THE UNIVERSITY of EDINBURGH

Edinburgh Research Explorer

Some notes on Lie ideals in division rings

Citation for published version:
Aghabali, M, Ariannejad, M & Madadi, A 2018, 'Some notes on Lie ideals in division rings' Journal of algebra and its applications, vol 17, no. 03, 1850049. DOI: 10.1142/S0219498818500494

Digital Object Identifier (DOI):
10.1142/S0219498818500494

Document Version:
Peer reviewed version

Published In:
Journal of algebra and its applications

M. Aaghabali [a], M. Ariannejad [b], A. Madadi [c]

[a] School of Mathematics, The University of Edinburgh, James Clerk Maxwell Building,

The Kings Buildings, Mayfield Road EH9 3JZ, Edinburgh, UK

[b] Department of Mathematics, University of Zanjan, Zanjan, Iran

[c] Department of Mathematics, Zanjan Branch Azad Islamic University, Zanjan, Iran

Abstract A Lie ideal of a division ring A is an additive subgroup L of A such that the Lie product $[l, a] = la - al$ of any two elements $l \in L, a \in A$ is in L, or $[l, a] \in L$. The main concern of this paper is to present some properties of Lie ideals of A which may be interpreted as being dual to known properties of normal subgroups of A^*. In particular, we prove that if A is a finite-dimensional division algebra with center F and char$F \neq 2$, then any finitely generated $\mathbb{Z}$-module Lie ideal of A is central. We also show that the additive commutator subgroup $[A, A]$ of A is not a finitely generated $\mathbb{Z}$-module. Some other results about maximal additive subgroups of A and $M_n(A)$ are also presented.

AMS Classification: 17A01, 17A35.

Keywords: Lie ideal, division ring, finitely generated.

1 Preliminary

Let A be an associative algebra over a field F with dimension $[A : F]$. If we replace the usual multiplication ab of two elements a and b of A by their *Lie product* $[a, b] = ab - ba$, then at the same time we have a non-associative structure of a Lie ring on A, usually denoted by $\overline{A}$. A *Lie ideal* in A is a regular ideal of $\overline{A}$ with its Lie multiplication; in other words, an additive subgroup I of A is called a Lie ideal if for all $i \in I, a \in A$ we have $[i, a] \in I$. The main concern of this paper is to present some dual properties of these two structures on A: considering A as an associative algebra and as a Lie ring. These properties represent very similar roles of normal subgroups in A^* (the multiplicative group of unit elements) and (Lie) ideals in $\overline{A}$. In particular, when A is a division ring we give some properties of Lie ideals of A which are analogous to similar properties of normal subgroups of A^*.

A *derivation* on A is an additive group homomorphism $d : A \to A$ satisfying $d(ab) = d(a)b + ad(b)$. A derivation $d_a : A \to A$ that is defined by $d_a(b) = ab - ba$, for some fixed $a \in A$ is called *inner derivation*; the set $\{d_a | a \in A\}$ of all inner derivations of A is denoted by $InnDer(A)$. In group theory, the normal subgroups of a group G usually are defined as the subgroups which are invariant under all inner automorphisms of G (denoted by $Inn(G)$). Equivalently, in the theory of Lie algebras, Lie ideals of an algebra A are defined as the submodules which are invariant under all inner derivations of A. Let $Z(A)$ denote the center of A; then we have the following similar isomorphisms: $Inn(A^*) \simeq A^*/Z(A^*)$ and as a dual version $InnDer(A) \simeq \overline{A}/Z(A)$ [10, p. 73].

The following two main theorems give some more important signs in identifying a connection between the concepts of normal subgroups and Lie ideals: Let F be a field, then the SkolemNoether theorem (in particular) states that if A is a finite-dimensional central simple F-algebra, then any F-automorphism of A is inner [3, p. 93]. A dual version of this theorem states that if A is a finite-dimensional central simple F-algebra, then any F-linear derivation of A is inner [3, p. 105].

The other one is the CartanBrauerHua theorem which states that if A is a division ring and B is a subdivision ring of A such that B^* is a normal subgroup of A^*, then either $B = A$ or $B \subseteq Z(A)$ [6, p. 211]. A dual version of this theorem states that if A is a division ring and B is a subdivision ring of A such that B is a Lie ideal in A and char$A \neq 2$, then either $B = A$ or $B \subseteq Z(A)$ [6, p. 205].

We consider some results about the structure of normal subgroups of a division ring and examine their dual versions in terms of Lie ideals of the division ring. As an example, Akbari et al. [1, 2] proved that "If

E-mail Addresses: mehdi.aaghabali@ed.ac.uk, arian@znu.ac.ir, as.madadi@yahoo.com
The authors are indebted to the referee for careful reading and detailed valuable comments, which improved the paper. The research of the first author was supported with ERC [grant number 320974].

A is a finite-dimensional division algebra with center F, then any finitely generated normal subgroup of A^* is central". Also, they proved that "If A is an infinite division ring with center F such that $[A : F] < \infty$, then A^* contains no finitely generated maximal subgroups". Here, as an analogous statement, we show that "If A is a finite-dimensional division algebra with center F such that $\text{char}F \neq 2$, then any finitely generated $\mathbb{Z}$-module Lie ideal of A is central". We also show that "If A is an infinite division ring with center F such that $[A : F] < \infty$, then A contains no finitely generated $\mathbb{Z}$-module maximal additive subgroup". We show that the additive commutator subgroup $[A, A]$ of A is not a finitely generated $\mathbb{Z}$-module. To sum up, the applicability of similar arguments we used to prove these dual properties reveals similar roles of these two substructures: the normal subgroups and Lie ideals in division rings [7, 8, 9].

2 Main Results

We begin by recalling the following theorem:

Theorem 1.[4, p. 5] *Let A be a division algebra with center F and $\text{char}(A) \neq 2$. Assume that L is a Lie ideal of A. Then either $L \subseteq F$ or $[A, A] \subseteq L$.*

This result allows us to present our first main result:

Theorem 2. *Let A be a division ring which is finite-dimensional over its center F and $\text{char}F \neq 2$. If A contains a non-central Lie ideal which is a finitely generated $\mathbb{Z}$-module, then F is finitely generated over its prime subfield P.*

Proof. Let L be a non-central finitely generated $\mathbb{Z}$-module Lie ideal of A. By Theorem 1, $[A, A] \subseteq L$. Let T be the F-subdivision algebra generated by L. Since any non-commutative division ring is generated as a division ring by all of its additive commutators together with its center [6, p. 205], we conclude that $T = A$. Note that since A is a finite-dimensional division ring, L generates A as an algebra, too. If $[A : F] = n$, then A has a faithful matrix representation θ of degree n [5, p. 82] (usually called the regular representation). Since L is a finitely generated $\mathbb{Z}$-module, there exist a finite number k of matrices $M_1, \ldots, M_k$ in $GL_n(F)$ which generate $\theta(L)$ in $M_n(F)$ as a $\mathbb{Z}$-module. Let $\Gamma \subseteq F$ be the set of elements of F that appear as entries in the matrices $M_1, \ldots, M_k$. Since L builds A as an algebra, invoking θ one can see that this set of matrices first builds $\theta(L)$ and then builds $A \simeq \theta(A)$ in $M_n(F)$. Since θ is an embedding, we may consider $A = T \subseteq M_n(P(\gamma)) \subseteq M_n(F)$, where $P(\Gamma)$ is the subfield of F generated by $P \cup \Gamma$. Consequently, for all $a \in F$, its representation aI is in $M_n(P(\Gamma))$, where I is the identity matrix and so $a \in P(\Gamma)$ or $F = P(\Gamma)$. □

We need the following lemma to present our next results.

Lemma 3. *Let D be a UFD with infinitely many prime ideals and let T be its field of fractions. Let A be a T-subalgebra of $M_n(T)$. Then any Lie ideal of A which is finitely generated as a $\mathbb{Z}$-module is central.*

Proof. For the sake of contradiction, assume that L is a non-central Lie ideal of A which is finitely generated as a $\mathbb{Z}$-module. Let $a \in A$ and $l \in L$ be such that $[a, l] \neq 0$. Since L is finitely generated as a $\mathbb{Z}$-module, there is a nonzero $d \in D$ such that $L \subseteq M_n(D[\frac{1}{d}])$. Since T is a field, for any $x \in T$ we have $x[a, l] = [xa, l] \in L$. Hence $x[a, l] \in M_n(D[\frac{1}{d}])$. Since $[a, l] \in M_n(T)$ is a nonzero matrix, one of its entries is nonzero, say b. Therefore, $xb \in D[\frac{1}{d}]$ for all $x \in T$, which is a contradiction, for if p is a prime element such that $p \nmid d$, then $b/p^n \notin D[\frac{1}{d}]$ for enough large positive integer n. □

Theorem 4. *Let A be a finite-dimensional division algebra with center F such that $\text{char}F \neq 2$. Then any finitely generated $\mathbb{Z}$-module Lie ideal of A is central.*

Proof. By Theorem 2, F is finitely generated over its prime subfield P. Hence we may write F as a finite extension of a purely transcendental extension $P(x_1, \ldots, x_d)$ of P, where d is the transcendence degree of F over P. We consider two cases:

Case 1. $d = 0$. If $P = \mathbb{F}_p$, then F is a finite field. Hence by the Wedderburns Little Theorem, A is commutative [6, p. 203]. If $P = \mathbb{Q}$, then $[F : \mathbb{Q}] < \infty$ allows us to view $A \in M_n(\mathbb{Q})$ via the regular representation. Now, using the above lemma, we are done.

Case 2. $d > 0$. Then $P[x_1, \ldots, x_d]$ is a UFD with infinitely many prime ideals. Let T be the field of fractions of D. Since $[F : T] < \infty$, again we may view $A \in M_n(T)$ via the regular representation. Now, applying the above Lemma completes the proof □

The following is our main result:

Corollary 5. *Let A be a non-commutative division algebra of finite dimension over its center F and $\text{char}F \neq 2$. Then the additive commutator subgroup $[A, A]$ of A is not finitely generated as a $\mathbb{Z}$-module.*

The Lie ideal structure we have considered above really is a kind of additive subgroup of algebras. In what follows, we turn our attention to another kind of additive subgroups. By a *maximal additive subgroup* of an algebra, we mean an additive subgroup which is *maximal* under inclusion among proper ones. Clearly, by a *maximal Lie ideal*, we mean a Lie ideal which is maximal under inclusion among Lie ideals.

Corollary 6. *Let A be a division ring with center F and $\mathrm{char}F \neq 2$. Assume that L is a proper maximal additive subgroup of A containing F. If the additive group index $[L : F]$ of L over F is finite, then $A = F$.*

Proof. First, consider the case $[A : F] < \infty$ and let $x_1, \ldots, x_t$ be the representations of the finite number of cosets of F in L, so $L = (F + x_1) \cup \cdots \cup (F + x_t)$. We have $L = F + \langle\{x_1, \ldots, x_t\}\rangle$, where $\langle\{x_1, \ldots, x_t\}\rangle$ is the additive subgroup generated by $x_1, \ldots, x_t$ in A. Suppose that $x \in A \setminus L$. By maximality of L, we obtain that $A = F + \langle\{x_1, \ldots, x_t, x\}\rangle$. Put $H = \langle\{x_1, \ldots, x_t, x\}\rangle$. Thus, $A = F + H$ and consequently $[A, A] = [H, H]$. This means that $[H, H]$ is a Lie ideal of A which is finitely generated as a $\mathbb{Z}$-module by the finite set $\{x_ix_j - x_jx_i, xx_i - x_ix; i, j = 1, \ldots, t\}$. By Theorem 4, we conclude that $[A, A] = [H, H] \subseteq F$ or $A = F$ as desired. Now, consider the case $[A : F] = \infty$. As in the above case, let $L = (F + x_1) \cup \cdots \cup (F + x_t)$ and take $x \in A \setminus L$. Let V be the vector space generated by the set $\{1, x_1, \ldots, x_t, x\}$ over F. Clearly $[V : F] < \infty$ and $L \subsetneq V$. Now, maximality of L implies that $V = A$, a contradiction. This completes the proof. □

We continue our study with the following two lemmas:

Lemma 7. *Let A be an F-algebra and L be a maximal Lie ideal of A. Then*

(i) L contains either F or $[A, A]$.

(ii) If A is a division ring, then either $A = F(L)$ or $L \setminus \{0\}$ is the multiplicative group $F(L) \setminus \{0\}$, where $F(L)$ is the division ring generated by $F \cup L$.

Proof. (i) Assume that L does not contain F. By maximality of L and since $F + L$ is a Lie ideal containing L, we have $A = F + L$. Consequently, we have $[A, A] = [L, L] \subseteq L$.

(ii) Consider the division ring $F(L)$ generated by L and F. By maximality of L and since $F(L)$ is a Lie ideal containing L, we have either $A = F(L)$ or $L = F(L)$. In the latter case, we obtain $F(L)^* = F(L) \setminus \{0\} = L \setminus \{0\}$ is a multiplicative group. □

Lemma 8. *Let A be a division ring with center F and assume that L is a maximal Lie ideal of A. Then either the multiplicative center of L is equal to $F \cap L$ or L is a maximal division subring of A.*

Proof. By first part of the previous lemma, either $F \subseteq L$ or $[A, A] \subseteq L$. If $[A, A] \subseteq L$, then $Z(L) = C_L(L) \subseteq C_A(L) \subseteq C_A([A, A]) = F$, where the latter inclusion is by [6, p. 205]. In other words, $Z(L) \subseteq F \cap L$ and so $Z(L) = F \cap L$. If $F \subseteq L$ and $[A, A]$ is not contained in L, then consider the division ring $F(L)$. Since $F(L)$ is a Lie ideal containing L, we have either $A = F(L)$ or $L = F(L)$ by the maximality of L in A. In the first case, it is easily checked that $Z(L) = Z(A) = F \cap L$. Otherwise, $L = F(L)$ which means that L is a maximal division subring of A. □

Now, we can show that Theorem 4 has an analogous statement which applies to maximal additive subgroups.

Theorem 9. *Let A be a non-commutative division ring with center F. Then A contains no finitely generated $\mathbb{Z}$-module maximal additive subgroup.*

Proof. Assume that L is a maximal additive subgroup of A that is finitely generated as a $\mathbb{Z}$-module. For each element $x \in A \setminus L$, we have $A = L + Zx$ which means that A is a finitely generated $\mathbb{Z}$-module and this is impossible: If $\mathrm{char}F \neq 0$, then A would be finite and so commutative. If $\mathrm{char}F = 0$, this condition makes A to be finite-dimensional over the center. So A is a finitely generated $\mathbb{Z}$-module Lie ideal of a finite-dimensional division ring A which by Theorem 4 is contained in F and thus is commutative, contradicting our assumption that A is non-commutative. □

Theorem 10. *Let A be a division algebra algebraic over its center F with $\mathrm{char}F \neq 2$ and let n be a natural number. Assume that L is a maximal Lie ideal of $M_n(A)$. If L is finite, then $A = F$.*

Proof. Let $I \in M_n(A)$ be the identity matrix. By Lemma 7(i), either $FI \subseteq L$ or $[M_n(A), M_n(A)] \subseteq L$. The latter case implies that $[A, A]$ is finite, so $A = F$ by Corollary 6. If $FI \subseteq L$, then F is finite and so A a division algebra algebraic over a finite field would be commutative and thus $A = F$. □

References

[1] S. Akbrai and M. Mahdavi-Hezavehi, Normal subgroups of $GL_n(D)$ are not finitely generated, *Proc. Amer. Math. Soc.* **128**(6) (2000) 16271632.

[2] S. Akbari, M. Mahdavi-Hezavehi and M. G. Mahmudi, Maximal subgroups of $GL_1(D)$, *J. Algebra* **217** (1999) 422433.

[3] B. Farb and R. K. Dennis, Noncommutative Algebra, Graduate Text in Mathematics, Vol. 144 (Springer-Verlag, 1993).

[4] I. N. Herstein, Topics in Ring Theory (The University of Chicago Press, 1972).

[5] R. S. Pierce, Associative Algebras (Springer-Verlag, New York, 1982).

[6] T. Y. Lam, A First Course in Noncommutative Rings, 2nd edn., Graduate Text in Mathematics, Vol. 131 (Springer-Verlag, New York, 2001).

[7] M. Mahdavi-Hezavehi, Free subgroups in maximal subgroups of $GL_1(D)$, *J. Algebra* **241** (2001) 720730.

[8] M. Mahdavi-Hezavehi and S. Akbari, Some special subgroups of $GL_n(D)$, *Algebra Colloq.* **5**(4) (1998) 361370.

[9] M. Mahdavi-Hezavehi, M. G. Mahmudi and S. Yasamin, Finitely generated subnormal subgroups of $GL_n(D)$ are central, *J. Algebra* **225** (2000) 517521.

[10] L. H. Rowen, Ring Theory, Student edition (Academic Press, Boston, 1991).