Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **81**, 1–20; https://doi.org/10.14232/ejqtde.2017.1.81 www.math.u-szeged.hu/ejqtde/

Perron's theorem for nondensely defined partial functional differential equations

Nadia Drisi, **Brahim Es-sebbar**✉ and **Khalil Ezzinbi**

Département de Mathématiques, Faculté des Sciences Semlalia
Université Cadi Ayyad, B.P. 2390
Marrakesh, Morocco

Received 12 July 2017, appeared 21 November 2017
Communicated by László Simon

Abstract. The aim of this work is to establish a Perron type theorem for some nondensely defined partial functional differential equations with infinite delay. More specifically, we show that if the nonlinear delayed part is "small" (in a sense to be made precise below), then the asymptotic behavior of solutions can be described in terms of the dynamic of the unperturbed linear part of the equation.

Keywords: functional differential equations, asymptotic behavior, Perron's theorem.

2010 Mathematics Subject Classification: 35B40, 35R10.

1 Introduction

The aim of this work is to study the asymptotic behavior of solutions for the following partial functional differential equation

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t, x_t) \quad \text{for } t \geq 0, \\ x_0 = \phi \in \mathcal{B}, \end{cases} \tag{1.1}$$

where A is a linear operator on a Banach space X satisfying the well-known Hille–Yosida condition, the domain is not necessarily dense, namely, we suppose that:

(H0) There exist $M_0 \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{M_0}{(\lambda - \omega)^n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \omega, \tag{1.2}$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$.

L is a bounded linear operator from $\mathcal{B}$ to X, where $\mathcal{B}$ is a normed linear space of functions mapping $(-\infty, 0]$ into X satisfying the fundamental axioms introduced by Hale and Kato in [14]. For every $t \in \mathbb{R}$, the history function x_t is defined by

$$x_t(\theta) = x(t + \theta) \quad \text{for } \theta \leq 0.$$

✉Corresponding author. Email: essebbar@live.fr

(H1) The nonlinearity $f : \mathbb{R}^+ \times \mathcal{B} \to X$ is continuous such that

$$|f(t,\phi)| \le q(t) \|\phi\|_{\mathcal{B}} \quad \text{for } (t,\phi) \in [0,\infty) \times \mathcal{B}, \tag{1.3}$$

where $q : [0,\infty) \to [0,\infty)$ is a continuous function.

(H2) The function q in (1.3) satisfies

$$\int_t^{t+1} q(s)\, ds \to 0 \quad \text{when } t \to \infty. \tag{1.4}$$

The conditions (1.3) and (1.4) means that the nonlinearity $f(t,x_t)$ in equation (1.1) is "small" as $t \to \infty$. As a consequence, the solutions of equation (1.1) are expected to have similar asymptotic properties as the solutions of the following unperturbed equation

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + L(x_t) & \text{for } t \ge 0, \\ x_0 = \phi \in \mathcal{B}. \end{cases}$$

The so-called Perron's theorem for the asymptotic behavior of solutions of differential equations have been the subject of many studies, see [7,22–25]. For ordinary differential equations, we refer the reader to the books [8, 9, 11, 16]. Let us recall Perron's theorem for ordinary differential equations, which is presented in a form given by Coppel [9].

Theorem. [9] *Consider the following ordinary differential equation*

$$\begin{cases} \dfrac{d}{dt}x(t) = Bx(t) + g(t,x(t)) & \text{for } t \ge 0 \\ x(0) = x_0 \in \mathbb{C}^n, \end{cases} \tag{1.5}$$

where B is an $n \times n$ constant complex matrix and $g : [0,\infty) \times \mathbb{C}^n \to \mathbb{C}^n$ is a continuous function such that

$$|g(t,z)| \le \gamma(t)\,|z| \quad \text{for } t \ge 0 \text{ and } z \in \mathbb{C}^n,$$

where $\gamma : [0,\infty) \to [0,\infty)$ is a continuous function satisfying

$$\int_t^{t+1} \gamma(s) ds \to 0 \quad \text{as } t \to \infty.$$

If $x(\cdot)$ is a solution of equation (1.5), then either

$$x(t) = 0 \quad \text{for all large } t,$$

or

$$\lim_{t\to\infty} \frac{\log|x(t)|}{t} = \operatorname{Re}\lambda_0,$$

where λ_0 is one of the eigenvalues of the matrix B.

Recently in [4, 5], the authors showed that the asymptotic exponential behavior of the solutions of the linear nonautonomous differential equation

$$\frac{d}{dt}x(t) = B(t)x(t) \quad \text{for } t \ge 0, \tag{1.6}$$

in $\mathbb{C}^n$, persists under sufficiently small nonlinear perturbations. More precisely, they showed that if all Lyapunov exponents of equation (1.6) are limits, then the same can be said about the solutions of the following perturbed nonlinear system

$$\frac{d}{dt}x(t) = B(t)x(t) + f(t, x(t)) \quad \text{for } t \geq 0.$$

In [24], the author proved a Perron theorem for equation (1.1), when $A = 0$, the delay is finite and the space X is finite dimensional. In [21], the authors treated the case when X is infinite dimensional and the delay is infinite. They assumed that A is densely defined in X and satisfies the Hille–Yosida condition **(H0)**, which is equivalent, by the Hille–Yosida theorem, to A being the infinitesimal generator of a strongly continuous semigroup on X. They assumed that this semigroup is compact and they used the variation of constants formula established in [18] for some specific phase space.

In this work, we are interested in studying the asymptotic behavior of solutions of equation (1.1) when A is not necessarily densely defined and for a general class of phase spaces $\mathcal{B}$. We use the variation of constants formula established in [3]. Since the nonlinear part of equation (1.1) is assumed to be "small" in some sense ((1.3) and (1.4)), we describe the asymptotic behavior of solutions in terms of the growth bound of the semigroup solution of the unperturbed linear system (Theorem 5.1). This result is then refined in the form of a Perron type theorem where the asymptotic behavior of solutions is compared to the essential growth bound of the semigroup solution of the unperturbed linear system. Unlike in [21, 24], we do not need to assume any kind of compactness (see Remark 5.4). A condition of compactness is, however, needed in order to describe the asymptotic behavior of solutions in terms of the parameters of the system (Corollary 5.8).

This work is organized as follows. In Section 2, we state the fundamental axioms on $\mathcal{B}$ that will be used in this work, and we recall some spectral properties of the semigroup solution of equation (1.1) with $f = 0$. In Section 3, we recall the variation of constants formula established in [3] and we give the spectral decomposition of the phase space which plays an important role in the whole of this work. Section 4 is devoted to study the asymptotic behavior of solutions of equation (1.1) with respect to the invariant subspaces corresponding to the spectral decomposition. In Section 5, we describe the asymptotic behavior of solutions in terms of the growth bound and the essential growth bound of the semigroup solution of the unperturbed linear system. An example of a Lotka–Volterra model with diffusion is given to illustrate our studies.

2 Phase space, integral solutions and spectral analysis

The choice of the phase space $\mathcal{B}$ plays an important role in the qualitative analysis of partial functional differential equations with infinite delay. In fact, the choice of $\mathcal{B}$ affects some properties of solutions. In this work, we employ an axiomatic definition of the phase space $\mathcal{B}$ which has been introduced at first by Hale and Kato [14]. We assume that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a normed space of functions mapping $(-\infty, 0]$ into a Banach space $(X, |\cdot|)$ and satisfying the following fundamental axioms.

(A) There exist a positive constant N and functions $K, \widetilde{K} : [0, \infty) \to [0, \infty)$, with K continuous and $\widetilde{K}$ locally bounded, such that if a function $x : (-\infty, a] \to X$ is continuous on $[\sigma, a]$ with $x_\sigma \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$:

(i) $x_t \in \mathcal{B}$;

(ii) $t \mapsto x_t$ is continuous with respect to $\|.\|_{\mathcal{B}}$ on $[\sigma, a]$;

(iii) $N\,|x\,(t)| \leq \|x_t\|_{\mathcal{B}} \leq K\,(t-\sigma) \sup_{\sigma \leq s \leq t} |x\,(s)| + \widetilde{K}\,(t-\sigma)\,\|x_\sigma\|_{\mathcal{B}}\,.$

(B) $\mathcal{B}$ is a Banach space.

(C) If $(\phi_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{B}$ which converges compactly to a function ϕ, then $\phi \in \mathcal{B}$ and $\|\phi_n - \phi\|_{\mathcal{B}} \to 0$ as $n \to \infty$.

As a consequence of Axiom **(A)**, we deduce the following result.

Lemma 2.1 ([19]). *Let $C_{00}\,((-\infty, 0], X)$ be the space of continuous functions mapping $(-\infty, 0]$ into X with compact supports. Then $C_{00}\,((-\infty, 0], X) \subset \mathcal{B}$. In addition, for $a < 0$, we have*

$$\|\phi\|_{\mathcal{B}} \leq K\,(-a) \sup_{\theta \leq 0} |\phi\,(\theta)|\,,$$

for any $\phi \in C_{00}\,((-\infty, 0], X)$ with the support included in $[a, 0]$.

In the sequel, we assume that the operator A satisfies the Hille–Yosida condition **(H0)**. Consider the following Cauchy problem

$$\begin{cases} \dfrac{d}{dt}x\,(t) = Ax\,(t) + L(x_t) + f(t) & \text{for } t \geq 0 \\ x_0 = \phi \in \mathcal{B}. \end{cases} \tag{2.1}$$

Definition 2.2 ([3]). Let $\phi \in \mathcal{B}$. A function $x : \mathbb{R} \to X$ is called an integral solution of equation (2.1) on $\mathbb{R}$ if the following conditions hold

(i) x is continuous on $[0, \infty)$,

(ii) $x_0 = \phi$,

(ii) $\int_0^t x\,(s)\,ds \in D(A)$ for $t \geq 0$,

(iv) $x\,(t) = \phi\,(0) + A\int_0^t x\,(s)\,ds + \int_0^t [L(x_s) + f(s)]ds$ for $t \geq 0$.

If x is an integral solution of equation (2.1), then from the continuity of x, we have $x\,(t) \in \overline{D(A)}$, for all $t \geq 0$. In particular, $\phi\,(0) \in \overline{D(A)}$. Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ defined by

$$\begin{cases} D(A_0) := \{x \in D(A) :\ Ax \in \overline{D(A)}\}, \\ A_0 x := Ax \text{ for } x \in D(A_0). \end{cases}$$

Lemma 2.3 ([26]). *Assume that* **(H0)** *holds. Then A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

For the existence of integral solutions, one has the following result:

Theorem 2.4 ([6,26]). *Assume that* **(H0)** *holds. Then, for all $\phi \in \mathcal{B}$ such that $\phi\,(0) \in \overline{D(A)}$, equation (2.1) has a unique integral solution x on $\mathbb{R}$. Moreover, x is given by*

$$\begin{cases} x\,(t) = T_0(t)\phi\,(0) + \lim\limits_{\lambda \to \infty} \displaystyle\int_0^t T_0(t-s)B_\lambda\,[L(x_s) + f(s)]\,ds & \text{for } t \geq 0, \\ x_0 = \phi, \end{cases}$$

where $B_\lambda := \lambda R(\lambda, A)$ for $\lambda > \omega$.

In the sequel of this work, for simplicity, integral solutions are called solutions. A solution of equation (2.1) is denoted by $x(\cdot, \phi, L, f)$. The phase space $\mathcal{B}_A$ of equation (2.1) is given by

$$\mathcal{B}_A := \{\phi \in \mathcal{B} : \phi(0) \in \overline{D(A)}\}.$$

For each $t \geq 0$, $V(t)$ is the bounded linear operator defined on $\mathcal{B}_A$ by

$$V(t)\phi = x_t(\cdot, \phi, L, 0),$$

where $x(\cdot, \phi, L, 0)$ is the solution of the homogeneous equation

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + L(x_t) & \text{for } t \geq 0, \\ x_0 = \phi. \end{cases}$$

We have the following result:

Proposition 2.5 ([2, Proposition 2])**.** *Assume that* **(H0)** *holds. Then* $(V(t))_{t\geq 0}$ *is a strongly continuous semigroup on* $\mathcal{B}_A$*. Moreover,* $(V(t))_{t\geq 0}$ *satisfies the following translation property*

$$(V(t)\phi)(\theta) = \begin{cases} V(t+\theta)\phi(0) & \textit{for } t+\theta \geq 0, \\ \phi(t+\theta) & \textit{for } t+\theta \leq 0. \end{cases}$$

Let $\mathcal{A}_V$ denote the infinitesimal generator of the semigroup $(V(t))_{t\geq 0}$ on $\mathcal{B}_A$.

For a bounded subset B of a Banach space Y, the Kuratowski measure of noncompactness $\alpha(B)$ is defined by

$$\alpha(B) := \inf\{d > 0 : \text{ there exist finitely many sets of diameter at most } d \text{ which cover } B\}.$$

Moreover, for a bounded linear operator K on Y, we define $\alpha(K)$ by

$$\alpha(K) := \inf\{k > 0 : \alpha(K(B)) \leq k\alpha(B) \text{ for any bounded set } B \text{ of } Y\}.$$

Definition 2.6 ([12])**.** Let $\mathcal{C}$ be a densely defined operator on a Banach space Y. Let $\sigma(\mathcal{C})$ denote the spectrum of the operator $\mathcal{C}$. The essential spectrum of $\mathcal{C}$ denoted by $\sigma_{ess}(\mathcal{C})$ is the set of $\lambda \in \sigma(\mathcal{C})$ such that one of the following conditions holds:

(i) $\operatorname{Im}(\lambda I - \mathcal{C})$ is not closed,

(ii) the generalized eigenspace $M_\lambda(\mathcal{C}) := \bigcup_{k\geq 1} \operatorname{Ker}(\lambda I - \mathcal{C})^k$ is of infinite dimension,

(iii) λ is a limit point of $\sigma(\mathcal{C}) \setminus \{\lambda\}$.

The essential radius of $\mathcal{C}$ is defined by

$$r_{ess}(\mathcal{C}) = \sup\{|\lambda| : \lambda \in \sigma_{ess}(\mathcal{C})\}.$$

Let $(R(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space Y and $\mathcal{A}_R$ its infinitesimal generator.

Definition 2.7 ([12])**.** The growth bound $\omega_0(R)$ of the C_0-semigroup $(R(t))_{t\geq 0}$ is defined by

$$\omega_0(R) := \inf\left\{\omega \in \mathbb{R} : \sup_{t\geq 0} e^{-\omega t}|R(t)| < \infty\right\}.$$

Definition 2.8 ([19]). The essential growth bound $\omega_{ess}(R)$ of the C_0-semigroup $(R(t))_{t\geq 0}$ is defined by

$$\omega_{ess}(R) := \lim_{t\to\infty} \frac{\log \alpha (R(t))}{t} = \inf_{t>0} \frac{\log \alpha (R(t))}{t}. \tag{2.2}$$

The relation between $r_{ess}(R(t))$ and $\omega_{ess}(R)$ is given by the following formula

$$r_{ess}(R(t)) = e^{t\omega_{ess}(R)}. \tag{2.3}$$

From the spectral mapping inclusion $e^{t\sigma_{ess}(\mathcal{A}_R)} \subset \sigma_{ess}(R(t))$ and the formula (2.3), one can see that

$$\sigma_{ess}(\mathcal{A}_R) \subset \{\lambda \in \sigma(\mathcal{A}_R) : \mathrm{Re}\,\lambda \leq \omega_{ess}(R)\}.$$

This means that if $\lambda \in \sigma(\mathcal{A}_R)$ and $\mathrm{Re}\,\lambda > \omega_{ess}(R)$, then λ does not belong to $\sigma_{ess}(\mathcal{A}_R)$. Therefore λ is an isolated eigenvalue of $\mathcal{A}_R$.

The spectral bound $s(\mathcal{A}_R)$ of the infinitesimal generator $\mathcal{A}_R$ is defined by

$$s(\mathcal{A}_R) := \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(\mathcal{A}_R)\}.$$

Recall the following formula

$$\omega_0(R) = \max\{\omega_{ess}(R), s(\mathcal{A}_R)\}.$$

3 Variation of constants formula and spectral decomposition of the phase space $\mathcal{B}_A$

We introduce the following sequence of linear operators Θ^n mapping X into $\mathcal{B}_A$ defined for $n > \omega$ and $y \in X$ by

$$(\Theta^n y)(\theta) = \begin{cases} (n\theta + 1) B_n y & \text{for } -\frac{1}{n} \leq \theta \leq 0, \\ 0 \quad \text{for } \theta < -\frac{1}{n}, \end{cases}$$

where B_n is the bounded operator defined for sufficiently large n by $B_n := nR(n, A)$. For $y \in X$, the function $\Theta^n y$ belongs to $C_{00}((-\infty, 0], X)$ with the support included in $[-1, 0]$. By Lemma 2.1, we deduce that $\Theta^n y \in \mathcal{B}$ and

$$\|\Theta^n y\|_{\mathcal{B}} \leq \widetilde{N} K(1) |y|, \tag{3.1}$$

where $\widetilde{N} := \sup_{n>\omega} |B_n|$. In addition we have for each $y \in X$

$$(\Theta^n y)(0) = B_n y = nR(n, A) y \in D(A).$$

It follows that $\Theta^n y \in \mathcal{B}_A$.

Now we give the variation of constants formula for equation (2.1) established in [3].

Theorem 3.1 ([3]). *Assume that* **(H0)** *holds. Then, for all $\phi \in \mathcal{B}_A$, the solution $x(\cdot, \phi, L, f)$ of equation (2.1) satisfies the following variation of constants formula*

$$x_t(\cdot, \phi, L, f) = V(t)\phi + \lim_{n\to\infty} \int_0^t V(t-s)\Theta^n f(s)\, ds \quad \textit{for } t \geq 0. \tag{3.2}$$

The spectral decomposition of the phase space provides a powerful tool to analyze the asymptotic behavior of solutions. We know that each $\lambda \in \sigma(\mathcal{A}_V)$ with $\operatorname{Re}\lambda > \omega_{ess}(V)$ is an isolated eigenvalue of the operator $\mathcal{A}_V$. Let $\rho > \omega_{ess}(V)$ be such that

$$\sigma(\mathcal{A}_V) \cap (i\mathbb{R} + \rho) = \emptyset.$$

Consider the set

$$\Sigma_\rho := \{\lambda \in \sigma(\mathcal{A}_V) : \operatorname{Re}\lambda \geq \rho\}. \tag{3.3}$$

From [12, Corollary 2.11, Chapter IV and Theorem 3.1, Chapter V], the set Σ_ρ is finite and we have the following decomposition of the phase space $\mathcal{B}_A$

$$\mathcal{B}_A = U_\rho \oplus S_\rho, \tag{3.4}$$

where U_ρ and S_ρ are closed subspaces of $\mathcal{B}_A$ which are invariant under $(V(t))_{t\geq 0}$. The subspace U_ρ is finite-dimensional. For every sufficiently small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{cases} \|V(t)\phi\|_{\mathcal{B}} \leq C_\varepsilon e^{(\rho-\varepsilon)t}\|\phi\|_{\mathcal{B}} & \text{for } t \geq 0 \text{ and } \phi \in S_\rho \\ \|V(t)\phi\|_{\mathcal{B}} \leq C_\varepsilon e^{(\rho+\varepsilon)t}\|\phi\|_{\mathcal{B}} & \text{for } t \leq 0 \text{ and } \phi \in U_\rho. \end{cases} \tag{3.5}$$

In what follows, $V^{U_\rho}(t)$ and $V^{S_\rho}(t)$ denote the restrictions of $V(t)$ on U_ρ and S_ρ respectively. Π^{U_ρ} and Π^{S_ρ} denote the projections on U_ρ and S_ρ respectively. $\left(V^{U_\rho}(t)\right)_{t\in\mathbb{R}}$ is a group of operators. Let $\varepsilon_\rho > 0$ be such that $\sigma(\mathcal{A}_V) \cap \{\lambda \in \mathbb{C} : \rho - \varepsilon_\rho \leq \operatorname{Re}\lambda \leq \rho + \varepsilon_\rho\} = \emptyset$. Put

$$\rho_1 := \rho - \varepsilon_\rho \quad \text{and} \quad \rho_2 := \rho + \varepsilon_\rho. \tag{3.6}$$

We deduce from (3.5) that there exists a constant $C_\rho > 0$ such that for each $t \geq 0$

$$\left\|V^{S_\rho}(t)\right\| \leq C_\rho e^{\rho_1 t} \quad \text{and} \quad \left\|V^{U_\rho}(-t)\right\| \leq C_\rho e^{-\rho_2 t}.$$

We introduce the new norm defined on $\mathcal{B}_A$ by

$$|\phi|_{\mathcal{B}} := \sup_{t\geq 0} e^{-\rho_1 t}\left\|V^{S_\rho}(t)\Pi^{S_\rho}\phi\right\|_{\mathcal{B}} + \sup_{t\geq 0} e^{\rho_2 t}\left\|V^{U_\rho}(-t)\Pi^{U_\rho}\phi\right\|_{\mathcal{B}}.$$

Lemma 3.2 ([10, 21, 24])**.** *The two norms $\|\cdot\|_{\mathcal{B}}$ and $|\cdot|_{\mathcal{B}}$ are equivalent, namely, for all $\phi \in \mathcal{B}_A$, we have*

$$\|\phi\|_{\mathcal{B}} \leq |\phi|_{\mathcal{B}} \leq C_2\|\phi\|_{\mathcal{B}}, \tag{3.7}$$

where $C_2 := C_\rho\left(\left\|\Pi^{S_\rho}\right\| + \left\|\Pi^{U_\rho}\right\|\right)$. In addition, for all $\phi \in \mathcal{B}_A$

$$|\phi|_{\mathcal{B}} = \left|\Pi^{S_\rho}\phi\right|_{\mathcal{B}} + \left|\Pi^{U_\rho}\phi\right|_{\mathcal{B}}. \tag{3.8}$$

The corresponding operator norms $\left|V^{S_\rho}(t)\right|$ and $\left|V^{U_\rho}(-t)\right|$ satisfy

$$\left|V^{S_\rho}(t)\right| \leq e^{\rho_1 t} \quad \textit{and} \quad \left|V^{U_\rho}(-t)\right| \leq e^{-\rho_2 t} \quad \textit{for } t \geq 0. \tag{3.9}$$

4 Asymptotic behavior of the solutions in the invariant subspaces

In this section, we give the lemmas that will be used for the proof of our main results. First, we give sufficient conditions which insure the existence of global solutions for equation (1.1).

Theorem 4.1. *Assume that* **(H0)** *and* **(H1)** *hold. Let $\phi \in \mathcal{B}_A$. If the nonlinearity $f : \mathbb{R}^+ \times \mathcal{B} \to X$ is locally Lipschitz with respect to the second argument, then equation* (1.1) *has a unique solution x which is defined on $\mathbb{R}$.*

Proof. Since the nonlinearity f is locally Lipschitz with respect to the second argument, then using the same argument as in [13, Theorem 3.4], one can prove that there exists a maximal interval of existence $(-\infty, b_\phi)$ and a unique solution $x(\cdot, \phi)$ of equation (1.1) defined on $(-\infty, b_\phi)$ and either $b_\phi = \infty$ or $\limsup_{t \to b_\phi^-} |x(t,\phi)| = \infty$. Now using the fact that the nonlinearity f satisfies (1.3), we deduce using the same approach as in [13, Corollary 3.5] that $b_\phi = \infty$. □

Theorem 4.2. *Assume that* **(H0)** *and* **(H1)** *hold. Let $\phi \in \mathcal{B}_A$. If the C_0-semigroup $(T_0(t))_{t\geq 0}$ is compact, then equation* (1.1) *has at least a solution x which is defined on $\mathbb{R}$.*

Proof. As in [1, Theorem 17], the proof uses the Schauder fixed point theorem to prove the existence of at least a solution defined on a maximal interval of existence $(-\infty, b_\phi)$. It follows again as in Theorem 4.1 that $b_\phi = \infty$. □

Remark 4.3. Unlike in Theorem 4.1, the global solution provided by Theorem 4.2 is not necessarily unique.

For the rest of this work, we assume the global existence of a solution x of equation (1.1). The following lemma is essential for the rest of the paper.

Lemma 4.4. *Suppose that* **(H0)–(H2)** *hold. Let x be a solution of equation* (1.1)*. Then for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) \geq 1$ such that*

$$\|x_t\|_{\mathcal{B}} \leq C(\varepsilon)\, e^{(\omega_0(V)+\varepsilon)(t-\sigma)} \exp\left(\widetilde{N}K(1)\,C(\varepsilon)\int_\sigma^t q(s)\,ds\right)\|x_\sigma\|_{\mathcal{B}} \quad \text{for } 0 \leq \sigma \leq t. \tag{4.1}$$

In particular, there exists a constant $C_1 \geq 0$ such that for $m \in \mathbb{N}$ and $m \leq t \leq m+1$, we have

$$\frac{1}{C_1}\|x_{m+1}\|_{\mathcal{B}} \leq \|x_t\|_{\mathcal{B}} \leq C_1\|x_m\|_{\mathcal{B}}. \tag{4.2}$$

Proof. Using the variation of constants formula (3.2), we have for $0 \leq \sigma \leq t$

$$x_t = V(t-\sigma)\,x_\sigma + \lim_{n\to\infty}\int_\sigma^t V(t-s)\,\Theta^n f(s, x_s)\,ds. \tag{4.3}$$

Let $\varepsilon > 0$. Then, there exists $C(\varepsilon) \geq 1$ such that

$$\|V(t)\| \leq C(\varepsilon)\, e^{(\omega_0(V)+\varepsilon)t} \quad \text{for } t \geq 0. \tag{4.4}$$

It follows from (3.1), (4.3) and (4.4) that

$$\begin{aligned}
\|x_t\|_{\mathcal{B}} &\leq \|V(t-\sigma)\|\,\|x_\sigma\|_{\mathcal{B}} + \lim_{n\to\infty}\int_\sigma^t \|V(t-s)\|\,\|\Theta^n f(s,x_s)\|_{\mathcal{B}}\,ds \\
&\leq C(\varepsilon)\, e^{(\omega_0(V)+\varepsilon)(t-\sigma)}\,\|x_\sigma\|_{\mathcal{B}} + \widetilde{N}K(1)\,C(\varepsilon)\int_\sigma^t e^{(\omega_0(V)+\varepsilon)(t-s)}\,|f(s,x_s)|\,ds \\
&\leq C(\varepsilon)\, e^{(\omega_0(V)+\varepsilon)(t-\sigma)}\,\|x_\sigma\|_{\mathcal{B}} + \widetilde{N}K(1)\,C(\varepsilon)\int_\sigma^t e^{(\omega_0(V)+\varepsilon)(t-s)} q(s)\,\|x_s\|_{\mathcal{B}}\,ds.
\end{aligned}$$

It follows that

$$e^{-(\omega_0(V)+\varepsilon)t}\left\|x_t\right\|_{\mathcal{B}} \leq C(\varepsilon)\, e^{-(\omega_0(V)+\varepsilon)\sigma}\left\|x_\sigma\right\|_{\mathcal{B}} + \widetilde{N}K(1)\,C(\varepsilon)\int_\sigma^t e^{-(\omega_0(V)+\varepsilon)s}\left\|x_s\right\|_{\mathcal{B}} q(s)\,ds.$$

Gronwall's lemma implies that for $0 \leq \sigma \leq t$

$$e^{-(\omega_0(V)+\varepsilon)t}\left\|x_t\right\|_{\mathcal{B}} \leq C(\varepsilon)\, e^{-(\omega_0(V)+\varepsilon)\sigma}\left\|x_\sigma\right\|_{\mathcal{B}} \exp\left(\widetilde{N}K(1)\,C(\varepsilon)\int_\sigma^t q(s)\,ds\right).$$

Therefore we get the inequality (4.1). Now let $m \in \mathbb{N}$ and $m \leq t \leq m+1$. By taking $\varepsilon = 1$ and $\sigma = m$ in (4.1), we get

$$\begin{aligned}\left\|x_t\right\|_{\mathcal{B}} &\leq C(1)e^{(\omega_0(V)+1)(t-m)}\left\|x_m\right\|_{\mathcal{B}} \exp\left(\widetilde{N}K(1)\,C(1)\int_m^t q(s)\,ds\right)\\ &\leq C_1\left\|x_m\right\|_{\mathcal{B}}.\end{aligned}$$

where $C_1 := C(1)\max\left\{1, e^{(\omega_0(V)+1)}\right\}e^{\widetilde{N}K(1)C(1)Q}$ and $Q := \sup_{m\geq 0}\int_m^{m+1} q(s)\,ds$ which is finite by (1.4). Similarly, we get

$$\left\|x_{m+1}\right\|_{\mathcal{B}} \leq C_1\left\|x_t\right\|_{\mathcal{B}}.$$

□

Remark 4.5. By (4.2) and (3.7), we can see that for $m \in \mathbb{N}$ and $m \leq t \leq m+1$

$$\frac{1}{C_3}\left|x_{m+1}\right|_{\mathcal{B}} \leq \left|x_t\right|_{\mathcal{B}} \leq C_3\left|x_m\right|_{\mathcal{B}}, \tag{4.5}$$

where $C_3 := C_1C_2$.

For the rest of this section, we suppose that **(H0)–(H2)** hold and we fix a real number ρ such that $\rho > \omega_{ess}(V)$ and $\sigma(\mathcal{A}_V) \cap (i\mathbb{R}+\rho) = \emptyset$. Recall that U_ρ and S_ρ are the subspaces in the decomposition of the phase space $\mathcal{B}_A$ given by (3.4). Let x be a solution of equation (1.1). Define for $m \in \mathbb{N}$

$$x^U(m) := \left|\Pi^{U_\rho}x_m\right|_{\mathcal{B}}, \qquad x^S(m) := \left|\Pi^{S_\rho}x_m\right|_{\mathcal{B}} \tag{4.6}$$

and

$$\widetilde{q}(m) := \widetilde{N}K(1)\,C_1C_2^2\max\left\{1, e^{\rho_1}, e^{\rho_2}\right\}\int_m^{m+1} q(s)\,ds, \tag{4.7}$$

where ρ_1 and ρ_2 are the real numbers defined by (3.6).

Lemma 4.6. *The following estimations hold*

$$x^S(m+1) \leq e^{\rho_1}x^S(m) + \widetilde{q}(m)\left(x^S(m) + x^U(m)\right), \tag{4.8}$$

and

$$x^U(m+1) \geq e^{\rho_2}x^U(m) - \widetilde{q}(m)\left(x^S(m) + x^U(m)\right). \tag{4.9}$$

Proof. Using the variation of constants formula (3.2), we obtain for each $m \in \mathbb{N}$

$$x_{m+1} = V(1)\,x_m + \lim_{n\to\infty}\int_m^{m+1} V(m+1-s)\,\Theta^n f(s, x_s)\,ds. \tag{4.10}$$

By projecting the formula (4.10) onto the subspace S_ρ and using (3.9), (3.7), (3.1), (1.3) and (4.2), we have

$$\begin{aligned}\left|\Pi^{S_\rho}x_{m+1}\right|_{\mathcal{B}} &\le \left|V^{S_\rho}(1)\Pi^{S_\rho}x_m\right|_{\mathcal{B}} + \lim_{n\to\infty}\int_m^{m+1}\left|V^{S_\rho}\left(m+1-s\right)\Pi^{S_\rho}\Theta^n f\left(s,x_s\right)\right|_{\mathcal{B}}ds\\
&\le e^{\rho_1}\left|\Pi^{S_\rho}x_m\right|_{\mathcal{B}} + \lim_{n\to\infty}\int_m^{m+1}e^{\rho_1(m+1-s)}\left|\Pi^{S_\rho}\Theta^n f\left(s,x_s\right)\right|_{\mathcal{B}}ds\\
&\le e^{\rho_1}\left|\Pi^{S_\rho}x_m\right|_{\mathcal{B}} + \lim_{n\to\infty}\int_m^{m+1}e^{\rho_1(m+1-s)}C_2\left\|\Pi^{S_\rho}\right\|\left\|\Theta^n f\left(s,x_s\right)\right\|_{\mathcal{B}}ds\\
&\le e^{\rho_1}\left|\Pi^{S_\rho}x_m\right|_{\mathcal{B}} + \widetilde{N}K\left(1\right)C_2^2\max\left\{1,e^{\rho_1}\right\}\int_m^{m+1}\left|f\left(s,x_s\right)\right|ds\\
&\le e^{\rho_1}\left|\Pi^{S_\rho}x_m\right|_{\mathcal{B}} + \widetilde{N}K\left(1\right)C_2^2\max\left\{1,e^{\rho_1}\right\}\int_m^{m+1}q\left(s\right)\left\|x_s\right\|_{\mathcal{B}}ds\\
&\le e^{\rho_1}\left|\Pi^{S_\rho}x_m\right|_{\mathcal{B}} + \widetilde{N}K\left(1\right)C_2^2\max\left\{1,e^{\rho_1}\right\}C_1\left\|x_m\right\|_{\mathcal{B}}\int_m^{m+1}q\left(s\right)ds\\
&\le e^{\rho_1}\left|\Pi^{S_\rho}x_m\right|_{\mathcal{B}} + \widetilde{N}K\left(1\right)C_1C_2^2\max\left\{1,e^{\rho_1}\right\}\int_m^{m+1}q\left(s\right)ds\left|x_m\right|_{\mathcal{B}}.\end{aligned}$$

Using (3.8) and the above inequality, we conclude that (4.8) holds.

Now from (3.9), we have for $\phi \in U_\rho$

$$\left|V^{U_\rho}(1)\phi\right|_{\mathcal{B}} \ge e^{\rho_2}\left|\phi\right|_{\mathcal{B}}.$$

By projecting the formula (4.10) onto the subspace U_ρ using (3.9), (3.7), (3.1), (1.3), (4.2) and (3.8), we deduce that

$$\begin{aligned}\left|\Pi^{U_\rho}x_{m+1}\right|_{\mathcal{B}} &= \left|V^{U_\rho}(1)\left(\Pi^{U_\rho}x_m + \lim_{n\to\infty}\int_m^{m+1}V^{U_\rho}\left(m-s\right)\Pi^{U_\rho}\Theta^n f\left(s,x_s\right)ds\right)\right|_{\mathcal{B}}\\
&\ge e^{\rho_2}\left|\Pi^{U_\rho}x_m + \lim_{n\to\infty}\int_m^{m+1}V^{U_\rho}\left(m-s\right)\Pi^{U_\rho}\Theta^n f\left(s,x_s\right)ds\right|_{\mathcal{B}}\\
&\ge e^{\rho_2}\left|\Pi^{U_\rho}x_m\right|_{\mathcal{B}} - e^{\rho_2}\lim_{n\to\infty}\int_m^{m+1}\left|V^{U_\rho}\left(m-s\right)\Pi^{U_\rho}\Theta^n f\left(s,x_s\right)\right|_{\mathcal{B}}ds\\
&\ge e^{\rho_2}x^U\left(m\right) - e^{\rho_2}\lim_{n\to\infty}\int_m^{m+1}e^{\rho_2(m-s)}\left|\Pi^{U_\rho}\Theta^n f\left(s,x_s\right)\right|_{\mathcal{B}}ds\\
&\ge e^{\rho_2}x^U\left(m\right) - e^{\rho_2}\lim_{n\to\infty}\int_m^{m+1}e^{\rho_2(m-s)}C_2\left\|\Pi^{U_\rho}\right\|\left\|\Theta^n f\left(s,x_s\right)\right\|_{\mathcal{B}}ds\\
&\ge e^{\rho_2}x^U\left(m\right) - \widetilde{N}K\left(1\right)e^{\rho_2}\max\left\{1,e^{-\rho_2}\right\}\int_m^{m+1}C_2^2\left|f\left(s,x_s\right)\right|ds\\
&\ge e^{\rho_2}x^U\left(m\right) - \widetilde{N}K\left(1\right)e^{\rho_2}C_2^2\max\left\{1,e^{-\rho_2}\right\}\int_m^{m+1}q\left(s\right)C_1\left\|x_m\right\|_{\mathcal{B}}ds\\
&\ge e^{\rho_2}x^U\left(m\right) - \widetilde{N}K\left(1\right)C_1C_2^2\max\left\{e^{\rho_2},1\right\}\int_m^{m+1}q\left(s\right)ds\left(x^U\left(m\right)+x^S\left(m\right)\right).\end{aligned}$$

Therefore, we get the estimation (4.9). □

In what follows, we assume that the solution x is nontrivial, that is, $\|x_t\|_{\mathcal{B}} > 0$ for $t \ge 0$. We have the following lemma.

Lemma 4.7. *Either*

$$\lim_{m\to\infty}\frac{x^U\left(m\right)}{x^S\left(m\right)} = 0 \tag{4.11}$$

or

$$\lim_{m\to\infty}\frac{x^S(m)}{x^U(m)}=0. \tag{4.12}$$

Proof. The proof follows the same approach as in [21,24]. From (3.7), one can see that $|x_t|_{\mathcal{B}}>0$ for $t\geq 0$. Suppose that (4.11) fails, then there exists $\varepsilon_0>0$ such that

$$\frac{x^U(m)}{x^S(m)}\geq\varepsilon_0,$$

for infinitely many m. Next we will show that (4.12) must hold. From (1.4) and (4.7) we can see that

$$\lim_{m\to\infty}\widetilde{q}(m)=0. \tag{4.13}$$

By (4.13), there exists $m_1\geq 0$ such that for $m\geq m_1$

$$e^{\rho_2}-\frac{1+\varepsilon_0}{\varepsilon_0}\widetilde{q}(m)>0$$

and

$$\frac{e^{\rho_1}+(1+\varepsilon_0)\,\widetilde{q}(m)}{\varepsilon_0e^{\rho_2}-(1+\varepsilon_0)\,\widetilde{q}(m)}<\frac{1}{\varepsilon_0}. \tag{4.14}$$

Since (4.11) fails then there exists $m_2\geq m_1$ such that

$$x^U(m_2)\geq\varepsilon_0x^S(m_2).$$

Next we will show that for all $m\geq m_2$

$$x^U(m)\geq\varepsilon_0x^S(m). \tag{4.15}$$

Suppose by induction that this inequality holds for some $m\geq m_2$. Then it follows from (4.8) that

$$\begin{aligned}x^S(m+1)&\leq e^{\rho_1}\frac{x^U(m)}{\varepsilon_0}+\widetilde{q}(m)\frac{x^U(m)}{\varepsilon_0}+\widetilde{q}(m)\,x^U(m)\\&=\left(\frac{e^{\rho_1}}{\varepsilon_0}+\frac{\widetilde{q}(m)}{\varepsilon_0}+\widetilde{q}(m)\right)x^U(m).\end{aligned}$$

Now from (4.9) we have

$$\begin{aligned}x^U(m+1)&\geq e^{\rho_2}x^U(m)-\widetilde{q}(m)\frac{x^U(m)}{\varepsilon_0}-\widetilde{q}(m)\,x^U(m)\\&=\left(e^{\rho_2}-\frac{\widetilde{q}(m)}{\varepsilon_0}-\widetilde{q}(m)\right)x^U(m).\end{aligned} \tag{4.16}$$

It follows that

$$\begin{aligned}x^S(m+1)&\leq\left(\frac{e^{\rho_1}}{\varepsilon_0}+\frac{\widetilde{q}(m)}{\varepsilon_0}+\widetilde{q}(m)\right)x^U(m)\\&\leq\left(\frac{e^{\rho_1}}{\varepsilon_0}+\frac{\widetilde{q}(m)}{\varepsilon_0}+\widetilde{q}(m)\right)\frac{1}{e^{\rho_2}-\widetilde{q}(m)-\frac{\widetilde{q}(m)}{\varepsilon_0}}x^U(m+1)\\&=\frac{e^{\rho_1}+\widetilde{q}(m)+\varepsilon_0\widetilde{q}(m)}{\varepsilon_0e^{\rho_2}-\varepsilon_0\widetilde{q}(m)-\widetilde{q}(m)}x^U(m+1).\end{aligned}$$

Now from (4.14), we deduce that

$$x^U(m+1) \geq \varepsilon_0 x^S(m+1).$$

Thus by induction, the inequality (4.15) holds for all $m \geq m_2$. From (4.8) and (4.16), we deduce that for $m \geq m_2$

$$\begin{aligned}\frac{x^S(m+1)}{x^U(m+1)} &\leq \frac{e^{\rho_1}x^S(m) + \widetilde{q}(m)\left(x^S(m) + x^U(m)\right)}{\left(e^{\rho_2} - \widetilde{q}(m) - \frac{\widetilde{q}(m)}{\varepsilon_0}\right)x^U(m)} \\ &= \frac{e^{\rho_1} + \widetilde{q}(m)}{\left(e^{\rho_2} - \widetilde{q}(m) - \frac{\widetilde{q}(m)}{\varepsilon_0}\right)}\frac{x^S(m)}{x^U(m)} + \frac{\widetilde{q}(m)}{\left(e^{\rho_2} - \widetilde{q}(m) - \frac{\widetilde{q}(m)}{\varepsilon_0}\right)}.\end{aligned}$$

It follows by (4.13) that

$$\limsup_{m\to\infty}\frac{x^S(m)}{x^U(m)} \leq \frac{e^{\rho_1}}{e^{\rho_2}}\limsup_{m\to\infty}\frac{x^S(m)}{x^U(m)}.$$

That is

$$\left(1 - e^{\rho_1-\rho_2}\right)\limsup_{m\to\infty}\frac{x^S(m)}{x^U(m)} \leq 0.$$

But since $\rho_1 < \rho_2$ and $\limsup_{m\to\infty}\frac{x^S(m)}{x^U(m)} \geq 0$, we deduce that $\limsup_{m\to\infty}\frac{x^S(m)}{x^U(m)} = 0$. Therefore

$$\lim_{m\to\infty}\frac{x^S(m)}{x^U(m)} = 0.$$

This ends the proof of Lemma 4.7. $\square$

The proof of Theorem 5.3 is based on the following principal lemma.

Lemma 4.8. *Either*

$$\limsup_{t\to\infty}\frac{\log\|x_t\|_{\mathcal{B}}}{t} < \rho \tag{4.17}$$

or

$$\liminf_{t\to\infty}\frac{\log\|x_t\|_{\mathcal{B}}}{t} > \rho. \tag{4.18}$$

Proof. By Lemma 4.7, we have to discuss two cases:

Case 1. Assume that (4.11) holds. Then we have $x^U(m) < x^S(m)$ for all large integers m, where $x^U(m)$ and $x^S(m)$ are given by (4.6). Let ε be a positive real number. Then by (4.13), there exists a large positive integer m_ε such that for $m \geq m_\varepsilon$,

$$\widetilde{q}(m) < \varepsilon \quad \text{and} \quad x^U(m) < x^S(m). \tag{4.19}$$

Using (4.8) and (4.19) we have $x^S(m+1) \leq (e^{\rho_1} + 2\varepsilon)\, x^S(m)$ for $m \geq m_\varepsilon$. It follows that

$$x^S(m) \leq (e^{\rho_1} + 2\varepsilon)^{m-m_\varepsilon} x^S(m_\varepsilon) = K_\varepsilon (e^{\rho_1} + 2\varepsilon)^m,$$

where $K_\varepsilon := (e^{\rho_1} + 2\varepsilon)^{-m_\varepsilon} x^S(m_\varepsilon) > 0$. For $t \geq m_\varepsilon$, we have $[t] \geq m_\varepsilon$, where $[\cdot]$ is the floor function. Since $[t] \leq t \leq [t] + 1$, it follows from (3.7), (3.8), (4.5) and (4.19) that

$$\|x_t\|_{\mathcal{B}} \leq |x_t|_{\mathcal{B}} \leq C_3\left|x_{[t]}\right|_{\mathcal{B}} \leq 2C_3 x^S([t]) \leq 2C_3 K_\varepsilon (e^{\rho_1} + 2\varepsilon)^{[t]}.$$

Hence,

$$\frac{\log \|x_t\|_{\mathcal{B}}}{t} \leq \frac{\log (2C_3K_\varepsilon)}{t} + \frac{[t]}{t} \log \left(e^{\rho_1} + 2\varepsilon\right).$$

By taking $t \to \infty$ we get

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} \leq \log \left(e^{\rho_1} + 2\varepsilon\right).$$

Now by taking $\varepsilon \to 0$, we obtain that

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} \leq \log \left(e^{\rho_1}\right) = \rho_1 < \rho,$$

that is, (4.17) holds.

Case 2. Suppose that (4.12) holds. Note that $x^S(m) < x^U(m)$ for all large integers m. Let ε such that $0 < \varepsilon < \frac{e^{\rho_2}}{2}$. By (4.13), there exists a large positive integer m_ε such that for $m \geq m_\varepsilon$,

$$\widetilde{q}(m) < \varepsilon \quad \text{and} \quad x^S(m) < x^U(m). \tag{4.20}$$

Using (4.9) and (4.20) we have $x^U(m+1) \geq \left(e^{\rho_2} - 2\varepsilon\right) x^U(m)$ for $m \geq m_\varepsilon$, which implies that

$$x^U(m) \geq \left(e^{\rho_2} - 2\varepsilon\right)^{m-m_\varepsilon} x^U(m_\varepsilon) = K_\varepsilon \left(e^{\rho_2} - 2\varepsilon\right)^m,$$

where $K_\varepsilon := \left(e^{\rho_2} - 2\varepsilon\right)^{-m_\varepsilon} x^U(m_\varepsilon) > 0$. For $t \geq m_\varepsilon$, we have $[t]+1 \geq m_\varepsilon$. Since $[t] \leq t \leq [t]+1$, it follows from (3.7), (4.5) that

$$\|x_t\|_{\mathcal{B}} \geq \frac{|x_t|_{\mathcal{B}}}{C_2} \geq \frac{\left|x_{[t]+1}\right|_{\mathcal{B}}}{C_2C_3} \geq \frac{x^U([t]+1)}{C_2C_3} \geq \frac{K_\varepsilon \left(e^{\rho_2} - 2\varepsilon\right)^{[t]+1}}{C_2C_3}.$$

Hence,

$$\frac{\log \|x_t\|_{\mathcal{B}}}{t} \geq \frac{\log \left(\frac{K_\varepsilon}{C_2C_3}\right)}{t} + \frac{[t]+1}{t} \log \left(e^{\rho_2} - 2\varepsilon\right).$$

By taking $t \to \infty$ we get

$$\liminf_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} \geq \log \left(e^{\rho_2} - 2\varepsilon\right).$$

Now by taking $\varepsilon \to 0$, we obtain that

$$\liminf_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} \geq \log \left(e^{\rho_2}\right) = \rho_2 > \rho,$$

that is, (4.18) holds. This completes the proof. □

5 Perron's theorem for equation (1.1)

Let x be a solution of equation (1.1). If there exists $t_0 \geq 0$ such that $\|x_{t_0}\|_{\mathcal{B}} = 0$, then it follows from Lemma 4.4 that $x_t = 0$ for all $t \geq t_0$. Thus, in what follows, we will only consider the case where $\|x_t\|_{\mathcal{B}} > 0$ for all $t \geq 0$.

The following result describes the asymptotic behavior of solutions of equation (1.1) in terms of the growth bound of the semigroup solution $(V(t))_{t\geq 0}$ of the unperturbed linear equation, which is natural given the smallness of the nonlinear term.

Theorem 5.1. *Suppose that* **(H0)–(H2)** *hold. Let x be a solution of equation* (1.1) *such that $\|x_t\|_{\mathcal{B}} > 0$ for $t \geq 0$. Then we have*

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} \leq \omega_0(V).$$

Proof. Let $\varepsilon > 0$, from Lemma 4.4, we deduce that for $t \geq 0$

$$\frac{\log \|x_t\|_{\mathcal{B}}}{t} \leq \frac{\log(C_0(\varepsilon)\|x_0\|_{\mathcal{B}})}{t} + \omega_0(V) + \varepsilon + \widetilde{N}K(1)C_0(\varepsilon)\frac{\int_0^t q(s)\,ds}{t}. \tag{5.1}$$

Using (1.4) we can see that $\frac{\int_0^t q(s)ds}{t} \to 0$ as $t \to \infty$. By taking $t \to \infty$ in (5.1), we obtain

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} \leq \omega_0(V) + \varepsilon. \tag{5.2}$$

Now by letting $\varepsilon \to 0$ in (5.2) we obtain the desired estimation. □

Corollary 5.2. *If $\omega_0(V) < 0$, then the equilibrium point 0 is globally asymptotically stable.*

Now we give the Perron's theorem for equation (1.1) which constitutes a refinement of Theorem 5.1.

Theorem 5.3. *Suppose that* **(H0)–(H2)** *hold. Let x be a solution of equation* (1.1) *such that $\|x_t\|_{\mathcal{B}} > 0$ for $t \geq 0$. Then either*

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} \leq \omega_{ess}(V), \tag{5.3}$$

or

$$\lim_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} = \operatorname{Re}\lambda_0 > \omega_{ess}(V), \tag{5.4}$$

where λ_0 is an eigenvalue of the operator $\mathcal{A}_V$.

Proof. We will show that if (5.3) fails, then (5.4) must hold. Suppose that

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} > \omega_{ess}(V).$$

It follows from Theorem 5.1 that

$$\omega_0(V) > \omega_{ess}(V).$$

Therefore

$$\omega_0(V) = \max\{s(\mathcal{A}_V), \omega_{ess}(V)\} = s(\mathcal{A}_V).$$

It follows that $\Lambda := \{\lambda \in \sigma(\mathcal{A}_V) : \operatorname{Re}\lambda > \omega_{ess}(V)\} \neq \emptyset$. We claim that there exist $\lambda_0 \in \Lambda$ such that

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} = \operatorname{Re}\lambda_0.$$

In fact, if $\limsup_{t\to\infty} \frac{\log\|x_t\|_{\mathcal{B}}}{t} = \rho \notin \{\operatorname{Re}\lambda : \lambda \in \Lambda\}$, with $\rho > \omega_{ess}(V)$, then the condition (4.17) in Lemma 4.8 fails. Hence we must have

$$\liminf_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} > \rho.$$

However this implies that

$$\rho = \limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} \geq \liminf_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} > \rho$$

which is a contradiction. Therefore, there exist $\lambda_0 \in \Lambda$ such that

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} = \operatorname{Re}\lambda_0.$$

Since $\operatorname{Re}\lambda_0 > \omega_{ess}(V)$, then there exists $\rho_0 \notin \{\operatorname{Re}\lambda : \lambda \in \Lambda\}$ such that $\operatorname{Re}\lambda_0 > \rho_0 > \omega_{ess}(V)$. That is

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} = \operatorname{Re}\lambda_0 > \rho_0. \tag{5.5}$$

By applying Lemma 4.8 to ρ_0 using (5.5), we obtain

$$\liminf_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} > \rho_0 > \omega_{ess}(V).$$

We claim that

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} = \liminf_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t}.$$

In fact if $\limsup_{t\to\infty} \frac{\log\|x_t\|_{\mathcal{B}}}{t} > \liminf_{t\to\infty} \frac{\log\|x_t\|_{\mathcal{B}}}{t}$, then there exists $\rho_1 \notin \{\operatorname{Re}\lambda : \lambda \in \Lambda\}$ with $\rho_1 > \omega_{ess}(V)$ such that

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} > \rho_1 \tag{5.6}$$

and

$$\liminf_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} < \rho_1. \tag{5.7}$$

By applying Lemma 4.8 to ρ_1 using (5.6) , we obtain

$$\liminf_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} > \rho_1,$$

which contradicts (5.7). Therefore, we have

$$\lim_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} = \limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} = \liminf_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} = \operatorname{Re}\lambda_0.$$

That is (5.4) holds. □

Remark 5.4. Unlike in [24], where the author assumed that $\dim(X) < \infty$ and in [21] where the authors assumed that the semigroup generated by A is compact, we do not assume any kind of compactness in Theorem 5.3. Thus, the asymptotic behavior of solutions is described in terms of the essential growth bound of the semigroup solution of the unperturbed linear system. That is why the approach we adopted in the proof of Theorem 5.3 differs from the one in [21, 24].

Consider the following operators defined for each $\phi \in \mathcal{B}_A$ by

$$(V_0(t)\phi)(\theta) = \begin{cases} T_0(t+\theta)\phi(0) & \text{for } t+\theta \geq 0 \\ \phi(t+\theta) & \text{for } t+\theta < 0 \end{cases}$$

and

$$(V_c(t)\phi)(\theta) = \begin{cases} \lim_{\lambda\to\infty} \int_0^{t+\theta} T_0(t+\theta-s)B_\lambda\left[L(x_t(.,\phi,L,0))\right] ds & \text{for } t+\theta \geq 0 \\ 0 & \text{for } t+\theta < 0. \end{cases} \tag{5.8}$$

We have the following decomposition of the semigroup solution $(V(t))_{t\in\mathbb{R}}$

$$V(t) = V_0(t) + V_c(t) \quad \text{for } t \geq 0.$$

Consider the following hypothesis:

(H3) The operator $T_0(t)$ is compact for each $t > 0$.

Lemma 5.5 ([6]). *Suppose that* **(H0)** *and* **(H3)** *hold. Then the operator $V_c(t)$ is compact for each $t > 0$.*

Lemma 5.6 ([6]). *Suppose that* **(H0)** *and* **(H3)** *hold. Then there exists a constant $\widetilde{C} > 0$ such that*

$$\alpha(V_0(t)) \leq \widetilde{C}\widetilde{K}(t-1) \quad \textit{for } t > 1,$$

where $\widetilde{K}(\cdot)$ is the function related to the phase space $\mathcal{B}$ in Axiom **(A)**.

From Lemma 5.5, Lemma 5.6 and the definition of the essential growth bound (2.2), we obtain the following result:

Proposition 5.7. *Suppose that* **(H0)** *and* **(H3)** *hold. Then the essential growth bound of the semigroup solution $(V(t))_{t\geq 0}$ satisfies the following estimate*

$$\omega_{ess}(V) = \omega_{ess}(V_0) \leq \underline{\widetilde{K}} := \liminf_{t\to\infty} \frac{\log \widetilde{K}(t)}{t}. \tag{5.9}$$

It is clear that the constant $\underline{\widetilde{K}}$ depends only on the phase space $\mathcal{B}$. The following result is a direct consequence of Theorem 5.3 and Proposition 5.7.

Corollary 5.8. *Suppose that* **(H0)–(H3)** *hold. Let x be a solution of equation (1.1) such that $\|x_t\|_{\mathcal{B}} > 0$ for $t \geq 0$. Then, we have*

$$\limsup_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} \leq \underline{\widetilde{K}} \tag{5.10}$$

or

$$\lim_{t\to\infty} \frac{\log \|x_t\|_{\mathcal{B}}}{t} = \operatorname{Re}\lambda_0 > \underline{\widetilde{K}}, \tag{5.11}$$

where λ_0 is an eigenvalue of the operator $\mathcal{A}_V$.

Remark 5.9. The main result in [21] can be retrieved from Corollary 5.8 if we assume that $\overline{D(A)} = X$ and $\mathcal{B} = C_\gamma$ (see Section 6 for the definition of the space C_γ). The corresponding result in [24] can be deduced directly from Theorem 5.3 if the delay is finite, $A = 0$ and $\dim(X) < \infty$. In fact, in this case, the operator solution $V(t)$ becomes compact for sufficiently large times t [15,27]. Thus from the definition of the essential growth bound, we get $\omega_{ess}(V) = -\infty$. It follows from Theorem 5.3 that some solutions x may satisfy $\limsup_{t\to\infty} \frac{\log\|x_t\|_{\mathcal{B}}}{t} = -\infty$. Those are called small solutions, they tend to zero faster than any exponential as $t \to \infty$. For more details about this kind of solutions, see [17] and [15, page 74].

6 Application

Let us consider the n-dimensional Lotka–Volterra equation with diffusion

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \displaystyle\sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 v}{\partial x_j \partial x_k}(t,x) + \sum_{k=1}^{n} b_k(x)\frac{\partial v}{\partial x_k}(t,x) + c(x)v(t,x) \\ \qquad\qquad + \displaystyle\int_{-\infty}^{0} h_1(\theta)v(t+\theta,x)\,d\theta \\ \qquad\qquad + \displaystyle\int_{-\infty}^{0} h_2(t,\theta,v(t+\theta,x))\,d\theta \quad \text{for } x \in \overline{\Omega} \text{ and } t \geq 0, \\ v(t,x) = 0 \quad \text{for } x \in \partial\Omega \text{ and } t \geq 0, \\ v(\theta,x) = v_0(\theta,x) \quad \text{for } x \in \overline{\Omega} \text{ and } \theta \leq 0, \end{cases} \tag{6.1}$$

where Ω is a C^2-bounded domain in $\mathbb{R}^n$, $a_{jk} = a_{kj}$, b_k, c are continuous functions in $\overline{\Omega}$, $h_1 \in C((-\infty,0])$, $h_2 \in C([0,\infty) \times (-\infty,0] \times \mathbb{R})$, $v_0 \in C\left((-\infty,0],\overline{\Omega}\right)$ and the uniform ellipticity condition:

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq \eta|\xi|^2 \quad \text{for } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^n,$$

holds with a constant $\eta > 0$.

It is known (see [20]) that the operator

$$Au = \sum_{j,k=1}^{n} a_{jk}(\cdot)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{k=1}^{n} b_k(\cdot)\frac{\partial u}{\partial x_k} + c(\cdot)u,$$

$$u \in D(A) = \left\{ u \in \bigcap_{p\geq 1} W^{2,p}_{\mathrm{loc}}(\Omega) : \, u,\, Au \in C(\overline{\Omega}),\, u|_{\partial\Omega} = 0 \right\},$$

satisfies the Hille–Yosida condition **(H0)** on $X := C(\overline{\Omega})$ and

$$\overline{D(A)} = \{u \in X : u|_{\partial\Omega} = 0\} \neq X.$$

Lemma 6.1 ([20]). *The semigroup $(T_0(t))_{t\geq 0}$ generated by the part of A in $\overline{D(A)}$ is compact for each $t > 0$.*

That is **(H3)** holds. Consider the phase space C_γ defined by

$$C_\gamma := \left\{ \phi \in C((-\infty,0],X) : \lim_{\theta\to-\infty} |\phi(\theta)|\, e^{\gamma\theta} = 0 \right\},$$

equipped with the following norm

$$\|\phi\|_\gamma = \sup_{\theta\leq 0} |\phi(\theta)|\, e^{\gamma\theta} \quad \text{for } \phi \in C_\gamma,$$

where γ is a real number. From [19, Theorem 3.7], the phase space C_γ satisfies Axioms **(A)**, **(B)** and **(C)**. The function $\widetilde{K}$ in Axiom **(A)** is given by $\widetilde{K}(t) = e^{-\gamma t}$, thus $\underline{\widetilde{K}} = -\gamma$. We assume in addition the following:

(M1) $\lim_{\theta\to-\infty} e^{\gamma\theta} v_0(\theta,x) = 0$ uniformly for $x \in \overline{\Omega}$ and $v_0(0,x) = 0$ for $x \in \partial\Omega$;

(M2) $\displaystyle\int_{-\infty}^{0} e^{-\gamma\theta}|h_1(\theta)|\,d\theta < \infty$;

(M3) for $t \geq 0$, $\theta \leq 0$ and $y \in \mathbb{R}$
$$|h_2(t,\theta,y)| \leq p(t)c(\theta)|y|,$$
where $p : [0,\infty) \to [0,\infty)$ is a bounded continuous function such that
$$\int_t^{t+1} p(s)ds \to 0 \quad \text{as } t \to \infty$$
and $c : (-\infty,0] \to [0,\infty)$ is a measurable function such that
$$\int_{-\infty}^0 e^{-\gamma\theta}c(\theta)\,d\theta < \infty.$$

In order to rewrite equation (6.1) in an abstract form, we introduce the following notation. For $t \geq 0$, $\theta \leq 0$ and $x \in \overline{\Omega}$
$$u(t)(x) = v(t,x) \quad \text{and} \quad \varphi(\theta)(x) = v_0(\theta,x).$$
For $\phi \in C_\gamma$, $t \geq 0$ and $x \in \overline{\Omega}$
$$L(\phi)(x) := \int_{-\infty}^0 h_1(\theta)\phi(\theta)(x)\,d\theta \quad \text{and} \quad f(t,\phi)(x) := \int_{-\infty}^0 h_2(t,\theta,\phi(\theta)(x))\,d\theta.$$
Then equation (6.1) takes the following form
$$\begin{cases} \dfrac{d}{dt}u\,(t) = Au\,(t) + L(u_t) + f(t,u_t) & \text{for } t \geq 0 \\ u_0 = \varphi. \end{cases} \tag{6.2}$$

(M1) implies that $\varphi \in C_\gamma$ and $\varphi(0) \in \overline{D(A)}$. From **(M2)** and **(M3)**, we can see that $L : C_\gamma \to X$ is a bounded operator and the function $f : [0,\infty) \times C_\gamma \to X$ satisfies **(H1)** and **(H2)** with $q(t) = \big(\int_{-\infty}^0 e^{-\gamma\theta}c(\theta)\,d\theta\big)p(t)$. Theorem 4.2 assures the existence of at least one global solution u of equation (6.2). By Corollary 5.8, we conclude that the global solutions have three possible asymptotic behaviors.

Proposition 6.2. *Either $u_t = 0$ for $t \geq t_0$ for some $t_0 \geq 0$,*
$$\limsup_{t\to\infty} \frac{\log|u_t|}{t} \leq -\gamma,$$
or
$$\lim_{t\to\infty} \frac{\log|u_t|}{t} = \operatorname{Re}\lambda_0 > -\gamma,$$
where λ_0 is an eigenvalue of the infinitesimal generator of the semigroup solution of the linear part of equation (6.2).

Remark 6.3. We note that since $\operatorname{Re}\lambda_0 > -\gamma$, then using the same approach as in [3, Lemma 2.13], there exists $w \in D(A)\backslash\{0\}$ solving the following characteristic equation
$$\lambda_0 w - Aw - L(e^{\lambda_0\cdot}w) = 0,$$
where $e^{\lambda_0\cdot}w$ is the element of C_γ defined for all $\theta \leq 0$ by $\left(e^{\lambda_0\cdot}w\right)(\theta) = e^{\lambda_0\theta}w$.

References

[1] M. Adimy, H. Bouzahir, K. Ezzinbi, Existence for a class of partial functional differential equations with infinite delay, *Nonlinear Anal.* **46**(2001), No. 1, 91–112. MR1845579; `https://doi.org/10.1016/S0362-546X(99)00447-2`

[2] M. Adimy, H. Bouzahir, K. Ezzinbi, Local existence and stability for some partial functional differential equations with infinite delay, *Nonlinear Anal.* **48**(2002), No. 3, 323–348. MR1869515; `https://doi.org/10.1016/S0362-546X(00)00184-X`

[3] M. Adimy, K. Ezzinbi, A. Ouhinou, Behavior near hyperbolic stationary solutions for partial functional differential equations with infinite delay, *Nonlinear Anal.* **68**(2008), No. 8, 2280–2302. MR2398650; `https://doi.org/10.1016/j.na.2007.01.051`

[4] L. Barreira, C. Valls, Nonautonomous equations with arbitrary growth rates: a Perron-type theorem, *Nonlinear Anal.* **75**(2012), No. 16, 6203–6215. MR2956137; `https://doi.org/10.1016/j.na.2012.06.027`

[5] L. Barreira, C. Valls, A Perron-type theorem for nonautonomous differential equations, *J. Differential Equations*, **258**(2015), No. 2, 339–361. MR3274761; `https://doi.org/10.1016/j.jde.2014.09.012`

[6] R. Benkhalti, H. Bouzahir, K. Ezzinbi, Existence of a periodic solution for some partial functional differential equations with infinite delay, *J. Math. Anal. Appl.* **256**(2001), No. 1, 257–280. MR1820080; `https://doi.org/10.1006/jmaa.2000.7321`

[7] T. A. Burton, Perron-type stability theorems for neutral equations, *Nonlinear Anal.* **55**(2003), No. 3, 285–297. MR2007475; `https://doi.org/10.1016/S0362-546X(03)00240-2`

[8] E. A. Coddington, N. Levinson, *Theory of ordinary differential equations*, Tata McGraw-Hill Education, 1955. MR0069338

[9] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, Heath Boston, 1965. MR0190463

[10] O. Diekmann, S. A. Van Gils, S. M. Verduyn Lunel, H.-O. Walther, *Delay equations: functional, complex, and nonlinear analysis*, Applied Mathematical Sciences, Vol. 110, Springer-Verlag, 1995. MR1345150; `https://doi.org/10.1007/978-1-4612-4206-2`

[11] M. S. P. Eastham, *The asymptotic solution of linear differential systems*, Oxford University Press, USA, 1989. MR1006434

[12] K.-J. Engel, R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, Vol. 194, Springer-Verlag, New York, 2000. MR1721989

[13] K. Ezzinbi, H. Toure, I. Zabsonre, Existence and regularity of solutions for some partial functional integrodifferential equations in Banach spaces, *Nonlinear Anal.* **70**(2009), No. 7, 2761–2771. MR2499744; `https://doi.org/10.1016/j.na.2008.04.001`

[14] J. K. Hale, J. Kato, Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21**(1978), No. 1, 11–41. MR0492721

[15] J. K. Hale, S. M. Verduyn Lunel, *Introduction to functional-differential equations*, Springer, 1993. MR1243878; https://doi.org/10.1007/978-1-4612-4342-7

[16] P. Hartman, *Ordinary differential equations*, Birkhaüser, Boston, 1982. MR658490

[17] D. Henry, Small solutions of linear autonomous functional differential equations, *J. Differential Equations* **8**(1970), No. 3, 494–501. MR0265713; https://doi.org/10.1016/0022-0396(70)90021-5

[18] Y. Hino, S. Murakami, T. Naito, N. Van Minh, A variation-of-constants formula for abstract functional differential equations in the phase space, *J. Differential Equations* **179**(2002), No. 1, 336–355. MR1883747; https://doi.org/10.1006/jdeq.2001.4020

[19] Y. Hino, S. Murakami, T. Naito, *Functional-differential equations with infinite delay*, Springer, 1991. MR1122588; https://doi.org/10.1007/BFb0084432

[20] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Springer, 2013. MR3012216

[21] K. Matsui, H. Matsunaga, S. Murakami, Perron type theorems for functional differential equations with infinite delay in a Banach space, *Nonlinear Anal.* **69**(2008), No. 111, 3821–3837. MR2463337; https://doi.org/10.1016/j.na.2007.10.017

[22] H. Matsunaga, S. Murakami, Asymptotic behavior of solutions of functional difference equations, *J. Math. Anal. Appl.* **305**(2005), No. 2, 391–410. MR2130709; https://doi.org/10.1016/j.jmaa.2004.10.065

[23] O. Perron, Über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen (in German), *Math. Z.* **29**(1929), 129–160. MR1544998; https://doi.org/10.1007/BF01180524

[24] M. Pituk, A Perron type theorem for functional differential equations, *J. Math. Anal. Appl.* **316**(2006), No. 1, 24–41. MR2201747; https://doi.org/10.1016/j.jmaa.2005.04.027

[25] M. Pituk, Asymptotic behavior and oscillation of functional differential equations, *J. Math. Anal. Appl.* **322**(2006), No. 2, 1140–1158. MR2250641; https://doi.org/10.1016/j.jmaa.2005.09.081

[26] H. R. Thieme, Semiflows generated by Lipschitz perturbations of non-densely defined operators, *Differential Integral Equations* **3**(1990), No. 6, 1035–1066. MR1073056

[27] C. Travis, G. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.* **200**(1974), 395–418. MR0382808; https://doi.org/10.2307/1997265