The NIEP for four dimensional Leslie and doubly–stochastic matrices with zero trace from the coefficients of the characteristic polynomial

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Abstract

The constraints on the coefficients of the characteristic polynomials of four dimensional doubly stochastic matrices and Leslie stochastic matrices with zero trace are determined. Then, on the basis of these constraints, the regions of the complex plane consisting of the individual eigenvalues of all four dimensional doubly stochastic matrices with zero trace and of all Leslie stochastic matrices with zero trace are characterized.

Keywords: Nonnegative inverse eigenvalue problem; doubly stochastic matrices.
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1 Introduction

A real square matrix is said to be *stochastic* if its elements are all nonnegative and all its rows sum to one. A real square matrix is said to be *doubly stochastic* if it is stochastic and all its columns sum to one. The classes of stochastic and doubly stochastic matrices have been the subject of several studies since the beginning of the last century, but very little is known about the spectral properties of these classes.

The regions Θ_n of the complex plane consisting of the individual eigenvalues of all n-dimensional stochastic matrices were first characterized by Dmitriev and Dynkin [3] in 1946, for $n = 2, \ldots, 5$, and later by Karpelevich [6] for all $n \geq 2$. A simplified description of the regions Θ_n has been provided by Ito [4] in 1997.

In 1965, Perfect and Mirsky [11] considered the analogous problem for doubly stochastic matrices, that is the characterization of the regions Ω_n of the complex plane consisting of the individual eigenvalues of all n-dimensional doubly stochastic matrices. They proved that, for $n \geq 2$,

$$\Pi_2 \cup \Pi_3 \cup \ldots \cup \Pi_n \subseteq \Omega_n \tag{1}$$

where Π_k denotes the convex hull of the k–th roots of unity. Moreover, they showed that the sign of inclusion in (1) can be replaced by the sign of equality for $n = 3$, that is $\Omega_3 = \Pi_2 \cup \Pi_3$. It has been then conjectured that

$$\Omega_n = \Pi_2 \cup \Pi_3 \cup \ldots \cup \Pi_n$$

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for all n. Recently, Levick, Pereira and Kribs [9] have proved that the conjecture holds true for $n = 4$, that is $\Omega_4 = \Pi_2 \cup \Pi_3 \cup \Pi_4$, while Mashreghi and Rivard [10], in 2007, exhibited a counterexample to the conjecture for $n = 5$. Anyway, there is computational evidence that suggests that the case $n = 5$ is either a rare exception, or the only exception, to the conjecture [5].

In 1992, Kirkland [7] characterized the regions L_n of the complex plane consisting of the individual eigenvalues of all n-dimensional Leslie stochastic matrices, that is stochastic matrices of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{pmatrix}$$

The case of stochastic and doubly stochastic matrices with zero trace has also been extensively studied. In 1996, Reams [12] characterized the regions Θ_4^0 of the complex plane consisting of the individual eigenvalues of all the four dimensional stochastic matrices with zero trace while the case $n = 5$ was solved, in 1999, by Laffey and Meehan [8].

In 1965, Perfect and Mirsky [11] considered also the regions Ω_n^0 consisting of those complex numbers which can serve as characteristic roots of n–dimensional doubly stochastic matrices with zero trace. They proved that, for $n \geq 2$,

$$\Pi_n^0 \subseteq \Omega_n^0 \tag{2}$$

where Π_n^0 denotes the union of the point 1 and the convex hull of the n–th roots of unity different from 1. Moreover, they showed that the sign of inclusion in (2) can be replaced by the sign of equality for $n = 3$, that is

$$\Omega_3^0 = \Pi_3^0.$$

Recently, in [1], Benvenuti provided a refinement of relation (2), showing that, for $n \geq 4$,

$$\Pi_2^0 \cup \Pi_3^0 \cup \ldots \cup \Pi_{n-2}^0 \cup [-1, 1] \cup \Pi_n^0 \subseteq \Omega_n^0 \tag{3}$$

There are no studies on the trace zero case for Leslie stochastic matrices. However, by definition, the following holds true

$$L_n^0 \subseteq \Omega_n^0 \cup L_n \tag{4}$$

where L_n^0 indicates the region of the complex plane consisting of the individual eigenvalues of all n-dimensional Leslie stochastic matrices with zero trace.

In this paper it is proved that, when $n = 4$, the sign of inclusion in (3) and in (4) can be replaced by the sign of equality, that is

$$\Omega_4^0 = [-1, 1] \cup \Pi_4^0$$

and

$$L_4^0 = \Omega_4^0 \cup L_4$$

Following the main idea of [13], the attention is firstly focused on the coefficients of the characteristic polynomial of a four dimensional doubly stochastic matrix (Leslie stochastic matrix) with zero trace determining the constraints that these coefficients have to obey to. Then, on the basis of these constraints, the spectra of this class of matrices is characterized. Regions Θ_4, Θ_4^0, L_4, L_4^0, Ω_4 and Ω_4^0 are depicted in Figure 1.

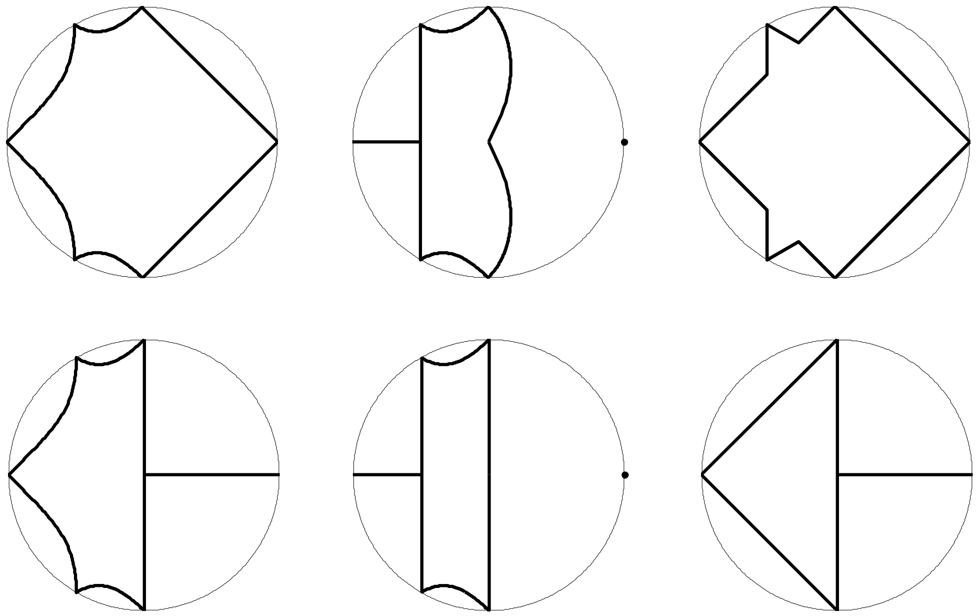

Figure 1: Regions Θ_4, L_4 and Ω_4 (from left to right, above), and Θ_4^0, L_4^0 and Ω_4^0 (from left to right, below)

2 Preliminary results

In order to prove the main result of the paper it is necessary to recall that, in view of the Birkhoffs theorem [2], any doubly stochastic matrix with zero trace can be written as a convex combination of some permutation matrices with zero trace. In this section, some preliminary results on permutation matrices are then provided.

Lemma 1 *Let P be a permutation matrix of dimension 4 having zero trace. Then*

(a) $tr(P^4) = 4$;

(b) $tr(P^3) = 0$;

Proof.

(a) The order of a permutation matrix is the smallest natural number k such that $P^k = I_4$. It can be computed from the weighted directed graph associated to the permutation matrix itself. In particular, when the dimension of the permutation matrix is 4 and the permutation matrix has zero trace, then the graph may consists of either only one cycle of length 4 or two cycles of length 2. In the first case, the order of the permutation is 4 while, in the second case, it is 2. Hence, in any case, $P^4 = I_4$, that is $tr(P^4) = 4$.

(b) Since the inverse of a permutation matrix is its transpose and $P^4 = I_4$, then $P^3 = P^T$. As a consequence the following holds true:

$$tr\left(P^3\right) = tr\left(P^T\right) = tr(P) = 0$$

■

Lemma 2 *Let P and Q be two permutation matrices of dimension 4 having zero trace such that $tr(P^2Q) = 4$. Then*

(a) $tr(PQ) = 0$;

(b) $tr(P^2) = 0$;

(c) $tr(Q^2) = 4$;

(d) $tr(PQ^2) = 0$.

Proof. Let first show that there exist two permutation matrices of dimension 4 having zero trace satisfying the hypothesis of the Lemma. To this end, for example, it is easy to verify that

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = I_4$$

In what follows it is necessary to take into account that the inverse of a permutation matrix is its transpose.

(a) If $tr(P^2Q) = 4$, then $P^2Q = I_4$. As a consequence, it follows that $P = (PQ)^T = Q^T P^T$. Hence, taking into account that the trace is invariant under cyclic permutations, the following holds true:

$$tr(PQ) = tr\left(Q^T P^T Q\right) = tr\left(Q Q^T P^T\right) = tr\left(P^T\right) = tr(P) = 0$$

(b) Since $P^2Q = I_4$, then $P^2 = Q^T$. As a consequence, the following holds true:

$$tr\left(P^2\right) = tr\left(Q^T\right) = tr(Q) = 0$$

(c) Since $P^2Q = I_4$, then $Q = \left(P^T\right)^2$. As a consequence, in view of Lemma 1, the following holds true:

$$tr\left(Q^2\right) = tr\left(\left(P^T\right)^4\right) = 4$$

(d) Since by condition (c) $tr(Q^2) = 4$, then $Q^2 = I_4$. A s a consequence, the following holds true:

$$tr\left(PQ^2\right) = tr(P) = 0$$

■

3 Four dimensional doubly stochastic matrix with zero trace

The following theorem provides the constraints for the coefficients of the characteristic polynomial of a four dimensional doubly stochastic matrix with zero trace:

Theorem 1 *Let* $p_A(\lambda) = \lambda^4 + k_1\lambda^3 + k_2\lambda^2 + k_3\lambda + k_4$ *be the characteristic polynomial of a doubly stochastic matrix* A *of dimension* 4 *having zero trace. Then:*

(a) $k_1 = 0$;

(b) $-2 \leq k_2 \leq 0$;

(c) $-4\sqrt{-\frac{k_2}{2}}\left(1-\sqrt{-\frac{k_2}{2}}\right)^2 \leq k_3 \leq 0$;

(d) $k_4 = -1 - k_2 - k_3$.

Proof. First note that the coefficients of the characteristic polynomial $p_A(\lambda)$ of a matrix A can be expressed in terms of the traces of powers of the matrix itself with the following recursive formula [14]:

$$k_n = -\frac{1}{n}\left[k_{n-1}\, tr(A) + k_{n-2}\, tr\left(A^2\right) + \ldots + k_1\, tr\left(A^{n-1}\right) + tr(A^n)\right] \tag{5}$$

(a) From the coefficients formula (5), it follows that $k_1 = -\, tr(A) = 0$.

(b) From the coefficients formula (5) and equality (a), it follows that $k_2 = -\frac{1}{2}\, tr(A^2)$. Since A^2 is a doubly stochastic matrix, then $0 \leq tr(A^2) \leq 4$ so that inequalities (b) immediately follow. The maximum and minimum values for k_2, as well as any intermediate value, can be obtained, for example, considering the following matrix with zero trace

$$A = \begin{pmatrix} 0 & 1+\frac{k_2}{2} & 0 & -\frac{k_2}{2} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{k_2}{2} & 0 & 1+\frac{k_2}{2} \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

which is doubly stochastic for $-2 \leq k_2 \leq 0$ and has characteristic polynomial equal to $p_A(\lambda) = \lambda^4 + k_2\lambda^2 + k_4$.

(c) From the coefficients formula (5) and equality (a), it follows that $k_3 = -\frac{1}{3}\, tr(A^3)$. Since A^3 is a doubly stochastic matrix, then $tr(A^3) \geq 0$ so that the right hand side of inequality (c) immediately follows. Note that k_3 may be equal to zero while k_2 takes every admissible value. This is clear, for example, when considering the matrix in (6). The lower bound for the coefficient k_3 can be obtained considering a matrix A which is a convex combination of two permutation matrices with zero trace, P and Q, that is

$$A = cP + (1-c)Q$$

In this case, taking into account that the trace is invariant under cyclic permutations, the coefficient k_3 is equal to

$$k_3 = -\frac{1}{3}\, tr\left(A^3\right) = -\frac{1}{3}\, tr\left((cP + (1-c)Q)^3\right) =$$

$$-\frac{c^3}{3}\, tr\left(P^3\right) - \frac{(1-c)^3}{3}\, tr\left(Q^3\right) - c^2(1-c)\, tr\left(P^2Q\right) - c(1-c)^2\, tr\left(PQ^2\right)$$

which, in view of Lemma 1, reduces to

$$k_3 = -c^2(1-c)\, tr\left(P^2Q\right) - c(1-c)^2\, tr\left(PQ^2\right)$$

In view of Lemma 2, the minimum value of k_3 is obtained when $tr\,(P^2Q) = 4$ and $tr\,(PQ^2) = 0$. In this case,

$$k_3^{min} = -4c^2(1-c) \tag{7}$$

On the other hand, in this case, the value of k_2 is equal to

$$k_2 = -\frac{1}{2}\,tr\left(A^2\right) = -\frac{1}{2}\,tr\left((cP + (1-c)Q)^2\right) =$$

$$-\frac{c^2}{2}\,tr\left(P^2\right) - \frac{(1-c)^2}{2}\,tr\left(Q^2\right) - c(1-c)\,tr\,(PQ)$$

Since $tr\,(P^2Q) = 4$, then, by virtue of Lemma 2, k_2 reduces to

$$k_2 = -2(1-c)^2$$

Hence, $c = 1 - \sqrt{-k_2/2}$ and equation (7) becomes

$$k_3^{min} = -4\sqrt{-\frac{k_2}{2}}\left(1 - \sqrt{-\frac{k_2}{2}}\right)^2$$

Note that k_3 may be equal to the minimum value k_3^{min} while k_2 takes every admissible value. This is clear, for example, when considering the following matrix

$$A = \begin{pmatrix} 0 & 0 & 1-\sqrt{-\frac{k_2}{2}} & \sqrt{-\frac{k_2}{2}} \\ 1-\sqrt{-\frac{k_2}{2}} & 0 & \sqrt{-\frac{k_2}{2}} & 0 \\ 0 & \sqrt{-\frac{k_2}{2}} & 0 & 1-\sqrt{-\frac{k_2}{2}} \\ \sqrt{-\frac{k_2}{2}} & 1-\sqrt{-\frac{k_2}{2}} & 0 & 0 \end{pmatrix}$$

which is doubly stochastic for $-2 \leq k_2 \leq 0$ and has characteristic polynomial equal to $p_A(\lambda) = \lambda^4 + k_2\lambda^2 + k_3^{min}\lambda + k_4$.

(d) Since the matrix A is doubly stochastic, then its characteristic polynomial $p_A(\lambda)$ has a root at $\lambda = 1$. Hence, taking into account equality (a),

$$p_A(1) = 1 + k_2 + k_3 + k_4 = 0$$

from which equality (d) immediately follows.

■

The set Ω_4^0 can be fully characterized on the basis of the previous result:

Theorem 2 *The set Ω_4^0 of individual eigenvalues of all four dimensional doubly stochastic matrices with zero trace is $\Omega_4^0 = [-1, 1] \cup \Pi_4^0$.*

Proof. All real spectra $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of four dimensional doubly stochastic matrices with zero trace have been fully determined in [11] by means of the following inequalities:

$$\lambda_1 = 1, \qquad -1 \leq \lambda_2, \lambda_3, \lambda_4 \leq 1, \qquad 1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$$

Hence, when considering only real spectra, $\Omega_4^0 = [-1, 1]$.

Consider then the case of complex spectra. The coefficients of the characteristic polynomial $p_A(\lambda)$ of a four dimensional doubly stochastic matrix with zero trace must satisfy the equalities and inequalities of Theorem 1. From conditions (a) and (d), that is $k_1 = 0$ and $k_4 = -1 - k_2 - k_3$, it follows that $p_A(1) = 0$. Hence, one of the eigenvalues, say λ_1, must be equal to 1. Moreover, since $k_1 = 0$, then

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0 \tag{8}$$

Let $\lambda_3 = a + ib$ and $\lambda_4 = a - ib$, with $b > 0$. Hence, taking into account equation (8), the spectrum of the matrix can be written as:

$$\Lambda = \{1, -1 - 2a, a + ib, a - ib\}$$

Then, the coefficients k_2 and k_3 of the characteristic polynomial

$$p_A(\lambda) = (\lambda - 1)(\lambda + 1 + 2a)(\lambda - a - ib)(\lambda - a + ib) = \lambda^4 + k_2\lambda^2 + k_3\lambda + k_4$$

are the following:

$$k_2 = b^2 - 3a^2 - 2a - 1, \qquad k_3 = 2a\left(b^2 + (1 + a)^2\right) \tag{9}$$

As a consequence, conditions (b) of Theorem 1 on coefficient k_2, that is $-2 \leq k_2 \leq 0$, become

$$3a^2 + 2a - 1 \leq b^2 \leq 3a^2 + 2a + 1 \tag{10}$$

Moreover, from non-positivity of the coefficient k_3, that is the right hand side of condition (c) of Theorem 1, immediately follows that

$$a \leq 0 \tag{11}$$

With easy calculations, the lower bound on k_3, that is the left hand side of condition (c) of Theorem 1, can be rewritten as

$$(k_3 + 4k_2)^2 \leq -2k_2\,(2 - k_2)^2 \tag{12}$$

Using equations (9) in inequality (12) gives

$$\left((a + 1)^2 - b^2\right)\left((5a^2 + 2a + b^2 + 1) - 16a^2b^2\right)^2 \geq 0$$

that is

$$b^2 \leq (a + 1)^2 \tag{13}$$

since $5a^2 + 2a + b^2 + 1 > 4ab$, for $a \leq 0$.

In summary, lower and upper bounds on the coefficients k_2 and k_3 can be written in terms of a and b as indicated by inequalities in (10), (11) and (13). Moreover, since $(a + 1)^2 = a^2 + 2a + 1 \leq 3a^2 + 2a + 1$, such inequalities reduce to

$$a \leq 0, \qquad 3a^2 + 2a - 1 \leq b^2 \leq (a + 1)^2 \tag{14}$$

Finally, taking into account that $3a^2 + 2a - 1 > (a + 1)^2 = a^2 + 2a + 1$ when $a < -1$, inequalities (14) further simplify to

$$-1 \leq a \leq 0, \qquad b \leq a + 1$$

that is $a \pm ib \in \Pi_4^0$. Moreover, since $-1 \le a \le 0$, then $-1 \le \lambda_2 = -1 - 2a \le 1$. Hence, the following holds true also when considering complex spectra

$$\Omega_4^0 \cap \mathbb{R} = [-1, 1]$$

■

A shown in [1], for $n = 5$, the sign of inclusion in (3) cannot be replaced by the sign of equality. As for the general case of doubly stochastic matrices, one may think that the $n = 5$ case is a rare exception, or the only exception, in which the sign of equality in (3) does not hold. The following example shows that this is not the case when considering doubly stochastic matrices with zero trace. Consider, in fact, the doubly stochastic matrix with zero trace

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8/28 & 20/28 & 0 & 0 & 0 \\ 10/28 & 17/28 & 1/28 & 0 & 0 & 0 \\ 18/28 & 3/28 & 7/28 & 0 & 0 & 0 \end{pmatrix}$$

whose complex conjugate eigenvalues do not belong to the set $\Pi_4^0 \cup \Pi_6^0$, as Figure 2 makes clear. Hence, also in the case $n = 6$, the inclusion in relation (3) is strict.

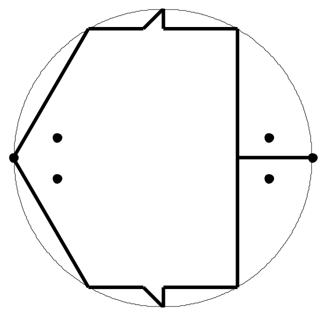

Figure 2: Eigenvalues of the matrix A and the region $\Pi_4^0 \cup [-1, 1] \cup \Pi_6^0$.

4 Four dimensional Leslie stochastic matrix with zero trace

The following theorem provides the constraints for the coefficients of the characteristic polynomial of a four dimensional Leslie stochastic matrix with zero trace:

Theorem 3 *Let $p_A(\lambda) = \lambda^4 + k_1\lambda^3 + k_2\lambda^2 + k_3\lambda + k_4$ be the characteristic polynomial of a Leslie stochastic matrix A of dimension 4 having zero trace. Then:*

(a) $k_1 = 0$;

(b) $-1 \le k_2 \le 0$;

(c) $-1 - k_2 \le k_3 \le 0$;

(d) $k_4 = -1 - k_2 - k_3$.

Proof.

(a) Consider a Leslie stochastic matrix A of dimension 4 having zero trace, that is:

$$A = \begin{pmatrix} 0 & a_2 & a_3 & a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Its characteristic polynomial is $p_A(\lambda) = \lambda^4 - a_2\lambda^2 - a_3\lambda - a_4$ so that $k_1 = 0$, and $k_i = -a_1$ for $i = 2, 3, 4$.

(b) Since $k_2 = -a_2$ and $0 \leq a_2 \leq 1$, then inequalities (b) immediately follow.

(c) Since $k_3 = -a_3$, then the right hand side of inequality (c) immediately follows from non negativity of a_3. Moreover, since $a_2 + a_3 \leq 1$, then $-1 - k_2 \leq k_3$, that is the left hand side of inequality (c).

(d) Since the matrix A is stochastic, then its first row sums to one. Hence,

$$a_2 + a_3 + a_4 = -k_2 - k_3 - k_4 = 1$$

from which equality (d) immediately follows.

■

The set L_4^0 can be fully characterized on the basis of the previous result:

Theorem 4 *The set L_4^0 of individual eigenvalues of all four dimensional Leslie stochastic matrices with zero trace is $L_4^0 = \Theta_4^0 \cap L_4$.*

Proof. First of all note that $L_4^0 \supseteq L_3^0$. This is obvious when considering a Leslie stochastic matrix A of dimension 4 having zero trace in which $a_4 = 0$, that is

$$A = \left(\begin{array}{ccc|c} 0 & a_2 & a_3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & 0 \end{array}\right)$$

In this case, in fact, $p_A(\lambda) = \lambda \cdot p_{A_{11}}(\lambda)$ where A_{11} is a Leslie stochastic matrix of dimension 3. Consider then the case of real spectra $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ of three dimensional Leslie stochastic matrices with zero trace. From stochasticity of the matrix, it follows that one eigenvalue, say λ_1, must be equal to 1. Moreover the trace zero assumption implies that $1 + \lambda_2 + \lambda_3 = 0$. Hence, the spectrum of the matrix can be written as:

$$\Lambda = \{1, \lambda_2, -1 - \lambda_2\}$$

and, the coefficient $k_3 = -a_3$ of the characteristic polynomial is

$$k_3 = \lambda_2(1 + \lambda_2). \qquad (15)$$

Since $0 \leq a_3 \leq 1$, then, from (15), it follows that $-1 \leq \lambda_2 \leq 0$. In summary, $L_4^0 \supseteq L_3^0 \supseteq [-1, 0]$. Moreover, from the Descartes' rule of signs, the polynomial

$$p_A(\lambda) = \lambda^4 - a_2\lambda^2 - a_3\lambda - a_4$$

has at most one positive real roots. Hence, $L_4^0 \cap \mathbb{R} = [-1, 0]$.

Consider then the case of complex spectra. The coefficients of the characteristic polynomial $p_A(\lambda)$ of a four dimensional Leslie stochastic matrix with zero trace must satisfy the equalities and inequalities of Theorem 3. Following the same steps as in the proof of Theorem 2, the spectrum of the matrix can be written as:

$$\Lambda = \{1, -1 - 2a, a + ib, a - ib\}$$

and the coefficients k_2 and k_3 of the characteristic polynomial are

$$k_2 = b^2 - 3a^2 - 2a - 1, \qquad k_3 = 2a\left(b^2 + (1 + a)^2\right)$$

As a consequence, conditions (b) of Theorem 3 on coefficient k_2, that is $-1 \leq k_2 \leq 0$, become

$$3a^2 + 2a \leq b^2 \leq 3a^2 + 2a + 1 \tag{16}$$

Moreover, from non-positivity of the coefficient k_3, that is the right hand side of condition (c) of Theorem 3, immediately follows that

$$a \leq 0 \tag{17}$$

while the lower bound on k_3, that is the left hand side of condition (c) of Theorem 3, gives

$$(1 + 2a)(a^2 + b^2) \geq 0$$

that is

$$a \geq -\frac{1}{2} \tag{18}$$

In summary, lower and upper bounds on the coefficients k_2 and k_3 can be written in terms of a and b as indicated by inequalities in (16), (17) and (18). Moreover, since $3a^2+2a \leq 0$, when $-1/2 \leq a \leq 0$, then such inequalities reduce to

$$-\frac{1}{2} \leq a \leq 0, \qquad b^2 \leq 3a^2 + 2a + 1$$

To conclude the proof it suffices to show that the extreme points of this set, that is the points

$$z = a + i\sqrt{3a^2 + 2a + 1} \qquad \text{with } -\frac{1}{2} \leq a \leq 0$$

belong to the boundary of the set Θ_4 (or Θ_4^0) . This can be done taking into account the characterization of the regions Θ_n given in [4]. In fact, these points lie on the border of the region Θ_4 consisting of the curvilinear arc connecting the points $i = e^{2\pi i/4}$ and $e^{2\pi i/3}$, since they are solutions of the equation

$$z(z^3 - s) = 1 - s$$

with $0 \leq s = -4a(2a^2 + 2a + 1) \leq 1$. ■

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