In particular, the class $\mathfrak{A}\mathfrak{F}$ is countably recognizable. Moreover, as

$$\mathfrak{N} = \bigcup_{n \in \mathbb{N}} \mathfrak{N}_n,$$

another application of Theorem 1.2.24 yields that $\mathfrak{N}\mathfrak{F}$ is a countably recognizable group class. □

Of course, an argument similar to that used in the proof of Corollary 1.2.25 shows that also the class $\mathfrak{S}\mathfrak{F}$ is countably recognizable.

Corollary 1.2.26 *The class $\mathfrak{F}\mathfrak{A}\mathfrak{F}$ of all finite-by-abelian-by-finite groups is countably recognizable.*

Proof — For each positive integer n, let $\mathfrak{F}_n$ be the class of all finite groups of order at most n. Then the class $\mathfrak{F}_n\mathfrak{A}$ is $\boldsymbol{S}$ and $\boldsymbol{L}$-closed for all n, and

$$\mathfrak{F}\mathfrak{A} = \bigcup_{n \in \mathbb{N}} \mathfrak{F}_n\mathfrak{A}.$$

Hence the class $\mathfrak{F}\mathfrak{A}\mathfrak{F}$ is countably recognizable by Theorem 1.2.24. □

Notice that the above statement can also be obtained as a consequence of a combinatorial result. In fact, it was proved in [14] that a group G is finite-by-abelian-by-finite if and only it has the *permutational property*, i.e. if and only if there exists a positive integer n such that for all elements $x_1, \ldots, x_n$ of G there is a non-trivial permutation σ of $\{1, \ldots, n\}$ such that

$$x_1 x_2 \ldots x_n = x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}.$$

Our purpose is now to show that the class $\mathfrak{F}\mathfrak{N}$ of all finite-by-nilpotent groups is countably recognizable. Of course, a group G is finite-by-nilpotent if and only if there is a positive integer k such that $\gamma_k(G)$ is finite. If G is any group, we shall denote by $\mathrm{Count}(G)$ the set of all countable subgroups of G.

Lemma 1.2.27 *Let G be a group, and let*

$$\delta : \mathrm{Count}(G) \longrightarrow \mathbb{N}$$

be an increasing function. Then there exists a positive integer m such that $\delta(X) \leqslant m$ for each countable subgroup X of G.

Proof — Assume for a contradiction that the function δ is unbounded, so that for each positive integer n there exists a countable subgroup X_n of G such that $\delta(X_n) > n$. Then

$$X = \langle X_n \mid n \in \mathbb{N} \rangle$$

is a countable subgroup of G, and $\delta(X_n) \leqslant \delta(X)$ for each n, which is obviously impossible. □

Theorem 1.2.28 *Let $\mathfrak{X}$ be a countably recognizable group class which is **S** and $\boldsymbol{R}_0$-closed. Then the class $\mathfrak{F}\mathfrak{X}$ is countably recognizable.*

Proof — Let G be a group whose countable subgroups belong to $\mathfrak{F}\mathfrak{X}$. If X is any countable subgroup of G, it follows from the $\boldsymbol{R}_0$-closure of $\mathfrak{X}$ that the $\mathfrak{X}$-residual $\rho^*_{\mathfrak{X}}(X)$ of X is finite and the factor group $X/\rho^*_{\mathfrak{X}}(X)$ lies in $\mathfrak{X}$. Moreover, if X and Y are countable subgroups of G such that $X \leqslant Y$, we have that $X/X \cap \rho^*_{\mathfrak{X}}(Y)$ belongs to $\mathfrak{X}$, and hence $\rho^*_{\mathfrak{X}}(X) \leqslant \rho^*_{\mathfrak{X}}(Y)$. Therefore the map

$$\delta : X \in \mathrm{Count}(G) \longmapsto |\rho^*_{\mathfrak{X}}(X)| \in \mathbb{N}$$

is bounded by Lemma 1.2.27. Let m be the smallest positive integer such that $\rho^*_{\mathfrak{X}}(X)$ has order at most m for each countable subgroup X of G, and choose a countable subgroup H such that $|\rho^*_{\mathfrak{X}}(H)| = m$.

Let U be any countable subgroup of G containing H. Then $U/\rho^*_{\mathfrak{X}}(U)$ is an $\mathfrak{X}$-group and $|\rho^*_{\mathfrak{X}}(U)| \leqslant m$; moreover $\rho^*_{\mathfrak{X}}(H) \leqslant \rho^*_{\mathfrak{X}}(U)$, and hence $\rho^*_{\mathfrak{X}}(U) = \rho^*_{\mathfrak{X}}(H)$. In particular, if g is any element of G, the subgroup $\langle H, g\rangle$ is countable, and so the subgroup $\rho^*_{\mathfrak{X}}(H) = \rho^*_{\mathfrak{X}}(\langle H, g\rangle)$ is normalized by g. Therefore $\rho^*_{\mathfrak{X}}(H)$ is normal in G. Let $U/\rho^*_{\mathfrak{X}}(H)$ be any countable subgroup of $G/\rho^*_{\mathfrak{X}}(H)$. Then U is countable, so that $\rho^*_{\mathfrak{X}}(U) = \rho^*_{\mathfrak{X}}(H)$ and hence $U/\rho^*_{\mathfrak{X}}(H)$ belongs to $\mathfrak{X}$. Hence $G/\rho^*_{\mathfrak{X}}(H)$ is an $\mathfrak{X}$-group and G belongs to $\mathfrak{F}\mathfrak{X}$. □

Corollary 1.2.29 *The class $\mathfrak{F}\mathfrak{N}$ is countably recognizable.*

The combination of the above statement with Theorem 1.2.22 yields that also the class $\mathfrak{F}(\boldsymbol{R}\mathfrak{N})$, consisting of all groups whose nilpotent residual is finite, is countably recognizable.

Two relevant theorems of Baer and P. Hall prove that for a group G there exists a positive integer h such that the subgroup $\gamma_h(G)$ is finite if and only if the index $|G : Z_k(G)|$ is finite for some integer $k \geqslant 0$ (see [61] Part 1, p.113 and p.117). Therefore Corollary 1.2.29 shows also that the property of being finite over some term with finite ordinal type of the upper central series is countably recognizable. Observe here that if G is a group such that $\gamma_h(X)$ is finite for each countable subgroup X of G and for a fixed positive integer h, then also $\gamma_h(G)$ is finite, because

$$\gamma_h(G) = \big\langle [x_1, \dots, x_h] \;\big|\; x_1, \dots, x_h \in G \big\rangle.$$

It can also be proved that the class of all groups G which are finite over $Z_k(G)$ for some fixed positive integer k is countably recognizable.

Let $\mathfrak{B}$ be a variety, and let W be a set of words such that $\mathfrak{B}(W) = \mathfrak{B}$. Recall that a normal subgroup N of a group G is said to be $\mathfrak{B}$-*marginal* if

$$\theta(g_1, \dots, g_{i-1}, g_i x, g_{i+1}, \dots, g_n) = \theta(g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_n)$$

for each word $\theta \in W$ in n variables and for all elements $g_1, \dots, g_n$ of G and x of N. Every group G contains a largest $\mathfrak{B}$-marginal subgroup, which is denoted by $W^*(G)$, and G belongs to $\mathfrak{B}$ if and only if $W^*(G) = G$.

Theorem 1.2.30 *Let $\mathfrak{B}$ be any variety, the class of groups containing a $\mathfrak{B}$-marginal subgroup of finite index is countably recognizable.*

Proof — Let W be a set of words such that $\mathfrak{B}(W) = \mathfrak{B}$, and let G be a group such that the index $|X : W^*(X)|$ is finite for every countable subgroup X. If X and Y are subgroups of G such that $X \leqslant Y$, we have obviously $X \cap W^*(Y) \leqslant W^*(X)$, and so the map

$$\delta : X \in \mathrm{Count}(G) \longmapsto |X : W^*(X)| \in \mathbb{N}$$

is bounded by Lemma 1.2.27. Let m be a positive integer with the property that $|X : W^*(X)| \leqslant m$ for each countable subgroup X of G. Assume for a contradiction that $W^*(G)$ has infinite index in G, so that there are elements $y_1, \dots, y_m, y_{m+1}$ such that $y_i W^*(G) \neq y_j W^*(G)$ if $i \neq j$. Then for all $i \neq j$ there exists a word $\theta(i,j) \in W$ in $s(i,j)$ variables such that

$$\begin{aligned} \theta(i,j)\big(g_1^{(i,j)}, \dots, g_{k-1}^{(i,j)}, g_k^{(i,j)} y_i^{-1} y_j, g_{k+1}^{(i,j)}, \dots, g_{s(i,j)}^{(i,j)}\big) \\ \neq \theta(i,j)\big(g_1^{(i,j)}, \dots, g_{k-1}^{(i,j)}, g_k^{(i,j)}, g_{k+1}^{(i,j)}, \dots, g_{s(i,j)}^{(i,j)}\big) \end{aligned}$$

for suitable elements $g_1^{(i,j)}, \dots, g_{s(i,j)}^{(i,j)}$ of G. Therefore

$$H = \langle y_1, \dots, y_m, y_{m+1}, g_1^{(i,j)}, \dots, g_{s(i,j)}^{(i,j)} \mid i \neq j \rangle$$

is a countable subgroup of G, and $y_i W^*(H) \neq y_j W^*(H)$ for $i \neq j$, which is impossible because $|H : W^*(H)| \leqslant m$. □

If k is any positive integer, and W_k is the set consisting of the single word $[x_1, \dots, x_k]$, then $\mathfrak{B}(W_k)$ is the variety of all nilpotent groups of class at most k and $W_k^*(G) = Z_k(G)$ for any group G. Therefore Theorem 1.2.30 has the following consequence.

Corollary 1.2.31 *If k is any positive integer, the class of groups which are finite over the k-th term of their upper central series is countably recognizable.*

It has been recently proved that a group G is finite over its hypercentre $\overline{Z}(G)$ if and only if G contains a finite normal subgroup N

such that G/N is hypercentral (see [18]). As the class of hypercentral groups is countably recognizable, it follows from Theorem 1.2.28 that also the class of groups which are finite over the hypercentre is countably recognizable.

Further Countably Recognizable Classes

For every group G, let $\Xi(G)$ be a set of subgroups of G containing the identity subgroup $\{1\}$; the elements of $\Xi(G)$ are called Ξ-*subgroups* of G. We shall say that Ξ is an *embedding subgroup property* if

$$\big(\Xi(G)\big)^{\varphi} = \Xi(G^*)$$

for every group isomorphism $\varphi : G \longrightarrow G^*$ and X belongs to $\Xi(Y)$, whenever $X \leqslant Y \leqslant G$ and $X \in \Xi(G)$. An embedding property Ξ is called *absolute* if it holds for each subgroup X^* of a group G^* such that $X^* \simeq X$, where X is a Ξ-subgroup of some group G; in particular, if $\mathfrak{X}$ is any group class, the property for a subgroup to belong to $\mathfrak{X}$ is an absolute property. On the other hand, there are many relevant embedding properties (like for instance normality and subnormality) which are not absolute.

An embedding property Ξ is said to have *countable character* when a subgroup X of an arbitrary group G is a Ξ-subgroup if and only if Ξ holds for all countable subgroups of X. In particular, if Ξ is an embedding property of countable character, and X is a Ξ-subgroup of a group G, then all subgroups of X have the property Ξ. Note also that, if $\mathfrak{X}$ is an $\boldsymbol{S}$-closed group class, the property for a subgroup to be in $\mathfrak{X}$ is an absolute property which has countable character if and only if $\mathfrak{X}$ is countably recognizable.

The first result of this paragraph shows in particular that if Ξ is an embedding property of countable character, then the class of group with a non-trivial normal Ξ-subgroup is countably recognizable.

Lemma 1.2.32 *Let Ξ be an embedding property of countable character, and let k be a positive integer. Then the class of all groups containing a non-trivial subnormal Ξ-subgroup of defect at most k is countably recognizable.*

Proof — Let G be a non-trivial group which has no non-trivial subnormal Ξ-subgroups of defect at most k. If $x \neq 1$ is an element of G, the subgroup $\langle x\rangle^{G,k}$ is subnormal in G with defect at most k, and so it cannot be a Ξ-subgroup. Since Ξ has countable character, there exists a countable subgroup Y_k of $\langle x\rangle^{G,k}$ for which the property Ξ does not hold, and it follows from Lemma 1.2.13 that G contains a countable subgroup U_k such that $Y_k \leqslant \langle x\rangle^{U_k,k}$. Note that Ξ does not hold for $\langle x\rangle^{U_k,k}$, because this latter subgroup contains Y_k. Put

$$H_1 = \langle x, U_k \mid k \in \mathbb{N}\rangle,$$

and suppose that a countable subgroup H_n of G has been defined for some positive integer n. For each element $h \neq 1$ of H_n and for each non-negative integer k, the subnormal subgroup $\langle h\rangle^{G,k}$ cannot have the property Ξ; as above there exists a countable subgroup $U_k(h)$ of G such that $\langle h\rangle^{U_k(h),k}$ is not a Ξ-subgroup, and we can put

$$H_{n+1} = \langle h, U_k(h) \mid h \in H_n,\ k \in \mathbb{N}\rangle.$$

In this way a chain $(H_n)_{n\in\mathbb{N}}$ of countable subgroups of G has been defined. Then

$$H = \bigcup_{n\in\mathbb{N}} H_n$$

is a countable subgroup of G. Moreover, for each element h of H the property Ξ does not hold for the subnormal subgroup $\langle h\rangle^{H,k}$ of H, and hence H has no non-trivial subnormal Ξ-subgroups of defect at most k. The statement is proved. □

It should be remarked here that the class of all groups having an abelian non-trivial normal subgroup is not ***L***-closed. In fact, consider an infinite collection $(p_n)_{n\in\mathbb{N}}$ of prime numbers, and for each positive integer n let C_n be a group of order p_n. Put $G_1 = C_1$ and for every n consider the standard wreath product $G_{n+1} = C_{n+1} \wr G_n$. Then

$$G = \bigcup_{n\in\mathbb{N}} G_n$$

is a periodic locally soluble group with trivial Hirsch-Plotkin radical HP(G) (i.e. its largest locally nilpotent normal subgroup), because

$$HP(G_n) \cap HP(G_{n+1}) = \{1\}$$

for all n. In particular, G cannot contain abelian non-trivial normal subgroups.

An argument similar to that used in the proof of the Lemma 1.2.32 shows that a corresponding result holds when there is no bound for the subnormal defect.

Lemma 1.2.33 *Let Ξ be an embedding property of countable character. Then the class of all groups containing a non-trivial subnormal Ξ-subgroup is countably recognizable.*

Lemma 1.2.34 *Let Ξ be an embedding property which is inherited by homomorphic images and such that the class of groups containing a non-trivial normal Ξ-subgroup is countably recognizable. Then the class of groups admitting an ascending normal series with Ξ-factors is countably recognizable.*

Proof — Let G be a group whose countable subgroups have an ascending normal series with Ξ-factors, and let N be any proper normal subgroup of G. If H/N is a countable (non-trivial) subgroup of G/N, there exists a countable subgroup X of H such that $H = XN$. Since $X \cap N \neq X$, there exists a normal subgroup Y of X properly containing $X \cap N$ such that $Y/X \cap N$ is a Ξ-subgroup of $X/X \cap N$. Therefore $Y \cap N = X \cap N$ and $YN/N \simeq Y/Y \cap N$ is a non-trivial normal Ξ-subgroup of H/N. It follows from the hypotheses that also G/N contains a non-trivial normal Ξ-subgroup, and hence G has an ascending normal series with Ξ-factors. □

Corollary 1.2.35 *Let Ξ be an embedding property of countable character which is inherited by homomorphic images. Then the class of groups admitting an ascending normal series with Ξ-factors is countably recognizable.*

Proof — It follow from Lemma 1.2.32 that the class of all groups containing a non-trivial normal Ξ-subgroup is countably recognizable, and so the statement is a consequence of Lemma 1.2.34. □

Of course, the above results can be specialized to the case of the absolute property determined by a group class. Recall that if $\mathfrak{X}$ is any class of groups, a group is said to be *hyper*-$\mathfrak{X}$ if it has an ascending normal series whose factors belong to $\mathfrak{X}$.

Corollary 1.2.36 *Let $\mathfrak{X}$ be an $\boldsymbol{S}$ and $\boldsymbol{H}$-closed group class. If $\mathfrak{X}$ is countably recognizable, then also the class of hyper-$\mathfrak{X}$ groups is countably recognizable.*

This corollary shows in particular that the class of hyperabelian groups and that of hypercyclic groups are countably recognizable. Moreover, if we choose as $\mathfrak{X}$ the class $\boldsymbol{L}\mathfrak{N}$ of all locally nilpotent groups, it follows that also the class of radical groups is countably recognizable; here a group is called *radical* if it has an ascending (normal) series with locally nilpotent factors. Finally, as the class $\mathfrak{A} \cup \mathfrak{F}$ is clearly countably recognizable, we have that groups with an ascending normal series whose factors are either abelian or finite form a countably recognizable group class.

Note that Corollary 1.2.36 also proves that the class of hyperfinite groups is countably recognizable. A group G is called a *Specht group* if it admits an ascending chain

$$\{1\} = G_0 < G_1 < \ldots < G_\alpha < G_{\alpha+1} < \ldots < G_\tau = G$$

such that G_α has finite index in $G_{\alpha+1}$ for all $\alpha < \tau$. These groups were introduced by W. Specht [74]. An easy transfinite induction proves that any Specht group is locally finite, and it is also clear that all countable locally finite groups have the Specht property. On the other hand, Hickin and Phillips [38] constructed an uncountable locally finite p-group which is not a Specht group. In particular, the class of Specht groups is not countably recognizable.

A group G is said to be *subsoluble* if it has an ascending series Σ with abelian factors such that all terms of Σ are subnormal. It is easy to see that in any group G the subgroup generated by all abelian subnormal subgroups is the largest normal Baer subgroup (the *Baer radical* of G); then a group is subsoluble if and only if it has an ascending normal series whose factors are Baer groups. Thus an application of Corollary 1.2.16 and Corollary 1.2.36 yields the followiing result.

Corollary 1.2.37 *The class of subsoluble groups is countably recognizable.*

Let Ξ be an embedding property. If $\mathfrak{X}$ is any class of groups, a new embedding property $\Xi \vee \mathfrak{X}$ can be defined, by requiring that a subgroup X of a group G has such property if and only if either X is a Ξ-subgroup of G or it belongs to $\mathfrak{X}$. Note that if Ξ has countable character and $\mathfrak{X}$ is $\boldsymbol{S}$-closed and countably recognizable, then also $\Xi \vee \mathfrak{X}$ has countable character. Therefore Corollary 1.2.35 has the following interesting consequence.

Corollary 1.2.38 *Let Ξ be an embedding property of countable character which is inherited by homomorphic images, and let $\mathfrak{X}$ be an $\boldsymbol{S}$ and $\boldsymbol{H}$-closed* *countably recognizable group class. Then the class of all groups admitting an ascending normal series with* $(\Xi \vee \mathfrak{X})$*-factors is countably recognizable.*

As a special case, we obtain the following result.

Corollary 1.2.39 *The class of all groups admitting a normal series whose factors are either central or finite is countably recognizable.*

It is easy to prove that the property of having a non-trivial central subgroup is countably recognizable. Next lemma shows that the same is true for the property of having a non-trivial normal abelian subgroup.

Lemma 1.2.40 *A group* G *contains an abelian non-trivial normal subgroup if and only if there exists an element* $a \neq 1$ *such that* $[x, a, a^y] = 1$ *for all elements* x, y *of* G.

Proof — Assume first that G contains an abelian non-trivial normal subgroup A, and let $a \neq 1$ be an element of A. If x and y are arbitrary elements of G, then $[x, a]$ and a^y belong to A, and hence $[x, a, a^y] = 1$.

Conversely, suppose that $a \neq 1$ is an element of G satisfying the condition of the statement. It can be assumed that $Z(G) = \{1\}$, so that there exists an element x of G such that $ax \neq xa$. As $[x, a, a^y] = 1$ for all $y \in G$, the commutator $[x, a]$ is a non-trivial element of $Z(\langle a \rangle^G)$, and hence $Z(\langle a \rangle^G)$ is an abelian non-trivial normal subgroup of G. □

We finally consider the class of residually soluble groups and that of hypoabelian groups. Recall that a group G is called *hypoabelian* if it has a descending (normal) series with abelian factors; it follows from the definition that a group is hypoabelian if and only if its derived series terminates with the identity subgroup (after infinitely many steps, eventually). Of course, every residually soluble group is hypoabelian.

Theorem 1.2.41 *The class* $\boldsymbol{R}\mathfrak{S}$ *of residually soluble groups is countably recognizable.*

Proof — For each positive integer n, the class $\mathfrak{S}_n$ of all soluble groups of derived length at most n is a variety, and hence the statement is a special case of Lemma 1.2.21. □

Our next lemma shows that the class consisting of all imperfect groups, with the addition of the identity group, is countably recognizable.

Lemma 1.2.42 *Let* G *be non-trivial group* G *whose countable non-trivial subgroups contain properly their commutator subgroup. Then* $G' \neq G$.

Proof — Assume for a contradiction that the group G is perfect, and let $x \neq 1$ be an element of G. Then there exists a finitely generated (and so countable) subgroup X_1 of G such that x belongs to X_1'. Suppose that a countable subgroup X_n has been chosen for some positive integer n. As G is perfect, each element of X_n is the product of finitely many commutators, and hence there exists a countable subgroup X_{n+1} of G such that X_n is contained in X_{n+1}'. Consider now the sequence of countable subgroups $(X_n)_{n\in\mathbb{N}}$. Then

$$X = \bigcup_{n\in\mathbb{N}} X_n$$

is a countable subgroup of G and $X' = X$. This contradiction proves the lemma. □

Our last result is a direct consequence of Lemma 1.2.42.

Theorem 1.2.43 *The class of hypoabelian groups is countably recognizable.*

Proof — The statement follows from Lemma 1.2.42, as clearly a group is hypoabelian if and only if all its non-trivial subgroups have a proper commutator subgroup. □

1.3 The Class of Minimax Groups

Recall that a group G satisfies the *minimal condition* on subgroups if there are no infinite descending chains of subgroups, and G satisfies the *maximal condition* on subgroups if it admits no infinite ascending chains of subgroups. It is almost obvious that both the class of groups satisfying the minimal condition and that of groups satisfying the maximal condition on subgroups are countably recognizable. A group G is called *minimax* if it has a series of finite length

$$\{1\} = G_0 < G_1 < \ldots < G_n = G$$

each of whose factors satisfies either the minimal or the maximal condition on subgroups. The structure of soluble minimax groups has been described by Robinson (see [61] Part 2, Chapter 10).

As the class of soluble groups of finite rank is countably recognizable, and all soluble groups of finite rank are countable, it follows that any group whose countable subgroups are soluble and minimax is countable, and so also minimax. Therefore the class of soluble minimax groups is countably recognizable. However, the situation is much more complicated in the insoluble case, and in particular V.N. Obraztsov [57] constructed an uncountable group satisfying the minimal condition on subgroups. The aim of this section is to prove that *the class of minimax groups is countably recognizable.*

Denote by $\vee$ and $\wedge$ the minimal and the maximal condition on subgroups, respectively, and for a positive integer n let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be any n-tuple whose entries belong to the set $\{\vee, \wedge\}$. We shall say that a group G is *minimax of type* σ (or σ-*minimax*) if it has a σ-series, i.e. a finite series

$$\{1\} = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_n = G$$

of length n such that the factor group G_i/G_{i-1} satisfies the condition σ_i for each positive integer $i \leqslant n$. Clearly, σ-minimax groups are minimax and every minimax group is σ-minimax for some σ, but for a minimax group the minimax type is not uniquely determined. Note also that any abelian minimax group is $(\wedge, \vee)$-minimax. We point out finally that the class of σ-minimax groups is closed with respect to subgroups and homomorphic images, and that if H and K are normal subgroups of a group G such that both G/H and G/K

are σ-minimax, then also the factor group $G/H \cap K$ is σ-minimax.

Let G be a minimax group of type $\sigma = (\sigma_1, \dots, \sigma_n)$. A subnormal subgroup X of G is called a *σ-subgroup* if it satisfies σ_1 and there exists a series

$$X = X_1 \leqslant \dots \leqslant X_n = G$$

such that X_i/X_{i-1} satisfies σ_i for each $i = 2, \dots, n$. Of course, a normal subgroup N of a group G is a σ-subgroup if and only if it satisfies σ_1 and the factor group G/N is σ'-minimax, where we put $\sigma' = (\sigma_2, \dots, \sigma_n)$.

Recall that if $\mathfrak{X}$ is a class of groups, the *residual* of a group G with respect to $\mathfrak{X}$ is the intersection of all normal subgroups N of G such that G/N belongs to $\mathfrak{X}$.

Lemma 1.3.44 *Let G be a σ-minimax group for some $\sigma = (\sigma_1, \dots, \sigma_n)$, where $n \geqslant 2$ and $\sigma_1 = \vee$. Then G contains a normal subgroup N satisfying the minimal condition on subgroups and such that the factor group G/N is $(\sigma_2, \dots, \sigma_n)$-minimax.*

Proof — Let

$$\{1\} = G_0 \leqslant G_1 \leqslant \dots \leqslant G_n = G$$

be a σ-series of G. As the statement is obvious if $n = 2$, we may suppose $n \geqslant 3$. It can be assumed by induction on n that G_{n-1} has a normal subgroup K with the minimal condition and such that G_{n-1}/K is σ'-minimax, where $\sigma' = (\sigma_2, \dots, \sigma_{n-1})$. As K contains the residual R of G_{n-1} with respect to the class of σ'-minimax groups, it follows that also the group G_{n-1}/R is σ'-minimax. Clearly, R is a normal subgroup of G and G/R is a $(\sigma_2, \dots, \sigma_n)$-minimax group, and so the proof is complete. □

Next lemma is the crucial point in the proof of countably recognizablity of minimax groups.

Lemma 1.3.45 *Let G be a group whose countable subgroups are σ-minimax for a fixed minimax type $\sigma = (\sigma_1, \dots, \sigma_n)$. Then G is minimax.*

Proof — Assume for a contradiction that the statement is false, and choose a counterexample for which the minimax type σ has shortest length n. Then $n > 1$, because the class of groups with the minimal condition and that of groups satisfying the maximal condition are countably recognizable.

Put $\sigma' = (\sigma_2, \ldots, \sigma_n)$, and suppose first $\sigma_1 = \vee$. Let $\mathcal{C}$ be the set of all countable subgroups of G, and for each element X of $\mathcal{C}$ denote by X_0 the residual of X with respect to the class of σ'-minimax groups. Write

$$G_0 = \bigcup_{X \in \mathcal{C}} X_0.$$

If X and Y are arbitrary elements of $\mathcal{C}$, we have

$$\langle X_0, Y_0 \rangle \leqslant \langle X, Y \rangle_0$$

and hence G_0 is a subgroup of G, which is obviously normal. Let H be any countable subgroup of G_0, and for each element h of H choose a countable subgroup $X(h)$ of G such that h belongs to $X(h)_0$. Then

$$K = \langle X(h) \mid h \in H \rangle$$

is a countable subgroup of G and

$$H \leqslant \langle X(h)_0 \mid h \in H \rangle \leqslant K_0.$$

Moreover, since K is σ-minimax, it follows from Lemma 1.3.44 that K_0 satisfies the minimal condition on subgroups, and hence also H has the minimal condition. Therefore G_0 satisfies the minimal condition on subgroups. Let V/G_0 be any countable subgroup of G_0, and let W be a countable subgroup of G such that $V = G_0W$. Then V/G_0 is a homomorphic image of W/W_0, and so it is a σ'-minimax group by Lemma 1.3.44. It follows now from the minimal assumption on n that the factor group G/G_0 is minimax, so that G itself is minimax, and this contradiction shows that $\sigma_1 = \wedge$.

Let K be any countable subgroup of G, and let $\mathcal{E}(K)$ be the set of all σ-subgroups of K. Clearly, $\mathcal{E}(K)$ is countable, because all its elements are finitely generated. For each element E of $\mathcal{E}(K)$, we will define a suitable countable subgroup $U_1(E)$ of G containing K.

If E is not subnormal in G of defect at most $n-1$, it follows Lemma 1.2.14 that there exists a countable subgroup V of G containing K such that $E^{V,n-1} \neq E$, and in this case we put $U_1(E)=V$. Suppose now that E is subnormal in G of defect at most $n-1$, so that $E^{G,n-1} = E$. As the group G is not minimax, there is a non-negative integer $i < n-1$ such that $E^{G,i}/E^{G,i+1}$ is not minimax, and so the minimal assumption on n yields that $E^{G,i}$ contains a countable subgroup X such that $XE^{G,i+1}/E^{G,i+1}$ is not σ'-minimax. In this situation, Lemma 1.2.13 can be applied to obtain a countable subgroup W containing K such that X lies in $E^{W,i}$. Note that the group $E^{W,i}/E^{W,i+1}$ is not σ'-minimax, because it admits a section isomorphic to $XE^{G,i+1}/E^{G,i+1}$. In this second case, we put $U_1(E)=W$.

As the subgroup

$$U_1 = \big\langle U_1(E) \mid E \in \mathcal{E}(K)\big\rangle$$

is clearly countable, the above argument can be iterated to construct an ascending sequence $(U_n)_{n\in\mathbb{N}}$ of countable subgroups of G. Then

$$U_\infty = \bigcup_{n\in\mathbb{N}} U_n$$

is a countable subgroup of G, so that it is σ-minimax and we may consider an element E_∞ in the set $\mathcal{E}(U_\infty)$. In particular, E_∞ is a finitely generated subgroup of U_∞, and hence it is contained in U_m for some positive integer m. Moreover, E_∞ is subnormal in U_∞ of defect $\leqslant n-1$, and so it follows from the definition of U_{m+1} that E_∞ must be even subnormal in G of defect at most $n-1$. Therefore the group

$$E_\infty^{U_{m+1}(E_\infty),i}/E_\infty^{U_{m+1}(E_\infty),i+1}$$

is not σ'-minimax for some i, which is impossible because E_∞ belongs to the set $\mathcal{E}\big(U_{m+1}(E_\infty)\big)$. This last contradiction completes the proof of the lemma. □

Theorem 1.3.46 *The class of minimax groups is countably recognizable.*

Proof — Let G be a group whose countable subgroups are minimax, and assume for a contradiction that G is not minimax. Then it follows from Lemma 1.3.45 that for each minimax type σ there exists a countable subgroup G_σ of G which is not σ-minimax. As the set Σ of all minimax types is obviously countable, the subgroup

$$G_\infty = \langle G_\sigma \mid \sigma \in \Sigma\rangle$$

is countable and it cannot be minimax. This contradiction proves the theorem. □

A group G is said to satisfy the *weak minimal condition* on subgroups if it has no infinite descending chains of subgroups

$$X_1 > X_2 > \ldots > X_n > \ldots$$

such that the index $|X_n : X_{n+1}|$ is infinite for all n. The *weak maximal condition* on subgroups is defined replacing descending chains by ascending chains. It was independently proved by Baer [4] and D.I. Zaicev [79] that for soluble groups the weak minimal condition, the weak maximal condition and the property of being minimax are equivalent. Our last result shows that also the weak minimal and the weak maximal conditions can be detected from the behaviour of countable subgroups.

Proposition 1.3.47 *The class of groups satisfying the weak minimal condition and that of groups satisfying the weak maximal condition are countably recognizable.*

Proof — Let G be a group whose countable subgroups satisfy the weak minimal condition, and assume for a contradiction that G admits an infinite descending chain of subgroups

$$X_1 > X_2 > \ldots > X_n > \ldots$$

such that the index $|X_n : X_{n+1}|$ is infinite for all positive integers n. Then for each n we can choose a countably infinite subset U_n of X_n such that $uX_{n+1} \neq vX_{n+1}$ whenever u and v are elements of U_n and $u \neq v$. Then

$$U = \langle U_n \mid n \in \mathbb{N} \rangle$$

is a countable subgroup of G and U_n lies in $U \cap X_n$ for all n. It follows that

$$U \cap X_1 > U \cap X_2 > \ldots > U \cap X_n > \ldots$$

is an infinite descending chain of subgroups of U such that the index $|U \cap X_n : U \cap X_{n+1}|$ is infinite for each n. This contradiction proves that the class of groups satisfying the weak minimal condition is countably recognizable. A similar argument proves that also the class of groups satisfying the weak maximal condition is countably recognizable. □

1.4 Subgroups Closed in the Profinite Topology

Let G be any group, and let $\mathcal{J}(G)$ be the set of all normal subgroups of finite index of G. The *profinite topology* on G can be defined by choosing the set $\mathcal{J}(G)$ as a base of neighbourhoods of the identity; if X is any subgroup of G, the closure $\widehat{X}$ of X with respect to this topology is the intersection of all subgroups of finite index of G containing X, i.e.

$$\widehat{X} = \bigcap_{H \in \mathcal{J}(G)} XH.$$

In particular, a subgroup X is *closed* (with respect to the profinite topology) if and only if it is the intersection of a collection of subgroups of finite index, and a group G is residually finite if and only if the trivial subgroup $\{1\}$ is closed. It is also well-known that every subgroup of an arbitrary polycyclic group is closed. The structure of nilpotent groups in which all subgroups are closed was studied by M. Menth [51], while Robinson, A. Russo and G. Vincenzi [67] recently characterized groups with the same property within the universe of groups with finite conjugacy classes, and B.A.F. Wehrfritz [76] investigated the case of linear groups.

The aim of the first part of this section is to show that closure properties with respect to profinite topology can be detected from the behavior of countable subgroups. In particular, if G is a group and X is a subgroup of G, we will prove that for X being closed in G, is equivalent to require that $X \cap K$ is closed in K for each countable subgroup K of G. This result, obtained as a special case of a more general result, has a number of interesting consequences, the most striking being that the class of groups in which all subgroups are closed is countably recognizable. Furthermore, it follows that residually supersoluble groups form a countably recognizable class; moreover, if $\mathfrak{F}_\pi$ denotes the class of finite π-groups, it turns out that $\boldsymbol{R}\mathfrak{F}_\pi$ is countably recognizable for each set π of prime numbers.

We have seen that the class $\mathfrak{F}(\boldsymbol{R}\mathfrak{F})$, consisting of all groups G whose finite residual $J(G)$ is finite is countably recognizable (see Theorem 1.2.28). In the final part of the section, among other results on properties of the finite residual, it will be shown that in the above statement the class $\mathfrak{F}$ can be replaced by any subgroup closed and countably recognizable group class.

Closure Properties

Let $\mathfrak{X}$ be a class of groups, and for any group G let $\mathfrak{I}_{\mathfrak{X}}(G)$ be the set of all normal subgroups N of G such that G/N belongs to $\mathfrak{X}$. A subgroup X of a group G is said to be $\mathfrak{X}$-*closed* in G if

$$X = \bigcap_{N \in \mathfrak{I}_{\mathfrak{X}}(G)} XN.$$

In particular, if $\mathfrak{F}$ is the class of all finite groups, we have

$$\mathfrak{I}_{\mathfrak{F}}(G) = \mathfrak{I}(G)$$

and so the subgroup X is $\mathfrak{F}$-closed in G if and only if it is the intersection of a collection of subgroups of finite index of G, i.e. if and only if it is a closed subgroup of G. Note also that, if the group class $\mathfrak{X}$ is closed under homomorphic images, then a normal subgroup H of a group G is $\mathfrak{X}$-closed if and only if the factor group G/H is residually $\mathfrak{X}$.

Lemma 1.4.48 *Let $\mathfrak{X}$ be a subgroup closed class of groups, and let X be an $\mathfrak{X}$-closed subgroup of a group G. Then $X \cap K$ is $\mathfrak{X}$-closed in K for each subgroup K of G. In particular, X is $\mathfrak{X}$-closed in H, whenever H is a subgroup of G containing X.*

Proof — As the class $\mathfrak{X}$ is subgroup closed, the intersection $N \cap K$ belongs to $\mathfrak{I}_{\mathfrak{X}}(K)$ for every $N \in \mathfrak{I}_{\mathfrak{X}}(G)$, so that

$$\begin{aligned} X \cap K &= \Big(\bigcap_{N \in \mathfrak{I}_{\mathfrak{X}}(G)} XN \Big) \cap K = \bigcap_{N \in \mathfrak{I}_{\mathfrak{X}}(G)} (XN \cap K) \\ &\geqslant \bigcap_{N \in \mathfrak{I}_{\mathfrak{X}}(G)} (X \cap K)(N \cap K) \geqslant \bigcap_{L \in \mathfrak{I}_{\mathfrak{X}}(K)} (X \cap K)L \end{aligned}$$

and hence

$$X \cap K = \bigcap_{L \in \mathfrak{I}_{\mathfrak{X}}(K)} (X \cap K)L$$

is an $\mathfrak{X}$-closed subgroup of K. □

To the purposes of this paragraph, we will need the following classical theorem of Kurosh on inverse systems of finite sets (see for instance [46], Theorem 1.K.1).

Lemma 1.4.49 *The inverse limit of an inverse system of finite non-empty sets is non-empty.*

Theorem 1.4.50 *Let $\mathfrak{X}$ be a subgroup closed class of finite groups, and let X be a subgroup of a group G. If $X \cap K$ is $\mathfrak{X}$-closed in K for every countable subgroup K of G, then X is $\mathfrak{X}$-closed in G.*

Proof — Let g be any element of $G \setminus X$, and for each countable subgroup K of G containing g, let $H(K)$ be a normal subgroup of K such that g does not belong to $(X \cap K)H(K)$ and $K/H(K)$ is an $\mathfrak{X}$-group whose order $h(K)$ is smallest possible under these conditions. Consider the set $\mathscr{E}$ of all finitely generated subgroups of G containing g, and assume that there exists an infinite sequence $(E_n)_{n\in\mathbb{N}}$ of elements of $\mathscr{E}$ such that

$$h(E_1) < h(E_2) < \ldots < h(E_n) < \ldots$$

Clearly,

$$U = \langle E_n \mid n \in \mathbb{N} \rangle$$

is a countable subgroup of G, and $|E_n : H(U) \cap E_n| \leqslant |U : H(U)|$, so that we have $h(E_n) \leqslant h(U)$ for all n. This contradiction shows that the set of positive integers

$$\{h(E) \mid E \in \mathscr{E}\}$$

is finite, and so it has a largest element m.

For each element E of $\mathscr{E}$, let $\mathscr{L}(E)$ be the set of all normal subgroups L of E such that $g \notin (X \cap E)L$ and E/L is an $\mathfrak{X}$-group of order at most m. Clearly, the set $\mathscr{L}(E)$ is finite, because any finitely generated group contains only finitely many subgroups of a given finite index; moreover, it follows from the choice of m that every $\mathscr{L}(E)$ is non-empty. If E and F are elements of $\mathscr{E}$ such that $F \leqslant E$, the intersection $L \cap F$ belongs to $\mathscr{L}(F)$ for each subgroup $L \in \mathscr{L}(E)$, and so we may consider the intersection map $\alpha_{E,F}$ of $\mathscr{L}(E)$ into $\mathscr{L}(F)$. Then

$$\{\mathscr{L}(E), \alpha_{E,F} \mid E, F \in \mathscr{E},\ F \leqslant E\}$$

is an inverse system of finite non-empty sets, and so its inverse limit

$$\mathscr{L} = \varprojlim \mathscr{L}(E)$$

is not empty by Lemma 1.4.49. Let $(Y_E)_{E\in\mathscr{E}}$ be an element of $\mathscr{L}$.

If E and F are arbitrary elements of $\mathscr{E}$, we have $\langle Y_E, Y_F\rangle \leqslant Y_{\langle E,F\rangle}$ and hence

$$Y = \bigcup_{E\in\mathscr{E}} Y_E$$

is a subgroup of G. Moreover, Y is normal in G, because if y is any element of Y and x is an arbitrary element of G, then y^x lies in $Y_{\langle g,y,x\rangle} \leqslant Y$.

If F and F^* are arbitrary elements of $\mathscr{E}$, we have $F \cap Y_{\langle F,F^*\rangle} = Y_F$, so that

$$F \cap Y = F \cap \Big(\bigcup_{E\in\mathscr{E}} Y_{\langle E,F\rangle}\Big) = \bigcup_{E\in\mathscr{E}} (F \cap Y_{\langle E,F\rangle}) = Y_F.$$

Assume now for a contradiction that $|G : Y|$ is infinite, and let

$$g_1, \dots, g_m, g_{m+1}$$

be $m+1$ different elements of a transversal to Y in G. Then

$$E = \langle g, g_1, \dots, g_m, g_{m+1}\rangle$$

is an element of $\mathscr{E}$ and

$$|E : Y_E| = |E : Y \cap E| > m,$$

which is impossible because Y_E belongs to $\mathscr{L}(E)$. Therefore the index $|G : Y|$ is finite. Consider the element W of $\mathscr{E}$ generated by g and by a transversal to Y in G. Then $WY = G$ and hence

$$G/Y \simeq W/W \cap Y = W/Y_W$$

is an $\mathfrak{X}$-group.

Assume finally that g belongs to

$$XY = \bigcup_{E\in\mathscr{E}} XY_E,$$

so that there exist an element E of $\mathscr{E}$ and a finitely generated subgroup X_0 of X such that g lies in X_0Y_E. Then the subgroup $F = \langle X_0, E\rangle$ is an element of $\mathscr{E}$ and g belongs to $(X \cap F)Y_F$. This contradiction proves that g is not in XY, so that X is $\mathfrak{X}$-closed in G because g is an arbitrary element of $G \setminus X$. $\square$

The following result is obtained as an easy combination of Lemma 1.4.48 and Theorem 1.4.50.

Corollary 1.4.51 *Let $\mathfrak{X}$ be a subgroup closed class of finite groups, and let X be a subgroup of a group G. Then the following statements are equivalent:*

(a) X *is* $\mathfrak{X}$*-closed in* G*;*

(b) X *is* $\mathfrak{X}$*-closed in* $\langle X, K\rangle$ *for each countable subgroup* K *of* G*;*

(c) $X \cap K$ *is* $\mathfrak{X}$*-closed in* K *for each countable subgroup* K *of* G*.*

It follows from the above statement that if $\mathfrak{X}$ is any subgroup closed class of finite groups, then the class of groups in which all subgroups are $\mathfrak{X}$-closed is countably recognizable. In particular, groups all of whose subgroups are closed in the profinite topology form a countably recognizable class, although it is clear that such class is not local. Another special case is the following interesting fact.

Corollary 1.4.52 *Let* G *be a group whose countable subgroups are closed. Then all subgroups of* G *are closed.*

Actually, it can be remarked that for a single subgroup the embedding property of being closed is countably detectable.

Corollary 1.4.53 *Let* X *be a subgroup of a group* G*. If all countable subgroups of* X *are closed in* G*, then* X *itself is closed in* G*.*

Proof — Let K be any countable subgroup of G. Then the intersection $X \cap K$ is obviously countable and so it is closed in G. In particular, $X \cap K$ is closed in K, and hence X is a closed subgroup of G by Corollary 1.4.51. □

Notice also that the proof of Corollary 1.4.53 can be used to prove that a corresponding more general statement holds for the property of being $\mathfrak{X}$-closed, where $\mathfrak{X}$ is any subgroup closed class of finite groups.

Corollary 1.4.54 *Let $\mathfrak{X}$ be a subgroup closed class of finite groups. Then the class $\boldsymbol{R}\mathfrak{X}$ of residually $\mathfrak{X}$ groups is countably recognizable.*

In particular, the latter statement improves Neumann's theorem on residually finite groups, showing for instance that the class $\boldsymbol{R}\mathfrak{F}_{\pi}$ is countably recognizable, for any set π of prime numbers.

Corollary 1.4.55 *Let $\mathfrak{X}$ be a group class which is closed with respect to subgroups and homomorphic images. If $\mathfrak{X}$ is contained in $\boldsymbol{R}\mathfrak{F}$, then the class $\boldsymbol{R}\mathfrak{X}$ is countably recognizable.*

Proof — Since the class $\mathfrak{X}$ is closed with respect to homomorphic images, we have

$$\boldsymbol{R}\mathfrak{X} = \boldsymbol{R}(\mathfrak{X} \cap \mathfrak{F}),$$

and hence the statement follows from Corollary 1.4.54. □

Since any supersoluble group is residually finite, the above statement has the following special case.

Corollary 1.4.56 *The class of residually supersoluble groups is countably recognizable.*

Theorem 1.4.50 can be used to prove that also some other relevant group classes defined by closure properties in the profinite topology are countably recognizable. In fact, if Θ is any subgroup property such that $X \cap H$ is a Θ-subgroup of H whenever X is a Θ-subgroup of a group G and $H \leqslant G$, it follows that the class of groups whose Θ-subgroups are closed is countably recognizable. For instance, we have that the class of groups whose abelian subgroups are closed is countably recognizable, while if we apply this remark to the property of being a normal subgroup, we obtain the following interesting result.

Corollary 1.4.57 *The class of groups whose homomorphic images are residually finite is countably recognizable.*

Note that the above corollary can also be obtained as a special case of the following result.

Lemma 1.4.58 *Let $\mathfrak{X}$ be a countably recognizable class of groups. Then also the subclass $\mathfrak{X}^{\boldsymbol{H}}$ of $\mathfrak{X}$, consisting of all groups whose homomorphic images belong to $\mathfrak{X}$, is countably recognizable.*

Proof — Let G be any group whose countable subgroups belong to the class $\mathfrak{X}^{\boldsymbol{H}}$, and let N be any normal subgroup of G. If H/N is any countable subgroup of G/N, there exists a countable subgroup X of G such that $H = XN$, and so

$$H/N \simeq X/X \cap N$$

is an $\mathfrak{X}$-group. As $\mathfrak{X}$ is countably recognizable, it follows that G/N belongs to $\mathfrak{X}$. Therefore the class $\mathfrak{X}^{\boldsymbol{H}}$ is countably recognizable. □

Corollary 1.4.59 *Let $\mathfrak{X}$ be a subgroup closed class of finite groups. Then the class of groups whose homomorphic images are residually $\mathfrak{X}$ is countably recognizable.*

Proof — The class of residually $\mathfrak{X}$ is countably recognizable by Corollary 1.4.54. Therefore, the statement follows directly from Lemma 1.4.58. □

The Finite Residual

This paragraph is devoted to the study of countably detectable properties of the finite residual. For any group G, we shall denote by $J(G)$ the finite residual of G.

Theorem 1.4.60 *Let $\mathfrak{X}$ be a subgroup closed countably recognizable class of groups. Then the class $\mathfrak{X}(\boldsymbol{R}\mathfrak{F})$, consisting of all groups whose finite residual belongs to $\mathfrak{X}$, is countably recognizable.*

Proof — Let G be a group such that the finite residual of every countable subgroup of G belongs to $\mathfrak{X}$, and let $\mathcal{C}$ be the set of all countable subgroups of G. For each countable subgroup H of G, the set-theoretic union

$$L(H) = \bigcup_{C\in\mathcal{C}} \big(H \cap J(\langle H, C\rangle)\big)$$

is obviously a subgroup of H. If h is any element of $L(H)$, there exists a countable subgroup K_h of G containing H such that h lies in $H \cap J(K_h)$. Then

$$K = \langle K_h \mid h \in L(H)\rangle$$

is a countable subgroup of G and $L(H)$ is contained in the finite residual $J(K)$ of K. It follows that $L(H) = H \cap J(K)$ is the largest element of the set

$$\big\{H \cap J(\langle H, C\rangle) \mid C \in \mathcal{C}\big\}.$$

Moreover $L(H_1) \leqslant L(H_2)$ whenever H_1 and H_2 are countable subgroups of G such that $H_1 \leqslant H_2$, and so

$$L = \bigcup_{H\in\mathcal{C}} L(H)$$

is a subgroup of G. Let X be any countable subgroup of L, and let x be an arbitrary ele ment of X. Then there exist countable subgroups V_x and W_x of G such that $V_x \leqslant W_x$ and x lies in $L(V_x) = V_x \cap J(W_x)$, and hence $X \leqslant J(W)$, where

$$W = \langle W_x \mid x \in X \rangle$$

is countable. Therefore X belongs to the subgroup closed class $\mathfrak{X}$, and so L itself is an element of $\mathfrak{X}$, because $\mathfrak{X}$ is countably recognizable.

Let Y be any countable subgroup of G. Then $L \cap Y$ is a countable subgroup of L, and so there exists a countable subgroup E of G containing Y such that $L \cap Y \leqslant L(E)$. Then $L(E) = E \cap J(H)$ for some countable subgroup $H \geqslant E$, and hence

$$L(Y) \leqslant L \cap Y \leqslant L(E) \cap Y = E \cap J(H) \cap Y = J(H) \cap Y \leqslant L(Y).$$

It follows that $L \cap Y = J(H) \cap Y$ is a closed subgroup of Y, and an application of Theorem 1.4.50 yields that L is a closed subgroup of G. Therefore the finite residual J(G) of G is contained in L, and hence it belongs to $\mathfrak{X}$. □

Corollary 1.4.61 *The class $\boldsymbol{P}(\boldsymbol{R}\mathfrak{F})$ of all groups admitting a finite series with residually finite factors is countably recognizable.*

Proof — For each positive integer n, let $(\boldsymbol{R}\mathfrak{F})^n$ be the class of all groups admitting a finite series of length at most n whose factors are residually finite. An obvious induction argument and Theorem 1.4.60 yield that $(\boldsymbol{R}\mathfrak{F})^n$ is countably recognizable for every n. Since it is also clear that each $(\boldsymbol{R}\mathfrak{F})^n$ is subgroup closed, it follows that also the class

$$\boldsymbol{P}(\boldsymbol{R}\mathfrak{F}) = \bigcup_{n \in \mathbb{N}} (\boldsymbol{R}\mathfrak{F})^n$$

is countably recognizable (see for instance [31], Lemma 2.1). □

We have seen that the class of imperfect groups is countably recognizable (see Lemma 1.2.42). Our next result shows that also the class of groups which are not $\mathfrak{F}$-perfect can be countably detected; recall here that a group is called *$\mathfrak{F}$-perfect* if it has no proper subgroups of finite index.

Theorem 1.4.62 *Let G be a non-trivial group in which every countable non-trivial subgroup has a proper subgroup of finite index. Then G itself contains a proper subgroup of finite index.*

Proof — Assume that the statement is false, and suppose first that each element of G belongs to the finite residual of some countable subgroup of G. Fix a non-trivial element x of G, and let X_1 be a countable subgroup of G such that x belongs to the finite residual $J(X_1)$ of X_1. Assume now that a countable subgroup X_n has been chosen for some positive integer n. If y is any element of X_n, there exists a countable subgroup H_y of G such that $X_n \leqslant H_y$ and y lies in $J(H_y)$, and

$$X_{n+1} = \langle H_y \mid y \in X_n \rangle$$

is a countable subgroup of G such that $X_n \leqslant J(X_{n+1})$. It follows that

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

is a countable subgroup of G which coincides with its finite residual, i.e. which has no proper subgroups of finite index. This contradiction shows that there exists a non-trivial element g of G such that every countable subgroup K of G contains a subgroup of finite index $H(K)$ such that $g \notin K$ and the index $h(K) = |K : H(K)|$ is smallest possible.

Let $\mathscr{E}$ be the set of all finitely generated subgroups of G containing g, and let $\mathscr{F}$ be any countable subset of $\mathscr{E}$. Then

$$F = \langle X \mid X \in \mathscr{F} \rangle$$

is a countable subgroup of G and $h(X) \leqslant h(F)$ for all $X \in \mathscr{F}$. It follows that there exists a positive integer m such that $h(E) \leqslant m$ for all $E \in \mathscr{E}$. For each element E of $\mathscr{E}$, consider the set $\mathscr{L}(E)$ of all subgroups L of E such that $g \notin L$ and $|E : L| \leqslant m$. Then each $\mathscr{L}(E)$ is a finite non-empty set, and if E and F are elements such that $F \leqslant E$, the intersection map $\alpha_{E,F}$ goes from $\mathscr{L}(E)$ into $\mathscr{L}(F)$. Therefore

$$\{\mathscr{L}(E), \alpha_{E,F} \mid E, F \in \mathscr{E},\ F \leqslant E\}$$

is an inverse system, and its inverse limit

$$\mathscr{L} = \varprojlim \mathscr{L}(E)$$

is not empty by Lemma 1.4.49. If $(Y_E)_{E \in \mathscr{E}}$ is an element of $\mathscr{L}$, it is easy to prove that

$$Y = \bigcup_{E \in \mathscr{E}} Y_E$$

is a subgroup of finite index of G, and $g \notin Y$. This contradiction completes the proof. $\square$

Observe that the argument of the above proof can also be used to show that, for any set π of prime numbers, the class of groups admitting a homomorphic image which is a finite non-trivial π-group is countably detectable.

Since a group G has a descending series with finite factors if and only if every non-trivial subgroup of G contains a proper subgroup of finite index, Theorem 1.4.62 has the following consequence.

Corollary 1.4.63 *The class $\acute{P}\mathfrak{F}$ of groups admitting a descending series with finite factors is countably recognizable.*

Notice that $\acute{P}_n(R\mathfrak{F}) = \acute{P}(R\mathfrak{F}) = \acute{P}\mathfrak{F}$, because the finite residual of any group is a characteristic subgroup, and so the above corollary should also be seen in relation to Corollary 1.4.61.

We shall say that an arbitrary group class $\mathfrak{X}$ has *countable type* (or that $\mathfrak{X}$ is $L_{\aleph_0}$*-closed*) if a group G belongs to $\mathfrak{X}$ whenever each countable subgroup of G is contained in some $\mathfrak{X}$-subgroup. Of course, every class of countable nature is countably recognizable, and for subgroup closed group classes these two concepts coincide. On the other hand, a countably recognizable class need not have countable nature: to see this, it is enough to consider the class $\mathfrak{A}_0^*$ formed by the trivial group and by all countable non-abelian groups, and observe that if G is any uncountable non-abelian group, then each countable subgroup of G lies in some countable non-abelian subgroup.

Recall that a topological group is said to be *profinite* if it is isomorphic to the inverse limit of an inverse system of finite groups endowed with discrete topologies. It is well known that a topological group is profinite if and only if it is compact and totally disconnected (see [78], Corollary 1.2.4). Of course, any profinite group is residually finite. The following example shows that the class of profinite groups does not have countable nature.

Let $C = \{0, 1\}$ be the group with two elements, and in the cartesian power C^I of C over a set I of cardinality $\aleph_1$ consider the subgroup G consisting of all elements with countable support, endowed with the topology induced by the product topology of C^I. If X is any countable subgroup of G, there exists a countable subset I_0 of I such that $X \leqslant C^{I_0} \leqslant G$, and C^{I_0} is compact by Tychonoff's theorem.

Since G is totally disconnected (see for instance [21], Theorem 1.8), it follows that every countable subgroup of G is contained in a profinite subgroup. On the other hand, the group G is not profinite because it is not compact.

Chapter 2

Uncountable in Group Theory

Let $\mathfrak{X}$ be a group theoretical property, let G be a group of uncountable cardinality and suppose that all its proper uncountable subgroups satisfy $\mathfrak{X}$. In this chapter we will try to answer the following question:

Is it true that all (proper) subgroups of G *satisfy* $\mathfrak{X}$*?*

The main obstacle here is a relevant result by S. Shelah [71], who proved (without appeal to the continuum hypothesis) that there exists a group of cardinality $\aleph_1$ whose proper subgroups (and even subsemigroups) have cardinality strictly smaller than $\aleph_1$. In such a way, he answered to a question, generalizing the classical Šmidt's problem, that A.G. Kuroš and S.N. Černikov asked in their seminal paper [47]. This question was later extended to the case of arbitrary algebras by B. Jónsson (see for instance [44], p.133) and such groups are now called *Jónsson groups*.

Observe that the situation is completely different in the case of fields, since it is well-known that any uncountable field contains a proper uncountable subfield (see for instance [10]).

In order to avoid Shelah's example and other similar obstructions, it will be often used the additional requirement that the group has no simple homomorphic images of cardinality $\aleph$, a condition which is obviously satisfied in the case of locally soluble groups, or some other generalized soluble condition.

In the the first two sections of this chapter, it will be proved that (in a suitable universe of groups) the class of FC-groups is uncountably recognizable as well as the class of nilpotent groups. Then, it will be studied the uncountable character of the class of groups in which the normality is transitive; and, finally, we turn our attention to some classes of groups defined by restriction on their lattice of subgroups.

We refer to the monograph [43] for terminology and properties concerning cardinal numbers.

Some Preliminaries

This paragraph collects some elementary results concerning the cardinality of subgroups of infinite abelian groups.

It is well-known that a torsion-free abelian group A has finite rank $r = r(A)$ if and only if the rational vector space

$$V(A) = A \otimes_{\mathbb{Z}} \mathbb{Q}$$

has dimension r. Moreover, an abelian p-group A (where p is a prime number) has finite rank $r = r(A)$ if and only if r is the dimension of the $\mathbb{Z}_p$-vector space

$$V(A) = \mathrm{Hom}(\mathbb{Z}_p, A).$$

These remarks enable us to define in both cases the rank $r(A)$ of A as the dimension of the vector space $V(A)$, even if this dimension is an infinite cardinal number. It turns out that $r(A)$ is the cardinality of any maximal linearly independent subset of A consisting of elements of infinite order, when A is torsion-free, and of elements of order p, if A is a p-group.

Consider now an abelian group A of finite rank, and let T be the subgroup of all elements of finite order of A. Then

$$r(A) = r(A/T) + \max_p r(A_p),$$

where p ranges over the set of all prime numbers and A_p is the p-component of A. This relation can be used to define the rank of an arbitrary abelian group as the sum of cardinal numbers

$$r(A) = r(A/T) + \sup_p r(A_p).$$

It follows easily that for an uncountable abelian group the rank and the cardinality coincide.

Let A be an abelian group. For each positive integer m, we shall denote by $A[m]$ the characteristic subgroup of A consisting of all elements a such that $a^m = 1$. In particular, if A is an abelian p-group for some prime number p, the subgroup $A[p]$ coincides obviously with the socle of A, and so its cardinality is precisely the rank of A. Therefore we have:

Lemma 2.64 *Let A be an uncountable abelian p-group (where p is a prime number). Then A and $A[p]$ have the same cardinality.*

Lemma 2.65 *Let A be a torsion-free abelian group, and let B be a subgroup of A such that A/B is periodic. Then A and B have the same cardinality.*

Proof — Let $\aleph$ be the cardinality of A, and assume for a contradiction that B has cardinality $\aleph' < \aleph$. Then the periodic group A/B has cardinality $\aleph$, and so there exists a prime number p such that the p-component P/B of A/B has cardinality strictly larger than $\aleph'$. As $\aleph'$ is an infinite cardinal number, it follows from Lemma 2.64 that also the socle S/B of P/B has cardinality stricly larger than $\aleph'$. On the other hand, S is isomorphic to the subgroup S^p of B, and this contradiction proves the lemma. $\square$

Lemma 2.66 *Let A be an infinite abelian group. Then A contains a proper subgroup B such that A/B is countable.*

Proof — As the statement is obvious if A has a proper subgroup of finite index, it can be assumed that A is divisible. Then A can be decomposed into a direct product of a collection of countable subgroups, and in particular it admits a countable non-trivial homomorphic image. $\square$

Our next result deals with large homomorphic images of uncountable abelian groups.

Lemma 2.67 *Let A be an uncountable abelian group of cardinality $\aleph$. Then A contains a subgroup B of cardinality $\aleph$ such that also A/B has cardinality $\aleph$.*

Proof — Let E be a maximal free abelian subgroup of A, and assume first that E has cardinality $\aleph$. Then also E/E^2 has cardinality $\aleph$, so that A/E^2 has cardinality $\aleph$, and it is enough to put $B = E^2$ because $E^2 \simeq E$. Suppose now that E has cardinality strictly smaller than $\aleph$, so that A/E has cardinality $\aleph$, and replacing A by the periodic group A/E it can be assumed that A is periodic.

For each prime number p, let A_p be the p-component of A, so that

$$A = \underset{p}{\mathrm{Dr}}\, A_p.$$

Assume that every A_p has cardinality strictly smaller than $\aleph$. As

$$\sup_p \mathrm{card}(A_p) = \aleph,$$

there exists a set π of prime numbers such that both A_π and $A_{\pi'}$ have cardinality $\aleph$, and we can choose $B = A_\pi$. Suppose finally that A_p has cardinality $\aleph$ for some prime p. Then Lemma 2.64 yields that also the socle S of A_p has cardinality $\aleph$ and so it is the direct product of $\aleph$ subgroups of order p. It follows that S can be decomposed into the direct product of two subgroups of cardinality $\aleph$, and the statement is proved also in this case. $\square$

Observe that Lemma 2.67 shows in particular that a Jónsson group cannot be abelian. It is known that if $\aleph$ is an uncountable regular cardinal, then a locally finite group of cardinality $\aleph$ cannot be a Jónsson group (see [46], Theorem 2.6). However, as we mentioned, Shelah constructed a Jónsson group of cardinality $\aleph_1$, and it was remarked by A. Macintyre that Shelah's example is simple over the centre.

Lemma 2.68 *Let G be a Jónsson group of cardinality $\aleph$. Then $\langle x\rangle^G = G$ for every non-central element x of G.*

Proof — Let x be any element of the set $G \setminus Z(G)$. Then the centralizer $C_G(x)$ is a proper subgroup of G, so that its cardinality is strictly smaller than $\aleph$, and so $|G : C_G(x)| = \aleph$. It follows that the conjugacy class of x has cardinality $\aleph$, and hence $\langle x\rangle^G = G$. $\square$

Corollary 2.69 *Let G be a Jónsson group of cardinality $\aleph$. Then G is perfect and $G/Z(G)$ is a simple group of cardinality $\aleph$.*

Proof — It follows from Lemma 2.68 that the factor group $G/Z(G)$ is simple. Since G cannot be abelian, the centre $Z(G)$ is a proper subgroup, and so its cardinality is strictly smaller than $\aleph$. Therefore the simple group $G/Z(G)$ has cardinality $\aleph$, and in particular G' cannot be contained in $Z(G)$. If x is an element of $G' \setminus Z(G)$, we have $\langle x\rangle^G \leqslant G'$, so that $G = G'$ is perfect. $\square$

2.1 Groups with Restricted Conjugacy Classes

The group G is said to be *minimal non-*FC if it is not an FC-group but all its proper subgroups have the FC-property. The structure of minimal non-FC groups has been completely described by V.V. Belyaev and N.F. Sesekin ([6],[7]) in the case of groups admitting a non-trivial homomorphic image which is either finite or abelian. Since the property FC is countably recognizable, it follows that minimal non-FC group must be countable. Moreover, it has been proved in [20] that if G is a soluble group of infinite rank whose proper subgroups of infinite rank have the FC-property, then G is an FC-group. The main result of this section shows that if $\aleph$ is an uncountable regular cardinal and G is a group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the FC-property, then G itself is an FC-group, provided that it has no simple homomorphic images of cardinality $\aleph$. It follows that similar results hold for groups whose proper subgroups of large cardinality are either central-by-finite or finite-by-abelian.

We begin with the following lemma.

Lemma 2.1.70 *Let* G *be an uncountable group of cardinality* $\aleph$*. If* G *contains a proper normal subgroup* N *of cardinality* $\aleph$ *such that*

$$\aleph^* = \sup\Big\{|H : C_H(h)| \;\Big|\; N \leqslant H < G,\, h \in H\Big\} < \aleph\,,$$

then every proper subgroup of G *is contained in a proper subgroup of cardinality* $\aleph$*.*

Proof — Let X be any subgroup of G whose cardinality $\aleph'$ is strictly smaller than $\aleph$; since XN has cardinality $\aleph$, it can obviously be assumed that $XN = G$. Suppose first that N has finite index in G. As

$$(X \cap N)^G = (X \cap N)^N = \big\langle \langle x\rangle^N \mid x \in X \cap N\big\rangle,$$

it follows from the hypotheses that the cardinality of the normal subgroup $(X \cap N)^G$ is bounded by $\max\{\aleph_0, \aleph', \aleph^*\}$, and so it is strictly smaller than $\aleph$. Thus the factor group $G/(X \cap N)^G$ has cardinality $\aleph$, and so replacing G by $G/(X \cap N)^G$ it can be assumed without loss of generality that $X \cap N = \{1\}$, so that in particular X is finite. If N is not abelian, and a is a non-central element of N, the centralizer $C_N(a)$ is a proper subgroup of N and the index $|N : C_N(a)|$ is strictly smaller than $\aleph$; this remark and Lemma 2.66 show that in any case N contains a proper subgroup K such that $|N : K| < \aleph$. Then

$$K_0 = \bigcap_{x \in X} K^x$$

is a proper X-invariant subgroup of N and $|N : K_0| < \aleph$, so that XK_0 is a proper subgroup of G of cardinality $\aleph$.

Suppose now that the index $|G : N|$ is infinite, so that in particular X is infinite. If N is contained in a proper subgroup H of finite index, then N is also contained in the core H_G of H, and the statement follows from the above argument applied to H_G. Assume that N is not contained in any proper subgroup of finite index of G. Then $\langle x, N \rangle$ is a proper subgroup of G for each element x of X, so that

$$|\langle x, N \rangle : C_{\langle x, N \rangle}(x)| \leqslant \aleph^*$$

and hence the normal subgroup

$$\langle x \rangle^G = \langle x \rangle^{\langle x, N \rangle X} = \big\langle x^g \mid g \in \langle x, N \rangle \big\rangle^X$$

has cardinality at most $\max\{\aleph', \aleph^*\}$. It follows that the subgroup X^G has cardinality strictly smaller than $\aleph$, and so G/X^G has cardinality $\aleph$. Since $G = NX^G$, the factor group G/X^G is either abelian or contains a non-central element admitting less than $\aleph$ conjugates, and so it follows that G/X^G has a proper subgroup L/X^G of cardinality $\aleph$. The lemma is proved. □

Theorem 2.1.71 *Let $\aleph$ be an uncountable regular cardinal, and let G be a group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the FC-property. If G has no simple homomorphic images of cardinality $\aleph$, then G is an FC-group.*

Proof — Clearly, it follows from Lemma 2.66 that every uncountable FC-group contains a proper subgroup with the same cardinality. Assume for a contradiction that G has no proper normal subgroups of cardinality $\aleph$. If K is any proper normal subgroup of G, the factor group G/K has cardinality $\aleph$, so that it cannot be simple and it follows from Corollary 2.69 that G/K contains a proper subgroup of cardinality $\aleph$, i.e. K is contained in a proper subgroup of G of cardinality $\aleph$. Therefore all proper normal subgroups of G have finite conjugacy classes. Let M be the subgroup generated by all proper normal subgroups of G. As the factor group G/M is simple, it cannot have cardinality $\aleph$ and hence $G = M$ is the join of its proper normal subgroups. Moreover, as the product of two proper normal subgroups of G is also properly contained in G, we obtain that G can be decomposed into a set-theoretic union

$$G = \bigcup_{V \in \Omega} V,$$

where Ω is a chain of proper normal subgroups. Let X be any subgroup of G of cardinality strictly smaller than $\aleph$. For each element x of X there exists a subgroup $W_x \in \Omega$ such that x belongs to W_x, and hence

$$X = \bigcup_{x \in X} (X \cap W_x)$$

is contained in the normal subgroup

$$W = \bigcup_{x \in X} W_x$$

of G. Clearly, W is properly contained in G, because $\aleph$ is a regular cardinal, and so it is an FC-group. Therefore all proper subgroups of the uncountable group G have the FC-property, and hence G itself is an FC-group. This contradiction shows that G contains a proper normal subgroup N of cardinality $\aleph$.

Consider any proper subgroup H of G containing N. Then H is an FC-group, so that the index $|H : C_H(h)|$ is finite for each element h of H and hence

$$\sup\Big\{|H : C_H(h)| \;\Big|\; N \leqslant H < G,\, h \in H\Big\} \leqslant \aleph_0 < \aleph\,.$$

It follows from Lemma 2.1.70 that every proper subgroup of G is contained in a proper subgroup of cardinality $\aleph$, and so it is an FC-group. Therefore G itself is an FC-group. □

As a consequence of the theorem, we obtain corresponding results for some special classes of FC-groups, namely that of groups which are finite over the centre and that of groups with finite commutator subgroup. These two group classes are related by the celebrated theorem of I. Schur on the finiteness of the commutator subgroup of a group whose centre has finite index (see for instance [61] Part 1, Theorem 4.12).

Corollary 2.1.72 *Let $\aleph$ be an uncountable regular cardinal, and let* G *be a group of cardinality* $\aleph$ *which has no simple homomorphic images of cardinality* $\aleph$*.*

(a) *If all proper subgroups of* G *of cardinality* $\aleph$ *are abelian, then* G *is abelian.*

(b) *If all proper subgroups of* G *of cardinality* $\aleph$ *are central-by-finite, then the index* $|G : Z(G)|$ *is finite.*

(c) *If all proper subgroups of* G *of cardinality* $\aleph$ *are finite-by-abelian, then the commutator subgroup* G' *of* G *is finite.*

Proof — The group G has the FC-property by Theorem 2.1.71. In particular, if G is not abelian, it contains a proper normal subgroup of finite index, so that G satisfies the hypotheses of Lemma 2.1.70 and hence every proper subgroup of G is contained in a proper subgroup of cardinality $\aleph$. Then statement (a) follows immediately. Moreover, it is well-known that if an FC-group contains an abelian subgroup of finite index, then it is finite over the centre, and this remark proves statement (b).

Suppose finally that G is a non-abelian group satisfying the condition of statement (c), and let N be a proper normal subgroup of G such that G/N is finite. Then G/N' is an abelian-by-finite FC-group, so that it is finite over the centre and it follows from Schur's theorem that G'/N' is finite. As N' is finite by hypothesis, also the commutator subgroup G' of G is finite. □

The additional assumption in the statements of the above theorem and its corollary is imposed in order to avoid Jónsson groups and other pathologies. In fact, it is also enough to assume that the group contains some large proper normal subgroup. For instance, if we suppose that the group G has a non-trivial homomorphic image whose cardinality is strictly smaller than $\aleph$, the arguments used in the above proofs can be applied, even in the case of an arbitrary uncountable cardinal number.

In the next section, it will be clear that all the results we stated here are susceptible of improvements with respect to the cardinality of the group. In fact, we can replace the regularity condition on the cardinality, with the request of having uncountable cofinality. Furthermore, it will be shown how to remove all the requests when the generalized continuum hypothesis holds.

2.2 Generalized Nilpotency Properties

A group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Thus locally graded groups form a large class of generalized soluble groups, containing in particular all locally (soluble-by-finite) and all residually finite groups.

The aim of this section is to show the following main results.

THEOREM 2.2.6 *Let $\aleph$ be a regular cardinal number, and let G be a locally graded group of cardinality $\aleph$ which has no simple homomorphic images of cardinality $\aleph$. If all proper subgroups of G of cardinality $\aleph$ are nilpotent-by-finite, then G itself is nilpotent-by-finite.*

This first theorem is susceptible of the same improvements we spoke of at the end of the previous section. However, we still state (and prove) it in these terms to get the reader acquainted with these kind of results. Starting from this on, results will be stated (and proved) directly in the most general hypotheses.

THEOREM 2.2.14 *Let $\aleph$ be a cardinal number whose cofinality is strictly larger than $\aleph_0$, and let G be a group of cardinality $\aleph$ which has no infinite simple homomorphic images. If all proper subgroups of G of cardinality $\aleph$ are nilpotent, then G itself is nilpotent.*

Of course, any uncountable regular cardinal number has cardinality strictly larger than $\aleph_0$, and so the above theorems hold in particular for such cardinals. On the other hand, there exist cardinals with cofinality strictly larger than $\aleph_0$ which are not regular, like for instance $\aleph_{\aleph_1}$. However it will be proved that the assumption on the cofinality of the cardinal number $\aleph$ in Theorem 2.2.13 can be dropped out under the assumption of GCH, the *generalized continuum hypothesis*.

As a consequence of Theorem 2.2.14, it turns out that, under the same hypotheses, if all proper subgroups of cardinality $\aleph$ of G have nilpotency class bounded by a positive integer c, then also G has nilpotency class at most c.

Our third relevant result deals with uncountable locally graded groups whose proper uncountable subgroups are locally nilpotent.

THEOREM 2.2.12 *Let G be an uncountable locally graded group of cardinality $\aleph$ which has no simple homomorphic images of cardinality $\aleph$. If all* *proper subgroups of cardinality* $\aleph$ *of* G *are locally nilpotent, then* G *itself is locally nilpotent.*

It is known that in many problems concerning (generalized) supersoluble groups the main obstacle is the behaviour of the commutator subgroup. For instance, it was proved in [34] that if G is a group of infinite rank whose proper subgroups of infinite rank are locally supersoluble, then G itself is locally supersoluble, provided that its commutator subgroup G' is locally nilpotent. A corresponding result holds for groups of large cardinality.

Theorem 2.2.13 *Let* G *be a group of uncountable cardinality* $\aleph$ *whose proper subgroups of cardinality* $\aleph$ *are locally supersoluble. If the commutator subgroup* G' *of* G *is locally nilpotent, then* G *is locally supersoluble.*

The final part of the section is dedicated to the study of uncountable groups whose proper subgroups of large cardinality are soluble, and in this case the following result has been proved.

Theorem 2.2.23 *Let* G *be an uncountable group of cardinality* $\aleph$ *which has no simple non-abelian homomorphic images. If all proper subgroups of cardinality* $\aleph$ *are soluble with derived length at most* k *(where* k *is a fixed positive integer), then* G *itself is soluble with derived length at most* k.

We leave as an open question whether results similar to Theorems 2.2.6 and 2.2.14 hold when nilpotency is replaced by solubility. An obstacle here is caused by the fact that the structure of unsoluble locally soluble groups whose proper subgroups are soluble seems to be unknown; although such groups must be countable, they may occur as homomorphic images of uncountable groups whose proper subgroups of large cardinality are soluble.

Uncountable Recognizability of Nilpotent-by-Finite Groups

We begin with the following lemma.

Lemma 2.2.73 *Let* G *be an uncountable group of cardinality* $\aleph$ *whose proper subgroups of cardinality* $\aleph$ *are nilpotent-by-finite. If* G *is not nilpotent-by-finite, then either* G *is perfect or* G/G' *is a group of type* p^∞ *for some prime number* p.

Proof — Since G is not nilpotent-by-finite and the class of nilpotent-by-finite groups is countably recognizable, G must contain a proper subgroup X which is not nilpotent-by-finite. In particular, X cannot be contained in a proper subgroup of G of cardinality $\aleph$, and hence the cardinality of the abelian group G/G' is strictly smaller than $\aleph$ by Lemma 2.67. As G' has cardinality $\aleph$, it follows that the divisible group G/G' cannot be decomposed into the product of two proper subgroups, and so G/G' is either trivial or a group of type p^∞ for some prime number p. □

We will also need the following result, that was proved by B. Bruno and R.E. Phillips (see [8], Lemma 2.3).

Lemma 2.2.74 *Let G be a periodic group, and let A be a G-module whose additive group is torsion-free. If π is any finite set of prime numbers, there exists a G-submodule B of A such that the group A/B is periodic and π is contained in the set $\pi(A/B)$.*

Our next lemma put together results by F. Napolitani and E. Pegoraro [53] and by A.O. Asar [1]. It shows that within the universe of locally graded groups there are no minimal non-(nilpotent-by-Černikov) groups.

Lemma 2.2.75 *Let G be a locally graded group whose proper subgroups are nilpotent-by-Černikov. Then G itself is nilpotent-by-Černikov.*

Finally, we need information on the existence of large submodules in uncountable modules over a Prüfer group, which can be obtained using the following elementary lemma concerning endomorphisms of vector spaces (see for instance [77], Theorem 5.4.6).

Lemma 2.2.76 *Let V be a vector space over a field F, and φ an endomorphism of V such that $h(\varphi) = 0$ for some non-zero polynomial $h \in F[x]$. If f is the minimal polynomial of φ over F, and $f = g_1 \cdots g_t$ is the product of pairwise coprime polynomials $g_1, \dots, g_t$, then*

$$V = W_1 \oplus \dots \oplus W_t,$$

where W_i is the kernel of $g_i(\varphi)$ for every $i = 1, \dots, t$.

Lemma 2.2.77 *Let P be a group of type p^∞ for some prime number p, and let A be a P-module whose additive group has prime exponent q and uncountable regular cardinality $\aleph$. Then A contains a proper P-submodule of cardinality $\aleph$.*

Proof — Assume for a contradiction that every proper P-submodule of A has cardinality strictly smaller than $\aleph$, and let

$$\{y_n \mid n \in \mathbb{N}_0\}$$

be a set of generators of P with the usual relations

$$y_0 = 1 \text{ and } y_{n+1}^p = y_n$$

for each non-negative integer n. Since every subgroup of A containing $[A, P]$ is a P-submodule, we have $[A, P] = A$, and so also

$$A = \bigcup_{n\in\mathbb{N}} [A, y_n].$$

But $\aleph$ is a regular cardinal, so that the P-submodule $[A, y_k]$ has cardinality $\aleph$ for some positive integer k, and hence $[A, y_k] = A$. In particular, the group $\langle y_k, A\rangle$ cannot be nilpotent, and hence $q \neq p$ by a result of Baumslag (see [61] Part 2, Lemma 6.34).

Consider the elementary abelian q-group A as a vector space over the field $\mathbb{F}_q$ with q elements, and for each positive n, let φ_n be the automorphism of A determined by y_n. As $y_n^{p^n} = 1$, the automorphism φ_n is a root of the polynomial

$$x^{p^n} - 1$$

over the field $\mathbb{F}_q$. Thus φ_n admits a minimal polynomial f_n over $\mathbb{F}_q$; moreover, f_n has only simple roots, because it divides $x^{p^n} - 1$ and $p \neq q$. Let

$$f_n = g_1 \cdots g_t$$

be a decomposition of f_n into the product of polynomials $g_1, \dots, g_t$ which are irreducible over $\mathbb{F}_q$. As f_n has only simple roots, the polynomials $g_1, \dots, g_t$ are pairwise coprime, and hence it follows by using Lemma 2.2.76 that

$$A = W_1 \oplus \cdots \oplus W_t,$$

where the subspace W_i is the kernel of $g_i(\varphi_n)$ for each $i = 1, \dots, t$. On the other hand, every $g_i(\varphi_n)$ is a P-endomorphism of A, because P is abelian, and hence $W_1, \dots, W_t$ are P-submodules of A.

But all proper P-submodules of A have cardinality strictly smaller than $\aleph$, and so that $A = W_i$ for some $i \leqslant t$. Thus $g_i(\varphi_n) = 0$, and hence $f_n = g_i$ is an irreducible polynomial over $\mathbb{F}_q$. It follows that the subring R_n generated by φ_n in the endomorphism ring of the $\mathbb{F}_q$-vector space A is a finite field. Moreover, $R_n \leqslant R_{n+1}$ for all n, so that also

$$R = \bigcup_{n \in \mathbb{N}} R_n$$

is a countable subfield of the endomorphism ring of A, and P is contained in the multiplicative group of R. Let

$$A = \bigoplus_{i \in I} A_i$$

be a decomposition of A into the direct sum of R-subspaces of dimension 1. Then every A_i is a countable P-submodule of A, and hence

$$\bigoplus_{i \neq j} A_i$$

is a proper P-submodule of A of cardinality $\aleph$ for any fixed index j. This contradiction completes the proof of the lemma. □

We are now in a position to prove the first main theorem of the section.

Proof of Theorem 2.2.6 — Assume for a contradiction that we have a group G which is not nilpotent-by-finite, so that in particular G has no proper subgroups of finite index. Moreover, the uncountable group G contains a proper subgroup X which is not nilpotent-by-finite, because the class of nilpotent-by-finite groups is countably recognizable.

Suppose that all proper normal subgroups of G have cardinality strictly smaller than $\aleph$. Since G has no simple homomorphic images of cardinality $\aleph$, we have that in this case G is covered by its proper normal subgroups; as in the proof of Theorem 2.1.71, it can be shown that every proper subgroup of G is contained in a proper subgroup of cardinality $\aleph$, and so it is nilpotent-by-finite. This contradiction proves that G has a proper normal subgroup N of cardinality $\aleph$, and G/N has cardinality strictly smaller than $\aleph$ since $G = XN$.

As N is nilpotent-by-finite, its Fitting subgroup F has finite index, and it is known that the factor group G/F is likewise locally graded (see [49]). It follows that also G/N is locally graded. Moreover, all proper subgroups of G/N are nilpotent-by-finite, and so Lemma 2.2.75 yields that G/N contains a nilpotent normal subgroup K/N such that G/K is a Černikov group. Since G has no proper subgroups of finite index, we have that G/K is abelian, so that G/N is soluble and $G' \neq G$. Moreover, G/G' is a group of type p^∞ for some prime number p by Lemma 2.2.73, and G' is nilpotent-by-finite. It follows that G is nilpotent-by-abelian-by-finite, and so even nilpotent-by-abelian. Thus G' is a nilpotent group of cardinality $\aleph$. On the other hand, the tensor product of two abelian groups of cardinality strictly smaller than $\aleph$ has likewise cardinality strictly smaller than $\aleph$, and hence G'/G'' must have cardinality $\aleph$ (see [61] Part 1, Theorem 2.26). Moreover, the factor group G/G'' is not nilpotent (see [61] Part 1, Theorem 2.27), so that it is also a counterexample, and so replacing G by G/G'' it can be assumed without loss of generality that G is metabelian.

Let T be the subgroup consisting of all elements of finite order of G', and assume that $T \neq G'$. An application of Lemma 2.2.74 to the G/G'-module G'/T yields that there exists a normal subgroup L of G such that $T < L < G'$, G'/L is periodic and the set $\pi(G'/L)$ contains more than two prime numbers. Moreover, L has cardinality $\aleph$ by Lemma 2.65, so that all proper subgroups of G/L are nilpotent-by-finite. As $\pi(G/L)$ contains more than two elements, it follows from the structure of minimal non-(nilpotent-by-finite) groups that G is either locally nilpotent or nilpotent-by-finite (see [8], Corollary 2.7). But G has no proper subgroups of finite index, so that G/L is locally nilpotent, a contradiction, because the periodic group G/L cannot be decomposed into the product of two proper normal subgroups. Therefore G' is periodic.

Since $\aleph$ is an uncountable regular cardinal, and G' is a periodic abelian group of cardinality $\aleph$, there exists a prime number q such that also the q-component Q of G' has cardinality $\aleph$. Then the socle A of Q has exponent q and cardinality $\aleph$. But the subgroup X is not nilpotent-by-finite, so that $G = XA$ and $A \cap X$ is a normal subgroup of G. The action by conjugation of G on A induces a structure of G/G'-module on $A/A \cap X$, and hence an application of Lemma 2.2.77 yields that there exists a normal subgroup B of G with cardinality $\aleph$ with the property that $A \cap X < B < A$. Then $XB = G$, and so $A = XB \cap A = B$. This last contradiction completes the proof of the theorem. $\square$

It is clear from results in the first part of the thesis that the class of abelian-by-finite groups is countably recognizable. Therefore any minimal non-(abelian-by-finite) group is countable and we obtain the following corollary (which is still true if we substitute the term abelian with nilpotent of fixed nilpotency class).

Corollary 2.2.7 *Let $\aleph$ be an uncountable regular cardinal, and let G be a locally graded group of cardinality $\aleph$ which has no simple homomorphic images of cardinality $\aleph$. If all proper subgroups of G of cardinality $\aleph$ are abelian-by-finite, then G itself is abelian-by-finite.*

Proof — It can clearly be assumed that the group G has no proper subgroups of finite index. Moreover, G is nilpotent-by-finite by Theorem 2.2.6, and so even nilpotent. It follows that the factor group G/G' has cardinality $\aleph$, and hence Lemma 2.67 yields that there exists a subgroup K of G containing G' such that both K and G/K have cardinality $\aleph$. If X is any subgroup of G of cardinality strictly smaller than $\aleph$, the product XK is a proper subgroup of G, and hence X is abelian-by-finite. Therefore all proper subgroups of G are abelian-by-finite, and so G itself is abelian-by-finite. □

Corollary 2.2.8 *Let $\aleph$ be an uncountable regular cardinal, and G be a group of cardinality $\aleph$ admitting a non-trivial homomorphic image whose cardinality is strictly smaller than $\aleph$.*

(a) *If all proper subgroups of G of cardinality $\aleph$ are nilpotent-by-finite, then G is nilpotent-by-finite.*

(b) *If all proper subgroups of G of cardinality $\aleph$ are abelian-by-finite, then G is abelian-by-finite.*

Proof — Assume for a contradiction that the statement is false, so that it follows from Theorem 2.2.6 and Corollary 2.2.7 that G contains a normal subgroup N such that G/N is a simple group of cardinality $\aleph$. Let K be a proper normal subgroup of G such that the factor group G/K has cardinality strictly smaller than $\aleph$, so that in particular K has cardinality $\aleph$, and hence it is nilpotent-by-finite. Obviously, K is not contained in N, and so $KN = G$. Then the infinite simple group is nilpotent-by-finite, which is clearly impossible. □

Uncountable Recognizability of Locally Nilpotent Groups

We start with two easy lemmas.

Lemma 2.2.9 *Let $\mathfrak{X}$ be a subgroup closed class of groups, and let G be an uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ belong to $\mathfrak{X}$. If G has no Jónsson homomorphic images of cardinality $\aleph$, then every proper normal subgroup of G is an $\mathfrak{X}$-group.*

Proof — Let N be any normal subgroup of G of cardinality strictly smaller than $\aleph$. Then G/N has cardinality $\aleph$, and so it cannot be a Jónsson group. It follows that N is contained in a proper subgroup of G of cardinality $\aleph$, and hence N belongs to $\mathfrak{X}$. □

Our next lemma shows in particular that any uncountable abelian group admits a countable homomorphic image which is not finitely generated.

Lemma 2.2.10 *Let G be a nilpotent-by-finite group whose countable homomorphic images are finitely generated. Then G is finitely generated, and so it satisfies the maximal condition.*

Proof — Suppose first that G is abelian-by-finite. Let A be an abelian subgroup of finite index of G, and consider a countable homomorphic image A/H of A. Clearly, H has only finitely many conjugates in G, so that G/H_G is countable and hence even finitely generated. Thus A/H is finitely generated, and so all countable homomorphic images of A are finitely generated.

Obviously, A has no divisible non-trivial homomorphic images, and in particular it is reduced. Suppose first that A is periodic. Since every non-trivial primary component of A has a non-trivial cyclic direct factor, it is clear that A has only finitely many non-trivial primary components. Let P be any primary component of A, and let B be a basic subgroup of P. Then P/B is a divisible homomorphic image of A, so that it is trivial and $P = B$ is the direct product of a collection of cyclic subgroups and hence it is finite. It follows that A itself is finite. In the general case, let T be the subgroup consisting of all elements of finite order of A. Since A has no divisible non-trivial homomorphic images, a maximal free abelian subgroup U/T of A/T must be finitely generated. Moreover, it follows from the first part of the proof that the periodic group A/U is finite, so that A/T is finitely generated and A splits over T. Thus T is a homomorphic image of A

and so it is finite. Therefore A is finitely generated, and hence also G is finitely generated.

Suppose finally that G is nilpotent-by-finite, and let N be a nilpotent normal subgroup of finite index of G. Then the factor group G/N' is abelian-by-finite, and it follows from the first part of the proof that G/N' is finitely generated. In particular, N/N' is finitely generated, so that N itself is finitely generated. Therefore G is finitely generated. □

Recall that a group class $\mathfrak{X}$ is $\boldsymbol{N}_0$-closed if in any group the product of two normal $\mathfrak{X}$-subgroups is likewise an $\mathfrak{X}$-subgroup. Generalizing this concept, if $\mathfrak{U}$ is a class of groups, we shall say that a group class $\mathfrak{X}$ is *$\boldsymbol{N}_0$-closed* in the universe $\mathfrak{U}$ if, whenever G is an $\mathfrak{U}$-group and X and Y are normal $\mathfrak{X}$-subgroups of G, also the product XY belongs to $\mathfrak{X}$. Of course, a group class is $\boldsymbol{N}_0$-closed in the ordinary sense if and only if it is $\boldsymbol{N}_0$-closed in the universe of all groups.

In our considerations we will need the well-known facts that the order of any finite minimal non-supersoluble group is divisible by at most three prime numbers (see [24]), and that a polycyclic group whose finite homomorphic images are supersoluble is likewise supersoluble (see [5]). Combining these two results, it is easy to show that if G is a finitely generated soluble group whose proper subgroups are supersoluble, then G is either finite or supersoluble (see for instance [34, Lemma 3.1]).

Theorem 2.2.11 *Let $\mathfrak{X}$ be a class of locally supersoluble groups which is S- and $\boldsymbol{N}_0$-closed in a universe $\mathfrak{U}$, and let G be an uncountable $\mathfrak{U}$-group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ belong to $\mathfrak{X}$. If G is locally graded and has no simple homomorphic images of cardinality $\aleph$, then G is locally $\mathfrak{X}$.*

Proof — Assume for a contradiction that the statement is false, so that G contains a finitely generated subgroup E which is not in $\mathfrak{X}$. By Lemma 2.2.9 all proper normal subgroups of G belong to $\mathfrak{X}$. As the class $\mathfrak{X}$ is $\boldsymbol{N}_0$-closed in $\mathfrak{U}$, the subgroup E cannot be contained in the product of finitely many proper normal subgroups of G, and so in particular G cannot be the join of its proper normal subgroups. Since G has no simple homomorphic images of cardinality $\aleph$, it follows that G contains a proper normal subgroup of cardinality $\aleph$. If N is any such normal subgroup, all proper subgroups of G containing N belong to $\mathfrak{X}$, and in particular they are locally supersoluble.

On the other hand, the product EN is a subgroup of cardinality $\aleph$ which is not in $\mathfrak{X}$, so that $EN = G$ and G/N is finitely generated. As N is locally supersoluble, its commutator subgroup N' is locally nilpotent, and hence the factor group G/N is locally graded (see [49]). Then G/N has a finite non-trivial homomorphic image, which is soluble because all its proper subgroups are supersoluble. Therefore the commutator subgroup G' is properly contained in G.

Since $EN = G$ for each normal subgroup N of G of cardinality $\aleph$, all countable homomorphic images of G are finitely generated, and an application of Lemma 2.2.10 yields that any abelian-by-finite homomorphic image of G is finitely generated. In particular, G/G' is finitely generated. As G cannot be the product of two proper normal subgroups, it follows that G/G' is cyclic of prime-power order, and so G is locally polycyclic, because G' is locally supersoluble.

Suppose that $G/G^{(i)}$ is finite for some positive integer i, and assume that the next factor $G^{(i)}/G^{(i+1)}$ of the derived series of G is infinite. Since the abelian-by-finite group $G/G^{(i+1)}$ is finitely generated, the subgroup $G^{(i+1)}$ has cardinality $\aleph$, and so all proper subgroups of $G/G^{(i+1)}$ are supersoluble. Therefore $G/G^{(i+1)}$ is an infinite supersoluble group whose commutator subgroup has finite index, and hence it admits an infinite dihedral homomorphic image, which is impossible because G/G' has prime-power order. This contradiction shows that the group $G/G^{(n)}$ is finite for each non-negative integer n, i.e. all soluble homomorphic images of G are finite.

Let J be the finite residual of G. As $G = NE$ for each normal subgroup of finite index N of G, the factor group G/J is soluble, and so also finite. Moreover, $J = J'$ and J cannot contain proper G-invariant subgroups which have cardinality $\aleph$. Let M be the join of all proper G-invariant subgroups of J. If M is properly contained in J, we have that J/M is a chief factor of G, and hence it is abelian, a contradiction. Therefore $J = M$ is the join of its proper G-invariant subgroups. As G is locally polycyclic, the subgroup $E \cap J$ is finitely generated, and so its normal closure $(E \cap J)^G$ is a proper subgroup of J. Put $\overline{G} = G/(E \cap J)^G$. Since J is perfect, there exists a proper G-invariant subgroup $\overline{K}$ of $\overline{J}$ containing an element $\overline{x}$ such that $[\overline{x}, \overline{J}] \neq \{1\}$. Now, the index of $\overline{C} = C_{\overline{G}}(\overline{x})$ in $\overline{G}$ is strictly smaller than $\aleph$, because all conjugates of $\overline{x}$ belong to $\overline{K}$. If $\overline{y}$ is any element of the finite subgroup $\overline{E}$, then

$$|\overline{C} : \overline{C} \cap \overline{C}^{\overline{y}}| \leqslant |\overline{G} : \overline{C}^{\overline{y}}| < \aleph$$

and hence

$$\overline{W} = \overline{J} \cap \Big(\bigcap_{\overline{y} \in \overline{E}} \overline{C}^{\overline{y}}\Big)$$

is a proper $\overline{E}$-invariant subgroup of $\overline{J}$ of cardinality $\aleph$. It follows that the product EW is a proper subgroup of G of cardinality $\aleph$, and this contradiction completes the proof of the theorem. □

Observe that the class of locally nilpotent groups is $\boldsymbol{N}_0$-closed by the theorem of Hirsch and Plotkin, while it follows easily from a result of Baer that the class of locally supersoluble groups is $\boldsymbol{N}_0$-closed in the universe of groups with locally nilpotent commutator subgroup (see for instance [34, Lemma 2.2]). Therefore both Theorem 2.2.12 and Theorem 2.2.13 are special cases of Theorem 2.2.11.

Uncountable Recognizability of Nilpotency

Now we are able to prove that the class of nilpotent groups is uncountably recognizable.

Proof of Theorem 2.2.14 — Assume for a contradiction that the statement is false. As the class of nilpotent groups is countably recognizable, there exists in G a countable non-nilpotent subgroup X. Moreover, all proper normal subgroups of G are nilpotent by Lemma 2.2.9, and so X cannot be contained in a proper normal subgroup of G, i.e. $X^G = G$.

Let H be a normal subgroup of G of cardinality strictly smaller than $\aleph$, and suppose that G has no proper normal subgroups of cardinality $\aleph$ containing H. Since G has no infinite simple homomorphic images, it follows that G is generated by its proper normal subgroups containing H. Moreover, there exists clearly a sequence $(K_n)_{n\in\mathbb{N}}$ of proper normal subgroups of G containing H such that X lies in

$$\langle K_n \mid n \in \mathbb{N}\rangle,$$

and this latter is a proper normal subgroup of G, because $\aleph$ has cofinality strictly larger than $\aleph_0$. This contradiction proves that H is contained in a proper normal subgroup of G of cardinality $\aleph$, and in particular G has proper normal subgroups of cardinality $\aleph$. If N is any such normal subgroup, the product NX is not nilpotent, so that $NX = G$ and hence G/N is countable. Moreover, N/N' likewise has cardinality $\aleph$, and then the commutator subgroup N' of N must have cardinality strictly smaller than $\aleph$.

Let a be any element of G such that $\langle a\rangle^G \neq G$. The above argument shows that a belongs to a proper normal subgroup N of G of cardinality $\aleph$. As N' has cardinality $\aleph' < \aleph$, the element a has less than $\aleph$ conjugates in N, and so also in G. Then the normal closure $\langle a\rangle^G$ has cardinality strictly smaller than $\aleph$. On the other hand, X is countable and $X^G = G$, so that by the cofinality assumption on $\aleph$ there exists an element x of X such that $\langle x\rangle^G = G$. It follows in particular that G properly contains the join M of all its proper normal subgroups. Then M is nilpotent and G/M is finite.

The group G is locally nilpotent by Theorem 2.2.12, and so its elements of finite order form a subgroup T. Suppose that T is properly contained in G. As G is nilpotent-by-finite, it follows from the theory of isolators in torsion-free locally nilpotent groups that the factor group G/T is nilpotent (see for instance [48, Section 2.3]); on the other hand, G cannot be the product of two proper normal subgroups, and so it has no torsion-free abelian non-trivial homomorphic images. Then $G = T$ is a periodic group, and hence it is the direct product of its Sylow subgroups. But all proper normal subgroups of G are nilpotent, and so it follows that G is a p-group for some prime number p.

As $\overline{M} = M/M'$ has cardinality $\aleph$, also its socle $\overline{S}$ has cardinality $\aleph$, and so G/S is countable. Let $\overline{U}$ be a G-invariant subgroup of $\overline{S}$ which is maximal with respect to the condition of being the direct product of a collection of finite G-invariant subgroups, and assume that $\overline{U}$ has cardinality strictly smaller than $\aleph$. Then

$$\overline{S} = \overline{U} \times \overline{V}$$

for a suitable subgroup $\overline{V}$, and the index of $|\overline{G} : \overline{V}|$ is strictly smaller than $\aleph$. On the other hand, $\overline{V}$ has finitely many conjugates in $\overline{G}$, so that also the index $|\overline{G} : \overline{V}_{\overline{G}}|$ is strictly smaller than $\aleph$, and hence the core $\overline{V}_{\overline{G}}$ of $\overline{V}$ in G has cardinality $\aleph$. If $\overline{y}$ is a non-trivial element of $\overline{V}_{\overline{G}}$, the normal closure $\langle\overline{y}\rangle^{\overline{G}}$ is finite and

$$\langle \overline{U}, \langle\overline{y}\rangle^{\overline{G}}\rangle = \overline{U} \times \langle\overline{y}\rangle^{\overline{G}}.$$

This contradiction shows that $\overline{U}$ has cardinality $\aleph$, and so $\overline{U} = \overline{U}_1 \times \overline{U}_2$, where both $\overline{U}_1 = U_1/M'$ and $\overline{U}_2 = U_2/M'$ have cardinality $\aleph$. It follows that XU_1 is a proper subgroup of G of cardinality $\aleph$, and this last contradiction completes the proof of the theorem. □

Our next result deals with groups whose proper subgroups of large cardinality are nilpotent, but under the assumption of (GCH).

THEOREM 2.2.14′ *Assume that the generalized continuum hypothesis holds, and let* G *be an uncountable group of cardinality* $\aleph$ *which has no infinite simple homomorphic images. If all proper subgroups of* G *of cardinality* $\aleph$ *are nilpotent, then* G *itself is nilpotent.*

PROOF — Assume for a contradiction that the statement is false, so that G contains a countable non-nilpotent subgroup X, and Theorem 2.2.14 yields that $\aleph > \aleph_1$. By Lemma 2.2.9 all proper normal subgroups of G are nilpotent, so that in particular G cannot be the product of two proper normal subgroups. Then G/G' is a locally cyclic p-group for some prime number p, and hence all nilpotent homomorphic images of G are countable. It follows that if K is any normal subgroup of G of cardinality strictly smaller than $\aleph$, the factor group G/K is not nilpotent, and so it is a counterexample to the statement.

Suppose that all proper normal subgroups of G have cardinality strictly smaller than $\aleph$, and let N be a proper normal subgroup of G containing a non-central element x. The factor group $G/C_G(\langle x\rangle^G)$ embeds into the automorphism group of $\langle x\rangle^G$, and so it has cardinality at most $2^{\aleph'}$, where $\aleph' < \aleph$ is the cardinality of N. On the other hand, also the cardinality of $C_G(\langle x\rangle^G)$ is strictly smaller than $\aleph$, so that $G/C_G(\langle x\rangle^G)$ has cardinality $\aleph$, and hence $2^{\aleph'} = \aleph$ by (GCH). It follows that each proper normal subgroup of G has cardinality at most $\aleph'$. Moreover, G cannot contain proper subgroups of finite index, so that it has no simple homomorphic images, and hence by Zorn's Lemma G can be decomposed as the set-theoretic union of a chain $(N_\lambda)_{\lambda\in\Lambda}$ of proper normal subgroups. Let Λ_0 be a countable subset of Λ such that X is contained in the normal subgroups

$$W = \bigcup_{\lambda\in\Lambda_0} N_\lambda,$$

which of course has cardinality at most $\aleph'$. This is a contradiction, because all proper normal subgroups of G are nilpotent. Therefore G contains a proper normal subgroup M of cardinality $\aleph$.

As M is nilpotent, the group M/M' likewise has cardinality $\aleph$. Then the product XM' is a proper subgroup of G, so that M' has cardinality strictly smaller than $\aleph$ and hence G/M' is not nilpotent. Thus G may be replaced by G/M', and so without loss of generality it can be assumed that M is abelian. Clearly $XM = G$, so that $X \cap M$ is a normal subgroup of G and the factor group $G/X \cap M$ is not nilpotent. A further replacement of G by $G/X \cap M$ allows now to suppose that $X \cap M = \{1\}$. Clearly, M has a countable non-trivial homomorphic image M/L, and the subgroup L has only countably many conjugates in G. Therefore there exists a countable subset Y of G such that the factor group M/L_G embeds into the cartesian product of the collection of groups $(M/L^y)_{y \in Y}$, and hence M/L_G has cardinality at most $\aleph_1$. But $\aleph > \aleph_1$, and so the normal subgroup L_G has cardinality $\aleph$. Then $XL_G = G$ and hence

$$M = XL_G \cap M = L_G(X \cap M) = L_G.$$

This last contradiction completes the proof of the theorem. □

Corollary 2.2.15 *Let G be an uncountable group of cardinality $\aleph$ which has no infinite simple homomorphic images. If all proper subgroups of G of cardinality $\aleph$ are nilpotent with class at most c (where c is a fixed positive integer), then G itself is nilpotent with class at most c, provided that either the cofinality of $\aleph$ is strictly larger than $\aleph_0$ or the generalized continuum hypothesis is assumed to hold.*

Proof — The group G is nilpotent either by Theorem 2.2.14 or by Theorem 2.2.14′, and so the factor group G/G' has cardinality $\aleph$. It follows from Lemma 2.2.10 that G contains a normal subgroup N such that G/N is a countable abelian group which is not finitely generated. Let E be any finitely generated subgroup of G. Then EN is a proper subgroup of G of cardinality $\aleph$, and hence it has class at most c. Therefore also G has nilpotency class at most c. □

Uncountable Recognizability of Soluble Groups

We turn now to the solubility. The first lemma shows that if G is an uncountable group whose proper subgroups of large cardinality belong to a group class $\mathfrak{X}$, then G contains a large normal $\mathfrak{X}$-subgroup, under suitable closure conditions on the class $\mathfrak{X}$.

Lemma 2.2.16 *Let $\mathfrak{X}$ be a class of groups which is **S** and **L**-closed, and let G be an uncountable group of cardinality $\aleph$ whose proper subgroups* *of* G *of cardinality* $\aleph$ *belong to* $\mathfrak{X}$*. Then either* $G/Z(G)$ *is a simple group of cardinality* $\aleph$ *or* G *contains a normal* $\mathfrak{X}$*-subgroup* N *such that the factor group* G/N *is simple.*

Proof — Assume for a contradiction that the group G does not contain any normal $\mathfrak{X}$-subgroup N such that G/N is simple. Then it follows from Zorn's Lemma and from the $\boldsymbol{L}$-closure of the class $\mathfrak{X}$ that G contains proper normal subgroups which are not in $\mathfrak{X}$. Let K be any such subgroup, and let $E = \langle x_1, \dots, x_t \rangle$ be any finitely generated subgroup of K which is not in $\mathfrak{X}$. It follows from the $\boldsymbol{S}$-closure of $\mathfrak{X}$ that E cannot be contained in a proper subgroup of G of cardinality $\aleph$. The conjugacy class of any element of K in G has cardinality strictly smaller than $\aleph$, and so in particular $|G : C_G(x_i)| < \aleph$ for each $i = 1, \dots, t$. Thus $|G : C_G(E)| < \aleph$, and so the centralizer $C_G(E)$ has cardinality $\aleph$. On the other hand, the product $EC_G(E)$ cannot belong to $\mathfrak{X}$, so that $G = EC_G(E)$ and the subgroup $C_G(E)$ is normal in G. If $C_G(E) \neq G$, the centralizer $C_G(E)$ is contained in a maximal normal subgroup M of G, which of course has cardinality $\aleph$ and so belongs to $\mathfrak{X}$, contrary to our assumptions. Therefore E is contained in $Z(G)$, and so also K lies in $Z(G)$. The factor group G/E has cardinality $\aleph$, while all its proper subgroups have cardinality strictly smaller than $\aleph$, i.e. G/E is a Jónsson group, and so G/C is simple of cardinality $\aleph$, where $C/E = Z(G/E)$. Moreover, the normal subgroup C of G is not in $\mathfrak{X}$, and hence the same argument used above for K yields that C is contained in $Z(G)$. Therefore $C = Z(G)$, and $G/Z(G)$ is a simple group of cardinality $\aleph$. □

The consideration of Jónsson groups shows that in the above results the group G can be far from being in $\mathfrak{X}$. However, something more can be said if the group has no large simple homomorphic images.

Corollary 2.2.17 *Let* $\mathfrak{X}$ *be a class of groups which is* $\boldsymbol{S}$ *and* $\boldsymbol{L}$*-closed and contains all abelian groups. If* G *is an uncountable group of cardinality* $\aleph$ *whose proper subgroups of cardinality* $\aleph$ *belong to* $\mathfrak{X}$*, then* G *contains a normal* $\mathfrak{X}$*-subgroup* N *such that the factor group* G/N *is simple.*

Corollary 2.2.18 *Let* G *be an uncountable group of cardinality* $\aleph$ *whose proper subgroups of cardinality* $\aleph$ *locally satisfy the maximal condition. If* G *has no infinite simple homomorphic images, then it locally satisfies the maximal condition.*

Proof — It follows from Corollary 2.2.17 that G contains a normal subgroup N locally satisfying the maximal condition such that G/N is simple. Then G/N is finite, and so G locally satisfies the maximal condition. □

Corollary 2.2.19 *Let $\mathfrak{X}$ be a class of groups which is **S** and **L**-closed, and let G be an uncountable group of cardinality $\aleph$ which has no simple homomorphic images of cardinality $\aleph$. If every proper subgroup of cardinality $\aleph$ of G belongs to $\mathfrak{X}$, then G contains a normal $\mathfrak{X}$-subgroup N such that G/N is a finitely generated simple group.*

Proof — The statement is obvious when G lies in $\mathfrak{X}$, so that we may suppose that G is not an $\mathfrak{X}$-group, and hence it contains a finitely generated subgroup E which is not in $\mathfrak{X}$. Since G has no simple homomorphic images of cardinality $\aleph$, it follows from Lemma 2.2.16 that there exists a normal $\mathfrak{X}$-subgroup N of G such that the factor group G/N is simple and has cardinality strictly smaller than $\aleph$. Then N has cardinality $\aleph$, so that $EN = G$ and hence G/N is finitely generated. □

Corollary 2.2.20 *Let G be an uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ are locally soluble. If G has no simple non-abelian homomorphic images, then it is locally soluble.*

Corollary 2.2.21 *Let G be an uncountable group of cardinality $\aleph$ which has no simple non-abelian homomorphic images. If all proper subgroups of G of cardinality $\aleph$ are locally polycyclic, then G itself is locally polycyclic.*

If in the statements of Lemma 2.2.16 and Corollary 2.2.19 we choose for $\mathfrak{X}$ the class of soluble groups with derived length at most k, where k is a fixed positive integer, we obtain the following consequence.

Corollary 2.2.22 *Let G be an uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ are soluble with derived length at most k, where k is a fixed positive integer. Then G contains a soluble normal subgroup N of derived lenght at most k such that the factor group G/N is simple. Moreover, if G is locally graded and has no simple homomorphic images of cardinality $\aleph$, then G/N is finite and in particular G is soluble-by-finite.*

Proof — The first part of the statement is just a special case of Corollary 2.2.17. Suppose now in addition that G is locally graded and has no simple homomorphic images of cardinality $\aleph$. Then the simple factor group G/N is finitely generated by Corollary 2.2.19. Moreover, as N is soluble, G/N is locally graded (see [49]) and hence it must be finite. □

We can finally prove the last theorem of the section.

Proof of Theorem 2.2.23 — The group G is soluble by Corollary 2.2.20 and Corollary 2.2.22. Assume for a contradiction that G has derived length $n > k$, and consider a finitely generated subgroup E of G with derived length n. Let i be the largest non-negative integer such that the term $G^{(i)}$ of the derived series of G has cardinality $\aleph$. In particular, the subgroup $G^{(i)}E$ has cardinality $\aleph$, and so $G = G^{(i)}E$. Suppose that $E/E \cap G^{(i)}$ is infinite. Then $E \cap G^{(i)}$ is properly contained in a proper subgroup X of E with derived length n (see [22], Theorem 1), so that $G^{(i)}X = G$ and hence

$$E = E \cap G^{(i)}X = (E \cap G^{(i)})X = X.$$

This contradiction shows that $E/E \cap G^{(i)}$ must be finite, so that also the factor group $G/G^{(i)}$ is finite. However, the subgroup $G^{(i+1)}$ has cardinality strictly smaller than $\aleph$, and therefore the abelian factor group $G^{(i)}/(EG^{(i+1)} \cap G^{(i)})$ is infinite and hence it contains a proper subgroup $U/(EG^{(i+1)} \cap G^{(i)})$ such that $G^{(i)}/U$ is countable. Then the factor group $G^{(i)}/U_G$ is likewise countable, and so U_G has cardinality $\aleph$. Therefore $U_GE = G$, so that

$$G^{(i)} = U_GE \cap G^{(i)} = U_G(E \cap G^{(i)}) \leqslant U$$

and this last contradiction completes the proof. □

2.3 Groups with a Normal Transitive Property

The aim of this section is to investigating uncountable groups of cardinality $\aleph$ in which all proper subgroups of cardinality $\aleph$ have a transitive normality relation. The corresponding problem in the case of groups of infinite rank has been solved in [16].

We say that a group G has the T-*property* (or is a T-*group*) if normality in G is a transitive relation, i.e. if all subnormal subgroups of G are normal. The structure of soluble T-groups has been described by W. Gaschütz [27] in the finite case and, for arbitrary groups, by Robinson [62]. It turns out in particular that soluble groups with the T-property are metabelian and hypercyclic, and that finitely generated soluble T-groups are either finite or abelian. Although the class of T-groups is not subgroup closed (because any simple group is obviously a T-group), it is known that subgroups of finite soluble T-groups likewise have the T-property. A group G is called a $\overline{\mathrm{T}}$-*group* if all its subgroups have the T-property. It follows easily from the properties of T-groups, that any finite $\overline{\mathrm{T}}$-group is soluble, and so even supersoluble, while soluble non-periodic groups with the $\overline{\mathrm{T}}$-property are abelian.

It turns out that an uncountable soluble group in which all proper normal subgroups have the T-property need not be a T-group. However, our first main result shows in particular that a group of this type has finite conjugacy classes of subnormal subgroups. The structure of soluble groups with this latter property was investigated by C. Casolo [11].

Theorem 2.3.8 *Let* G *be an uncountable soluble group of cardinality* $\aleph$ *whose proper normal subgroups of cardinality* $\aleph$ *have the* T-*property. Then every subnormal subgroup of* G *has only finitely many conjugates.*

Recall that a group is *subsoluble* if it has an ascending series with abelian factors consisting of subnormal subgroups; in particular, all hyperabelian groups are subsoluble. Our second main result deals with uncountable subsoluble groups in which every proper large subgroup is a T-group, and proves that these groups have the $\overline{\mathrm{T}}$-property.

Theorem 2.3.15 *Let* G *be an uncountable subsoluble group of cardinality* $\aleph$ *whose proper subgroups of cardinality* $\aleph$ *have the* T-*property. Then* G *is a* $\overline{\mathrm{T}}$-*group.*

The T-Property for Large Normal Subgroups

It seems to be unclear whether a subsoluble (or even hyperabelian) uncountable group of cardinality $\aleph$ must contain at least one proper normal subgroup of cardinality $\aleph$. However, this property obviously holds in the case of abelian groups, and hence also for uncountable groups which properly contain their commutator subgroup.

Lemma 2.3.23 *Let G be an uncountable group with cardinality $\aleph$ and such that $G' \neq G$. Then G contains a proper normal subgroup of cardinality $\aleph$.*

The imposition of the T-property to large proper normal subgroups of uncountable groups has a strong effect, at least when the commutator subgroup is not large. In fact, it turns out in particular that if G is an uncountable group whose proper normal uncountable subgroups have the T-property, then G itself is a T-group, provided that its commutator subgroup G' is countable.

Lemma 2.3.24 *Let G be an uncountable group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property. If G/G' has cardinality $\aleph$, then G is a T-group.*

Proof — Let X be a subnormal subgroup of G such that XG'/G' has cardinality strictly smaller than $\aleph$, and let g be any element of G. Then the abelian group $G/\langle g, X, G'\rangle$ has cardinality $\aleph$, and so it contains a proper subgroup $H/\langle g, X, G'\rangle$ of cardinality $\aleph$. Then H is a normal subgroup of cardinality $\aleph$, so that it is a T-group and X is normal in H. It follows that $X^g = X$, and hence X is a normal subgroup of G.

Suppose now that X is any subnormal subgroup of G. If x is any element of X, the subgroup $\langle x, X'\rangle$ is subnormal in G, and the factor group $\langle x, X'\rangle G'/G' = \langle x\rangle G'/G'$ is countable, so that $\langle x, X'\rangle$ is normal in G by the first part of the proof. Therefore also

$$X = \big\langle \langle x, X'\rangle \mid x \in X \big\rangle$$

is a normal subgroup of G, and hence G is a T-group. □

Corollary 2.3.25 *Let G be an uncountable group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property. If G' has cardinality strictly smaller than $\aleph$, then G is a T-group.*

Notice that in the latter two statements the assumption that the cardinal number $\aleph$ is uncountable cannot be omitted. In fact, in the direct product $G = \mathrm{Alt}(4) \times P$, where $\mathrm{Alt}(4)$ is the alternating group of degree 4 and P is a group of type p^∞ for some prime number $p > 3$, all infinite proper normal subgroups have the T-property but G is not a T-group, although its commutator subgroup is finite.

As we mentioned, a soluble group in which all proper normal subgroups have the T-property need not be a T-group. To see this, let M be an abelian group of exponent 7 and let α be the automorphism of M defined by putting $a^\alpha = a^2$ for all $a \in M$. Then $\alpha^3 = 1$, and we may consider a homomorphism

$$\theta : \mathrm{Alt}(4) \longrightarrow \mathrm{Aut}(M)$$

such that $\mathrm{Im}\theta = \langle \alpha \rangle$. The semidirect product

$$G = \mathrm{Alt}(4) \ltimes_\theta M$$

is a periodic metabelian group whose proper normal subgroups are abelian, but clearly G is not a T-group. If the group M is chosen to be uncountable, this example also shows that under the assumptions of Theorem 2.3.8 the T-property may not hold for the group G. The above construction can be slighty modified in order to produce a soluble group of derived length 3 whose proper normal subgroups have the T-property; it is enough to replace M by the direct product $K = Q_8 \times M$, where Q_8 is a quaternion group of order 8, and extend α to K in such a way that it acts on Q_8 as an automorphism of order 3.

It is clear that if all proper normal subgroups of a group G have the T-property, then such property also holds for all proper subnormal subgroups of G. Our next lemma shows that this situation also occurs whenever G is an uncountable soluble group whose proper normal subgroups of high cardinality have the T-property.

Lemma 2.3.26 *Let G be an uncountable soluble group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property. Then all proper subnormal subgroups of G have the T-property.*

Proof — Let X be any proper subnormal subgroup of G. If the normal closure X^G of X has cardinality strictly smaller than $\aleph$, the factor group G/X^G has cardinality $\aleph$, and so by Lemma 2.3.23 it contains a proper normal subgroup of cardinality $\aleph$. Thus in any case X is contained in a proper normal subgroup H of G of cardinality $\aleph$. As H is a T-group and X is subnormal, it follows that also X has the T-property. □

Corollary 2.3.27 *Let G be an uncountable soluble group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the T-property. Then the group G has derived length at most 3.*

Lemma 2.3.28 *Let G be a group whose proper normal subgroups have the T-property, and let X be a subgroup of G such that $[X, G'] \leqslant X$. Then X has only finitely many conjugates in G.*

Proof — As the normalizer $N_G(X)$ contains G', it is normal in G and so the subgroup X is subnormal in G. Of course, it can be assumed that X is not normal in G. As all proper subgroups of G containing $N_G(X)$ have the T-property and so normalize X, it follows that $N_G(X)$ is a maximal subgroup of G. Hence the index $|G : N_G(X)|$ is finite, and so X has finitely many conjugates in G. □

Note that the proof of Lemma 2.3.28 actually shows that under the same assumptions the subgroup X either is normal or has a prime number of conjugates.

If G is any uncountable soluble group of cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ are T-groups, it follows from Lemma 2.3.26 that each proper subnormal subgroup of G has the T-property, and so Theorem 2.3.8 is a special case of our next result.

Theorem 2.3.29 *Let G be a soluble group whose proper normal subgroups have the T-property. Then every subnormal subgroup of G has only finitely many conjugates.*

Proof — Assume for a contradiction that the statement is false, and let X be a subnormal subgroup of G admitting infinitely many conjugates. As the commutator subgroup G' of G is a T-group, it follows from Lemma 2.3.28 that each subnormal subgroup of G' has finitely many conjugates in G. In particular, the normalizer $N_G(X \cap G')$ has finite index in G, so that X has infinitely many conjugates in the subgroup $N_G(X \cap G')$. Moreover, $N_G(X \cap G')$ is normal in G, because it contains G', and so all its proper normal subgroups have the T-property. Therefore the factor group $N_G(X \cap G')/X \cap G'$ is also a counterexample to the statement, and hence it can be assumed without loss of generality that $X \cap G' = \{1\}$.

Another application of Lemma 2.3.28 yields that X is not normalized by G', so that XG' is not a T-group and hence $XG' = G$. It follows from the Dedekind's modular law that the subnormal subgroup XG'' is properly contained in G, so that it has the T-property and X is normalized by G''. Then

$$G'' \leqslant N_G(X) \cap G' < G',$$

and so $N_G(X) \cap G'$ is normal in $G = XG'$. Let K be any normal subgroup of G such that

$$N_G(X) \cap G' \leqslant K < G'.$$

Thus XK is a proper subnormal subgroup of G, so that it is a T-group and X is normalized by K, and hence $K = N_G(X) \cap G'$. Therefore $G'/N_G(X) \cap G'$ is a chief factor of G. Let g be an element of G' such that $X^g \neq X$, so that

$$G' = \left\langle g, N_G(X) \cap G' \right\rangle^G.$$

On the other hand, $\left\langle g, N_G(X) \cap G' \right\rangle$ is a normal subgroup of G', so that it has finitely many conjugates in G. It follows that the chief factor $G'/N_G(X) \cap G'$ is finitely generated, and hence even finite. Therefore

$$|G : N_G(X)| = |G' : G' \cap N_G(X)|$$

is finite, and this contradiction completes the proof of the theorem. □

As soluble groups with finite conjugacy classes of subnormal subgroups are metabelian-by-finite (see [11]), the following statement is a consequence of Theorem 2.3.29; it applies in particular to the case of an uncountable soluble group of cardinality ℵ whose proper normal subgroups of cardinality ℵ have the T-property.

Corollary 2.3.9 *Let G be a soluble group whose proper normal subgroups have the T-property. Then G has derived length at most 3 and contains a metabelian subgroup of finite index.*

The T-Property for Large Subgroups

Let $\mathfrak{X}$ be a class of groups. A group G is said to be *minimal non-$\mathfrak{X}$* if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. The structure of minimal non-$\mathfrak{X}$ groups has been studied for several different choices of the group class $\mathfrak{X}$. As the T-property is local, it is clear that any minimal non-T group is countable. Moreover, finite minimal non-T groups are soluble, since it is well-known that any finite group with only supersoluble proper subgroups is soluble. It seems to be an open question whether there exist infinite minimal non-T groups. On the other hand, it was proved that a locally finite group whose proper subgroups have the T-property is either finite or a T-group (see [63]), and a similar result holds also in the case of groups with no infinite simple sections [15].

It is also clear that every group with no infinite simple sections is locally graded. Therefore the following theorem extends both the above quoted results about minimal non-T groups.

Theorem 2.3.10 *Let G be an infinite locally graded group whose proper subgroups have the T-property. Then G is a T-group.*

Proof — Assume for a contradiction that G is not a T-group, so that G is minimal non-T, and hence it is finitely generated, because the T-property is local. If J is the finite residual of G, it follows that the group G/J is infinite.

Let N be any normal subgroup of finite index of G. Then the factor group G/N either is minimal non-T or has the $\overline{\mathrm{T}}$-property, so that in any case it is soluble and has derived length at most 3. Therefore also the infinite group G/J is soluble, so that it cannot be minimal non-T (see [15]) and hence it has the $\overline{\mathrm{T}}$-property. Moreover, G/J cannot be periodic, and so it is a finitely generated abelian group. In particular, there exist two maximal subgroups M_1 and M_2 of G both containing J. As M_1 and M_2 have the T-property, it follows that every subnormal subgroup of J is normal in G.

Let X be a subnormal non-normal subgroup of G. Clearly, G can be generated by elements having infinite order with respect to J, and so there exists an element g of G such that $X^g \neq X$ and the coset gJ has infinite order. The intersection $X \cap J$ is a normal subgroup of G, and the factor group $G/X \cap J$ is likewise a counterexample, so that without loss of generality we may suppose that $X \cap J = \{1\}$. Then X is abelian, and hence it is contained in the Baer radical B of G. Moreover, the subgroup $\langle B, g\rangle$ is subsoluble, so that it is properly contained in G and hence it is a non-periodic group with the $\overline{T}$-property. Therefore $\langle B, g\rangle$ is abelian, which is impossible as $X^g \neq X$. This contradiction proves the statement. □

We turn now to the proof of Theorem 2.3.15. The first lemma shows in particular that any uncountable abelian group of cardinality $\aleph$ has a residual system consisting of normal subgroups with indices strictly smaller than $\aleph$.

Lemma 2.3.11 *Let A be an uncountable abelian group of cardinality $\aleph$, and let B be a subgroup of A of cardinality strictly smaller than $\aleph$. Then A contains a subgroup C such that $B \cap C = \{1\}$ and $|A : C| < \aleph$.*

Proof — Let A^* be the divisible hull of A. Then A^* has cardinality $\aleph$, and so

$$A^* = \operatorname*{Dr}_{i \in I} D_i^*$$

where each D_i^* is isomorphic either to the additive group of rational numbers or to a Prüfer group, and the index set I has cardinality $\aleph$. Clearly, I has a subset I' of cardinality strictly smaller than $\aleph$ such that

$$B^* = \operatorname*{Dr}_{i \in I'} D_i^*$$

contains B. If

$$C^* = \operatorname*{Dr}_{i \in I \setminus I'} D_i^*$$

we have that $C = A \cap C^*$ is a subgroup of A such that $B \cap C = \{1\}$ and

$$|A : C| \leqslant |A^* : C^*| < \aleph.$$

The statement is proved. □

Lemma 2.3.12 *Let G be a group, and let A be an uncountable abelian normal subgroup of G of cardinality $\aleph$. If each subgroup of A has only finitely many conjugates in G, then there exists a collection $(a_\alpha)_{\alpha \in \aleph}$ of non-trivial elements of A such that*

$$\langle a_\alpha \mid \alpha \in \aleph\rangle^G = \operatorname*{Dr}_{\alpha \in \aleph} \langle a_\alpha\rangle^G.$$

Proof — Choose any non-trivial element a_0 of A, and suppose that λ is an element of $\aleph$ for which a_α has been defined for all $\alpha < \lambda$

in such a way that

$$B = \langle a_\alpha \mid \alpha < \lambda \rangle^G = \operatorname*{Dr}_{\alpha<\lambda} \langle a_\alpha \rangle^G.$$

Since every $\langle a_\alpha \rangle^G$ is countable, the subgroup B has cardinality strictly smaller than $\aleph$, and so Lemma 2.3.11 yields that A contains a subgroup C such that $B \cap C = \{1\}$ and $|A : C| < \aleph$. But C has only finitely many conjugates in G, and hence we have also $|A : W| < \aleph$, where W is the core of C in G. In particular, W cannot be trivial and so we can choose an element a_λ in W. The normal closure $\langle a_\lambda \rangle^G$ is contained in W, so that $B \cap \langle a_\lambda \rangle^G = \{1\}$ and hence

$$\langle a_\alpha \mid \alpha \leqslant \lambda \rangle = B \times \langle a_\lambda \rangle^G = \operatorname*{Dr}_{\alpha\leqslant\lambda} \langle a_\alpha \rangle^G.$$

The proof of the statement can now be completed by transfinite induction on λ. □

Corollary 2.3.13 *Let G be a subsoluble uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the T-property. Then G is metabelian.*

Proof — It is well-known that any subsoluble T-group is metabelian, so that all proper subgroups of G of cardinality $\aleph$ are metabelian. Therefore the group G is metabelian by Theorem 2.2.3. □

Next lemma is the main step in the proof of Theorem 2.3.15.

Lemma 2.3.14 *Let G be an uncountable subsoluble group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the T-property. Then G is a T-group.*

Proof — The group G is metabelian by Corollary 2.3.13, so that in particular each subgroup of G' has only finitely many conjugates in G by Lemma 2.3.28. Moreover, by Corollary 2.3.25 it can be assumed without loss of generality that the commutator subgroup G' of G has cardinality $\aleph$, so that it follows from Lemma 2.3.12 that G' contains a G-invariant subgroup of the form

$$U = \operatorname*{Dr}_{i\in I} U_i,$$

where each U_i is a countable non-trivial normal subgroup of G and the set I has cardinality $\aleph$.

Let X be any subnormal subgroup of G, and suppose first that X has cardinality strictly smaller than $\aleph$. The above decomposition of U shows the existence of two G-invariant subgroups V and W of U of cardinality $\aleph$ such that

$$V \cap W = \langle V, W \rangle \cap X = \{1\}.$$

The factor groups G/V and G/W are T-groups by Theorem 2.3.10. It follows that the subnormal subgroups XV and XW of G are normal, so that also $X = XV \cap XW$ is normal in G. In particular, all cyclic subnormal subgroups of G are normal, and hence every subgroup of G' is normal in G.

Suppose now that the subnormal subgroup X has cardinality $\aleph$. If the commutator subgroup X' has cardinality strictly smaller than $\aleph$, the above argument yields that the subgroup $\langle x, X' \rangle$ is normal in G for each element x of X, so that also

$$X = \big\langle \langle x, X' \rangle \mid x \in X \big\rangle$$

is normal in G. Suppose finally that the abelian subgroup X' has cardinality $\aleph$, so that by Lemma 2.2.10 it contains a proper subgroup Y such that the index $|X' : Y|$ is countably infinite. Then Y is a normal subgroup of G of cardinality $\aleph$, so that all proper subgroups of the infinite group G/Y have the T-property and hence G/Y itself is a T-group. Therefore X is normal in G, and G is a T-group. □

Proof of Theorem 2.3.15 — The group G is a T-group by Lemma 2.3.14, and in particular it is metabelian. Assume for a contradiction that G is not a $\overline{T}$-group, so that it contains a subgroup H of cardinality strictly smaller than $\aleph$, which has a subnormal non-normal subgroup K. If G contains an abelian normal subgroup A of cardinality $\aleph$, an application of Lemma 2.3.12 yields that A contains two G-invariant subgroups V and W of cardinality $\aleph$ such that

$$V \cap W = \langle V, W \rangle \cap K = \{1\}.$$

Then the subgroups HV and HW have the T-property, so that the subgroups KV and KW are normal in HV and HW, respectively, and hence

$$K = KV \cap KW$$

is normal in H. This contradiction shows that G has no abelian normal subgroups of cardinality $\aleph$. In particular, the Fitting subgroup F of G has cardinality strictly smaller than $\aleph$, so that the index $|G:F|$ is infinite. It follows that F is periodic (see [62]), and hence also G' is a periodic subgroup of cardinality strictly smaller than $\aleph$. Of course, the intersection $K \cap G'$ is normal in G, and the replacement of G by $G/K \cap G'$ allows to assume that $K \cap G' = \{1\}$.

Suppose that G is not periodic, so that G' is divisible and the equality $[G', G] = G'$ holds. Write

$$G' = \underset{i \in I}{\mathrm{Dr}}\, P_i$$

where each P_i is a group of type p_i^∞ for some prime number p_i and put

$$P_i^* = \underset{j \neq i}{\mathrm{Dr}}\, P_j$$

for each index i. Then

$$K = \bigcap_{i \in I} KP_i^*$$

and so there is an index j such that KP_j^* is not normalized by H. It follows that the factor group G/P_j^* is also a counterexample, and hence it can be assumed without loss of generality that G' is a group of type p^∞ for some prime p. As $[G', G] = G'$, the cohomology class of the extension

$$G' \rightarrowtail G \twoheadrightarrow G/G'$$

has finite order and hence G nearly splits over G' (see [65] and [66]). This means that there exists a subgroup L of G such that $G = LG'$ and $L \cap G'$ is finite. Clearly, L contains an element a of infinite order, and $[G', a] \neq \{1\}$ because $C_G(G') = F$ is periodic. Let x be an element of G' such that $L \cap G' \leqslant \langle x \rangle$ and $[x, a] \neq 1$. As $\langle x, L \rangle$ is a subgroup of G of cardinality $\aleph$, it has the T-property and hence also its normal subgroup $\langle x, a \rangle$ is a T-group. But finitely generated soluble T-groups are abelian, so that $xa = ax$ and this contradiction proves that the counterexample G must be periodic.

For each prime number p, let $G'(p')$ be the subgroup consisting of all elements of G' whose order is prime to p. As

$$K = \bigcap_{p} KG'(p')$$

there exists a prime number q such that H does not normalize the subgroup $KG'(q')$ and hence a further replacement of G by $G/G'(q')$ allows to assume that G' is a q-group. It follows that the factor group $G/C_G(G')$ is finite, because it is isomorphic to a group of power automorphisms of G' and all periodic groups of power automorphisms of an abelian q-group are finite. This last contradiction completes the proof of Theorem 2.3.15. □

2.4 Lattice Properties

The aim of this section is to investigating uncountable groups of cardinality $\aleph$ in which all proper subgroups of cardinality $\aleph$ have modular subgroup lattice. The corresponding problem in the case of groups of infinite rank has been solved in [19]. Recall here that a lattice $\mathfrak{L}$ is *modular* if the identity

$$(x \vee y) \wedge z = x \vee (y \wedge z)$$

holds for all elements x, y, z of $\mathfrak{L}$ such that $x \leqslant z$. Obviously, any abelian group has modular subgroup lattice, and hence groups with modular subgroup lattice naturally arise in the study of projectivities (i.e. lattice isomorphisms) between groups; on the other hand, there exist also infinite simple groups with modular subgroup lattice, like for instance Tarski groups (i.e. infinite simple groups all of whose proper non-trivial subgroups have prime order). Our first main result is the following.

Theorem 2.4.4 *Let* G *be an uncountable group of cardinality* $\aleph$ *whose proper subgroups of cardinality* $\aleph$ *have modular subgroup lattice. If* G *has no simple homomorphic images of cardinality* $\aleph$*, then the lattice of subgroups of* G *is modular.*

Our second main result deals with uncountable groups whose proper uncountable subgroups are quasihamiltonian. A group G is said to be *quasihamiltonian* if all its subgroups are permutable, i.e. if $XY = YX$ for all subgroups X and Y of G. Of course, all quasihamiltonian groups have modular subgroup lattice, and in fact it turns out that a group is quasihamiltonian if and only if it is locally nilpotent and the lattice of its subgroups is modular.

Theorem 2.4.5 *Let* G *be an uncountable group of cardinality* $\aleph$ *whose proper subgroups of cardinality* $\aleph$ *are quasihamiltonian. If* G *has no simple homomorphic images of cardinality* $\aleph$*, then it is quasihamiltonian.*

The consideration of Jónsson groups shows that the assumption on the simple homomorphic images cannot be dropped out from our statements.

Proofs of the Theorems

The structure of groups with modular subgroup lattice has been completely described by K. Iwasawa ([41],[42]) and R. Schmidt [69]. We give here a short account of their results, and refer to [70] for a detailed treatment of groups with such property.

Let G be any non-periodic group with modular subgroup lattice. Then the set T of all elements of finite order of G is an abelian subgroup and the factor group G/T is abelian; moreover, if G is not abelian, the torsion-free abelian group G/T has rank 1 (see [70], Theorem 2.4.11). It follows that the commutator subgroup of any group with modular subgroup lattice is periodic, and so torsion-free groups with modular subgroup lattice are abelian.

We consider now periodic groups with modular subgroup lattice, describing their structure first in the locally finite case. It is necessary to recall here that an automorphism θ of a group G is a *power automorphism* if θ maps every subgroup of G onto itself. The set $\mathrm{PAut}(G)$ of all power automorphisms of a group G is a subgroup of the full automorphism group of G, and it is easy to see that $\mathrm{PAut}(G)$ is residually finite (i.e. the intersection of all its subgroups of finite index is trivial). Power automorphisms play a crucial role in many relevant problems of group theory.

Let p and q be two different prime numbers. A group is called a P^*-*group* of type (p, q) if it is a semidirect product $\langle x\rangle \ltimes A$, where A is an abelian normal subgroup of exponent p and x is an element of order q^n acting on A as a power automorphism of order q. The lattice of subgroups of any P^*-group is modular (see [70], Lemma 2.4.1), and it turns out that a locally finite group G has modular subgroup lattice if and only if it is a direct product

$$G = \mathop{\mathrm{Dr}}_{i\in I} G_i,$$

where each factor G_i is either a P^*-group or a locally finite p-group with modular subgroup lattice and elements in different factors have coprime orders (see [70], Theorem 2.4.13). Note also that, if p is any prime number, a locally finite p-group G has modular subgroup lattice if and only if either G is a Dedekind group or it contains an abelian normal subgroup A of finite exponent p^k and an element x such that $G/A = \langle xA\rangle$ has order p^m, and $a^x = a^{1+p^s}$ for all $a \in A$, where s is a positive integer such that $s < k \leqslant s + m$, and $s \geqslant 2$

if $p = 2$ (see [70], Theorem 2.4.14).

As we mentioned, the lattice of all subgroups of a Tarski group is obviously modular, and groups of this type must occur in the structure of periodic groups with modular subgroup lattice which are not locally finite. A group G is an *extended Tarski group* if it contains a cyclic normal subgroup N with prime-power order such that G/N is a Tarski group and each subgroup of G either contains or is contained in N; it is easy to show that extended Tarski groups must be primary and have modular subgroup lattice. The classification of groups with modular subgroup lattice was completed by R. Schmidt, who proved that a periodic group has modular subgroup lattice if and only if it is a direct product of Tarski groups, extended Tarski groups and locally finite groups with modular subgroup lattice such that elements in different factors have coprime orders (see [70], Theorem 2.4.16).

Our first elementary lemma shows that in our considerations the attention can be restricted to the case of groups with small centre.

Lemma 2.4.15 *Let $\mathfrak{X}$ be a subgroup closed class of groups, and let G be an uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ belong to $\mathfrak{X}$. If the centre $Z(G)$ has cardinality $\aleph$, then all proper subgroups of G are $\mathfrak{X}$-groups.*

Proof — As $Z(G)$ is an abelian group of cardinality $\aleph$, it contains a subgroup C such that both C and $Z(G)/C$ have cardinality $\aleph$ by Lemma 2.3.11. If X is any proper subgroup of G of cardinality strictly smaller than $\aleph$, the product XC is a proper subgroup of G of cardinality $\aleph$. Then XC belongs to $\mathfrak{X}$, and so also X is an $\mathfrak{X}$-group. □

Our next result provides a further information on uncountable groups with modular subgroup lattice, that will be relevant in the proof of Theorem 2.2.4.

Lemma 2.4.16 *Let G be an uncountable non-abelian group of cardinality $\aleph$. If G has modular subgroup lattice, then for some prime number p it contains a normal Sylow p-subgroup P of cardinality $\aleph$ which is a finite extension of an abelian subgroup A of finite exponent such that all subgroups of A are normal in G.*

Proof — Since G is a non-abelian group with modular subgroup lattice, its elements of finite order form a subgroup T and the factor group G/T is countable. Then T has cardinality $\aleph$, and it can be decomposed into a direct product

$$T = \operatorname*{Dr}_{i \in I} T_i,$$

where each factor T_i is either a Tarski group or an extended Tarski group or locally finite, and elements of different factors have coprime orders. In particular, only countably many factors which are not locally finite can occur in such decomposition, and so $T = H \times K$, where H is countable, K is locally finite and $\pi(H) \cap \pi(K) = \emptyset$. It follows now from the structure of locally finite groups with modular subgroup lattice that $K = U \times V$, where the factor U has cardinality $\aleph$ and it is either a p-group for some prime p or a P^*-group of type (p, q) for two different primes p and q, and $\pi(U) \cap \pi(V) = \emptyset$. In both cases, U has a unique Sylow p-subgroup P, which has cardinality $\aleph$ and is a finite extension of an abelian subgroup A of finite exponent, such that U induces on A a group of power automorphisms. Then all subgroups of A are normal in T. On the other hand, it is well known that if G is not periodic, i.e. if $T \neq G$, all subgroups of T are normal in G (see [70], Lemma 2.4.8), and so it follows that in any case every subgroup of A is normal in G. □

Proof of Theorem 2.2.4 — Assume for a contradiction that the statement is false. As the class of groups with modular subgroup lattice is local, G contains a finitely generated subgroup $E = \langle x_1, \dots, x_t \rangle$ whose subgroup lattice is not modular, and E cannot be contained in a proper subgroup of G of cardinality $\aleph$.

Suppose first that there exists in G a proper normal subgroup N of finite index. Then N has modular subgroup lattice, and so it follows from Lemma 2.4.16 that N contains an abelian characteristic subgroup A of cardinality $\aleph$ and prime exponent p such that every subgroup of A is normal in N, and so has only finitely many conjugates in G. As E is finitely generated, A contains a subgroup A_0 such that A/A_0 is countable and $E \cap A_0 = \{1\}$. Obviously, A_0 has cardinality $\aleph$, and contains a proper subgroup B such that A_0/B is countable. Let $C = B_G$ be the core of B in G. Then also A/C is countable, because B has finitely many conjugates in G, and so C has cardinality $\aleph$. It follows that $EC = G$, and hence

$$A_0 = EC \cap A_0 = C(E \cap A_0) = C.$$

This contradiction shows that G has no proper subgroups of finite index.

All proper normal subgroups of G have modular subgroup lattice by Lemma 2.2.9, so that $E^G = G$ and hence the group G cannot be the join of a chain of proper normal subgroups. It follows from Zorn's lemma that G contains a maximal normal subgroup M. By hypothesis the simple group G/M has cardinality strictly smaller than $\aleph$, so that M has cardinality $\aleph$ and hence $EM = G$. Let $i \leqslant t$ be any positive integer such that the subgroup $\langle x_i, M\rangle$ is not abelian. As $\langle x_i, M\rangle$ is properly contained in G, it has modular subgroup lattice, and by Lemma 2.4.16 we can find a prime number p_i such that $\langle x_i, M\rangle$ has a normal Sylow p_i-subgroup P_i of cardinality $\aleph$ which is a finite extension of an abelian subgroup U_i of finite exponent such that all subgroups of U_i are normal in $\langle x_i, M\rangle$. Then also M has a normal Sylow p_i-subgroup $Q_i = P_i \cap M$, and Q_i has cardinality $\aleph$ because G/M is countable. Clearly, Q_i is a normal subgroup of G, so that $EQ_i = G$ and in particular M/Q_i is countable; note also that the subgroup $V_i = U_i \cap M$ has finite index in Q_i, and hence M/V_i is countable (although not necessary, it can be noted here that $p_i = p_j$ whenever $\langle x_i, M\rangle$ and $\langle x_j, M\rangle$ are both non-abelian). Whenever $k \leqslant t$ is a positive integer such that $\langle x_k, M\rangle$ is abelian, put also $V_k = M$. Then the abelian subgroup

$$V = \bigcap_{i=1}^{t} V_i$$

is normal in $G = EM$, and the factor group M/V is countable, so that V has cardinality $\aleph$. Moreover, all subgroups of V are normal in G, and hence $G/C_G(V)$ is isomorphic to a group of power automorphisms of V. Since any group of power automorphisms is residually finite, it follows that $G/C_G(V)$ is trivial, so that V lies in the centre of G, and hence $Z(G)$ has cardinality $\aleph$. Then all proper subgroups of G have modular subgroup lattice by Lemma 2.4.15, and this last contradiction completes the proof of the theorem. □

Similar cardinality problems can of course be studied for other lattice properties of groups, like for instance the (much) stronger property of having distributive subgroup lattice. On the other hand, it is well known that a group has distributive subgroup lattice if and only if it is locally cyclic (see [70], Theorem 1.2.3), and so it follows that if G is any uncountable group whose proper uncountable subgroups have distributive subgroup lattice, then G has the Jónsson property.

Proof of Theorem 2.2.5 — The group G has modular subgroup lattice by Theorem 2.2.4, so that in particular the commutator subgroup G' of G is periodic. Moreover, all proper normal subgroups of G are quasihamiltonian by Lemma 2.2.9. Suppose first that G is not periodic, so that it is generated by its elements of infinite order. If a is any element of infinite order of G, we have $\langle a \rangle = \langle a^p, a^q \rangle$ for all different primes p and q, and hence

$$G = \langle H \mid G' \leqslant H < G \rangle.$$

As all proper subgroups of G containing G' are locally nilpotent, it follows from the well known theorem of Hirsch and Plotkin that G itself is locally nilpotent, and hence quasihamiltonian.

Suppose now that G is periodic, so that it is a direct product of Tarski groups, extended Tarski groups and a locally finite group of cardinality $\aleph$. On the other hand, since all proper normal subgroups of G are quasihamiltonian, Tarski and extended Tarski factors cannot occur in this decomposition. Then G is locally finite and all its proper subgroups of cardinality $\aleph$ are locally nilpotent, so that G itself is locally nilpotent by Theorem 2.2.12. Therefore G is a quasihamiltonian group. $\square$

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