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Variational Methods in Continuum Damage and Fracture Mechanics

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Synonyms

Variational approach to damage and fracture mechanics; Variational formulation of damage and fracture mechanics

Definitions

Damage is defined as the loss of material stiffness under loading conditions. This process is intrinsically irreversible and, therefore, dissipative. When the stiffness vanishes, fracture is achieved. In order to derive governing equations, variational methods have been employed. Standard variational methods for non-dissipative systems are here formulated in order to contemplate dissipative systems as the ones considered in continuum damage mechanics.

Principle of Least Action for Dissipative Systems

Variational principles and calculus of variations have always been important tools for formulating mathematical models of physical phenomena (dell'Isola and Placidi 2011). Indeed, they are the main tool for the axiomatization of physical theories because they provide an efficient and elegant way to formulate and solve mathematical problems which are of interest for scientists and engineers. If the action functional is well behaving, variational principles always give rise to intrinsically well-posed mathematical problems, allowing also to find straightforwardly boundary conditions that guarantee uniqueness of the solution (dell'Isola et al. 2015b, 2016; Carcaterra et al. 2015). Thus, in order to formulate the governing equations of nonstandard models, it is natural to use a variational procedure.

However, it is often argued that dissipation cannot be handled by means of a least action principle. Indeed, it is usually pointed out that a limit of the modeling procedure based on variational principles consists in their impossibility of encompassing nonconservative phenomena. First

H. Altenbach, A. Öchsner (eds.), *Encyclopedia of Continuum Mechanics*,
https://doi.org/10.1007/978-3-662-53605-6_199-1

of all, this is not exactly true, as it is possible to find some action functionals for a large class of dissipative systems. This would be enough to contradict the thesis for that variational principles can be used only for non-dissipative systems.

Another possibility to answer to this criticism is to assume a slightly different point of view, usually attributed to Hamilton and Rayleigh (dell'Isola et al. 2009). Once the quantities which expend power on the considered velocity fields are known in terms of the postulated action, a suitable positive definite Rayleigh dissipation function is introduced that is related to the first variation of the action functional.

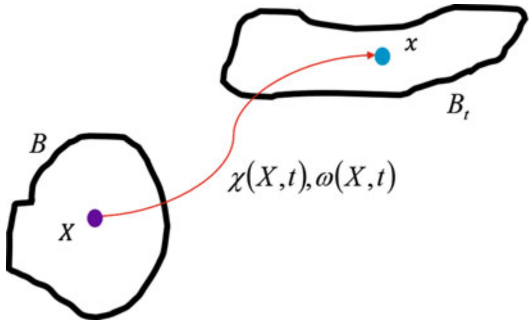
In continuum damage mechanics, see, e.g., Chaboche (1988), Misra and Singh (2013, 2015), and Poorsolhjouy and Misra (2016), the point of view is different. This is due to the monolateral behavior of damage kinematic descriptors. In general, in order to find a mathematical model for a class of natural phenomena by the use of variational principles, the first ingredient is to establish the right kinematics, i.e., the kinematic descriptors modeling the state of the considered physical systems. The second ingredient is to establish the set of admissible motions for the system under description, i.e., to the correct model for the admissible evolution of the system. In standard continuum mechanics, the kinematics is given by a single placement function χ that is defined on the reference configuration $\mathscr{B}$ and on a given time interval $\mathscr{I}$. The simplest way to treat continuum damage mechanics is to complement such a function with a scalar function ω, defined on the same reference configuration $\mathscr{B}$ and on the same interval of time $\mathscr{I}$.

The set of kinematic descriptors, therefore, does not contain, as usual, the placement field $\chi = \chi(X, t)$ only, but it also contains the damage field $\omega = \omega(X, t)$, see Fig. 1. Thus, the strain energy density reads as

$$\mathscr{E}(u, \omega)\,, \qquad (1)$$

where $\mathscr{E}$ is the total deformation energy functional. The damage state of a material point X is therefore characterized, at time t, by a scalar internal variable ω, that is assumed to be within the range $[0, 1]$. The cases $\omega = 0$ and $\omega = 1$ are customarily taken to correspond, respectively, to the undamaged state and to failure (Cuomo et al. 2014). Fracture is clearly assumed to be initiated at those points where $\omega = 1$ (Anderson 2017). The material is generally assumed to be not self-healing, and, hence, ω is assumed to be a non-decreasing function of time. This implies that the transition from undamaged to damaged states is irreversible and, roughly speaking, the total deformation energy is dissipated as far as the damage increases its value. Thus, if the damage ω is assumed to be one of the fundamental kinematic descriptors of the system (first ingredient), the set of its admissible motions (second ingredient) is intrinsically nonstandard. Keeping this in mind, the principle of least action should be generalized for those dissipative systems which possess kinematic descriptors with monolateral constraints. First of all, the variation $\delta\mathscr{E}$ of the total deformation energy functional $\mathscr{E}$ represented not only as a function of the kinematic descriptors χ and ω but also of their admissible variations $\delta\chi$ and $\delta\omega$, i.e.:

$$\delta\mathscr{E}(\chi, \omega, \delta\chi, \delta\omega) = A(\chi, \omega)\delta\chi + B(\chi, \omega)\delta\omega, \qquad (2)$$



Variational Methods in Continuum Damage and Fracture Mechanics, Fig. 1 Basic kinematics in damage mechanics. For each point of the domain, and therefore for each point of the reference configuration $\mathscr{B}$ and of the time interval $\mathscr{I}$, the kinematic is defined by the placement function χ and by a scalar function ω. χ is the placement of each point of the reference domain and ω is the state of damage. Herein, ω is assumed to be within the range $[0, 1]$ and the cases $\omega = 0$ and $\omega = 1$ correspond, respectively, to the undamaged state and to failure where it is made explicit that $\delta\mathscr{E}$ is, by definition, linear with respect to both $\delta\chi$ and $\delta\omega$.

For standard, bilateral, admissible motion, the principle of least action is expressed by imposing that the variation (2) is zero for any bilateral, admissible motion. This is made explicit in Fig. 2a, where a bilateral admissible variation of the solution, i.e., of the minimum of the represented graphic, gives that the correct minimum condition is a null variation of the functional to be minimized. In the case of monolateral admissible motion, the principle of least action must be made explicit differently. In Fig. 2b it is clear, in fact, that monolateral admissible motions do not necessarily imply that the variation of the functional to be minimized must always be assumed to vanish. In this case, it is better to assume that any admissible variation $\delta\mathscr{E}(\chi, \omega, \delta\chi, \delta\omega)$ is always greater than (better not lower than) the variation $\delta\mathscr{E}(u, \omega, \dot{u}, \dot{\omega})$ that is calculated in correspondence of the solution of the problem. Thus, from a mathematical point of view, the principle of least action is expressed by assuming that

$$\delta\mathscr{E}(u, \omega, \dot{u}, \dot{\omega}) \leq \delta\mathscr{E}(u, \omega, \upsilon, \beta), \quad \forall\upsilon,\ \forall\beta \geq 0, \qquad (3)$$

where υ and β are compatible virtual velocities starting from the configuration χ and ω, and dots represent derivation with respect to time. Thus, $\dot{\chi}$ and $\dot{\omega}$ are, respectively, the standard velocity field and the rate of damage that are calculated on the basis of the solutions $\chi(X, t)$ and $\omega(X, t)$ of the problem.

As commented in Marigo (1989), inequality (3) says that the true energy release rate (i.e., $-\delta\mathscr{E}(u, \omega, \dot{u}, \dot{\omega})$) is not smaller than any possible one (i.e., $-\delta\mathscr{E}(u, \omega, \upsilon, \beta)$). It constitutes, therefore, a kind of principle of maximum energy release rate. It is worth to be noted that such a principle was shown also by Hill in 1948 (Hill 1948), see also Maier (1970), in order to express a variational principle of maximum plastic work. Among others, it is worth to be mentioned the contributions due to Bourdin et al. (2008), Fleck and Willis (2009), Kuczma and Whiteman (1995), Rokoš et al. (2016), and Reddy (2011a,b).

Reduction to the Standard Variational Principle

In this section it is verified that the variational principle expressed in (3) reduces to the usual one, i.e., to $\delta\mathscr{E} = 0$, for arbitrary variations $\delta\chi$, when no variation δ is considered ($\delta\omega = 0$). Namely, it is checked that

$$\delta\mathscr{E}(u, \omega, \delta u, 0) = 0, \qquad \forall\delta u. \qquad (4)$$

Let the virtual velocity field υ be $\upsilon = \dot{u} + \overline{\upsilon}$, with arbitrary $\overline{\upsilon}$, and the other virtual velocity β to be $\beta = \dot{\omega}$ in (3). Since β is an arbitrary positive field, the choice $\beta = \dot{\omega}$ is admissible because also $\dot{\omega}$ is a nonnegative (nonarbitrary!) field. This yields

$$\delta\mathscr{E}(u, \omega, \dot{u}, \dot{\omega}) \leq \delta\mathscr{E}(u, \omega, \dot{u} + \overline{\upsilon}, \dot{\omega}). \qquad (5)$$

Let now the virtual velocity field υ be $\upsilon = \dot{u} - \overline{\upsilon}$, with the same field $\overline{\upsilon}$ of (5), and again $\beta = \dot{\omega}$ in (3). We get

$$\delta\mathscr{E}(u, \omega, \dot{u}, \dot{\omega}) \leq \delta\mathscr{E}(u, \omega, \dot{u} - \overline{\upsilon}, \dot{\omega}). \qquad (6)$$

Since the first variation of a functional is linear with respect to the admissible variations, see the representation (2), inequality (5) implies

$$\delta\mathscr{E}(u, \omega, \overline{\upsilon}, 0) \geq 0 \qquad (7)$$

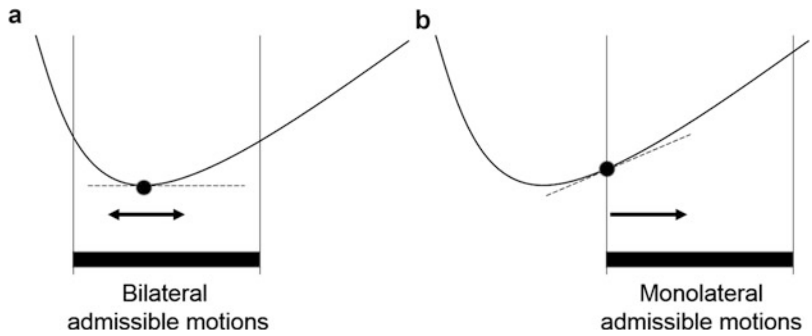
and inequality (6) implies

$$\delta\mathscr{E}(u, \omega, \overline{\upsilon}, 0) \leq 0. \qquad (8)$$

Combining (7) and (8)

$$\delta\mathscr{E}(u, \omega, \bar{\upsilon}, 0) = 0, \qquad \forall\overline{\upsilon} \qquad (9)$$

is obtained, which has the same desired form as (4). This is a very important result. It tells that the principle of least action in the form of the variational inequality (3) is a generalization of the same principle that is generally expressed as in (9), for the case of monolateral kinematic descriptors.



Variational Methods in Continuum Damage and Fracture Mechanics, Fig. 2 (**a**) Bilateral admissible motions imply that the minimum condition is expressed by assuming that the first variation of the functional to be minimized vanishes. (**b**) Monolateral admissible motions do not necessarily imply that the minimum condition is expressed by assuming that the first variation of the functional to be minimized vanishes

The Derivations of KKT Conditions

The formulation (3) of the principle of least action does not only give back the standard formulation (9), but it also furnishes further conditions, the so-called KKT conditions. In the previous section, we have exploited the cases with the virtual velocity $\upsilon = \dot{u} \pm \overline{\upsilon}$. It is clear that for monolateral admissible virtual velocities, this is not immediately generalizable because the condition $\beta \geq 0$ must always be satisfied. To do this, the choice $\upsilon = \dot{u}$ and $\beta = 0$ is firstly used in (3). It yields

$$\delta\mathscr{E}\left(u, \omega, \dot{u}, \dot{\omega}\right) \leq \delta\mathscr{E}\left(u, \omega, \dot{u}, 0\right). \tag{10}$$

A second choice $\upsilon = \dot{u}$ and $\beta = 2\dot{\omega}$ has been made in (3). It yields

$$\delta\mathscr{E}\left(u, \omega, \dot{u}, \dot{\omega}\right) \leq \delta\mathscr{E}\left(u, \omega, \dot{u}, 2\dot{\omega}\right). \tag{11}$$

Since the first variation of a functional is linear with respect to virtual variations, see the representation (2), the inequality (10) implies

$$\delta\mathscr{E}\left(u, \omega, 0, \dot{\omega}\right) \leq 0, \tag{12}$$

and the inequality (11) implies

$$\delta\mathscr{E}\left(u, \omega, 0, \dot{\omega}\right) \geq 0. \tag{13}$$

Combining (12) and (13)

$$\delta\mathscr{E}\left(u, \omega, 0, \dot{\omega}\right) = 0 \tag{14}$$

is obtained, which is an integral form of the KKT conditions. A suitable localization of (14) gives the KKT conditions in their standard form. However, it is worth to be noted that the formulation (14) is different with respect to that represented in (9). In fact, (9) is valid for any admissible virtual velocity $\overline{\upsilon}$, while (14) is valid only for one single rate of damage $\dot{\omega}$. Such a localization can be achieved, therefore, only after a further exploitation of the principle of least action. Thus, the choice $\upsilon = \dot{u}$ in (3) implies

$$\delta\mathscr{E}\left(u, \omega, \dot{u}, \dot{\omega}\right) \leq \delta\mathscr{E}\left(u, \omega, \dot{u}, \beta\right) \ \forall \beta \geq 0. \tag{15}$$

By the linear representation in (2), it is easily shown that

$$\delta\mathscr{E}\left(u, \omega, 0, \beta\right) \geq \delta\mathscr{E}\left(u, \omega, 0, \dot{\omega}\right) \ \forall \beta \geq 0. \tag{16}$$

Reminding (14) and (16) reads as

$$\delta\mathscr{E}\left(u, \omega, 0, \beta\right) \geq 0 \ \forall \beta \geq 0. \tag{17}$$

The integral form (17) is now suitable for localization purposes because of the arbitrariness of the virtual velocity β. Thus, the so-called Karush-Kuhn-Tucker (KKT) conditions for damage mechanics have been derived simply from the principle of least action in the form of the variational inequality stated in (3).

In order to get governing equations with this method, this variational principle is generally presented as in Placidi (2015, 2016) in the next section.

The Definition of the Total Deformation Energy Functional in Nonlocal Continuum Mechanics

The total deformation energy functional $\mathscr{E}$ is the state function of the problem. It is generally decomposed into an elastic part $\mathscr{E}_e$:

$$\mathscr{E}_e = \mathscr{E}_e^{\text{int}} - \mathscr{E}_e^{\text{ext}}, \tag{18}$$

that is decomposed into an internal part $\mathscr{E}_e^{\text{int}}$:

$$\mathscr{E}_e^{\text{int}} = \int_{\mathscr{B}} U, \tag{19}$$

due to the material, and an external part $\mathscr{E}_e^{\text{ext}}$, due to the interaction with the external world, and a dissipation $\mathscr{E}_d$ part:

$$\mathscr{E}_d = \int_{\mathscr{B}} w(\omega), \tag{20}$$

where U is the density of the internal energy and w is the density of the dissipation energy.

Localizations of the deformation process are always preferential from an energetic viewpoint. Accordingly one must introduce some characteristic lengths in order to penalize the deformations that are too localized. This leads to the concept of nonlocal damage models. The nonlocal approach, for controlling the size of the localization zone, implies nonlocal terms either in the internal part of the total deformation energy functional or in the dissipated part.

Usually, the nonlocal terms are given by the dependence of the density of the total deformation energy functional upon not only the damage ω but also upon the first gradient of it, i.e., of $\nabla\omega$. From this point of view, it is worth to be noted, among others, the contributions of the group of Marigo (Marigo 1989; Pham et al. 2011; Bourdin et al. 2008; Amor et al. 2009; Pham and Marigo 2010a,b), Perego (Comi and Perego 1995) and Miehe (Miehe et al. 2016). A fully nonlocal approach (i.e., an integral procedure which is based on integration of the state variables over a typical domain whose size is related to the characteristic length of the localization) is due to the group of Bažant (Pijaudier-Cabot and Bažant 1987; Bažant and Jirásek 2002; Bazant and Pijaudier-Cabot 1988). As commented in dell'Isola et al. (2015a), it is possible to trace back such a fully nonlocal approach to the pioneering ideas of G. Piola (dell'Isola et al. 2014) that were also exploited in Silling (2000). A micromorphic approach is used by the group of Forest (Forest 2009; Aslan et al. 2011; Dillard et al. 2006). Strain gradient formulation is also used in the literature (Yang and Misra 2012; Yang et al. 2011; Peerlings et al. 2001). In the next section, a strain gradient formulation for damage continuum 1D bodies will be shown as an example of damage continuum mechanics with the variational approach that is here illustrated. The first variational formulation of this kind for strain gradient materials has been presented in Placidi (2015, 2016), from where the notation of the next section has been taken.

Damage Strain Gradient Formulation for the 1D Case

As an example, we consider, in the reference configuration, a body that it is modeled as a one-dimensional straight line of length L, with an abscissa $X \in [0, L]$. Let us further assume the quasi-static approximation. Thus, the inertia, i.e., the kinetic energy, is neglected. Since we deal with infinitesimal deformations, the total deformation energy functional $\mathscr{E}$ will be now expressed in terms of the displacement field $u(X,t) = \chi(X,t) - X$ and not of the placement $\chi(X,t)$.

An explicit form for the second gradient case of the total deformation energy functional is, therefore,

$$\begin{aligned}
&\mathscr{E}\left(u\left(X,t\right),\omega\left(X,t\right)\right)\\
&=\int_0^L\left[K_0(X)\omega(X,t)+\tfrac{1}{2}K(X)\omega(X,t)^2\right]dX\\
&+\int_0^L\left[\tfrac{1}{2}C\left(X,\omega\left(X,t\right)\right)\left[u'\left(X,t\right)\right]^2+\tfrac{1}{2}P\left(\omega\left(X\right)\right)\left[u''\left(X,t\right)\right]^2\right]dX\\
&-\int_0^L\left[b_\sigma(X)u(X,t)+b_m(X)u'(X,t)\right]dX\\
&-\sigma_0\,u(0,t)-\sigma_L\,u(L,t)-m_0\,u'(0,t)-m_L\,u'(L,t),
\end{aligned} \tag{21}$$

where $K\left(X\right)$ is the resistance to damage that is assumed to be independent of damage, $K_0(X)$ is another independent damage constitutive field that will be interpreted as the initial damage threshold, $C\left(X,\omega\left(X,t\right)\right)$ is the standard stiffness (that is assumed to depend on damage), and $P\left(\omega\left(X,t\right)\right)$ is the second gradient stiffness (that is also assumed to depend on damage). $b_\sigma(X)$ and $b_m(X)$ are the distributed external actions that expend work, respectively, on the displacement and on the gradient of the displacement. $b_\sigma(X)$ is also called the distributed external force and $b_m(X)$ the distributed external double force. σ_0, σ_L, m_0, and m_L are the concentrated external actions on the boundaries, $X = 0$ and $X = L$, of the domain $[0, L]$: σ_0 and σ_L are the concentrated external actions that make work on the displacement, respectively, on the left- and on the right-hand side of the one-dimensional body (also called external forces at the boundaries), and m_0 and m_L are the concentrated external actions that make work on the gradient of the displacement, respectively, on the left- and on the right-hand side of the one-dimensional body (also called external double forces at the boundaries).

An explicit form of the standard elastic formulation (9) for the strain gradient case expressed in (21) is

$$\begin{aligned}
&\int_0^L\left[\delta u\left(-\left(\sigma-m'-b_m\right)'-b_\sigma\right)dX+\left[\delta u\left(\sigma-b_m-m'\right)+\delta u' m\right]_{X=0}^{X=L}\right.\\
&-\sigma_0\delta u(0,t)-\sigma_L\delta u(L,t)-m_0\delta u'(0,t)-m_L\delta u'(L,t),\quad\forall\delta u
\end{aligned} \tag{22}$$

where integration by parts has been performed and where the contact force σ and the contact double force m are involved in the following form:

$$\sigma = C\left(X,\alpha\left(X,t\right)\right)u'\left(X,t\right),$$
$$m = P\left(\alpha\left(X,t\right)\right)u''\left(X,t\right). \tag{23}$$

The integral form (22) is suitable for the following localization:

$$\left(\sigma-m'-b_m\right)'+b_\sigma=0. \tag{24}$$

Insertion of (23) into (24) gives the standard partial differential equation (PDE) for a second gradient 1D continuum:

$$\left(Cu'-\left(Pu''\right)'-b_m\right)'+b_\sigma=0,$$
$$\forall X\in\left[0,L\right]. \tag{25}$$

Besides, the following duality conditions are derived from (22), i.e.:

$$\delta u(L)\left[Cu'-\left(Pu''\right)'-b_m\right]_{X=L}=\sigma_L, \tag{26}$$
$$\delta u(0)\left[Cu'-\left(Pu''\right)'-b_m\right]_{X=0}=-\sigma_0, \tag{27}$$
$$\delta u'(L)Pu''(L)=m_L, \tag{28}$$
$$\delta u'(0)Pu''(0)=-m_0, \tag{29}$$

where the boundary conditions (BCs) can be derived from the explicit form of the constraints, which are assumed to be expressed in terms of the displacement field.

The integral form (14), with the total deformation energy functional (21), has the following explicit form:

$$\int_0^L \dot{\omega}\left[K_0(X)+K(X)\,\omega(X,t)+\frac{1}{2}\frac{\partial C(X,\omega)}{\partial \omega}\left[u'(X,t)\right]^2+\frac{1}{2}\frac{\partial P(\omega)}{\partial \omega}\left[u''(X,t)\right]^2\right]dX=0. \tag{30}$$

The global form of the KKT (17) has the following other form:

$$\int_0^L \beta\left[K_0(X)+K(X)\,\omega(X,t)+\frac{1}{2}\frac{\partial C(X,\omega)}{\partial \omega}\left[u'(x,t)\right]^2+\frac{1}{2}\frac{\partial P(\omega)}{\partial \omega}\left[u''(X,t)\right]^2\right]dX\geq 0,\quad \forall\beta\geq 0. \tag{31}$$

In order to localize (31), let $\Omega_\gamma(X)\subset R$ be a family, parameterized over $\gamma\in R^+$, of bounded neighborhoods of $X\in[0,L]$, such that their diameters are diam $\Omega_\gamma(X)=\gamma$. Besides, let $\beta_\gamma:[0,L]\to R^+$ be a family of functions, parameterized over $\gamma\in R^+$, defined as

$$\beta_\gamma(X)=\begin{cases}0 & \text{if } X\notin\Omega_\gamma(X)\\ 1 & \text{if } X\in\Omega_\gamma(X).\end{cases} \tag{32}$$

Clearly, for each $\gamma\in R^+$, β_γ defined in (32) fulfills the positive definiteness required to β in (31). Hence, (31), with the specification of β as in (32), yields

$$\int_0^L \beta_\gamma\left[K_0(X)+K(X)\,\omega(X,t)+\frac{1}{2}\frac{\partial C(X,\omega)}{\partial \omega}\left[u'(X,t)\right]^2+\frac{1}{2}\frac{\partial P(\omega)}{\partial \omega}\left[u''(X,t)\right]^2\right]dX=0,\quad \gamma\in R^+. \tag{33}$$

and, letting $\gamma\to 0^+$, we finally get, $\forall X\in[0,L]$

$$K_0(X)+K(X)\,\omega(X,t)+\frac{1}{2}\frac{\partial C(X,\omega)}{\partial \omega}\left[u'(X,t)\right]^2+\frac{1}{2}\frac{\partial P(\omega)}{\partial \omega}\left[u''(X,t)\right]^2\geq 0. \tag{34}$$

Since by hypothesis we have $\dot{\omega}\geq 0$, keeping in mind (34), in order to fulfill the relation (30) we have that, $\forall X\in[0,L]$,

$$K_0(X)+K(X)\,\omega(X,t)+\frac{1}{2}\frac{\partial C(X,\omega)}{\partial \omega}\left[u'(X,t)\right]^2+\frac{1}{2}\frac{\partial P(\omega)}{\partial \omega}\left[u''(X,t)\right]^2=0, \tag{35}$$

and/or

$$\dot{\omega} = 0, \quad \forall X \in [0, L]. \tag{36}$$

The combination of (35) and (36) gives, $\forall X \in [0, L]$, the desired localform of the so-called Karush-Kuhn-Tucker (KKT) conditions for damage mechanics

$$\dot{\omega}\left(K_0(X) + K(X)\,\omega(X,t) + \frac{1}{2}\frac{\partial C(X,\omega)}{\partial \omega}\left[u'(X,t)\right]^2 + \frac{1}{2}\frac{\partial P(\omega)}{\partial \omega}\left[u''(X,t)\right]^2\right) = 0 \tag{37}$$

that has been derived simply from the variational inequality given in (3).

According to previous results in the literature, see, e.g., Yang and Misra (2012), the stiffness $C(X, \omega(X,t))$ is generally assumed to decrease with damage growth. The most simple relation of this kind that fulfills this condition is the linear one, i.e.:

$$C(X, \omega(X,t)) = C_0(X)\,(1 - \omega(X,t)). \tag{38}$$

Besides, also the most simple constitutive relation for the second gradient stiffness $P(\omega(X,t))$ is of linear type:

$$P(\omega(X,t)) = P_0\,(1 - n\omega(X,t)), \tag{39}$$

where, on the one hand, $n = 1$ indicates that $P = 0$ at the failure condition $\omega = 1$ and, on the other hand, $n = -1$, as proposed in Placidi (2015), indicates that the micro-structure represented by second gradient terms in (21) is enlarged by the presence of damage. By insertion of (38) and (39) into (37),

$$\dot{\omega}\left(K_0(X) + K(X)\,\omega(X,t) - \frac{1}{2}C_0(X)\left[u'(X,t)\right]^2 - \frac{1}{2}nP_0\left[u''(X,t)\right]^2\right) = 0. \tag{40}$$

Assuming $K(X) > 0$, (40) is rewritten in another form:

$$\dot{\omega}\,(\omega(X,t) - \omega_T(X,t)) = 0. \tag{41}$$

where the damage threshold $\omega_T(X,t)$ has been defined as follows:

$$\omega_T(X,t) = -\frac{K_0(X)}{K(X)} + \frac{C_0(X)}{2\,K(X)}\left[u'(X,t)\right]^2 + \frac{nP_0}{2\,K(X)}\left[u''(X,t)\right]^2. \tag{42}$$

Equation (42) is of interest. It gives an analytical expression of the damage evolution that has been derived from the variational inequality (3). Because of the local form (41) of the KKT conditions, the damage field $\omega(X,t)$ is given by its threshold in (42) only if the condition $\dot{\omega} \geq 0$ is satisfied. Otherwise, the (41) implies $\dot{\omega} = 0$. It is worth to be noted that if an initial undamaged condition, i.e., $\omega(X,0) = 0$, with no displacement field in an unstressed reference configuration, i.e., $u(X,0) = 0\ \forall X$, is selected, then, since $K_0(X) > 0$ and $K(X) > 0$, the threshold $\omega_T(X,0)$, from (42), is negative. Thus, in order to fulfill condition (41), the rate of damage, and therefore also damage, must be zero before time $t = t^*$, when the condition

$$\omega_T(X, t^*) = 0 \tag{43}$$

is satisfied. This means that damage starts to increase its value from the condition $\omega = 0$ only if the displacement field guarantees the occurrence of (43), i.e.,

$$K_0(X) = \frac{1}{2}C_0(X)\left[u'(X,t^*)\right]^2 + nP_0\frac{1}{2}\left[u''(X,t)\right]^2. \qquad (44)$$

Such a condition gives a clear interpretation of the constitutive function $K_0(X)$.

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