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The Effects of Linear and Nonlinear Characteristic Parameters on the Output Frequency Responses of Nonlinear Systems: The Associated Output Frequency Response Function

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Abstract

In the present study, a new concept known as the Associated Output Frequency Response Function (AOFRF) is introduced to facilitate the analysis of the effects of both linear and nonlinear characteristic parameters on the output frequency responses of nonlinear systems. Based on the AOFRF concept, the study has shown, for the first time, that the output frequency responses of a wide class of nonlinear systems that are described by the NARX (Nonlinear Auto Regressive with eXegenous input) model can be represented by a polynomial function of both the system linear and nonlinear characteristic parameters of interests to the system analysis. Moreover, an efficient algorithm is derived to determine the structure and coefficients of the AOFRF based representation for system output frequency responses. Finally, a case study is provided to demonstrate the effectiveness and advantages of the new AOFRF based representation and the implication of the result to the analysis and design of nonlinear systems in the frequency domain.

Key words: Associated Output Frequency Response Function; NARX model; Nonlinear system; Analysis and design; Frequency domain.

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1. Introduction

Because the linear system frequency domain analysis and design have a wide range of scientific and engineering applications, the analysis and design of nonlinear systems in the frequency domain have also received considerable interests (Dobrowiecki and Schoukens 2007). Compared with the time domain methods such as, e.g., the traditional harmonic balance and multi-scale methods, the frequency domain approaches have shown the capability to deal with a wide class of nonlinear systems, rather than the systems with specific model descriptions (Novara et al, 2013).

Based on the Volterra series theory of nonlinear systems, the concept of Generalized Frequency Response Functions (GFRFs) was proposed in 1959 (George; 1959), which are a series of multi-dimensional functions. The multi-dimensional nature makes the GFRFs difficult to be applied in practice. To address this challenge, some one-dimensional frequency domain representations of nonlinear systems have been proposed. These include, for example, Nonlinear Output Frequency Response Functions (NOFRFs) (Lang et al, 2005), Output Frequency Response Function (OFRF) (Lang et al, 2007), Higher Order Sinusoidal Input Describing Functions (HOSIDF) (Nuij et al, 2006), and the nonlinear Bode plots (Pavlov et al, 2007). The NOFRFs represent the relationship between the input and output spectra of nonlinear systems in a way similar to the Frequency Response Function (FRF) based representation of linear systems and have found applications in the areas of structural health monitoring and fault diagnosis (Zhao et al, 2015). The OFRF reveals an analytical relationship between the output spectrum of nonlinear systems and the parameters which define system nonlinearities, and provide an effective approach to the design of the system nonlinear properties in the frequency domain (Ho et al, 2014). The HOSIDF can be considered to be a special case of the OFRF of a static polynomial nonlinear system (Rijlaarsdam et al, 2011). Considering the wide application of the FRF in linear system analysis, Pavlov et al (2007) proposed the concept of nonlinear Bode plots for the analysis and design of nonlinear convergent systems. However, in most cases, the nonlinear Bode plots cannot be analytically studied (Rijlaarsdam et al, 2017) and are, consequently, difficult to be used to understand the properties of underlying systems.

It is well known that the output frequency responses of nonlinear systems are affected by both the linear and nonlinear characteristic parameters of the system. The OFRF shows an analytical relationship between the output spectra of nonlinear systems and the system's nonlinear characteristic parameters, but this relationship is only valid under the condition that the system linear characteristic parameters are fixed. Very recently, the issue associated with the effect of linear characteristic parameters on the nonlinear system output frequency responses have been studied (Xiao and Jing, 2016). However, the result is, so far, only a conceptual polynomial approximation for the system output spectrum, and there are still no results that can systematically relate the output frequency response of nonlinear systems to both system linear and nonlinear characteristic parameters so as to facilitate the system analysis and design.

In the present study, motivated by the need of the analysis and design of the effects of any parameters of a nonlinear system on the output frequency response, a new concept known as Associated Output Frequency Response Function (AOFRF) is introduced for the NARX model of nonlinear systems. Based on the novel AOFRF concept, it is rigorously shown that the output frequency response of nonlinear systems can be represented by a polynomial function of both the system linear and non-linear characteristic parameters. Effective algorithms are derived to determine the structure and coefficients of the AOFRF based representation of the output frequency response of nonlinear systems. Finally, a case study is used to show the application and verify the effectiveness of the algorithms in the analysis of output frequency responses of nonlinear systems to both deterministic and random inputs. The results demonstrate the significance of the new AOFRF concept and associated techniques in the revelation of the effects of both linear and nonlinear characteristic parameters on the output frequency responses of a wide class of nonlinear systems.

2. The output frequency responses of nonlinear systems

Consider the nonlinear systems described by a polynomial NARX model (Peyton-Jones and Billings, 1989)

$$y(k)=\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=1}^{K}\left[c_{p,q}\left(k_1,\cdots,k_{p+q}\right)\prod_{i=1}^{p}y\left(k-k_i\right)\times\prod_{i=p+1}^{p+q}u\left(k-k_i\right)\right] \tag{1}$$

where $y(.)$ and $u(.)$ are the outputs and inputs of the system; k represents the discrete time; $c_{p,q}\left(k_1,\cdots,k_{p+q}\right)$ with $p+q=m$ represents the model coefficients of the NARX model and $\sum_{k_1,k_{p+q}=1}^{K}=\sum_{k_1=1}^{K}\cdots\sum_{k_{p+q}=1}^{K}$; M and K are integers.

The NARX model (1) is a determinstic representation of nonlinear systems. When the physical model of nonlinear systems under study is not avaliable, a NARX model can be obtained from a NARMAX (Nonlinear Auto Regressive Moving Average with eXegenous input) model by removing the terms representing modelling error and noise in the model. The NARMAX model can be identified from the system input/output data using the NARMAX method of nonlinear system identification (Billings, 2013).

Under the condition that system (1) is stable at zero equilibrium, the output of system (1) can be described by the discrete time Volterra series

$$y(k)=\sum_{n=1}^{N}y_n(k)=\sum_{n=1}^{N}\sum_{\tau_1=-\infty}^{+\infty}\cdots\sum_{\tau_n=-\infty}^{+\infty}h_n\left(\tau_1,\ldots,\tau_n\right)\prod_{i=1}^{n}u\left(k-\tau_i\right) \tag{2}$$

where $h_n\left(\tau_1,\ldots,\tau_n\right)$ is the n th order Volterra kernel, and N is the maximum nonlinear order of the Volterra series. Moreover, the output spectrum of the system can be represented as (Lang and Billings, 1996)

$$Y(j\omega)=\sum_{n=1}^{N}Y_n(j\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}\left(2\pi\right)^{n-1}}\times\int_{\omega_1+\cdots+\omega_n=\omega}H_n\left(\omega_1,\ldots,\omega_n\right)\prod_{i=1}^{n}U\left(j\omega_i\right)d\sigma_\omega \tag{3}$$

where $-\pi f_s\le\omega\le\pi f_s$, $f_s=1/\Delta t$ is the sampling frequency; $U(j\omega)$ and $Y(j\omega)$ are the spectra of the system input and output, respectively, and

$$H_n\left(\omega_1,\ldots,\omega_n\right)=\sum_{\tau_1=-\infty}^{+\infty}\cdots\sum_{\tau_n=-\infty}^{+\infty}h_n\left(\tau_1,\ldots,\tau_n\right)\times\exp\left(-j\left(\omega_1\tau_1+\cdots+\omega_n\tau_n\right)\Delta t\right) \tag{4}$$

is the n th order GFRF of the system.

According to the concept of the OFRF proposed by Lang et al (2007), the output spectrum of a wide class of nonlinear systems can rigously be represented by a polynomial function of the system nonlinear characteristic parameters.

Recently, Xiao and Jing (2016) have conceptually indicated that the output spectrum can also be represented by a polynomial function of the system linear characteristic parameters. These results imply that the output spectrum of nonlinear systems could be represented by a polynomial function of both the system linear and nonlinear characteristic parameters. However, there are neither results about the conditions under which this representation is valid nor algorithms that can be used to determine the detailed structure of this polynomial.

The present study is motivated by the successful application of the OFRF and associated techniques (Lang et al, 2009, 2013, Ho et al, 2014) and the need to study the effects of both the linear and nonlinear characteristic parameters on the output responses of nonlinear systems. The work will be based on a new concept known as the Associated Output Frequency Response Function (AOFRF).

3. The concept of the Associated Output Frequency Response Function (AOFRF)

The concept of the AOFRF of the NARX model of nonlinear systems is introduced in Proposition 1 to facilitate the representation of the system output frequency response in terms of both the NARX model linear and nonlinear characteristic parameters.

Proposition 1. The output spectrum of the nonlinear system (1)/(2) can be described as

$$Y(j\omega)=\sum_{r=0}^{N}\tilde{Y}_r(j\omega) \tag{5}$$

where

$$\tilde{Y}_r(j\omega)=\sum_{n=r}^{N}\int_{\omega_1+\cdots+\omega_n=\omega}H_1^{r}\left(\omega_1,\ldots,\omega_r\right)\times\left[\mathbf{L}_{(n:r)}\left(\omega_1,\ldots,\omega_n\right)\circ\boldsymbol{\Phi}_{(n:r)}\left(\omega_1,\ldots,\omega_n\right)\right]\boldsymbol{C}^{T}_{(n:r)}\,d\sigma_\omega \tag{6}$$

is referred to as the r th order AOFRF with $r=0,\ldots,n$; *"* $\circ$ *" represents the Hadamard product,*

$$\boldsymbol{\Phi}_{(n:r)}(\omega_1,\ldots,\omega_n)=\frac{\boldsymbol{\Phi}_{H(n:r)}(\omega_1,\ldots,\omega_n)\prod_{j=1}^{n}U(j\omega_j)}{\sqrt{n}(2\pi)^{n-1}} \quad (7)$$

with $\boldsymbol{\Phi}_{H(n:r)}(.)$ representing the functions of frequencies $\omega_1,\cdots,\omega_n$; $\mathbf{L}_{(n:r)}(.)$ are the functions of the system linear characteristic parameters; and $\mathbf{C}_{(n:r)}$ are determined by the system nonlinear characteristic parameters.

Proof of Proposition 1. It can be shown by following the results in Peyton-Jones and Billings (1989) and Jing et al (2009) that

$$\begin{aligned}H_n(\omega_1,\ldots,\omega_n)&=\sum_{r=0}^{n}H_1^{\,r}(\omega_1,\ldots,\omega_r)\\&\times\left[\mathbf{L}_{(n:r)}(\omega_1,\ldots,\omega_n)\circ\boldsymbol{\Phi}_{H(n:r)}(\omega_1,\ldots,\omega_n)\right]\boldsymbol{C}^{T}_{(n:r)}\end{aligned} \quad (8)$$

Proposition 1 can then be obtained by substituting (7) into (3).

Remark 1. $\mathbf{L}_{(n:r)}(.)$ and $\mathbf{C}_{(n:r)}$ in Proposition 1 can be obtained by using a recursive algorithm as demonstrated in Appendix A.

Remark 2. Given the system linear characteristic parameters and input spectrum, (5) and (6) become the OFRF of system (1)/(2), a polynomial function of the system nonlinear characteristic parameters.

Remark 3. Generally, the AOFRF based representation (5) and (6) explicitly decouples the effects of the system linear and nonlinear characteristic parameters on the output spectrum. This will facilitate the study of the relationship between the system output spectrum and both the linear and nonlinear characteristic parameters.

4. The AOFRF based representation of the output frequency response of nonlinear systems

4.1. The AOFRF based representation

Denote $L_1^{-1}(.)$ for $i=1,\ldots,n$ as the elements composing $\mathbf{L}_{(n:r)}(.)$ as shown in Appendix A. It will be shown in Propositon 2 below that under certain conditions $H_1(.)$ and $L_1^{-1}(.)$ can be expanded into a power series in terms of the system linear characteristic parameters $c_{1,0}(.)$ and $c_{0,1}(.)$, and the AOFRF based representation (5) and (6) of the system output spectrum can be described as a polynomial function of both the system linear and nonlinear characteristic parameters.

Proposition 2. If there exist a set of constants $\mathbf{C}_z$ = $\left[c_z(k)\middle|k=1,\ldots,K\right]$ such that

$$\sum_{k=1}^{K}\left|c_{1,0}(k)-c_z(k)\right|<1 \quad (9)$$

(5) and (6) can be expanded into a polynomial function

$$\begin{aligned}Y(j\omega)&=\sum_{r=0}^{N}\tilde{Y}_r(j\omega)=\sum_{r=0}^{N}\boldsymbol{\Lambda}_r\boldsymbol{\Theta}_r^{\,T}\\&=\sum_{(j_1,\cdots,j_S)\in\mathbf{J}_S}\lambda_{j_1,\cdots,j_S}(j\omega)\theta_1^{\,j_1}\cdots\theta_S^{\,j_S}\end{aligned} \quad (10)$$

where $\boldsymbol{\Lambda}_r$ is a vector composed of the coefficients $\lambda_{j_1,\cdots,j_S}(j\omega),(j_1,\cdots,j_S)\in\mathbf{J}_S$, which are the functions of frequency variable ω; $\boldsymbol{\Theta}_r$, $r=0,\ldots,N$ are the vectors of the monomials in the polynomial representation;

$$\theta_1,\ldots,\theta_S\in\left[c_{p,q}(k_1,\cdots,k_{p+q})\middle|\begin{cases}p+q\geq 1\\k_1,\cdots,k_{p+q}=1,\ldots,K\end{cases}\right] \quad (11)$$

are the linear or nonlinear characteristic parameters of the NARX model (1); S is an integer and $\mathbf{J}_S$ is the set containing all indices of $j_1,\ldots,j_S$.

Proof of Proposition 2. Omitted due to space limitation.

Remark 4. The order of the polynomial representation (10) is determined by the order N of the Volterra series expansion and the order of the power series expansion for $H_1(.)$ and $L_1^{-1}(.)$ for $i=1,\ldots,n$. When the order for the power series expansion of $H_1(.)$ and $L_1^{-1}(.)$ are given as n_L, the maximum order of the linear characteristic parameters in any specific term in (6) can be directly obtained by referring to the expansion order of $H_1^{\,r}(.)$ and $\mathbf{L}_{(n:r)}(.)$ as rn_L and $(n-1)n_L$, respectively. This explains the importance of the introduction of the AOFRF concept which can, as simply illustrated above and described in more details in Section 4.2 below, significantly facilitate the determination of the structure of a polynomial representation for the system output spectrum in terms of both the linear and nonlinear characteristic parameters.

4.2. Determination of the AOFRF based representation

The determination of the AOFRF based polynomial representation (10) of the system output spectrum is concerned with determining both the structure and the coefficients of the polynomial. In order to determine the structure, the components in $\boldsymbol{\Theta}_r$, which are the monomials in the polynomial representation of the r th order AOFRF, can be determined by using Proposition 3.

Proposition 3. The monomials in the polynomial representation of the r th order AOFRF can be determined as

$$\boldsymbol{\Theta}_r=\bigcup_{n=r}^{N}\left[\boldsymbol{\Theta}^{L}_{(n:r)}\otimes\boldsymbol{\Theta}^{NL}_{(n:r)}\right] \quad (12)$$

In (12), "$\otimes$" denotes the Kronecker product; $\boldsymbol{\Theta}^{L}_{(n:r)}$ represents a vector only consisting of the system linear characteristic parameters such that

$$\boldsymbol{\Theta}^{L}_{(n:r)}=\left[\boldsymbol{\theta}_H\right]^{r}\otimes\bigcup_{i=1}^{n-1}\left[\boldsymbol{\theta}_L\right]^{i} \quad (13)$$

where $\boldsymbol{\theta}_H$ and $\boldsymbol{\theta}_L$ are the vectors of the monomials in the power series expansion of $H_1(\omega_i)$ and $L_1^{-1}(\omega_1,\cdots,\omega_i)$ for $i=1,\ldots,n$ in terms of system linear characteristic parameters with

$$\left[\boldsymbol{\theta}_H\right]^{r}=\underbrace{\boldsymbol{\theta}_H\otimes\cdots\otimes\boldsymbol{\theta}_H}_{r}\ \text{and}\ \left[\boldsymbol{\theta}_L\right]^{i}=\underbrace{\boldsymbol{\theta}_L\otimes\cdots\otimes\boldsymbol{\theta}_L}_{i} \quad (14)$$

$\boldsymbol{\Theta}_{(n:r)}^{NL}$ represents a vector only consisting of the system nonlinear characteristic parameters such that

$$\boldsymbol{\Theta}_{(n:r)}^{NL} = \left[\bigcup_{k_1,\ldots,k_n=1}^{K} c_{0,n}(k_1,\cdots,k_n) \right] \cup \left[\bigcup_{q=1}^{n-1} \bigcup_{p=1}^{n-q} \bigcup_{k_1,\ldots,k_n=1}^{K} \left(c_{p,q}(k_1,\cdots,k_{p+q}) \boldsymbol{\Theta}_{([n-q,p]:r)}^{NL} \right) \right] \cup \left[\bigcup_{p=2}^{n} \bigcup_{k_1,\ldots,k_n=1}^{K} \left(c_{p,0}(k_1,\cdots,k_p) \boldsymbol{\Theta}_{([n,p]:r)}^{NL} \right) \right] \quad (15)$$

where

$$\begin{cases} \boldsymbol{\Theta}_{([n,p]:r)}^{NL} = \bigcup_{i=1}^{n-p+1} \bigcup_{R=0}^{r} \boldsymbol{\Theta}_{(i:R)}^{NL} \otimes \boldsymbol{\Theta}_{([n-i,p-1]:r-R)}^{NL} ; \\ \boldsymbol{\Theta}_{([n,1]:r)}^{NL} = \boldsymbol{\Theta}_{(n:r)}^{NL} \end{cases} \quad (16)$$

$\boldsymbol{\Theta}_{(1:1)}^{NL} = [1]$ and $\boldsymbol{\Theta}_{(1:0)}^{NL} = \boldsymbol{\Theta}_{(n:r)}^{NL} = [\text{NULL}]$ for $r > n$

Proof of Proposition 3. Omitted due to space limitation.

Remark 5. Proposition 3 provides an algorithm that can be implemented using computer codes to automatically produce all the monomials in the polynomial form representation of the system output spectrum (10).

When $\boldsymbol{\Theta}_r$, $r = 0,\ldots,N$ have been obtained, the coefficients $\lambda_{j_1,\cdots,j_s}(j\omega)$ in the AOFRF based representation (10) of the output spectrum can be evaluated using a Weighted Least Squares (WLS) method from the simulated responses of system (1) to a given input in the case where the system linear and nonlinear characteristic parameters of concern vary over a selected range of values of interest (Zhu and Lang, 2017).

5. A case study

Consider the Duffing equation with nonlinear damping

$$\ddot{y}(t) + c_{n1}\dot{y}(t) + k_{n1}y(t) + k_{n3}y(t)^3 + c_{n3}\dot{y}(t)^3 = u(t) \quad (17)$$

where $c_{n1} = 80\ \text{N/ms}^{-1}$ and $c_{n3} = 200\ \text{N/m}^3\text{s}^{-3}$.

Approximating the first and the second derivatives in (17) as:

$$\dot{y}(t) = \frac{y(k) - y(k-1)}{\Delta t},\ \ddot{y}(t) = \frac{y(k+1) - 2y(k) + y(k-1)}{\Delta t^2} \quad (18)$$

and substituting (18) into (17) with $\Delta t = 1/256\,\text{s}$ yields a NARX model of system (17) as

$$\begin{aligned} y(k) &= c_{0,1}(1)u(k-1) + c_{1,0}(1)y(k-1) + c_{1,0}(2)y(k-2) \\ &+ c_{3,0}(1,1,1)y^3(k-1) + c_{3,0}(1,1,2)y^2(k-1)y(k-2) \\ &+ c_{3,0}(1,2,2)y(k-1)y^2(k-2) + c_{3,0}(2,2,2)y^3(k-2) \end{aligned} \quad (19)$$

where

$$\begin{gathered} c_{0,1}(1) = 1.526\times10^{-5};\ c_{1,0}(2) = -0.687; \\ c_{3,0}(1,1,2) = 1.024\times10^{5};\ c_{3,0}(1,2,2) = -1.024\times10^{5}; \quad (20a) \\ c_{3,0}(2,2,2) = 0.512\times10^{5}; \end{gathered}$$

$$\begin{gathered} c_{3,0}(1,1,1) = -\left(1.526\times10^{-5}k_{n3} + 0.512\times10^{5}\right); \\ c_{1,0}(1) = -\left(1.526\times10^{-5}k_{n1} - 1.687\right);\ \text{else } c_{p,q}(.) = 0 \quad (20b) \end{gathered}$$

The objective now is to investigate how the physical parameters k_{n1}, k_{n3} of system (17) affect the system output frequency response when the values of the two parameters vary over the range of

$$k_{n1} \in [0.5, 1.4]\times10^4\ \text{N/m and } k_{n3} \in [0,1]\times10^9\ \text{N/m}^3 \quad (21)$$

Denote $c_L = c_{1,0}(1)$ and $c_{NL} = c_{3,0}(1,1,1)$. It is known from the relationship between k_{n1} and k_{n3} and $c_{1,0}(1)$ and $c_{3,0}(1,1,1)$ that

$$\begin{cases} c_L = c_{1,0}(1) \in [1.473, 1.611] \\ c_{NL} = c_{3,0}(1,1,1) \in [0.512, 0.665]\times10^5 \end{cases} \quad (22)$$

Take $N = 5$, then the AOFRF based representation for the output spectrum of the system can be determined by the following steps.

Step 1: Determine a polynomial representation of $H_1(\omega_i)$ and $L_1^{-1}(\omega_1,\cdots,\omega_i)$ for $i = 1,\ldots,n$ in terms of the system linear characteristic parameter of concern, which is $c_L = c_{1,0}(1)$, in this case.

For NARX model (19), over the range of values of parameters $c_{1,0}(1)$ and $c_{3,0}(1,1,1)$ given by (21), it can be shown that the converge condition (9) is satisfied by chosen, for example, $c_z(1) = (1.473 + 1.611)/2 = 1.542$ and $c_z(2) = c_{1,0}(2) = -0.687$. Therefore, $H_1(\omega)$ can be expanded into a convergent polynomial function of c_L whose second order approximation can be written as

$$H_1(\omega) \approx \varphi_0(j\omega) + \varphi_1(j\omega)c_L + \varphi_2(j\omega)c_L^{\ 2} \quad (23)$$

Moreover, $L_1^{-1}(.)$ for $i = 1,\ldots,n$ can also be approximately expanded into a polynomial function of c_L of the same form as (23).

Step 2: Determine the structure of the AOFRF based representation of the system output frequency response.

According to Proposition 3, the monomial vector associated with the AOFRF based representation can be written as

$$\boldsymbol{\Theta} = \bigcup_{r=1}^{N} \left[\boldsymbol{\Theta}_{(r:r)}^{L} \otimes \boldsymbol{\Theta}_{(r:r)}^{NL} \right] = \left[\boldsymbol{\Theta}_{(1:1)}^{L} \otimes \boldsymbol{\Theta}_{(1:1)}^{NL} \right] \cup \left[\boldsymbol{\Theta}_{(3:3)}^{L} \otimes \boldsymbol{\Theta}_{(3:3)}^{NL} \right] \cup \left[\boldsymbol{\Theta}_{(5:5)}^{L} \otimes \boldsymbol{\Theta}_{(5:5)}^{NL} \right] \quad (24)$$

In this case, it is known from Step 1 that $\boldsymbol{\theta}_H = \boldsymbol{\theta}_L = [1, c_L, c_L^{\ 2}]$ and, from Proposition 3, it is known that

$$\begin{cases} \boldsymbol{\Theta}_{(1:1)}^{L} = \boldsymbol{\theta}_H = [1, c_L, c_L^{\ 2}] \\ \boldsymbol{\Theta}_{(1:1)}^{NL} = [1] \end{cases} \quad (25a)$$

$$\begin{cases} \boldsymbol{\Theta}_{(3:3)}^{L} = [\boldsymbol{\theta}_H]^3 \otimes \bigcup_{i=1}^{2} [\boldsymbol{\theta}_L]^i = [1, c_L, c_L^{\ 2}, \ldots, c_L^{\ 10}] \\ \boldsymbol{\Theta}_{(3:3)}^{NL} = [c_{NL}] \end{cases} \quad (25b)$$

and

$$\begin{cases} \boldsymbol{\Theta}_{(5:5)}^{L} = [\boldsymbol{\theta}_H]^5 \otimes \bigcup_{i=1}^{4} [\boldsymbol{\theta}_L]^i = [1, c_L, c_L^{\ 2}, \ldots, c_L^{\ 18}] \\ \boldsymbol{\Theta}_{(5:5)}^{NL} = [c_{NL}^{\ 2}] \end{cases} \quad (25c)$$

Consequently, substituting (25) into (24) yields the structure of the AOFRF based representation of the output spectrum of system (17) as:

$$\begin{aligned} Y(j\omega) \approx\ & \lambda_{0,0}(j\omega) + \lambda_{1,0}(j\omega)c_L + \lambda_{2,0}(j\omega)c_L^{\,2} \\ & + c_{NL}\sum_{l_1=0}^{10}\lambda_{l_1,1}(j\omega)c_L^{\,l_1} + c_{NL}^{\,2}\sum_{l_2=0}^{18}\lambda_{l_2,1}(j\omega)c_L^{\,l_2} \end{aligned} \tag{26}$$

Step 3: Determination of the coefficients in the polynomial representation of $Y(j\omega)$.

Equation (26) can be rewritten as

$$\begin{aligned} Y(j\omega) \approx\ & \tilde{\lambda}_{0,0}(j\omega) + \tilde{\lambda}_{1,0}(j\omega)\frac{c_L}{l_L} + \tilde{\lambda}_{2,0}(j\omega)\frac{c_L^{\,2}}{l_L^{\,2}} \\ & + \frac{c_{NL}}{l_{NL}}\sum_{l_1=0}^{10}\tilde{\lambda}_{l_1,1}(j\omega)\frac{c_L^{\,l_1}}{l_L^{\,l_1}} + \frac{c_{NL}^{\,2}}{l_{NL}^{\,2}}\sum_{l_2=0}^{18}\tilde{\lambda}_{l_2,1}(j\omega)\frac{c_L^{\,l_2}}{l_L^{\,l_2}} \end{aligned} \tag{27}$$

where $\lambda_{i,j}(j\omega) = \tilde{\lambda}_{i,j}(j\omega)/l_L^{\,i} l_{NL}^{\,j}$, $i, j = 0, 1, \ldots$ with $l_L = 1$ and $l_{NL} = 10^4$ are introduced as weights to transform the frequency related polynomial coefficients from $\lambda_{i,j}(j\omega)$ in (26) to $\tilde{\lambda}_{i,j}(j\omega)$ in (27). The objective is to circumvent possible numerical issues with evaluation of these coefficients (Zhu and Lang, 2017).

Now consider the situation where the system input is $u(t) = 3\cos(\omega t)$ with $\omega = 110$ rad/s. The coefficients $\lambda_{i,j}(j\omega)$, $i, j = 0, 1, \ldots$ in (27) were evaluated from the system output frequency responses to this input when the system linear and nonlinear characteristic parameters c_L and c_{NL} vary over the following range of values

$$\begin{cases} c_L/l_L = [1.54 : 0.003 : 1.57] \\ c_{NL}/l_{NL} = [5.80 : 0.05 : 6.30] \end{cases} \tag{28}$$

The result is a specific case of (27), which is a polynomial function of the system parameters c_L and c_{NL} containing 33 terms which are omitted here due to space limitation. A comparison of the system output spectrum evaluated using (27) thus determined and the result determined from the simulated system output response is shown in Fig.1.

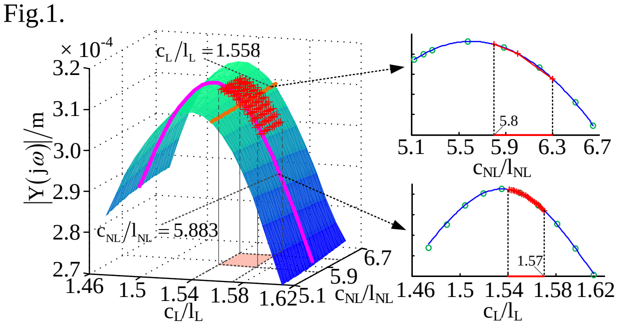

Fig.1. A comparison of the simulated system output spectrum with the result evaluated using the AOFRF based representation. Circle: Simulated results; Surface: Results evaluated using the AOFRF based representation; Cross: Data of the output spectra used to determine the AOFRF based representation.

It is worth noting that, if the system linear characteristic parameters are fixed, then the AOFRF based representation (27) will become the OFRF of the system, which is a 2nd order polynomial function of the system nonlinear characteristic parameter c_{NL}. If the system nonlinear characteristic parameters are fixed, the AOFRF based representation (27) becomes a 18th order polynomial function of the system linear characteristic parameter c_L. These results are also illustrated in Fig.1.

Fig.1 clearly indicates that the AOFRF based representation for the system output spectrum is valid over a wide range of values of the system linear and nonlinear characteristic parameters, including the values which are outside the parameter ranges (28), over which the polynomial representation was determined. This is because the AOFRF based representation is capable to capture inherent system dynamics rather than simply fit the data.

Now consider another case where a random band limited signal over the frequency range of $\omega \in [50, 200]$ rad/s with magnitude varying over $[-30, 30]$ N is applied as input to system (17). The AOFRF based representation for the system output spectrum was determined over the same range of the values of the system parameters c_L and c_{NL} as in (28). Fig.2 shows a comparison of simulated output spectra of system (17) to this random input with the results evaluated using the AOFRF based representation under three different sets of values of c_L and c_{NL}, indicating that the AOFRF based representation can also accurately des-cribe the system output spectra to a random input.

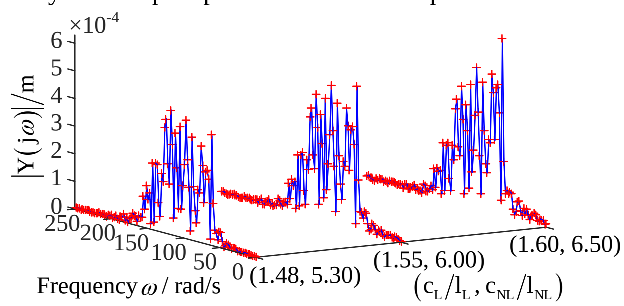

Fig.2. The output spectrum and its AOFRF based representation under a random input

Line: Simulation results; Cross: The AOFRF based representation.

It is worth noting that many terms in the AOFRF based representation of nonlinear system output spectra as determined in the case study above are often redundant. An optimal selection of the terms (monomials) in the polynomial representation is being investigated and will be discussed in a future publication.

6. Conclusions

The OFRF based representation for output frequency responses of nonlinear systems (Lang et al, 2007) has demonstrated significant advantages in both the system analysis and design. However, the OFRF only shows a polynomial relationship between the system's output spectrum and nonlinear characteristic parameters; it can't explicitly reveal the effect of system linear characteristic parameters on output spectra. In order to address this issue, a new concept known as the AOFRF of nonlinear systems, has been proposed. The AOFRF enables an explicit separation of the system linear and nonlinear characteristic parameters in the representation of the system output spectrum and, consequently, facilitates the derivation of a polynomial representation in terms of both the system linear and nonlinear characteristic parameters. A case study has demonstrated how to determine the AOFRF based representation for the output response of a nonlinear system to a harmonic and a random input, respectively. The results show that the new AOFRF based representation has potential to be used for the analysis and design of nonlinear systems in a wide range of applications.

Appendix A. The recursive algorithm

Let "$\oplus$" represent the concatenation between two vectors, then

$$\begin{aligned}\mathbf{L}_{(n:r)}(\omega_1,\ldots,\omega_n) = \mathrm{L}_n^{-1}(\omega_1,\ldots,\omega_n)\Bigg[&\mathop{\oplus}_{k_1,\ldots,k_n=1}^{K}\boldsymbol{\delta} \\ &\oplus\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}\mathop{\oplus}_{k_1,\ldots,k_{p+q}=1}^{K}\mathbf{L}_{([n-q,p]:r)}(\omega_1,\cdots,\omega_{n-q}) \\ &\oplus\mathop{\oplus}_{p=2}^{n}\mathop{\oplus}_{k_1,\ldots,k_p=1}^{K}\mathbf{L}_{([n,p]:r)}(\omega_1,\cdots,\omega_n)\Bigg]\end{aligned} \tag{A1}$$

and $\boldsymbol{\delta} = \begin{cases}[1] & r=0\\ [\mathrm{NULL}] & r>0\end{cases}$ with

$$\begin{cases}\mathbf{L}_{([n,p]:r)}(\omega_1,\cdots,\omega_n) = \mathop{\oplus}_{i=1}^{n-p+1}\mathop{\oplus}_{R=0}^{r}\mathbf{L}_{(i:R)}(\omega_1,\cdots,\omega_i) \\ \quad\otimes\mathbf{L}_{([n-i,p-1]:r-R)}(\omega_{i+1},\cdots,\omega_n) \\ \mathbf{L}_{([n,1]:r)}(\omega_1,\cdots,\omega_n) = \mathbf{L}_{(n:r)}(\omega_1,\cdots,\omega_n)\end{cases} \tag{A2}$$

$$\begin{cases}\mathbf{L}_{(1:1)}(\omega_1) = [1] \\ \mathbf{L}_{(1:0)}(\omega_1) = \mathbf{L}_{(n:r)}(\omega_1,\cdots,\omega_n) = [\mathrm{NULL}] \text{ for } r>n\end{cases}$$

only related to the system linear characteristic parameters, where

$$\mathrm{L}_n(\omega_1,\cdots,\omega_n) = 1-\sum_{k_1=1}^{K}c_{1,0}(k_1)\exp\left(-j(\omega_1+\cdots+\omega_n)k_1\Delta t\right)$$

$$\begin{aligned}\mathbf{C}_{(n:r)} = &\mathop{\oplus}_{k_1,\ldots,k_n=1}^{K}c_{0,n}(k_1,\ldots,k_n)\boldsymbol{\delta} \\ &\oplus\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}\mathop{\oplus}_{k_1,\ldots,k_{p+q}=1}^{K}\left(c_{p,q}(k_1,\ldots,k_{p+q})\mathbf{C}_{([n-q,p]:r)}\right) \\ &\oplus\mathop{\oplus}_{p=2}^{n}\mathop{\oplus}_{k_1,\ldots,k_p=1}^{K}\left(c_{p,0}(k_1,\ldots,k_p)\mathbf{C}_{([n,p]:r)}\right)\end{aligned} \tag{A3}$$

with

$$\begin{cases}\mathbf{C}_{([n,p]:r)} = \mathop{\oplus}_{i=1}^{n-p+1}\mathop{\oplus}_{R=0}^{r}\mathbf{C}_{(i:R)}\otimes\mathbf{C}_{([n-i,p-1]:r-R)}; \\ \mathbf{C}_{([n,1]:r)} = \mathbf{C}_{(n:r)}\end{cases} \tag{A4}$$

$$\mathbf{C}_{(1:1)} = [1] \text{ and } \mathbf{C}_{(1:0)} = \mathbf{C}_{(n:r)} = [\mathrm{NULL}] \text{ for } r>n$$

determined by the system nonlinear characteristic parameters.

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