To link to this article : DOI : 1080/11663081.2015.1011489
URL : http://dx.doi.org/10.1080/11663081.2015.1011489

To cite this version : Balbiani, Philippe *Putting right the wording and the proof of the Truth Lemma for APAL.* (2015) Journal of Applied Non-Classical Logics, vol. 25. pp. 2-19. ISSN 1166-3081

RESEARCH ARTICLE

Putting right the wording and the proof of the Truth Lemma for *APAL*

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(*Received 20 December 2013; accepted 5 January 2015*)

$APAL$ is an extension of public announcement logic. It is based on a modal operator that expresses what is true after any arbitrary announcement. An incorrect Truth Lemma has been stated and 'demonstrated' in Balbiani et al. (2008). In this paper, we put right the wording and the proof of the Truth Lemma for $APAL$.

Keywords: dynamic epistemic logic; public announcements; arbitrary announcements; axiomatisation; completeness

1. Introduction

Public announcement logic (PAL) is an extension of epistemic logic with modal operators that express what is true after such and such announcement Plaza (2007). The modal operator $[\phi]$ means 'if ϕ holds, then after the announcement of $\phi, \ldots$', whereas the dual modal operator $\langle\phi\rangle$ means 'ϕ holds and after the announcement of $\phi, \ldots$'. Within the context of PAL, it becomes possible to reason about information flow. Formally, in PAL, a formula such as $\langle\phi\rangle K_a\psi$ stands for 'ϕ holds and after the announcement of ϕ, agent a knows that ψ holds'. Membership in PAL is known to be $PSPACE$-complete (Lutz, 2006). Further examples of announcement-based extensions of epistemic logic abound (for details, see Van Ditmarsch, van der Hoek, & Kooi, 2007).

Arbitrary public announcement logic ($APAL$) is an extension of PAL with a modal operator that expresses what is true after any arbitrary announcement (Balbiani et al., 2008). The modal operator $\Box$ means 'after every announcement, ...' whereas the dual modal operator $\Diamond$ 'after some announcement, ...'. Within the context of $APAL$, formulas such as $\Box\phi$ and $\Diamond\phi$ are semantically equivalent, respectively, to the infinite conjunction of all formulas of the form $[\psi]\phi$ in which ψ is a purely epistemic formula and to the infinite disjunction of all formulas of the form $\langle\psi\rangle\phi$ in which ψ is a purely epistemic formula. Membership in $APAL$ is known to be undecidable (French & van Ditmarsch, 2008).

PAL is completely axiomatised by the ordinary laws of epistemic logic plus the so-called 'reduction axioms' which allow us to eliminate modal operators of announcement, one by one, from any PAL formula. To completely axiomatise $APAL$ is more difficult. Nevertheless, an axiomatic system has been considered (see Balbiani et al., 2008, Table 2). A peculiar derivation rule of this axiomatic system is the derivation rule $R(\Box)$ concerning the dynamic modal operator that expresses what is true after any arbitrary announcement (see Balbiani et al., 2008, Definition 4.6). Such rules have been called 'non-structural rules' and 'context dependent rules'(Goranko, 1998; Marx & Venema,1997). Whether they can be replaced by more orthodox rules is a research subject in itself.

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Presented in Balbiani et al. (2008, Sections 4.4–4.5), the proof of completeness for $APAL$ is based on the traditional tools and techniques of the canonical model construction: the Lindenbaum Lemma and the Truth Lemma. The main effect of the infinitary variant $R^{\omega}(\Box)$ of the derivation rule $R(\Box)$ considered in Balbiani et al. (2008, Section 4.3) is that it makes the canonical model (consisting of all maximal consistent sets of formulas closed under $R^{\omega}(\Box)$) standard for the modal operators of arbitrary announcement. Concerning the Lindenbaum Lemma considered in Balbiani et al., it happens that its wording and its proof are correct. In this introduction, let us see why the same cannot be said for the wording and the proof of the Truth Lemma considered in Balbiani et al.

The Truth Lemma considered in Balbiani et al. (2008, p. 327) can be worded in the following way: (A) for all non-negative integers m, for all formulas $\psi_1, \ldots, \psi_m$ and for all possible worlds x in the canonical model $\mathcal{M}_c$ of $APAL$, if $\psi_1 \in x, \ldots, [\psi_1] \ldots [\psi_{m-1}]\psi_m \in x$, then $x \in \|\phi\|^{\mathcal{M}_{c|\psi_1 \ldots|\psi_m}}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in x$. In the property (A), the expression '$\mathcal{M}_{c|\psi_1 \ldots|\psi_m}$' denotes the restriction of $\mathcal{M}_c$ determined by the formulas $\psi_1, \ldots, \psi_m$. The problem is that the assumptions $\psi_1 \in x, \ldots, [\psi_1] \ldots [\psi_{k-1}]\psi_k \in x$ are not strong enough to justify the existence of this restriction. It follows that the property (A) is a piece of nonsense when the restriction of $\mathcal{M}_c$ determined by the formulas $\psi_1, \ldots, \psi_m$ does not exist.

Regardless of this first mistake in the wording of the Truth Lemma, a second mistake concerns the proof of the Truth Lemma. In the case $\phi = K_a\phi'$ of the proof of (A) that has been done by induction on ϕ, the assumption $x \notin \|K_a\phi'\|^{\mathcal{M}_{c|\psi_1 \ldots|\psi_m}}$ considered in Balbiani et al. (2008), p. 328) only implies the existence of a maximal consistent theory y such that $y \in \|\psi_1\|^{\mathcal{M}_c}, \ldots, y \in \|\psi_m\|^{\mathcal{M}_{c|\psi_1 \ldots|\psi_{m-1}}}$, $xR_a y$ and $y \notin \|\phi'\|^{\mathcal{M}_{c|\psi_1 \ldots|\psi_m}}$. The problem is that the facts $y \in \|\psi_1\|^{\mathcal{M}_c}, \ldots, y \in \|\psi_m\|^{\mathcal{M}_{c|\psi_1 \ldots|\psi_{m-1}}}$ are not strong enough to imply that $\psi_1 \in y, \ldots, [\psi_1] \ldots [\psi_{m-1}]\psi_m \in y$. And the facts $\psi_1 \in y, \ldots, [\psi_1] \ldots [\psi_{m-1}]\psi_m \in y$ are the ones we need in order to infer $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \notin y$ from $y \notin \|\phi'\|^{\mathcal{M}_{c|\psi_1 \ldots|\psi_m}}$.

In this paper, we put right the Truth Lemma for *APAL* by proving a new property worded in the following way: (B) for all formulas $\psi_1, \ldots, \psi_m$, if $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi) \leq i$, then for all possible worlds x in the canonical model $\mathcal{M}_c$ of $APAL$, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in x$. In Section 6, the proof of (B) will be done by induction on (i, m, ϕ). This means that a well-founded strict partial order between triples such as (i, m, ϕ) will be used to provide the induction hypothesis. We believe that this new wording of the Truth Lemma for $APAL$ and its proof by induction on (i, m, ϕ) can be successfully applied to other $APAL$-like dynamic epistemic logics as well (Agotnes, Balbiani, van Ditmarsch, & Seban, 2010; Balbiani, van Ditmarsch, & Kudinov, 2013; Kuijer, 2014).

The breakdown of this paper is as follows. Section 2 defines the syntax and Section 3 introduces the semantics. In Section 4, an axiomatic system is given. To carry out the proof of its completeness in Section 6, we need to learn about theories in Section 5.

2. Syntax

Let Atm be a countably infinite set of atoms (with typical members denoted p, q, etc.) and Agt be a finite set of agents (with typical members denoted a, b, etc.). The set $\mathcal{L}_{apal}$ of all formulas (with typical members denoted ϕ, ψ, etc.) is inductively defined as follows:

- $\phi ::= p \mid \neg\phi \mid (\phi \wedge \psi) \mid K_a\phi \mid [\phi]\psi \mid \Box\phi$,

where p is an atom and a is an agent. We define the other Boolean constructs as usual. The formulas $\hat{K}_a\phi$, $\langle\phi\rangle\psi$ and $\Diamond\phi$ are obtained as abbreviations: $\hat{K}_a\phi$ for $\neg K_a\neg\phi$, $\langle\phi\rangle\psi$

for $\neg[\phi]\neg\psi$ and $\Diamond\phi$ for $\neg\Box\neg\phi$. We adopt the standard rules for omission of parentheses. A derivation rule is a pair consisting of a set of formulas and a formula. The set of all subformulas of a formula ϕ, represented by $Sub(\phi)$, is the set of formulas inductively defined as follows:

- $Sub(p) = \{p\}$,
- $Sub(\neg\phi) = \{\neg\phi\} \cup Sub(\phi)$,
- $Sub(\phi \wedge \psi) = \{\phi \wedge \psi\} \cup Sub(\phi) \cup Sub(\psi)$,
- $Sub(K_a\phi) = \{K_a\phi\} \cup Sub(\phi)$,
- $Sub([\phi]\psi) = \{[\phi]\psi\} \cup Sub(\phi) \cup Sub(\psi)$,
- $Sub(\Box\phi) = \{\Box\phi\} \cup Sub(\phi)$.

We will say that a formula ϕ is $\Box$-free iff $Sub(\phi)$ contains no formula of the form $\Box\psi$. A formula ϕ is said to be $[\cdot]$-free iff $Sub(\phi)$ contains no formula of the form $[\psi]\chi$. We will say that a formula ϕ is epistemic iff ϕ is both $\Box$-free and $[\cdot]$-free. The set $\mathcal{L}_{pal}$ considered in Balbiani et al. (2008) is nothing but the set of all $\Box$-free formulas. As for the set $\mathcal{L}_{el}$ considered in Balbiani et al., it is nothing but the set of all epistemic formulas. The size of a formula ϕ, represented by $Size(\phi)$, is the non-negative integer inductively defined as follows:

- $Size(p) = 2$,
- $Size(\neg\phi) = Size(\phi) + 1$,
- $Size(\phi \wedge \psi) = Size(\phi) + Size(\psi) + 3$,
- $Size(K_a\phi) = Size(\phi) + 3$,
- $Size([\phi]\psi) = Size(\phi) + Size(\psi) + 2$,
- $Size(\Box\phi) = Size(\phi) + 1$.

We define the binary relation $<^{Size}$ between formulas in the following way:

- $\phi <^{Size} \psi$ iff $Size(\phi) < Size(\psi)$.

Proposition 1. *$<^{Size}$ is a well-founded strict partial order between formulas.*

Proof. By the well-foundedness of the standard linear order between non-negative integers. □

The $\Box$-depth of a formula ϕ, represented by $d_\Box(\phi)$, is the non-negative integer inductively defined as follows:

- $d_\Box(p) = 0$,
- $d_\Box(\neg\phi) = d_\Box(\phi)$,
- $d_\Box(\phi \wedge \psi) = \max\{d_\Box(\phi), d_\Box(\psi)\}$,
- $d_\Box(K_a\phi) = d_\Box(\phi)$,
- $d_\Box([\phi]\psi) = \max\{d_\Box(\phi), d_\Box(\psi)\}$,
- $d_\Box(\Box\phi) = d_\Box(\phi) + 1$.

We define the binary relation $<_{d_\Box}$ between formulas in the following way:

- $\phi <_{d_\Box} \psi$ iff $d_\Box(\phi) < d_\Box(\psi)$.

Proposition 2. *$<_{d_\Box}$ is a well-founded strict partial order between formulas.*

Proof. By the well-foundedness of the standard linear order between non-negative integers.

The binary relation $<^{Size}_{d_\Box}$ between formulas is defined in the following way:

- $\phi <^{Size}_{d_\square} \psi$ iff either $d_\square(\phi) < d_\square(\psi)$, or $d_\square(\phi) = d_\square(\psi)$ and $Size(\phi) < Size(\psi)$.

Proposition 3. *$<^{Size}_{d_\square}$ is a well-founded strict partial order between formulas.*

Proof. By Propositions 1 and 2. □

Proposition 4. *Let ϕ be a formula. If ϕ is epistemic, then $d_\square(\phi) = 0$.*

Proof. By $<^{Size}$-induction on ϕ. □

Lemma 5. *Let ϕ, ψ be formulas and $a \in Agt$.*

- $\phi <^{Size}_{d_\square} \neg\phi$,
- $\phi <^{Size}_{d_\square} \phi \wedge \psi$ *and* $\psi <^{Size}_{d_\square} \phi \wedge \psi$,
- $\phi <^{Size}_{d_\square} K_a\phi$,
- $\phi <^{Size}_{d_\square} [\phi]\psi$ *and* $\psi <^{Size}_{d_\square} [\phi]\psi$,
- *if ψ is epistemic, then* $[\psi]\phi <^{Size}_{d_\square} \square\phi$.

Proof. Leaving to the reader the task of proving the first items, we only pay attention to the last one. Suppose ψ is epistemic and not $[\psi]\phi <^{Size}_{d_\square} \square\phi$. Hence, $d_\square([\psi]\phi) \geq d_\square(\square\phi)$. Thus, $\max\{d_\square(\psi), d_\square(\phi)\} \geq d_\square(\phi)+1$. Since ψ is epistemic, by Proposition 4, $d_\square(\psi) = 0$. Since $\max\{d_\square(\psi), d_\square(\phi)\} \geq d_\square(\phi) + 1$, $d_\square(\phi) \geq d_\square(\phi) + 1$: a contradiction. □

Now, let us consider a new atom denoted $\sharp$. The set NF of all necessity forms (with typical members denoted $\varphi(\sharp)$, $\varphi'(\sharp)$, etc.) is inductively defined as follows:

- $\varphi(\sharp) ::= \sharp \mid \phi \to \varphi(\sharp) \mid K_a\varphi(\sharp) \mid [\phi]\varphi(\sharp)$,

where ϕ is a formula and a is an agent. The size of a necessity form $\varphi(\sharp)$, represented by $Size(\varphi(\sharp))$, is a non-negative integer that can be inductively defined in the same way as $Size(\phi)$ of a formula ϕ. A well-founded strict partial order $<^{Size}$ between necessity forms can be defined in the same way as the well-founded strict partial order $<^{Size}$ between formulas.

Lemma 6. *Let $\varphi(\sharp) \in NF$, ϕ be a formula and a be an agent.*

- $\varphi(\sharp) <^{Size} \phi \to \varphi(\sharp)$,
- $\varphi(\sharp) <^{Size} K_a\varphi(\sharp)$,
- $\varphi(\sharp) <^{Size} [\phi]\varphi(\sharp)$.

Proof. Left to the reader. □

It is well worth noting that in each necessity form $\varphi(\sharp)$, $\sharp$ has a unique occurrence. The result of the replacement of $\sharp$ in its place in $\varphi(\sharp)$ with a formula ϕ is a formula which will be denoted as $rep(\varphi(\sharp), \phi)$. It is inductively defined as follows:

- $rep(\sharp, \phi) = \phi$,
- $rep(\psi \to \varphi(\sharp), \phi) = \psi \to rep(\varphi(\sharp), \phi)$,
- $rep(K_a\varphi(\sharp), \phi) = K_a rep(\varphi(\sharp), \phi)$,
- $rep([\psi]\varphi(\sharp), \phi) = [\psi]rep(\varphi(\sharp), \phi)$.

3. Semantics

A model is an ordered triple $\mathcal{M} = (W, R, V)$ where W is a non-empty set of possible worlds (with typical members denoted x, y, etc.), R is a function assigning to each agent a

an equivalence relation $R(a)$ on W and V is a function assigning to each atom p a subset $V(p)$ of W. For all agents a and for all $S \subseteq W$, let $[R_a]S = \{x\colon \text{for all } y \in W, \text{if } x R_a y, \text{then } y \in S\}$ and $\langle R_a \rangle S = \{x\colon \text{there exists } y \in W \text{ such that } x R_a y \text{ and } y \in S\}$. For all $S \subseteq W$, if $S \neq \emptyset$, then $\mathcal{M}(S)$ will denote the model obtained as the restriction of $\mathcal{M}$ to the possible worlds in S. Since this restriction is not a model when $S = \emptyset$, the expression $\mathcal{M}(S)$ is a piece of nonsense when $S = \emptyset$. The truth-set of a formula ϕ in a model $\mathcal{M} = (W, R, V)$, represented by $\|\phi\|^{\mathcal{M}}$, is inductively defined as follows:

- $\|p\|^{\mathcal{M}} = V(p)$,
- $\|\neg\phi\|^{\mathcal{M}} = W \setminus \|\phi\|^{\mathcal{M}}$,
- $\|\phi \wedge \psi\|^{\mathcal{M}} = \|\phi\|^{\mathcal{M}} \cap \|\psi\|^{\mathcal{M}}$,
- $\|K_a\phi\|^{\mathcal{M}} = [R_a]\|\phi\|^{\mathcal{M}}$,
- $\|[\phi]\psi\|^{\mathcal{M}} = \text{if } \|\phi\|^{\mathcal{M}} = \emptyset, \text{then } W, \text{else } (W \setminus \|\phi\|^{\mathcal{M}}) \cup \|\psi\|^{\mathcal{M}(\|\phi\|^{\mathcal{M}})}$,
- $\|\Box\phi\|^{\mathcal{M}} = \bigcap\{\|[\psi]\phi\|^{\mathcal{M}}\colon \psi \text{ is epistemic}\}$.

The above definition of the truth-set of a formula in a model is implicitly based on the well-founded strict partial order $<^{Size}_{d_\Box}$ between formulas and Lemma 5.

Proposition 7. *Let ϕ, ψ be formulas and $a \in Agt$.*

- $\|\hat{K}_a\phi\|^{\mathcal{M}} = \langle R_a \rangle \|\phi\|^{\mathcal{M}}$,
- $\|\langle\phi\rangle\psi\|^{\mathcal{M}} = \textit{if } \|\phi\|^{\mathcal{M}} = \emptyset, \textit{then } \emptyset, \textit{else } \|\phi\|^{\mathcal{M}} \cap \|\psi\|^{\mathcal{M}(\|\phi\|^{\mathcal{M}})}$,
- $\|\Diamond\phi\|^{\mathcal{M}} = \bigcup\{\|\langle\chi\rangle\phi\|^{\mathcal{M}}\colon \chi \textit{ is epistemic}\}$.

Proof. Left to the reader. □

We shall say that a formula ϕ is valid iff for all models $\mathcal{M} = (W, R, V)$, $\|\phi\|^{\mathcal{M}} = W$. A set Γ of formulas is said to be valid iff for all formulas $\phi \in \Gamma$, ϕ is valid. We shall say that a derivation rule (Γ, ϕ) is admissible iff if Γ is valid, then ϕ is valid.

Lemma 8. *Let ϕ, ψ, χ be formulas, $p \in Atm$ and $a \in Agt$. The following formulas are valid:*

(A0) *all instantiations of propositional tautologies,*
(A1) $K_a(\phi \to \psi) \to (K_a\phi \to K_a\psi)$,
(A2) $K_a\phi \to \phi$,
(A3) $K_a\phi \to K_aK_a\phi$,
(A4) $\phi \to K_a\hat{K}_a\phi$,
(A5) $[\phi]p \leftrightarrow (\phi \to p)$,
(A6) $[\phi]\neg\psi \leftrightarrow (\phi \to \neg[\phi]\psi)$,
(A7) $[\phi](\psi \wedge \chi) \leftrightarrow [\phi]\psi \wedge [\phi]\chi$,
(A8) $[\phi]K_a\psi \leftrightarrow (\phi \to K_a[\phi]\psi)$,
(A9) $[\phi][\psi]\chi \leftrightarrow [\phi \wedge [\phi]\psi]\chi$,
(A10) *if ψ is epistemic, then $\Box\phi \to [\psi]\phi$.*

Proof. Leaving to the reader the task of proving the first items, we only pay attention to the last one. Suppose ψ is epistemic and $\Box\phi \to [\psi]\phi$ is not valid. Hence, there exists a model $\mathcal{M} = (W, R, V)$ such that $\|\Box\phi \to [\psi]\phi\|^{\mathcal{M}} \neq W$. Thus, there exists $x \in W$ such that $x \in \|\Box\phi\|^{\mathcal{M}}$ and $x \notin \|[\psi]\phi\|^{\mathcal{M}}$. Since ψ is epistemic, $x \in \|[\psi]\phi\|^{\mathcal{M}}$: a contradiction. □

Lemma 9. *Let ϕ, ψ be formulas, $a \in Agt$ and $\varphi(\sharp) \in NF$. The following derivation rules are admissible:*

($R0$) $(\{\phi, \phi \to \psi\}, \psi)$,
($R1$) $(\{\phi\}, K_a\phi)$,
($R2$) $(\{\phi\}, [\psi]\phi)$,
($R3$) $(\{rep(\varphi(\sharp), [\psi]\phi)$: *$\psi$ is epistemic*$\}, rep(\varphi(\sharp), \Box\phi))$.

Proof. Leaving to the reader the task of proving the first items, we only pay attention to the last one. Suppose $(\{rep(\varphi(\sharp), [\psi]\phi)$: ψ is epistemic$\}, rep(\varphi(\sharp), \Box\phi))$ is not admissible. Hence, $\{rep(\varphi(\sharp), [\psi]\phi)$: ψ is epistemic$\}$ is valid and $rep(\varphi(\sharp), \Box\phi)$ is not valid. Thus, there exists a model $\mathcal{M} = (W, R, V)$ such that $\|rep(\varphi(\sharp), \Box\phi)\|^{\mathcal{M}} \neq W$. Therefore, there exists $x \in W$ such that $x \notin \|rep(\varphi(\sharp), \Box\phi)\|^{\mathcal{M}}$. By Proposition 13 below, there exists an epistemic formula θ such that $x \notin \|rep(\varphi(\sharp), [\theta]\phi)\|^{\mathcal{M}}$. Since $\{rep(\varphi(\sharp), [\psi]\phi)$: ψ is epistemic$\}$ is valid, $rep(\varphi(\sharp), [\theta]\phi)$ is valid. Consequently, $\|rep(\varphi(\sharp), [\theta]\phi)\|^{\mathcal{M}} = W$. Hence, $x \in \|rep(\varphi(\sharp), [\theta]\phi)\|^{\mathcal{M}}$: a contradiction. □

Lemma 10. *Let ϕ, ψ be formulas. For all models $\mathcal{M} = (W, R, V)$ and for all $x \in W$, $x \in \|\langle\phi\rangle\psi\|^{\mathcal{M}}$ iff $x \in \|\phi\|^{\mathcal{M}}$ and $x \in \|\psi\|^{\mathcal{M}(\|\phi\|^{\mathcal{M}})}$.*

Proof. By Proposition 7. □

Proposition 11. *Let m be a non-negative integer. If $m \geq 1$, then for all formulas $\psi_1, \ldots, \psi_m$, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \leftrightarrow \langle\psi_1\rangle \ldots \langle\psi_{m-1}\rangle\psi_m$ is valid.*

Proof. Suppose $m \geq 1$. Let $\psi_1, \ldots, \psi_m$ be formulas. By Lemma 10, $\langle\psi_m\rangle\top \leftrightarrow \psi_m$ is valid. Hence, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \leftrightarrow \langle\psi_1\rangle \ldots \langle\psi_{m-1}\rangle\psi_m$ is valid. □

Proposition 12. *For all non-negative integers m,*

$P(m)$: *for all formulas $\psi_1, \ldots, \psi_m$, for all $p \in Atm$, for all formulas ϕ, ψ, for all $a \in Agt$, for all models $\mathcal{M} = (W, R, V)$ and for all $x \in W$,*

- $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle p\|^{\mathcal{M}}$ *iff* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}}$ *and* $x \in V(p)$,
- $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\neg\phi\|^{\mathcal{M}}$ *iff* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}}$ *and* $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi\|^{\mathcal{M}}$,
- $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle(\phi \wedge \psi)\|^{\mathcal{M}}$ *iff* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi\|^{\mathcal{M}}$ *and* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\psi\|^{\mathcal{M}}$,
- $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle K_a\phi\|^{\mathcal{M}}$ *iff* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}}$ *and for all* $y \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}}$, *if* xR_ay, *then* $y \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi\|^{\mathcal{M}}$,
- $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle[\phi]\psi\|^{\mathcal{M}}$ *iff* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}}$ *and if* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi\|^{\mathcal{M}}$, *then* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi\rangle\psi\|^{\mathcal{M}}$,
- $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\Box\phi\|^{\mathcal{M}}$ *iff* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}}$ *and for all epistemic formulas* χ, *if* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi\|^{\mathcal{M}}$, *then* $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi\|^{\mathcal{M}}$.

Proof. See the Appendix. □

Proposition 13. *For all $\varphi(\sharp) \in NF$,*

$Q(\varphi(\sharp))$: *for all formulas ϕ, for all models $\mathcal{M} = (W, R, V)$ and for all $x \in W$, $x \in \|rep(\varphi(\sharp), \Box\phi)\|^{\mathcal{M}}$ iff for all epistemic formulas ψ, $x \in \|rep(\varphi(\sharp), [\psi]\phi)\|^{\mathcal{M}}$.*

Proof. See the Appendix. □

4. Axiomatisation

An axiomatic system consists of a collection of formulas and a collection of derivation rules. Let us consider the axiomatic system consisting of formulas ($A0$)–($A10$) and derivation rules $(R0)$–$(R3)$ considered in Lemmas 8 and 9 and let $APAL^{\omega}$ be the least subset of $\mathcal{L}_{apal}$ containing $(A0)$–$(A10)$ and closed under $(R0)$–$(R3)$.

Lemma 14. *For all formulas* ψ,

$R(\psi)$: *for all formulas* ϕ, $\neg\phi \to [\phi]\psi$ *is in* $APAL^{\omega}$.

Proof. By $<^{Size}_{d_{\Box}}$-induction on ψ. □

Lemma 15. *Let* ϕ, ψ, χ *be formulas. The following formulas are in* $APAL^{\omega}$:

(1) $[\phi]\bot \leftrightarrow \neg\phi$.
(2) $[\phi](\psi \vee \chi) \leftrightarrow ([\phi]\psi \vee [\phi]\chi)$.
(3) $[\phi](\psi \to \chi) \leftrightarrow ([\phi]\psi \to [\phi]\chi)$.

Proof. (1) Since $\bot$ is an abbreviation for $p \wedge \neg p$, the following formulas are deductively equivalent in $APAL^{\omega}$: $[\phi]\bot$, $[\phi]p \wedge [\phi]\neg p$, $(\phi \to p) \wedge (\phi \to \neg[\phi]p)$, $(\phi \to p) \wedge (\phi \to \neg(\phi \to p))$, $(\phi \to p) \wedge (\phi \to \neg p)$, $\neg\phi$.

(2) Since $\psi \vee \chi$ is an abbreviation for $\neg(\neg\psi \wedge \neg\chi)$, the following formulas are deductively equivalent in $APAL^{\omega}$: $[\phi](\psi \vee \chi)$, $\phi \to \neg[\phi](\neg\psi \wedge \neg\chi)$, $\phi \to \neg([\phi]\neg\psi \wedge [\phi]\neg\chi)$, $\phi \to \neg((\phi \to \neg[\phi]\psi) \wedge (\phi \to \neg[\phi]\chi))$, $\phi \to [\phi]\psi \vee [\phi]\chi$. By Lemma 14, $\neg\phi \to [\phi]\psi \vee [\phi]\chi$ is in $APAL^{\omega}$. Since $[\phi](\psi \vee \chi)$ and $\phi \to [\phi]\psi \vee [\phi]\chi$ are deductively equivalent in $APAL^{\omega}$, $[\phi](\psi \vee \chi)$ and $[\phi]\psi \vee [\phi]\chi$ are deductively equivalent in $APAL^{\omega}$.

(3) Since $\psi \to \chi$ is an abbreviation for $\neg(\psi \wedge \neg\chi)$, the following formulas are deductively equivalent in $APAL^{\omega}$: $[\phi](\psi \to \chi)$, $\phi \to \neg[\phi](\psi \wedge \neg\chi)$, $\phi \to \neg([\phi]\psi \wedge [\phi]\neg\chi)$, $\phi \to \neg([\phi]\psi \wedge (\phi \to \neg[\phi]\chi))$, $\phi \to ([\phi]\psi \to [\phi]\chi)$. By Lemma 14, $\neg\phi \to ([\phi]\psi \to [\phi]\chi)$ is in $APAL^{\omega}$. Since $[\phi](\psi \to \chi)$ and $\phi \to ([\phi]\psi \to [\phi]\chi)$ are deductively equivalent in $APAL^{\omega}$, $[\phi](\psi \to \chi)$ and $[\phi]\psi \to [\phi]\chi$ are deductively equivalent in $APAL^{\omega}$. □

Proposition 16. *Let* m *be a non-negative integer. If* $m \geq 1$, *then for all formulas* $\psi_1, \ldots, \psi_m$, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \leftrightarrow \langle\psi_1\rangle \ldots \langle\psi_{m-1}\rangle\psi_m \in APAL^{\omega}$.

Proof. By Lemma 15. □

Proposition 17. *For all formulas* ϕ, *if* $\phi \in APAL^{\omega}$, *then* ϕ *is valid.*

Proof. By Lemmas 8 and 9. □

Proposition 18. *For all* $\varphi(\sharp) \in NF$,

$S(\varphi(\sharp))$: *for all formulas* ϕ *and for all epistemic formulas* ψ, $rep(\varphi(\sharp), \Box\phi) \to rep(\varphi(\sharp), [\psi]\phi) \in APAL^{\omega}$.

Proof. See the Appendix. □

Looking at our axiomatic system attentively, the reader will notice that it contains the same formulas and derivation rules as the axiomatic system $APAL^{\omega}$ considered in Balbiani, Baltag, van Ditmarsch, Herzig, Hoshi, & de Lima (2008, p. 325).

5. Canonical model

A set x of formulas is called a theory iff it satisfies the following conditions:

- x contains $APAL^{\omega}$,
- x is closed under $(R0)$ and $(R3)$.

Obviously, the least theory is $APAL^{\omega}$ whereas the greatest theory is $\mathcal{L}_{apal}$. A theory x is said to be consistent iff $\bot \notin x$. Let us remark that the only inconsistent theory is $\mathcal{L}_{apal}$. Moreover, the reader may easily verify that a theory x is consistent iff for all formulas ϕ, $\phi \notin x$ or $\neg\phi \notin x$. We shall say that a theory x is maximal iff for all formulas ϕ, $\phi \in x$ or $\neg\phi \in x$.

Lemma 19. *Let ϕ, ψ be formulas. For all maximal consistent theories x,*

- $\bot \notin x$,
- $\neg\phi \in x$ *iff* $\phi \notin x$,
- $(\phi \vee \psi) \in x$ *iff* $\phi \in x$ *or* $\psi \in x$.

Proof. Left to the reader. □

Let x be a theory. For all formulas ϕ and for all $a \in Agt$, let $x + \phi = \{\psi : \phi \to \psi \in x\}$, $K_a x = \{\phi : K_a\phi \in x\}$ and $[\phi]x = \{\psi : [\phi]\psi \in x\}$.

Lemma 20. *Let ϕ be a formula and $a \in Agt$. For all theories x,*

- $x + \phi$ *is a theory containing* x *and* ϕ,
- $[\phi]x$ *is a theory,*
- $K_a x$ *is a theory.*

Proof. By Balbiani et al. (2008, Lemma 4.11). □

Lemma 21. *Let ϕ be a formula. For all theories x, $x + \phi$ is consistent iff $\neg\phi \notin x$.*

Proof. By Balbiani et al. (2008, Lemma 4.11). □

Lemma 22 (Lindenbaum Lemma)**.** *Each consistent theory can be extended to a maximal consistent theory.*

Proof. By Balbiani et al. (2008, Lemma 4.12). □

Lemma 23. *Let $a \in Agt$. For all maximal consistent theories x, y, z,*

- $K_a x \subseteq x$,
- *if* $K_a x \subseteq y$ *and* $K_a y \subseteq z$, *then* $K_a x \subseteq z$,
- *if* $K_a x \subseteq y$, *then* $K_a y \subseteq x$.

Proof. Left to the reader. □

Lemma 24. *Let ϕ be a formula and $a \in Agt$. For all theories x, if $K_a\phi \notin x$, then there exists a maximal consistent theory y such that $K_a x \subseteq y$ and $\phi \notin y$.*

Proof. Suppose $K_a\phi \notin x$. Hence, $\phi \notin K_a x$. By Lemmas 21 and 22, there exists a maximal consistent theory y such that $K_a x + \neg\phi \subseteq y$. Thus, $K_a x \subseteq y$ and $\phi \notin y$. □

Lemma 25. *Let ϕ be a formula. For all maximal consistent theories x, if $\phi \in x$, then $[\phi]x$ is a maximal consistent theory.*

Proof. Suppose $\phi \in x$. If $[\phi]x$ is not consistent, then $\bot \in [\phi]x$. Hence, $[\phi]\bot \in x$. Thus, $\neg\phi \in x$. Since x is consistent, $\phi \notin x$: a contradiction. If $[\phi]x$ is not maximal, then there exists a formula ψ such that $\psi \notin [\phi]x$ and $\neg\psi \notin [\phi]x$. Therefore, $[\phi]\psi \notin x$ and $[\phi]\neg\psi \notin x$. Since x is maximal, $\neg[\phi]\psi \in x$ and $\neg[\phi]\neg\psi \in x$. Consequently, $\neg([\phi]\psi \vee [\phi]\neg\psi) \in x$. Hence, $\neg[\phi](\psi \vee \neg\psi) \in x$. Since x is consistent, $[\phi](\psi \vee \neg\psi) \notin x$. Since $\psi \vee \neg\psi \in APAL^{\omega}$, $[\phi](\psi \vee \neg\psi) \in APAL^{\omega}$. Thus, $[\phi](\psi \vee \neg\psi) \in x$: a contradiction. □

Lemma 26. *Let ϕ, ψ be formulas. For all maximal consistent theories x, $\langle\phi\rangle\psi \in x$ iff $\phi \in x$ and $\psi \in [\phi]x$.*

Proof. ($\Rightarrow$) Suppose $\langle\phi\rangle\psi \in x$. Hence, $\langle\phi\rangle\top \in x$. By Proposition 16, $\phi \in x$. By Lemma 25, $[\phi]x$ is a maximal consistent theory. Suppose $\psi \notin [\phi]x$. Since $[\phi]x$ is maximal, $\neg\psi \in [\phi]x$. Thus, $[\phi]\neg\psi \in x$. Therefore, $\neg\langle\phi\rangle\psi \in x$. Since x is consistent, $\langle\phi\rangle\psi \notin x$: a contradiction.

($\Leftarrow$) Suppose $\phi \in x$ and $\psi \in [\phi]x$. By Lemma 25, $[\phi]x$ is a maximal consistent theory. Suppose $\langle\phi\rangle\psi \notin x$. Since x is maximal, $\neg\langle\phi\rangle\psi \in x$. Hence, $[\phi]\neg\psi \in x$. Thus, $\neg\psi \in [\phi]x$. Since $[\phi]x$ is consistent, $\psi \notin [\phi]x$: a contradiction. □

Lemma 27. *Let ϕ be a formula and $a \in Agt$. For all theories x, if $\phi \in x$, then $K_a[\phi]x = [\phi]K_a x$.*

Proof. Suppose $\phi \in x$. For all formulas ψ, the reader may easily verify that the following conditions are equivalent:

(1) $\psi \in K_a[\phi]x$,
(2) $K_a\psi \in [\phi]x$,
(3) $[\phi]K_a\psi \in x$,
(4) $\phi \to K_a[\phi]\psi \in x$,
(5) $K_a[\phi]\psi \in x$,
(6) $[\phi]\psi \in K_a x$,
(7) $\psi \in [\phi]K_a x$.

□

Lemma 28. *Let m be a non-negative integer. If $m \geq 1$, then for all formulas $\psi_1, \ldots, \psi_m$, for all formulas ϕ and for all maximal consistent theories x, the following conditions are equivalent:*

- *$\psi_1 \in x$, $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top \in [\psi_1]x$ and for all maximal consistent theories y containing $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top$, if $K_a[\psi_1]x \subseteq y$, then $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\phi \in y$,*
- *$\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and for all maximal consistent theories y containing $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top$, if $K_a x \subseteq y$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in y$.*

Proof. Suppose $m \geq 1$. ($\Rightarrow$) Suppose $\psi_1 \in x$, $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top \in [\psi_1]x$ and for all maximal consistent theories y containing $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top$, if $K_a[\psi_1]x \subseteq y$, then $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\phi \in y$. By Lemma 26, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$. Let z be a maximal consistent theory containing $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top$. Suppose $K_a x \subseteq z$. Hence, $[\psi_1]K_a x \subseteq [\psi_1]z$. Since $\psi_1 \in x$, by Lemma 27, $K_a[\psi_1]x = [\psi_1]K_a x$. Since $[\psi_1]K_a x \subseteq [\psi_1]z$, $K_a[\psi_1]x \subseteq [\psi_1]z$. Since $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in z$, by Lemma 26, $\psi_1 \in z$ and $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top \in [\psi_1]z$. Since for all maximal consistent theories y containing $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top$, if $K_a[\psi_1]x \subseteq y$, then $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\phi \in y$ and $K_a[\psi_1]x \subseteq [\psi_1]z$, $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\phi \in [\psi_1]z$. Since $\psi_1 \in z$, by Lemma 26, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in z$.

($\Leftarrow$) Suppose $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and for all maximal consistent theories y containing $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top$, if $K_a x \subseteq y$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in y$. By Lemma 26, $\psi_1 \in x$ and $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top \in [\psi_1]x$. Let z be a maximal consistent theory containing $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top$. Suppose $K_a[\psi_1]x \subseteq z$. Since $\psi_1 \in x$, by Lemma 27, $K_a[\psi_1]x = [\psi_1]K_a x$. Since $K_a[\psi_1]x \subseteq z$, $[\psi_1]K_a x \subseteq z$. We consider the following 2 cases.

Case $K_a[\psi_1] \ldots [\psi_m]\phi \in x$**.** Hence, $[\psi_1] \ldots [\psi_m]\phi \in K_a x$. Thus, $[\psi_2] \ldots [\psi_m]\phi \in [\psi_1]K_a x$. Since $[\psi_1]K_a x \subseteq z$, $[\psi_2] \ldots [\psi_m]\phi \in z$. Since $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top \in z$, $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\phi \in z$.

Case $K_a[\psi_1]\dots[\psi_m]\phi \notin x$**.** Hence, there exists a maximal consistent theory t such that $K_a x \subseteq t$ and $[\psi_1]\dots[\psi_m]\phi \notin t$. Thus, $\langle\psi_1\rangle\dots\langle\psi_m\rangle\top \in t$ and $\langle\psi_1\rangle\dots\langle\psi_m\rangle\phi \notin t$. Since for all maximal consistent theories y containing $\langle\psi_1\rangle\dots\langle\psi_m\rangle\top$, if $K_a x \subseteq y$, then $\langle\psi_1\rangle\dots\langle\psi_m\rangle\phi \in y$, $\langle\psi_1\rangle\dots\langle\psi_m\rangle\phi \in t$: a contradiction. □

Proposition 29. *For all non-negative integers m,*

$T(m)$: *for all formulas* $\psi_1, \dots, \psi_m$, *for all* $p \in Atm$, *for all formulas* ϕ, ψ, *for all* $a \in Agt$ *and for all maximal consistent theories* x,

- $\langle\psi_1\rangle\dots\langle\psi_m\rangle p \in x$ *iff* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\top \in x$ *and* $p \in x$,
- $\langle\psi_1\rangle\dots\langle\psi_m\rangle\neg\phi \in x$ *iff* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\top \in x$ *and* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\phi \notin x$,
- $\langle\psi_1\rangle\dots\langle\psi_m\rangle(\phi \wedge \psi) \in x$ *iff* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\phi \in x$ *and* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\psi \in x$,
- $\langle\psi_1\rangle\dots\langle\psi_m\rangle K_a\phi \in x$ *iff* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\top \in x$ *and for all maximal consistent theories* y *containing* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\top$, *if* $K_a x \subseteq y$, *then* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\phi \in y$,
- $\langle\psi_1\rangle\dots\langle\psi_m\rangle[\phi]\psi \in x$ *iff* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\top \in x$ *and if* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\phi \in x$, *then* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\langle\phi\rangle\psi \in x$,
- $\langle\psi_1\rangle\dots\langle\psi_m\rangle\Box\phi \in x$ *iff* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\top$ *and for all epistemic formulas* ψ, *if* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\psi \in x$, *then* $\langle\psi_1\rangle\dots\langle\psi_m\rangle\langle\psi\rangle\phi \in x$.

Proof. See the Appendix. □

Lemma 30. *Let* $\varphi(\sharp) \in NF$ *and* ϕ *be a formula. For all theories* x, *if* $rep(\varphi(\sharp), \Box\phi) \in x$, *then for all epistemic formulas* ψ, $rep(\varphi(\sharp), [\psi]\phi) \in x$.

Proof. By Proposition 18. □

Now, the canonical structure $\mathcal{M}_c = (W_c, R_c, V_c)$ is defined as follows:

- W_c is the set of all maximal consistent theories,
- R_c is the function assigning to each agent a the binary relation $R_c(a)$ on W_c defined as follows:
  - $x R_c(a) y$ iff $K_a x \subseteq y$,
- V_c is the function assigning to each atom p the subset $V_c(p)$ of W_c defined as follows:
  - $x \in V_c(p)$ iff $p \in x$.

This is obviously a model: by Lemma 22, W_c is a non-empty set and by Lemma 23, the binary relation $R_c(a)$ is an equivalence relation on W_c for each agent a.

6. Completeness

Let $X = \mathbb{N} \times \mathbb{N} \times \mathcal{L}_{apal}$. We define the binary relation $\ll$ on X in the following way:

- $(i, m, \phi) \ll (j, n, \psi)$ iff either $i < j$, or $i = j$ and $m < n$, or $i = j$, $m = n$ and $\phi <^{Size} \psi$.

Lemma 31. $\ll$ *is a well-founded strict partial order on* X.

Proof. By the well-foundedness of the standard linear order between non-negative integers and Proposition 1. □

The degree of a formula ϕ, represented by $\deg(\phi)$, is the non-negative integer inductively defined as follows:

- $\deg(p) = 0$,
- $\deg(\neg\phi) = \deg(\phi)$,
- $\deg(\phi \wedge \psi) = \max\{\deg(\phi), \deg(\psi)\}$,
- $\deg(K_a\phi) = \deg(\phi)$,
- $\deg([\phi]\psi) = \deg(\phi) + \deg(\psi) + 2$,
- $\deg(\Box\phi) = \deg(\phi) + 2$.

Lemma 32 (Truth Lemma)**.** *For all $(i, m, \phi) \in X$,*

$U(i, m, \phi)$: *For all formulas $\psi_1, \ldots, \psi_m$, if $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi) \leq i$, then for all maximal consistent theories x, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in x$.*

Proof See the Appendix. □

The following result is a direct consequence of Lemma 32.

Proposition 33. *Let ϕ be a formula. For all maximal consistent theories x, $x \in \|\phi\|^{\mathcal{M}_c}$ iff $\phi \in x$.*

Proof. By Lemma 32. □

Now, we are ready to prove the completeness of $APAL^{\omega}$.

Proposition 34. *For all formulas ϕ, if ϕ is valid, then $\phi \in APAL^{\omega}$.*

Proof. Suppose ϕ is valid and $\phi \notin APAL^{\omega}$. By Lemmas 21 and 22, there exists a maximal consistent theory x containing $\neg\phi$. By Proposition 33, $x \notin \|\phi\|^{\mathcal{M}_c}$. Hence, $\|\phi\|^{\mathcal{M}_c} \neq W_c$. Thus, ϕ is not valid: a contradiction. □

7. Remarks

In this paper, the set of agents is finite. In the absence of modal operators that express mutual knowledge (everybody knows) or common knowledge (everybody knows that everybody knows that ...), the change to a countably infinite set of agents would affect neither the axiomatisation, nor the proof of completeness.

In other respects, the second component in a model $\mathcal{M} = (W, R, V)$ is a function R assigning to each agent a an equivalence relation $R(a)$. The change to relations $R(a)$ satisfying no specific conditions would affect neither the axiomatisation nor the proof of completeness – apart from the formulas $(A2)$–$(A4)$ and Lemma 23.

Several $APAL$-like dynamic epistemic logics have been proposed (for details, see Agotnes et al., 2010; Balbiani et al., 2013; Kuijer, 2014). We believe that their axiomatisation/completeness can be successfully based on variants of the axiomatic system and the proof of completeness developed for $APAL$ in this paper.

Acknowledgements

Special acknowledgement is earnestly granted to Hans van Ditmarsch, Jie Fan and my colleagues at the *Institut de recherche en informatique de Toulouse* who made several helpful comments for improving the correctness and readability of this paper. I also wish to thank the anonymous referees for their useful comments and their detailed suggestions about the overall organisation of the paper and for pointing out typos that had been missed.

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Appendix

Proof of Proposition 12. Let m be a non-negative integer. Suppose (H) for all non-negative integers n, if $n < m$, then $P(n)$. If $m = 0$, then the reader may easily verify that $P(m)$. Otherwise, let $\psi_1, \dots, \psi_m$ be formulas, $p \in Atm$, ϕ, ψ be formulas, $a \in Agt$, $\mathcal{M} = (W, R, V)$ be a model and $x \in W$. Since (H), $P(m-1)$.

About $\langle\psi_1\rangle \dots \langle\psi_m\rangle p$, the reader may easily verify that the following conditions are equivalent:

(1) $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle p\|^{\mathcal{M}}$,
(2) $x \in \|\psi_1\|^{\mathcal{M}}$ and $x \in \|\langle\psi_2\rangle \dots \langle\psi_m\rangle p\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(3) $x \in \|\psi_1\|^{\mathcal{M}}$, $x \in \|\langle\psi_2\rangle \dots \langle\psi_m\rangle\top\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$ and $x \in V(p)$,
(4) $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\top\|^{\mathcal{M}}$ and $x \in V(p)$.

Hence, $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle p\|^{\mathcal{M}}$ iff $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\top\|^{\mathcal{M}}$ and $x \in V(p)$.

About $\langle\psi_1\rangle \dots \langle\psi_m\rangle\neg\phi$, the reader may easily verify that the following conditions are equivalent:

(1) $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\neg\phi\|^{\mathcal{M}}$,
(2) $x \in \|\psi_1\|^{\mathcal{M}}$ and $x \in \|\langle\psi_2\rangle \dots \langle\psi_m\rangle\neg\phi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(3) $x \in \|\psi_1\|^{\mathcal{M}}$, $x \in \|\langle\psi_2\rangle \dots \langle\psi_m\rangle\top\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$ and $x \not\in \|\langle\psi_2\rangle \dots \langle\psi_m\rangle\phi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(4) $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\top\|^{\mathcal{M}}$ and $x \not\in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\phi\|^{\mathcal{M}}$.

Hence, $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\neg\phi\|^{\mathcal{M}}$ iff $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\top\|^{\mathcal{M}}$ and $x \not\in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\phi\|^{\mathcal{M}}$.

About $\langle\psi_1\rangle \dots \langle\psi_m\rangle(\phi \wedge \psi)$, the reader may easily verify that the following conditions are equivalent:

(1) $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle(\phi \wedge \psi)\|^{\mathcal{M}}$,
(2) $x \in \|\psi_1\|^{\mathcal{M}}$ and $x \in \|\langle\psi_2\rangle \dots \langle\psi_m\rangle(\phi \wedge \psi)\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(3) $x \in \|\psi_1\|^{\mathcal{M}}$, $x \in \|\langle\psi_2\rangle \dots \langle\psi_m\rangle\phi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$ and $x \in \|\langle\psi_2\rangle \dots \langle\psi_m\rangle\psi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(4) $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\phi\|^{\mathcal{M}}$ and $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\psi\|^{\mathcal{M}}$.

Hence, $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle(\phi \wedge \psi)\|^{\mathcal{M}}$ iff $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\phi\|^{\mathcal{M}}$ and $x \in \|\langle\psi_1\rangle \dots \langle\psi_m\rangle\psi\|^{\mathcal{M}}$.

About $\langle\psi_1\rangle\ldots\langle\psi_m\rangle K_a\phi$**,** the reader may easily verify that the following conditions are equivalent:

(1) $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle K_a\phi\|^{\mathcal{M}}$,
(2) $x \in \|\psi_1\|^{\mathcal{M}}$ and $x \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle K_a\phi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(3) $x \in \|\psi_1\|^{\mathcal{M}}$, $x \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$ and for all $y \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$, if xR_ay, then $y \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle\phi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(4) $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}}$ and for all $y \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}}$, if xR_ay, then $y \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi\|^{\mathcal{M}}$.

Hence, $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle K_a\phi\|^{\mathcal{M}}$ iff $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}}$ and for all $y \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}}$, if xR_ay, then $y \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi\|^{\mathcal{M}}$.

About $\langle\psi_1\rangle\ldots\langle\psi_m\rangle[\phi]\psi$**,** the reader may easily verify that the following conditions are equivalent:

(1) $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle[\phi]\psi\|^{\mathcal{M}}$,
(2) $x \in \|\psi_1\|^{\mathcal{M}}$ and $x \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle[\phi]\psi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(3) $x \in \|\psi_1\|^{\mathcal{M}}$, $x \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$ and if $x \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle\phi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$, then $x \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle\langle\phi\rangle\psi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(4) $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}}$ and if $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi\|^{\mathcal{M}}$, then $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\langle\phi\rangle\psi\|^{\mathcal{M}}$.

Hence, $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle[\phi]\psi\|^{\mathcal{M}}$ iff $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}}$ and if $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi\|^{\mathcal{M}}$, then $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\langle\phi\rangle\psi\|^{\mathcal{M}}$.

About $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\Box\phi$**,** the reader may easily verify that the following conditions are equivalent:

(1) $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\Box\phi\|^{\mathcal{M}}$,
(2) $x \in \|\psi_1\|^{\mathcal{M}}$ and $x \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle\Box\phi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(3) $x \in \|\psi_1\|^{\mathcal{M}}$, $x \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$ and for all epistemic formulas χ, if $x \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle\chi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$, then $x \in \|\langle\psi_2\rangle\ldots\langle\psi_m\rangle\langle\chi\rangle\phi\|^{\mathcal{M}(\|\psi_1\|^{\mathcal{M}})}$,
(4) $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}}$ and for all epistemic formulas χ, if $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\chi\|^{\mathcal{M}}$, then $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\langle\chi\rangle\phi\|^{\mathcal{M}}$.

Hence, $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\Box\phi\|^{\mathcal{M}}$ iff $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}}$ and for all epistemic formulas χ, if $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\chi\|^{\mathcal{M}}$, then $x \in \|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\langle\chi\rangle\phi\|^{\mathcal{M}}$.

Remark. *In all cases, the equivalence between* (1) *and* (2) *follows from the definition of truth-sets, the equivalence between* (2) *and* (3) *follows from* $P(m-1)$ *and the equivalence between* (3) *and* (4) *follows from the definition of truth-sets.*

□

Proof of Proposition 13. Let $\varphi(\sharp) \in NF$. Supose (H) for all $\varphi'(\sharp) \in NF$, if $\varphi'(\sharp) <^{Size} \varphi(\sharp)$, then $Q(\varphi'(\sharp))$. Let ϕ be a formula, $\mathcal{M} = (W, R, V)$ be a model and $x \in W$. We consider the following 4 cases.

Case $\varphi(\sharp) = \sharp$**.** The reader may easily verify that the following conditions are equivalent:

(1) $x \in \|rep(\sharp, \Box\phi)\|^{\mathcal{M}}$,
(2) $x \in \|\Box\phi\|^{\mathcal{M}}$,
(3) for all epistemic formulas ψ, $x \in \|[\psi]\phi\|^{\mathcal{M}}$,
(4) for all epistemic formulas ψ, $x \in \|rep(\sharp, [\psi]\phi)\|^{\mathcal{M}}$.

Hence, $x \in \|rep(\sharp, \Box\phi)\|^{\mathcal{M}}$ iff for all epistemic formulas ψ, $x \in \|rep(\sharp, [\psi]\phi)\|^{\mathcal{M}}$.

Case $\varphi(\sharp) = \psi \to \varphi'(\sharp)$**.** The reader may easily verify that the following conditions are equivalent:

(1) $x \in \|rep(\psi \to \varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$,
(2) $x \in \|\psi \to rep(\varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$,
(3) if $x \in \|\psi\|^{\mathcal{M}}$, then $x \in \|rep(\varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$,

(4) if $x \in \|\psi\|^{\mathcal{M}}$, then for all epistemic formulas χ, $x \in \|rep(\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$,
(5) for all epistemic formulas χ, if $x \in \|\psi\|^{\mathcal{M}}$, then $x \in \|rep(\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$,
(6) for all epistemic formulas χ, $x \in \|\psi \to rep(\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$,
(7) for all epistemic formulas χ, $x \in \|rep(\psi \to \varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$.

Hence, $x \in \|rep(\psi \to \varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$ iff for all epistemic formulas χ, $x \in \|rep(\psi \to \varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$.

Case $\varphi(\sharp) = K_a\varphi'(\sharp)$**.** The reader may easily verify that the following conditions are equivalent:

(1) $x \in \|rep(K_a\varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$,
(2) $x \in \|K_a rep(\varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$,
(3) for all $y \in W$, if $xR_a y$, then $y \in \|rep(\varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$,
(4) for all $y \in W$, if $xR_a y$, then for all epistemic formulas χ, $y \in \|rep(\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$,
(5) for all epistemic formulas χ and for all $y \in W$, if $xR_a y$, then $x \in \|rep(\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$,
(6) for all epistemic formulas χ, $x \in \|K_a rep(\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$,
(7) for all epistemic formulas χ, $x \in \|rep(K_a\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$.

Hence, $x \in \|rep(K_a\varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$ iff for all epistemic formulas χ, $x \in \|rep(K_a\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$.

Case $\varphi(\sharp) = [\psi]\varphi'(\sharp)$**.** The reader may easily verify that the following conditions are equivalent:

(1) $x \in \|rep([\psi]\varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$,
(1) $x \in \|[\psi]rep(\varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$,
(2) if $x \in \|\psi\|^{\mathcal{M}}$, then $x \in \|rep(\varphi'(\sharp), \Box\phi)\|^{\mathcal{M}(\|\psi\|^{\mathcal{M}})}$,
(3) if $x \in \|\psi\|^{\mathcal{M}}$, then for all epistemic formulas χ, $x \in \|rep(\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}(\|\psi\|^{\mathcal{M}})}$,
(4) for all epistemic formulas χ, if $x \in \|\psi\|^{\mathcal{M}}$, then $x \in \|rep(\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}(\|\psi\|^{\mathcal{M}})}$,
(5) for all epistemic formulas χ, $x \in \|[\psi]rep(\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$,
(6) for all epistemic formulas χ, $x \in \|rep([\psi]\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$.

Hence, $x \in \|rep([\psi]\varphi'(\sharp), \Box\phi)\|^{\mathcal{M}}$ iff for all epistemic formulas χ, $x \in \|rep([\psi]\varphi'(\sharp), [\chi]\phi)\|^{\mathcal{M}}$. □

Remark. *In the first case, the equivalence between* (1) *and* (2) *follows from the definition of* $rep(\cdot,\cdot)$*, the equivalence between* (2) *and* (3) *follows from the definition of truth-sets and the equivalence between* (3) *and* (4) *follows from the definition of* $rep(\cdot,\cdot)$*. Note that in all other cases, the equivalence between* (1) *and* (2) *follows from the definition of* $rep(\cdot,\cdot)$*, the equivalence between* (2) *and* (3) *follows from the definition of truth-sets, the equivalence between* (3) *and* (4) *follows from* (H)*, the equivalence between* (4) *and* (5) *follows from logical reasoning, the equivalence between* (5) *and* (6) *follows from the definition of truth-sets and the equivalence between* (6) *and* (7) *follows from the definition of* $rep(\cdot,\cdot)$. □

Proof of Proposition 18. Let $\varphi(\sharp) \in NF$. Suppose (H) for all $\varphi'(\sharp) \in NF$, if $\varphi'(\sharp) <^{Size} \varphi(\sharp)$, then $S(\varphi'(\sharp))$. Let ϕ be a formula and ψ be an epistemic formula. We consider the following 4 cases.

Case $\varphi(\sharp) = \sharp$**.** Since ψ is epistemic, $\Box\phi \to [\psi]\phi \in APAL^{\omega}$. Hence, $rep(\sharp, \Box\phi) \to rep(\sharp, [\psi]\phi) \in APAL^{\omega}$.

Case $\varphi(\sharp) = \chi \to \varphi'(\sharp)$**.** If $rep(\chi \to \varphi'(\sharp), \Box\phi) \to rep(\chi \to \varphi'(\sharp), [\psi]\phi) \notin APAL^{\omega}$, then $\chi \to (rep(\varphi'(\sharp), \Box\phi) \to rep(\varphi'(\sharp), [\psi]\phi)) \notin APAL^{\omega}$. Hence, $rep(\varphi'(\sharp), \Box\phi) \to rep(\varphi'(\sharp), [\psi]\phi) \notin APAL^{\omega}$. Now, note that $\varphi'(\sharp) <^{Size} \varphi(\sharp)$. By (H), $rep(\varphi'(\sharp), \Box\phi) \to rep(\varphi'(\sharp), [\psi]\phi) \in APAL^{\omega}$: a contradiction.

Case $\varphi(\sharp) = K_a\varphi'(\sharp)$**.** If $rep(K_a\varphi'(\sharp), \Box\phi) \to rep(K_a\varphi'(\sharp), [\psi]\phi) \notin APAL^{\omega}$, then $K_a(rep(\varphi'(\sharp), \Box\phi) \to rep(\varphi'(\sharp), [\psi]\phi)) \notin APAL^{\omega}$. Hence, $rep(\varphi'(\sharp), \Box\phi) \to rep(\varphi'(\sharp), [\psi]\phi) \notin APAL^{\omega}$. Now, note that $\varphi'(\sharp) <^{Size} \varphi(\sharp)$. By (H), $rep(\varphi'(\sharp), \Box\phi) \to rep(\varphi'(\sharp), [\psi]\phi) \in APAL^{\omega}$: a contradiction.

Case $\varphi(\sharp) = [\chi]\varphi'(\sharp)$**.** If $rep([\chi]\varphi'(\sharp), \Box\phi) \to rep([\chi]\varphi'(\sharp), [\psi]\phi) \notin APAL^{\omega}$, then $[\chi](rep(\varphi'(\sharp), \Box\phi) \to rep(\varphi'(\sharp), [\psi]\phi)) \notin APAL^{\omega}$. Hence, $rep(\varphi'(\sharp), \Box\phi) \to rep(\varphi'(\sharp), [\psi]\phi) \notin APAL^{\omega}$. Now, note that $\varphi'(\sharp) <^{Size} \varphi(\sharp)$. By (H), $rep(\varphi'(\sharp), \Box\phi) \to rep(\varphi'(\sharp), [\psi]\phi) \in APAL^{\omega}$: a contradiction. □

Proof of Proposition 29. Let m be a non-negative integer. Suppose (H) for all non-negative integers n, if $n < m$, then $T(n)$. If $m = 0$, then the reader may easily verify that $T(m)$. Otherwise, let $\psi_1, \ldots, \psi_m$ be formulas, $p \in Atm$, ϕ, ψ be formulas, $a \in Agt$ and x be a maximal consistent theory. Since (H), $T(m-1)$.

About $\langle\psi_1\rangle \ldots \langle\psi_m\rangle p$, the reader may easily verify that the following conditions are equivalent:

(1) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle p \in x$,
(2) $\psi_1 \in x$ and $\langle\psi_2\rangle \ldots \langle\psi_m\rangle p \in [\psi_1]x$,
(3) $\psi_1 \in x$, $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top \in [\psi_1]x$ and $p \in [\psi_1]x$,
(4) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and $p \in x$.

Hence, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle p \in x$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and $p \in x$.

About $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\neg\phi$, the reader may easily verify that the following conditions are equivalent:

(1) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\neg\phi \in x$,
(2) $\psi_1 \in x$ and $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\neg\phi \in [\psi_1]x$,
(3) $\psi_1 \in x$, $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top \in [\psi_1]x$ and $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\phi \notin [\psi_1]x$,
(4) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \notin x$.

Hence, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\neg\phi \in x$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \notin x$.

About $\langle\psi_1\rangle \ldots \langle\psi_m\rangle(\phi \wedge \psi)$, the reader may easily verify that the following conditions are equivalent:

(1) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle(\phi \wedge \psi) \in x$,
(2) $\psi_1 \in x$ and $\langle\psi_2\rangle \ldots \langle\psi_m\rangle(\phi \wedge \psi) \in [\psi_1]x$,
(3) $\psi_1 \in x$, $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\phi \in [\psi_1]x$ and $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\psi \in [\psi_1]x$,
(4) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\psi \in x$.

Hence, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle(\phi \wedge \psi) \in x$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\psi \in x$.

About $\langle\psi_1\rangle \ldots \langle\psi_m\rangle K_a\phi$, the reader may easily verify that the following conditions are equivalent:

(1) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle K_a\phi \in x$,
(2) $\psi_1 \in x$ and $\langle\psi_2\rangle \ldots \langle\psi_m\rangle K_a\phi \in [\psi_1]x$,
(3) $\psi_1 \in x$, $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top \in [\psi_1]x$ and for all maximal consistent theories y containing $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top$, if $K_a[\psi_1]x \subseteq y$, then $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\phi \in y$,
(4) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and for all maximal consistent theories y containing $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top$, if $K_a x \subseteq y$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in y$.

Hence, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle K_a\phi \in x$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and for all maximal consistent theories y containing $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top$, if $K_a x \subseteq y$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in y$.

About $\langle\psi_1\rangle \ldots \langle\psi_m\rangle[\phi]\psi$, the reader may easily verify that the following conditions are equivalent:

(1) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle[\phi]\psi \in x$,
(2) $\psi_1 \in x$ and $\langle\psi_2\rangle \ldots \langle\psi_m\rangle[\phi]\psi \in [\psi_1]x$,
(3) $\psi_1 \in x$, $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top \in [\psi_1]x$ and if $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\phi \in [\psi_1]x$, then $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\langle\phi\rangle\psi \in [\psi_1]x$,
(4) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and if $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in x$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi\rangle\psi \in x$.

Hence, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle[\phi]\psi \in x$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and if $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi \in x$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi\rangle\psi \in x$.

About $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\Box\phi$, the reader may easily verify that the following conditions are equivalent:

(1) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\Box\phi$,
(2) $\psi_1 \in x$ and $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\Box\phi \in [\psi_1]x$,
(3) $\psi_1 \in x$, $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\top \in [\psi_1]x$ and for all epistemic formulas ψ, if $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\psi \in [\psi_1]x$, then $\langle\psi_2\rangle \ldots \langle\psi_m\rangle\langle\psi\rangle\phi \in [\psi_1]x$,
(4) $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top$ and for all epistemic formulas ψ, if $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\psi \in x$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\psi\rangle\phi \in x$.

Hence, $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\Box\phi$ iff $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top$ and for all epistemic formulas ψ, if $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\psi\in x$, then $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\langle\psi\rangle\phi\in x$.

Remark. *In the case $\langle\psi_1\rangle\ldots\langle\psi_m\rangle K_a\phi$, the equivalence between* (1) *and* (2) *follows from Lemma* (26)*, the equivalence between* (2) *and* (3) *follows from (H) and the equivalence between* (3) *and* (4) *follows from Lemmas* (26) *and* (28)*. Note that in all the other cases, the equivalence between* (1) *and* (2) *follows from Lemma* (26)*, the equivalence between* (2) *and* (3) *follows from (H) and the equivalence between* (3) *and* (4) *follows from Lemma* (26)*.*

□

Proof of Lemma 32. Let $(i,m,\phi)\in X$. Suppose (H) for all $(j,n,\psi)\in X$, if $(j,n,\psi)\ll(i,m,\phi)$, then $U(j,n,\psi)$. Let $\psi_1,\ldots,\psi_m$ be formulas such that $m+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\phi)\leq i$. □

Claim 35. *For all maximal consistent theories x, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\in x$.*

Proof. Let x be a maximal consistent theory. The case $m=0$ is obvious. Hence, we only pay attention to the case $m\geq 1$.

($\Rightarrow$) Suppose $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}_c}$. By Proposition 11, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_{m-1}\rangle\psi_m\|^{\mathcal{M}_c}$. Now, note that $(i,m-1,\psi_m)\ll(i,m,\phi)$. By (H), $U(i,m-1,\psi_m)$. Since $m+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\phi)\leq i$, $m-1+\deg(\psi_1)+\ldots+\deg(\psi_{m-1})+\deg(\psi_m)\leq i$. Since $U(i,m-1,\psi_m)$, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_{m-1}\rangle\psi_m\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle\ldots\langle\psi_{m-1}\rangle\psi_m\in x$. Since $x\in\|\langle\psi_1\rangle\ldots\langle\psi_{m-1}\rangle\psi_m\|^{\mathcal{M}_c}$, $\langle\psi_1\rangle\ldots\langle\psi_{m-1}\rangle\psi_m\in x$. By Proposition 16, $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\in x$.

($\Leftarrow$) Suppose $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\in x$. By Proposition 16, $\langle\psi_1\rangle\ldots\langle\psi_{m-1}\rangle\psi_m\in x$. Again, note that $(i,m-1,\psi_m)\ll(i,m,\phi)$. By (H), $U(i,m-1,\psi_m)$. Since $m+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\phi)\leq i$, $m-1+\deg(\psi_1)+\ldots+\deg(\psi_{m-1})+\deg(\psi_m)\leq i$. Since $U(i,m-1,\psi_m)$, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_{m-1}\rangle\psi_m\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle\ldots\langle\psi_{m-1}\rangle\psi_m\in x$. Since $\langle\psi_1\rangle\ldots\langle\psi_{m-1}\rangle\psi_m\in x$, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_{m-1}\rangle\psi_m\|^{\mathcal{M}_c}$. By Proposition 11, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}_c}$. □

Claim 36. *For all maximal consistent theories x, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi\in x$.*

Proof. Let x be a maximal consistent theory. We consider the following 7 cases.

Case $\phi=p$.

($\Rightarrow$) Suppose $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle p\|^{\mathcal{M}_c}$. By Proposition 12, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}_c}$ and $x\in V_c(p)$. By Claim 35, $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\in x$. Moreover, $p\in x$. By Proposition 29, $\langle\psi_1\rangle\ldots\langle\psi_m\rangle p\in x$.

($\Leftarrow$) Suppose $\langle\psi_1\rangle\ldots\langle\psi_m\rangle p\in x$. By Proposition 29, $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\in x$ and $p\in x$. By Claim 35, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}_c}$. Moreover, $x\in V_c(p)$. By Proposition 12, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle p\|^{\mathcal{M}_c}$.

Case $\phi=\neg\phi'$.

($\Rightarrow$) Suppose $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\neg\phi'\|^{\mathcal{M}_c}$. By Proposition 12, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}_c}$ and $x\notin\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$. By Claim 35, $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\in x$. Now, note that $(i,m,\phi')\ll(i,m,\phi)$. By (H), $U(i,m,\phi')$. Since $m+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\phi)\leq i$, $m+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\phi')\leq i$. Since $U(i,m,\phi')$, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\in x$. Since $x\notin\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$, $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\notin x$. Since $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\in x$, by Proposition 29, $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\neg\phi'\in x$.

($\Leftarrow$) Suppose $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\neg\phi'\in x$. By Proposition 29, $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\in x$ and $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\notin x$. By Claim 35, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}_c}$. Again, note that $(i,m,\phi')\ll(i,m,\phi)$. By (H), $U(i,m,\phi')$. Since $m+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\phi)\leq i$, $m+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\phi')\leq i$. Since $U(i,m,\phi')$, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\in x$. Since $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\notin x$, $x\notin\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$. Since $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\top\|^{\mathcal{M}_c}$, by Proposition 12, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\neg\phi'\|^{\mathcal{M}_c}$.

Case $\phi=\phi'\wedge\phi''$.

($\Rightarrow$) Suppose $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle(\phi'\wedge\phi'')\|^{\mathcal{M}_c}$. By Proposition 12, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ and $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi''\|^{\mathcal{M}_c}$. Now, note that $(i,m,\phi')\ll(i,m,\phi)$ and $(i,m,\phi'')\ll(i,m,\phi)$. By (H), $U(i,m,\phi')$ and $U(i,m,\phi'')$. Since $m+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\phi)\leq i$, $m+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\phi')\leq i$ and $m+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\phi'')\leq i$. Since $U(i,m,\phi')$ and $U(i,m,\phi'')$, $x\in\|\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle\ldots\langle\psi_m\rangle\phi'\in x$ and

$x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi''\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'' \in x$. Since $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ and $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi''\|^{\mathcal{M}_c}$, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'' \in x$. By Proposition 29, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle(\phi' \wedge \phi'') \in x$.

($\Leftarrow$) Suppose $\langle\psi_1\rangle \ldots \langle\psi_m\rangle(\phi' \wedge \phi'') \in x$. By Proposition 29, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'' \in x$. Again, note that $(i, m, \phi') \ll (i, m, \phi)$ and $(i, m, \phi'') \ll (i, m, \phi)$. By (H), $U(i, m, \phi')$ and $U(i, m, \phi'')$. Since $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi) \leq i$, $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi') \leq i$ and $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi'') \leq i$. Since $U(i, m, \phi')$ and $U(i, m, \phi'')$, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in x$ and $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi''\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'' \in x$. Since $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'' \in x$, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ and $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi''\|^{\mathcal{M}_c}$. By Proposition 12, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle(\phi' \wedge \phi'')\|^{\mathcal{M}_c}$.

Case $\phi = K_a\phi'$.

($\Rightarrow$) Suppose $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle K_a\phi'\|^{\mathcal{M}_c}$. By Proposition 12, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$ and for all $y \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$, if xR_ay, then $y \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$. By Claim 35, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$. Suppose $\langle\psi_1\rangle \ldots \langle\psi_m\rangle K_a\phi' \notin x$. Since $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$, by Proposition 29, there exists a maximal consistent theory z containing $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top$ such that $K_ax \subseteq z$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \notin z$. By Claim 35, $z \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$. Moreover, xR_az. Since for all $y \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$, if xR_ay, then $y \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$, $z \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$. Now, note that $(i, m, \phi') \ll (i, m, \phi)$. By (H), $U(i, m, \phi')$. Since $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi) \leq i$, $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi') \leq i$. Since $U(i, m, \phi')$, $z \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in z$. Since $z \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in z$: a contradiction.

($\Leftarrow$) Suppose $\langle\psi_1\rangle \ldots \langle\psi_m\rangle K_a\phi' \in x$. By Proposition 29, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and for all maximal consistent theories y containing $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top$, if $K_ax \subseteq y$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in y$. By Claim 35, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$. Suppose $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle K_a\phi'\|^{\mathcal{M}_c}$. Since $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$, by Proposition 12, there exists $z \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$ such that xR_az and $z \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$. By Claim 35, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in z$. Moreover, $K_ax \subseteq z$. Since for all maximal consistent theories y containing $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top$, if $K_ax \subseteq y$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in y$, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in z$. Now, note that $(i, m, \phi') \ll (i, m, \phi)$. By (H), $U(i, m, \phi')$. Since $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi) \leq i$, $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi') \leq i$. Since $U(i, m, \phi')$, $z \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in z$. Since $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in z$, $z \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$: a contradiction.

Case $\phi = [\phi']\phi''$.

($\Rightarrow$) Suppose $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle[\phi']\phi''\|^{\mathcal{M}_c}$. By Proposition 12, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}}$ and if $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}}$, then $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi\rangle\phi''\|^{\mathcal{M}}$. By Claim 35, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$. Suppose $\langle\psi_1\rangle \ldots \langle\psi_m\rangle[\phi']\phi'' \notin x$. Since $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$, by Proposition 29, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi'' \notin x$. Now, note that $(i-1, m, \phi') \ll (i, m, \phi)$ and $(i-1, m+1, \phi'') \ll (i, m, \phi)$. By (H), $U(i-1, m, \phi')$ and $U(i-1, m+1, \phi'')$. Since $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi) \leq i$, $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi') \leq i-1$ and $m + 1 + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi') + \deg(\phi'') \leq i-1$. Since $U(i-1, m, \phi')$ and $U(i-1, m+1, \phi'')$, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in x$ and $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi''\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi'' \in x$. Since $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi'' \notin x$, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ and $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi''\|^{\mathcal{M}_c}$. Since $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$, by Proposition 12, $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle[\phi']\phi''\|^{\mathcal{M}_c}$: a contradiction.

($\Leftarrow$) Suppose $\langle\psi_1\rangle \ldots \langle\psi_m\rangle[\phi']\phi'' \in x$. By Proposition 29, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and if $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in x$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi'' \in x$. By Claim 35, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$. Suppose $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle[\phi']\phi''\|^{\mathcal{M}_c}$. Since $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$, by Proposition 12, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ and $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi''\|^{\mathcal{M}_c}$. Now, note that $(i-1, m, \phi') \ll (i, m, \phi)$ and $(i-1, m+1, \phi'') \ll (i, m, \phi)$. By (H), $U(i-1, m, \phi')$ and $U(i-1, m+1, \phi'')$. Since $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi) \leq i$, $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi') \leq i-1$ and $m + 1 + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi') + \deg(\phi'') \leq i-1$. Since $U(i-1, m, \phi')$ and $U(i-1, m+1, \phi'')$, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in x$ and $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi''\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi'' \in x$. Since $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi'\|^{\mathcal{M}_c}$ and $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi''\|^{\mathcal{M}_c}$, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\phi' \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\phi'\rangle\phi'' \notin x$. Since $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$, by Proposition 29, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle[\phi']\phi'' \notin x$: a contradiction.

Case $\phi = \Box\phi'$.

($\Rightarrow$) Suppose $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\Box\phi'\|^{\mathcal{M}_c}$. By Proposition 12, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$ and for all epistemic formulas ψ, if $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\psi\|^{\mathcal{M}_c}$, then $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\psi\rangle\phi'\|^{\mathcal{M}_c}$. By Claim 35, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$. Suppose $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\Box\phi' \notin x$. Since $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$, by Proposition 29, there exists an epistemic formula χ such that $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi' \notin x$. Now, note that $(i-1, m, \chi) \ll (i, m, \phi)$ and $(i-1, m+1, \phi') \ll (i, m, \phi)$. By (H), $U(i-1, m, \chi)$ and $U(i-1, m+1, \phi')$. Moreover, note that since χ is epistemic, by Proposition 4, $\deg(\chi) = 0$. Since $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi) \leq i$, $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\chi) \leq i-1$ and $m+1+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\chi)+\deg(\phi') \leq i-1$. Since $U(i-1, m, \chi)$ and $U(i-1, m+1, \phi')$, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi \in x$ and $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi'\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi' \in x$. Since $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi' \notin x$, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi\|^{\mathcal{M}_c}$ and $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi'\|^{\mathcal{M}_c}$. Since for all epistemic formulas ψ, if $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\psi\|^{\mathcal{M}_c}$, then $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\psi\rangle\phi'\|^{\mathcal{M}_c}$ and χ is epistemic, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi'\|^{\mathcal{M}_c}$: a contradiction.

($\Leftarrow$) Suppose $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\Box\phi' \in x$. By Proposition 29, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top \in x$ and for all epistemic formulas ψ, if $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\psi \in x$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\psi\rangle\phi' \in x$. By Claim 35, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$. Suppose $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\Box\phi'\|^{\mathcal{M}_c}$. Since $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\top\|^{\mathcal{M}_c}$, by Proposition 12, there exists an epistemic formula χ such that $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi\|^{\mathcal{M}_c}$ and $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi'\|^{\mathcal{M}_c}$. Now, note that $(i-1, m, \chi) \ll (i, m, \phi)$ and $(i-1, m+1, \phi') \ll (i, m, \phi)$. By (H), $U(i-1, m, \chi)$ and $U(i-1, m+1, \phi')$. Moreover, note that since χ is epistemic, by Proposition 4, $\deg(\chi) = 0$. Since $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\phi) \leq i$, $m + \deg(\psi_1) + \ldots + \deg(\psi_m) + \deg(\chi) \leq i-1$ and $m+1+\deg(\psi_1)+\ldots+\deg(\psi_m)+\deg(\chi)+\deg(\phi') \leq i-1$. Since $U(i-1, m, \chi)$ and $U(i-1, m+1, \phi')$, $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi \in x$ and $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi'\|^{\mathcal{M}_c}$ iff $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi' \in x$. Since $x \in \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi\|^{\mathcal{M}_c}$ and $x \notin \|\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi'\|^{\mathcal{M}_c}$, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\chi \in x$ and $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi' \notin x$. Since for all epistemic formulas ψ, if $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\psi \in x$, then $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\psi\rangle\phi' \in x$ and χ is epistemic, $\langle\psi_1\rangle \ldots \langle\psi_m\rangle\langle\chi\rangle\phi' \in x$: a contradiction.

From Claim 36, we infer $U(i, m, \phi)$. $\Box$