# Sum of exit times in a series of two metastable states

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**Abstract.** The problem of not degenerate in energy metastable states forming a series in the framework of reversible finite state space Markov chains is considered. Metastability has been widely studied both in the mathematical and physical literature. Metastable states arises close to a first order phase transition, when the system can be trapped for a long time (exponentially long with respect to the inverse of the temperature) before switching to the thermodynamically stable phase. In this paper, under rather general conditions, we give a sharp estimate of the exit time from a metastable state in a presence of a second metastable state that must be necessarily visited by the system before eventually reaching the stable phase. In this framework we give a sharp estimate of the exit time from the metastable state at higher energy and, on the proper exponential time scale, we prove an addition rule. As an application of the theory, we study the Blume–Capel model in the zero chemical potential case.

## 1 Introduction

Metastable states in finite volume Statistical Mechanics lattice systems, in which stochastic transition between states are controlled by an energy function, is a well understood phenomenon. Different mathematical theories have been developed in the last decades. The pioneering *pathwise approach* [1–3], further developed for non–reversible dynamics in [4], and the more recent *potential theoretic approach* [5–7], further developed in [8] via the use of the trace process, allow a thorough description of the phenomenon.

The first theory gives a handy definition of the metastable state and a physically clear interpretation of the associated exit time. In the low temperature limit, it has been proven indeed that the time scale on which the system leaves the metastable state is exponentially large with the inverse temperature, with a mass given by the smallest energy barrier that the system has to overcome along the paths connecting

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the metastable state to the stable state. Moreover, the theory gives information about the paths followed by the system during its transition to the stable state. In particular it ensures that with high probability the system visits one of the configurations, namely, the critical droplets, where the smallest energy barrier is attained before reaching the stable state.

The latter theory, on the other hand, allows a precise estimate of the mean exit time. In particular it is proven that, measured in terms of the exponential time scale, such a mean time can be written in terms of a quantity, called *capacity*, which can be bounded from above and from below by using effective variational principles. Under suitable hypotheses on the energy landscape, the theory allows the computation of the prefactor and, what is very relevant on physical grounds, shows that it does depend on entropic effects. Indeed, in many specific models the prefactor has been computed explicitly and it turns out to be connected, loosely speaking, to the number of possible ways in which the system can perform its transition to the stable state [1, 9]. More precisely, it depends on the number of critical droplets that the system can use as a gate towards the stable state.

In this framework general results are proven under suitable hypotheses on the energy landscape ruling out the possibility to have multiple non degenerate in energy metastable states. In this paper we approach precisely such a problem and assume that two metastable states are indeed present. Moreover, we chose a peculiar structure of the energy landscape such that the two states form a "series", in the sense that, starting from the metastable state at higher energy, the system has to visit necessarily (in probability in the low temperature limit) the second one in its way towards the ground state. We prove a sort of addition rule for the exit time and compute, on the exponential scale, a sharp estimate for the exit time.

We approach this problem in the framework of rather general reversible Markov chains. Our aim is that of introducing a model covering both the standard Statistical Mechanics stochastic lattice models (e.g., the Metropolis dynamics) and the reversible Probabilistic Cellular Automata. In this framework we assume a minimal characterization of the energy landscape sufficient to ensure both the presence of two non degenerate in energy metastable state and their serial structure.

In the last part of the paper we discuss an application of this theory. Indeed, we approach the Blume–Capel model [10, 11], whose metastability behavior has already been studied from different point of views and in different limits in [12–15], and we discuss how to derive, with a different method, the same result recently proven in [14] on the sharp estimate of the exit time from the metastable minus one state in the zero chemical potential case. We mention that the application of our results to the Probabilistic Cellular Automaton studied in [16] is reported in [17].

The paper is organized as follows: in Section 2 the general model is introduced and our main results are stated. Those results are then proved in Section 3. Finally, the application to the Blume–Capel model is discussed in Section 4.

## 2 Model and results

In this section we first introduce a general reversible Markov chain and specify the conditions on the energy landscape in order to have a series of metastable states. We next state our main results in this framework.

### 2.1 Reversible Markov chains

We want to give the notion of reversible Markov chain [3, Condition R, page 335] in a quite general setup so that the theory will apply to different and relevant examples

such as Statistical Mechanics Lattice models and the reversible Probabilistic Cellular Automata.

Consider a finite state space $X$ and a family of irreducible and aperiodic Markov chain $x(t)$, with $t \in \mathbb{Z}_+$ parametrized by the parameter $\beta > 0$, called *inverse temperature*. We let $p_\beta(x, y)$ and $\mu_\beta(x)$, for $x, y \in X$ be respectively the transition matrix and the stationary measure. We assume that the Markov chains are reversible with respect to $\mu_\beta$, namely,

$$\mu_\beta(x)p_\beta(x, y) = p_\beta(y, x)\mu_\beta(y) \tag{1}$$

for any $x, y \in X$. We also assume that if a jump is not allowed, that does not depend on $\beta$, namely, if $p_{\beta'}(x, y) = 0$ then $p_{\beta''}(x, y) = 0$ for any $\beta''$.

The definition of the model will be completed by assuming a slightly enforced version of the well known Wentzel–Friedlin condition and by requiring that the stationary measure is "close" to a Gibbs one[1]. More precisely, we assume that there exists $\Delta : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ and $r : X \times X \to \mathbb{R}$ such that, for any $x, y \in X$, $\Delta(x, y) = \infty$ if $p_\beta(x, y) = 0$ and

$$\lim_{\beta \to \infty} [-\log p_\beta(x, y) - \beta \Delta(x, y)] = r(x, y) \quad \text{if } p_\beta(x, y) > 0. \tag{2}$$

Note that $\Delta$ and $r$ do not depend on the inverse temperature. We shall call $\Delta$ the *cost function*.

Moreover, we assume that there exist two functions $s, H : X \to \mathbb{R}$ and a family of functions $G_\beta : X \to \mathbb{R}$ parametrized by $\beta > 0$ such that

$$\mu_\beta(x) = \frac{1}{\sum_{y \in X} e^{-G_\beta(y)}} e^{-G_\beta(x)} \quad \text{and} \quad \lim_{\beta \to \infty} [G_\beta(x)) - \beta H(x)] = s(x) \tag{3}$$

for $x \in X$ and $\max_{x \in X} s(x) = \bar{s} < \infty$. The normalization factor in $\mu_\beta$ is denoted by $Z_\beta$ and called *partition function*. Note that the function $H$ does not depend on the inverse temperature. We shall call $H$ the *Hamiltonian* or *energy* of the model.

From (2) and (3) it follows immediately that

$$\lim_{\beta \to \infty} \frac{1}{\beta} \log p_\beta(x, y) = -\Delta(x, y)$$

for all $x, y \in X$ such that $p_\beta(x, y) > 0$ and

$$\lim_{\beta \to \infty} \frac{1}{\beta} G_\beta(x) = H(x)$$

for $x \in X$. From (1) and the conditions above it follows also that

$$H(x) + \Delta(x, y) = \Delta(y, x) + H(y) \tag{4}$$

for all $x, y \in X$.

For any $x \in X$, we denote by $\mathbb{P}_x(\cdot)$ and $\mathbb{E}_x[\cdot]$ respectively the probability and the average along the trajectories of the process started at $x$.

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[1]  The fact that the stationary measure is close to a Gibbs one at low temperature is generally valid in the framework of Wentzell–Freidlin dynamics, see, e.g., [18, Proposition 4.1]. What we assume in this paper, see (3), is, on the other hand, slightly stronger.

## 2.2 Examples

In this section we discuss two important examples of dynamics fitting in the general scheme depicted above. The *Metropolis* dynamics on the states space $X$ with *energy* $K : X \to \mathbb{R}$, *inverse temperature* $\beta$, and *connection matrix* $q(x, y)$ is defined by letting

$$p_\beta(x, y) = q(x, y)\, e^{-\beta[K(y) - K(x)]_+} \quad \text{if } x \neq y$$

where, for any real $a$, we let $[a]_+ = a$ if $a > 0$ and $[a]_+ = 0$ otherwise be the positive part of $a$, and

$$p_\beta(x, x) = 1 - \sum_{y \neq x} p_\beta(x, y).$$

It is well known that the Metropolis dynamics has as stationary measure the Gibbs measure with Hamiltonian $K$, so that the condition (3) is satisfied with $G_\beta = \beta K$ and $s = 0$.

Now, we let $r(x, x) = 0$ and $r(x, y) = -\log q(x, y)$ for $x \neq y$ and $q(x, y) > 0$. For any $x \neq y$, we let also $\Delta(x, y) = \infty$ if $q(x, y) = 0$ and $\Delta(x, y) = [K(y) - K(x)]_+$ otherwise. Finally, we let $\Delta(x, x) = \infty$ if $p_\beta(x, x) = 0$ and $\Delta(x, x) = -(1/\beta) \log p_\beta(x, x)$ otherwise. It is immediate to verify that condition (2) is satisfied for any $x, y \in X$.

A second important example is that of reversible Probabilistic Cellular Automata (PCA). Reversible PCA have been introduced in [19], see also [20] for a detailed discussion, and provide a very interesting example of dynamics with a parallel updating rule which are reversible with respect to a stationary measure which is very close to a Gibbs measure. The metastable behavior of some reversible Probabilistic Cellular Automata has been firstly studied in [21].

Let $\Lambda \subset \mathbb{Z}^2$ be a finite cube with periodic boundary conditions. Associate with each site $i \in \Lambda$ the state variable $x_i \in \{-1, +1\}$ and denote by $X = \{-1, +1\}^\Lambda$ the *state space*. For any $x \in X$ we consider on $\{-1, +1\}$ the probability measure

$$f_{x, \beta}(s) = \frac{1}{2}\Big\{1 + s \tanh\Big[\beta\Big(\sum_{j \in \Lambda} k(j) x_j + h\Big)\Big]\Big\}$$

for $s \in \{-1, +1\}$, where $\beta > 0$ and $h \in \mathbb{R}$ are called *inverse temperature* and *magnetic field* respectively. The function $k : \mathbb{Z}^2 \to \mathbb{R}$ is such that its support is a subset of $\Lambda$ and $k(j) = k(-j)$ for all $j \in \Lambda$. Recall that, by definition, the support of the function $k$ is the subset of $\Lambda$ where the function $k$ is different from zero. We assume, also, that

$$\sum_{j \in \Lambda} k(j - i) x_j + h \neq 0 \tag{5}$$

for any $x \in X$ and $i \in \Lambda$.

We finally introduce the shift $\Theta_i$ on the torus, for any $i \in \Lambda$, defined as the map $\Theta_i : X \to X$ such that $(\Theta_i x)_j = x_{i+j}$. A reversible PCA is the Markov chain on $X$ with transition matrix

$$p_\beta(x, y) = \prod_{i \in \Lambda} f_{\Theta_i x, \beta}(y_i)$$

for $x, y \in X$. We remark that the character of the evolution is parallel, in the sense that at each time all the spins are potentially flipped.

It is not difficult to prove [19] that the above specified PCA dynamics is reversible with respect to the finite–volume Gibbs–like measure

$$\mu_\beta(x) = \frac{1}{Z_\beta}\, e^{-F_\beta(x)}$$

with

$$F_\beta(x) = -\beta h \sum_{i \in \Lambda} x_i - \sum_{i \in \Lambda} \log \cosh \left[ \beta \Big( \sum_{j \in \Lambda} k(j-i)x_j + h \Big) \right]$$

with $Z_\beta$ the normalization constant.

The low–temperature behavior of the stationary measure of the PCA can be guessed by looking at the function

$$K(x) = \lim_{\beta \to \infty} \frac{1}{\beta} F_\beta(x) = -h \sum_{i \in \Lambda} x_i - \sum_{i \in \Lambda} \left| \sum_{j \in \Lambda} k(j-i)x_j + h \right|$$

The difference between $F_\beta$ and $\beta K$ can be computed explicitly, indeed in [20] it is proven that

$$F_\beta(x) - \beta K(x) = -\sum_{i \in \Lambda} \log \left( 1 + \exp \left\{ -2\beta \left| \sum_{j \in \Lambda} k(j-i)x_j + h \right| \right\} \right) + |\Lambda| \log 2$$

for each $\beta > 0$ and $x \in X$.

From the remarks above, recall also the assumption (5), it follows immediately that the reversible PCA satisfies condition (3) with $G_\beta = F_\beta$, $H = K$, and $s(x) = |\Lambda| \log 2$ for any $x \in X$.

As for the transition rates, we set

$$V(x, y) = -\lim_{\beta \to \infty} \frac{1}{\beta} \log p_\beta(x, y) = \sum_{\substack{i \in \Lambda: \\ y_i(\sum_{j \in \Lambda} k(j-i)x_j + h) < 0}} 2 \left| \sum_{j \in \Lambda} k(j-i)x_j + h \right|$$

We shall prove that

$$-\log p_\beta(x, y) - \beta V(x, y) = \sum_{i \in \Lambda} \log(1 + e^{-2\beta| \sum_{j \in \Lambda} k(j-i)x_i + h|}) \qquad (6)$$

Thus, the reversible PCA satisfies condition (2) with $\Delta = V$ and $r(x, y) = 0$ for any $x, y \in X$.

For completeness, we finally prove (6). In the following computation we shall use many times the assumption (5). First note that

$$-\log p_\beta(x, y) - \beta V(x, y) = \sum_{i \in \Lambda} \log(1 + e^{-2\beta y_i[\sum_{j \in \Lambda} k(j-i)x_j + h]})$$
$$+ \beta \sum_{\substack{i \in \Lambda: \\ y_i(\sum_{j \in \Lambda} k(j-i)x_j + h) < 0}} 2y_i \Big( \sum_{j \in \Lambda} k(j-i)x_j + h \Big)$$

Hence

$$-\log p_\beta(x, y) - \beta V(x, y) = \sum_{\substack{i \in \Lambda: \\ y_i(\sum_{j \in \Lambda} k(j-i)x_j + h) > 0}} \log(1 + e^{-2\beta y_i[\sum_{j \in \Lambda} k(j-i)x_j + h]})$$
$$+ \sum_{\substack{i \in \Lambda: \\ y_i(\sum_{j \in \Lambda} k(j-i)x_j + h) < 0}} \log(1 + e^{-2\beta y_i[\sum_{j \in \Lambda} k(j-i)x_j + h]})$$
$$+ \beta \sum_{\substack{i \in \Lambda: \\ y_i(\sum_{j \in \Lambda} k(j-i)x_j + h) < 0}} 2y_i \Big( \sum_{j \in \Lambda} k(j-i)x_j + h \Big)$$

Finally,

$$-\log p_\beta(x,y) - \beta V(x,y) = \sum_{\substack{i \in \Lambda: \\ y_i(\sum_{j \in \Lambda} k(j-i)x_j+h)>0}} \log(1 + e^{-2\beta y_i[\sum_{j \in \Lambda} k(j-i)x_j+h]})$$
$$+ \sum_{\substack{i \in \Lambda: \\ y_i(\sum_{j \in \Lambda} k(j-i)x_j+h)<0}} \log(e^{2\beta y_i[\sum_{j \in \Lambda} k(j-i)x_j+h]} + 1)$$

yielding (6).

### 2.3 Energy landscape

After the short "intermezzo" on the Metropolis and the reversible PCA models, we come back to the general setup of Section 2.1. Let $Q$ be the set of pairs $(x,y) \in X \times X$ such that $p_\beta(x,y) > 0$ or, equivalently, $\Delta(x,y) < \infty$. The quadruple $(X, Q, H, \Delta)$ is then a reversible energy landscape [12].

Given $Y \subset X$ we let its *external boundary* $\partial Y$ be the collection of states $z \in X \setminus Y$ such that there exists $y \in Y$ such that $(y, z) \in Q$. In words, the external boundary is made of those states outside $Y$ such that there exists a state in $Y$ where the system can jump.

Given $Y \subset X$ such that $H(y) = H(y')$ for any $y, y' \in Y$, we shall denote by $H(Y)$ the energy of the states in $Y$. For any $Y \subset X$ we shall denote by $F(Y)$ the set of the minima of the energy inside $Y$, that is to say $y \in F(Y)$ if and only if $H(y') \geq H(y)$ for any $y' \in Y$. We let $X_s := F(X)$ be the set of *ground states* of $H$, namely, the set of the absolute minima of the energy.

For any positive integer $n$, $\omega \in X^n$ such that $(\omega_i, \omega_{i+1}) \in Q$ for all $i = 1, \ldots, n-1$ is called a *path* joining $\omega_0$ to $\omega_n$; we also say that $n$ is the length of the path. For any path $\omega$ of length $n$, we let

$$\Phi_\omega := \max_{i=1,\ldots,n-1} [H(\omega_i) + \Delta(\omega_i, \omega_{i+1})] \tag{7}$$

be the *height* of the path[2]. For any $y, z \in X$ we denote by $\Omega(y, z)$ the set of the paths joining $y$ to $z$. For any $y, z \in X$ we define the *communication height* between $y$ and $z$ as

$$\Phi(y, z) := \min_{\omega \in \Omega(y,z)} \Phi_\omega \tag{8}$$

From (4), (7), and (8) it follows immediately that

$$\Phi(y, z) = \Phi(z, y) \tag{9}$$

for all $y, z \in X$. For any $Y, Z \subset X$ we let

$$\Phi(Y, Z) := \min_{\omega \in \Omega(Y,Z)} \Phi_\omega = \min_{y \in Y, z \in Z} \Phi(y, z) \tag{10}$$

where we have used the notation $\Omega(Y, Z)$ for the set of paths joining a state in $Y$ to a state in $Z$.

For any $y, z \in X$ we define also the *communication cost* from $y$ to $z$ as the quantity $\Phi(y, z) - H(y)$. Note that in general the communication cost from $y$ to $z$ differs from that from $z$ to $y$.

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[2] Since the energy landscape is reversible, the energy of the state $\omega_n$ is implicitly taken into account in (7), indeed (4) implies $H(\omega_n) \leq \Delta(\omega_{n-1}, \omega_n) + H(\omega_{n-1})$.

## 2.4 Metastable states

For any $x \in X$ we denote by $I_x$ the set of states $y \in X$ such that $H(y) < H(x)$. Note that $I_x = \emptyset$ if $x \in X_s$. We then define the *stability level* of any $x \in X \setminus X_s$

$$V_x := \Phi(x, I_x) - H(x) \geq 0 \tag{11}$$

Note that the stability level $V_x$ of $x$ is the minimal communication cost that, starting from $x$, has to be payed in order to reach states at energy lower than $H(x)$.

Following [1] we now introduce the notion of maximal stability level. Assume $X \setminus X_s \neq \emptyset$, we let the *maximal stability level* be

$$\Gamma_{\mathrm{m}} := \sup_{x \in X \setminus X_s} V_x \tag{12}$$

**Definition 1** *We call metastable set $X_{\mathrm{m}}$, the set*

$$X_{\mathrm{m}} := \{x \in X \setminus X_s : V_x = \Gamma_{\mathrm{m}}\} \tag{13}$$

Note that, since the state space is finite, the maximal stability level $\Gamma_{\mathrm{m}}$ is a finite number. Following [1], that is to say by assuming the so called *pathwise* point of view, we shall call $X_{\mathrm{m}}$ the set of *metastable* states of the system. Each state $x \in X_{\mathrm{m}}$ is called *metastable*.

A different, even if strictly related, notion of metastable states is that given in [5] in the framework of the Potential Theoretic Approach. First recall that the *Dirichlet form* associated with the reversible Markov chain is defined as the functional

$$\mathscr{D}_\beta[f] := \frac{1}{2} \sum_{y,z \in X} \mu_\beta(y) p_\beta(y, z)[f(y) - f(z)]^2 \tag{14}$$

where $f : X \to \mathbb{R}$ is a generic function.

Thus, given two not empty disjoint sets $Y, Z \subset X$ the *capacity* of the pair $Y$ and $Z$ can be defined as

$$\mathrm{cap}_\beta(Y, Z) := \min_{\substack{f : X \to [0,1] \\ f|_Y = 1, f|_Z = 0}} \mathscr{D}_\beta[f] \tag{15}$$

Note that the capacity is a *symmetric* function of the sets $Y$ and $Z$. It can be proven that the right hand side of (15) has a unique minimizer called *equilibrium potential* of the pair $Y$ and $Z$.

A nice interpretation of the equilibrium potential in terms of hitting times can be given. For $x \in X$ and $Y \subset X$ we shall denote by $\tau_Y^x$ the *first hitting* time to $Y$ of the chain started at $x$. Whenever possible we shall drop the superscript denoting the starting point from the notation. Then, it can be proven that the *equilibrium potential* of the pair $Y$ and $Z$ is equal to the function $h_{Y,Z}$ defined as follows

$$h_{Y,Z}(x) = \begin{cases} \mathbb{P}_x(\tau_Y < \tau_Z) & \text{for } x \in X \setminus (Y \cup Z) \\ 1 & \text{for } x \in Y \\ 0 & \text{for } x \in Z \end{cases} \tag{16}$$

where $\tau_Y$ and $\tau_Z$ are, respectively, the first hitting time to $Y$ and $Z$ for the chain started at $x$. It can be also proven that, for any $Y \subset X$ and $z \in X \setminus Y$,

$$\mathrm{cap}_\beta(z, Y) = \mu_\beta(z) \mathbb{P}_z(\tau_Y < \tau_z) \tag{17}$$

see [22, equation (3.10)].

**Definition 2** *A set $M \subset X$ is said to be* p.t.a.–metastable *if*

$$\lim_{\beta \to \infty} \frac{\max_{x \notin M} \mu_\beta(x)[\mathrm{cap}_\beta(x, M)]^{-1}}{\min_{x \in M} \mu_\beta(x)[\mathrm{cap}_\beta(x, M \setminus \{x\})]^{-1}} = 0 \tag{18}$$

The prefix p.t.a. stands for potential theoretic approach. We used this expression in order to avoid confusion with the set of metastable states $X_\mathrm{m}$ introduced in (13). The physical meaning of the above definition can be understood once one remarks that the quantity $\mu_\beta(x)/\mathrm{cap}_\beta(x, y)$, for any $x, y \in X$, is strictly related to the communication cost between the states $x$ and $y$, see Proposition 1. Thus, condition (18) ensures that the communication cost between any state outside $M$ and $M$ itself is smaller than the communication cost between any two states in $M$. In other words, it states that getting to $M$ starting from any state outside $M$ is "much" easier than going from any point in $M$ to any other point in $M$.

Finally, given a p.t.a.–metastable set $M \subset X$, for any $x \in M$ we let

$$A(x) := \{y \in X : \mathbb{P}_y(\tau_x = \tau_M) = \sup_{z \in M} \mathbb{P}_y(\tau_z = \tau_M)\} \tag{19}$$

be the *valley* associated with $x$. For any $y \in X$ and any $z \in M$ the quantity $\mathbb{P}_y(\tau_z = \tau_M)$ measures the probability that, starting from $y$, the system touches $M$ for the first time in $z$. Thus, by computing $\sup_{z, \in M} \mathbb{P}_y(\tau_z = \tau_M)$, one detects the best way to touch $M$ for the system started at $y$. Hence, the condition $\mathbb{P}_y(\tau_x = \tau_M) = \sup_{z, \in M} \mathbb{P}_y(\tau_z = \tau_M)$ selects all the sites $y$ such that, for the system started at $y$, the best way to touch $M$ is that of touching it for the first time at $x$.

### 2.5 Series of metastable states

The aim of this paper is that of proving an addition formula for the exit time from metastable states in the case in which they form a series. With this expression we mean that the structure of the energy landscape is such that the system has two non degenerate in energy metastable states and the system, started at the one having higher energy, must necessarily pass through the second one before relaxing to the stable state. See Fig. 1 for a schematic description of the situation we have in mind and that will be formalized through the following conditions.

**Condition 1** *Recall (12) and (13). We assume that the energy landscape $(X, Q, H, \Delta)$ is such that there exist three states $x_2$, $x_1$, and $x_0$ such that $X_\mathrm{s} = \{x_0\}$, $X_\mathrm{m} = \{x_1, x_2\}$, and $H(x_2) > H(x_1)$.*

Note that, by recalling the definition of the set of ground states $X_\mathrm{s}$, we immediately have that

$$H(x_1) > H(x_0) \tag{20}$$

Moreover, from the definition (12) of maximal stability level it follows that (see [12, Theorem 2.3]) the communication cost from $x_2$ to $x_0$ is equal to that from $x_1$ to $x_0$, that is to say

$$\Phi(x_2, x_0) - H(x_2) = \Phi(x_1, x_0) - H(x_1) = \Gamma_\mathrm{m} \tag{21}$$

Note that, since $x_2$ is a metastable state, its stability level cannot be lower than $\Gamma_\mathrm{m}$. Then, recalling that $H(x_2) > H(x_1)$, one has that $\Phi(x_2, x_1) - H(x_2) \geq \Gamma_\mathrm{m}$. On the other hand, (21) implies that there exists a path $\omega \in \Omega(x_2, x_1)$ such that $\Phi_\omega = H(x_2) + \Gamma_\mathrm{m}$ and, hence, $\Phi(x_2, x_1) - H(x_2) \leq \Gamma_\mathrm{m}$. The two bounds finally imply that

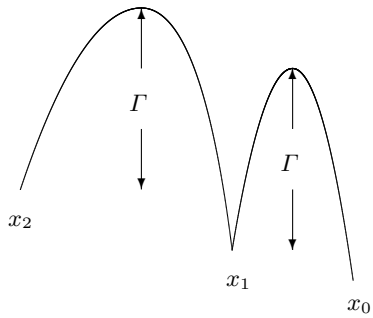$$\Phi(x_2, x_1) - H(x_2) = \Gamma_\mathrm{m} \tag{22}$$

**Fig. 1.** Schematic description of the energy landscape for a series of metastable states.

Note that the communication cost from $x_0$ to $x_2$ and that from $x_1$ to $x_2$ are larger than $\Gamma_m$, that is to say,

$$\Phi(x_0, x_2) - H(x_0) > \Gamma_m \quad \text{and} \quad \Phi(x_1, x_2) - H(x_1) > \Gamma_m \tag{23}$$

Indeed, by recalling the reversibility property (9) we have

$$\begin{aligned}
\Phi(x_1, x_2) - H(x_1) &= \Phi(x_2, x_1) - H(x_2) + H(x_2) - H(x_1) \\
&= \Gamma_m + H(x_2) - H(x_1) > \Gamma_m
\end{aligned}$$

where in the last two steps we have used (22) and Condition 1, which proves the second of the two equations (23). The first of them can be proved similarly.

We want to implement in the model the series structure depicted in Fig. 1. With this we mean that when the system is started at $x_2$ with high probability it will visit $x_1$ before $x_0$. For this reason we shall assume the following condition.

**Condition 2** *Condition 1 is satisfied and*

$$\lim_{\beta \to \infty} \mathbb{P}_{x_2}(\tau_{x_0} < \tau_{x_1}) = 0 \tag{24}$$

We remark that the Condition 2 is indeed a condition on the equilibrium potential $h_{x_0, x_1}$ evaluated at $x_2$.

The most important goal of this paper is that of proving the formula (28) for the expectation of the escape time $\tau_{x_0}$ for the chain started at $x_2$. Such an expectation, hence, will be of order $\exp\{\beta \Gamma_m\}$ and the prefactor will be that given in (28). At the level of logarithmic equivalence, namely, by renouncing to get sharp estimate, this result can be proven by the methods in [1]. More precisely, one gets that $(1/\beta) \log \mathbb{E}_{x_2}[\tau_{x_0}]$ tends to $\Gamma_m$ in the large $\beta$ limit.

We can thus formulate the further assumptions that we shall need in the sequel in order to discuss the problem from the point of view of the Potential Theoretic Approach.

**Condition 3** *Condition 1 is satisfied and there exists two positive constants $k_1, k_2 < \infty$ such that*

$$\frac{\mu_\beta(x_2)}{\text{cap}_\beta(x_2, \{x_1, x_0\})} = \frac{1}{k_2} e^{\beta \Gamma_m}[1 + o(1)], \quad \frac{\mu_\beta(x_1)}{\text{cap}_\beta(x_1, x_0)} = \frac{1}{k_1} e^{\beta \Gamma_m}[1 + o(1)] \tag{25}$$

*where $o(1)$ denotes a function tending to zero in the limit $\beta \to \infty$.*

## 2.6 Main results

We shall prove the addition rule for the exit times from the metastable states by using the sharp estimates provided by the Potential Theoretic Approach to metastability originally developed in [5].

**Theorem 1** *Assume Conditions 1 is satisfied. Then $\{x_0, x_1, x_2\} \subset X$ is a p.t.a.–metastable set.*

By means of the theory in [5] it is possible to write asymptotic estimates of the first hitting time to a subset of a p.t.a.–metastable set when the dynamics is started in state of the same p.t.a.–metastable set not belonging to the considered subset. These results, see for instance [5, Theorem 1.3], are typically proven under suitable not degeneracy conditions [5, Definition 1.2] that are not satisfied in our case, due to the presence of multiple metastable states. In the following theorem we state two results that, for the reasons outlined above, can be deduced directly from those in [5]. On the other hand, as we shall discuss in detail in Section 3, they can be deduced by some of the results proven in [22] (see, also, [6]). But, since we assumed strong hypotheses on the energy landscape of the model, it will be possible to prove the theorem directly by means of simple estimates. This "ad hoc" proof is also given in Section 3.

**Theorem 2** *Assume Conditions 1 is satisfied. Then*

$$\mathbb{E}_{x_2}[\tau_{\{x_1,x_0\}}] = \frac{\mu_\beta(x_2)}{\mathrm{cap}_\beta(x_2, \{x_1, x_0\})}[1 + o(1)], \ \mathbb{E}_{x_1}[\tau_{x_0}] = \frac{\mu_\beta(x_1)}{\mathrm{cap}_\beta(x_1, x_0)}[1 + o(1)] \quad (26)$$

**Theorem 3** *Assume Conditions 1 and 3 are satisfied. Then*

$$\mathbb{E}_{x_2}[\tau_{\{x_1,x_0\}}] = e^{\beta\Gamma_\mathrm{m}}\frac{1}{k_2}[1 + o(1)] \quad and \quad \mathbb{E}_{x_1}[\tau_{x_0}] = e^{\beta\Gamma_\mathrm{m}}\frac{1}{k_1}[1 + o(1)] \quad (27)$$

**Theorem 4** *Assume Conditions 1, 2, and 3 are satisfied. Then*

$$\mathbb{E}_{x_2}[\tau_{x_0}] = e^{\beta\Gamma_\mathrm{m}}\Big(\frac{1}{k_1} + \frac{1}{k_2}\Big)[1 + o(1)] \quad (28)$$

We remark that Theorem 4 gives an addition formula for the mean first hitting time to $x_0$ starting from $x_2$. Neglecting terms of order $o(1)$, such a mean time can be written as the sum of the mean hitting time to the pair $\{x_1, x_0\}$ when the chain is started at $x_2$ and of the mean hitting time to $x_0$ when the chain is started at $x_1$. It is very interesting to note that no role is plaid in this decomposition by the mean hitting time to $x_1$ for the chain started at $x_2$. Indeed, on the $\exp\{\beta\Gamma_\mathrm{m}\}$ time scale no control can be proven for such a mean time. For instance, in the case of the Blume–Capel model that will be studied in Section 4, in [14, Proposition 2.5] it is proven that $\mathbb{E}_{x_2}[\tau_{x_1}]/e^{\beta\Gamma_\mathrm{m}}$ diverges in the limit $\beta \to \infty$.

## 3 Proof of results

In this section we proof the theorems stated above and related to the general setup given in Section 2.5.

*Proof of Theorem 1.* The theorem follows immediately by Condition 1, (22), and [12, Theorem 3.6].  □

We just note that the Theorem 3.6 in [12] has been proved in a slightly different context, but the proof given there applies also to the more general case studied here.

Theorem 2 can be deduced by using the structure provided by Theorem 1 above, and the general results [22, Eq. (4.14) and Lemma 4.3]. Alternatively, one can use [6, Eq. (8.1.6), Eq. (8.3.3), and Lemma 8.13]. Since we assumed strong hypotheses on the energy landscape of the model, it is possible to prove directly equation (26) by means of simple estimates. Before discussing such a proof we state two useful lemmas. Recall Condition 1, in the first of the two lemmas we collect two bounds to the energy cost that has to be payed to go from any state $x \neq x_1$ to $x_1$ or to $x_0$. The second lemma is similar.

**Lemma 1** *Assume Condition 1 is satisfied. For any $x \in X$ and $x \neq x_1$. If $H(x) \leq H(x_1)$, we have that*

$$\Phi(x, x_0) - H(x) < \Gamma_{\mathrm{m}} \quad and \quad \Phi(x, x_1) - H(x_1) \geq \Gamma_{\mathrm{m}} \tag{29}$$

*Proof.* Let us prove the first inequality. By Theorem 2.3 in [12] we have that $\Phi(x, x_0) \leq \Gamma_{\mathrm{m}} + H(x)$. If by contradiction $\Phi(x, x_0) = \Gamma_{\mathrm{m}} + H(x)$ then, by the same Theorem 2.3 in [12], $x \in X_{\mathrm{m}}$ which is in contradiction with Condition 1.

As regards the proof of the second inequality we distinguish two cases. Case $H(x) < H(x_1)$: we have that $x \in I_{x_1}$. By Definition 1 of metastable state and by (11), we get

$$\Phi(x_1, x) \geq \Phi(x_1, I_{x_1}) = \Gamma_{\mathrm{m}} + H(x_1)$$

that proves the inequality.

Case $H(x) = H(x_1)$: let us define the set

$$\mathcal{C} := \{y \in X : \Phi(y, x_1) < H(x_1) + \Gamma_{\mathrm{m}}\}$$

and show that $x \notin \mathcal{C}$. Since $H(x) = H(x_1)$, the identity $I_x = I_{x_1}$ follows. Furthermore, being $x_1 \in X_{\mathrm{m}}$, we have $\mathcal{C} \cap I_{x_1} = \emptyset$; hence, $\mathcal{C} \cap I_x = \emptyset$ as well. Moreover, if $x \in \mathcal{C}$ then $V_x = \Phi(x, I_x) - H(x) \geq H(x_1) + \Gamma_{\mathrm{m}} - H(x) = \Gamma_{\mathrm{m}}$. By the Definition 1, $x$ would be a metastable state, in contradiction with Condition 1. Hence, since $x \notin \mathcal{C}$, we have that

$$\Phi(x, x_1) \geq \Gamma_{\mathrm{m}} + H(x_1)$$

that proves the inequality. the inequality. $\square$

**Lemma 2** *Assume Condition 1 is satisfied. For any $x \in X$ and $x \notin \{x_2, x_1, x_0\}$. If $H(x) \leq H(x_2)$, we have that*

$$\Phi(x, \{x_1, x_0\}) - H(x) < \Gamma_{\mathrm{m}} \quad and \quad \Phi(x, x_2) - H(x_2) \geq \Gamma_{\mathrm{m}} \tag{30}$$

*Proof.* Let us prove the first inequality. By Theorem 2.3 in [12] we have $\Phi(x, \{x_1, x_0\}) \leq \Phi(x, x_0) \leq \Gamma_{\mathrm{m}} + H(x)$. If by absurdity $\Phi(x, x_0) = \Gamma_{\mathrm{m}} + H(x)$ then, by the same Theorem 2.3 in [12], $x \in X_{\mathrm{m}}$ which is in contradiction with Condition 1.

As regards the proof of the second inequality we distinguish two cases. Case $H(x) < H(x_2)$: we have that $x \in I_{x_2}$. By Definition 1 of metastable state and by (11), we get

$$\Phi(x_2, x) \geq \Phi(x_2, I_{x_2}) = \Gamma_{\mathrm{m}} + H(x_2)$$

that proves the inequality.

Case $H(x) = H(x_2)$: let us define the set

$$\mathcal{C} := \{y \in X : \Phi(y, x_2) < H(x_2) + \Gamma_{\mathrm{m}}\}$$

and show that $x \notin C$. Since $H(x) = H(x_2)$, the identity $I_x = I_{x_2}$ follows. Furthermore, being $x_2 \in X_{\mathrm{m}}$, we have $C \cap I_{x_2} = \emptyset$; hence, $C \cap I_x = \emptyset$ as well. Moreover, if $x \in C$ then $V_x = \Phi(x, I_x) - H(x) \geq H(x_2) + \Gamma_{\mathrm{m}} - H(x) = \Gamma$. By the Definition 1, $x$ would be a metastable state, in contradiction with Condition 1. Hence, since $x \notin C$, we have that

$$\Phi(x, x_2) \geq \Gamma_{\mathrm{m}} + H(x_2)$$

that proves the inequality.   □

*Proof of Theorem 2.* We prove in details the right of equation (26). The proof is based on Lemma 1. The equation on the left can be deduced with precisely the same arguments and using the bounds in Lemma 2. The only general results used is the representation of the expected mean time in terms of the Green function given in [5, Corollary 3.3] (see also equation (3.18) in the proof of the Theorem 3.5 in [5] or [23, Eq. (4.29)]). Indeed, recalling (17) above, we have:

$$\mathbb{E}_{x_1}[\tau_{x_0}] = \frac{1}{\mathrm{cap}_\beta(x_1, x_0)} \sum_{x \in X} \mu_\beta(x)\, h_{x_1, x_0}(x) \tag{31}$$

Considering the contribution of $x_1$ in the sum and recalling (16), we get the following lower bound:

$$\mathbb{E}_{x_1}[\tau_{x_0}] \geq \frac{1}{\mathrm{cap}(x_1, x_0)} \mu_\beta(x_1) h_{x_1, x_0}(x_1) = \frac{1}{\mathrm{cap}(x_1, x_0)} \mu_\beta(x_1) \tag{32}$$

In order to provide un upper bound, we first use the boundary conditions in (16) to rewrite (31) as follows:

$$\mathbb{E}_{x_1}[\tau_{x_0}] = \frac{1}{\mathrm{cap}(x_1, x_0)} \Big[ \sum_{\substack{x \in X \setminus x_0, \\ H(x) \leq H(x_1)}} \mu_\beta(x) h_{x_1, x_0}(x) + \sum_{\substack{x \in X \setminus x_0, \\ H(x) > H(x_1)}} \mu_\beta(x) h_{x_1, x_0}(x) \Big]$$

Recalling that $h_{x_1, x_0}(x_1) = 1$, the equilibrium potential is not bigger than one, the configuration space is finite, and $\mu_\beta(x) = \mu_\beta(x_1) \exp\{-\beta\delta\}$ for some positive $\delta$ and for any $x \in X$ such that $H(x) > H(x_1)$, we get

$$\mathbb{E}_{x_1}[\tau_{x_0}] = \frac{1}{\mathrm{cap}(x_1, x_0)} \Big[ \sum_{\substack{x \in X \setminus x_0, \\ H(x) \leq H(x_1),\, x \neq x_1}} \mu_\beta(x)\, h_{x_1, x_0}(x) + \mu_\beta(x_1)[1 + o(1)] \Big] \tag{33}$$

By (16) and (2) we can give the following upper bound for the equilibrium potential $h_{x_1, x_0}(x)$, for any $x \neq x_1, x_0$

$$h_{x_1, x_0}(x) \leq \frac{\mathrm{cap}(x, x_1)}{\mathrm{cap}(x, x_0)} \ .$$

Thus, if $H(x) \leq H(x_1)$, we have

$$h_{x_1, x_0}(x) \leq C \frac{e^{-\beta\Phi(x, x_1)}}{e^{-\beta\Phi(x, x_0)}} \leq C \frac{e^{-\beta(\Gamma_{\mathrm{m}} + H(x_1))}}{e^{-\beta(\Gamma_{\mathrm{m}} + H(x) - \delta)}} = C e^{-\beta\delta} \frac{\mu_\beta(x_1)}{\mu_\beta(x)}$$

where in the first inequality we used Proposition 1, in the second Lemma 1, and $C, \delta$ are suitable positive constants. By using (33) we get

$$\mathbb{E}_{x_1}[\tau_{x_0}] \leq \frac{1}{\mathrm{cap}(x_1, x_0)} \Big[ \sum_{\substack{x \in X \setminus x_0, \\ H(x) \leq H(x_1),\, x \neq x_1}} C \mu_\beta(x) e^{-\beta\delta} \frac{\mu_\beta(x_1)}{\mu_\beta(x)} + \mu_\beta(x_1)[1 + o(1)] \Big]$$

Which implies

$$\mathbb{E}_{x_1}[\tau_{x_0}] \leq \frac{\mu_\beta(x_1)}{\text{cap}(x_1, x_0)}[1 + o(1)] \tag{34}$$

where we have used that the configuration space is finite. The Theorem finally follows by (32) and (34). $\square$

*Proof of Theorem 3.* The theorem follows immediately by exploiting Condition 3 and applying Theorem 2. $\square$

The proof of Theorem 4 is based on the following lemma.

**Lemma 3** *Given three states $y, w, z \in X$ pairwise mutually different, we have that the following holds*

$$\mathbb{E}_y[\tau_z] = \mathbb{E}_y[\tau_{\{w,z\}}] + \mathbb{E}_w[\tau_z]\mathbb{P}_y(\tau_w < \tau_z) \tag{35}$$

*Proof.* First of all we note that

$$\mathbb{E}_y(\tau_z) = \mathbb{E}_y[\tau_z \mathbb{I}_{\tau_w < \tau_z}] + \mathbb{E}_y[\tau_z \mathbb{I}_{\tau_w \geq \tau_z}]$$

We now rewrite the first term as follows

$$\mathbb{E}_y[\tau_z \mathbb{I}_{\{\tau_w < \tau_z\}}] = \mathbb{E}_y[\mathbb{E}_y[\tau_z \mathbb{I}_{\{\tau_w < \tau_z\}}|\mathcal{F}_{\tau_w}]] = \mathbb{E}_y[\mathbb{I}_{\{\tau_w < \tau_z\}}(\tau_w + \mathbb{E}_w[\tau_z])]$$
$$= \mathbb{E}_y[\tau_w \mathbb{I}_{\{\tau_w < \tau_z\}}] + \mathbb{P}_y(\tau_w < \tau_z)\mathbb{E}_w(\tau_z).$$

where we have used the fact that $\tau_w$ is a stopping time, that $\mathbb{I}_{\{\tau_w < \tau_z\}}$ is measurable with respect to the pre–$\tau_w$–$\sigma$–algebra $\mathcal{F}_{\tau_w}$ and the strong Markov property which gives $\mathbb{E}_y[\tau_z|\mathcal{F}_{\tau_w}] = \tau_w + \mathbb{E}_w[\tau_z]$ on the event $\{\tau_w \leq \tau_z\}$. Since $(\tau_w \mathbb{I}_{\{\tau_w < \tau_z\}} + \tau_z \mathbb{I}_{\{\tau_w \geq \tau_z\}}) = \tau_{\{w,z\}}$, (35) follows. $\square$

*Proof of Theorem 4.* By (35) we have that

$$\mathbb{E}_{x_2}[\tau_{x_0}] = \mathbb{E}_{x_2}[\tau_{\{x_1,x_0\}}] + \mathbb{E}_{x_1}[\tau_{x_0}]\mathbb{P}_{x_2}(\tau_{x_1} < \tau_{x_0})$$

By Theorem 3 and Condition 2 it follows that

$$\mathbb{E}_{x_2}[\tau_{x_0}] = e^{\beta \Gamma_m}\left(\frac{1}{k_1} + \frac{1}{k_2}\right)[1 + o(1)]$$

which completes the proof. $\square$

## 4 Application to the Blume–Capel model

In this section, as a possible application of the theory described above, we apply our results to the case of the Blume–Capel model [10,11]. In particular, we consider the model with null chemical potential, which has two metastable states non degenerate in energy. We shall then derive, in a different way, the results already appeared in [14].

Let us consider a square lattice $\Lambda \subset \mathbb{Z}^2$ with periodic boundary conditions and side length $L$. Let $\{-1, 0, +1\}$ be the single spin state space and $\mathcal{X} := \{-1, 0, +1\}^\Lambda$ be the configuration space. The Hamiltonian of the model [12] is

$$H(\sigma) = \sum_{\langle i,j \rangle}(\sigma(i) - \sigma(j))^2 - h\sum_{i \in \Lambda}\sigma(i) \tag{36}$$
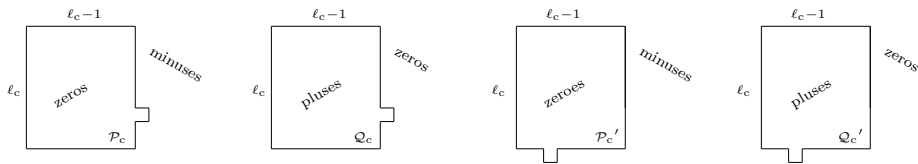
**Fig. 2.** Schematic representation of the configurations $\mathcal{P}_c$ and $\mathcal{Q}_c$. The protuberance can be either on the left or on the right vertical edge (the longest ones in the picture) and there it can be shifted freely. Note that the two configurations $\mathcal{P}_c{}'$ and $\mathcal{Q}_c{}'$, with the protuberance on the shortest side, do not play any role in the transition from the metastable to the stable state.

for any $\sigma \in \mathcal{X}$, where the first sum runs over the pairs of nearest neighbors and $h \in \mathbb{R}$ is the magnetic field. We denote by $\mu_\beta$ the corresponding Gibbs measure

$$\mu_\beta(\sigma) = \exp\{-\beta H(\sigma)\} / \sum_{\eta \in \mathcal{X}} \exp\{-\beta H(\eta)\}$$

with inverse temperature $\beta$. We shall study the zero chemical potential Blume–Capel model for the following choice of the parameters: the magnetic field $h$ and the torus $\Lambda$ are such that $0 < h < 1$, $2/h$ is not integer, and $|\Lambda| \geq 49/h^4$ finite, where, for any positive real $a$, we let $\lfloor a \rfloor$ be the largest integer smaller than or equal to $a$.

The time evolution of the model is defined by the Metropolis Markov chain $\sigma_t$ with $t = 0, 1, \ldots$ the discrete time variable, see Section 2.2, with Hamiltonian $H$ and connectivity matrix

$$q(\sigma, \eta) := \begin{cases} 0 & \text{if } \sigma, \eta \text{ differ at more than one site} \\ 1/(2|\Lambda|) & \text{otherwise} \end{cases}.$$

As already remarked in Section 2.2, the dynamics above is an example of the dynamics defined in Section 2.1, provided we let $i)$ $r(x, x) = 0$ and $r(x, y) = -\log q(x, y)$ for $x \neq y$ and $q(x, y) > 0$; $ii)$ for any $x \neq y$, $\Delta(x, y) = \infty$ if $q(x, y) = 0$ and $\Delta(x, y) = [H(y) - H(x)]_+$ otherwise; $iii)$ $\Delta(x, x) = \infty$ if $p_\beta(x, x) = 0$ and $\Delta(x, x) = -(1/\beta) \log p_\beta(x, x)$ otherwise. The notation introduced in Section 2.1–2.4 is then trivially particularized to the Blume–Capel case.

Given $V \subset \Lambda$ and $\sigma \in \mathcal{X}$, we let $\sigma_V$ be the restriction of $\sigma$ to $V$, namely, $\sigma_V \in \{-1, +1\}^V$ such that $\sigma_V(i) = \sigma(i)$ for any $i \in V$.

We let $\mathbf{u} \in \mathcal{X}$ to be the configuration such that $\mathbf{u}(i) = +1$ for all $i \in \Lambda$. Other two very relevant configurations are $\mathbf{d}$ and $\mathbf{0}$, that is the configuration in which all the spin are minus one and the one in which all the spins are zero. Note that in these configurations the exchange part of the energy is minimal, although the magnetic part is not.

We now define the *critical length* of the model as

$$\ell_c := \left\lfloor \frac{2}{h} \right\rfloor + 1 \tag{37}$$

We denote by $\mathcal{P}_c$ the set of configurations in which all the spins are minus excepted those, which are zeros, in a rectangle of sides long $\ell_c$ and $\ell_c - 1$ and in a site adjacent to one of the longest sides of the rectangle (see Fig. 2). We denote by $\mathcal{Q}_c$ the set of configurations in which all the spins are zeros excepted those, which are pluses, in a rectangle of sides long $\ell_c$ and $\ell_c - 1$ and in a site adjacent to one of the longest sides of the rectangle (see the caption of Fig. 2). We have:

$$H(\mathcal{P}_c) - H(\mathbf{d}) = H(\mathcal{Q}_c) - H(\mathbf{0}) = 4\ell_c - h[\ell_c(\ell_c - 1) + 1]$$

We then set

$$\Gamma_{\mathrm{c}} := H(\mathcal{P}_{\mathrm{c}}) - H(\mathbf{d}) = H(\mathcal{Q}_{\mathrm{c}}) - H(\mathbf{0})$$

A simple direct computation shows that for $h$ small one has $\Gamma_{\mathrm{c}} \sim 4/h$.

In order to give a result in the spirit of Theorem 4 in the case of the Blume–Capel model, we first have to prove preliminary Lemmas ensuring that the Conditions assumed in the general discussion in Section 2.5 are satisfied in the Blume–Capel case.

**Lemma 4** *With the parameters chosen as below* (36), *we have that* $\mathcal{X}_{\mathrm{s}} = \{\mathbf{u}\}$, $\mathcal{X}_{\mathrm{m}} = \{\mathbf{d}, \mathbf{0}\}$, *and* $H(\mathbf{d}) > H(\mathbf{0})$. *Moreover, the maximal stability level* $\Gamma_{\mathrm{m}}$ *is equal to* $\Gamma_{\mathrm{c}}$.

**Lemma 5** *With the parameters chosen as below* (36), *we have that there exists* $\kappa > 0$ *and* $\beta_0 > 0$ *such that for any* $\beta > \beta_0$

$$\mathbb{P}_{\mathbf{d}}(\tau_{\mathbf{u}} < \tau_{\mathbf{0}}) \le e^{-\beta\kappa} \tag{38}$$

**Lemma 6** *With the parameters chosen as below* (36), *we have that*

$$\frac{\mu_\beta(\mathbf{d})}{\mathrm{cap}_\beta(\mathbf{d}, \{\mathbf{0}, \mathbf{u}\})} = \frac{3e^{\beta\Gamma_{\mathrm{c}}}}{2(2\ell_{\mathrm{c}} - 1)}\left[1 + o(1)\right], \quad \frac{\mu_\beta(\mathbf{0})}{\mathrm{cap}_\beta(\mathbf{0}, \mathbf{u})} = \frac{3e^{\beta\Gamma_{\mathrm{c}}}}{2(2\ell_{\mathrm{c}} - 1)}\left[1 + o(1)\right] \tag{39}$$

For shortness we skip the proof of the model dependent results in Lemmas 4–6 that, although not straitghforward, can be achieved by methods similar to those developed for the Ising model (see, arXiv:1603.03483). We stress that analogous results have been proven in [14, Proposition 2.3] for the continuous time version of the same model.

We finally state our main results about the sharp estimate in the exit time in the Blume–Capel model with zero chemical potential.

**Theorem 5** *With the parameters chosen as below* (36), *we have that*

$$\mathbb{E}_{\mathbf{d}}[\tau_{\{\mathbf{0}, \mathbf{u}\}}] = e^{\beta\Gamma_{\mathrm{c}}} \frac{3}{2(2\ell_{\mathrm{c}} - 1)}[1 + o(1)], \quad \mathbb{E}_{\mathbf{0}}[\tau_{\mathbf{u}}] = e^{\beta\Gamma_{\mathrm{c}}} \frac{3}{2(2\ell_{\mathrm{c}} - 1)}[1 + o(1)] \tag{40}$$

**Theorem 6** *With the parameters chosen as below* (36), *we have that*

$$\mathbb{E}_{\mathbf{d}}[\tau_{\mathbf{u}}] = e^{\beta\Gamma_{\mathrm{c}}} \frac{3}{(2\ell_{\mathrm{c}} - 1)}[1 + o(1)] \tag{41}$$

The proof of the theorems is achieved by applying the general results discussed in Section 2.6 and the model dependent lemmas given above. Indeed, Theorem 5 follows by Theorem 3 and Lemmas 4–6, whereas Theorem 6 follows by Theorem 4, and Lemmas 4–6.

## A General bounds

In this appendix we summarize some general results whose statement and proof can already be found in the literature but, sometimes, in slightly different contexts.

**Proposition 1** ([24, Lemma 3.1.1]) *Consider the Markov chain defined in Section 2.1. For every not empty disjoint sets $Y, Z \subset X$ there exist constants $0 < C_1 < C_2 < \infty$ such that*

$$C_1 \leq e^{\beta \Phi(Y,Z)} Z_\beta \operatorname{cap}_\beta(Y, Z) \leq C_2 \tag{1}$$

*for all $\beta$ large enough.*

*Proof.* The upper bound can be obtained by choosing $f = \mathbb{I}_{K(Y,Z)}$ in (15) with

$$K(Y, Z) := \{x \in X \setminus Y : \Phi(x, Y) \leq \Phi(Y, Z)\}$$

For any pair $u, v \in X$ such that $u \in K(Y, Z)$ and $v \in X \setminus K(Y, Z)$, we have that $H(u) + \Delta(u, v) \geq \Phi(Y, Z)$. In fact, if by absurdity it were $H(u) + \Delta(u, v) < \Phi(Y, Z)$, it would be possible to construct a path $\omega$ starting at $v$ and ending in $Y$ such that $\Phi_\omega < \Phi(Y, Z)$, which is in contradiction with $v \in X \setminus K(Y, Z)$. Hence, by (14) and (15)

$$Z_\beta \operatorname{cap}_\beta(Y, Z) \leq Z_\beta \mathscr{D}_\beta[\mathbb{I}_{K(Y)}] = \frac{1}{2} \sum_{\substack{u \in K(Y,Z) \\ v \in X \setminus K(Y,Z)}} p_\beta(u, v) e^{-G_\beta(u)}$$

Recalling (2) and (3), we get that there exists $C$ such that for $\beta$ large enough

$$Z_\beta \operatorname{cap}_\beta(Y, Z) \leq \sum_{\substack{u \in K(Y) \\ v \in X \setminus K(Y)}} C e^{-\beta [H(u) + \Delta(u,v)]}$$

Finally, the upper bound in (1) follows from the fact that $H(u) + \Delta(u, v) \geq \Phi(Y, Z)$ for any $u \in K(Y, Z)$ and $v \in X \setminus K(Y, Z)$.

The lower bound can be proven by picking any path $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ that realizes the minimax in $\Phi(Y, Z)$ and ignore all the transitions that are not the path and using the same argument as in the proof of [Lemma 3.1.1, [24]]. An alternative proof can be given applying the Berman–Konsowa lemma [Proposition 2.4, [25]] which provides a complementary variational principle, in the sense that any test flow will give a lower bound. Hence, the lower bound can be obtained by picking any path $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$, with $\omega_0 \in Y$, $\omega_n \in Z$, such that it realizes the minimax in $\Phi(Y, Z)$ and such that $H(\omega_i) + \Delta(\omega_i, \omega_{i+1}) \leq \Phi(Y, Z)$ for $i \in \{0, \ldots, n - 1\}$ (recall (7) and (10)). If we choose a unitary flow for the edges in the path $\omega$ and null otherwise, the induced Markov chains is a deterministic chain along the path, so that the expectation in [Proposition 2.4, [25]] is just the contribution of the deterministic path. Hence, for the chosen flow we have:

$$\operatorname{cap}_\beta(Y, Z) \geq \left[ \sum_{k=0}^{n-1} \frac{1}{\mu_\beta(\omega_k) p_\beta(\omega_k, \omega_{k+1})} \right]^{-1} \geq C_1 \frac{1}{Z_\beta} e^{-\beta \Phi(Y,Z)} [1 + o(1)]$$

where in the last inequality we used (2) and (3).  □

**Proposition 2** *Consider the Markov chain defined in Section 2.1. We have that*

$$\mathbb{P}_y(\tau_{y_1} < \tau_{y_2}) \leq \frac{\operatorname{cap}_\beta(y, y_1)}{\operatorname{cap}_\beta(y, y_2)} \tag{2}$$

*for any $y \neq y_1$, $y_1 \neq y_2$, $y \neq y_2$, and $y, y_1, y_2 \in X$.*

*Proof.* Given $y, y_1, y_2 \in X$, a renewal argument and the strong Markov property yield:

$$
\begin{aligned}
\mathbb{P}_y(\tau_{y_1} < \tau_{y_2}) &= \mathbb{P}_y(\tau_{y_1} < \tau_{y_2}, \tau_{\{y_1,y_2\}} > \tau_y) + \mathbb{P}_y(\tau_{y_1} < \tau_{y_2}, \tau_{\{y_1,y_2\}} < \tau_y) \\
&= \mathbb{P}_y(\tau_{y_1} < \tau_{y_2} | \tau_{\{y_1,y_2\}} > \tau_y)\mathbb{P}_y(\tau_{\{y_1,y_2\}} > \tau_y) \\
&\quad + \mathbb{P}_y(\tau_{y_1} < \tau_{y_2}, \tau_{\{y_1,y_2\}} < \tau_y) \\
&= \mathbb{P}_y(\tau_{y_1} < \tau_{y_2})\mathbb{P}_y(\tau_{\{y_1,y_2\}} > \tau_y) + \mathbb{P}_y(\tau_{y_1} < \tau_{y_2}, \tau_{y_1} < \tau_y) \\
&= \mathbb{P}_y(\tau_{y_1} < \tau_{y_2})\mathbb{P}_y(\tau_{\{y_1,y_2\}} > \tau_y) + \mathbb{P}_y(\tau_{y_1} < \tau_{\{y_2,y\}})
\end{aligned}
$$

Therefore

$$
\mathbb{P}_y(\tau_{y_1} < \tau_{y_2}) = \frac{\mathbb{P}_y(\tau_{y_1} < \tau_{\{y_2,y\}})}{1 - \mathbb{P}_y(\tau_{\{y_1,y_2\}} > \tau_y)} = \frac{\mathbb{P}_y(\tau_{y_1} < \tau_{\{y_2,y\}})}{\mathbb{P}_y(\tau_{\{y_1,y_2\}} < \tau_y)} \leq \frac{\mathbb{P}_y(\tau_{y_1} < \tau_y)}{\mathbb{P}_y(\tau_{y_2} < \tau_y)}
$$

Recalling (17), we can rewrite the ratio in terms of ratio of capacities:

$$
\frac{\mathbb{P}_y(\tau_{y_1} < \tau_y)}{\mathbb{P}_y(\tau_{y_2} < \tau_y)} = \frac{\mathrm{cap}_\beta(y, y_1)}{\mathrm{cap}_\beta(y, y_2)}
$$

Hence, we get (2).   □

## References

1. F. Manzo, F.R. Nardi, E. Olivieri, E. Scoppola, Journ. Stat. Phys. **115**, (2004) 591–642.
2. E. Olivieri, E. Scoppola, Journ. Stat. Phys. **79**, (1995) 613–647.
3. E. Olivieri, M.E. Vares, *Large deviations and metastability* (Cambridge University Press, UK, 2004).
4. E.N.M. Cirillo, F.R. Nardi, J. Sohier, Journ. Stat. Phys. **161**, (2015) 365–403.
5. A. Bovier, M. Eckhoff, V. Gayrard, M. Klein, Comm. Math. Phys. **228**, (2002) 219–255.
6. A. Bovier, F. den Hollander, *Metastability: a potential–theoretic approach* (Grundlehren der mathematischen Wissenschaften, Springer, 2015).
7. M. Slowik, *Metastability in Stochastic Dynamics: Contributions to the Potential Theoretic Approach* (ISBN-13: 978-3838134123, Südwestdeutscher Verlag für Hochschulschriften August 31, 2012).
8. J. Beltrán, C. Landim, Journ. Stat. Phys. **140**, (2010) 1065–1114.
9. A. Bovier, F. Manzo, Journ. Stat. Phys. **107**, (2002) 757–779.
10. M. Blume, Phys. Rev. **141**, (1966) 517.
11. H.W. Capel, Physica **32**, (1966) 966.
12. E.N.M. Cirillo, F.R. Nardi, Journ. Stat. Phys. **150**, (2013) 1080–1114.
13. E.N.M. Cirillo, E. Olivieri, Journ. Stat. Phys. **83**, (1996) 473–554.
14. C. Landim, P. Lemire, Journ. Stat. Phys. **164**, (2016) 346–376.
15. F. Manzo, E. Olivieri, Journ. Stat. Phys. **104**, (2001) 1029–1090.
16. E.N.M. Cirillo, F.R. Nardi, Journ. Stat. Phys. **110**, (2003) 183–217.
17. E.N.M. Cirillo, F.R. Nardi, C. Spitoni, *Sum of exit times in a series of two metastable states in Probabilistic Cellular Automata*, Cellular Automata and Discrete Complex Systems, 22nd IFIP WG 1.5 International Workshop, AUTOMATA 2016, Zurich, Switzerland, June 15-17, 2016, Proceedings, Springer (2016).
18. O. Catoni. *Simulated annealing algorithms and Markov chains with rare transitions*, in Séminaire de Probabilités, XXXIII, volume 1709 of Lecture Notes in Math., pages 69–119. Springer, Berlin, 1999.
19. G. Grinstein, C. Jayaprakash, Y. He, Phys. Rev. Lett. **55**, (1985) 2527–2530.
20. E.N.M. Cirillo, P.–Y. Louis, W. Ruszel, C. Spitoni, Automata." Chaos, Solitons, and Fractals **64**, (2014) 36–47.
21. S. Bigelis, E.N.M. Cirillo, J.L. Lebowitz, E.R. Speer, Phys. Rev. E **59**, (1999) 3935.

22. A. Bovier, *Metastability: a potential theoretic approach* Proceedings of ICM 2006, EMS
    Publishing House, 2006, pp. 499–518.
23. A. Gaudillière, *Condenser physics applied to Markov Chains*, XII Escola Brasileira de
    Probabilidade, Ouro Preto, Minas Gerais, Brazil, 2008.
24. A. Bovier, F. den Hollander, and F.R. Nardi, Probab. Theory Relat. Fields **135**, (2006)
    265–310.
25. A. Bovier, F. den Hollander and C. Spitoni, Ann. Prob. **38**, (2010) 661–713.