OPTIMAL ACTUATOR LOCATION FOR SEMI-LINEAR SYSTEMS

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Abstract. Actuator location and design are important choices in controller design for distributed parameter systems. Semi-linear partial differential equations model a wide spectrum of physical systems with distributed parameters. It is shown that under certain conditions on the nonlinearity and the cost function, an optimal control input together with an optimal actuator choice exist. First order necessary optimality conditions are derived. The results are applied to optimal actuator location and controller design in a nonlinear railway track model.

Key words. Actuator location, Semilinear partial differential equations, Optimal control, Flexible structures

AMS subject classifications. 49J27, 49K27, 49J10

1. Introduction. Actuator location and design are important design variables in controller synthesis for distributed parameter systems. Finding the best actuator location to control a distributed parameter system can significantly reduce the cost of the control and improve its effectiveness; see for example, [14, 30, 31]. The optimal actuator location problem has been discussed by many researchers in various contexts; see [17, 41] for a review of applications and [35] for optimal location of actuators to maximize controllability in the wave equation. In [29] it was proved that an optimal actuator location exists for linear-quadratic control. Conditions under which using approximations in optimization yield the optimal location are also established. Similar results have been obtained for H_2 and H_∞ controller design objectives with linear models [21, 32]. Results for optimal design have been obtained [33] and also characterizing the derivatives for shape optimization in a diffusion equation [20]. There are results on the related problem of optimal sensor location for linear PDE's; see [36] for locations of maximum observability in the wave equation and [43] for concurrent sensor choice/estimator design to minimize the error variance.

Nonlinearities can have a significant effect on dynamics and such systems cannot be accurately modelled by linear differential equations. Optimal control of systems modelled by nonlinear partial differential equations (PDE's) has been studied for a number of applications, including wastewater treatment systems [27], steel cooling plants [40], oil extraction through a reservoir [25], solidification models in metallic alloys [7], thermistors [19], the Schlögl model [8], static elastoplasticity [12], and the Fokker-Planck equation [16]. A review of PDE-constrained optimization theory can be found in the books [18, 24, 39]. In [9, 37] first-order optimality conditions are investigated for parabolic partial differential equations.

Optimal actuator location has been addressed for some applications modelled by nonlinear distributed parameter systems using a finite dimensional approximation of the original partial differential equation model. In [3], authors investigated the optimal actuator and sensor location problem for a transport-reaction process using a finite-dimensional model. Similarly, in [26], the optimal actuator and sensor location of Kuramoto-Sivashinsky equation was studied using a finite-dimensional approximation. Other research concerned with optimal actuator location in problems with nonlinear

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distributed parameter dynamics can be found in [4, 28, 38]. To our knowledge, there are no theoretical results on optimal actuator location of nonlinear PDE's.

Theory for concurrent optimal control and actuator design of a class of controlled semi-linear PDE's is described in this paper. The research described extends previous work on optimal control of PDE's in that the linear part of the partial differential equation is not constrained to be the generator of an analytic semigroup. The input operator of the system is parametrized by the possible actuator designs. A general class of PDE's with weakly continuous nonlinear part is considered. Optimality equations explicitly characterizing the optimal control and actuator are obtained.

Location of actuators on structures has been one of the motivators for research into optimal actuator location [17]. Various models have been studied. Classical results in the literature concern control of linear and nonlinear Euler Bernoulli and Timoshenko beam models [22, 23, e.g.]. In recent years, non-classical models of flexible beams such as micro-beam models have also attracted attention [13, e.g.]. In nonlinear flexible structures, the nonlinearity typically is on the deformation, not on the rate of deformation. The space in which deformations evolve is compactly embedded in that of rate of deformation. As a result, the nonlinear terms are weakly continuous in the underlying state space. One important application of the results in this paper is to the development of an optimal control strategy for the vibration suppression of railway tracks [11]. Vibrations in a railway track with the interaction with the foundation included in the model lead to a nonlinear PDE with a weakly continuous nonlinearity.

The paper is organized as follows. After a short paragraph on notation, the problem definition as well as the main results are stated in section 2. Section 3 discusses the existence of a solution to the semi-linear partial differential equation. The existence of an optimizer is established in section 4. First-order necessary condition for the optimizer are provided in section 5. In section 6, the results of the previous sections are applied to the railway track model. It is shown that the problem has a optimal control and actuator location.

Notation. Throughout this paper, the letters c, t, and x denote a generic positive constant, temporal variable, and spatial variable, respectively. The blackboard letters as in $\mathbb{Z}$ denote Banach spaces, the calligraphic letters as in $\mathcal{A}$ denote operators on a Banach space. If an operator is nonlinear its argument is shown in parenthesis as in $\mathcal{F}(\cdot)$. The Fraktur letters as in $\mathfrak{z}$ refer to states evolving in a Banach space; the rest of letters represent physical constants, generic constants, or a matrix. The adjoint of an operator is denoted by $\mathcal{A}^*$. The superscript $\cdot^o$ shows that a state or an input is optimal, and the tilde overscript $\tilde{\cdot}$ is reserved for the state of a linearized system unless otherwise stated. Norms and inner products on the underlying state space are typed without any subscript, but on any other spaces, they are shown with a suitable subscript to avoid confusion. Strong convergences on a Banach space are shown by $\to$, whereas a weak convergence is shown by $\rightharpoonup$. If the Banach space $\mathbb{Z}_1$ is continuously embedded in $\mathbb{Z}_2$, we write $\mathbb{Z}_1 \hookrightarrow \mathbb{Z}_2$. The Banach space $C([0,T];\mathbb{Z})$ will often be indicated $C(0,T;\mathbb{Z})$ for simplicity of notation.

2. Main Results. Consider a semi-linear system with state $\mathfrak{z}(t)$ on a separable reflexive Banach space $\mathbb{Z}$:

$$\dot{\mathfrak{z}}(t) = \mathcal{A}\mathfrak{z}(t) + \mathcal{F}(\mathfrak{z}(t)) + \mathcal{B}(r)u(t), \quad \mathfrak{z}(0) = \mathfrak{z}_0 \in D(\mathcal{A}), \tag{1}$$

The function $u(t)$ is the input to the system, and takes values in a reflexive Banach space $\mathbb{U}$. The control operator $\mathcal{B}(\cdot)$ depends on a parameter r that takes values in a set K_{ad} in a topological space $\mathbb{K}$. The parameter r typically has interpretation as possible actuator locations. The operators $\mathcal{A}$, $\mathcal{F}(\cdot)$, and $\mathcal{B}(\cdot)$ satisfy the following assumptions.

Assumption A.

1. *The state operator $\mathcal{A}$ with domain $D(\mathcal{A})$ generate a strongly continuous semi-group $\mathcal{T}(t)$ on $\mathbb{Z}$.*
2. *The nonlinear operator $\mathcal{F}(\cdot)$ is locally Lipschitz continuous on $\mathbb{Z}$; that is, for every positive number δ, there exists $L_{\mathcal{F}\delta} > 0$ such that*

$$\|\mathcal{F}(\mathfrak{z}_2) - \mathcal{F}(\mathfrak{z}_1)\| \leq L_{\mathcal{F}\delta} \|\mathfrak{z}_2 - \mathfrak{z}_1\|,$$

   for all $\|\mathfrak{z}_2\| \leq \delta$ and $\|\mathfrak{z}_1\| \leq \delta$.
3. *For each $r \in K_{ad}$, the input operator $\mathcal{B}(r)$ is a linear bounded operator that maps the input space $\mathbb{U}$ into the state space $\mathbb{Z}$. This family of operators is uniformly bounded over K_{ad}, i.e., there exist a positive number $M_{\mathcal{B}}$ such that $\|\mathcal{B}(r)\| \leq M_{\mathcal{B}}$ for all $r \in K_{ad}$.*

In some cases, due to lack of regularity of the input u, a classical solution to the IVP (1) is not assured.

Definition 2.1. *If $\mathfrak{z} \in C(0,T;\mathbb{Z})$ satisfies*

$$\mathfrak{z}(t) = \mathcal{T}(t)\mathfrak{z}_0 + \int_0^t \mathcal{T}(t-s)\mathcal{F}(\mathfrak{z}(s))ds + \int_0^t \mathcal{T}(t-s)\mathcal{B}(r)u(s)ds, \tag{2}$$

for every $\mathfrak{z}_0 \in \mathbb{Z}$, it is said to be a mild solution *to the IVP (1).*

In section 3, the existence of a unique mild solution to the initial value problem (IVP) (1) is proven assuming that $u \in L^p(0,T;\mathbb{U})$.

Theorem 3.1: *Under assumption A, for each $\mathfrak{z}_0 \in \mathbb{Z}$ and positive number R, there exists $T > 0$ such that the IVP (1) admits a unique local mild solution $\mathfrak{z} \in C(0,T;\mathbb{Z})$ for all $u \in L^p(0,T;\mathbb{U})$, $\|u\|_p \leq R$, and all $r \in K_{ad}$.*

For functionals $\phi(\mathfrak{z})$ on $\mathbb{Z}$ and $\psi(u)$ on $\mathbb{U}$, consider the cost function

$$J(u,r;\mathfrak{z}_0) = \int_0^T \phi(\mathfrak{z}(t)) + \psi(u(t))\, dt,$$

where the admissible control input $u(t)$ belongs to the set

$$U_{ad} = \{u \in L^p(0,T;\mathbb{U}) |\ \|u\|_p \leq R\}.$$

The optimization problem is to minimize $J(u,r;\mathfrak{z}_0)$ over all admissible control inputs $u \in U_{ad}$, and also over all admissible actuator locations $r \in K_{ad}$, subject to the IVP (1) with a fixed initial condition $\mathfrak{z}_0 \in \mathbb{Z}$. That is,

$$\text{(P)} \qquad \begin{cases} \min & J(u,r;\mathfrak{z}_0) \\ \text{s.t.} & \dot{\mathfrak{z}}(t) = \mathcal{A}\mathfrak{z}(t) + \mathcal{F}(\mathfrak{z}(t)) + \mathcal{B}(r)u(t), \quad \forall t \in (0,T] \\ & \mathfrak{z}(0) = \mathfrak{z}_0, \\ & u(t) \in U_{ad}, \\ & r \in K_{ad}. \end{cases}$$

To guarantee the existence of a unique optimizer, further assumptions are needed on the operators $\mathcal{F}(\cdot)$, $\mathcal{B}(\cdot)$, the set K_{ad}, and the cost function $J(u,r;\mathfrak{z}_0)$.

Assumption B.

1. *The nonlinear operator* $\mathcal{F}(\cdot)$ *is weakly sequentially continuous, i.e., if* $\mathfrak{z}_n \rightharpoonup \mathfrak{z}$*, then* $\mathcal{F}(\mathfrak{z}_n) \rightharpoonup \mathcal{F}(\mathfrak{z})$ *on* $\mathbb{Z}$*.*
2. *Let* K_{ad} *be a compact set in the actuator location space* $\mathbb{K}$*. The family of input operators* $\mathcal{B}(r) : K_{ad}(\subset \mathbb{K}) \to \mathcal{L}(\mathbb{U}, \mathbb{Z})$ *are continuous with respect to* r *in the operator norm topology:*

$$\lim_{r_2 \to r_1} \|\mathcal{B}(r_2) - \mathcal{B}(r_1)\| = 0.$$

3. *The functionals* $\phi(\cdot)$ *and* $\psi(\cdot)$ *are weakly lower semi-continuous positive functionals on* $\mathbb{Z}$ *and* $\mathbb{U}$*, respectively.*

It is shown in section 4 that under these assumptions, an optimal control and actuator location exist.

Theorem 4.1: *For initial condition* $\mathfrak{z}_0 \in \mathbb{Z}$ *let* T *be such that the mild solution exists for all* $u \in U_{ad}$*, and all* $r \in K_{ad}$*. Under assumptions A and B, there exists a control input* $u^o \in U_{ad}$ *together with an actuator location* $r^o \in K_{ad}$*, that solve the optimization problem P.*

To characterize an optimizer to the optimization problem, further assumptions on differentiability of the nonlinear operator $\mathcal{F}(\cdot)$ and the cost function are needed.

Assumption C.

1. *The nonlinear operator* $\mathcal{F}(\cdot)$ *is Fréchet differentiable on* $\mathbb{Z}$*. Indicate the Fréchet derivative of* $\mathcal{F}(\cdot)$ *at* $\mathfrak{z}$ *by* $F'_{\mathfrak{z}}$*.*
2. *The mapping* $\mathfrak{z} \mapsto \mathcal{F}'_{\mathfrak{z}}$ *is bounded, i.e., bounded sets in* $\mathbb{Z}$ *are mapped into bounded sets in* $\mathcal{L}(\mathbb{Z})$*.*
3. *The control operator* $\mathcal{B}(r)$ *is Fréchet differentiable with respect to* r *in* $\mathcal{L}(\mathbb{U}, \mathbb{Z})$*. Indicate the Fréchet derivative of* $\mathcal{B}(r)$ *at* r *by* $\mathcal{B}'_r$*.*
4. *The actuator location space* $\mathbb{K}$ *is a Hilbert space.*
5. *The state space* $\mathbb{Z}$ *and* $\mathbb{U}$ *are Hilbert spaces. Also, in the cost function, set*

$$\phi(\mathfrak{z}) = \langle \mathcal{Q}\mathfrak{z}, \mathfrak{z} \rangle, \quad \psi(u) = \langle \mathcal{R}u, u \rangle_{\mathbb{U}},$$

where the linear operator $\mathcal{Q}$ *is a positive semi-definite, self-adjoint bounded operator on* $\mathbb{Z}$*, and the linear operator* $\mathcal{R}$ *is a positive definite, self-adjoint bounded operator on* $\mathbb{U}$*.*

The following theorem is proved in section 5. In this theorem $\mathfrak{z} = \mathcal{S}(u; r, \mathfrak{z}_0)$ denotes the control-to-state map (see Definition 5.1), and the operator $(\mathcal{B}'_{r^o}u)^* : \mathbb{Z} \to \mathbb{K}$ is defined as

$$\langle (\mathcal{B}'_{r^o}u)^*\mathfrak{p}, r \rangle_{\mathbb{K}} = \langle \mathfrak{p}, (\mathcal{B}'_{r^o}r)u \rangle, \quad \forall (u, \mathfrak{p}, r) \in \mathbb{U} \times \mathbb{Z} \times \mathbb{K}.$$

Theorem 5.7: *Suppose assumptions A, B1, B2, C hold. For any initial condition* $\mathfrak{z}_0 \in \mathbb{Z}$*, let the pair* (u^o, r^o) *be a local minimizer of the optimization problem P with the optimal trajectory* $\mathfrak{z}^o = \mathcal{S}(u^o; r^o, \mathfrak{z}_0)$*. Also, let* $\mathfrak{p}^o(t)$*, the adjoint state, indicate the mild solution of the final value problem*

$$\dot{\mathfrak{p}}^o(s) = -(\mathcal{A}^* + \mathcal{F}'^*_{\mathfrak{z}^o(t)})\mathfrak{p}^o(s) - \mathcal{Q}\mathfrak{z}^o(s), \quad \mathfrak{p}^o(T) = 0.$$

Then, if (u^o, r^o) is an interior point of $U_{ad} \times K_{ad}$, it satisfies

$$u^o(t) = -\mathcal{R}^{-1}\mathcal{B}^*(r^o)\mathfrak{p}^o(t),$$
$$\int_0^T (\mathcal{B}'_{r^o} u^o(s))^* \mathfrak{p}^o(s)\, ds = 0.$$

3. Existence of a Solution to the IVP. In the existing literature, the existence of a unique local solution to (1) is guaranteed for continuesly differentiable control inputs (see e.g. [34, Thm. 6.1.5]). This condition on the input to be continuously differentiable is too restrictive. The following theorem guarantees the existence of a unique local mild solution. In this theorem $\|\cdot\|_p$ refers to the norm on $L^p(0,T;\mathbb{U})$, $1 < p < \infty$.

THEOREM 3.1. *Under assumption A, for each $\mathfrak{z}_0 \in \mathbb{Z}$ and positive number R, there exists $T > 0$ such that the IVP (1) admits a unique local mild solution $\mathfrak{z} \in C(0,T;\mathbb{Z})$ for all $u \in L^p(0,T;\mathbb{U})$, $\|u\|_p \le R$, and all $r \in K_{ad}$.*

Proof. The idea of the proof is similar to [34, Thm. 6.1.4], with a slight modification that here u is in $L^p(0,T;\mathbb{U})$. For any $\mathfrak{z}_0 \in \mathbb{Z}$ choose constants $\delta_0 > 0$ and $T > 0$ such that for $t \in [0,T]$

$$\|\mathcal{T}(t)\mathfrak{z}_0 - \mathfrak{z}_0\| \le \delta_0.$$

Let $\mathbb{S}$ be the closed bounded subset of $C(0,T;\mathbb{Z})$ defined as

$$\mathbb{S} = \{\mathfrak{z}(t) \in C(0,T;\mathbb{Z}) |\ \mathfrak{z}(0) = \mathfrak{z}_0,\ \|\mathfrak{z}(t) - \mathfrak{z}_0\| \le 2\delta_0\}. \tag{4}$$

Define the operator $\mathcal{G}$ on $\mathbb{S}$ to be

$$\mathcal{G}(\mathfrak{z}(t)) = \mathcal{T}(t)\mathfrak{z}_0 + \int_0^t \mathcal{T}(t-s)\mathcal{F}(\mathfrak{z}(s))\, ds + \int_0^t \mathcal{T}(t-s)\mathcal{B}(r)u(s)\, ds. \tag{5}$$

It will be shown that for sufficiently small T, $\mathcal{G}$ maps $\mathbb{S}$ into $\mathbb{S}$ and is a contraction on $\mathbb{S}$.

Use the triangle inequality and write

$$\begin{aligned}\|\mathcal{G}(\mathfrak{z}(t)) - \mathfrak{z}_0\| \le \|\mathcal{T}(t)\mathfrak{z}_0 - \mathfrak{z}_0\| &+ \left\| \int_0^t \mathcal{T}(t-s)\mathcal{F}(\mathfrak{z}(s))\, ds \right\| \\ &+ \left\| \int_0^t \mathcal{T}(t-s)\mathcal{B}(r)u(s)\, ds \right\|. \end{aligned} \tag{6}$$

There exist numbers $M > 0$ and ω such that $\|\mathcal{T}(t)\| \le Me^{\omega t}$. Also, recall from assumption A2 that there is $L_{\mathcal{F}\delta} > 0$ so that $\|\mathcal{F}(\mathfrak{z}(s))\| \le L_{\mathcal{F}\delta}\|\mathfrak{z}(s)\|$ on a ball of radius $\delta = \|\mathfrak{z}_0\| + 2\delta_0$ centered at the origin. This gives a bound for the second term on the left hand side of the inequality (6)

$$\left\| \int_0^t \mathcal{T}(t-s)\mathcal{F}(\mathfrak{z}(s))\, ds \right\| \le M \max(1, e^{\omega T}) L_{\mathcal{F}\delta}\delta T. \tag{7}$$

Using assumption A3, an upper bound for the third right hand side term is

$$\left\| \int_0^t \mathcal{T}(t-s)\mathcal{B}(r)u(s)\, ds \right\| \le M \max(1, e^{\omega T}) M_{\mathcal{B}} \|u\|_p T^{(p-1)/p}. \tag{8}$$

Applying these bounds to inequality (6) it follows for all u with $\|u\|_p \leq R$ that

$$\|\mathcal{G}(\mathfrak{z}(t)) - \mathfrak{z}_0\| \leq \delta_0 + M\max(1, e^{\omega T})L_{\mathcal{F}\delta}\delta T + M\max(1, e^{\omega T})M_{\mathcal{B}}RT^{(p-1)/p}. \tag{9}$$

Choose T small enough that the right hand side in (9) is less than $2\delta_0$. For such T, $\mathcal{G} : \mathbb{S} \to \mathbb{S}$.

Because of the local Lipschitz continuity of $\mathcal{F}(\cdot)$

$$\begin{aligned}\|\mathcal{G}(\mathfrak{z}_2(t)) - \mathcal{G}(\mathfrak{z}_1(t))\|_{C(0,T;\mathbb{Z})} &\leq \left\| \int_0^t \mathcal{T}(t-s)\left(\mathcal{F}(\mathfrak{z}_2(s)) - \mathcal{F}(\mathfrak{z}_1(s))\right) ds \right\|_{C(0,T;\mathbb{Z})} \\ &\leq M\max(1, e^{\omega T})L_{\mathcal{F}\delta}T\, \|\mathfrak{z}_2(t) - \mathfrak{z}_1(t)\|_{C(0,T;\mathbb{Z})}. \end{aligned} \tag{10}$$

Choosing T so $M\max(1, e^{\omega T})L_{\mathcal{F}\delta}T < 1$ yields that $\mathcal{G}$ is a contraction on $\mathbb{S}$. Thus, the operator $\mathcal{G}$ has a unique fixed point in $\mathbb{S}$ that satisfies

$$\mathfrak{z}(t) = \mathcal{T}(t)\mathfrak{z}_0 + \int_0^t \mathcal{T}(t-s)\mathcal{F}(\mathfrak{z}(s))\, ds + \int_0^t \mathcal{T}(t-s)\mathcal{B}(r)u(s)\, ds\,. \tag{11}$$

Thus, $\mathfrak{z}(t)$ is the unique local mild solution of IVP (1). ∎

COROLLARY 3.2. *Under assumption A, for each $\mathfrak{z}_0 \in \mathbb{Z}$ and $R > 0$, there exists a positive number c_T such that the mild solution to the IVP (1) satisfies, for all $\|u\|_p \leq R$,*

$$\|\mathfrak{z}\|_{C(0,T;\mathbb{Z})} \leq c_T \left(\|\mathfrak{z}_0\| + T^{(p-1)/p} \|\mathcal{B}(r)\| \, \|u\|_p \right). \tag{12}$$

Proof. Choose a sufficiently small time T such that the mild solution exists. Take the norm of both sides of (2) and apply assumption A together with the triangle inequality to obtain

$$\begin{aligned}\|\mathfrak{z}(t)\| &\leq \|\mathcal{T}(t)\mathfrak{z}_0\| + \left\| \int_0^t \mathcal{T}(t-s)\mathcal{F}(\mathfrak{z}(s))\, ds \right\| + \left\| \int_0^t \mathcal{T}(t-s)\mathcal{B}(r)u(s)\, ds \right\| \\ &\leq M\max(1, e^{\omega T})\, \|\mathfrak{z}_0\| + M\max(1, e^{\omega T})L_{\mathcal{F}\delta} \int_0^t \|\mathfrak{z}(t)\|\, dt \\ &\quad + M\max(1, e^{\omega T})T^{(p-1)/p} \|\mathcal{B}(r)\| \, \|u\|_p\,. \end{aligned} \tag{13}$$

Defining the constant

$$c_T = M\max(1, e^{\omega T})e^{M\max(1, e^{\omega T})L_{\mathcal{F}\delta}T},$$

and applying Grönwall's lemma [42, Thm. 1.4.1] to the above inequality yield

$$\|\mathfrak{z}(t)\| \leq c_T \left(\|\mathfrak{z}_0\| + T^{(p-1)/p} \|\mathcal{B}(r)\| \, \|u\|_p \right). \tag{14}$$

∎

4. Existence of an Optimizer. The following theorem ensures that the optimization problem P admits an optimal control input $u^o \in U_{ad}$ together with an optimal actuator location $r^o \in K_{ad}$.

THEOREM 4.1. *For initial condition $\mathfrak{z}_0 \in \mathbb{Z}$ let T be such that the mild solution exists for all $u \in U_{ad}$, and all $r \in K_{ad}$. Under assumptions A and B, there exists a control input $u^o \in U_{ad}$ together with an actuator location $r^o \in K_{ad}$, that solve the optimization problem P.*

Proof. The cost function $J(u, r; \mathfrak{z}_0)$ is bounded from below, and thus it has an infimum, say $j(\mathfrak{z}_0)$. This infimum is finite by assumption. As a result, there is a sequence of inputs $u_n \in U_{ad}$ and actuator location $r_n \in K_{ad}$ such that

$$\lim_{n\to\infty} J(u_n, r_n; \mathfrak{z}_0) \to j(\mathfrak{z}_0). \tag{15}$$

Moreover, by Theorem 3.1, for every pair (u_n, r_n), there exists a state $\mathfrak{z}_n(t) \in C(0, T; \mathbb{Z})$. The sequence $\{\mathfrak{z}_n(t)\}$ is also uniformly bounded in $C(0, T; \mathbb{Z})$ by Corollary 3.2; that is

$$\|\mathfrak{z}_n\|_{C(0,T;\mathbb{Z})} \le c_T \left(\|\mathfrak{z}_0\| + T^{(p-1)/p} R M_{\mathcal{B}} \right). \tag{16}$$

The set U_{ad} is a bounded subset of the reflexive space $L^p(0, T; \mathbb{U})$, $1 < p < \infty$, and hence it is weakly sequentially compact [39, Thm. 2.10.]. Since U_{ad} is closed and convex, it is also weakly closed [39, Thm. 2.11.]. These statements mean that there is a subsequence of u_n that converges weakly to some element $u^o \in U_{ad}$. To simplify the notation, we denote the weakly convergent subsequence by u_n, and indicate the limit by u^o in U_{ad}:

$$u_n(t) \rightharpoonup u^o(t) \quad \text{as} \quad n \to \infty. \tag{17}$$

The compactness of K_{ad} implies that there is also a subsequence of r_n that converges to some r^o in K_{ad}. This subsequence is also indicated by r_n; that is

$$r_n \to r^o \quad \text{as} \quad n \to \infty. \tag{18}$$

Using assumption B2, it follows that

$$\mathcal{B}(r_n) u_n(t) \rightharpoonup \mathcal{B}(r^o) u^o(t) \quad \text{in} \quad L^p(0, T; \mathbb{Z}) \quad \text{as} \quad n \to \infty \tag{19}$$

The sequence $\mathfrak{z}_n(t)$ is bounded in $C(0, T; \mathbb{Z})$, and thus, in $L^p(0, T; \mathbb{Z})$ as well. The latter is a reflexive Banach space; this means that a subsequence of $\mathfrak{z}_n(t)$, denote it by $\mathfrak{z}_n(t)$ for simplicity, weakly converges to an element of $\mathfrak{z}^o$ in $L^p(0, T; \mathbb{Z})$. Since the operator $\mathcal{F}(\cdot)$ is weakly sequentially continuous, the sequence $\mathcal{F}(\mathfrak{z}_n(t))$ weakly converges to $\mathcal{F}(\mathfrak{z}^o(t))$ in $L^p(0, T; \mathbb{Z})$:

$$\mathcal{F}(\mathfrak{z}_n(t)) \rightharpoonup \mathcal{F}(\mathfrak{z}^o(t)). \tag{20}$$

Now for each (u_n, r_n) consider the linear inhomogeneous initial value problem

$$\dot{\mathfrak{y}}_n(t) = \mathcal{A}\mathfrak{y}_n(t) + \mathcal{F}(\mathfrak{z}_n(t)) + \mathcal{B}(r_n) u_n(t), \quad \mathfrak{y}_n(0) = \mathfrak{z}_0, \tag{21}$$

which has mild solution

$$\mathfrak{y}_n(t) = \mathcal{T}(t)\mathfrak{z}_0 + \int_0^t \mathcal{T}(t-s) \left(\mathcal{F}(\mathfrak{z}_n(s)) + \mathcal{B}(r_n) u_n(s) \right) ds. \tag{22}$$

Since the inhomogeneous part $\mathcal{F}(\mathfrak{z}_n(s)) + \mathcal{B}(r_n) u_n(s)$ is in $L^p(0, T; \mathbb{Z})$ for each n, the sequence

$$\mathfrak{g}_n(t) = \int_0^t \mathcal{T}(t-s) \left(\mathcal{F}(\mathfrak{z}_n(s)) + \mathcal{B}(r_n) u_n(s) \right) ds \tag{23}$$

belongs to $C(0,T;\mathbb{Z})$ by Theorem 3.1. The limits (19) and (20) imply that this inhomogeneous part is a weakly convergent sequence in $L^p(0,T;\mathbb{Z})$. The convolution operation (23) defines a continuous linear operation from $L^p(0,T;\mathbb{Z})$ to $C(0,T;\mathbb{Z})$. Since every continuous linear map is also weakly continuous [6, Proposition 1.84], the sequence $\mathfrak{g}_n(t)$ weakly converges to some element $\mathfrak{g}^o(t)$ in $C(0,T;\mathbb{Z})$ satisfying

$$\mathfrak{g}^o(t) = \int_0^t \mathcal{T}(t-s)\left(\mathcal{F}(\mathfrak{z}^o(s)) + \mathcal{B}(r^o)u^o(s)\right) ds. \tag{24}$$

As a result, the sequence $\mathfrak{y}_n(t)$ weakly converges to $\mathfrak{y}^o(t) = \mathcal{T}(t)\mathfrak{z}_0 + \mathfrak{g}^o(t)$ in $C(0,T;\mathbb{Z})$. Since the mild solution is unique, $\mathfrak{y}^o(t) = \mathfrak{z}^o(t)$,

$$\mathfrak{z}^o(t) = \mathcal{T}(t)\mathfrak{z}_0 + \int_0^t \mathcal{T}(t-s)\left(\mathcal{F}(\mathfrak{z}^o) + \mathcal{B}(r^o)u^o(s)\right)\, ds. \tag{25}$$

It remains to show that $(\mathfrak{z}^o(t), u^o(t), r^o)$ minimizes $J(u,r;\mathfrak{z}_0)$. Recall from definition of the sequence u_n and r_n that

$$\begin{aligned} j(\mathfrak{z}_0) &= \liminf_{n\to\infty} J(u_n, r_n; \mathfrak{z}_0) \\ &= \liminf_{n\to\infty} \int_0^T \phi(\mathfrak{z}_n(t))\, dt + \liminf_{n\to\infty} \int_0^T \psi(u_n(t))\, dt. \end{aligned} \tag{26}$$

From assumption B3, the cost function is weakly lower semicontinuous in $\mathfrak{z}$ and u. This implies

$$j(\mathfrak{z}_0) \geq \int_0^T \phi(\mathfrak{z}^o(t))\, dt + \int_0^T \psi(u^o(t))\, dt = J(u^o, r^o; \mathfrak{z}_0). \tag{27}$$

Since $j(\mathfrak{z}_0)$ was defined to be the infimum,

$$j(\mathfrak{z}_0) = J(u^o, r^o; \mathfrak{z}_0).$$

Therefore, for every initial condition $\mathfrak{z}_0 \in \mathbb{Z}$, there exists an control input $u^o(t)$ together with an actuator location r^o, with corresponding mild solution $\mathfrak{z}^o(t)$ that achieve the minimum value of the cost function . ∎

For a linear partial differential equation and quadratic cost, the optimal actuator problem may not be convex; see for example [29, Fig. 7]. Uniqueness of the optimal control and actuator is not guaranteed.

5. Optimality Conditions. In order to establish the first order optimality condition for an optimizer (u^o, r^o), further regularity on the control-to-state map is needed. In next two theorems, it is proved that under certain assumptions, the control-to-state map is Lipschitz continuous in both u and r. For the Lipschitz continuity with respect to the actuator location, a stronger assumption on the input operator $\mathcal{B}(r)$ than continuity in r is needed.

DEFINITION 5.1. *For each initial condition $\mathfrak{z}_0 \in \mathbb{Z}$, and actuator design $r \in K_{ad}$, the control-to-state operator is the operator $\mathcal{S}(u; r, \mathfrak{z}_0) : U_{ad} \subset (L^p(0,T;\mathbb{U})) \to L^p(0,T;\mathbb{Z})$ that maps every input $u \in U_{ad}$ to the state $\mathfrak{z} \in L^p(0,T;\mathbb{Z})$. It is described by*

$$\mathfrak{z}(t) = \mathcal{T}(t)\mathfrak{z}_0 + \int_0^t \mathcal{T}(t-s)\mathcal{F}(\mathfrak{z}(s))ds + \int_0^t \mathcal{T}(t-s)\mathcal{B}(r)u(s)ds.$$

In the next proposition, the Lipschitz continuity of this operator in r is used to establish the Lipschitz continuity of the control-to-state map with respect to r.

PROPOSITION 5.2. *Under assumption A and B2, for any initial condition $\mathfrak{z}_0 \in \mathbb{Z}$ and actuator location $r \in K_{ad}$, the control-to-state map $\mathcal{S}(u; r, \mathfrak{z}_0)$ is Lipschitz continuous in u, i.e., there exists a positive constant L_u such that*

$$\|\mathcal{S}(u_2; r, \mathfrak{z}_0) - \mathcal{S}(u_1; r, \mathfrak{z}_0)\|_p \leq L_u \|u_2 - u_1\|_p, \tag{28}$$

for all u_1 and u_2 in U_{ad}.

For a fixed control input $u \in U_{ad}$ and extra assumptions C3 and C4, the control-to-state map $\mathcal{S}(u; r, \mathfrak{z}_0)$ is Lipschitz continuous in r, i.e., there exists a positive constant L_r such that

$$\|\mathcal{S}(u; r_2, \mathfrak{z}_0) - \mathcal{S}(u; r_1, \mathfrak{z}_0)\|_p \leq L_r \|r_2 - r_1\|_{\mathbb{K}}, \tag{29}$$

for all r_1 and r_2 in K_{ad}.

The proof of this proposition is straightforward; a proof is provided in Appendix A.

Fréchet differentiability of the control-to-state map as well as its derivatives need to be formulated in order to characterize an optimizer. For any $\mathfrak{z}^o \in C(0, T; \mathbb{Z})$ define the time-varying operator operator $\mathcal{F}'_{\mathfrak{z}^o(t)}$. At any $t > 0$ this operator is linear in $\mathfrak{z}$. Consider the time-varying IVP

$$\dot{\tilde{\mathfrak{z}}}(t) = (\mathcal{A} + \mathcal{F}'_{\mathfrak{z}^o(t)})\tilde{\mathfrak{z}}(t) + \mathcal{B}(r)\tilde{u}(t), \quad \tilde{\mathfrak{z}}(0) = 0. \tag{30}$$

The mild solution is described by a two parameter family of operators, say $\mathcal{U}(t, s)$, known as an evolution operator.

The following lemma relies on the existence results: Theorem 5.5.6 and Theorem 5.5.10 in [15].

LEMMA 5.3. *The mild solution of IVP problem (30) is described by*

$$\tilde{\mathfrak{z}}(t) = \int_0^t \mathcal{U}(t, s)\mathcal{B}(r)u(s)\, ds, \tag{31}$$

in which $\mathcal{U}(t, s)$ is a strongly continuous evolution operator on $\mathbb{Z}$ for $0 \leq s \leq t \leq T$. In addition, let $f \in L^1(0, T; \mathbb{Z})$, and consider the following final value problem (FVP) backward in time

$$\dot{\tilde{\mathfrak{p}}}(s) = -(\mathcal{A}^* + \mathcal{F}'^*_{\mathfrak{z}^o(t)})\mathfrak{p}(s) - f(s), \quad \tilde{\mathfrak{p}}(T) = 0, \tag{32}$$

then the mild solution of this evolution equation satisfies

$$\tilde{\mathfrak{p}}(s) = \int_s^T \mathcal{U}^*(s, t) f(s)\, ds, \tag{33}$$

where $\mathcal{U}^*(s, t)$ is the adjoint of $\mathcal{U}(s, t)$ on $\mathbb{Z}$ for every $0 \leq s \leq t \leq T$.

Proof. The time-invariant part of the state operator in (30), $\mathcal{A}$, is the generator of an strongly continuous semigroup. According to [15, Thm. 5.5.6], in order for a strongly continuous evolution operator $\mathcal{U}(t, s)$ to exist so that (31) is the mild solution to the IVP (30), it is sufficient that for every $\tilde{\mathfrak{z}} \in \mathbb{Z}$ the mapping $t \mapsto \mathcal{F}'_{\mathfrak{z}^o(t)}\tilde{\mathfrak{z}}$ is strongly measurable and that a function $\alpha(t) \in L^1(0, T)$ exists such that

$$\left\|\mathcal{F}'_{\mathfrak{z}^o(t)}\right\| \leq \alpha(t), \quad t \in [0, T]. \tag{34}$$

By assumption C2, since the state $\mathfrak{z}^o(t)$ is uniformly bounded, the operator norm of $\mathcal{F}'_{\mathfrak{z}^o(t)}$ admits an upper bound for all $t \in [0,T]$. In inequality (34), a simple choice for $\alpha(t)$ can be the operator norm of $\mathcal{F}'_{\mathfrak{z}^o(t)}$ itself. Consequently, a strongly continuous evolution operator $\mathcal{U}(t,s)$ exists so that (31) is the mild solution to the IVP (30).

Since the state space $\mathbb{Z}$ is a separable reflexive Banach space, Theorem 5.5.10 in [15] implies that the mild solution of the FVP (32) is described by an evolution operator. Moreover, for every $0 \le s \le t \le T$, this evolution operator is the adjoint on $\mathbb{Z}$ of the evolution operator $\mathcal{U}(t,s)$. ∎

Proposition 5.4. *Under assumption A, B2, C1, and C2, for every initial condition $\mathfrak{z}_0 \in \mathbb{Z}$ and actuator location $r \in K_{ad}$, the control-to-state map $\mathcal{S}(u;r,\mathfrak{z}_0)$ is Fréchet differentiable in u in the interior of U_{ad}. The Fréchet derivative of $\mathcal{S}(u;r,\mathfrak{z}_0)$ at u^o is*

$$\mathcal{S}'_{u^o}\tilde{u} = \tilde{\mathfrak{z}}, \quad \forall \tilde{u} \in L^p(0,T;\mathbb{U}), \tag{35}$$

where, defining $\mathfrak{z}^o(t) = \mathcal{S}(u^o;r,\mathfrak{z}_0)$, $\tilde{\mathfrak{z}}$ is the mild solution to the IVP

$$\dot{\tilde{\mathfrak{z}}}(t) = (\mathcal{A} + \mathcal{F}'_{\mathfrak{z}^o(t)})\tilde{\mathfrak{z}}(t) + \mathcal{B}(r)\tilde{u}(t), \quad \tilde{\mathfrak{z}}(0) = 0. \tag{36}$$

The mild solution to this equation is provided by the evolution operator $\mathcal{U}(t,s)$ in (31).

Proof. If the Fréchet derivative of the control-to-state map $\mathcal{S}(u;r,\mathfrak{z}_0)$ with respect to u at u^o is given by (35), then it needs to satisfy

$$\lim_{\|\tilde{u}\|_p \to 0} \frac{\|\mathcal{S}(\tilde{u}+u^o;r,\mathfrak{z}_0) - \mathcal{S}(u^o;r,\mathfrak{z}_0) - \mathcal{S}'_{u^o}\tilde{u}\|_p}{\|\tilde{u}\|_p} = 0. \tag{37}$$

Denote by $\mathfrak{z}_p = \mathcal{S}(\tilde{u}+u^o;r,\mathfrak{z}_0)$ the mild solution to the IVP

$$\dot{\mathfrak{z}}_p(t) = \mathcal{A}\mathfrak{z}_p(t) + \mathcal{F}(\mathfrak{z}_p(t)) + \mathcal{B}(r)(\tilde{u}(t) + u^o(t)), \quad \mathfrak{z}_p(0) = \mathfrak{z}_0. \tag{38}$$

The state $\mathfrak{z}^o = \mathcal{S}(u^o;r,\mathfrak{z}_0)$ is by definition the mild solution of the IVP

$$\dot{\mathfrak{z}}^o(t) = \mathcal{A}\mathfrak{z}^o(t) + \mathcal{F}(\mathfrak{z}^o(t)) + \mathcal{B}(r)u^o(t), \quad \mathfrak{z}^o(0) = \mathfrak{z}_0. \tag{39}$$

Define $\mathfrak{z}_r = \mathfrak{z}_p - \mathfrak{z}^o - \tilde{\mathfrak{z}}$, notice that

$$\mathfrak{z}_r = \mathcal{S}(\tilde{u}+u^o;r,\mathfrak{z}_0) - \mathcal{S}(u^o;r,\mathfrak{z}_0) - \mathcal{S}'_{u^o}\tilde{u}. \tag{40}$$

Subtract the equations (39) and (36) from (38) to obtain

$$\dot{\mathfrak{z}}_r(t) = \mathcal{A}\mathfrak{z}_r(t) + \mathcal{F}(\mathfrak{z}_p(t)) - \mathcal{F}(\mathfrak{z}^o(t)) - \mathcal{F}'_{\mathfrak{z}^o(t)}\tilde{\mathfrak{z}}(t), \quad \mathfrak{z}_r(0) = 0, \tag{41}$$

where $\mathcal{F}'_{\mathfrak{z}^o(t)}$ is the Fréchet derivative of $\mathcal{F}(\cdot)$ at $\mathfrak{z}^o(t)$ for every $t \in [0,T]$, noting that $\mathfrak{z}^o(t) \in C(0,T;\mathbb{Z})$. The Fréchet derivative of $\mathcal{F}(\cdot)$ by definition satisfies

$$\mathcal{F}(\mathfrak{z}_p(t)) = \mathcal{F}(\mathfrak{z}^o(t)) + \mathcal{F}'_{\mathfrak{z}^o(t)}(\mathfrak{z}_p(t) - \mathfrak{z}^o(t)) + \mathfrak{r}_{\mathcal{F}}(t). \tag{42}$$

where the remainder $\mathfrak{r}_{\mathcal{F}}(t)$ is in $\mathbb{Z}$ satisfying

$$\lim_{\|\mathfrak{z}_p(t)-\mathfrak{z}^o(t)\| \to 0} \frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\mathfrak{z}_p(t) - \mathfrak{z}^o(t)\|} = 0, \tag{43}$$

for every $t \in [0,T]$. To further prove the proposition, the function inside the limit in (43) needs to be bounded on $[0,T]$. Note that, by Corollary 3.2, the norm of the states $\mathfrak{z}_p(t)$ and $\mathfrak{z}^o(t)$ is uniformly bounded over $[0,T]$ by some number δ,

$$\delta \leq c_T \left(\|\mathfrak{z}_0\| + T^{(p-1)/p} M_{\mathcal{B}} R\right). \tag{44}$$

Use the local Lipschitz continuity of $\mathcal{F}(\cdot)$ (assumption A2) to write

$$\|\mathcal{F}(\mathfrak{z}_p(t)) - \mathcal{F}(\mathfrak{z}^o(t))\| \leq L_{\mathcal{F}\delta} \|\mathfrak{z}_p(t) - \mathfrak{z}^o(t)\| \tag{45}$$

Also apply assumption C2 on $\mathcal{F}'_{\mathfrak{z}^o}(t)$ to find

$$\left\|\mathcal{F}'_{\mathfrak{z}^o(t)}(\mathfrak{z}_p(t) - \mathfrak{z}^o(t))\right\| \leq \sup_{\|\mathfrak{z}^o(t)\| \leq \delta} \left\|\mathcal{F}'_{\mathfrak{z}^o(t)}\right\| \cdot \|\mathfrak{z}_p(t) - \mathfrak{z}^o(t)\|, \tag{46}$$

Incorporate these inequalities into (42), and apply the triangle inequality. A uniform upper bound on $\mathfrak{r}_{\mathcal{F}}(t)$ over $[0,T]$ is derived as

$$\frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\mathfrak{z}_p(t) - \mathfrak{z}^o(t)\|} \leq L_{\mathcal{F}\delta} + \sup_{\|\mathfrak{z}^o(t)\| \leq \delta} \left\|\mathcal{F}'_{\mathfrak{z}^o(t)}\right\|. \tag{47}$$

By substituting (42) into (41), the state $\mathfrak{z}_r$ becomes the mild solution to IVP

$$\dot{\mathfrak{z}}_r(t) = (\mathcal{A} + \mathcal{F}'_{\mathfrak{z}^o(t)})\mathfrak{z}_r(t) + \mathfrak{r}_{\mathcal{F}}(t), \quad \mathfrak{z}_r(0) = 0. \tag{48}$$

In this equation, the operator $\mathcal{A}$ is perturbed by the time-dependent operator $\mathcal{F}'_{\mathfrak{z}^o(t)}$. According to Lemma 5.3, the mild solution of this evolution equation is described by an evolution operator $\mathcal{U}(t,s)$. Let M be an upper bound for the operator norm of $\mathcal{U}(t,s)$ over $0 \leq t \leq s \leq T$, then the mild solution to (48) satisfies the estimate

$$\|\mathfrak{z}_r\|_p \leq M T^{(p-1)/p} \left(\int_0^T \|\mathfrak{r}_{\mathcal{F}}(t)\|^p\right)^{1/p}. \tag{49}$$

Divide both sides of this inequality by $\|\tilde{u}\|_p$ and rewrite it as

$$\frac{\|\mathfrak{z}_r\|_p}{\|\tilde{u}\|_p} \leq M T^{(p-1)/p} \int_0^T \left(\frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|^p}{\|\tilde{u}\|_p}\right)^{1/p} dt. \tag{50}$$

The function inside the integral converges pointwise to zero for every $t \in [0,T]$ and is uniformly bounded over $[0,T]$. To see this, recall Proposition 5.2, the mapping $\mathcal{S}(\cdot; r, \mathfrak{z}_0)$ is Lipschitz continuous, giving

$$\|\mathfrak{z}_p - \mathfrak{z}^o\|_p \leq L_u \|\tilde{u}\|_p. \tag{51}$$

This can be used to write

$$\begin{aligned} \lim_{\|\tilde{u}\|_p \to 0} \frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\tilde{u}\|_p} &= \lim_{\|\tilde{u}\|_p \to 0} \frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\mathfrak{z}_p(t) - \mathfrak{z}^o(t)\|} \frac{\|\mathfrak{z}_p(t) - \mathfrak{z}^o(t)\|}{\|\tilde{u}\|_p} \\ &= L_u \lim_{\|\mathfrak{z}_p(t) - \mathfrak{z}^o(t)\| \to 0} \frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\mathfrak{z}_p(t) - \mathfrak{z}^o(t)\|} = 0. \end{aligned} \tag{52}$$

and also by inequality (47),

$$\frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\tilde{u}\|_p} \leq \left(L_{\mathcal{F}\delta} + \sup_{\|\mathfrak{z}^o(t)\| \leq \delta} \left\| \mathcal{F}'_{\mathfrak{z}^o(t)} \right\| \right) L_u. \tag{53}$$

Applying the bounded convergence theorem to the integral in (50) results in

$$\lim_{\|\tilde{u}\|_p \to 0} \frac{\|\mathfrak{z}_r\|_p}{\|\tilde{u}\|_p} = 0. \tag{54}$$

Recall (40), the previous limit is in fact the limit in (37) we aimed to prove. ∎

Proposition 5.5. *Under assumption A, B2, C1-C4, for every initial condition $\mathfrak{z}_0 \in \mathbb{Z}$ and control input $u(t) \in U_{ad}$, the control-to-state map $\mathcal{S}(u; r, \mathfrak{z}_0)$ is Fréchet differentiable in r in the interior of K_{ad}. The Fréchet derivative of $\mathcal{S}(u; r, \mathfrak{z}_0)$ at r^o is*

$$S'_{r^o}\tilde{r} = \tilde{\mathfrak{h}}, \quad \forall \tilde{r} \in \mathbb{K}, \tag{55}$$

where, defining $\mathfrak{z}^o(t) = \mathcal{S}(u; r^o, \mathfrak{z}_0)$, $\tilde{\mathfrak{h}}$ is the mild solution to the IVP:

$$\dot{\tilde{\mathfrak{h}}}(t) = (\mathcal{A} + \mathcal{F}'_{\mathfrak{z}^o(t)})\tilde{\mathfrak{h}}(t) + (\mathcal{B}'_{r^o}\tilde{r})\, u(t), \quad \tilde{\mathfrak{h}}(0) = 0. \tag{56}$$

The proof of this proposition is similar to that of Proposition 5.4; a proof is provided in Appendix B.

Now that differentiability and derivatives of the control-to-state map has been established, the first order necessary conditions for a pair (u^o, r^o) to be a local optimizer can be derived. In order to place the problem in a Hilbert space, assumptions C4 and C5 are used, assuming that the spaces are Hilbert spaces and defining a cost function. It will also be assumed that $p = 2$, considering control inputs in $L^2(0, T; \mathbb{U})$. It is shown in the following lemma that this cost function is consistent with previous assumptions on the cost function (assumption B3).

Lemma 5.6. *The cost function in assumption C5 satisfies assumption B3; that is, it is weakly lower semicontinuous in $\mathfrak{z}$ and u.*

Proof. The cost function $J(u, r; \mathfrak{z}_0)$ in assumption C5 is continuous and convex function in both $\mathfrak{z}$ and u; that is

$$\int_0^T \langle \mathcal{Q}\mathfrak{z}_n(t), \mathfrak{z}_n(t) \rangle \, dt \to \int_0^T \langle \mathcal{Q}\mathfrak{z}(t), \mathfrak{z}(t) \rangle \, dt \quad \text{as} \quad \mathfrak{z}_n \to \mathfrak{z} \quad \text{in} \quad L^p(0, T; \mathbb{Z})$$
$$\langle \lambda \mathcal{Q}\mathfrak{z}_1 + (1-\lambda)\mathcal{Q}\mathfrak{z}_2, \lambda\mathfrak{z}_1 + (1-\lambda)\mathfrak{z}_2 \rangle \leq \lambda \langle \mathcal{Q}\mathfrak{z}_1, \mathfrak{z}_1 \rangle + (1-\lambda) \langle \mathcal{Q}\mathfrak{z}_2, \mathfrak{z}_2 \rangle,$$

and a similar argument for u. According to Theorem 2.12 in [39], if a functional defined on a Banach space is continuous and convex; then, it is also weakly lower semicontinuous. Therefore, the cost function $J(u, r; \mathfrak{z}_0)$ is weakly lower semicontinuous in both $\mathfrak{z}$ and u. ∎

The next theorem drives the first order necessary conditions for an optimizer of the optimization problem P.

Theorem 5.7. *Suppose assumptions A, B1, B2, C hold. For any initial condition $\mathfrak{z}_0 \in \mathbb{Z}$, let the pair (u^o, r^o) be a local minimizer of the optimization problem P with the optimal trajectory $\mathfrak{z}^o = \mathcal{S}(u^o; r^o, \mathfrak{z}_0)$. Also, let $\mathfrak{p}^o(t)$, the adjoint state, indicate the mild solution of the final value problem*

$$\dot{\mathfrak{p}}^o(s) = -(\mathcal{A}^* + \mathcal{F}'^*_{\mathfrak{z}^o(t)})\mathfrak{p}^o(s) - \mathcal{Q}\mathfrak{z}^o(s), \quad \mathfrak{p}^o(T) = 0. \tag{57}$$

Then, if (u^o, r^o) *is an interior point of* $U_{ad} \times K_{ad}$*, it satisfies*

$$u^o(t) = -\mathcal{R}^{-1}\mathcal{B}^*(r^o)\mathfrak{p}^o(t), \tag{58a}$$

$$\int_0^T (\mathcal{B}'_{r^o} u^o(s))^* \mathfrak{p}^o(s)\, ds = 0. \tag{58b}$$

Proof. Since the optimal pair (u^o, r^o) belongs to the interior of $U_{ad} \times K_{ad}$, the cost function satisfies

$$J\left(u^o + \tau h_u, r^o + \tau h_r; \mathfrak{z}_0\right) \geq J(u^o, r^o; \mathfrak{z}_0), \tag{59}$$

for sufficiently small numbers τ along any direction $(h_u, h_r) \in L^2(0,T;\mathbb{U}) \times \mathbb{K}$. This implies that the directional derivative of $J(\cdot, r^o; \mathfrak{z}_0)$ at u^o along any direction h_u is non-negative as well as the directional derivative of $J(u^o, \cdot; \mathfrak{z}_0)$ at r^o along any direction h_r. Use assumption C5, the cost function is sum of two inner products in the Hilbert spaces $L^2(0,T;\mathbb{Z})$ and $L^2(0,T;\mathbb{U})$; that is

$$J(u, r; \mathfrak{z}_0) = \langle \mathcal{Q}\mathfrak{z}, \mathfrak{z}\rangle_{L^2(0,T;\mathbb{Z})} + \langle \mathcal{R}u, u\rangle_{L^2(0,T;\mathbb{U})}. \tag{60}$$

Thus, the directional derivative at u^o along h_u is

$$\begin{aligned} J'_{u^o} h_u &= 2\left\langle \mathcal{Q}\mathcal{S}(u^o; r^o, \mathfrak{z}_0), \mathcal{S}'_{u^o} h_u\right\rangle_{L^2(0,T;\mathbb{Z})} + 2\left\langle \mathcal{R}u^o, h_u\right\rangle_{L^2(0,T;\mathbb{U})} \\ &= 2\left\langle \mathcal{S}'^*_{u^o}\mathcal{Q}\mathcal{S}(u^o; r^o, \mathfrak{z}_0) + \mathcal{R}u^o, h_u\right\rangle_{L^2(0,T;\mathbb{U})}. \end{aligned} \tag{61}$$

This inner product must be non-negative for any direction $h_u \in L^2(0,T;\mathbb{U})$, and so

$$u^o = -\mathcal{R}^{-1}\mathcal{S}'^*_{u^o}\mathcal{Q}\mathcal{S}(u^o; r^o, \mathfrak{z}_0). \tag{62}$$

To calculate the adjoint operator $\mathcal{S}'^*_{u^o}$, let $\tilde{u}(t) \in L^2(0,T;\mathbb{U})$, $\tilde{\mathfrak{z}}(t) \in L^2(0,T;\mathbb{Z})$ be arbitrary. Using Lemma 5.3,

$$\begin{aligned} \int_0^T \langle \tilde{\mathfrak{z}}(t), \mathcal{S}'_{u^o}\tilde{u}(t)\rangle\, dt &= \int_0^T \left\langle \tilde{\mathfrak{z}}(t), \int_0^t \mathcal{U}(t,s)\mathcal{B}(r^o)\tilde{u}(s)\, ds\right\rangle dt \\ &= \int_0^T \int_s^T \langle \tilde{\mathfrak{z}}(t), \mathcal{U}(t,s)\mathcal{B}(r^o)\tilde{u}(s)\rangle\, dtds \\ &= \int_0^T \left\langle \mathcal{B}^*(r^o)\int_s^T \mathcal{U}^*(t,s)\tilde{\mathfrak{z}}(t)dt, \tilde{u}(s)\right\rangle_{\mathbb{U}} ds. \end{aligned} \tag{63}$$

Thus,

$$\mathcal{S}'^*_{u^o}\tilde{\mathfrak{z}}(t) = \mathcal{B}^*(r^o)\int_s^T \mathcal{U}^*(t,s)\tilde{\mathfrak{z}}(t)dt.$$

Define $\tilde{\mathfrak{p}}(t) = \int_s^T \mathcal{U}^*(t,s)\tilde{\mathfrak{z}}(t)dt$. From the second part of Lemma 5.3, $\tilde{\mathfrak{p}}(t)$ is the mild solution of the following FVP solved backward in time

$$\dot{\tilde{\mathfrak{p}}}(s) = -(\mathcal{A}^* + \mathcal{F}'^*_{\mathfrak{z}^o(s)})\tilde{\mathfrak{p}}(s) - \tilde{\mathfrak{z}}(s), \quad \tilde{\mathfrak{p}}(T) = 0. \tag{64}$$

It follows that

$$\mathcal{S}'^*_{u^o}\tilde{\mathfrak{z}}(t) = \mathcal{B}^*(r^o)\tilde{\mathfrak{p}}(t). \tag{65}$$

This implies that (62) can be written

$$u^o(t) = -\mathcal{R}^{-1}\mathcal{B}^*(r^o)\mathfrak{p}^o(t) \tag{66}$$

where $\mathfrak{p}^o(t)$ solves

$$\dot{\mathfrak{p}}^o(s) = -(\mathcal{A}^* + \mathcal{F}'^*_{\mathfrak{z}^o(s)})^*\mathfrak{p}^o(s) - \mathcal{Q}\mathfrak{z}^o(s), \quad \mathfrak{p}^o(T) = 0. \tag{67}$$

Taking the directional derivative of $J(u^o, \cdot\,; \mathfrak{z}_0)$ at r^o along h_r yields

$$\begin{aligned} J'_{r^o}h_r &= 2\left\langle \mathcal{Q}\mathcal{S}(u^o; r^o, \mathfrak{z}_0), \mathcal{S}'_{r^o}h_r\right\rangle_{L^2(0,T;\mathbb{Z})} \\ &= 2\left\langle \mathcal{S}'^*_{r^o}\mathcal{Q}\mathcal{S}(u^o; r^o, \mathfrak{z}_0), h_r\right\rangle_{\mathbb{K}}. \end{aligned} \tag{68}$$

Similarly, this inner product must be non-negative for any direction $h_r \in \mathbb{K}$ yielding the optimality condition in $\mathbb{K}$:

$$\mathcal{S}'^*_{r^o}\mathcal{Q}\mathcal{S}(u^o; r^o, \mathfrak{z}_0) = 0. \tag{69}$$

To calculate the adjoint operator $\mathcal{S}'^*_{r^o}$, use Lemma 5.3, and proceed as follows

$$\begin{aligned} \left\langle \mathcal{Q}\mathcal{S}(u^o; r^o, \mathfrak{z}_0), \mathcal{S}'_{r^o}h_r\right\rangle_{L^2(0,T;\mathbb{Z})} &= \int_0^T \left\langle \mathcal{Q}\mathfrak{z}^o(t), \int_0^t \mathcal{U}(t,s)(\mathcal{B}'_{r^o}h_r)u^o(s)\,ds\right\rangle dt \\ &= \int_0^T \left\langle \int_s^T \mathcal{U}^*(t,s)\mathcal{Q}\mathfrak{z}^o(t)\,dt, (\mathcal{B}'_{r^o}h_r)u^o(s)\right\rangle ds \\ &= \int_0^T \left\langle \mathfrak{p}^o(s), (\mathcal{B}'_{r^o}h_r)u^o(s)\right\rangle ds. \end{aligned} \tag{70}$$

For each $u \in \mathbb{U}$, $(\mathcal{B}'_{r^o}h_r)u$ is an element of $\mathbb{Z}$. This defines a bounded linear map from $h_r \in \mathbb{K}$ to $\mathbb{Z}$. There exists a bounded linear operator $(\mathcal{B}'_{r^o}u)^*\colon \mathbb{Z} \to \mathbb{K}$ satisfying

$$\left\langle (\mathcal{B}'_{r^o}u)^*\mathfrak{p}, h_r\right\rangle_{\mathbb{K}} = \left\langle \mathfrak{p}, (\mathcal{B}'_{r^o}h_r)u\right\rangle. \tag{71}$$

Incorporate this into (70) to obtain

$$\begin{aligned} \left\langle \mathcal{Q}\mathcal{S}(u^o; r^o, \mathfrak{z}_0), \mathcal{S}'_{r^o}h_r\right\rangle_{L^2(0,T;\mathbb{Z})} &= \int_0^T \left\langle (\mathcal{B}'_{r^o}u^o(s))^*\mathfrak{p}^o(s), h_r\right\rangle_{\mathbb{K}} ds \\ &= \left\langle \int_0^T (\mathcal{B}'_{r^o}u^o(s))^*\mathfrak{p}^o(s)\,ds, h_r\right\rangle_{\mathbb{K}}. \end{aligned} \tag{72}$$

This gives an explicit form of the adjoint operator $\mathcal{S}'^*_{r^o}$, and hence that of the optimality condition (69):

$$\mathcal{S}'^*_{r^o}\mathcal{Q}\mathcal{S}(u^o; r^o, \mathfrak{z}_0) = \int_0^T (\mathcal{B}'_{r^o}u^o(s))^*\mathfrak{p}^o(s)\,ds = 0, \tag{73}$$

where p^o solves (67). This completes the proof. ∎

Together with the original PDE, Theorem 5.7 provides the following system of equations characterizing an optimizer $(\mathfrak{z}^o, \mathfrak{p}^o, u^o, r^o)$:

$$\begin{cases} \dot{\mathfrak{z}}^o(t) = \mathcal{A}\mathfrak{z}^o(t) + \mathcal{F}(\mathfrak{z}^o(t)) + \mathcal{B}(r^o)u^o(t), & \mathfrak{z}^o(0) = \mathfrak{z}_0, \\ \dot{\mathfrak{p}}^o(t) = -(\mathcal{A}^* + \mathcal{F}'^*_{\mathfrak{z}^o(t)})\mathfrak{p}^o(t) - \mathcal{Q}\mathfrak{z}^o(t), & \mathfrak{p}^o(T) = 0, \\ u^o(t) = -\mathcal{R}^{-1}\mathcal{B}^*(r^o)\mathfrak{p}^o(t), \\ \int_0^T (\mathcal{B}'_{r^o}u^o(s))^*\mathfrak{p}^o(s)\,ds = 0. \end{cases} \tag{74}$$

If the control space $\mathbb{U}$ and actuator location space $\mathbb{K}$ are separable Hilbert spaces, the optimizing control and actuator can be characterized further. Let $e_j^{\mathbb{K}}$, $e_i^{\mathbb{U}}$, and $e_k^{\mathbb{Z}}$ be orthonormal bases for $\mathbb{K}$, $\mathbb{U}$, and $\mathbb{Z}$, respectively. Then there exists $b_i(r) \in \mathbb{Z}$, $r \in \mathbb{K}$ so that for any $u \in \mathbb{U}$,

$$\mathcal{B}(r)u = \sum_{i=1}^{\infty} \left\langle u, e_i^{\mathbb{U}} \right\rangle_{\mathbb{U}} b_i(r). \tag{75}$$

Since the operator $\mathcal{B}(\cdot)u : \mathbb{K} \to \mathbb{Z}$ is Fréchet differentiable with respect to r, each $b_i(\cdot)$ is a Fréchet differentiable map from $\mathbb{K}$ to $\mathbb{Z}$. Denote the Fréchet derivative of $b_i(r)$ at r^o by $b'_{i,r^o} : \mathbb{K} \to \mathbb{Z}$, then

$$(\mathcal{B}'_{r^o} r)u = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\langle u, e_i^{\mathbb{U}} \right\rangle_{\mathbb{U}} \left\langle r, e_j^{\mathbb{K}} \right\rangle_{\mathbb{K}} b'_{i,r^o} e_j^{\mathbb{K}}. \tag{76}$$

Corollary 5.8. *Assume further that the control space $\mathbb{U}$ and actuator location space $\mathbb{K}$ are separable. Let $e_i^{\mathbb{U}}$, $e_j^{\mathbb{K}}$ and $e_k^{\mathbb{Z}}$ be orthonormal bases for $\mathbb{K}$, $\mathbb{U}$, and $\mathbb{Z}$, respectively. Define $u_j^o(t)$ and $\mathfrak{p}_k(t)$ as*

$$u_j^o(t) := \left\langle u^o(t), e_j^{\mathbb{U}} \right\rangle_{\mathbb{U}}, \tag{77a}$$

$$\mathfrak{p}_k^o(t) := \left\langle \mathfrak{p}^o(t), e_k^{\mathbb{Z}} \right\rangle. \tag{77b}$$

The optimality conditions (58) can be written

$$u_j^o(t) + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \left\langle b_i(r^o), e_k^{\mathbb{Z}} \right\rangle \left\langle \mathcal{R}^{-1} e_i^{\mathbb{U}}, e_j^{\mathbb{U}} \right\rangle_{\mathbb{U}} \mathfrak{p}_k^o(t) = 0, \quad \text{for each } j, \tag{78a}$$

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \left\langle b'_{i,r^o} e_j^{\mathbb{K}}, e_k^{\mathbb{Z}} \right\rangle \int_0^T u_i^o(s) \mathfrak{p}_k^o(s)\, ds = 0, \quad \text{for each } j. \tag{78b}$$

Proof. For every $\mathfrak{p} \in \mathbb{Z}$, the element $\mathcal{B}^*(r^o)\mathfrak{p} \in \mathbb{U}$ can by obtained by using (75), and doing the calculation

$$\begin{aligned} \langle \mathcal{B}^*(r^o)\mathfrak{p}, u \rangle_{\mathbb{U}} &= \langle \mathfrak{p}, \mathcal{B}(r^o)u \rangle \\ &= \sum_{i=1}^{\infty} \left\langle u, e_i^{\mathbb{U}} \right\rangle_{\mathbb{U}} \langle \mathfrak{p}, b_i(r^o) \rangle \\ &= \left\langle \sum_{i=1}^{\infty} \langle b_i(r^o), \mathfrak{p} \rangle\, e_i^{\mathbb{U}}, u \right\rangle_{\mathbb{U}}. \end{aligned} \tag{79}$$

This yields

$$\mathcal{B}^*(r^o)\mathfrak{p} = \sum_{i=1}^{\infty} \langle b_i(r^o), \mathfrak{p} \rangle\, e_i^{\mathbb{U}}. \tag{80}$$

Similarly, using (76), for every $u \in \mathbb{U}$, the operator $(\mathcal{B}'_{r^o} u)^*$ maps $\mathfrak{p} \in \mathbb{Z}$ to $\mathbb{K}$ as follows

$$(\mathcal{B}'_{r^o} u^o)^* \mathfrak{p} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left\langle u^o, e_i^{\mathbb{U}} \right\rangle_{\mathbb{U}} \left\langle b'_{i,r^o} e_j^{\mathbb{K}}, \mathfrak{p} \right\rangle e_j^{\mathbb{K}}. \tag{81}$$

Substituting these elements into the optimality conditions (58) and using (77) leads to (78). ∎

In the next section, an application of these results to railway track models is described.

6. Nonlinear Railway Track Model. Railway tracks are rested on ballast which are known for exhibiting nonlinear viscoelastic behavior [2]. If a track beam is made of a Kelvin-Voigt material, then the railway track model will be a parabolic semi-linear partial differential equation on $x \in [0, \ell]$ as follows:

$$
\begin{aligned}
&\rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x^2}(EI\frac{\partial^2 w}{\partial x^2} + C_d \frac{\partial^3 w}{\partial x^2 \partial t}) + \mu \frac{\partial w}{\partial t} + kw + \alpha w^3 = b(x;r)u(t),\\
&w(x,0) = w_0(x), \quad \frac{\partial w}{\partial t}(x,0) = v_0(x),\\
&w(0,t) = w(\ell,t) = 0,\\
&EI\frac{\partial^2 w}{\partial x^2}(0,t) + C_d \frac{\partial^3 w}{\partial x^2 \partial t}(0,t) = EI\frac{\partial^2 w}{\partial x^2}(\ell,t) + C_d \frac{\partial^3 w}{\partial x^2 \partial t}(\ell,t) = 0,
\end{aligned} \tag{82}
$$

where the positive constants E, I, ρ, A, and ℓ are the modulus of elasticity, second moment of inertia, density of the beam, cross-sectional area, and length of the beam, respectively. The linear and nonlinear parts of the foundation elasticity correspond to the coefficients k and α, respectively. The constant μ is the damping coefficient of the foundation, and C_d is the coefficient of Kelvin-Voigt damping in the beam. Denoted by $u(t)$ is the external force exerted on the railway track; it will further be considered as a scalar control input to manipulate the system; also $b(x;r)$ is a piece-wise continuous function in x parametrized by r, the actuator locations. Since the function $b(x;r)$ will accommodate the effect of actuators on the system; it needs be sufficiently smooth function of the actuator location (see assumption B2 and C3).

Let us define the closed self-adjoint positive operator $\mathcal{A}_0$ on $L^2(0,\ell)$ as:

$$
\begin{aligned}
&\mathcal{A}_0 w := w_{xxxx},\\
&D(\mathcal{A}_0) := \left\{ w \in H^4(0,\ell) |\, w(0) = w(\ell) = 0,\ w_{xx}(0) = w_{xx}(\ell) = 0 \right\},
\end{aligned} \tag{83}
$$

where subscripts denote the derivative with respect to spatial variable. As a result, the state operator associated with the Kelvin-Voigt beam is

$$
\mathcal{A}_{KV}(w,v) := \left(v, -\frac{1}{\rho A}\mathcal{A}_0(EIw + C_d v) \right), \tag{84}
$$

which is defined on the state space $\mathbb{Z} = H^2(0,\ell) \cap H^1_0(0,\ell) \times L^2(0,\ell)$ equipped with the norm

$$
\|(w,v)\|^2 = \int_0^L EIw_{xx}^2 + kw^2 + \rho A v^2 \, dx \tag{85}
$$

Accordingly, the domain of the state operator is

$$
D(\mathcal{A}_{KV}) := \left\{ (w,v) \in \mathbb{Z} |\, v \in H^2(0,\ell) \cap H^1_0(0,\ell),\ EIw + C_d v \in D(\mathcal{A}_0) \right\}, \tag{86}
$$

which is dense on $\mathbb{Z}$. The underlying state space $\mathbb{Z}$ is separable since the spaces $H^2(0,\ell) \cap H^1_0(0,\ell)$ and $L^2(0,\ell)$ are separable. Furthermore, define the linear operators $\mathcal{K}$, $\mathcal{B}(r)$, and the nonlinear operator $\mathcal{F}(\cdot)$ as

$$
\mathcal{K}(w,v) := (0, -\frac{1}{\rho A}(\mu v + kw)), \tag{87}
$$

$$
\mathcal{B}(r)u := (0, \frac{1}{\rho A} b(x;r)u), \tag{88}
$$

$$
\mathcal{F}(w,v) := (0, \frac{\alpha}{\rho A} w^3). \tag{89}
$$

The operator $\mathcal{K}$ is a bounded linear operator on $\mathbb{Z}$. For each r, operator $\mathcal{B}(r)$ is also a bounded operator that maps an input $u \in \mathbb{R}$ to the state space $\mathbb{Z}$. Since the space $H^2(0,\ell)$ is contained in the space of continuous functions over $[0,\ell]$, the the nonlinear term w^3 is in $L^2(0,\ell)$. Thus, the nonlinear operator $\mathcal{F}(\cdot)$ is well-defined on $\mathbb{Z}$. Lastly, define the operator $\mathcal{A} = \mathcal{A}_{KV} + \mathcal{K}$, with the same domain as $\mathcal{A}_{KV}$. With these definition and by setting $\mathfrak{z} = (w, v)$, the state space representation of the railway model (6) is

$$\dot{\mathfrak{z}}(t) = \mathcal{A}\mathfrak{z}(t) + \mathcal{F}(\mathfrak{z}(t)) + \mathcal{B}(r)u(t), \quad \mathfrak{z}(0) = \mathfrak{z}_0 \in D(\mathcal{A}). \tag{90}$$

It is straightforward to show that the operator $\mathcal{A}_0$ is closed, densely-defined, self-adjoint, and positive; it also has a compact resolvent. As a result, the operator $\mathcal{A}_{KV}$ will be a special case of the operator $\mathscr{A}_B$ in [10] with $\alpha = 1$. According to Theorem 1.1 in [10], such operators are generator of an analytic semigroup (also see [5, Sec. 3] for a different approach). Furthermore, the operator $\mathcal{A}_{KV} + \mathcal{K}$ is a bounded perturbation of the operator $\mathcal{A}_{KV}$. By Corollary 3.2.2 in [34], $\mathcal{A}_{KV} + \mathcal{K}$ also generates an analytic semi-group.

The railway track model in [2] neglects the Kelvin-Voigt damping in the beam (i.e. $C_d = 0$), and only includes Kelvin-Voigt damping in the ballast. In this case, the semigroup generated by $\mathcal{A}$ is not analytic. The results of this paper hold true for both models.

To guarantee the existence of a unique solution to the PDE (82), the nonlinear operator $\mathcal{F}(\cdot)$ needs to fall into assumption A2, B1, C1, and C2.

LEMMA 6.1. *The nonlinear operator $\mathcal{F}(\cdot)$ is continuously Fréchet differentiable on $\mathbb{Z}$. This operator is also weakly sequentially continuous in $\mathbb{Z}$.*

Proof. A candidate for Fréchet derivative of $\mathcal{F}$ at $\mathfrak{z}_0$ is the linear operator

$$\mathcal{F}'_{\mathfrak{z}_0}\mathfrak{z} = (0, \frac{3\alpha}{\rho A} w_0^2 w), \quad \forall \mathfrak{z}_0 = (w_0, v_0) \in \mathbb{Z}. \tag{91}$$

To be the unique Fréchet derivative, this linear operator must satisfy

$$\lim_{\|\mathfrak{z}\| \to 0} \frac{\|\mathcal{F}(\mathfrak{z}_0 + \mathfrak{z}) - \mathcal{F}(\mathfrak{z}_o) - \mathcal{F}'_{\mathfrak{z}_0}\mathfrak{z}\|}{\|\mathfrak{z}\|} = 0. \tag{92}$$

Recall the definition of the operator $\mathcal{F}$ and that of norm on the space $\mathbb{Z}$, above limit simplifies to

$$\lim_{\|w\|_{H^2} \to 0} \frac{\|w^3 + 3w^2 w_0\|_{L^2}}{\|w\|_{H^2}} = 0. \tag{93}$$

Notice that functions w and w_0 are in $H^2(0,\ell)$, and thus, continuous on $[0,\ell]$. Use triangle inequality, and Hölder's inequality to obtain

$$\begin{aligned} \|w^3 + 3w^2 w_0\|_{L^2} &\le \|w^3\|_{L^2} + \|3w^2 w_0\|_{L^2} \\ &\le \|w\|_{L^6}^3 + 3\|w\|_{L^8}^2 \|w_0\|_{L^4}. \end{aligned} \tag{94}$$

Apply the Sobolev embedding results $H^2(0,\ell) \hookrightarrow L^4(0,\ell)$ and $H^2(0,\ell) \hookrightarrow L^8(0,\ell)$

$$\|w^3 + 3w^2 w_0\|_{L^2} \le c_1 \|w\|_{H^2}^3 + 3c_2 \|w\|_{H^2}^2 \|w_0\|_{H^2}, \tag{95}$$

for some positive constants c_1 and c_2. As a result, the expression in (93) is bounded above according to

$$\frac{\|w^3 + 3w^2 w_0\|_{L^2}}{\|w\|_{H^2}} \le c_1 \|w\|_{H^2}^2 + 3c_2 \|w_0\|_{H^2} \|w\|_{H^2}. \tag{96}$$

This shows that the limit in (93) holds, and the operator $\mathcal{F}(\cdot)$ is indeed Fréchet differentiable.

Now, select $\mathfrak{z}_1 = (w_1, v_1)$, $\mathfrak{z}_2 = (w_2, v_2)$, and $\tilde{\mathfrak{z}} = (\tilde{w}, \tilde{v})$ as generic elements of $\mathbb{Z}$. From Lemma 6.1, the Fréchet derivative of $\mathcal{F}(\cdot)$ at $\mathfrak{z}_2 - \mathfrak{z}_1$ is

$$\mathcal{F}'_{\mathfrak{z}_2 - \mathfrak{z}_1} \tilde{\mathfrak{z}} = (0, \frac{3\alpha}{\rho A}(w_2 - w_1)^2 \tilde{w}).$$

Now, take the norm of $\mathcal{F}'_{\mathfrak{z}_2 - \mathfrak{z}_1} \tilde{\mathfrak{z}}$, and use Hölder inequality to obtain

$$\begin{aligned} \left\| \mathcal{F}'_{\mathfrak{z}_2 - \mathfrak{z}_1} \tilde{\mathfrak{z}} \right\| &= \frac{3\alpha}{\sqrt{\rho A}} \left(\int_0^L (w_2 - w_1)^4 \tilde{w}^2 \, dx \right)^{\frac{1}{2}} \\ &\le \frac{3\alpha}{\sqrt{\rho A}} \left(\int_0^L (w_2 - w_1)^8 dx \right)^{\frac{2}{8}} \left(\int_0^L \tilde{w}^4 dx \right)^{\frac{1}{4}}. \end{aligned} \tag{97}$$

Apply the Sobolev embedding results $H^2(0,\ell) \hookrightarrow L^4(0,\ell)$ and $H^2(0,\ell) \hookrightarrow L^8(0,\ell)$; for some positive number c_e, this yields

$$\begin{aligned} \left\| \mathcal{F}'_{\mathfrak{z}_2 - \mathfrak{z}_1} \tilde{\mathfrak{z}} \right\| &\le \frac{3\alpha}{\sqrt{\rho A}} c_e \left(\int_0^L (w_{2,xx} - w_{1,xx})^2 \, dx \right) \left(\int_0^L \tilde{w}_{xx}^2 dx \right)^{\frac{1}{2}} \\ &\le \frac{3\alpha}{\sqrt{\rho A}} c_e \left\| \mathfrak{z}_2 - \mathfrak{z}_1 \right\|^2 \left\| \tilde{\mathfrak{z}} \right\|. \end{aligned} \tag{98}$$

The last inequality indicates that the operator norm of $\mathcal{F}'_{\mathfrak{z}}$ continuously depends on $\mathfrak{z}$.

To show that the nonlinear operator $\mathcal{F}(\cdot)$ is weakly sequentially continuous on $\mathbb{Z}$, consider a sequence $\mathfrak{z}_n = (w_n, v_n)$ weakly converging to some element $\mathfrak{z} = (w, v)$ in $\mathbb{Z}$. The nonlinear operator $\mathcal{F}(\cdot)$ maps this sequence to

$$\mathcal{F}(w_n, v_n) = (0, \frac{\alpha}{\rho A} w_n^3). \tag{99}$$

The space $H^2(0,\ell) \cap H_0^1(0,\ell)$ is compactly embedded in $L^6(0,\ell)$ by Rellich-Kondrachov compact embedding theorem [1, Ch. 6]. The sequence w_n is weakly convergent in $H^2(0,\ell) \cap H_0^1(0,\ell)$, and thus, it is strongly convergent to some element w_0 in $L^6(0,\ell)$. A weak limit is unique; thus, $w_0 \in H^2(0,\ell) \cap L^2(0,\ell)$, and $w = w_0$. Also, the sequence w_n^3 strongly converges to w_0^3 in $L^2(0,\ell)$. Therefore, the sequence $\mathcal{F}(w_n, v_n)$ strongly converges to $\mathcal{F}(w, v)$ in $\mathbb{Z}$; which also implies weak convergence. This proves that the nonlinear operator $\mathcal{F}(\cdot)$ is weakly sequentially continuous. ∎

The previous lemma ensures that the nonlinear operator $\mathcal{F}(\cdot)$ satisfies assumption A2. By Theorem 3.1, for control inputs $u \in L^p(0,T)$, $1 < p < \infty$, the existence of a unique local mild solution is guaranteed.

To address the optimization problem P for the railway track model, assumption B and C need to be satisfied. In Lemma 5.6, it was shown that assumption B3 will hold for the particular choice of the cost function in assumption C5. As a result, the existence of an optimal pair (u^o, r^o) together with an optimal trajectory $\mathfrak{z}^o$ follows from Theorem 4.1.

Accordingly, using Theorem 5.7, the optimal pair (u^o, r^o) satisfies the equation (74). In order to characterize the optimum (74) some adjoint operators needs to be calculated. Calculation of the operator $\mathcal{A}^*$ is straightforward; it is

$$\mathcal{A}^*(f,g) = \left(-g, \frac{1}{\rho A}\mathcal{A}_0(EIf - C_d g) + \frac{k}{\rho A}f - \frac{\mu}{\rho A}g\right), \tag{100}$$

for all (f, g) in the domain

$$D(\mathcal{A}^*) = \left\{(f,g) \in \mathbb{Z} |\, g \in H^2(0,\ell) \cap H_0^1(0,\ell),\ EIf - C_d g \in D(\mathcal{A}_0)\right\}. \tag{101}$$

Let $\mathfrak{z}^o(t) = (w^o, v^o)$ be the optimal trajectory evaluated at time $t \in [0,T]$. To calculate the adjoint of the operator $\mathcal{F}'_{\mathfrak{z}^o(t)}$ for every $t \in [0,T]$ on $\mathbb{Z}$, take the inner product $\mathcal{F}'_{\mathfrak{z}^o(t)}(w, v)$ with $(f, g) \in \mathbb{Z}$; that is

$$\left\langle \mathcal{F}'_{\mathfrak{z}^o(t)}(w,v), (f,g)\right\rangle = \int_0^\ell \frac{3\alpha}{\rho A}(w^o(x))^2 w(x) g(x)\, dx. \tag{102}$$

To calculate the adjoint, for any $g \in L^2(0,\ell)$, consider the function $\zeta(x) \in H^2(0,\ell) \cap H_0^1(0,\ell)$ satisfying the differential equation

$$\begin{aligned} &\zeta_{xxxx}(x) + \zeta(x) = \frac{3\alpha}{\rho A}(w^o(x))^2 g(x), \\ &\zeta(0) = \zeta(\ell) = 0, \\ &\zeta_{xx}(0) = \zeta_{xx}(\ell) = 0. \end{aligned} \tag{103}$$

An explicit solution $\zeta(x)$ to (103) can be calculated using a Green's function:

$$\begin{aligned} \zeta(x) &= \frac{3\alpha}{\rho A}\int_0^\ell G(x,y)(w^o(y))^2 g(y)\, dy, \\ G(x,y) &= \frac{1}{6\ell}\begin{cases} (2\ell^2 y - 3\ell y^2 + y^3)x + (y-\ell)x^3, & x \le y \\ (y^3 - \ell^2 y)x + yx^3, & x > y\,. \end{cases} \end{aligned} \tag{104}$$

With this calculation, for any $(w, v) \in \mathbb{Z}$,

$$\begin{aligned} \langle (w,v), (\zeta, 0)\rangle &= \int_0^\ell EIw''\zeta'' + \rho A w\zeta\, dx \\ &= [\zeta_{xx} w_x]_0^\ell - [\zeta_{xxx} w]_0^\ell + \int_0^\ell (\zeta_{xxxx}(x) + \zeta(x)) w(x)\, dx \\ &= \int_0^\ell \frac{3\alpha}{\rho A}(w^o(x))^2 w(x) g(x)\, dx. \end{aligned} \tag{105}$$

Comparing this equation to (102); the adjoint of $\mathcal{F}'_{\mathfrak{z}^o(t)}$ is defined by

$$\mathcal{F}'^*_{\mathfrak{z}^o(t)}(f,g) = (\zeta, 0). \tag{106}$$

The adjoint of the operator $\mathcal{B}(r)$ for every $(f, g) \in \mathbb{Z}$ is

$$\mathcal{B}^*(r)(f,g) = \rho A \int_0^\ell b(x;r) g\, dx. \tag{107}$$

Let $(q_1, q_2) \in \mathbb{Z}$, set $\mathcal{Q}(w, v) = (q_1 w, q_2 v)$ and $\mathcal{R} = 1$ in the cost function of assumption C5, and denote $b_r(x; r)$ to be the derivative of $b(x; r)$ with respect to r. In conclusion, the following set of equations yields an optimizer for every initial condition $\mathfrak{z}_0 = (w_0, v_0) \in \mathbb{Z}$:

$$\begin{cases} \rho A w^o_{tt} + (EI w^o_{xx} + C_d w^o_{txx})_{xx} + \mu w^o_t + k w^o + \alpha (w^o)^3 = b(x; r^o) u^o(t), \\ w^o(0,t) = w^o(\ell, t) = 0, \\ EI w^o_{xx}(0,t) + C_d w^o_{txx}(0,t) = EI w^o_{xx}(\ell,t) + C_d w^o_{txx}(\ell,t) = 0, \\ w^o(x,0) = w_0(x),\ w^o_t(x,0) = v_0(x), \\ \rho A f^o_t - \rho A g^o + 3\alpha \int_0^\ell G(x,y)(w^o(y))^2 g^o(y)\, dy = -\rho A q_1(x) w^o, \\ \rho A g^o_t + (EI f^o_{xx} - C_d g^o_{xx})_{xx} - \mu g^o + k f^o = -\rho A q_2(x) w^o_t, \\ f^o(0,t) = f^o(\ell,t) = 0,\ g^o(0,t) = g^o(\ell,t) = 0, \\ EI f^o_{xx}(0,t) - C_d g^o_{xx}(0,t) = EI f^o_{xx}(\ell,t) - C_d g^o_{xx}(\ell,t) = 0, \\ f^o(x,T) = 0,\ g^o(x,T) = 0, \\ u^o(t) = -\rho A \int_0^\ell b(x; r^o) g^o(x,t)\, dx, \\ \int_0^T \int_0^\ell u^o(t) b_r(x; r^o) g^o(x,t)\, dx dt = 0. \end{cases} \tag{108}$$

7. Conclusions. A semi-linear infinite dimensional system was considered in this paper where the optimal controller design involves both the controlled input and the actuator design. It was shown that the existence of an optimal control input together with an optimal actuator design are guaranteed under some assumptions. Moreover, first-order necessary optimality conditions were obtained. As a novel application of the theory, a nonlinear railway track model was studied.

Current work is concerned with developing numerical methods for solution of the optimality equations and also the consideration of a wider class of nonlinearities.

Appendix A. Proof of Proposition 5.2. For $\mathfrak{z}_0 \in \mathbb{Z}$ and $r \in K_{ad}$, consider $\mathfrak{z}_1(t)$ and $\mathfrak{z}_2(t)$ as the mild solutions to the IVP (1) corresponding to the inputs $u_1(t)$ and $u_2(t)$, respectively. The inputs are in a ball of radius R contained in $L^p(0, T; \mathbb{U})$, $1 < p < \infty$; consequently, by Corollary 3.2 and assumption A3, the states $\mathfrak{z}_1(t)$ and $\mathfrak{z}_2(t)$ are contained in a ball of radius

$$\delta = c_T(\|\mathfrak{z}_0\| + T^{(p-1)/p} M_{\mathcal{B}} R). \tag{109}$$

From (2), it follows that

$$\begin{aligned} \mathfrak{z}_2(t) - \mathfrak{z}_1(t) = &\int_0^t \mathcal{T}(t-s)\left(\mathcal{F}(\mathfrak{z}_2(s)) - \mathcal{F}(\mathfrak{z}_1(s))\right) ds \\ &+ \int_0^t \mathcal{T}(t-s)\mathcal{B}(r)\left(u_2(s) - u_1(s)\right) ds. \end{aligned} \tag{110}$$

Recall that $\mathcal{T}(t)$ satisfies $\|\mathcal{T}(t)\| \le M \max(1, e^{\omega T})$ for all $t \in [0, T]$ and some number $M > 0$ and ω. Also, remember that the operator $\mathcal{F}(\cdot)$ is locally Lipschitz continuous, and $\mathcal{B}(r)$ is uniformly bounded in $\mathbb{Z}$ for all $r \in K_{ad}$. Taking the norm in $\mathbb{Z}$ of both sides of this equation yields

$$\begin{aligned}\|\mathfrak{z}_2(t)-\mathfrak{z}_1(t)\| \leq & M\max(1,e^{\omega T})L_{\mathcal{F}\delta}\int_0^t \|\mathfrak{z}_2(s)-\mathfrak{z}_1(s)\|\,ds \\ & + M\max(1,e^{\omega T})T^{(p-1)/p}M_{\mathcal{B}}\|u_2-u_1\|_p .\end{aligned} \tag{111}$$

Define the constant L_u as

$$L_u = T^{1/p}e^{M\max(1,e^{\omega T})L_{\mathcal{F}\delta}T}M\max(1,e^{\omega T})T^{(p-1)/p}M_{\mathcal{B}}. \tag{112}$$

By Grönwall's lemma [42, Thm. 1.4.1], it follows that

$$\|\mathfrak{z}_2-\mathfrak{z}_1\|_p \leq L_u\|u_2-u_1\|_p . \tag{113}$$

This is in fact the inequality (28).

Similarly, consider the mild solutions $\mathfrak{z}_1(t)$ and $\mathfrak{z}_2(t)$ corresponding to the actuator locations r_1 and r_2 and the fixed initial condition $\mathfrak{z}_0$ and control input u. Use local Lipschitz continuity of $\mathcal{F}(\cdot)$ and growth condition on semigroup $\mathcal{T}(t)$ and obtain

$$\begin{aligned}\|\mathfrak{z}_2(t)-\mathfrak{z}_1(t)\| \leq & M\max(1,e^{\omega T})L_{\mathcal{F}\delta}\int_0^t \|\mathfrak{z}_2(s)-\mathfrak{z}_1(s)\|\,ds \\ & + M\max(1,e^{\omega T})T^{(p-1)/p}\|u\|_p\|\mathcal{B}(r_2)-\mathcal{B}(r_1)\| .\end{aligned} \tag{114}$$

Assumption C3 implies that the control operator $\mathcal{B}(r)$ is Lipschitz continuous with respect to r in operator norm topology, i.e., there exist a positive constant $L_{\mathcal{B}}$ such that

$$\|\mathcal{B}(r_2)-\mathcal{B}(r_1)\| \leq L_{\mathcal{B}}\|r_2-r_1\|_{\mathbb{K}}, \tag{115}$$

for all r_1 and r_2 in K_{ad}. Accordingly, the inequality (114) can be re-written as

$$\begin{aligned}\|\mathfrak{z}_2(t)-\mathfrak{z}_1(t)\| \leq & M\max(1,e^{\omega T})L_{\mathcal{F}\delta}\int_0^t \|\mathfrak{z}_2(s)-\mathfrak{z}_1(s)\|\,ds \\ & + M\max(1,e^{\omega T})T^{(p-1)/p}\|u\|_p L_{\mathcal{B}}\|r_2-r_1\|_{\mathbb{K}} .\end{aligned} \tag{116}$$

Denote the constant L_r by

$$L_r = T^{1/p}e^{M\max(1,e^{\omega T})L_{\mathcal{F}\delta}T}M\max(1,e^{\omega T})T^{(p-1)/p}RL_{\mathcal{B}}. \tag{117}$$

Use Grönwall's lemma [42, Thm. 1.4.1], and apply $\|u\|_p \leq R$ to derive

$$\|\mathfrak{z}_2-\mathfrak{z}_1\|_p \leq L_r\|r_2-r_1\|_{\mathbb{K}} . \tag{118}$$

This is in fact the inequality (29).

Appendix B. Proof of Proposition 5.5. Assume that S'_{r^o} is the Fréchet derivative of the control-to-state map $\mathcal{S}(u;r,\mathfrak{z}_0)$ with respect to r at r^o, then it needs to satisfy

$$\lim_{\|\tilde{r}\|_{\mathbb{K}}\to 0}\frac{\|\mathcal{S}(u;\tilde{r}+r^o,\mathfrak{z}_0)-\mathcal{S}(u;r^o,\mathfrak{z}_0)-\mathcal{S}'_{r^o}\tilde{r}\|_p}{\|\tilde{r}\|_{\mathbb{K}}} = 0. \tag{119}$$

Denote by $\mathfrak{y}_p = \mathcal{S}(u;\tilde{r}+r^o,\mathfrak{z}_0)$ the mild solution to the IVP

$$\dot{\mathfrak{y}}_p(t) = \mathcal{A}\mathfrak{y}_p(t) + \mathcal{F}(\mathfrak{y}_p(t)) + \mathcal{B}(\tilde{r}+r^o)u(t), \quad \mathfrak{y}_p(0) = \mathfrak{y}_0. \tag{120}$$

The state $\mathfrak{y}^o = \mathcal{S}(u;r^o,\mathfrak{z}_0)$ is the mild solution of the IVP

$$\dot{\mathfrak{y}}^o(t) = \mathcal{A}\mathfrak{y}^o(t) + \mathcal{F}(\mathfrak{y}^o(t)) + \mathcal{B}(r^o)u(t), \quad \mathfrak{y}^o(0) = \mathfrak{y}_0. \tag{121}$$

Define $\mathfrak{y}_r = \mathfrak{y}_p - \mathfrak{y}^o - \tilde{\mathfrak{y}}$, notice that

$$\mathfrak{y}_r = \mathcal{S}(u;\tilde{r}+r^o,\mathfrak{z}_0) - \mathcal{S}(u;r^o,\mathfrak{z}_0) - \mathcal{S}'_{r^o}\tilde{r}. \tag{122}$$

Subtract the equations (121) and (56) from (120), incorporate the Fréchet derivative of the operators $\mathcal{F}(\cdot)$ and $\mathcal{B}(\cdot)$ (see assumption C1 and C3), obtain

$$\dot{\mathfrak{y}}_r(t) = (\mathcal{A}+\mathcal{F}'_{\mathfrak{y}^o(t)})\mathfrak{y}_r(t) + \mathfrak{r}_{\mathcal{F}}(t) + \mathfrak{r}_{\mathcal{B}}(\tilde{r})u(t), \quad \mathfrak{y}_r(0) = 0. \tag{123}$$

in which the remainders $\mathfrak{r}_{\mathcal{F}}(t) \in \mathbb{Z}$ and $\mathfrak{r}_{\mathcal{B}}(\tilde{r}) \in \mathcal{L}(\mathbb{U},\mathbb{Z})$ satisfy

$$\lim_{\|\mathfrak{y}_p(t)-\mathfrak{y}^o(t)\|\to 0} \frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\mathfrak{y}_p(t)-\mathfrak{y}^o(t)\|} = 0, \quad \forall t\in[0,T], \tag{124a}$$

$$\lim_{\|\tilde{r}\|_{\mathbb{K}}\to 0} \frac{\|\mathfrak{r}_{\mathcal{B}}(\tilde{r})\|}{\|\tilde{r}\|_{\mathbb{K}}} = 0, \tag{124b}$$

In the evolution equation (19), the operator $\mathcal{A}$ is perturbed by the time-dependent operator $\mathcal{F}'_{\mathfrak{y}^o(t)}$. According to Lemma 5.3, the mild solution of the evolution equation (123) is described by an evolution operator $\mathcal{U}(t,s)$. Let M be an upper bound for the operator norm of $\mathcal{U}(t,s)$ over $0\le t\le s\le T$, then the mild solution to (48) satisfies the estimate

$$\|\mathfrak{y}_r\|_p \le MT^{(p-1)/p}\left(\int_0^T \|\mathfrak{r}_{\mathcal{F}}(t)+\mathfrak{r}_{\mathcal{B}}(\tilde{r})u(t)\|^p \, dt\right)^{1/p}. \tag{125}$$

Divide both side of this inequality by $\|\tilde{r}\|_{\mathbb{K}}$ and use the triangle inequity to rewrite it as

$$\frac{\|\mathfrak{y}_r\|_p}{\|\tilde{r}\|_{\mathbb{K}}} \le MT^{(p-1)/p}\left(\int_0^T \frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|^p}{\|\tilde{r}\|_{\mathbb{K}}^p}\, dt\right)^{1/p} + MT^{(p-1)/p}\frac{\|\mathfrak{r}_{\mathcal{B}}(\tilde{r})\|}{\|\tilde{r}\|_{\mathbb{K}}}\|u\|_p. \tag{126}$$

From Proposition 5.2, the mapping $\mathcal{S}(u;\cdot,\mathfrak{z}_0)$ is Lipschitz continuous in r; this can be used to write

$$\begin{aligned}\lim_{\|\tilde{r}\|_{\mathbb{K}}\to 0}\frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\tilde{r}\|_{\mathbb{K}}} &= \lim_{\|\tilde{r}\|_{\mathbb{K}}\to 0}\frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\mathfrak{z}_p(t)-\mathfrak{z}^o(t)\|}\frac{\|\mathfrak{z}_p(t)-\mathfrak{z}^o(t)\|}{\|\tilde{r}\|_{\mathbb{K}}}\\ &= L_r \lim_{\|\tilde{r}\|_{\mathbb{K}}\to 0}\frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\mathfrak{z}_p(t)-\mathfrak{z}^o(t)\|} = 0\end{aligned} \tag{127}$$

pointwise for every $t\in[0,T]$. In addition, similar to inequality (53) in the proof of Proposition 5.4, the following uniform upper bound holds

$$\frac{\|\mathfrak{r}_{\mathcal{F}}(t)\|}{\|\tilde{r}\|_{\mathbb{K}}} \le \left(L_{\mathcal{F}\delta} + \sup_{\|\mathfrak{z}^o(t)\|\le\delta}\left\|\mathcal{F}'_{\mathfrak{z}^o(t)}\right\|\right)L_r. \tag{128}$$

By the bounded convergence theorem, these two statements ensure that the integral in (126) converges to zero as $\tilde{r}$ tends to zero. Therefore, the limits in (124) result in

$$\lim_{\|\tilde{r}\|_{\mathbb{K}}\to 0} \frac{\|\mathfrak{z}_r\|_p}{\|\tilde{r}\|_{\mathbb{K}}} = 0. \tag{129}$$

Recall (122), the previous limit implies (119).

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