ARRANGEMENTS OF HOMOTHETS OF A CONVEX BODY II

MÁRTON NASZÓDI AND KONRAD J. SWANEPOEL

Abstract. A family of homothets of an o-symmetric convex body K in d-dimensional Euclidean space is called a Minkowski arrangement if no homothet contains the center of any other homothet in its interior. We show that any pairwise intersecting Minkowski arrangement of a d-dimensional convex body has at most $2 \cdot 3^d$ members. This improves a result of Polyanskii (Discrete Mathematics **340** (2017), 1950–1956).

Using similar ideas, we also give a proof the following result of Polyanskii: Let $K_1, \dots, K_n$ be a sequence of homothets of the o-symmetric convex body K, such that for any $i < j$, the center of K_j lies on the boundary of K_i. Then $n = O(3^d d)$.

1. Introduction

We use the notation $[n] = \{1, 2, \dots, n\}$. A *convex body* K in the d-dimensional Euclidean space $\mathbb{R}^d$ is a compact convex set with non-empty interior, and is *o-symmetric* if $K = -K$. A (positive) *homothet* of K is a set of the form $\lambda K + v := \{\lambda k + v : k \in K\}$, where $\lambda > 0$ is the homothety ratio, and $v \in \mathbb{R}^d$ is a translation vector. If K is o-symmetric, we also call v the *center* of the homothet $\lambda K + v$. An *arrangement of homothets of* K is a collection $\{\lambda_i K + v_i : i \in [n]\}$. A *Minkowski arrangement* of an o-symmetric convex body K is a family $\{v_i + \lambda_i K\}$ of homothets of K such that none of the homothets contains the center of any other homothet in its interior. This notion was introduced by L. Fejes Tóth [3] in the context of Minkowski's fundamental theorem on the minimal determinant of a packing lattice for a symmetric convex body, and was further studied by him in [4, 5], by Böröczky and Szabó in [2], and in connection with the Besicovitch covering theorem by Füredi and Loeb [6]. Recently, Minkowski arrangements have been used to study a problem arising in the design of wireless networks [10]. In [9] it was shown that the largest cardinality of a pairwise intersecting Minkowski arrangement of homothets of an o-symmetric convex body in $\mathbb{R}^d$

Date: (date1), and in revised form (date2).

2010 *Mathematics Subject Classification.* 52C17, 51N20, 51K05, 52A21.

Key words and phrases. homothets, convex bodies, Minkowski arrangements, packing.

M.N. was partially supported by the National Research, Development and Innovation Fund grant K119670, the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and the ÚNKP-17-4 New National Excellence Program of the Ministry of Human Capacities.

is $O(3^d d \log d)$. This was improved to 3^{d+1} by Polyanskii [11]. We make the following slight improvement.

Theorem 1.1. *For any o-symmetric convex body K in $\mathbb{R}^d$, a pairwise intersecting Minkowski arrangement has at most $2 \cdot 3^d$ members.*

Note that the d-cube has 3^d pairwise intersecting translates that form a Minkowski arrangement. The proof uses ideas from [8] and [7].

In [9], bounds on pairwise intersecting Minkowski arrangements were used to give an upper bound of $O(6^d d^2 \log d)$ on the length of a sequence of homothets $v_i + \lambda_i K$ of an o-symmetric convex body K such that $v_j \in \operatorname{bd}(v_i + \lambda_i K)$ whenever $j > i$. This bound was improved to $O(3^d d)$ by Polyanskii [11]. We use some similar ideas to the proof of Theorem 1.1 to give a short proof of this result of Polyanskii.

Theorem 1.2 (Polyanskii [11])**.** *Let K be an o-symmetric convex body, and $v_1, v_2, \dots, v_n \in \mathbb{R}^d$. Let $\lambda_1, \lambda_2, \dots, \lambda_{n-1} > 0$, and assume that for any $1 \le i < j \le n$ we have $v_j \in \operatorname{bd}(v_i + \lambda_i K)$. Then $n = O(3^d d)$.*

Clearly, when K is the cube, $n = 2^d$ is attained. It would be interesting to find better bounds for the maximum size of a family satisfying the conditions of Theorem 1.2.

The interest in this result is that it gives the upper bound $k^{O(3^d d)}$ to the cardinality of a set in a d-dimensional normed space in which only k non-zero distances occur between pairs of points. This is currently the best known upper bound if $k = \Omega(3^d d)$ (see [12] for a survey of this problem).

2. Proof of Theorem 1.1

Theorem 2.1. *Let $d \ge 1$. Suppose that there exists an o-symmetric convex body K in $\mathbb{R}^d$ which has a pairwise intersecting Minkowski arrangement of n homothets. Then there exists a set $\{x_1, \dots, x_n\}$ of n points in $\mathbb{R}^{d+1}$ such that $o \notin \operatorname{conv}\{x_1, \dots, x_n\}$, and for any distinct $i, j \in [n]$, $i < j$, there exists a non-zero linear functional $f_{ij} \colon \mathbb{R}^{d+1} \to \mathbb{R}$ with*

$$|f_{ij}(x_k)| \le |f_{ij}(x_i) - f_{ij}(x_j)| \quad \text{for all } k \in [n]. \tag{2.1}$$

We remark that the converse of the above theorem does not hold. We describe a simple counterexample for $d = 1$. On the one hand, clearly, a pairwise intersecting Minkowski arrangement of intervals in $\mathbb{R}$ has at most two members. On the other hand, there is a set of 5 points on the plane satisfying the conclusion of Theorem 2.1. Indeed, let $\{x_1, \dots, x_5\}$ be the vertex set of a regular pentagon, with o just outside the pentagon, close to the midpoint of an edge. It is easy to see that for any pair x_i, x_j of vertices, there is a line through o such that the projections $\pi(x_k)$ of the vertices onto the line are all within distance $|\pi(x_i) - \pi(x_j)|$ of o.

The above remark is to be contrasted with the equivalence in the following result, which generalizes part of Theorem 1.4 of [7].

Theorem 2.2. *Given $\lambda \geq 1$, and $D \in \mathbb{Z}, D \geq 1$. Then the following statements are equivalent.*

(i) There exists a set $\{x_1, \ldots, x_n\}$ of n points in $\mathbb{R}^D$, such that $o \notin \operatorname{conv}\{x_1, \ldots, x_n\}$, and for any distinct $i, j \in [n], i < j$ there exists a non-zero linear functional $f_{ij} : \mathbb{R}^D \to \mathbb{R}$ with

$$|f_{ij}(x_k)| \leq \frac{\lambda}{2} |f_{ij}(x_i) - f_{ij}(x_j)| \quad \text{for all } k \in [n]. \tag{2.2}$$

(ii) There is an o-symmetric convex set L in $\mathbb{R}^D$ that has n non-overlapping translates $L + t_1, \ldots, L + t_n$, each intersecting $(\lambda - 1)L$, with $o \notin \operatorname{conv}\{t_1, \ldots, t_n\}$.

We note that the equivalence between (ii) and (iv) of Theorem 1.4 in [7] is exactly the above theorem in the case $\lambda = 1$.

Theorem 2.3. *Let K be an o-symmetric convex set in $\mathbb{R}^D$ with $D \geq 2$, and let $\alpha K + t_1, \ldots, \alpha K + t_n$ be n non-overlapping translates of αK with $\alpha > 0$ such that each translate intersects K, and $o \notin \operatorname{int}(\operatorname{conv}\{t_1, \ldots, t_n\})$. Then*

$$n \leq \frac{(1 + 2\alpha)^{D-1}(1 + 3\alpha)}{2\alpha^D}. \tag{2.3}$$

This theorem is a slight modification of Theorem 1.5 of [7]. There the translates of αK touch K, whereas here they may overlap with K. Theorem 2.3 is sharp for $\alpha = 1$. Indeed, let K be the cube $[-1, 1]^D$, and consider the $2 \cdot 3^{D-1}$ translation vectors $\{t \in \{-2, 0, 2\}^D \ : \ t^{(1)} \geq t^{(2)}\}$.

Combining Theorems 2.1, 2.2 and 2.3 (with $\lambda = 2$, $K = (\lambda - 1)L = L$, $\alpha = \frac{1}{\lambda - 1} = 1$), we immediately obtain Theorem 1.1.

3. Proof of Theorem 2.1

Let the Minkowski arrangement by $\{v_i + \lambda_i K : i \in [n]\}$, where $\lambda_i > 0$ and $v_i \in \mathbb{R}^d$ for each $i \in [n]$. Let $x_i = (\lambda_i^{-1} v_i, \lambda_i^{-1}) \in \mathbb{R}^d \times \mathbb{R}$, $i \in [n]$. Fix distinct $i, j \in \{1, \ldots, n\}$. We will find a linear $f \colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ that satisfies (2.1). Let $\varphi \colon \mathbb{R}^d \to \mathbb{R}$ be a linear functional such that $\varphi(x) \leq \|x\|_K$ for all $x \in \mathbb{R}^d$ and $\varphi(v_j - v_i) = \|v_j - v_i\|_K$. (Thus, $\varphi^{-1}(1)$ is a hyperplane that supports K at $\|v_j - v_i\|_K^{-1} (v_j - v_i)$.)

Since any two homothets $v_k + \lambda_k K$ and $v_\ell + \lambda_\ell K$ intersect, any two of the compact intervals $\varphi(v_k + \lambda_k K)$ and $\varphi(v_\ell + \lambda_\ell K)$ intersect in $\mathbb{R}$. By Helly's Theorem in $\mathbb{R}$, there exists $\alpha \in \bigcap_{t=1}^{n} \varphi(v_t + \lambda_t K)$. Since $\varphi(v_i + \lambda_i K) = [\varphi(v_i) - \lambda_i, \varphi(v_i) + \lambda_i]$ and $\varphi(v_j + \lambda_j K) = [\varphi(v_j) - \lambda_j, \varphi(v_j) + \lambda_j]$, we have

$$\varphi(v_j) - \lambda_j \leq \alpha \leq \varphi(v_i) + \lambda_i.$$

By the Minkowski property,

$$\varphi(v_j - v_i) = \|v_j - v_i\|_K \geq \max\{\lambda_i, \lambda_j\}.$$

It follows that

$$\varphi(v_i) \leq \alpha \leq \varphi(v_j). \tag{3.1}$$

We set $f = (\varphi, -\alpha) \in (\mathbb{R}^d \times \mathbb{R})^*$, that is, define $f(x) = \varphi(v) - \alpha\mu$, where $x = (v, \mu) \in \mathbb{R}^d \times \mathbb{R}$. We show that $f(x_j - x_i) \geq 1$, and $|f(x_k)| \leq 1$ for all $k \in \{1, \dots, n\}$. This will show that (2.1) is satisfied, which will finish the proof.

$$\begin{aligned} f(x_j - x_i) &= \varphi(\lambda_j^{-1} v_j - \lambda_i^{-1} v_i) - \alpha(\lambda_j^{-1} - \lambda_i^{-1}) \\ &= \frac{\varphi(v_j) - \alpha}{\lambda_j} + \frac{\alpha - \varphi(v_i)}{\lambda_i} \\ &\overset{(3.1)}{\geq} \frac{\varphi(v_j) - \alpha + \alpha - \varphi(v_i)}{\max\{\lambda_i, \lambda_j\}} \\ &= \frac{\|v_j - v_i\|_K}{\max\{\lambda_i, \lambda_j\}} \geq 1. \end{aligned}$$

Since $\alpha \in \varphi(v_k + \lambda_k K)$, there exists $x \in K$ such that $\varphi(v_k + \lambda_k x) = \alpha$. Therefore,

$$|f(x_k)| = \left|\varphi(\lambda_k^{-1} v_k) - \alpha\lambda_k^{-1}\right| = |\varphi(x)| \leq \|x\|_K \leq 1.$$

□

4. Proof of Theorem 1.2

The following proof is very similar to the proof of Theorem 2.1.

Without loss of generality, $\min_i \lambda_i = 1$. Denote the unit ball of $\|\cdot\|$ by K. Let $x_i = (\lambda_i^{-1} v_i, \lambda_i^{-1}) \in \mathbb{R}^d \times \mathbb{R}$, $i = 1, \dots, n-1$. Let $N \geq 1$, to be fixed later. For each $m = 0, \dots, N$, let

$$X_m = \{x_i : i \in [n-1], \lfloor N \log_2 \lambda_i \rfloor \equiv m \pmod{N+1}\}.$$

Then $X_0, \dots, X_N$ partition $\{x_1, \dots, x_{n-1}\}$ into $N+1$ parts. Fix $x_i, x_j \in X_m$ such that $1 \leq i < j < n$. We will find a linear $f\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ such that (2.2) is satisfied for all $x_k \in X_m$ and $\lambda = 2 - 2^{1/N}$. Let $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ be a linear functional such that $\varphi(v) \leq \|v\|$ for all $v \in \mathbb{R}^d$ and

$$\varphi(v_j - v_i) = \|v_j - v_i\| = \lambda_i. \tag{4.1}$$

(Thus, $\varphi^{-1}(1)$ is a hyperplane that supports K at $\|v_j - v_i\|_K^{-1}(v_j - v_i)$.)

Since any two homothets $v_k + \lambda_k K$ and $v_\ell + \lambda_\ell K$ intersect in their interiors, any two of the open intervals $\varphi(v_k + \lambda_k \operatorname{int} K)$ and $\varphi(v_\ell + \lambda_\ell \operatorname{int} K)$ intersect in $\mathbb{R}$. By Helly's Theorem in $\mathbb{R}$, there exists $\alpha \in \bigcap_{t=1}^n \varphi(v_t + \lambda_t \operatorname{int} K)$. Since $\varphi(v_i + \lambda_i \operatorname{int} K) = (\varphi(v_i) - \lambda_i, \varphi(v_i) + \lambda_i)$ and $\varphi(v_j + \lambda_j \operatorname{int} K) = (\varphi(v_j) - \lambda_j, \varphi(v_j) + \lambda_j)$, we have

$$\varphi(v_j) - \lambda_j < \alpha < \varphi(v_i) + \lambda_i.$$

By (4.1), we can rewrite this as

$$-\lambda_i < \varphi(v_i) - \alpha < \lambda_j - \lambda_i. \tag{4.2}$$

We set $f = (\varphi, -\alpha) \in (\mathbb{R}^d \times \mathbb{R})^*$, that is, for $x = (v, \mu) \in \mathbb{R}^d \times \mathbb{R}$, we let $f(x) = \varphi(v) - \alpha\mu$. It remains to show that $f(x_j - x_i) > 2 - 2^{1/N}$, and $|f(x_k)| \leq 1$ for all $k \in \{0, \dots, n\}$, since this will show that (2.2) is satisfied with $\lambda = 2 - 2^{1/N}$. By applying Theorems 2.2 and 2.3 with $\lambda = 2/(2 - 2^{1/N}) = 2 + \frac{\log 4}{N} + O(N^{-2})$, $K = (\lambda - 1)L$ and $\alpha = 1/(\lambda - 1) = 2^{1-1/N} - 1$, we obtain $|X_m| \leq (1 + \lambda/2)(1 + \lambda)^d$, and it follows that

$$n - 1 \leq (N + 1)(1 + \lambda/2)(1 + \lambda)^d.$$

If we choose $N = d$, we obtain $\lambda = 2 + \frac{\log 4}{d} + O(d^{-2})$ and $n = 3^d O(d)$, which would finish the proof.

By definition of X_m,

$$\lfloor N \log_2 \lambda_j \rfloor - \lfloor N \log_2 \lambda_i \rfloor = kN \quad \text{for some } k \in \mathbb{Z}.$$

If $k \geq 1$, then $N \log_2 \lambda_j - N \log_2 \lambda_i > N$, hence $\lambda_j/\lambda_i > 2$. However, we also have

$$\lambda_i = \|v_i - v_j\| \geq \|v_j - v_n\| - \|v_n - v_i\| = \lambda_j - \lambda_i,$$

a contradiction. Therefore, $k \leq 0$, that is, $\lfloor N \log_2 \lambda_j \rfloor - \lfloor N \log_2 \lambda_i \rfloor \leq 0$. This gives $N \log_2 \lambda_j - N \log_2 \lambda_i < 1$ and

$$\frac{\lambda_j}{\lambda_i} < 2^{1/N}. \tag{4.3}$$

It follows that

$$\begin{aligned}
f(x_j - x_i) &= \varphi(\lambda_j^{-1} v_j - \lambda_i^{-1} v_i) - \alpha(\lambda_j^{-1} - \lambda_i^{-1}) \\
&= \frac{\varphi(v_j) - \alpha}{\lambda_j} + \frac{\alpha - \varphi(v_i)}{\lambda_i} \\
&= \frac{\varphi(v_i) + \lambda_i - \alpha}{\lambda_j} + \frac{\alpha - \varphi(v_i)}{\lambda_i} \\
&\overset{(4.2),(4.3)}{>} \frac{2^{-1/N}(\varphi(v_i) + \lambda_i - \alpha) + \alpha - \varphi(v_i)}{\lambda_i} \\
&= 2^{-1/N} + \frac{(1 - 2^{-1/N})(\alpha - \varphi(v_i))}{\lambda_i} \\
&\overset{(4.2)}{>} 2^{-1/N} + \frac{(1 - 2^{-1/N})(\lambda_i - \lambda_j)}{\lambda_i} \\
&= 1 - (1 - 2^{-1/N}) \frac{\lambda_j}{\lambda_i} \\
&\overset{(4.2)}{>} 1 - (1 - 2^{-1/N}) 2^{1/N} \\
&= 2 - 2^{1/N}.
\end{aligned}$$

Since $\alpha \in \varphi(v_k + \lambda_k K)$, there exists $x \in K$ such that $\varphi(v_k + \lambda_k x) = \alpha$. Therefore,

$$|f(x_k)| = \left|\varphi(\lambda_k^{-1} v_k) - \alpha \lambda_k^{-1}\right| = |\varphi(x)| \leq \|x\|_K \leq 1.$$

□

5. Proof of Theorem 2.2

Assume that (i) holds. Let $C := \bigcap_{i\neq j} S_{ij}$ be the intersection of the o-symmetric slabs $S_{ij} := \left\{p \in \mathbb{R}^D : |f_{ij}(p)| \leq \frac{\lambda}{2}\, |f_{ij}(x_i) - f_{ij}(x_j)|\right\}$. By assumption, $C \supseteq \{x_1, \ldots, x_n\}$. For each $i \in [n]$, let $C_i := \frac{\lambda x_i + C}{\lambda+1}$ be the homothetic copy of C with center of homothety x_i, and of ratio $\frac{1}{\lambda+1}$. It is an easy exercise that the C_is are non-overlapping. Moreover, by the symmetry of C, we have $\frac{\lambda-1}{\lambda+1}x_i \in C_i \cap \frac{\lambda-1}{\lambda+1}C$. Thus, for $L := \frac{1}{\lambda+1}C$, and $t_i := \frac{\lambda}{\lambda+1}x_i$, (ii) holds as promised.

Next, assume that (ii) holds. Fix $i, j \in [n], i \neq j$. Since $L + t_i$ and $L + t_j$ are non-overlapping, there is a linear functional f such that the two real intervals $s_i := f(L + t_i)$ and $s_j := f(L + t_i)$ do not overlap. These two intervals are of equal length, which we denote by w. Thus, we have

$$w \leq |f(t_i) - f(t_j)|\,. \tag{5.1}$$

On the other hand, $s_k := f(L + t_k)$ is also a real interval of length w for any $k \in [n]$; and $s_0 := f((\lambda - 1)L)$ is a 0-symmetric real interval of length $(\lambda - 1)w$, which intersects each s_k. Thus, for the center $f(t_k)$ of s_k, we have $|f(t_k)| \leq \frac{(\lambda-1)w}{2} + \frac{w}{2} = \frac{\lambda w}{2}$. Now, (5.1) yields $|f(t_k)| \leq \frac{\lambda}{2}\, |f(t_i) - f(t_j)|$. Thus, we may set $f_{ij} := f$. This argument is valid for any i and j, thus, with $x_i := t_i$, we obtain (i).

6. Proof of Theorem 2.3

The proof is an almost verbatim copy of the proof of Theorem 1.5 of [7]. There are two points of difference, which we will note.

We recall Lemma 3.1. of [7], which is a slightly more general version of the Lemma of [1].

Lemma 6.1. *Let f be a function on $[0, 1]$ with the properties $f(0) \geq 0$, f is positive and monotone increasing on $(0, 1]$, and $f(x) = (g(x))^k$ for some concave function g and $k > 0$. Then*

$$F(y) := \frac{1}{f(y)} \int_0^y f(x)\,\mathrm{d}x$$

is strictly increasing on $(0, 1]$.

Proof of Theorem 2.3. Clearly, we may assume that K is bounded, otherwise, by a projection, we can reduce the dimension. Let $\alpha K + t_1$, $\alpha K + t_2, \ldots, \alpha K + t_n$ be pairwise non-overlapping translates of αK that intersect K. By the assumptions of the theorem, there is a non-zero vector $v \in \mathbb{R}^D$ such that $a_i := \langle t_i, v\rangle \geq 0$ for $i \in [n]$. Set $h(x) := \{p \in \mathbb{R}^D : \langle p, v\rangle = x\}$. Without loss of generality, we may assume that $h(-1)$ and $h(1)$ are supporting hyperplanes of K.

Clearly, $\alpha K + t_i$ is between $h(-\alpha)$ and $h(1 + 2\alpha)$, and it is contained in $(1 + 2\alpha)K$, for $i \in [n]$.

$$\int_{-\alpha}^{1+2\alpha} \mathrm{V}_{D-1}\left(\left(\bigcup_{i=1}^{n} \alpha K + t_i\right) \cap h(x)\right) \mathrm{d}x = n\alpha^D \mathrm{V}_D(K). \tag{6.1}$$

$$\int_{0}^{1+2\alpha} \mathrm{V}_{D-1}\left(\left(\bigcup_{i=1}^{n} \alpha K + t_i\right) \cap h(x)\right) \mathrm{d}x \tag{6.2}$$

$$\leq \int_{0}^{1+2\alpha} \mathrm{V}_{D-1}\left((1+2\alpha)K \cap h(x)\right) \mathrm{d}x = \frac{(1+2\alpha)^d}{2} \mathrm{V}_D(K).$$

We note that this was the first point of difference from the proof in [7]: here, we do not subtract the contribution of K in the total volume on the right hand side of the inequality.

Set $f(x) := \mathrm{V}_{D-1}\left(\alpha K \cap h(x-\alpha)\right)$, and observe that the conditions of Lemma 6.1 are satisfied by f (with $k = D-1$, by the Brunn–Minkowski inequality). We may assume that $a_1, \ldots, a_m \leq \alpha < a_{m+1}, \ldots, a_n$. By Lemma 6.1,

$$\int_{-\alpha}^{0} \mathrm{V}_{D-1}\left(\left(\bigcup_{i=1}^{n} (\alpha K + t_i)\right) \cap h(x)\right) \mathrm{d}x = \sum_{i=1}^{m} \int_{0}^{\alpha - a_i} f(x)\, \mathrm{d}x$$

$$\leq \sum_{i=1}^{m} \int_{0}^{\alpha} f(x)\, \mathrm{d}x \frac{f(\alpha - a_i)}{f(\alpha)} = \frac{\alpha^d \mathrm{V}_D(K)}{2f(\alpha)} \sum_{i=1}^{m} \mathrm{V}_{D-1}\left((\alpha K + t_i) \cap h(0)\right)$$

$$= \frac{\alpha^d \mathrm{V}_D(K)}{2f(\alpha)} \mathrm{V}_{D-1}\left(\left(\bigcup_{i=1}^{m} (\alpha K + t_i)\right) \cap h(0)\right)$$

$$\leq \frac{\alpha^d \mathrm{V}_D(K)}{2f(\alpha)} \left[\mathrm{V}_{D-1}\left((1+2\alpha)K \cap h(0)\right)\right] = \frac{\alpha(1+2\alpha)^{D-1}}{2} \mathrm{V}_D(K).$$

We note that this was the second point of difference from the proof in [7]: again, the contribution of K to the volume is not subtracted.

This inequality, combined with (6.1) and (6.2), yields (2.3). □

References

1. Károly Bezdek and Peter Brass, *On k^+-neighbour packings and one-sided Hadwiger configurations*, Beiträge Algebra Geom. **44** (2003), no. 2, 493–498. MR MR2017050 (2004i:52017)
2. Károly Böröczky and László Szabó, *Minkowski arrangements of spheres*, Monatsh. Math. **141** (2004), no. 1, 11–19. MR 2109518
3. L. Fejes Tóth, *Minkowskian distribution of discs*, Proc. Amer. Math. Soc. **16** (1965), 999–1004. MR 0180921
4. ______, *Minkowskian circle-aggregates*, Math. Ann. **171** (1967), 97–103. MR 0221386

5. ______, *Minkowski circle packings on the sphere*, Discrete Comput. Geom. **22** (1999), no. 2, 161–166. MR 1698538
6. Zoltán Füredi and Peter A. Loeb, *On the best constant for the Besicovitch covering theorem*, Proc. Amer. Math. Soc. **121** (1994), no. 4, 1063–1073. MR 1249875 (95b:28003)
7. Zsolt Lángi and Márton Naszódi, *On the Bezdek-Pach conjecture for centrally symmetric convex bodies*, Canad. Math. Bull. **52** (2009), no. 3, 407–415. MR 2547807
8. Márton Naszódi, *On a conjecture of Károly Bezdek and János Pach*, Period. Math. Hungar. **53** (2006), no. 1-2, 227–230. MR MR2286473
9. Márton Naszódi, János Pach, and Konrad Swanepoel, *Arrangements of homothets of a convex body*, arXiv preprint arXiv:1608.04639 (2017).
10. Márton Naszódi, Leonardo Martínez Sandoval, and Shakhar Smorodinsky, *Bounding a global red-blue proportion using local conditions*, Proceedings of the 33rd European Workshop on Computational Geometry (EuroCG2017), Malmö University, 2017, pp. 213–217.
11. Alexandr Polyanskii, *Pairwise intersecting homothets of a convex body*, Discrete Math. **340** (2017), 1950–1956.
12. Konrad Swanepoel, *Combinatorial distance geometry in normed spaces*, New trends in intuitive geometry (Gergely Ambrus, Imre Bárány, Károly J. Böröczky, Gábor Fejes Tóth, and János Pach, eds.), Bolyai Soc. Math. Stud., vol. to appear, Springer, 2017.

Department of Geometry, Lorand Eötvös University, Pazmány Péter Sétány 1/C Budapest, Hungary 1117
E-mail address: marton.naszodi@math.elte.hu

Department of Mathematics, London School of Economics and Political Science, Houghton Street, London WC2A 2AE, United Kingdom
E-mail address: k.swanepoel@lse.ac.uk