Spin actions in Euclidean and Hermitian Clifford analysis in superspace

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Abstract

In [4] we studied the group invariance of the inner product of supervectors as introduced in the framework of Clifford analysis in superspace. The fundamental group SO_0 leaving invariant such an inner product turns out to be an extension of $\mathrm{SO}(m)\times \mathrm{Sp}(2n)$ and gives rise to the definition of the spin group in superspace through the exponential of the so-called extended superbivectors, where the spin group can be seen as a double covering of SO_0 by means of the representation $h(s)[x]=sx\overline{s}$. In the present paper, we study the invariance of the Dirac operator in superspace under the classical H and L actions of the spin group on superfunctions. In addition, we consider the Hermitian Clifford setting in superspace, where we study the group invariance of the Hermitian inner product of supervectors introduced in [3]. The group of complex supermatrices leaving this inner product invariant constitutes an extension of $\mathrm{U}(m)\times \mathrm{U}(n)$ and is isomorphic to the subset SO_0^J of SO_0 of elements that commute with the complex structure J. The realization of SO_0^J within the spin group is studied together with the invariance under its actions of the super Hermitian Dirac system. It is interesting to note that the spin element leading to the complex structure can be expressed in terms of the n-dimensional Fourier transform.

Keywords. Group invariance, Hermitian Clifford analysis, superspace, Dirac operator
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1 Introduction

In a previous paper, [4], we have introduced the spin group $\mathrm{Spin}(m|2n)(\Lambda_N)$ in the framework of Clifford analysis in superspace, where we consider m bosonic dimensions, $2n$ fermionic dimensions and the Grassmann algebra $\mathbb{R}\Lambda_N$ as the set of coefficients. Such a definition aims, as is the case in the classical setting, at describing the set of rotations in the space $\mathbb{R}^{m|2n}(\mathcal{V})$ of supervectors in terms of Clifford multiplication. However, here a more complicated situation arises since *supervector reflections* do not suffice for describing the whole set of supermatrices leaving the inner product of supervectors invariant. This is due to the structure of the real projection of the group $\mathrm{SO}_0(m|2n)(\mathbb{R}\Lambda_N)$ (SO_0 for short) of superrotations, which is given by $\mathrm{SO}(m)\times \mathrm{Sp}(2n)$ and clearly contains a real, non-nilpotent symplectic part. The non-nilpotent part of the supervector variables is given only by the classical Clifford vector part. Therefore the reflections generated by the supervectors only include the real rotation group $\mathrm{SO}(m)$.

This issue may be solved by means of the extension $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$ of the Lie algebra of superbivectors, which turns out to be isomorphic to the Lie algebra $\mathfrak{so}_0(m|2n)(\mathbb{R}\Lambda_N)$ ($\mathfrak{so}_0$ for short) of SO_0. The Lie group SO_0 is connected, whence it can be fully described through finite products of exponentials of $\mathfrak{so}_0$–elements. Hence, the exponentials of the extended superbivectors in $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$ can be seen as generators of the spin group in this setting. In fact, such a group fully covers SO_0 through the representation $h(s)[x]=sx\overline{s}$, $s\in \mathrm{Spin}(m|2n)(\Lambda_N)$, $x\in\mathbb{R}^{m|2n}(\mathcal{V})$. In particular, SO_0 can be decomposed as the product of three exponential maps acting in some specific subspaces $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_3$ of $\mathfrak{so}_0$ where $\mathfrak{so}_0=\mathfrak{s}_1\oplus\mathfrak{s}_2\oplus\mathfrak{s}_3$. The corresponding isomorphic decomposition for $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$ leads to a subset S of $\mathrm{Spin}(m|2n)(\Lambda_N)$ which was proven to be a double covering of SO_0, see [4].

In this paper, we study the operator actions on superfunctions, associated to the h-representation, given by $[H(s)[F(x)] = sF(\bar{s}xs)\bar{s}$ and $L(s)[F(x)] = sF(\bar{s}xs)$, $s \in \mathrm{Spin}(m|2n)(\Lambda_N)$. As in the classical case, the super Dirac operator ∂_x is invariant under those actions. We explicit this invariance through the commutation of ∂_x with the infinitesimal representation $dL(B) = \frac{d}{dt}L(e^{tB})\big|_{t=0}$, for every extended superbivector B. We recall that the action of the representation dL on the basis elements of $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$ gives rise to the super angular momentum operators.

In addition, we also study these spin actions within the Hermitian Clifford system in superspace. The basics of Hermitian Clifford analysis in superspace were introduced in [3] following the notion of an abstract complex structure in the Hermitian radial algebra, developed in [5, 8]. The radial algebra approach has been proven to be an efficient tool for giving meaning to vector spaces of negative dimension, abstractly defining the fundamental objects of Clifford analysis, such as vector variables and vector derivatives, and describing their main commutation properties. But, as it has been shown in [4], different realizations of the radial algebra setting may differ on some representation theory issues. For example, the compactness of $\mathrm{SO}(m)$, which is the group acting on the Clifford polynomial realization of the radial algebra, does not hold for SO_0, which is the group acting on the supersymmetry realization.

For that reason, we first study the group of complex supermatrices $U_0(m|n)(\mathbb{C}\Lambda_N)$ leaving the Hermitian inner product $\{z, u^\dagger\}$ in the space of complex supervectors $\mathbb{C}^{m|n}(\mathcal{V})$ invariant. The real realization $\mathrm{SO}_0^J(2m|2n)(\mathbb{R}\Lambda_N)$ of this group is composed by all $\mathrm{SO}_0(2m|2n)(\mathbb{R}\Lambda_N)$ supermatrices which commute with the complex structure $J \in \mathrm{SO}_0(2m|2n)(\mathbb{R}\Lambda_N)$. The corresponding projections operators $1/2(1 \pm iJ)$ establish the transition from the initial basis $\{e_j, j = 1, \ldots, 2m\}$ and $\{\grave{e}_j, j = 1, \ldots, 2n\}$ to the Witt basis. These projections also give rise to a direct sum decomposition of $\mathbb{C}^{2m|2n}(\mathcal{V})$ into two components, where $\mathrm{SO}_0^J(2m|2n)(\mathbb{R}\Lambda_N)$ is the group leaving those subspaces invariant. In addition, it turns out that $\mathrm{U}_0(m|n)(\mathbb{C}\Lambda_N)$ contains the group $\mathrm{U}(m) \times \mathrm{U}(n)$ as complex projection and preserves the connectedness from its classical antecedent $U(m)$. But, as is the case for SO_0, also $\mathrm{U}_0(m|n)(\mathbb{C}\Lambda_N)$ does not preserve the compactness and cannot be described by a single action of the exponential map on its Lie algebra. Nevertheless, its isomorphic copy $\mathrm{SO}_0^J(2m|2n)(\mathbb{R}\Lambda_N)$ may be decomposed as the product of only two exponentials, respectively acting on the elements of $\mathfrak{s}_1$ and $\mathfrak{s}_3$ which commute with J.

The subgroup $\mathrm{SO}_0^J(2m|2n)(\mathbb{R}\Lambda_N)$ is covered by a subgroup of $\mathrm{Spin}(2m|2n)(\Lambda_N)$, which is denoted by $\mathrm{Spin}_J(2m|2n)(\Lambda_N)$ and generated by the exponentials of extended superbivectors that are invariant under the action of the complex structure. Using the above-mentioned decomposition for $\mathrm{SO}_0^J(2m|2n)(\mathbb{R}\Lambda_N)$, we construct a subset S_J of $S \cap \mathrm{Spin}_J(m|2n)(\Lambda_N)$ which constitutes a double covering of $\mathrm{SO}_0^J(2m|2n)(\mathbb{R}\Lambda_N)$. Those properties allow to prove the invariance of the twisted super Dirac operator $\partial_{J(x)}$ under the $\mathrm{Spin}_J(2m|2n)(\Lambda_N)$-actions.

The paper is organized as follows. In section 2 we start with some preliminaries on super analysis, Grassmann envelopes and supermatrices. In section 3 we first describe the basics of Clifford analysis in superspace. We recall some results obtained in [4] related to the group of superrotations SO_0, the spin group and the algebra of extended superbivectors. Then, we study the invariance of the super Dirac operator under the H and L spin actions by checking its commutation with the infinitesimal representation dL of L. Section 4 is devoted to study of the group actions in the Hermitian Clifford setting in superspace. We start with the introduction of the complex structure J, defined in [3] in the radial algebra setting, as an element of $\mathrm{SO}_0(2m|2n)(\mathbb{R}\Lambda_N)$ when acting on $\mathbb{R}^{2m|2n}(\mathcal{V})$. This allows to determine the subspace of real supermatrices of order $(2m|2n)$ leaving the subspaces $1/2(1 \pm iJ)[\mathbb{R}^{2m|2n}(\mathcal{V})]$ invariant. Such a subspace of real $(2m|2n)$ supermatrices is isomorphic to a space of complex $(m|n)$ supermatrices, leading to the corresponding isomorphism of the connected groups $\mathrm{SO}_0^J(2m|2n)(\mathbb{R}\Lambda_N)$ and $\mathrm{U}_0(m|n)(\mathbb{C}\Lambda_N)$. Finally, in section 5 we study the spin realization $\mathrm{Spin}_J(2m|2n)(\Lambda_N)$ of $\mathrm{SO}_0^J(2m|2n)(\mathbb{R}\Lambda_N)$ through different characterizations of the extended superbivectors whose exponentials generate $\mathrm{Spin}_J(2m|2n)(\Lambda_N)$. A fundamental rôle is played here by the extended superbivector $\mathbf{B} = \frac{1}{2}\partial_x[J(x)]$ which generates the spin element $s_J = \exp(-\frac{\pi}{4}\mathbf{B})$ associated to J. Through the corresponding identifications, the fermionic part of s_J may be identified with the classical Fourier transform in n-bosonic dimensions.

2 Preliminaries

In our approach to super analysis we consider two sufficiently large sets $VAR := \{v, w, \ldots\}$ and $\grave{VAR} := \{\grave{v}, \grave{w}, \ldots\}$ of commuting (bosonic) and anti–commuting (fermionic) variables, respectively. In general, those variables can be defined in every graded-commutative Banach superalgebra $\Lambda = \Lambda_{\overline{0}} \oplus \Lambda_{\overline{1}}$ as explained in [13]. However, it is worth noticing some particular issues related to the choice made. For example, two polynomials in fermionic variables define the same Λ-valued function if and only if all coefficients of their difference belong to the annihilator of $\Lambda_{\overline{1}}$ defined by

$$^{\perp}\Lambda_{\overline{1}} := \{f \in \Lambda : \; fa = af = 0, \, \forall a \in \Lambda_{\overline{1}}\}.$$

One of the important consequences of this fact is the non-uniqueness of the odd derivatives whenever $^{\perp}\Lambda_{\overline{1}} \neq \{0\}$, see [13]. On the other hand, if $^{\perp}\Lambda_{\overline{1}} = \{0\}$, then the following conclusions hold:

- two polynomials define the same function if and only if they are identical;
- the odd derivatives are unique according to the definition given in [13].

The discussion about a suitable choice for the underlying superalgebra Λ is beyond the scope of this paper. In what follows, we consider the variables in VAR, $\grave{VAR}$ as independent symbols satisfying the first property stated above. In the next section, the partial bosonic (even) and fermionic (odd) derivatives will be defined as endomorphisms in the algebra of polynomials in $VAR \cup \grave{VAR}$ in accordance with the second property. This approach turns out to be suitable for any application and corresponding choice of Λ since every superalgebra can be embedded in another one with annihilator $^{\perp}\Lambda_{\overline{1}} = \{0\}$.

Next to the set of co-ordinate variables $VAR \cup \grave{VAR}$, we consider a set of coefficients containing the real numbers and some anti–commuting elements. In this paper, we consider the most simple set of such coefficients: the Grassmann algebra $\mathbb{K}\Lambda_N$ generated over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) by the canonical odd (anti–commuting) elements $f_1, \ldots, f_N$, $N \in \mathbb{N}$. In this way, we obtain the algebra of superpolynomials

$$\mathbb{K}\mathcal{V} := \mathbb{K}\Lambda_N \otimes \mathrm{Alg}_{\mathbb{K}}(VAR \cup \grave{VAR})$$

governed by the rules

$$f_j f_k = -f_k f_j, \qquad f_j v = v f_j \qquad f_j \grave{v} = -\grave{v} f_j, \qquad vw = wv, \qquad \grave{v}\grave{w} = -\grave{w}\grave{v}, \qquad v\grave{w} = \grave{w}v,$$

where the f_j's are the odd Grassmann generators of $\mathbb{K}\Lambda_N$, $v, w \in VAR$, and $\grave{v}, \grave{w} \in \grave{VAR}$. In this case, the set of constant polynomials (degree 0) is given by the Grassman algebra $\mathbb{K}\Lambda_N$. Every coefficient $a \in \mathbb{K}\Lambda_N$ can be written as $a = \sum_{A \subset \{1,\ldots,N\}} a_A f_A$, where $a_A \in \mathbb{K}$ and $f_A = f_{j_1} \cdots f_{j_k}$ for $A = \{j_1, \ldots, j_k\}$, with $1 \leq j_1 < \ldots < j_k \leq N$; moreover $f_\emptyset = 1$. The space of homogeneous Grassmann coefficients of degree k is defined by $\mathbb{K}\Lambda_N^{(k)} = \mathrm{span}_{\mathbb{K}}\{f_A : \; |A| = k\}$; in particular $\mathbb{K}\Lambda_N^{(k)} = \{0\}$ for $k > N$. It easily follows that

$$\mathbb{K}\Lambda_N = \bigoplus_{k=0}^{N} \mathbb{K}\Lambda_N^{(k)} \qquad \text{and} \qquad \mathbb{K}\Lambda_N^{(k)}\mathbb{K}\Lambda_N^{(\ell)} \subset \mathbb{K}\Lambda_N^{(k+\ell)}. \tag{1}$$

The projection of $\mathbb{K}\Lambda_N$ on its k-homogeneous part is denoted by $[\cdot]_k : \mathbb{K}\Lambda_N \to \mathbb{K}\Lambda_N^{(k)}$, i.e. $[a]_k = \sum_{|A|=k} a_A f_A$, and in particular $[a]_0 = a_\emptyset =: a_0$. The algebra $\mathbb{K}\mathcal{V}$ can be decomposed as the direct sum

$$\mathbb{K}\mathcal{V} = \mathbb{K}\mathcal{V}_{\overline{0}} \oplus \mathbb{K}\mathcal{V}_{\overline{1}}$$

where $\mathbb{K}\mathcal{V}_{\overline{0}}$ is the even subalgebra (generated by commuting elements) and $\mathbb{K}\mathcal{V}_{\overline{1}}$ is the odd subalgebra (generated by anticommuting elements). This decomposition defines a natural $\mathbb{Z}_2$-grading, turning $\mathbb{K}\mathcal{V}$ into a graded commutative superalgebra, since $\mathbb{K}\mathcal{V}_{\overline{j}}\mathbb{K}\mathcal{V}_{\overline{k}} \subset \mathbb{K}\mathcal{V}_{\overline{j+k}}$ for $\overline{j}, \overline{k} \in \mathbb{Z}_2$. This is compatible with the $\mathbb{Z}_2$-grading of the coefficient algebra $\mathbb{K}\Lambda_N$ since $\mathbb{K}\mathcal{V}_{\overline{j}} \cap \mathbb{K}\Lambda_N =: \mathbb{K}\Lambda_{N,\overline{j}}$ ($\overline{j} \in \mathbb{Z}_2$) where

$$\mathbb{K}\Lambda_{N,\overline{0}} = \bigoplus_{k\geq 0} \mathbb{K}\Lambda_N^{(2k)} \quad \text{and} \quad \mathbb{K}\Lambda_{N,\overline{1}} = \bigoplus_{k\geq 0} \mathbb{K}\Lambda_N^{(2k+1)}$$

are the even and odd subalgebras of $\mathbb{K}\Lambda_N$, respectively.

From (1) it follows that every $a \in \mathbb{K}\Lambda_N$ is the sum of a number $a_0 \in \mathbb{K}\Lambda_N^{(0)} = \mathbb{K}$ and a nilpotent element, since, in fact, every $\mathbf{a} \in \mathbb{K}\Lambda_N^+ := \bigoplus_{k=1}^N \mathbb{K}\Lambda_N^{(k)}$ satisfies $\mathbf{a}^{N+1} = 0$. It is easily seen that the projection $[\cdot]_0 : \mathbb{K}\Lambda_N \to \mathbb{K}$ is an algebra homomorphism, i.e., $[ab]_0 = a_0 b_0$ for every $a, b \in \mathbb{K}\Lambda_N$.

The algebra $\mathbb{K}\Lambda_N$ is a $\mathbb{K}$-vector space of dimension 2^N. As is the case for every finite dimensional $\mathbb{K}$-vector space, we can introduce an arbitrary norm (all norms being equivalent), turning $\mathbb{K}\Lambda_N$ into a Banach space. We consider the norm given by $|a| = \sum_A |a_A|$, $a \in \mathbb{K}\Lambda_N$, for which it is easily proven that

$$|ab| \leq |a||b|, \qquad \forall a, b \in \mathbb{K}\Lambda_N. \tag{2}$$

For every $a \in \mathbb{K}\Lambda_N$ we define the exponential of a, denoted e^a or $\exp(a)$, by the power series

$$\exp(a) = \sum_{j=0}^{\infty} \frac{a^j}{j!}$$

On account of the norm inequality above, we obtain the uniform convergence of the series, thence the continuity of the exponential map in $\mathbb{K}\Lambda_N$.

2.1 Grassmann envelopes and supermatrices

Consider the graded vector space $\mathbb{K}^{p|q}$ with standard basis $e_1, \ldots, e_p, \grave{e}_1, \ldots, \grave{e}_q$, where $\{e_1, \ldots, e_p\}$ is a basis for $\mathbb{K}^{p|0}$ and $\{\grave{e}_1, \ldots, \grave{e}_q\}$ is a basis for $\mathbb{K}^{0|q}$, with $\mathbb{K}^{p|q} = \mathbb{K}^{p|0} \oplus \mathbb{K}^{0|q}$. The *Grassmann envelope* $\mathbb{K}^{p|q}(\Lambda_N)$ is defined as the set of formal linear combinations

$$x = \underline{x} + \grave{\underline{x}} = \sum_{j=1}^{p} x_j e_j + \sum_{j=1}^{q} \grave{x}_j \grave{e}_j, \qquad \text{where } \; x_j \in \mathbb{K}\Lambda_{N,\bar{0}}, \quad \grave{x}_j \in \mathbb{K}\Lambda_{N,\bar{1}}, \tag{3}$$

constituting a $\mathbb{K}$-vector space of dimension $2^{N-1}(p+q)$ which inherits the $\mathbb{Z}_2$-grading of $\mathbb{K}^{p|q}$, i.e., $\mathbb{K}^{p|q}(\Lambda_N) = \mathbb{K}^{p|0}(\Lambda_N) \oplus \mathbb{K}^{0|q}(\Lambda_N)$ where $\mathbb{K}^{p|0}(\Lambda_N)$ denotes the subspace of vectors of the form (3) with $\grave{x}_j = 0$, and $\mathbb{K}^{0|q}(\Lambda_N)$ denotes the subspace of vectors of the form (3) with $x_j = 0$; $\mathbb{K}^{p|0}(\Lambda_N)$ and $\mathbb{K}^{0|q}(\Lambda_N)$ are called the Grassmann envelopes of $\mathbb{K}^{p|0}$ and $\mathbb{K}^{0|q}$, respectively. In $\mathbb{K}^{p|q}(\Lambda_N)$, there exists a subspace which is naturally isomorphic to $\mathbb{K}^{p|0}$: it consists of vectors (3) of the form $x = \sum_{j=1}^m x_j e_j$ with $x_j \in \mathbb{K}$. This leads to the important projection $[\cdot]_0 : \mathbb{K}^{p|q}(\Lambda_N) \to \mathbb{K}^{p|0}$ defined by: $[x]_0 = \sum_{j=1}^p [x_j]_0 \, e_j$.

We will represent elements of the standard basis of $\mathbb{K}^{p|q}$ also by column matrices, i.e.

$$\begin{aligned} e_j &= (0, \ldots, 1, \ldots, 0)^T && \text{(1 on the } j\text{-th place from the left)}, \\ \grave{e}_j &= (0, \ldots, 1, \ldots, 0)^T && \text{(1 on the } (p+j)\text{-th place from the left)}. \end{aligned}$$

A general element of $\mathbb{K}^{p|q}(\Lambda_N)$ then is represented by the column $x = (x_1, \ldots, x_p, \grave{x}_1, \ldots, \grave{x}_q)^T$.

A natural generalization of $\mathbb{K}^{p|q}(\Lambda_N)$ is the $\mathcal{V}$-envelope $\mathbb{K}^{p|q}(\mathcal{V})$ defined by elements of the form (3) but with $x_j \in \mathbb{K}\mathcal{V}_{\bar{0}}$ and $\grave{x}_j \in \mathbb{K}\mathcal{V}_{\bar{1}}$.

The Grassmann envelope of the space of endomorphisms on $\mathbb{K}^{p|q}$ is known as the space of supermatrices, meant to produce linear operators on $\mathbb{K}^{p|q}(\mathcal{V})$ or on its restriction $\mathbb{K}^{p|q}(\Lambda_N)$. The space of supermatrices is denoted by $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$ and consists of block matrices of the form

$$M = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & \grave{B} \\ \grave{C} & 0 \end{pmatrix} \tag{4}$$

where[1] $A \in (\mathbb{K}\Lambda_{N,\bar{0}})^{p \times p}$, $\grave{B} \in (\mathbb{K}\Lambda_{N,\bar{1}})^{p \times q}$, $\grave{C} \in (\mathbb{K}\Lambda_{N,\bar{1}})^{q \times p}$ and $D \in (\mathbb{K}\Lambda_{N,\bar{0}})^{q \times q}$. The $\mathbb{Z}_2$-grading of $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$, inherited from $\mathbb{K}\Lambda_N$, together with the usual matrix multiplication, provides a superalgebra structure to this Grassmann envelope. In this structure the first term in (4) is called the even part of M, while the second term is called the odd one.

[1] Given a set $\mathcal{S}$, we use the notation $\mathcal{S}^{p \times q}$ to refer to the set of matrices of order $p \times q$ with entries in $\mathcal{S}$.

For arbitrary $k \in \mathbb{N}$, let $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{(k)})$ be the space of homogeneous supermatrices of degree k. Then $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{(2k)})$ consists of all diagonal block matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

with entries in $\mathbb{K}\Lambda_N^{(2k)}$, while $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{(2k+1)})$ consists of all off-diagonal block matrices

$$\begin{pmatrix} 0 & \grave{B} \\ \grave{C} & 0 \end{pmatrix}$$

with entries in $\mathbb{K}\Lambda_N^{(2k+1)}$. These subspaces define a grading in $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$ in the sense that

$$\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N) = \bigoplus_{k=0}^{N} \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{(k)})$$

and

$$\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{(k)})\ \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{(\ell)}) \subset \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{(k+\ell)}).$$

It follows that every supermatrix M can be written as the sum of a numeric (real or complex) matrix $M_0 \in \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{(0)})$ and a nilpotent supermatrix $\mathbf{M} \in \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{+}) := \bigoplus_{k=1}^{N} \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{(k)})$. In accordance with the general principles for Grassmann algebras and Grassmann envelopes we define the algebra homomorphism $[\cdot]_0 : \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N) \to \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^{(0)})$ as the projection:

$$M = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} \longrightarrow \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix} = M_0 = [M]_0$$

where A_0 and D_0 are the numeric projections of A and D on $\mathbb{K}^{p\times p}$ and $\mathbb{K}^{q\times q}$, respectively. Furthermore, given a set of supermatrices $\mathbf{S}$ we define $[\mathbf{S}]_0 = \{[M]_0 : M \in \mathbf{S}\}$.

Every supermatrix M defines a linear operator on $\mathbb{K}^{p|q}(\mathcal{V})$ which acts on a vector $x = \underline{x} + \underline{\grave{x}}$ by left multiplication with its column representation:

$$Mx = \begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{\grave{x}} \end{pmatrix} = \begin{pmatrix} A\underline{x} + \grave{B}\underline{\grave{x}} \\ \grave{C}\underline{x} + D\underline{\grave{x}} \end{pmatrix} \in \mathbb{K}^{p|q}(\mathcal{V}).$$

In order to study some group structures in $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$ we start from the Lie group $\mathrm{GL}(p|q)(\mathbb{K}\Lambda_N)$ of all invertible elements of $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$. The following theorem states a well-known characterization of this group, see [1].

Theorem 1. *Let $M \in \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$, as in (4). Then the following statements are equivalent:*

(i) $M \in \mathrm{GL}(p|q)(\mathbb{K}\Lambda_N)$*;*

(ii) A*,* D *are invertible;*

(iii) A_0*,* D_0 *are invertible.*

The notions of transpose, trace and determinant of a matrix need to be redefined in the graded case. The *supertranspose* operation is defined by

$$M^{ST} = \begin{pmatrix} A^T & \grave{C}^T \\ -\grave{B}^T & D^T \end{pmatrix},$$

where $\cdot^T$ denotes the usual matrix transpose. The following properties can be easily checked, see [1].

Proposition 1. *Let $M, L \in \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$ and $x \in \mathbb{K}^{p|q}(\mathcal{V})$. Then*

(i) $(ML)^{ST} = L^{ST}M^{ST}$*;*

(ii) $(Mx)^T = x^T M^{ST}$*;*

(iii) $\left(M^{-1}\right)^{ST} = \left(M^{ST}\right)^{-1}$ *for every* $M \in \mathrm{GL}(p|q)(\mathbb{K}\Lambda_N)$*.*

The *supertrace* is defined as the map str : $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N) \to \mathbb{K}\Lambda_{N,\bar{0}}$, given by

$$\mathrm{str}\begin{pmatrix} A & \grave{B} \\ \grave{C} & D \end{pmatrix} = \mathrm{tr}(A) - \mathrm{tr}(D).$$

Proposition 2. *Let $M, L \in \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$. Then*

(i) $\mathrm{str}(ML) = \mathrm{str}(LM)$*;*
(ii) $\mathrm{str}\left(M^{ST}\right) = \mathrm{str}(M)$.

The *superdeterminant* or *Berezinian* is a function from $\mathrm{GL}(p|q)(\mathbb{K}\Lambda_N)$ to $\mathbb{K}\Lambda_{N,\bar{0}}$, defined by

$$\mathrm{sdet}(M) = \frac{\det(A - \grave{B}D^{-1}\grave{C})}{\det(D)} = \frac{\det(A)}{\det(D - \grave{C}A^{-1}\grave{B})}.$$

Some of its basic properties are given in the following proposition, see [1].

Proposition 3. *Let $M, L \in \mathrm{GL}(p|q)(\mathbb{K}\Lambda_N)$, then*

(i) $\mathrm{sdet}(ML) = \mathrm{sdet}(M)\,\mathrm{sdet}(L)$*;*
(ii) $\mathrm{sdet}\left(M^{ST}\right) = \mathrm{sdet}(M)$.

In the vector space $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$ we introduce the norm $\|M\| = \sum_{j,k=1}^{p+q} |m_{j,k}|$, where $m_{j,k} \in \mathbb{K}\Lambda_N$ $(j, k = 1, \dots, p+q)$ are the entries of M. As was the case in $\mathbb{K}\Lambda_N$, also this norm satisfies the inequality $\|ML\| \leq \|M\|\|L\|$ for every pair $M, L \in \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$, leading to the absolute convergence of the series

$$\exp(M) = \sum_{j=0}^{\infty} \frac{M^j}{j!}$$

and hence, the continuity of the exponential map in $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$. It is easily seen that also the supertranspose, the supertrace and the superdeterminant are continuous maps. Some properties of the exponential are gathered in the following proposition.

Proposition 4. *Let $M, L \in \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$. Then*

(i) $e^0 = I_{p+q}$*, where I_k denotes the identity matrix of order $k \in \mathbb{N}$;*
(ii) $\left(e^M\right)^{ST} = e^{M^{ST}}$*;*
(iii) if $ML = LM$, then $e^{M+L} = e^M\, e^L$;
(iv) $e^M \in \mathrm{GL}(p|q)(\mathbb{K}\Lambda_N)$ *and* $\left(e^M\right)^{-1} = e^{-M}$*;*
(v) $e^{(a+b)M} = e^{aM}e^{bM}$*, for every pair $a, b \in \mathbb{K}\Lambda_{N,\bar{0}}$;*
(vi) if $C \in \mathrm{GL}(p|q)(\mathbb{K}\Lambda_N)$ then, $e^{CMC^{-1}} = Ce^MC^{-1}$;
(vii) e^{tM} *$(t \in \mathbb{R})$ is a smooth curve in $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$, with*

$$\frac{d}{dt}e^{tM} = Me^{tM} = e^{tM}M, \quad \textit{and} \quad \frac{d}{dt}e^{tM}\Big|_{t=0} = M;$$

(viii) $\mathrm{sdet}\left(e^M\right) = e^{\mathrm{str}(M)}$.

Remark 2.1. *The proofs (i)-(vii) can be established by straightforward computation. A detailed proof for (viii) can be found in [1]. Similar properties to (i) and (iii)–(vii) can be obtained for the exponential map in $\mathbb{K}\Lambda_N$.*

In this setting we can also define the notion of logarithm of a supermatrix. For a supermatrix $M \in \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$ we define $\ln(M)$ by:

$$\ln(M) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{(M - I_{p+q})^j}{j}, \tag{5}$$

wherever the series converges. Following standard procedures it is possible to prove the next classical properties of the logarithmic map.

Proposition 5. *In* $\mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N)$ *it holds that:*

(i) the series (5) converges and yields a continuous function near I_{p+q}*;*

(ii) given a neighbourhood U *of* I_{p+q} *on which* $\ln$ *is defined and a neighbourhood* V *of* 0 *such that* $\exp(V) := \{\exp(M) \mid M \in V\} \subset U$*, then* $\exp(\ln(M)) = M$*,* $\forall M \in U$*, and* $\ln(\exp(L)) = L$*,* $\forall L \in V$*.*

With the above definitions of the exponential and logarithmic maps, it is possible to obtain all the classical results known for Lie groups and Lie algebras of real and complex matrices. The exponential of a nilpotent matrix $\mathbf{M} \in \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^+)$ reduces to a finite sum, yielding the bijective mapping

$$\exp : \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^+) \to I_{p+q} + \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^+)$$

with inverse

$$\ln : I_{p+q} + \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^+) \to \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^+),$$

since also the second expansion only has a finite number of non-zero terms, whence problems of convergence do not arise. We recall that a supermatrix M belongs to $\mathrm{GL}(p|q)(\mathbb{K}\Lambda_N)$ if and only if its numeric projection M_0 has an inverse. Then $M = M_0(I_{p+q} + M_0^{-1}\mathbf{M}) = M_0 \exp(\mathbf{L})$, for some unique $\mathbf{L} \in \mathrm{Mat}(p|q)(\mathbb{K}\Lambda_N^+)$.

3 Clifford analysis and spin actions in superspace

After presenting the above preliminary facts on Grassmann algebras, Grassmann envelopes and supermatrices, in this section we will briefly show how the Clifford setting is established in superspace. In particular, we pay attention to the special form adopted by the spin group in this case, see [4], and study the invariance of the super Dirac operator under the corresponding spin actions.

3.1 Clifford setting in superspace

Clifford algebras in superspace are obtained by considering $p = m$, $q = 2n$ $(m, n \in \mathbb{N})$ and $\mathbb{K} = \mathbb{R}$. The canonical basis $e_1, \ldots, e_m, \grave{e}_1, \ldots, \grave{e}_{2n}$ of the graded vector space $\mathbb{R}^{m|2n}$ is endowed with both an orthogonal and a symplectic structure by means of the multiplication relations

$$e_j e_k + e_k e_j = -2\delta_{j,k}, \quad e_j \grave{e}_k + \grave{e}_k e_j = 0, \quad \grave{e}_j \grave{e}_k - \grave{e}_k \grave{e}_j = g_{j,k},$$

where $g_{j,k}$ is a symplectic form defined by $g_{2j,2k} = g_{2j-1,2k-1} = 0$ and $g_{2j-1,2k} = -g_{2k,2j-1} = \delta_{j,k}$, $j, k = 1, \ldots, n$. In the study of Clifford analysis in superspace with Grassmann coefficients, the algebra of interest is

$$\mathcal{A}_{m,2n}(\mathbb{R}\mathcal{V}) = \mathrm{Alg}_{\mathbb{R}}(f_1, \ldots, f_N, \mathit{VAR}, \grave{\mathit{VAR}}, e_1, \ldots, e_m, \grave{e}_1, \ldots, \grave{e}_{2n}) = \mathbb{R}\mathcal{V} \otimes \mathcal{C}_{m,2n}$$

where $\mathcal{C}_{m,2n}$ is the infinite dimensional algebra generated by all elements of $\mathbb{R}^{m|2n}$, and the elements of $\mathcal{C}_{m,2n}$ are supposed to commute with the elements of $\mathbb{R}\mathcal{V}$. The set of Clifford coefficients in this setting is the subalgebra of $\mathcal{A}_{m,2n}(\mathbb{R}\mathcal{V})$ given by

$$\mathcal{A}_{m,2n}(\mathbb{R}\Lambda_N) = \mathrm{Alg}_{\mathbb{R}}(f_1, \ldots, f_N, e_1, \ldots, e_m, \grave{e}_1, \ldots, \grave{e}_{2n}) = \mathbb{R}\Lambda_N \otimes \mathcal{C}_{m,2n}.$$

Following [1, 13], the bosonic and fermionic partial derivatives ∂_v and $\partial_{\grave{v}}$ $(v \in \mathit{VAR}, \grave{v} \in \grave{\mathit{VAR}})$ can be uniquely defined as endomorphisms of $\mathcal{A}_{m,2n}(\mathbb{R}\mathcal{V})$ by means of the following recursive formulae:

$$\begin{cases} \partial_v[1] = 0, \\ \partial_v f_k = f_k \partial_v, \\ \partial_v w - w\partial_v = \delta_{v,w}, \\ \partial_v \grave{w} = \grave{w}\partial_v, \ \ \partial_v e_j = e_j \partial_v, \ \ \partial_v \grave{e}_j = \grave{e}_j \partial_v, \end{cases} \qquad \begin{cases} \partial_{\grave{v}}[1] = 0, \\ \partial_{\grave{v}} f_k = -f_k \partial_{\grave{v}} \\ \partial_{\grave{v}} \grave{w} + \grave{w}\partial_{\grave{v}} = \delta_{\grave{v},\grave{w}}, \\ \partial_{\grave{v}} w = w\partial_{\grave{v}}, \ \ \partial_{\grave{v}} e_j = e_j \partial_{\grave{v}}, \ \ \partial_{\grave{v}} \grave{e}_j = \grave{e}_j \partial_{\grave{v}}, \end{cases}$$

which are valid for the left and the right actions of $w, \grave{w}, \partial_v, \partial_{\grave{v}}$. From this definition it immediately follows that $\partial_v \partial_w = \partial_w \partial_v$, $\partial_{\grave{v}} \partial_{\grave{w}} = -\partial_{\grave{w}} \partial_{\grave{v}}$ and $\partial_v \partial_{\grave{w}} = \partial_{\grave{w}} \partial_v$.

As usual, a supervector variable x of $\mathbb{R}^{m|2n}(\mathcal{V})$ is then defined as

$$x = \underline{x} + \grave{\underline{x}} = \sum_{j=1}^{m} x_j e_j + \sum_{j=1}^{2n} \grave{x}_j \grave{e}_j, \qquad x_j \in VAR, \quad \grave{x}_j \in \grave{VAR}.$$

Supervector variables generate the realization $\mathbb{R}_{m|2n}(\mathcal{V})$ of the so-called radial algebra in this setting, see [2, 5, 9], and the anti-commutator $\{a, b\} = ab + ba$ between supervectors can be used to define a generalized inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ by

$$\langle x, y \rangle_{\mathbb{R}} = -\frac{1}{2}\{x, y\} = \sum_{j=1}^{m} x_j y_j - \frac{1}{2} \sum_{j=1}^{n} (\grave{x}_{2j-1} \grave{y}_{2j} - \grave{x}_{2j} \grave{y}_{2j-1}) \in \mathbb{R}\mathcal{V}_{\bar{0}}. \tag{6}$$

We consider functions of the supervector variable $x = \underline{x} + \grave{\underline{x}}$ which are of the form

$$F(x) = F(\underline{x}, \grave{\underline{x}}) = \sum_{A \subset \{1,\ldots,2n\}} F_A(\underline{x})\, \grave{x}_{j_1} \ldots \grave{x}_{j_k}, \qquad A = \{j_1, \ldots, j_k\},\ 1 \leq j_1 < \ldots < j_k \leq 2n,$$

where $F_A(\underline{x})$ is a function depending on the bosonic variables $x_1, \ldots, x_m \in VAR$. The most basic set of such functions is the super-polynomial algebra in the variables $x_1, \ldots, x_m, \grave{x}_1, \ldots, \grave{x}_{2n}$,

$$\mathcal{P}_{m|2n} = \mathcal{A}_{m,2n}(\mathbb{R}\Lambda_N) \otimes \mathbb{R}[x_1, \ldots, x_m] \otimes \Lambda(\grave{x}_1, \ldots, \grave{x}_{2n}),$$

where $\Lambda(\grave{x}_1, \ldots, \grave{x}_{2n})$ is the Grassmann algebra generated by the odd variables $\grave{x}_1, \ldots, \grave{x}_{2n} \in \grave{VAR}$. Superpolynomials play an important rôle in superanalysis since they form a dense set in other interesting function spaces. For that reason we restrict our attention to $\mathcal{P}_{m|2n}$ in this paper.

The vector derivative with respect to x is obtained in this setting through the bosonic and fermionic Dirac operators

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}, \qquad \partial_{\grave{\underline{x}}} = 2 \sum_{j=1}^{n} \left(\grave{e}_{2j} \partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1} \partial_{\grave{x}_{2j}} \right),$$

which lead to the left super Dirac operator $\partial_x \cdot = \partial_{\grave{\underline{x}}} \cdot - \partial_{\underline{x}} \cdot$.

The corresponding realization of the radial algebra in the set of Clifford coefficients $\mathcal{A}_{m,2n}(\mathbb{R}\Lambda_N)$ is the subalgebra generated by the Grassmann envelope of constant supervectors $\mathbb{R}^{m|2n}(\Lambda_N)$. This algebra is denoted by $\mathbb{R}_{m|2n}(\Lambda_N)$. Observe that $\mathbb{R}_{m|2n}(\Lambda_N)$ is a finite dimensional vector space since it is generated by the set

$$\{f_A e_j \mid A \subset \{1, \ldots, N\}, |A| \text{ even }, j = 1, \ldots, m\} \cup \{f_A \grave{e}_j \mid A \subset \{1, \ldots, N\}, |A| \text{ odd }, j = 1, \ldots, 2n\},$$

and there is a finite number of possible products amongst these generators. For more details about this kind of radial algebra realization we refer to [2, 3, 5, 10, 11, 12].

The definition of the spin group in superspace requires the definition of the exponential map over the algebra $\mathcal{A}_{m,2n}(\mathbb{R}\Lambda_N)$, see [4]. However, since this algebra is infinite dimensional, the power series $\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!}$ lies in general out of $\mathcal{A}_{m,2n}(\mathbb{R}\Lambda_N)$. For a correct definition of the exponential map in this setting its is necessary to introduce the corresponding tensor algebra. Let $T(V)$ be the tensor algebra of the vector space V spanned by the basis $B_V = \{f_1, \ldots, f_N, e_1, \ldots, e_m, \grave{e}_1, \ldots, \grave{e}_{2n}\}$, i.e. $T(V) = \bigoplus_{j=0}^{\infty} T^j(V)$ where $T^j(V) = \mathrm{span}_{\mathbb{R}}\{v_1 \otimes \cdots \otimes v_j : v_\ell \in B_V\}$ is the j-fold tensor product of V with itself. When considering the twosided ideal I generated by the elements

$$\begin{array}{lll} f_j \otimes f_k + f_k \otimes f_j, & f_j \otimes e_k - e_k \otimes f_j, & f_j \otimes \grave{e}_k - \grave{e}_k \otimes f_j, \\ e_j \otimes e_k + e_k \otimes e_j + 2\delta_{j,k}, & e_j \otimes \grave{e}_k + \grave{e}_k \otimes e_j, & \grave{e}_j \otimes \grave{e}_k - \grave{e}_k \otimes \grave{e}_j - g_{j,k}, \end{array}$$

$\mathcal{A}_{m,2n}(\mathbb{R}\Lambda_N)$ is clearly seen to be a subalgebra of $T(V)/I$. Indeed, $T(V)/I$ is isomorphic to the extension of $\mathcal{A}_{m,2n}(\mathbb{R}\Lambda_N)$ which also contains infinite sums of terms of the form $a e_{j_1} \cdots e_{j_k} \grave{e}_1^{\alpha_1} \cdots \grave{e}_{2n}^{\alpha_{2n}}$ where $a \in \mathbb{R}\Lambda_N$, $1 \leq j_1 \leq \ldots \leq j_k \leq m$ and $(\alpha_1, \ldots, \alpha_{2n}) \in (\mathbb{N} \cup \{0\})^{2n}$ is a multi-index.

This enables us to consider the exponential map $\exp(F) = \sum_{j=0}^{\infty} \frac{F^j}{j!}$ for $F \in \mathcal{A}_{m,2n}(\mathbb{R}\Lambda_N)$ since exp is well–defined on the tensor algebra $T(V)$ and in consequence also on $T(V)/I$, see [6]. In particular, we have the following mapping properties

$$\exp : \mathcal{A}_{m,2n}(\Lambda_N) \to T(V)/I, \qquad \exp : \mathbb{R}_{m|2n}(\Lambda_N) \to \mathbb{R}_{m|2n}(\Lambda_N).$$

Following the radial algebra approach it is possible to extend some important involutions to $\mathcal{A}_{m,2n}(\mathbb{R}\mathcal{V})$ and $T(V)/I$, see [3]. In particular, the conjugation can be defined on $\mathcal{A}_{m,2n}(\mathbb{R}\mathcal{V})$ as the linear map satisfying

$$\overline{a\, e_{j_1} \dots e_{j_k} \grave{e}_{\ell_1} \dots \grave{e}_{\ell_s}} = a\, \overline{e_{j_1} \dots e_{j_k} \grave{e}_{\ell_1} \dots \grave{e}_{\ell_s}}, \quad a \in \mathbb{R}\mathcal{V};$$

where

$$\overline{e_{j_1} \dots e_{j_k} \grave{e}_{\ell_1} \dots \grave{e}_{\ell_s}} = (-1)^{k+\frac{s(s+1)}{2}} \grave{e}_{\ell_s} \dots \grave{e}_{\ell_1} e_{j_k} \dots e_{j_1}.$$

Observe that $\overline{FG} = \overline{G}\,\overline{F}$ for every pair $F, G \in \mathbb{R}_{m|2n}(\mathcal{V})$, while this relation is not fulfilled in general in $\mathcal{A}_{m,2n}(\mathbb{R}\mathcal{V})$. The conjugation map can be continuously extended from $\mathcal{A}_{m,2n}(\mathbb{R}\Lambda_N)$ to $T(V)/I$. This leads, amongst others, to relations of the type $\overline{e^F} = e^{\overline{F}}$.

The extension of the set of Clifford coefficients to $T(V)/I$ allows to consider polynomials in the space

$$\mathcal{TP}_{m|2n} = T(V)/I \otimes \mathbb{R}[x_1, \dots, x_m] \otimes \Lambda(\grave{x}_1, \dots, \grave{x}_{2n}),$$

i.e. superpolynomials with coefficients in $T(V)/I$. Every linear action on $\mathcal{P}_{m|2n}$ can be extended by continuity to $\mathcal{TP}_{m|2n}$.

3.2 The spin group

The invariance of the bilinear form (6) was studied in [4] in order to properly define the spin group in superspace. The Lie group of supermatrices in $\mathrm{Mat}(m|2n)(\mathbb{R}\Lambda_N)$ leaving the "inner product" $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ invariant is given by

$$\mathrm{O}_0 = \mathrm{O}_0(m|2n)(\mathbb{R}\Lambda_N) = \big\{ M \in \mathrm{Mat}(m|2n)(\mathbb{R}\Lambda_N) : M^{ST}\mathbf{Q}M - \mathbf{Q} = 0 \big\},$$

where

$$\mathbf{Q} = \begin{pmatrix} I_m & 0 \\ 0 & -\frac{1}{2}\Omega_{2n} \end{pmatrix} \quad \text{and} \quad \Omega_{2n} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

Direct computations show that $M \in \mathrm{Mat}(m|2n)(\mathbb{R}\Lambda_N)$ (see (4)) belongs to O_0 if and only if the following conditions are simultaneously fulfilled:

$$A^T A - \frac{1}{2}\grave{C}^T \Omega_{2n} \grave{C} = I_m, \quad A^T \grave{B} - \frac{1}{2}\grave{C}^T \Omega_{2n} D = 0, \quad \grave{B}^T \grave{B} + \frac{1}{2} D^T \Omega_{2n} D = \frac{1}{2}\Omega_{2n}.$$

These conditions show that the real projection of O_0 is the group $\mathrm{O}(m) \times \mathrm{Sp}_\Omega(2n)$ where $\mathrm{O}(m)$ is the classical orthogonal group in dimension m and $\mathrm{Sp}_\Omega(2n)$ is the symplectic group defined through the antisymmetric matrix Ω_{2n}, i.e. $\mathrm{Sp}_\Omega(2n) = \{D_0 \in \mathbb{R}^{2n\times 2n} : D_0^T \Omega_{2n} D_0 = \Omega_{2n}\}$. In addition, it was shown in [4] that $\mathrm{sdet}(M) = \pm 1$ for every $M \in \mathrm{O}_0$, whence O_0 can be seen as a generalization of $\mathrm{O}(m)$ to superspace, see [4] for more details.

As in the classical case, it now is crucial to study the set of *superrotations*, i.e. the identity component of O_0, which is given by

$$\mathrm{SO}_0 = \mathrm{SO}_0(m|2n)(\mathbb{R}\Lambda_N) = \{ M \in \mathrm{O}_0 \, : \, \mathrm{sdet}(M) = 1 \}.$$

Clearly, the real projection of this group is $\mathrm{SO}(m) \times \mathrm{Sp}_\Omega(2n)$ where $\mathrm{SO}(m)$ is the special orthogonal group in dimension m, which makes it possible to prove that SO_0 is connected but non-compact, see [4].

The Lie algebra $\mathfrak{so}_0$ of O_0 and SO_0 plays an important rôle in the construction of the spin group; it is given by

$$\mathfrak{so}_0 = \mathfrak{so}_0(m|2n)(\mathbb{R}\Lambda_N) = \{X \in \mathrm{Mat}(m|2n)(\mathbb{R}\Lambda_N) \,:\, X^{ST}\mathbf{Q} + \mathbf{Q}X = 0\}.$$

Writing $X \in \mathrm{Mat}(m|2n)(\mathbb{R}\Lambda_N)$ in the block form (4) of a supermatrix, the above condition splits into the following requirements for its respective blocks:

$$A^T + A = 0, \quad \grave{B} - \grave{C}^T\Omega_{2n} = 0, \quad D^T\Omega_{2n} + \Omega_{2n}D = 0. \tag{7}$$

The real projection of $\mathfrak{so}_0$ is the Lie algebra $\mathfrak{so}(m)\times\mathfrak{sp}_\Omega(2n)$ where $\mathfrak{so}(m) = \{A_0 \in \mathbb{R}^{m\times m} : A_0^T + A_0 = 0\}$ is the special orthogonal Lie algebra and $\mathfrak{sp}_\Omega(2n) = \{D_0 \in \mathbb{R}^{2n\times 2n} : D_0^T\Omega_{2n} + \Omega_{2n}D_0 = 0\}$ is the symplectic Lie algebra defined through the antisymmetric matrix Ω_{2n}.

The symplectic structure present in SO_0, and its consequent non-compactness, make it impossible to write every element of SO_0 as a single exponential of a supermatrix in $\mathfrak{so}_0$. However, it was proven in [4] that the following decomposition holds.

Theorem 2. *Every supermatrix $M \in \mathrm{SO}_0$ can be written as*

$$M = e^X e^Y e^{\mathbf{Z}} \quad \textit{with} \quad \begin{cases} X \in \mathfrak{so}(m)\times[\mathfrak{sp}_\Omega(2n)\cap\mathfrak{so}(2n)],\\ Y \in \{0_m\}\times[\mathfrak{sp}_\Omega(2n)\cap\mathrm{Sym}(2n)],\\ \mathbf{Z} \in \mathfrak{so}_0(m|2n)(\Lambda_N^+),\end{cases}$$

where $\mathrm{Sym}(p)$ *is the subspace of symmetric matrices in* $\mathbb{R}^{p\times p}$ *and* $\mathfrak{so}_0(m|2n)(\Lambda_N^+) := \mathfrak{so}_0\cap\mathrm{Mat}(m|2n)(\Lambda_N^+)$. *In addition, the elements Y and $\mathbf{Z}$ are unique.*

The algebra $\mathfrak{so}_0$ is related to the algebra of superbivectors but they are not isomorphic in this setting. Indeed, following the radial algebra approach, superbivectors are elements generated by the wedge product of supervectors of $\mathbb{R}^{m|2n}(\Lambda_N)$ and in consequence, they have the form

$$B = \sum_{1\leq j<k\leq m} b_{j,k}\, e_je_k + \sum_{\substack{1\leq j\leq m\\ 1\leq k\leq 2n}} \grave{b}_{j,k}\, e_j\grave{e}_k + \sum_{1\leq j\leq k\leq 2n} B_{j,k}\, \grave{e}_j\odot\grave{e}_k, \tag{8}$$

where $b_{j,k} \in \mathbb{R}\Lambda_{N,\bar{0}}$, $\grave{b}_{j,k} \in \mathbb{R}\Lambda_{N,\bar{1}}$ and $B_{j,k} \in \mathbb{R}\Lambda_{N,\bar{0}}\cap\mathbb{R}\Lambda_{N,}^+$. This form has an important limitation with respect to $\mathfrak{so}_0$ since the symplectic part in (8), which is generated by $\grave{e}_j\odot\grave{e}_k$, is nilpotent, while the symplectic part of an $\mathfrak{so}_0$–element may be invertible. For that reason, it is necessary to consider an algebra of *extended superbivectors* $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$, containing elements of the form (8), however with $B_{j,k} \in \mathbb{R}\Lambda_{N,\bar{0}}$. In [4], it was proven that $\mathfrak{so}_0(m|2n)(\mathbb{R}\Lambda_N)$ and $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$ are isomorphic Lie algebras through the map $\phi : \mathbb{R}^{(2)E}_{m|2n}(\Lambda_N) \to \mathfrak{so}_0$ given by:

$$\phi(B)x = [B, x], \qquad B \in \mathbb{R}^{(2)E}_{m|2n}(\Lambda_N),\ x \in \mathbb{R}^{m|2n}(\mathcal{V}).$$

For the basis elements of $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$ we easily get

$$\begin{aligned} [b\, e_je_k,\, x] &= 2b\,(x_je_k - x_ke_j), & [b\,\grave{e}_{2j}\odot\grave{e}_{2k},\, x] &= -b\,(\grave{x}_{2j-1}\grave{e}_{2k} + \grave{x}_{2k-1}\grave{e}_{2j}),\\ [\grave{b}\, e_j\grave{e}_{2k-1},\, x] &= \grave{b}\,(2x_j\grave{e}_{2k-1} + \grave{x}_{2k}e_j), & [b\,\grave{e}_{2j-1}\odot\grave{e}_{2k-1},\, x] &= b\,(\grave{x}_{2j}\grave{e}_{2k-1} + \grave{x}_{2k}\grave{e}_{2j-1}),\\ [\grave{b}\, e_j\grave{e}_{2k},\, x] &= \grave{b}\,(2x_j\grave{e}_{2k} - \grave{x}_{2k-1}e_j), & [b\,\grave{e}_{2j-1}\odot\grave{e}_{2k},\, x] &= b\,(\grave{x}_{2j}\grave{e}_{2k} - \grave{x}_{2k-1}\grave{e}_{2j-1}), \end{aligned} \tag{9}$$

where b and $\grave{b}$ are arbitrary basis elements of $\mathbb{R}\Lambda_{N,\bar{0}}$ and $\mathbb{R}\Lambda_{N,\bar{1}}$ respectively.

The spin group in superspace then is introduced as follows:

$$\mathrm{Spin}(m|2n)(\Lambda_N) := \left\{e^{B_1}\cdots e^{B_k} : B_1,\ldots,B_k \in \mathbb{R}^{(2)E}_{m|2n}(\Lambda_N), k\in\mathbb{N}\right\},$$

and its action on $\mathbb{R}^{m|2n}(\mathcal{V})$ is given by the Lie group homomorphism $h : \mathrm{Spin}(m|2n)(\Lambda_N) \to \mathrm{SO}_0$, defined by

$$h(e^B)[x] = e^Bxe^{-B}, \qquad B \in \mathbb{R}^{(2)E}_{m|2n}(\Lambda_N),\ x \in \mathbb{R}^{m|2n}(\mathcal{V}). \tag{10}$$

This spin action covers the whole group SO_0, see [4]. This statement can be proved using the connectedness of SO_0 and the fact that ϕ is the derivative at the origin (or infinitesimal representation) of h, i.e.

$$e^{\phi(B)} = h(e^B), \qquad B \in \mathbb{R}^{(2)E}_{m|2n}(\Lambda_N). \tag{11}$$

The decomposition of SO_0 in Theorem 2 shows the exact number of exponentials of extended superbivectors needed in $\mathrm{Spin}(m|2n)(\Lambda_N)$ to cover the whole group SO_0. Considering the direct sum decomposition $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N) = S_1 \oplus S_2 \oplus S_3$ where the subspaces S_1, S_2, S_3 of $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$ are given by

$$\begin{aligned} S_1 &= \phi^{-1}\big(\mathfrak{so}(m) \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)]\big), && \dim S_1 = \frac{m(m-1)}{2} + n^2, \\ S_2 &= \phi^{-1}\big(\{0_m\} \times [\mathfrak{sp}_\Omega(2n) \cap \mathrm{Sym}(2n)]\big), && \dim S_2 = n^2 + n, \\ S_3 &= \phi^{-1}\big(\mathfrak{so}_0(m|2n)(\Lambda_N^+)\big) && \dim S_3 = \dim \mathfrak{so}_0 - \frac{m(m-1)}{2} - n(2n+1), \end{aligned} \tag{12}$$

we get the subset $S = \exp(S_1)\exp(S_2)\exp(S_3) \subset \mathrm{Spin}(m|2n)(\Lambda_N)$, which suffices for describing SO_0. In [4], it was moreover proven that S doubly covers SO_0. Indeed, for every $M = e^X e^Y e^{\mathbf{Z}} \in \mathrm{SO}_0$ (see Theorem 2), the number of elements $s \in S$ for which $h(s) = M$ only depends on the number of elements $B_1 \in S_1$ for which $e^{\phi(B_1)} = e^X$, seen the uniqueness of Y and $\mathbf{Z}$. Hence, it suffices to determine the kernel of the restricted group homomorphism $h|_{\exp(S_1)} : \exp(S_1) \to \mathrm{SO}(m) \times [\mathrm{Sp}(2n) \cap \mathrm{SO}(2n)]$, which was found to be $\{-1, 1\}$, using

$$\exp(S_1) = \mathrm{Spin}(m) \cdot \exp\left(\phi^{-1}\big(\{0_m\} \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)]\big)\right)$$

and the following observations:

- the classical spin group $\mathrm{Spin}(m)$ constitutes a double covering for $\mathrm{SO}(m)$;
- the kernel of $h : \exp\left(\phi^{-1}\big(\{0_m\} \times [\mathfrak{sp}_\Omega(2n) \cap \mathfrak{so}(2n)]\big)\right) \to [\mathrm{Sp}(2n) \cap \mathrm{SO}(2n)]$ is composed of the products of all integer powers of the elements $\exp\left[\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right)\right] = -1$.

For more details, we refer to [4]. The explicit forms of the superbivectors in each of the subspaces S_1, S_2 and S_3 were established in [4] as well.

Proposition 6. *The following statements hold:*

(i) a basis for S_1 is given by $\begin{cases} e_j e_k, & 1 \le j < k \le m, \\ \grave{e}_{2j-1} \odot \grave{e}_{2k-1} + \grave{e}_{2j} \odot \grave{e}_{2k}, & 1 \le j \le k \le n, \\ \grave{e}_{2j-1} \odot \grave{e}_{2k} - \grave{e}_{2j} \odot \grave{e}_{2k-1}, & 1 \le j < k \le n; \end{cases}$

(ii) a basis for S_2 is given by $\begin{cases} \grave{e}_{2j-1} \odot \grave{e}_{2j}, & 1 \le j \le n, \\ \grave{e}_{2j-1} \odot \grave{e}_{2k-1} - \grave{e}_{2j} \odot \grave{e}_{2k}, & 1 \le j \le k \le n, \\ \grave{e}_{2j-1} \odot \grave{e}_{2k} + \grave{e}_{2j} \odot \grave{e}_{2k-1}, & 1 \le j < k \le n; \end{cases}$

(iii) S_3 consists of all elements of the form (8) with $b_{j,k}, B_{j,k} \in \Lambda_{N,\bar{0}} \cap \Lambda_{N,}^+$ and $\grave{b}_{j,k} \in \Lambda_{N,\bar{1}}$.

Remark 3.1. *It is interesting to point out that within the algebra $Alg_{\mathbb{R}}\{\grave{e}_1, \ldots, \grave{e}_{2n}\}$ the elements $\grave{e}_{2j-1}, \grave{e}_{2j}$ may be identified with the operators $e^{\frac{\pi}{4}i}\partial_{a_j}, e^{-\frac{\pi}{4}i}a_j$ respectively, where the a_j's are real variables and i is the usual imaginary unit. Observe that these identifications immediately lead to the Weyl algebra defining relations*

$$e^{\frac{\pi}{4}i}\partial_{a_j}\, e^{-\frac{\pi}{4}i}a_k - e^{-\frac{\pi}{4}i}a_k\, e^{\frac{\pi}{4}i}\partial_{a_j} = \partial_{a_j}a_k - a_k\partial_{a_j} = \delta_{j,k}.$$

Hence every extended superbivector of the form $B = \sum_{j=1}^n \theta_j \pi \left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right) \in S_1$, $\theta_j \in \mathbb{R}$, may be identified with an operator $\sum_{j=1}^n \theta_j \pi i \left(\partial_{a_j}^2 - a_j^2\right)$, leading to the correspondence:

$$e^B \quad \longrightarrow \quad \exp\left[\sum_{j=1}^n \theta_j \pi i \left(\partial_{a_j}^2 - a_j^2\right)\right] = \prod_{j=1}^n \exp\left[\theta_j \pi i (\partial_{a_j}^2 - a_j^2)\right].$$

We recall that the classical Fourier transform in the variable a_j can be written as an operator exponential

$$\mathcal{F}_{a_j}[f] = \exp\left(\frac{\pi}{4}i\right) \exp\left(\frac{\pi}{4}i\left(\partial_{a_j}^2 - a_j^2\right)\right)[f].$$

Then, through the previous identifications we can see all operators

$$\prod_{j=1}^{n} \exp\left[\theta_j \pi i(\partial_{a_j}^2 - a_j^2)\right] = \prod_{j=1}^{n} \exp\left(-\theta_j \pi i\right) \mathcal{F}_{a_j}^{4\theta_j},$$

as elements of the spin group, where $\mathcal{F}_{a_j}^{4\theta_j}$ denotes the one-dimensional fractional Fourier transform of order $4\theta_j$ in the variable a_j. In particular, for $\theta_j = 1$ we get the element $\exp\left[\pi\left(\grave{e}_{2j-1}^2 + \grave{e}_{2j}^2\right)\right]$, identified with $e^{-\pi i}\mathcal{F}_{a_j}^4 = -id$, id denoting the identity operator.

3.3 Invariance of the super Dirac operator under the spin group action

Similarly to the classical case, we define the following spin actions on functions:

$$H(s)[F(x)] = sF(\bar{s}xs)\bar{s}, \qquad L(s)[F(x)] = sF(\bar{s}xs), \quad s \in \mathrm{Spin}(m|2n)(\Lambda_N),$$

which are Lie group homomorphisms from the spin group to the group of automorphisms on $\mathcal{TP}_{m|2n}$, i.e. H and L are Lie group representations of $\mathrm{Spin}(m|2n)(\Lambda_N)$. It is our aim to show that both representations commute with the super Dirac operator ∂_x, whence ∂_x can be called an invariant operator under the action of the spin group. The proof can be easily reduced to showing the invariance under the L action, i.e. to showing that $[\partial_x, L(s)] = 0$, $s \in \mathrm{Spin}(m|2n)(\Lambda_N)$.

Let $\mathrm{End}(\mathcal{TP}_{m|2n})$ be the space of endomorphisms on $\mathcal{TP}_{m|2n}$ and $dL : \mathbb{R}_{m|2n}^{(2)E}(\Lambda_N) \to \mathrm{End}(\mathcal{TP}_{m|2n})$ the infinitesimal representation of L, defined, as above, by $L(e^B) = e^{dL(B)}$, for all $B \in \mathbb{R}_{m|2n}^{(2)E}(\Lambda_N)$. It then suffices to prove that ∂_x commutes with $dL(B)$ for every basis element of $\mathbb{R}_{m|2n}^{(2)E}(\Lambda_N)$. To this end, we need the following result.

Proposition 7. *Let $B \in \mathbb{R}_{m|2n}^{(2)E}(\Lambda_N)$. Then*

$$dL(B) = B - \sum_{j=1}^{m}([B,x])_j \partial_{x_j} - \sum_{j=1}^{2n}([B,x])_{m+j}\partial_{\grave{x}_j},$$

where $([B,x])_j$ denotes the j-th co-ordinate of the supervector $[B,x]$.

Proof.
For every superbivector B we have by definition that $dL(B) = \frac{d}{dt}L(e^{tB})\big|_{t=0}$. Then

$$dL(B)[F(x)] = \frac{d}{dt}\left[e^{tB}F(e^{-tB}xe^{tB})\right]\Big|_{t=0} = BF(x) + \frac{d}{dt}\left[F(e^{-tB}xe^{tB})\right]\Big|_{t=0}.$$

Writing $y = e^{-tB}xe^{tB} = e^{-t\phi(B)}x$, we have that $\frac{dy}{dt}\big|_{t=0} = -\phi(B)x = -[B,x]$. Hence, the chain rule in superanalysis (see [2]) yields

$$\begin{aligned} dL(B)[F(x)] &= BF(x) + \left(\sum_{j=1}^{m}\frac{dy_j}{dt}\frac{\partial F}{\partial y_j}(e^{-tB}xe^{tB}) + \sum_{j=1}^{2n}\frac{d\grave{y}_j}{dt}\frac{\partial F}{\partial \grave{y}_j}(e^{-tB}xe^{tB})\right)\Bigg|_{t=0} \\ &= BF(x) - \sum_{j=1}^{m}\Big([B,x]\Big)_j \partial_{x_j}[F](x) - \sum_{j=1}^{2n}\Big([B,x]\Big)_{m+j}\partial_{\grave{x}_j}[F](x). \end{aligned}$$

□

Using (9) the following results are now easily obtained:

$$\begin{aligned} dL(b\, e_j e_k) &= -2b\,(x_j\partial_{x_k} - x_k\partial_{x_j} - \frac{1}{2}e_j e_k), && 1 \leq j < k \leq m, \\ dL(\grave{b}\, e_j \grave{e}_{2k-1}) &= -\grave{b}\,(2x_j\partial_{\grave{x}_{2k-1}} + \grave{x}_{2k}\partial_{x_j} - e_j\grave{e}_{2k-1}), && 1 \leq j \leq m, \quad 1 \leq k \leq n, \end{aligned}$$

$$\begin{aligned}
dL(\grave{b}\, e_j \grave{e}_{2k}) &= -\grave{b}\,(2x_j \partial_{\grave{x}_{2k}} - \grave{x}_{2k-1}\partial_{x_j} - e_j \grave{e}_{2k}), && 1 \leq j \leq m, \quad 1 \leq k \leq n,\\
dL(b\, \grave{e}_{2j-1} \odot \grave{e}_{2k-1}) &= -b(\grave{x}_{2j}\partial_{\grave{x}_{2k-1}} + \grave{x}_{2k}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1} \odot \grave{e}_{2k-1}), && 1 \leq j \leq k \leq n,\\
dL(b\, \grave{e}_{2j} \odot \grave{e}_{2k}) &= b(\grave{x}_{2j-1}\partial_{\grave{x}_{2k}} + \grave{x}_{2k-1}\partial_{\grave{x}_{2j}} + \grave{e}_{2j} \odot \grave{e}_{2k}), && 1 \leq j \leq k \leq n,\\
dL(b\, \grave{e}_{2j-1} \odot \grave{e}_{2k}) &= -b(\grave{x}_{2j}\partial_{\grave{x}_{2k}} - \grave{x}_{2k-1}\partial_{\grave{x}_{2j-1}} - \grave{e}_{2j-1} \odot \grave{e}_{2k}), && 1 \leq j \leq k \leq n,\\
dL(b\, \grave{e}_{2j} \odot \grave{e}_{2k-1}) &= -b(-\grave{x}_{2j-1}\partial_{\grave{x}_{2k-1}} + \grave{x}_{2k}\partial_{\grave{x}_{2j}} - \grave{e}_{2j} \odot \grave{e}_{2k-1}), && 1 \leq j < k \leq n.
\end{aligned}$$

whence we now are in the condition of proving the desired property.

Proposition 8. *For every* $B \in \mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$ *it holds that* $[\partial_x, dL(B)] = 0$.

Proof.
We now prove the commutation relation of ∂_x with $dL(B)$ for every basis element B of $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$. As an example, take $B = \grave{b}\, e_j \grave{e}_{2k-1}$; it is easily obtained that:

$$\begin{aligned}
[\grave{e}_{2\ell}\, \partial_{\grave{x}_{2\ell-1}}, \grave{b}\, x_j \partial_{\grave{x}_{2k-1}}] &= [\grave{e}_{2\ell}\, \partial_{\grave{x}_{2\ell-1}}, \grave{b}\, \grave{x}_{2k}\partial_{x_j}] = 0, \quad [\grave{e}_{2\ell}\, \partial_{\grave{x}_{2\ell-1}}, \grave{b}\, e_j \grave{e}_{2k-1}] = -\delta_{k,\ell}\, \grave{b}\, e_j \partial_{\grave{x}_{2\ell-1}},\\
[\grave{e}_{2\ell-1}\, \partial_{\grave{x}_{2\ell}}, \grave{b}\, x_j \partial_{\grave{x}_{2k-1}}] &= [\grave{e}_{2\ell-1}\, \partial_{\grave{x}_{2\ell}}, \grave{b}\, e_j \grave{e}_{2k-1}] = 0, \quad [\grave{e}_{2\ell-1}\, \partial_{\grave{x}_{2\ell}}, \grave{b}\, \grave{x}_{2k}\partial_{x_j}] = -\delta_{k,\ell}\, \grave{b}\, \grave{e}_{2\ell-1}\partial_{x_j}.
\end{aligned}$$

Then, $[\grave{e}_{2\ell}\, \partial_{\grave{x}_{2\ell-1}}, dL(\grave{b}\, e_j \grave{e}_{2k-1})] = -\delta_{k,\ell}\, \grave{b}\, e_j \partial_{\grave{x}_{2\ell-1}}$ and $[\grave{e}_{2\ell-1}\, \partial_{\grave{x}_{2\ell}}, dL(\grave{b}\, e_j \grave{e}_{2k-1})] = \delta_{k,\ell}\, \grave{b}\, \grave{e}_{2\ell-1}\partial_{x_j}$, whence

$$[\partial_{\grave{\underline{x}}}, dL(\grave{b}\, e_j \grave{e}_{2k-1})] = -2\sum_{\ell=1}^{n} \delta_{k,\ell}\, \grave{b} \left(e_j \partial_{\grave{x}_{2\ell-1}} + \grave{e}_{2\ell-1}\partial_{x_j}\right) = -2\grave{b} \left(e_j \partial_{\grave{x}_{2k-1}} + \grave{e}_{2k-1}\partial_{x_j}\right).$$

On the other hand, $[e_\ell\, \partial_{x_\ell}, \grave{b}\, x_j \partial_{\grave{x}_{2k-1}}] = \delta_{\ell,j}\, \grave{b}\, e_\ell \partial_{\grave{x}_{2k-1}}$, $[e_\ell\, \partial_{x_\ell}, \grave{b}\, \grave{x}_{2k}\partial_{x_j}] = 0$ and $[e_\ell\, \partial_{x_\ell}, \grave{b}\, e_j \grave{e}_{2k-1}] = -2\delta_{j,\ell}\, \grave{b}\, \grave{e}_{2k-1}\partial_{x_\ell}$, whence

$$[\partial_{\underline{x}}, dL(\grave{b}\, e_j \grave{e}_{2k-1})] = -2\sum_{\ell=1}^{m} \delta_{j,\ell}\, \grave{b} \left(e_\ell \partial_{\grave{x}_{2k-1}} + \grave{e}_{2k-1}\partial_{x_\ell}\right) = -2\grave{b} \left(e_j \partial_{\grave{x}_{2k-1}} + \grave{e}_{2k-1}\partial_{x_j}\right).$$

Finally, we obtain that $[\partial_x, dL(\grave{b}\, e_j \grave{e}_{2k-1})] = [\partial_{\grave{\underline{x}}}, dL(\grave{b}\, e_j \grave{e}_{2k-1})] - [\partial_{\underline{x}}, dL(\grave{b}\, e_j \grave{e}_{2k-1})] = 0$. The proof proceeds in a similar way for all other basis elements of $\mathbb{R}^{(2)E}_{m|2n}(\Lambda_N)$. $\square$

The above result implies the invariance of the super Dirac operator under the spin actions H and L.

Corollary 1. *For every* $s \in \mathrm{Spin}(m|2n)(\Lambda_N)$ *it holds that* $[\partial_x, L(s)] = 0 = [\partial_x, H(s)]$.

4 Hermitian super space

In this section we study some fundamental aspects of Hermitian Clifford analysis in superspace. In particular, we are interested in the group of supermatrices leaving the Hermitian inner product invariant, leading to a restriction of the spin group, depending on the so-called complex structure J.

4.1 Complex structure and Hermitian radial algebra in superspace

The fundaments of Hermitian Clifford analysis in superspace were introduced in [3] through the notion of a complex structure, which is defined by some axioms on the radial algebra level, see [5], but can also be seen as a specific element J of $\mathrm{SO}_0(m|2n)(\mathbb{R}\Lambda_N)$ satisfying $J^2 = -I_{m+2n}$. This condition implies that $\mathrm{sdet}(J)^2 = (-1)^{m-2n} = (-1)^m$, forcing m to be even.

The complex structure J chosen in [3] is the algebra automorphism over $\mathcal{A}_{2m,2n}(\mathbb{R}\mathcal{V})$ defined by

- J is the identity on $\mathbb{R}\mathcal{V}$;
- $J_1(e_j) = -e_{m+j}$, $J_1(e_{m+j}) = e_j$, $j = 1, \ldots, m$;
  $J(\grave{e}_{2j-1}) = -\grave{e}_{2j}$, $J(\grave{e}_{2j}) = \grave{e}_{2j-1}$, $j = 1, \ldots, n$;
- $J(FG) = J(F)J(G)$ for all $F, G \in \mathcal{A}_{2m,2n}(\mathbb{R}\mathcal{V})$.

Thence the action of J on every supervector has the form

$$J(x) = J(\underline{x}) + J(\grave{\underline{x}}) = \sum_{j=1}^{m} (x_{m+j} e_j - x_j e_{m+j}) + \sum_{j=1}^{n} (\grave{x}_{2j} \grave{e}_{2j-1} - \grave{x}_{2j-1} \grave{e}_{2j}),$$

and its estriction to $\mathbb{R}^{2m|2n}(\mathcal{V})$ can be written in a supermatrix form as

$$J = \begin{pmatrix} J_{2m} & 0 \\ 0 & \Omega_{2n} \end{pmatrix}, \qquad \text{where } J_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

showing that indeed $J \in SO_0(2m|2n)(\mathbb{R}\Lambda_N)$ and $J^2 = -I_{2m+2n}$.

Remark 4.1. *We recall that for $m = n$, the antisymmetric matrices J_{2n} and Ω_{2n} are used to define two different copies of the symplectic group, i.e. $\mathrm{Sp}_J(2n) = \{A_0 \in \mathbb{R}^{2n\times 2n} : A_0^T J_{2n} A_0 = J_{2n}\}$ and $\mathrm{Sp}_\Omega(2n) = \{D_0 \in \mathbb{R}^{2n\times 2n} : D_0^T \Omega_{2n} D_0 = \Omega_{2n}\}$. It is clear that J_{2n} and Ω_{2n} only differ by a permutation of the basis vectors, whence there exists an orthogonal matrix R such that $R^T J_{2n} R = \Omega_{2n}$. Hence,*

$$D_0^T \Omega_{2n} D_0 = \Omega_{2n} \iff D_0^T R^T J_{2n} R D_0 = R^T J_{2n} R \iff (R D_0 R^T)^T J_{2n} \, R D_0 R^T = J_{2n}.$$

meaning that $D_0 \in \mathrm{Sp}_\Omega(2n) \iff R D_0 R^T \in \mathrm{Sp}_J(2n)$, or equivalently, $R \, \mathrm{Sp}_\Omega(2n) \, R^T = \mathrm{Sp}_J(2n)$. The map $\gamma(D_0) = R D_0 R^T$ thus constitutes a Lie group isomorphism between $\mathrm{Sp}_\Omega(2n)$ and $\mathrm{Sp}_J(2n)$, whence also the corresponding Lie algebras $\mathfrak{so}_\Omega(2n) = \{D_0 \in \mathbb{R}^{2n\times 2n} : D_0^T \Omega_{2n} + \Omega_{2n} D_0 = 0\}$ and $\mathfrak{so}_J(2n) = \{A_0 \in \mathbb{R}^{2n\times 2n} : A_0^T J_{2n} + J_{2n} A_0 = 0\}$ are isomorphic. These observations allow us to speak of the "symplectic structure" independently of Ω_{2n} or J_{2n}.

The action of this complex structure produces a second super Dirac operator given by

$$\partial_{J(\underline{x})} := J(\partial_{\underline{x}}) = \sum_{j=1}^{m} \left(e_j \partial_{x_{m+j}} - e_{m+j} \partial_{x_j} \right), \qquad \partial_{J(\grave{\underline{x}})} := J(\partial_{\grave{\underline{x}}}) = 2 \sum_{j=1}^{n} \left(\grave{e}_{2j-1} \partial_{\grave{x}_{2j-1}} + \grave{e}_{2j} \partial_{\grave{x}_{2j}} \right),$$

and finally, $\partial_{J(x)} \cdot := J(\partial_x \cdot) = \partial_{J(\underline{x})} \cdot - \partial_{J(\grave{\underline{x}})} \cdot$. The complex structure J commutes with the partial derivatives ∂_{x_j} and $\partial_{\grave{x}_j}$, leading for the super Dirac operators ∂_x and $\partial_{J(x)}$ to the relations

$$J(\partial_x[F]) = \partial_{J(x)}[J(F)], \qquad J(\partial_{J(x)}[F]) = -\partial_x[J(F)], \qquad F \in \mathcal{TP}_{2m|2n}.$$

In accordance with the radial algebra axioms for J (see [3, 5]) the actions of the Dirac operators on the supervector variables x and $J(x)$ give two important defining elements: the superdimension M and the fundamental bivector $\mathbf{B} \in \mathbb{R}^{(2)E}_{2m|2n}(\Lambda_N)$. Indeed,

$$\partial_x[x] = 2m - 2n = \partial_{J(x)}[J(x)], \qquad \partial_x[J(x)] = 2\mathbf{B} = -\partial_{J(x)}[x],$$

where $\mathbf{B} = \sum_{j=1}^{m} e_j e_{m+j} - \sum_{j=1}^{2n} \grave{e}_j{}^2$. Since J also is an element of $\mathfrak{so}_0(2m|2n)(\mathbb{R}\Lambda_N)$, another important characterization of $\mathbf{B}$ is that

$$[\mathbf{B}, x] = -2J(x) \quad \text{or equivalently} \quad \phi(\mathbf{B}) = -2J. \tag{13}$$

A natural realization of the Hermitian radial algebra is easily found in the complexification $\mathcal{A}_{2m,2n}(\mathbb{C}\mathcal{V})$ of $\mathcal{A}_{2m,2n}(\mathbb{R}\mathcal{V})$, i.e. $\mathcal{A}_{m,2n}(\mathbb{C}\mathcal{V}) = \mathcal{A}_{2m,2n}(\mathbb{R}\mathcal{V}) \oplus i\mathcal{A}_{2m,2n}(\mathbb{R}\mathcal{V}) = \mathbb{C}\mathcal{V} \otimes \mathcal{C}_{2m,2n}$, where the imaginary unit i commutes with every element of $\mathcal{A}_{2m,2n}(\mathbb{R}\mathcal{V})$. Similarly the complexification of $T(V)/I$ is $T(V_\mathbb{C})/I$. As usual, the Hermitian conjugation is defined over $\mathcal{A}_{2m,2n}(\mathbb{C}\mathcal{V})$ by the rule $(a + ib)^\dagger = \overline{a} - i\overline{b}$, while the complex conjugation is denoted as $(a + ib)^c = a - ib$, where $a, b \in \mathcal{A}_{2m,2n}(\mathbb{R}\mathcal{V})$.

Letting act the projection operators $\frac{1}{2}(I_{2m+2n} \pm iJ)$ on $\mathbb{R}^{2m|2n}(\mathcal{V})$ we obtain the sets of complex supervectors generating the realization of the Hermitian radial algebra in this setting, given by the following subspaces of the $\mathcal{V}$-envelope $\mathbb{C}^{2m|2n}(\mathcal{V})$:

$$W = \left\{ z = \frac{1}{2}[x + iJ(x)] : x \in \mathbb{R}^{2m|2n}(\mathcal{V}) \right\}, \qquad W^\dagger = \left\{ z^\dagger = -\frac{1}{2}[x - iJ(x)] : x \in \mathbb{R}^{2m|2n}(\mathcal{V}) \right\}.$$

The Hermitian vector variables z, $z^\dagger$ may be decomposed as the sum of complex bosonic and fermionic vector variables $z = \underline{z} + \underline{\grave{z}}$, $z^\dagger = \underline{z}^\dagger + \underline{\grave{z}}^\dagger$ where

$$\begin{aligned}
\underline{z} = & \ \frac{1}{2}\Big(\underline{x} + iJ(\underline{x})\Big) = & \frac{1}{2}\sum_{j=1}^{m}(x_j + ix_{m+j})(e_j - ie_{m+j}) & = \sum_{j=1}^{m} z_j \mathfrak{f}_j,\\
\underline{z}^\dagger = & -\frac{1}{2}\Big(\underline{x} - iJ(\underline{x})\Big) = & -\frac{1}{2}\sum_{j=1}^{m}(x_j - ix_{m+j})(e_j + ie_{m+j}) & = \sum_{j=1}^{m} z_j^c \mathfrak{f}_j^\dagger,\\
\underline{\grave{z}} = & \ \frac{1}{2}\Big(\underline{\grave{x}} + iJ(\underline{\grave{x}})\Big) = & \frac{1}{2}\sum_{j=1}^{n}(\grave{x}_{2j-1} + i\grave{x}_{2j})(\grave{e}_{2j-1} - i\grave{e}_{2j}) & = \sum_{j=1}^{m} \grave{z}_j \grave{\mathfrak{f}}_j,\\
\underline{\grave{z}}^\dagger = & -\frac{1}{2}\Big(\underline{\grave{x}} - iJ(\underline{\grave{x}})\Big) = & -\frac{1}{2}\sum_{j=1}^{n}(\grave{x}_{2j-1} - i\grave{x}_{2j})(\grave{e}_{2j-1} + i\grave{e}_{2j}) & = \sum_{j=1}^{m} \grave{z}_j^c \grave{\mathfrak{f}}_j^\dagger,
\end{aligned}$$

where we have introduced the commuting and anticommuting variables $z_j = x_j + ix_{m+j} \in \mathbb{C}\mathcal{V}_{\overline{0}}$ and $\grave{z}_j = \grave{x}_{2j-1} + i\grave{x}_{2j} \in \mathbb{C}\mathcal{V}_{\overline{1}}$, together with their complex conjugates $z_j^c = x_j - ix_{m+j}$ and $\grave{z}_j^c = \grave{x}_{2j-1} - i\grave{x}_{2j}$. We have also introduced the Witt basis elements

$$\mathfrak{f}_j = \frac{1}{2}(e_j - ie_{m+j}), \qquad \mathfrak{f}_j^\dagger = -\frac{1}{2}(e_j + ie_{m+j}), \qquad \grave{\mathfrak{f}}_j = \frac{1}{2}(\grave{e}_{2j-1} - i\grave{e}_{2j}), \qquad \grave{\mathfrak{f}}_j^\dagger = -\frac{1}{2}(\grave{e}_{2j-1} + i\grave{e}_{2j}),$$

which submit to the multiplication rules

$$\begin{cases} \mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = 0,\\ \mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j^\dagger = 0,\\ \mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{j,k}, \end{cases} \quad \begin{cases} \grave{\mathfrak{f}}_j\grave{\mathfrak{f}}_k - \grave{\mathfrak{f}}_k\grave{\mathfrak{f}}_j = 0,\\ \grave{\mathfrak{f}}_j^\dagger\grave{\mathfrak{f}}_k^\dagger - \grave{\mathfrak{f}}_k^\dagger\grave{\mathfrak{f}}_j^\dagger = 0,\\ \grave{\mathfrak{f}}_j\grave{\mathfrak{f}}_k^\dagger - \grave{\mathfrak{f}}_k^\dagger\grave{\mathfrak{f}}_j = -\frac{i}{2}\delta_{j,k}, \end{cases} \quad \begin{cases} \mathfrak{f}_j\grave{\mathfrak{f}}_k + \grave{\mathfrak{f}}_k\mathfrak{f}_j = 0,\\ \mathfrak{f}_j\grave{\mathfrak{f}}_k^\dagger + \grave{\mathfrak{f}}_k^\dagger\mathfrak{f}_j = 0, \end{cases} \quad \begin{cases} \mathfrak{f}_j^\dagger\grave{\mathfrak{f}}_k + \grave{\mathfrak{f}}_k\mathfrak{f}_j^\dagger = 0,\\ \mathfrak{f}_j^\dagger\grave{\mathfrak{f}}_k^\dagger + \grave{\mathfrak{f}}_k^\dagger\mathfrak{f}_j^\dagger = 0. \end{cases}$$

The algebra generated by the vector variables $z, z^\dagger$ indeed is a suitable realization of the Hermitian radial algebra since the element $\{z, u^\dagger\}$ is a commuting object. In fact,

$$\langle z, u\rangle_{\mathbb{C}} =: \{z, u^\dagger\} = \sum_{j=1}^{m} z_j u_j^c - \frac{i}{2}\sum_{j=1}^{n} \grave{z}_j \grave{u}_j^c \in \mathbb{C}\mathcal{V}_{\overline{0}}. \tag{14}$$

Since $\{z, u^\dagger\}^c = \{u, z^\dagger\}$, the above expression can be used to define an "inner product" of the complex supervectors z and u, see [3].

4.2 Invariance of W and $W^\dagger$.

It is easily seen that the spaces W and $W^\dagger$ establish a direct sum decomposition of the complex $\mathcal{V}$-envelope $\mathbb{C}^{2m|2n}(\mathcal{V})$, i.e. $\mathbb{C}^{2m|2n}(\mathcal{V}) = W \oplus W^\dagger$. Every supermatrix $M \in \mathrm{Mat}(2m|2n)(\mathbb{R}\Lambda_N)$ acts on elements of W, $W^\dagger$ as it naturally does on elements of $\mathbb{C}^{2m|2n}(\mathcal{V})$:

$$z = \frac{1}{2}\left[x + iJ(x)\right] \longmapsto \frac{1}{2}\left[Mx + iMJ(x)\right]; \qquad z^\dagger = -\frac{1}{2}\left[x - iJ(x)\right] \longmapsto -\frac{1}{2}\left[Mx - iMJ(x)\right].$$

Hence, the subspace $\mathrm{Mat}_J(2m|2n)(\mathbb{R}\Lambda_N)$ of $\mathrm{Mat}(2m|2n)(\mathbb{R}\Lambda_N)$ leaving W and $W^\dagger$ invariant is given by

$$\mathrm{Mat}_J(2m|2n)(\mathbb{R}\Lambda_N) = \{M \in \mathrm{Mat}(2m|2n)(\mathbb{R}\Lambda_N) : MJ = JM\}.$$

Proposition 9. *Let $M \in \mathrm{Mat}(2m|2n)(\mathbb{R}\Lambda_N)$, as in (4). Then the following statements are equivalent:*

(i) $MJ = JM$;

(ii) the matrices, $A, \grave{B}, \grave{C}, D$ *satisfy* $AJ_{2m} = J_{2m}A$, $\grave{B}\Omega_{2n} = J_{2m}\grave{B}$, $\grave{C}J_{2m} = \Omega_{2n}\grave{C}$, $D\Omega_{2n} = \Omega_{2n}D$;

(iii) the matrices, $A, \grave{B}, \grave{C}, D$ *have the form*

$$A = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}, \quad \grave{B} = \begin{pmatrix} \grave{B}_1 \\ \grave{B}_1\Omega_{2m} \end{pmatrix}, \quad \grave{C} = \left(\grave{C}_1 \middle| -\Omega_{2n}\grave{C}_1\right)$$

and

$$D=\left\{D_{jk}\right\}_{j,k=1,\dots,n}\ \textit{with}\ D_{jk}=\left(\begin{array}{cc} a_{jk} & b_{jk}\\ -b_{jk} & a_{jk}\end{array}\right),$$

where $A_1,A_2\in\left(\mathbb{R}\Lambda_{N,\bar{0}}\right)^{m\times m}$, $\grave{B}_1\in\left(\mathbb{R}\Lambda_{N,\bar{1}}\right)^{m\times 2n}$, $\grave{C}_1\in\left(\mathbb{R}\Lambda_{N,\bar{1}}\right)^{2n\times m}$ *and* $a_{jk},b_{jk}\in\mathbb{R}\Lambda_{N,\bar{0}}$.

Proof.
The equivalence between (i) and (ii) easily follows from the block structure of the supermatrices M and J. To prove the equivalence between (ii) and (iii) we first write

$$\begin{array}{ll} A=\left(\begin{array}{cc} A_1 & A_2\\ A_3 & A_4\end{array}\right)\ A_j\in\left(\mathbb{R}\Lambda_{N,\bar{0}}\right)^{m\times m}, & \grave{B}=\left(\begin{array}{c}\grave{B}_1\\ \grave{B}_2\end{array}\right)\ \grave{B}_j\in\left(\mathbb{R}\Lambda_{N,\bar{1}}\right)^{m\times 2n},\\ \grave{C}=\left(\grave{C}_1\middle|\grave{C}_2\right)\ \grave{C}_j\in\left(\mathbb{R}\Lambda_{N,\bar{1}}\right)^{2n\times m}, & D=\left\{D_{jk}\right\}_{j,k=1,\dots,n}\ D_{jk}\in\left(\mathbb{R}\Lambda_{N,\bar{0}}\right)^{2\times 2}.\end{array}$$

Then, easy computations show that

$$\begin{array}{l} AJ_{2m}=J_{2m}A\iff\left(\begin{array}{cc}-A_2 & A_1\\ -A_4 & A_3\end{array}\right)=\left(\begin{array}{cc}A_3 & A_4\\ -A_1 & -A_2\end{array}\right)\iff A=\left(\begin{array}{cc}A_1 & A_2\\ -A_2 & A_1\end{array}\right),\\ \grave{B}\Omega_{2n}=J_{2m}\grave{B}\iff\left(\begin{array}{c}\grave{B}_1\Omega_{2n}\\ \grave{B}_2\Omega_{2n}\end{array}\right)=\left(\begin{array}{c}\grave{B}_2\\ -\grave{B}_1\end{array}\right)\iff\grave{B}=\left(\begin{array}{c}\grave{B}_1\\ \grave{B}_1\Omega_{2m}\end{array}\right),\\ \grave{C}J_{2m}=\Omega_{2n}\grave{C}\iff\left(-\grave{C}_2\middle|\grave{C}_1\right)=\left(\Omega_{2n}\grave{C}_1\middle|\Omega_{2n}\grave{C}_2\right)\iff\left(\grave{C}_1\middle|-\Omega_{2n}\grave{C}_1\right),\\ D\Omega_{2n}=\Omega_{2n}D\iff D_{jk}\Omega_2=\Omega_2D_{jk}\iff D_{jk}=\left(\begin{array}{cc}a_{jk} & b_{jk}\\ -b_{jk} & a_{jk}\end{array}\right).\end{array}$$

□

The subspaces W and $W^\dagger$ both are isomorphic to the $\mathcal{V}$-envelope $\mathbb{C}^{m|n}(\mathcal{V})$. For that reason we denote the Hermitian radial algebra generated by them as $\mathbb{C}_{m|n}(\mathcal{V})$. Every complex supervector $z=\sum_{j=1}^m z_j\mathfrak{f}_j+\sum_{j=1}^n \grave{z}_j\grave{\mathfrak{f}}_j$ and its Hermitian conjugate $z^\dagger=\sum_{j=1}^m z_j^c\mathfrak{f}_j^\dagger+\sum_{j=1}^n \grave{z}_j^c\,\grave{\mathfrak{f}}_j^\dagger$, can be written as column vectors of $\mathbb{C}^{m|n}(\mathcal{V})$ in the form

$$z=\left(\begin{array}{c}\underline{z}\\ \underline{\grave{z}}\end{array}\right)=\left(z_1,\dots,z_m,\grave{z}_1,\dots,\grave{z}_n\right)^T,\qquad z^c=\left(\begin{array}{c}\underline{z}^c\\ \underline{\grave{z}}^c\end{array}\right)=\left(z_1^c,\dots,z_m^c,\grave{z}_1^c,\dots,\grave{z}_n^c\right)^T.$$

In this way, the equalities $z=\frac{1}{2}\left[x+iJ(x)\right]$ and $z^\dagger=-\frac{1}{2}\left[x-iJ(x)\right]$ can be rewritten in the matrix[2] form:

$$z=\mathbf{P}x,\qquad z^c=\mathbf{P}^cx,\ \text{where}\ \mathbf{P}=\left(\begin{array}{cc}P & 0\\ 0 & Q\end{array}\right),$$

and with

$$P=\left(\begin{array}{cccc|cccc}1 & 0 & \dots & 0 & i & 0 & \dots & 0\\ 0 & 1 & \dots & 0 & 0 & i & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & i\end{array}\right)=\left(I_m\,\middle|\,iI_m\right)\in\mathbb{C}^{m\times 2m},\ Q=\left(\begin{array}{ccccccc}1 & i & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & i & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \dots & 1 & i\end{array}\right)\in\mathbb{C}^{n\times 2n}.$$

These complex matrices play a rôle in every computation involving the supervectors z and x. In particular, they satisfy the following relations.

$$\mathbf{P}\left(\mathbf{P}^T\right)^c=2I_{m+n},\qquad\left(\mathbf{P}^T\right)^c\mathbf{P}=I_{2m+2n}+iJ,\qquad\mathbf{P}J=-i\mathbf{P}.\tag{15}$$

Every linear transformation in $\mathbb{R}^{2m|2n}(\mathcal{V})$ defined by a supermatrix $M\in\mathrm{Mat}_J(2m|2n)(\mathbb{R}\Lambda_N)$ is associated to a transformation in $\mathbb{C}^{m|n}(\mathcal{V})$ determined by

$$u=\frac{1}{2}\left[Mx+iJ(Mx)\right],\quad x\in\mathbb{R}^{2m|2n}(\mathcal{V}).$$

[2]The complex conjugate M^c of a supermatrix $M\in\mathrm{Mat}(p|q)(\mathbb{C}\Lambda_N)$ is defined componentwise.

This transformation can be written in terms of a supermatrix $\psi(M) \in \mathrm{Mat}(m|n)(\mathbb{C}\Lambda_N)$ as $u = \psi(M)z$ with $z = \frac{1}{2}(x + iJ(x))$. Indeed, the above relation can be rewritten as $\mathbf{P}Mx = \psi(M)\mathbf{P}x$ for every $x \in \mathbb{R}^{2m|2n}(\mathcal{V})$, meaning that $\psi(M)\mathbf{P} = \mathbf{P}M$, or equivalently,

$$\psi(M) = \frac{1}{2}\mathbf{P}M\left(\mathbf{P}^T\right)^c. \tag{16}$$

Using Proposition 9 we easily get

$$\psi\left(\begin{array}{cc|c} A_1 & A_2 & \grave{B}_1 \\ -A_2 & A_1 & \grave{B}_1\Omega_{2n} \\ \hline \grave{C}_1 & -\Omega_{2n}\grave{C}_1 & D \end{array}\right) = \left(\begin{array}{cc} A_1 - iA_2 & \grave{B}_1\left(Q^T\right)^c \\ Q\grave{C}_1 & \frac{1}{2}QD\left(Q^T\right)^c \end{array}\right). \tag{17}$$

Remark 4.2. *It is known from Proposition 9 that the matrix D is composed of 2×2 blocks D_{jk}. It then easily follows that $\frac{1}{2}QD\left(Q^T\right)^c = \{a_{jk} - ib_{jk}\}_{j,k=1,\ldots,n}$.*

Proposition 10. *The map $\psi : \mathrm{Mat}_J(2m|2n)(\mathbb{R}\Lambda_N) \to \mathrm{Mat}(m|n)(\mathbb{C}\Lambda_N)$ is a real algebra isomorphism.*

Proof.
Using the properties of $\mathbf{P}$ given in (15) it easily follows that ψ is invertible and its inverse is given by

$$\psi^{-1}(L) = \frac{1}{2}\left[\left(\mathbf{P}^T\right)^c L\,\mathbf{P} + \mathbf{P}^T L^c\,\mathbf{P}^c\right], \qquad L \in \mathrm{Mat}(m|n)(\mathbb{C}\Lambda_N).$$

That ψ is an algebra isomorphism follows from its real-linearity and from

$$\begin{aligned}\psi(M_1)\psi(M_2) &= \frac{1}{4}\mathbf{P}M_1\left(\mathbf{P}^T\right)^c\mathbf{P}M_2\left(\mathbf{P}^T\right)^c = \frac{1}{4}\mathbf{P}M_1\left(I_{2m+2n} + iJ\right)M_2\left(\mathbf{P}^T\right)^c \\ &= \frac{1}{4}\left[\mathbf{P}M_1M_2\left(\mathbf{P}^T\right)^c + i\mathbf{P}JM_1M_2\left(\mathbf{P}^T\right)^c\right] \;=\; \frac{1}{2}\mathbf{P}M_1M_2\left(\mathbf{P}^T\right)^c \;=\; \psi(M_1M_2).\end{aligned}$$

□

4.3 Invariance of the Hermitian inner product. The group SO_0^J.

We are now interested in the invariance, under linear actions, of the inner product (14) between complex supervectors. In particular, we want to describe the set of supermatrices $M \in \mathrm{Mat}(m|n)(\mathbb{C}\Lambda_N)$ satisfying

$$\langle Mz, Mu\rangle_{\mathbb{C}} = \langle z, u\rangle_{\mathbb{C}}, \qquad \forall z, u \in \mathbb{C}^{m|n}(\mathcal{V}). \tag{18}$$

Observe that $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ can be written in a matricial form as

$$\langle z, u\rangle_{\mathbb{C}} = z^T\mathbf{H}u^c, \quad \text{ where } \mathbf{H} = \left(\begin{array}{cc} I_m & 0 \\ 0 & -\frac{i}{2}I_n \end{array}\right).$$

Then a super matrix $M \in \mathrm{Mat}(m|n)(\mathbb{C}\Lambda_N)$ satisfies the condition (18) if and only if

$$z^T\left(M^{ST}\mathbf{H}M^c - \mathbf{H}\right)u^c = 0 \quad \forall z, u \in \mathbb{C}^{m|n}(\mathcal{V}), \quad \text{ or equivalently } \quad M^{ST}\mathbf{H}M^c - \mathbf{H} = 0.$$

Hence the set of supermatrices leaving the inner product (14) invariant is given by

$$\mathrm{U}_0 = \mathrm{U}_0(m|n)(\mathbb{C}\Lambda_N) = \left\{M \in \mathrm{Mat}(m|n)(\mathbb{C}\Lambda_N) : \left(M^{ST}\right)^c\mathbf{H}^cM - \mathbf{H}^c = 0\right\},$$

which is a closed subgroup of $\mathrm{GL}(m|n)(\mathbb{C}\Lambda_N)$ and in consequence a Lie group.

Proposition 11. *The following statements hold:*

(i) a supermatrix $M \in \mathrm{Mat}(m|n)(\mathbb{C}\Lambda_N)$ of the form (4) belongs to U_0 if and only if

$$\left(A^T\right)^c A + \frac{i}{2}\left(\grave{C}^T\right)^c\grave{C} = I_m,\ \left(A^T\right)^c\grave{B} + \frac{i}{2}\left(\grave{C}^T\right)^c D = 0,\ -\left(\grave{B}^T\right)^c\grave{B} + \frac{i}{2}\left(D^T\right)^c D = \frac{i}{2}I_n;$$

(ii) $(\operatorname{sdet} M)^c \operatorname{sdet} M = 1$ *for every* $M \in \mathrm{U}_0$*;*

(iii) $[\mathrm{U}_0]_0 = \mathrm{U}(m) \times \mathrm{U}(n)$*, where* $\mathrm{U}(k)$ *denotes the unitary group of order* k*.*

Proof.

(i) The relation $\left(M^{ST}\right)^c \mathbf{H}^c M = \mathbf{H}^c$ can be written in terms of $A, \grave{B}, \grave{C}, D$ as

$$\left(\begin{array}{cc} \left(A^T\right)^c A + \frac{i}{2}\left(\grave{C}^T\right)^c \grave{C} & \left(A^T\right)^c \grave{B} + \frac{i}{2}\left(\grave{C}^T\right)^c D \\ -\left(\grave{B}^T\right)^c A + \frac{i}{2}\left(D^T\right)^c \grave{C} & -\left(\grave{B}^T\right)^c \grave{B} + \frac{i}{2}\left(D^T\right)^c D \end{array} \right) = \left(\begin{array}{cc} I_m & 0 \\ 0 & \frac{i}{2} I_n \end{array} \right).$$

(ii) It is easily obtained from $\left(M^{ST}\right)^c \mathbf{H}^c M = \mathbf{H}^c$.

(iii) Applying the homomorphism $[\cdot]_0$ to each of the relations in (i) we get for

$$[M]_0 = \left(\begin{array}{cc} A_0 & 0 \\ 0 & D_0 \end{array} \right)$$

that $\left(A_0^T\right)^c A_0 = I_m$ and $\left(D_0^T\right)^c D_0 = I_n$ or equivalently $A_0 \in \mathrm{U}(m)$ and $D_0 \in \mathrm{U}(n)$. □

As it was done in [4] for SO_0, it can be proved that this generalization of the unitary groups $\mathrm{U}(m)$ and $\mathrm{U}(n)$ is connected.

Proposition 12. U_0 *is a connected Lie group.*

Proof.
This proof is similar to the one in [4] showing the connectedness of SO_0. The strategy is to find for every $M \in \mathrm{U}_0$ a continuous path $M(t)$ $(0 \leq t \leq 1)$ connecting, inside U_0, M with its complex projection M_0, which belongs to the connected group $\mathrm{U}(m) \times \mathrm{U}(n)$. This continuous path is given by $M(t) = \sum_{j=0}^{N} t^j [M]_j$, where $[M]_j$ is the projection of M on $\mathrm{Mat}(m|n)(\mathbb{C}\Lambda_N^{(j)})$. □

As in the classical case, the Lie group $\mathrm{U}_0(m|n)(\mathbb{C}\Lambda_N)$ is related to $\mathrm{SO}_0(2m|2n)(\mathbb{R}\Lambda_N)$ through the complex structure J. Indeed, it is clear that supermatrices in the group

$$\mathrm{SO}_0^J = \mathrm{SO}_0^J(2m|2n)(\mathbb{R}\Lambda_N) = \mathrm{SO}_0(2m|2n)(\mathbb{R}\Lambda_N) \cap \mathrm{Mat}_J(2m|2n)(\mathbb{R}\Lambda_N),$$

have the double property of keeping the bilinear forms $\{x, y\}$ and $\{x, J(y)\}$ for $x, y \in \mathbb{R}^{2m|2n}(\mathcal{V})$ invariant. But for the Hermitian inner product (14) we have

$$\langle z, u \rangle_{\mathbb{C}} = \{z, u^\dagger\} = \left\{ \frac{1}{2}(x + iJ(x)), -\frac{1}{2}(y - iJ(y)) \right\} = -\frac{1}{2}\big[\{x, y\} - i\{x, J(y)\}\big],$$

whence the action of every supermatrix in SO_0^J leaves the Hermitian inner product invariant as well. This property is summarized below.

Proposition 13. *The map* ψ *defined in (16) is a Lie group isomorphism between* $\mathrm{SO}_0^J(2m|2n)(\mathbb{R}\Lambda_N)$ *and* $\mathrm{U}_0(m|n)(\mathbb{C}\Lambda_N)$*.*

Proof.
It is clear that ψ defines a smooth map and in addition, it was proved in Proposition 10 that ψ is a group homomorphism. It thus only remains to show that $\psi(\mathrm{SO}_0^J) = \mathrm{U}_0$. To that end, first observe that $\psi(\mathbf{Q}) = \mathbf{H}^c$ and that $\left(\psi(M)^{ST}\right)^c = \frac{1}{2}\mathbf{P} M^{ST} \left(\mathbf{P}^T\right)^c = \psi(M^{ST})$. Using Proposition 10 we get for every $M \in \mathrm{SO}_0^J$ that

$$\left(\psi(M)^{ST}\right)^c \mathbf{H}^c \psi(M) - \mathbf{H}^c = \psi(M^{ST})\psi(\mathbf{Q})\psi(M) - \psi(\mathbf{Q}) = \psi\left(M^{ST}\mathbf{Q}M - \mathbf{Q}\right) = 0,$$

meaning that $\psi(M) \in \mathrm{U}_0$, and in consequence $\psi(\mathrm{SO}_0^J) \subset \mathrm{U}_0$. On the other hand, for $L \in \mathrm{U}_0$ we get $\psi^{-1}(L)^{ST} = \frac{1}{2}\left[\mathbf{P}^T L^{ST} \mathbf{P}^c + \left(\mathbf{P}^T\right)^c \left(L^{ST}\right)^c \mathbf{P}\right] = \psi^{-1}\left(\left(L^{ST}\right)^c\right)$. Hence,

$$\psi^{-1}(L)^{ST}\mathbf{Q}\psi^{-1}(L) - \mathbf{Q} = \psi^{-1}(L)^{ST}\psi^{-1}(\mathbf{H}^c)\psi^{-1}(L) - \psi^{-1}(\mathbf{H}^c) = \psi^{-1}\left(\left(L^{ST}\right)^c \mathbf{H}^c L - \mathbf{H}^c\right) = 0.$$

This shows that the supermatrix $M = \psi^{-1}(L)$ belongs to O_0, but we still have to prove that $\mathrm{sdet}(M) = 1$. To that end it suffices to compute $\mathrm{sdet}(M_0)$, since $\mathrm{sdet}(M) = \mathrm{sdet}(M_0)$. From section 3 we know that

$$M_0 = \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$$

with $A_0 \in \mathrm{O}(2m)$ and $D_0 \in \mathrm{Sp}_\Omega(2n)$. In addition, $M_0 J = J M_0$ which implies in particular that $A_0 J_{2m} = J_{2m} A_0$. Combining this with $A_0^T A_0 = I_{2m}$, straightforward computations yield $A_0^T J_{2m} A_0 = J_{2m}$, or still $A_0 \in \mathrm{Sp}_J(2m)$. Hence, $\det(A_0) = \det(D_0) = 1$ since the determinant of a symplectic matrix always equals 1, implying that $\mathrm{sdet}(M_0) = 1$. Whence $M = \psi^{-1}(L) \in \mathrm{SO}_0^J$ and in consequence $\psi^{-1}(\mathrm{U}_0) \subset \mathrm{SO}_0^J$. □

The above proposition leads to the following result on the Lie algebra level.

Proposition 14. *The Lie algebras of* SO_0^J *and* U_0 *are given by*

$$\begin{aligned} \mathfrak{so}_0^J &= \mathfrak{so}_0^J(2m|2n)(\mathbb{R}\Lambda_N) = \{X \in \mathfrak{so}_0(2m|2n)(\mathbb{R}\Lambda_N) : XJ = JX\}, \\ \mathfrak{u}_0 &= \mathfrak{u}_0(m|n)(\mathbb{C}\Lambda_N) = \{X \in \mathrm{Mat}(m|n)(\mathbb{C}\Lambda_N) : \left(X^{ST}\right)^c \mathbf{H}^c + \mathbf{H}^c X = 0\}, \end{aligned} \tag{19}$$

respectively. In addition, $\psi : \mathfrak{so}_0^J \to \mathfrak{u}_0$ *is a Lie algebra isomorphism.*

Proof.
If $X \in \mathrm{Mat}(m|n)(\mathbb{C}\Lambda_N)$ belongs to the Lie algebra of U_0, it satisfies $e^{tX} \in \mathrm{U}_0$ for every $t \in \mathbb{R}$. Differentiating both sides of the equality $e^{t(X^{ST})^c}\mathbf{H}^c e^{tX} = \mathbf{H}^c$ and evaluating at $t = 0$, we get $\left(X^{ST}\right)^c \mathbf{H}^c + \mathbf{H}^c X = 0$. On the other hand, it is easily seen that every supermatrix satisfying the above condition is such that $e^{tX} \in \mathrm{U}_0$ for every $t \in \mathbb{R}$. Then, the Lie algebra of U_0 is the one given in (19). The Lie algebra of SO_0^J is similarly obtained: differentiating both sides of $e^{tM}J = Je^{tM}$, we get $MJ = JM$. Vice versa, if $MJ = JM$ then clearly $e^{tM}J = Je^{tM}$. Since $\psi : \mathrm{SO}_0^J \to \mathrm{U}_0$ is a Lie group isomorphism, its infinitesimal representation $d\psi : \mathfrak{so}_0^J \to \mathfrak{u}_0$ turns out to be a Lie algebra isomorphism, see [7]. The map $d\psi$ is obtained from ψ through the relation $e^{t\, d\psi(X)} = \psi(e^{tX})$, $t \in \mathbb{R}$, $X \in \mathfrak{so}_0^J$. Differentiating at $t = 0$ we get

$$d\psi(X) = \frac{d}{dt}\psi(e^{tX})\bigg|_{t=0} = \frac{d}{dt}\left[\frac{1}{2}\mathbf{P}e^{tX}\left(\mathbf{P}^T\right)^c\right]\bigg|_{t=0} = \frac{1}{2}\mathbf{P}X\left(\mathbf{P}^T\right)^c = \psi(X).$$

Hence, ψ is its own infinitesimal representation. □

Remark 4.3. *Straightforward computations show that the numeric projections of* SO_0^J, U_0, $\mathfrak{so}_0^J$ *and* $\mathfrak{u}_0$ *are given by the sets*

$$\begin{aligned} \left[\mathrm{SO}_0^J\right]_0 &= [\mathrm{SO}(2m) \cap \mathrm{Sp}_J(2m)] \times [\mathrm{SO}(2n) \cap \mathrm{Sp}_\Omega(2n)]\,, & [\mathrm{U}_0]_0 &= \mathrm{U}(m) \times \mathrm{U}(n), \\ \left[\mathfrak{so}_0^J\right]_0 &= [\mathfrak{so}(2m) \cap \mathfrak{sp}_J(2m)] \times [\mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)]\,, & [\mathfrak{u}_0]_0 &= \mathfrak{u}(m) \times \mathfrak{u}(n), \end{aligned}$$

where $\mathfrak{u}(k) = \{A_0 \in \mathbb{C}^{k\times k} : \left(A_0^T\right)^c + A_0 = 0\}$ *is the classical unitary Lie algebra in dimension* k*. Here,* ψ *is a Lie group isomorphism between* $[\mathrm{SO}_0^J]_0$ *and* $[\mathrm{U}_0]_0$ *(respectively a Lie algebra isomorphism between* $[\mathfrak{so}_0^J]_0$ *and* $[\mathfrak{u}_0]_0$*). The projections* $\psi_{J_{2m}}$ *and* $\psi_{\Omega_{2n}}$ *of* ψ *over* $[\mathrm{SO}(2m) \cap \mathrm{Sp}_J(2m)]$ *and* $[\mathrm{SO}(2n) \cap \mathrm{Sp}_\Omega(2n)]$ *(*$[\mathfrak{so}(2m) \cap \mathfrak{sp}_J(2m)]$ *and* $[\mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)]$*), respectively, are given by*

$$\psi_{J_{2m}}(A_0) = \frac{1}{2}PA_0\left(P^T\right)^c \qquad \psi_{\Omega_{2n}}(D_0) = \frac{1}{2}QD_0\left(Q^T\right)^c.$$

They define the Lie group (Lie algebra) isomorphisms

$$\begin{aligned} \psi_{J_{2m}} &: [\mathrm{SO}(2m) \cap \mathrm{Sp}_J(2m)] \to \mathrm{U}(m), & \psi_{J_{2m}} &: [\mathfrak{so}(2m) \cap \mathfrak{sp}_J(2m)] \to \mathfrak{u}(m), \\ \psi_{\Omega_{2n}} &: [\mathrm{SO}(2n) \cap \mathrm{Sp}_\Omega(2n)] \to \mathrm{U}(n), & \psi_{\Omega_{2n}} &: [\mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)] \to \mathfrak{u}(n). \end{aligned}$$

In the Euclidean Clifford setting in superspace, it has been shown that the fundamental symmetry group SO_0 is connected but non-compact, the non-compactness being due to the realization of $\mathrm{Sp}_\Omega(2n)$ in the real projection of SO_0. The introduction of the complex structure, and the consequent refinement of the symmetry group, causes the corresponding numeric projection $[\mathrm{U}_0]_0 = \mathrm{U}(m) \times \mathrm{U}(n)$ to be compact, while U_0 remains non-compact. Indeed, an example of an unbounded sequence $\{M(k)\}_{k\in\mathbb{N}}$ in U_0 is

$$M(k) = \begin{pmatrix} I_m & -\frac{i}{2} k\, \mathfrak{b}\, E_{m\times n} \\ k\, \mathfrak{b}\, E_{n\times m} & I_n \end{pmatrix}, \quad \text{where } \mathfrak{b} \in \mathbb{C}\Lambda_{N,\bar{1}},\ E_{p\times q} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{p\times q}.$$

As a consequence $\exp : \mathfrak{so}_0^J \to \mathrm{SO}_0^J$ may not be surjective. However, SO_0^J can be fully described by products of exponentials acting in some special subalgebras of $\mathfrak{so}_0^J$. Indeed, write $M \in \mathrm{SO}_0^J$ as $M = M_0 + \mathbf{M} = M_0(I_{2m+2n} + \mathbf{L})$ where $M_0 \in [\mathrm{SO}_0^J]_0$ is the real projection of M, $\mathbf{M} \in \mathrm{Mat}(2m|2n)(\Lambda_N^+)$ is its nilpotent projection and $\mathbf{L} = M_0^{-1}\mathbf{M}$. Since $[\mathrm{SO}_0^J]_0 \cong \mathrm{U}(m) \times \mathrm{U}(n)$ is connected and compact, the exponential map is surjective on this group, see Corollary 11.10 in [7], whence we can write $M_0 = e^X$ with $X \in [\mathfrak{so}(2m) \cap \mathfrak{sp}_J(2m)] \times [\mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)]$. As explained in Section 2, there is only one matrix in $\mathrm{Mat}(2m|2n)(\Lambda_N^+)$ whose exponential equals $I_{2m+2n} + \mathbf{L}$, viz $\mathbf{Z} = \ln(I_{2m+2n} + \mathbf{L})$. Following the decomposition of SO_0 in Theorem 2, we get that $\mathbf{Z} \in \mathfrak{so}_0(2m|2n)(\Lambda_N^+)$. In addition, we recall that $I_{2m+2n} + \mathbf{L}$ commutes with J, meaning $J\mathbf{L} = \mathbf{L}J$. Thence

$$\mathbf{Z} = \ln(I_{2m+2n} + \mathbf{L}) = \sum_{j=1}^{N} (-1)^{j+1} \frac{\mathbf{L}^j}{j}$$

commutes with J as well. In this way, we have obtained the following refinement of Theorem 2.

Theorem 3. *Every supermatrix $M \in \mathrm{SO}_0^J$ can be written as*

$$M = e^X e^{\mathbf{Z}} \quad \textit{where} \quad \begin{cases} X \in [\mathfrak{so}(2m) \cap \mathfrak{sp}_J(2m)] \times [\mathfrak{so}(2n) \cap \mathfrak{sp}_\Omega(2n)], \\ \mathbf{Z} \in \mathfrak{so}_0^J(2m|2n)(\Lambda_N^+), \end{cases}$$

with $\mathfrak{so}_0^J(2m|2n)(\Lambda_N^+) := \mathfrak{so}_0^J \cap \mathrm{Mat}(2m|2n)(\Lambda_N^+)$. In addition, the element $\mathbf{Z}$ is unique.

5 Spin realization of SO_0^J

The aim of this section is to find the spin realization of SO_0^J, i.e. the subgroup $\mathrm{Spin}_J(2m|2n)(\Lambda_N)$ of $\mathrm{Spin}(2m|2n)(\Lambda_N)$ containing all spin elements which correspond to elements of SO_0^J through the h-representation (10). To that end we must first find the realization of the Lie subalgebra $\mathfrak{so}_0^J \subset \mathfrak{so}_0$ in the algebra of extended superbivectors, i.e. $\phi^{-1}(\mathfrak{so}_0^J) \subset \mathbb{R}^{(2)E}_{2m|2n}(\Lambda_N)$. The exponentials of these bivectors yield all elements of $\mathrm{Spin}_J(2m|2n)(\Lambda_N)$, leaving the super Dirac operators ∂_x and $\partial_{J(x)}$ invariant.

5.1 The Lie algebras $\mathfrak{so}_0^J$ and $\phi^{-1}(\mathfrak{so}_0^J)$.

Propositions 9 and 14 show that $\mathfrak{so}_0^J$ can be described as the set of supermatrices of the form

$$M = \left(\begin{array}{cc|c} A_1 & A_2 & \grave{B}_1 \\ -A_2 & A_1 & \grave{B}_1\Omega_{2n} \\ \hline \grave{C}_1 & -\Omega_{2n}\grave{C}_1 & D \end{array}\right) \qquad \text{with} \quad \begin{cases} A_1^T + A_1 = 0, \\ A_2^T - A_2 = 0, \\ \grave{B}_1 - \grave{C}_1^{\,T}\Omega_{2n} = 0, \\ D^T + D = 0. \end{cases} \tag{20}$$

since the conditions for being an element of $\mathfrak{so}_0$ given in (7) can rewritten in this case as

$$A^T + A = \begin{pmatrix} A_1^T + A_1 & -A_2^T + A_2 \\ A_2^T - A_2 & A_1^T + A_1 \end{pmatrix} = 0, \quad \grave{B} - \grave{C}^T\Omega_{2n} = \begin{pmatrix} \grave{B}_1 \\ \grave{B}_1\Omega_{2n} \end{pmatrix} - \begin{pmatrix} \grave{C}_1^{\,T} \\ \grave{C}_1^{\,T}\Omega_{2n} \end{pmatrix}\Omega_{2n} = 0,$$

and $D^T\Omega_{2n} + \Omega_{2n}D = (D^T + D)\Omega_{2n} = 0$.

The relations in (20) provide an easy way of computing the dimension of $\mathfrak{so}_0^J$. Indeed, we can write $\mathfrak{so}_0^J$ as the direct sum of the real subspaces V_1, V_2, V_3, V_4 where

$$\begin{aligned}
V_1 &= \left\{ \left(\begin{array}{cc|c} A_1 & 0 & 0 \\ 0 & A_1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right) : A_1^T + A_1 = 0,\ A_1 \in \mathbb{R}\Lambda_{N,\bar{0}}^{m\times m} \right\}, \text{ with } \dim V_1 = 2^{N-1}\frac{m(m-1)}{2}, \\
V_2 &= \left\{ \left(\begin{array}{cc|c} 0 & A_2 & 0 \\ -A_2 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right) : A_2^T - A_2 = 0,\ A_2 \in \mathbb{R}\Lambda_{N,\bar{0}}^{m\times m} \right\}, \text{ with } \dim V_2 = 2^{N-1}\frac{m(m+1)}{2}, \\
V_3 &= \left\{ \left(\begin{array}{cc|c} 0 & 0 & \grave{C_1}^T\Omega_{2n} \\ 0 & 0 & -\grave{C_1}^T \\ \hline \grave{C_1} & -\Omega_{2n}\grave{C_1} & 0 \end{array} \right) : \grave{C_1} \in \mathbb{R}\Lambda_{N,\bar{1}}^{2n\times m} \right\}, \text{ with } \dim V_3 = 2^{N-1}2mn, \\
V_4 &= \left\{ \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & D \end{array} \right) : D\Omega_{2n} = \Omega_{2n}D,\ D^T + D = 0,\ D \in \mathbb{R}\Lambda_{N,\bar{0}}^{2n\times 2n} \right\}, \text{ with } \dim V_4 = 2^{N-1}n^2,
\end{aligned}$$

whence

$$\dim \mathfrak{so}_0^J = 2^{N-1}\left[\frac{m(m-1)}{2} + \frac{m(m+1)}{2} + 2mn + n^2\right] = 2^{N-1}(m+n)^2.$$

We now look for the realization of $\mathfrak{so}_0^J$ in the Lie algebra of extended bivectors. To that end, consider $M \in \mathfrak{so}_0$ and $B = \phi^{-1}(M) \in \mathbb{R}_{2m|2n}^{(2)E}(\Lambda_N)$. Then, $MJ = JM$ if and only if $MJx = JMx$ for every $x \in \mathbb{R}^{2m|2n}(\mathcal{V})$ or equivalently,

$$\phi(B)(J(x)) = J(\phi(B)(x)) \iff [B, J(x)] = J([B,x]) = [J(B), J(x)] \iff B = J(B).$$

The last equivalence is due to the fact that J is an automorphism over $\mathbb{R}^{2m|2n}(\mathcal{V})$. In this way, we have obtained that $\phi^{-1}(\mathfrak{so}_0^J) = \{B \in \mathbb{R}_{2m|2n}^{(2)E}(\Lambda_N) : B = J(B)\}$. In order to find the explicit form of the elements in $\phi^{-1}(\mathfrak{so}_0^J)$ we need the following computations:

$$\begin{cases} J(e_je_k) = e_{m+j}e_{m+k} & 1 \leq j < k \leq m, \\ J(e_je_{m+k}) = e_ke_{m+j} & 1 \leq j, k \leq m, \\ J(e_{m+j}e_{m+k}) = e_je_k & 1 \leq j < k \leq m, \end{cases} \qquad \begin{cases} J(e_j\grave{e}_{2k-1}) = e_{m+j}\grave{e}_{2k} & 1 \leq j \leq m,\ 1 \leq k \leq n, \\ J(e_j\grave{e}_{2k}) = -e_{m+j}\grave{e}_{2k-1} & 1 \leq j \leq m,\ 1 \leq k \leq n, \\ J(e_{m+j}\grave{e}_{2k-1}) = -e_j\grave{e}_{2k} & 1 \leq j \leq m,\ 1 \leq k \leq n, \\ J(e_{m+j}\grave{e}_{2k}) = e_j\grave{e}_{2k-1} & 1 \leq j \leq m,\ 1 \leq k \leq n, \end{cases}$$

$$\begin{cases} J(\grave{e}_{2j-1} \odot \grave{e}_{2k-1}) = \grave{e}_{2j} \odot \grave{e}_{2k} & 1 \leq j \leq k \leq n, \\ J(\grave{e}_{2j-1} \odot \grave{e}_{2k}) = -\grave{e}_{2j} \odot \grave{e}_{2k-1} & 1 \leq j \leq k \leq n, \\ J(\grave{e}_{2j} \odot \grave{e}_{2k-1}) = -\grave{e}_{2j-1} \odot \grave{e}_{2k} & 1 \leq j < k \leq n, \\ J(\grave{e}_{2j} \odot \grave{e}_{2k}) = \grave{e}_{2j-1} \odot \grave{e}_{2k-1} & 1 \leq j \leq k \leq n. \end{cases}$$

Applying J to both sides of (8) we obtain that $B = J(B)$ is equivalent to

$$\begin{cases} b_{j,k} = b_{m+j,m+k} & 1 \leq j < k \leq m, \\ b_{j,m+k} = b_{k,m+j} & 1 \leq j, k \leq m, \end{cases} \qquad \begin{cases} \grave{b}_{j,2k-1} = \grave{b}_{m+j,2k} & 1 \leq j \leq m,\ 1 \leq k \leq n, \\ \grave{b}_{j,2k} = -\grave{b}_{m+j,2k-1} & 1 \leq j \leq m,\ 1 \leq k \leq n, \end{cases}$$

and

$$\begin{cases} B_{2j-1,2k-1} = B_{2j,2k} & 1 \leq j \leq k \leq n, \\ B_{2j-1,2k} = -B_{2j,2k-1} & 1 \leq j < k \leq n, \\ B_{2j-1,2j} = 0 & 1 \leq j \leq n. \end{cases}$$

Hence, $B \in \phi^{-1}(\mathfrak{so}_0^J)$ if and only if $B = B_1 + B_2 + B_3$ where B_1, B_2, B_3 are of the form

$$\begin{aligned}
B_1 &= \sum_{1\leq j<k\leq m} b_{j,k}(e_j e_k + e_{m+j}e_{m+k}) + \sum_{j=1}^{m} b_{j,m+j}\, e_j e_{m+j} + \sum_{1\leq j<k\leq m} b_{j,m+k}(e_j e_{m+k} + e_k e_{m+j}),\\
B_2 &= \sum_{\substack{1\leq j\leq m\\ 1\leq k\leq n}} \grave{b}_{j,2k-1}(e_j \grave{e}_{2k-1} + e_{m+j}\grave{e}_{2k}) + \grave{b}_{j,2k}(e_j \grave{e}_{2k} - e_{m+j}\grave{e}_{2k-1}),\\
B_3 &= \sum_{1\leq j\leq k\leq n} B_{2j-1,2k-1}(\grave{e}_{2j-1}\odot \grave{e}_{2k-1} + \grave{e}_{2j}\odot \grave{e}_{2k}) + \sum_{1\leq j<k\leq n} B_{2j-1,2k}(\grave{e}_{2j-1}\odot \grave{e}_{2k} - \grave{e}_{2j}\odot \grave{e}_{2k-1}).
\end{aligned}$$

Summarizing, we have obtained the following result.

Proposition 15. *Let $\{b_1, \ldots, b_{2^{N-1}}\}$ and $\{\grave{b}_1, \ldots, \grave{b}_{2^{N-1}}\}$ be the canonical basis of $\mathbb{R}\Lambda_{N,\bar{0}}$ and $\mathbb{R}\Lambda_{N,\bar{1}}$ respectively. Then, a basis for $\phi^{-1}(\mathfrak{so}_0^J)$ is given by the elements*

$$\begin{aligned}
&b_r(e_j e_k + e_{m+j}e_{m+k}), && 1\leq r\leq 2^{N-1},\ 1\leq j<k\leq m,\\
&b_r(e_j e_{m+k} + e_k e_{m+j}), && 1\leq r\leq 2^{N-1},\ 1\leq j\leq k\leq m,\\
&\grave{b}_r(e_j \grave{e}_{2k-1} + e_{m+j}\grave{e}_{2k}), && 1\leq r\leq 2^{N-1},\ 1\leq j\leq m,\ 1\leq k\leq n,\\
&\grave{b}_r(e_j \grave{e}_{2k} - e_{m+j}\grave{e}_{2k-1}), && 1\leq r\leq 2^{N-1},\ 1\leq j\leq m,\ 1\leq k\leq n,\\
&b_r(\grave{e}_{2j-1}\odot \grave{e}_{2k-1} + \grave{e}_{2j}\odot \grave{e}_{2k}), && 1\leq r\leq 2^{N-1},\ 1\leq j\leq k\leq n,\\
&b_r(\grave{e}_{2j-1}\odot \grave{e}_{2k} - \grave{e}_{2j}\odot \grave{e}_{2k-1}), && 1\leq r\leq 2^{N-1},\ 1\leq j<k\leq n.
\end{aligned}$$

Remark 5.1. *Obviously the algebras $\phi^{-1}(\mathfrak{so}_0^J)$ and $\mathfrak{so}_0^J$ are isomorphic. This fact can be double checked through the previous result from which it follows that $\dim \phi^{-1}(\mathfrak{so}_0^J) = 2^{N-1}(m+n)^2$.*

5.2 The group $\mathrm{Spin}_J(2m|2n)(\mathbb{R}\Lambda_N)$

We may now introduce the group

$$\mathrm{Spin}_J \equiv \mathrm{Spin}_J(2m|2n)(\mathbb{R}\Lambda_N) := \{e^{B_1}\cdots e^{B_k} : B_1, \ldots, B_k \in \phi^{-1}(\mathfrak{so}_0^J),\ k\in\mathbb{N}\}.$$

This is a Lie subgroup of $\mathrm{Spin}(2m|2n)(\mathbb{R}\Lambda_N)$ which completely describes SO_0^J through the h-representation, as shown in the next result.

Proposition 16. *The group Spin_J covers SO_0^J.*

Proof.
The Lie group isomorphism $\mathrm{SO}_0^J \cong \mathrm{U}_0$ shows, in view of Proposition 12, that SO_0^J is connected. Hence, for every $M \in \mathrm{SO}_0^J$ there exist $X_1, \ldots, X_k \in \mathfrak{so}_0^J$ such that $e^{X_1}\cdots e^{X_k} = M$, see Corollary 3.47 in [7]. Taking $B_j = \phi^{-1}(X_j)$, $j = 1, \ldots, k$, and using the relations (10) and (11), we get for every $x \in \mathbb{R}^{2m|2n}(\mathcal{V})$ that

$$Mx = e^{X_1}\cdots e^{X_k}\,x = e^{\phi(B_1)}\cdots e^{\phi(B_k)}\,x = h(e^{B_1})\circ\cdots\circ h(e^{B_k})[x] = h(e^{B_1}\cdots e^{B_k})[x],$$

whence $M = h(s)$ with $s = e^{B_1}\cdots e^{B_k} \in \mathrm{Spin}_J$. □

The decomposition for SO_0^J given in Theorem 3 allows to describe the Spin_J-covering of SO_0^J more precisely. Indeed, following the decomposition given in (12) we obtain the Lie subalgebras of $\phi^{-1}(\mathfrak{so}_0^J)$

$$\begin{aligned}
S_1^J &= S_1 \cap \phi^{-1}(\mathfrak{so}_0^J) = \phi^{-1}\big([\mathfrak{so}(2m)\cap \mathfrak{sp}_J(2m)] \times [\mathfrak{so}(2n)\cap \mathfrak{sp}_\Omega(2n)]\big),\\
S_3^J &= S_3 \cap \phi^{-1}(\mathfrak{so}_0^J) = \phi^{-1}(\mathfrak{so}_0^J(2m|2n)(\Lambda_N^+)),
\end{aligned}$$

yielding the decomposition $\phi^{-1}(\mathfrak{so}_0^J) = S_1^J \oplus S_3^J$ (observe that $S_2 \cap \phi^{-1}(\mathfrak{so}_0^J) = \{0\}$). This leads to the subset $S_J = \exp(S_1^J)\exp(S_3^J) \subset S \cap \mathrm{Spin}_J$, which can be seen to constitute a double covering of SO_0^J.

It is easily seen that S_1^J is composed by all elements in $\phi^{-1}(\mathfrak{so}_0^J)$ with real coefficients, i.e. $S_1^J = \left[\phi^{-1}(\mathfrak{so}_0^J)\right]_0$, while S_3^J contains all the nilpotent elements of $\phi^{-1}(\mathfrak{so}_0^J)$. In this way, we obtain from Proposition 15 that a basis for S_1^J is given by

$$\begin{array}{llll} e_je_k + e_{m+j}e_{m+k}, & 1 \leq j < k \leq m, & \grave{e}_{2j-1} \odot \grave{e}_{2k-1} + \grave{e}_{2j} \odot \grave{e}_{2k}, & 1 \leq j \leq k \leq n, \\ e_je_{m+k} + e_ke_{m+j}, & 1 \leq j \leq k \leq m, & \grave{e}_{2j-1} \odot \grave{e}_{2k} - \grave{e}_{2j} \odot \grave{e}_{2k-1}, & 1 \leq j < k \leq n. \end{array}$$

We recall that an important element of SO_0^J is J itself. In order to find the spin element that represents J, or equivalently, an element $B_J \in S_1^J$ such that $e^{\phi(B_J)} = J$, we first compute

$$\ln(J) = \begin{pmatrix} \ln J_{2m} & 0 \\ 0 & \ln \Omega_{2n} \end{pmatrix}.$$

Observe that both J_{2m} and Ω_{2n} have eigenvalues $i, -i$, with multiplicity m and n, respectively. It easily follows that

$$\ln J_{2m} = \frac{\pi}{2} J_{2m}, \qquad \ln \Omega_{2m} = \frac{\pi}{2} \Omega_{2m}$$

Using the relations (13) we get $\ln J = \frac{\pi}{2} J = -\frac{\pi}{4}\phi(\mathbf{B})$, or equivalently $B_J = -\frac{\pi}{4}\mathbf{B}$. The spin element s_J associated to J thus is given by

$$\begin{aligned} s_J = \exp(-\frac{\pi}{4}\mathbf{B}) &= \prod_{j=1}^{m} \exp\left(-\frac{\pi}{4} e_j e_{m+j}\right) \prod_{j=1}^{n} \exp\left(\frac{\pi}{4}(\grave{e}_{2j-1}{}^2 + \grave{e}_{2j}{}^2)\right) \\ &= \frac{1}{2^{m/2}} \prod_{j=1}^{m} (1 - e_j e_{m+j}) \prod_{j=1}^{n} \exp\left(\frac{\pi}{4}(\grave{e}_{2j-1}{}^2 + \grave{e}_{2j}{}^2)\right). \end{aligned}$$

Remark 5.2. *In the purely fermionic case, i.e. $m = 0$, the element s_J may be identified with the operator*

$$\exp\left[\frac{\pi}{4} i \sum_{j=1}^{n} \left(\partial_{a_j}^2 - a_j^2\right)\right] = \exp\left(-n\frac{\pi}{4} i\right) \mathcal{F},$$

see remark 3.1. Here, $\mathcal{F}$ denotes the n-dimensional Fourier transform.

The fundamental extended bivector $\mathbf{B}$ provides other characterizations for $\phi^{-1}(\mathfrak{so}_0^J)$ and Spin_J.

Proposition 17. *Let $B \in \mathbb{R}^{(2)E}_{2m|2n}(\Lambda_N)$. Then $\phi(B) \in \mathfrak{so}_0^J$ if and only if $B\mathbf{B} = \mathbf{B}B$.*

Proof.
It suffices to observe that $\phi(B)J = J\phi(B) \iff [\phi(B), \phi(\mathbf{B})] = 0 \iff \phi([B, \mathbf{B}]) = 0 \iff [B, \mathbf{B}] = 0.$ □

Corollary 2. *Let $s \in \mathrm{Spin}_J$. Then $ss_J = s_Js$.*

5.3 Spin_J-invariance of $\partial_{J(x)}$

Our final goal is to show the invariance of the twisted Dirac operator $\partial_{J(x)}$ under the H and L actions of the group Spin_J, i.e.

$$[\partial_{J(x)}, L(s)] = 0 = [\partial_{J(x)}, H(s)], \qquad \forall s \in \mathrm{Spin}_J. \tag{21}$$

Following the same reasoning as in Section 3, it suffices to prove that $\partial_{J(x)}$ commutes with the infinitesimal representation $dL(B)$ of $L(e^B)$ for every $B \in \phi^{-1}(\mathfrak{so}_0^J)$. Using Proposition 7, we obtain for $B = J(B)$ that

$$\begin{aligned} J\left(dL(B)[F]\right) &= J\left(BF - \sum_{j=1}^{m} \Big([B,x]\Big)_j \partial_{x_j}[F] - \sum_{j=1}^{2n} \Big([B,x]\Big)_{m+j} \partial_{\grave{x}_j}[F]\right) \\ &= BJ(F) - \sum_{j=1}^{m} \Big([B,x]\Big)_j \partial_{x_j}[J(F)] - \sum_{j=1}^{2n} \Big([B,x]\Big)_{m+j} \partial_{\grave{x}_j}[J(F)] \;=\; dL(B)[J(F)]. \end{aligned}$$

Corollary 1 then yields that $\partial_x\left[dL(B)[G]\right] = dL(B)[\partial_x[G]]$, whence, applying J to both sides and writing $F = J(G)$, we get for every $B \in \phi^{-1}(\mathfrak{so}_0^J)$ that $\partial_{J(x)}\left[dL(B)[F]\right] = dL(B)[\partial_{J(x)}[F]]$. In this way, we have proved that $[\partial_{J(x)}, dL(B)] = 0$ for every $B \in \phi^{-1}(\mathfrak{so}_0^J)$ and in consequence (21) holds.

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