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Observer design for sampled-data systems with unknown inputs and uncertainties based on quasi sliding motion

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Abstract— In this paper, the state and unknown input estimation problem is addressed for a sampled-data system whose dynamics is affected by external signals and uncertainties. Unlike the numerous sliding mode observers for dynamical continuous-time systems which employ a nonlinear switching injection term to force the state errors to converge to zero in finite time, the observer design problem for sampled-data systems is often faced with limitations on the hardware, where the sampling time period cannot be made arbitrarily small. Hence, an approximate implementation of an observer, which is designed for a continuous-time system, is not always suitable in the sampled-data context. By exploiting the quasi-sliding motion concept, we propose an observer which takes into account the sampling time period. A theoretical analysis is provided to formally show the convergence of the observer. In the formulation, estimates of the unknown inputs are also given. Simulation results are shown to illustrate the efficacy of the proposed method.

I. INTRODUCTION

There have been a great number of papers concerning the problem of estimating state variables and unknown inputs for continuous-time systems using sliding mode approaches: see [1], [2], [3], [4], [5], [6], [7] and references therein. A typical property of these observers is that a sliding surface based on the output error is constructed and a nonlinear switching injection term is introduced to force the output errors to reach the sliding surface in finite time. The unknown inputs are then reliably reconstructed based on the nonlinear injection term (often passed through low-pass filters).

The problem of designing observers for discrete-time systems with unknown inputs has also been addressed extensively [8], [9], [10], [11], [12], [13]. In [8], a sampling delayed observer is constructed if the system is left-invertible without invariant zeros. In that scheme, some variables that are not affected by the unknown inputs are calculated and a change of coordinates is employed to make the transformed system well-suited for designing the delayed estimator. The states and unknown inputs of the linear systems are recovered after some finite number of sampling delays. In [9], a delayed observer is designed to reconstruct the unknown states and unknown inputs of a discrete-time system. A parameterization of the observer gain was developed to decouple the unknown inputs from the estimation error. In [10], [11],

the reconstruction of the state and the unknown input for a class of discrete-time linear systems is conducted based on geometric methods. A finite impulse response (FIR) system processes the output measurements to reconstruct the initial state and the subsequent state trajectory, with a possible delay related to the properties of the original system, from which the unknown input is recovered with a further delay of one step. In [12], a H_∞ filter approach was proposed for discrete-time linear system with unknown inputs, in which there are no similarity transformations. The description of the original model is modified to construct a linear filter, from which the original states and the unknown inputs of the system are extracted. In [13], a delayed observer was designed based on linear matrix inequalities (LMIs) from which the observer gains and peak-gain performance bounds on a pre-specified performance output of the observer are calculated. The unknown inputs can be reconstructed to a specified level of accuracy if the system is minimum-phase and the specified delay is sufficiently large.

As discussed above, numerous observers have been designed for continuous-time and discrete-time systems, in comparison, relatively little effort has been spent on sampled-data systems. In [14], an observer scheme was proposed for robust fault reconstruction for a class of continuous systems in which the output measurement is taken at the sampling times but the control input is still a continuous function. It should be noted that in sampled-data systems, switching actions at infinite frequency (required at least theoretically by sliding mode systems) cannot be achieved due to the sample/hold process. As a result, there is no ideal sliding mode. Instead, only "quasi sliding modes" are obtained, which keeps the system state in a boundary layer of the sliding surface ([15]). Since the measurement is taken at sampling times, there is no continuity in the output measurement. Furthermore, the system behavior between sampling times should be taken into account in sampled-data systems, which is different from discrete-time systems.

In this paper, we aim to design an observer to estimate the state and unknown input of a sampled-data system. The method is based on quasi-sliding motion ideas and requires that the system is minimum-phase and relative degree one. The difference between this work and other papers in the literature is that a one-step delay is used for the observer and our work is focused on sampled-data systems, which take into account the sampling time. Furthermore, the accuracy of the state and input estimates is analyzed.

The contributions of the paper are the following: a new sampled-data observer is proposed to estimate state variables

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and unknown signals simultaneously for a class of sampled-data systems which are under the influence of external disturbances and uncertainties. Furthermore, it is constructed based on quasi-sliding motion ideas to capture the behavior of the system states and external disturbances. A convergence and robustness analysis is provided to justify the proposed method.

Throughout the paper, $\lambda\{A\}$ denotes the spectrum of the matrix A. Matrix I_m stands for an identity matrix of order m. As in [16], a vector function $f(t,s) \in R^n$ is said to be $O(s)$ over an interval $[t_1,t_2]$, if there exist positive constants K and s^* such that $\|f(t,s)\| \leq Ks$, $\forall s \in [0,s^*]$, $\forall t \in [t_1,t_2]$. The notation $f[k]$ stands for $f(kT_s)$, where $k = 0,1,2,...$ denotes the index of the discrete-time sequence as a result of sampling where T_s is the sampling time period.

The organization of the paper is as follows. Some preliminaries are given in Section II and the estimator design is derived in Section III. Numerical results are employed to illustrate the efficacy of the proposed method in Section IV. Finally, Section V gives some conclusions.

II. PRELIMINARIES

Here we revisit the construction of a sliding mode observer for a continuous time linear system with external disturbances and uncertainties in [2].

Consider an uncertain linear dynamical system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ff(t) + M\xi(t,y,u) \quad (1) \\ y(t) &= Cx(t), \end{aligned}$$

where $x(t) \in R^n$ is the system state, $u(t) \in R^m$ is the system control input, $y(t) \in R^p$ is the system output, $f(t) \in R^q$ is an unknown bounded exogenous disturbance, $\xi(t,y,u) \in R^\kappa$ is the uncertainty, $q \leq p < n$, and the matrices C and F are of full rank. The system matrices A, B, C, F, and M are constant and of appropriate dimension. It is assumed that the state variables of the system (1) are unknown and only the control input $u(t)$ and output $y(t)$ are available.

We use the following assumptions for our design and analysis.

Assumption 1: System (1) has the following properties:

- rank$(CF) = q$
- The invariant zeros (if any) of (A,F,C) lie in the left half plane.

Assumption 2: The disturbance $f(t)$ and its first and second time derivatives are bounded.

Assumption 3: The uncertainty $\xi(t,u,y)$ is unknown but bounded subject to $\|\xi(t,y,u)\| < \nu$ where the positive scalar ν is known.

If $p > q$ and Assumption 1 is satisfied then the system matrices (A,F,C), through state transformation, can be brought into the following structure:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ F_0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & T_0 \end{bmatrix} \quad (2)$$

where $A_{11} \in R^{(n-p)\times(n-p)}$, $F_2 \in R^{p\times q}$ is nonsingular, $T_0 \in R^{p\times p}$ is orthogonal, and

$$F_0 = \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \quad (3)$$

where $F_2 \in R^{q\times q}$ and $\det F_2 \neq 0$. A state observer of the form

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t) - G_l e_y(t) + G_n v_n \\ y_0(t) &= Cz(t) \end{aligned} \quad (4)$$

was proposed in [2] where the discontinuous output error injection vector v_n is

$$v_n = \begin{cases} -\rho(t,y,u)\dfrac{P_0 e_y}{\|P_0 e_y\|} & \text{if } e_y \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where $e_y := y_0 - y$, and the scalar function $\rho : R_+ \times R^p \times R^m \to R$ is a design parameter. Matrices P_0, G_n, and G_l are determined from solving an H_∞ optimization problem [2], specifically:

Minimize γ with respect to the variables P, H, and E subject to

$$\begin{bmatrix} PA + A^T P - \gamma C^T (DD^T)^{-1}C & -PB_d & E^T \\ -Bd^T P & -\gamma I & H^T \\ E & H & -\gamma I \end{bmatrix} < 0 \quad (6)$$

where $B_d = \begin{bmatrix} 0 & M \end{bmatrix}$, $D = \begin{bmatrix} D_1 & 0 \end{bmatrix}$ with nonsingular $D_1 \in R^{p\times p}$, and $H = \begin{bmatrix} 0 & H_2 \end{bmatrix}$ with $H_2 \in R^{q\times\kappa}$. Once the solution of (6) is obtained, if P is decomposed as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad (7)$$

then from [2] a choice for the observer gains is

$$\begin{aligned} G_l &= \gamma P^{-1} C^T (DD^T)^{-1} & (8) \\ L &= P_{11}^{-1} P_{12} & (9) \\ G_n &= \begin{bmatrix} -LT_0^T \\ T_0^T \end{bmatrix} & (10) \\ P_0 &= T_0(P_{22} - L^T P_{11} L)T_0^T. & (11) \end{aligned}$$

For a sampled-data system, the observer in (4) may not be suitable as the control input $u(t)$ is kept constant between consecutive sampling periods and the measurement is taken at sampling times. Furthermore, the sample period T_s of the system may not be small due to limitations of the system hardware. Hence, we cannot obtain an infinite switching mechanism in (5) to estimate the unknown input $f(t)$. Therefore, in this paper an observer for a sampled-data counterpart of system (1) is considered, which estimates the system states and the unknown input $f(t)$.

To simplify the design and analysis process which follows, introduce the following transformation [2]:

$$T_L = \begin{bmatrix} I_{n-p} & L \\ 0 & T \end{bmatrix} \quad (12)$$

where L is calculated from (9) and T is an orthogonal matrix. In the new coordinates, the system matrices take the form

$$
\begin{aligned}
\mathscr{A} &= \begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{21} & \mathscr{A}_{22} \end{bmatrix}, \mathscr{F} = \begin{bmatrix} 0 \\ \mathscr{F}_2 \end{bmatrix}, \mathscr{C} = \begin{bmatrix} 0 & I_p \end{bmatrix}, \\
\mathscr{M} &= \begin{bmatrix} \mathscr{M}_1 \\ \mathscr{M}_2 \end{bmatrix}, \mathscr{B} = T_L B,
\end{aligned}
\tag{13}
$$

where in particular $\mathscr{A}_{11} = A_{11} + L A_{21}$ and $\mathscr{F}_2 = T F_2$. Note that L is such that $\mathscr{A}_{11}$ is Hurwitz i.e. its eigenvalues lie in the left half side of the complex plane [2]. Rewrite the original dynamical system (1) in the new coordinates as

$$
\begin{aligned}
\dot{\bar{x}}(t) &= \mathscr{A}\bar{x}(t) + \mathscr{B}u(t) + \mathscr{F}f(t) + \mathscr{M}\xi(t,y,u) \\
y(t) &= \mathscr{C}\bar{x}(t),
\end{aligned}
\tag{14}
$$

where $\bar{x}$ is the state variable in the new coordinates. The sampled-data counterpart of (14) is given by

$$
\begin{aligned}
\bar{x}[k+1] &= \Phi\bar{x}[k] + \Gamma u[k] + d[k] + \beta[k] \\
y[k] &= C\bar{x}[k],
\end{aligned}
\tag{15}
$$

where $\Phi = e^{\mathscr{A}T_s}$, $\Gamma = \int_0^{T_s} e^{\mathscr{A}\tau} d\tau \mathscr{B}$. The disturbance is

$$
d[k] = \int_0^{T_s} e^{\mathscr{A}\tau} \mathscr{F} f((k+1)T_s - \tau) d\tau,
\tag{16}
$$

the uncertainty is

$$
\begin{aligned}
\beta[k] &= \int_0^{T_s} e^{\mathscr{A}\tau} \mathscr{M}\xi(\bar{\tau}, y(\bar{\tau}), \\
& \qquad u(\bar{\tau})) d\tau,
\end{aligned}
\tag{17}
$$

where $\bar{\tau} = (k+1)T_s - \tau$ and T_s is the sampling time period. As in [17], [18], define $\bar{\mathscr{A}}$, $\bar{\bar{\mathscr{A}}}$ as

$$
\bar{\mathscr{A}} = \frac{1}{T_s}(\Phi - I_n)
\tag{18}
$$

$$
\bar{\bar{\mathscr{A}}} = \frac{1}{T_s^2}(\Phi - I_n - T_s\mathscr{A}).
\tag{19}
$$

Then,

$$
\bar{\mathscr{A}} = \mathscr{A} + T_s\bar{\bar{\mathscr{A}}},
\tag{20}
$$

and since

$$
\Phi = \sum_{k=0}^{\infty} T_s^k \frac{\mathscr{A}^k}{k!},
\tag{21}
$$

it follows

$$
\bar{\bar{\mathscr{A}}} = \sum_{k=2}^{\infty} T_s^{k-2} \frac{\mathscr{A}^k}{k!} = O(1),
\tag{22}
$$

and hence,

$$
\Phi = I_n + T_s(\mathscr{A} + T_s\bar{\bar{\mathscr{A}}}).
\tag{23}
$$

Let

$$
\Pi = \int_0^{T_s} e^{\mathscr{A}\tau} d\tau \mathscr{F}
\tag{24}
$$

and define

$$
\bar{\mathscr{F}} = \frac{\Pi}{T_s},
\tag{25}
$$

$$
\bar{\bar{\mathscr{F}}} = \frac{1}{T_s^2}(\Pi - T_s\mathscr{F}).
\tag{26}
$$

Then, as argued in [18],

$$
\Pi = T_s\bar{\mathscr{F}} = T_s(\mathscr{F} + T_s\bar{\bar{\mathscr{F}}}) = T_s(\mathscr{F} + T_s\bar{\bar{\mathscr{F}}}) = O(T_s).
\tag{27}
$$

Let $\bar{\mathscr{F}}$ and $\bar{\bar{\mathscr{F}}}$ be conformally partitioned into $\bar{\mathscr{F}}_1$, $\bar{\mathscr{F}}_2$, $\bar{\bar{\mathscr{F}}}_1$, $\bar{\bar{\mathscr{F}}}_2$ compatible with $\mathscr{F}$ in (13). Due to the structures of $\mathscr{F}$ and Π in (13) and (27) respectively,

$$
\Pi = \begin{bmatrix} T_s^2 \bar{\bar{\mathscr{F}}}_1 \\ T_s\bar{\mathscr{F}}_2 \end{bmatrix}.
\tag{28}
$$

Some properties of the disturbance $d[k]$ are described in the following lemma, [19], [20], [18].

Lemma 2.1: If Assumption 2 holds, then

$$
\begin{aligned}
d[k] &= \Pi f[k] + \frac{T_s}{2}\Pi v[k] + T_s^3 \Delta d_0[k] \\
&= \Pi \frac{(f[k] + f[k+1])}{2} + T_s^3 \Delta d_0[k] \\
&= \Pi f[k+1/2] + T_s^3 \Delta d[k],
\end{aligned}
\tag{29a}
$$

$$
d[k] - d[k-1] = O(T_s^2),
\tag{29b}
$$

$$
d[k] - 2d[k-1] + d[k-2] = O(T_s^3),
\tag{29c}
$$

where $v(t) = \dot{f}(t)$, and

$$
\Delta d[k] = \Delta d_0[k] - \frac{\Pi}{2}\int_{kT_s}^{(k+1)T_s}\int_{kT_s}^{\sigma}\dot{v}(\tau)d\sigma d\tau,
\tag{30}
$$

where

$$
\begin{aligned}
\Delta d_0[k] &= Gv[k] \\
&+ \frac{1}{T_s^3}\int_0^{T_s} e^{\mathscr{A}\tau}\mathscr{F}\int_{kT_s}^{(k+1)T_s - \tau}\int_{kT_s}^{\beta}\dot{v}(\sigma)d\sigma d\beta d\tau.
\end{aligned}
\tag{31}
$$

Note that $d_0[k] = O(1)$ and from (29a)

$$
G = (-\frac{1}{12}\mathscr{A} - \frac{T_s}{12}\bar{\bar{\mathscr{A}}})\mathscr{F} = O(1)
\tag{32}
$$

Proof: The proof follows the results presented in [18]. ∎

Our objective is to design a sliding mode observer for the sampled-data system (15) that generates the estimates of $x[k]$ and the unknown disturbance $f[k]$.

III. Main results

In this section, we will create an observer for the sampled-data system in (15). The convergence of the proposed observer will be discussed initially in the situation when the disturbance and uncertainty are not taken into account. Finally, the estimate of the disturbance and the bound on the state estimates will be analyzed.

A. Observer design

From the previous section, it is possible to write

$$
\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 \\ \Pi_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & C_2 \end{bmatrix},
\tag{33}
$$

where $\Phi_{11} \in R^{(n-p)\times(n-p)}$, $\Pi_2 \in R^{q\times q}$, and $C_2 \in R^{p\times p}$. Partition the observer state as

$$
z[k] = \begin{bmatrix} z_1[k] \\ z_2[k] \end{bmatrix}
\tag{34}
$$

where $z_1[k] \in R^{n-p}$ and $z_2[k] \in R^p$. Define an observer for (15) as

$$
\begin{aligned}
z_1[k+1] &= \Phi_{11}z_1[k] + \Phi_{12}y[k] + \Gamma_1 u[k] \\
z_2[k+1] &= \Phi_{21}z_1[k] + \Phi_{22}y[k] + \Gamma_2 u[k] + w[k] \quad (35)
\end{aligned}
$$

where the injection signal $w[k]$ is to be designed.

Define $e[k] = z[k] - \bar{x}[k]$ and write

$$
e[k] = \begin{bmatrix} e_1[k] \\ e_2[k] \end{bmatrix}. \quad (36)
$$

Let the output error be $e_y[k] = e_2[k] = z_2[k] - y[k]$. From (28) and Lemma 2.1,

$$
\begin{bmatrix} e_1[k+1] \\ e_2[k+1] \end{bmatrix} = \begin{bmatrix} \Phi_{11} & 0 \\ \Phi_{21} & 0 \end{bmatrix} \begin{bmatrix} e_1[k] \\ e_2[k] \end{bmatrix} - \begin{bmatrix} T_s^2 \bar{\bar{\mathscr{F}}}_2 \\ T_s \bar{\mathscr{F}}_2 \end{bmatrix} f[k+1/2]
$$
$$
- T_s^3 \begin{bmatrix} \delta d_1[k] \\ \delta d_2[k] \end{bmatrix} - \begin{bmatrix} \beta_1[k] \\ \beta_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ w[k] \end{bmatrix}. \quad (37)
$$

Note that from (23)

$$
\Phi_{11} = I_{n-p} + T_s(\mathscr{A}_{11} + T_s\bar{\mathscr{A}}_{11}). \quad (38)
$$

Since the eigenvalues of $\mathscr{A}_{11} = A_{11} + LA_{21}$ lie in the left hand side of the complex plane [2], there exists a small enough T_s such that the eigenvalues of Φ_{11} lie in the unit circle (for details, see the arguments in [18]).

At steady state, if $\beta_1[k] = O(T_s^2)$, then (37) implies that

$$
e_1[k] = O(T_s). \quad (39)
$$

Rewrite the dynamics of $e_2[k]$ as

$$
e_2[k+1] = w[k] + \theta[k] \quad (40)
$$

where

$$
\theta[k] = \Phi_{21}e_1[k] - T_s\bar{\mathscr{F}}f[k+1/2] - T_s^3\delta d_2[k] - \beta_2[k]. \quad (41)
$$

Note that $\theta[k]$ contains unknown components. In the continuous case, a nonlinear switching injection term would be introduced to make $e_2[k]$ converge to 0 in finite time and introduce a sliding mode. However, in the sampled-data context, this cannot be achieved. Instead, a quasi-sliding mode can be obtained to bring $e_2[k]$ as close to 0 as possible. Let

$$
w[k] = -\theta[k-1]. \quad (42)
$$

Then from (40) and (42), the function $w[k]$ satisfies

$$
w[k] = w[k-1] - e_2[k]. \quad (43)
$$

It also follows from (41) that

$$
\begin{aligned}
e_2[k+1] &= \theta[k] - \theta[k-1] \\
&= \Phi_{21}(e_1[k] - e_1[k-1]) \\
&\quad - T_s\bar{\mathscr{F}}(f[k+1/2] - f[k-1/2]) \\
&\quad - T_s^3(\delta d_2[k] - \delta d_2[k-1]) \\
&\quad - (\beta_2[k] - \beta_2[k-1]). \quad (44)
\end{aligned}
$$

Since

$$
f[k+1/2] = f[k-1/2] + \int_{(k-1/2)T_s}^{(k+1/2)T_s} v(\sigma)d\sigma
$$

and $v(t) = \dot{f}(t)$ is bounded,

$$
f[k+1/2] - f[k-1/2] = O(T_s). \quad (45)
$$

We can derive a similar lemma for $\beta[k]$ using arguments similar to those in Lemma 2.1 for $d[k]$. Following this reasoning, it is easy to show that

$$
\beta_2[k] - \beta_2[k-1] = O(T_s^2). \quad (46)
$$

Then arguing as in [18], the error dynamics in (37) implies $e_1[k+1] - e_1[k] = O(T_s)$, and hence, from (44), (45), (46)

$$
e_2[k+1] = O(T_s^2). \quad (47)
$$

This shows that the output estimation error is maintained in an $O(T_s^2)$ boundary layer of the output sliding surface. The presence of the past value of $w[k]$ constitutes dynamical behavior for this correction term. Hence, to study the dynamics of the estimation error overall, it is necessary to augment the dynamics of $w[k]$ to that of $e_1[k]$ and $e_2[k]$. The estimation error can be described by the following system:

$$
\begin{aligned}
e_1[k+1] &= \Phi_{11}e_1[k] - T_s^2\bar{\bar{\mathscr{F}}}_1 f[k+1/2] - T_s^3\delta d_1[k] \\
&\quad - \beta_1[k] \\
e_2[k+1] &= \Phi_{21}e_1[k] + w[k] - T_s\bar{\mathscr{F}}_2 f[k+1/2] - T_s^3\delta d_2[k] \\
&\quad - \beta_2[k] \\
w[k+1] &= -\Phi_{21}e_1[k] + T_s\bar{\mathscr{F}}_2 f[k+1/2] + T_s^3\delta d_2[k] \\
&\quad + \beta_2[k]. \quad (48)
\end{aligned}
$$

B. Convergence analysis

In this section, we investigate the convergence of the estimator (35) when the external disturbance and the uncertainty do not affect the system. In this specific case, the dynamics of the estimation errors in (48) can be rewritten as

$$
\begin{bmatrix} e_1[k+1] \\ e_2[k+1] \\ w[k+1] \end{bmatrix} = \begin{bmatrix} \Phi_{11} & 0 & 0 \\ \Phi_{21} & 0 & I_m \\ -\Phi_{21} & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1[k] \\ e_2[k] \\ w[k] \end{bmatrix} \quad (49)
$$

It can be seen from (49) that the eigenvalues of the system matrix in (49) are Φ_{11} and 0. Since the eigenvalues of Φ_{11} lie in the unit circle (as argued in the previous subsection), the eigenvalues of the system matrix of the estimation error dynamics in (49) lie in the unit circle. This implies the errors $e_1[k]$ and $e_2[k]$ converge to 0 exponentially. We conclude this subsection with the following theorem.

Theorem 3.1: In the absence of the external disturbances, the observer in (35) guarantees that the estimates of the state variables converge to the true values exponentially.

Proof: The proof follows from the analysis above. ∎

C. Reconstruction of the external disturbance

Now we consider the case when the external disturbance and uncertainty influences the dynamics of the system. From equation (27),

$$
\Phi_{21} = O(T_s).
$$

In addition, $e_1[k] = O(T_s)$ as pointed out in (39) provided that $\beta_1[k] = O(T_s^2)$. Hence, if $\beta_2[k] = O(T_s^2)$, from (48),

$$
w[k] = T_s\bar{\mathscr{F}}_2 f[k-1/2] + O(T_s^2), \quad (50)
$$

and hence, $f[k-1/2]$ can be calculated from $w[k]$. Note that

$$f[k] - f[k-1/2] = \int_{(k-1/2)T_s}^{kT_s} v(\sigma)d\sigma = O(T_s).$$

Hence, $f[k]$ can be approximated by $f[k-1/2]$ provided the sampling period T_s is small enough. From (50),

$$w[k] \approx T_s \bar{\mathscr{F}}_2 f[k-1/2],$$

or in other words,

$$f[k] \approx f[k-1/2] \approx \frac{\bar{\mathscr{F}}_2^\dagger}{T_s} w[k] \qquad (51)$$

where $\bar{\mathscr{F}}_2^\dagger$ is the left pseudo-inverse of $\bar{\mathscr{F}}_2$. We conclude the above analysis with the following theorem.

Theorem 3.2: Under the influence of the external disturbance $f[k]$ and the uncertainty, if Assumptions 1 and 2 hold and $\beta[k] = O(T_s^2)$, the closeness of the state estimates to their real values is guaranteed to be $O(T_s)$, and the unknown input signal is recovered as $f[k] \approx \frac{\bar{\mathscr{F}}_2^\dagger}{T_s} w[k]$ where $w[k]$ is defined in (43).

Proof: The proof follows from the analysis above. ∎

IV. NUMERICAL EXAMPLE

In this section, we employ the VTOL aircraft model model in [2] to illustrate our proposed method. The system states are horizontal velocity, vertical velocity, pitch rate, and pitch angle. The outputs are horizontal velocity, vertical velocity, and pitch angle. The control inputs are collective pitch control and longitudinal cyclic pitch control. The system matrices in (1) are:

$$A = \begin{bmatrix} -9.9477 & -0.7476 & 0.2632 & 5.0337 \\ 52.1659 & 2.7452 & 5.5532 & -24.4221 \\ 26.0922 & 2.6361 & -4.1975 & -19.2774 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix},$$

$$B = F = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T.$$

The parametric uncertainty ξ is given by

$$\xi = \begin{bmatrix} 0 & \Delta_{32} & \Delta_{34} \end{bmatrix} y$$

where $\Delta_{32} = 0.05$ and $\Delta_{34} = 0.02$. It is clear that CF is of full rank and it can be shown the triple (A, F, C) does not possess any invariant zeros. Hence, the method proposed in this paper is applicable. The system matrices are brought to the form in (1) by using the following transformation:

$$\mathscr{T}_0 = \begin{bmatrix} -6.5287 & -0.7428 & -1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \end{bmatrix}$$

Matrix L is calculated from the solution of (6) using the method described in [2]:

$$L = \begin{bmatrix} 3.1251 & 0 & 0 \end{bmatrix}. \qquad (52)$$

Since the state transition matrix A is stable, we do not need to use a controller to stabilize our system. In this paper, the sampling period is $T_s = 0.05\,s$. The control signals are chosen as

$$\begin{aligned} u_1[k] &= 0.1 + 0.1\sin(0.1\,T_s k) \\ u_2[k] &= 0.1\sin(0.5\,T_s k + \pi/2). \end{aligned}$$

The external signals are

$$\begin{aligned} f_1(t) &= 0.1 + 2\sin(t) \\ f_2(k) &= -0.5 + \sin(t - \pi/2). \end{aligned}$$

Fig. 1 illustrates the evolution of the state variables and their estimates. Here the estimate of x_3 takes a longer time than the others to converge, and the performance (as expected) is degraded due to the effect of the uncertainty. In Fig. 2, it can be seen that a quasi-sliding motion takes place in an $O(T_s^2)$ boundary layer around the origin. Fig. 3 shows that the estimates of the external signals converge to the true values.

The simulation results in this section illustrate our analysis and show that our method can reliably estimate the state variables and unknown inputs for sampled-data systems.
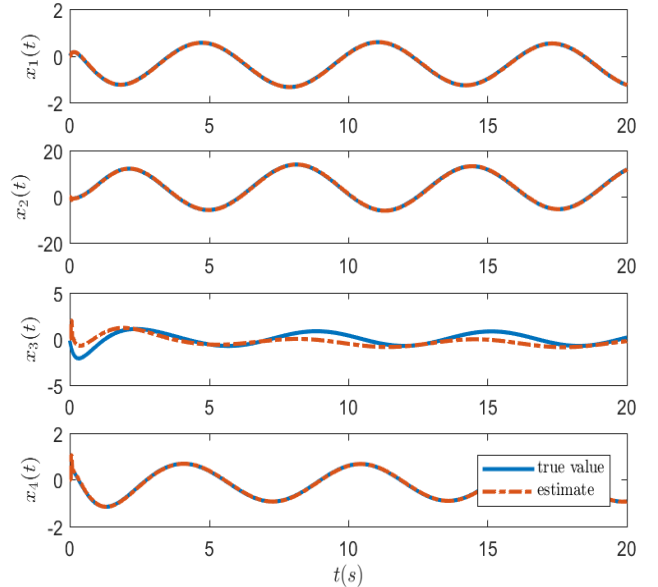

Fig. 1.   Evolution of the state estimates and their true values

V. CONCLUSION

This paper has presented a new observer to estimate the state variables and unknown inputs in a class of sampled-data systems in the presence of external signals and uncertainties.
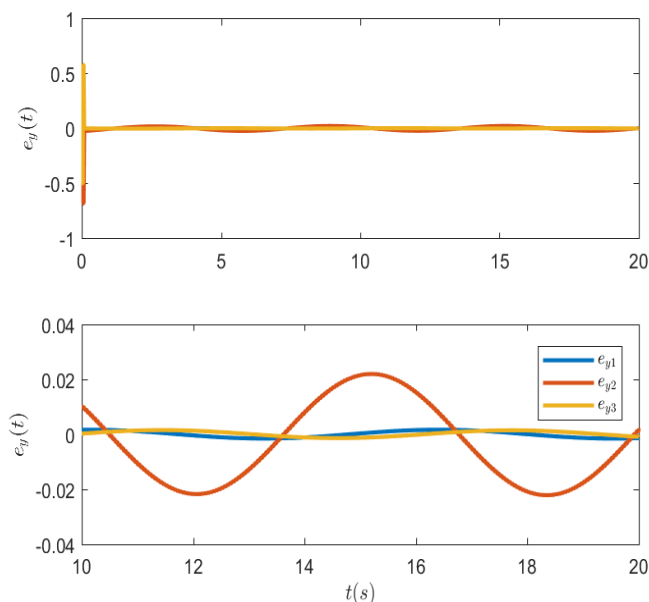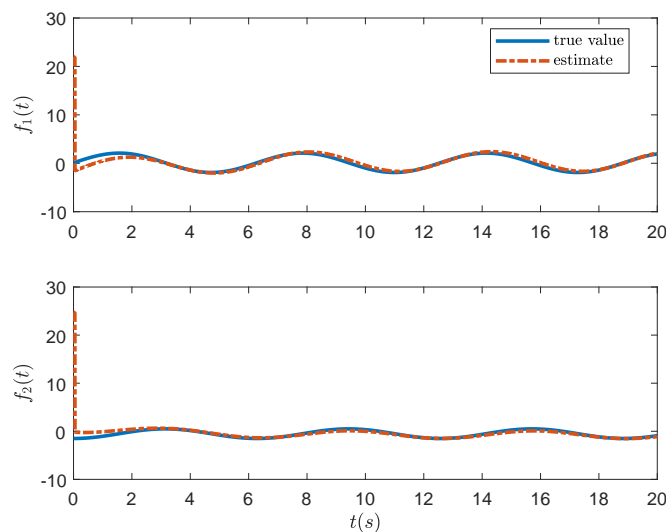
Fig. 2. Evolution of the output errors

Fig. 3. Evolution of the external disturbances and their estimates

The method is based on quasi-sliding motion concepts and the design is derived from the continuous time counterpart. A stability and robustness analysis was provided. Simulations were conducted to illustrate the effectiveness of the proposed method.

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