Original citation:
Auel, Asher, Boehning, Christian and Pirutka, Alena (2018) *Stable rationality of quadric and cubic surface bundle fourfolds.* European Journal of Mathematics . doi:10.1007/s40879-018-0233-1

Permanent WRAP URL:
http://wrap.warwick.ac.uk/99882

STABLE RATIONALITY OF QUADRIC AND CUBIC SURFACE BUNDLE FOURFOLDS

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Abstract. We study the stable rationality problem for quadric and cubic surface bundles over surfaces from the point of view of the specialization method for the Chow group of 0-cycles. Our main result is that a very general hypersurface X of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ is not stably rational. Via projections onto the two factors, $X \to \mathbb{P}^2$ is a cubic surface bundle and $X \to \mathbb{P}^3$ is a conic bundle, and we analyze the stable rationality problem from both these points of view. Also, we introduce, for any $n \geq 4$, new quadric surface bundle fourfolds $X_n \to \mathbb{P}^2$ with discriminant curve $D_n \subset \mathbb{P}^2$ of degree $2n$, such that X_n has nontrivial unramified Brauer group and admits a universally CH_0-trivial resolution.

1. Introduction

An integral variety X over a field k is *stably rational* if $X \times \mathbb{P}^m$ is rational, for some m. In recent years, failure of stable rationality has been established for many classes of smooth rationally connected projective complex varieties, see, for instance [1, 3, 4, 7, 15, 16, 23, 24, 25, 26, 27, 29, 31, 32, 33, 37, 36, 39, 40]. These results were obtained by the specialization method, introduced by Voisin [40] and developed in [16]. In many applications, one uses this method in the following form.

Theorem ([40, Theorem 2.1], [16, Théorème 2.3])**.** *Let k be an algebraically closed field, let B be a quasi-projective integral scheme over k and let $\mathcal{X} \to B$ be a generically smooth projective morphism with positive dimensional fibers. Assume there is a point $b_0 \in B(k)$ such that the fiber $Y = \mathcal{X}_{b_0}$ is an integral variety satisfying the following properties:*

(R) *Y admits a universally CH_0-trivial resolution $\widetilde{Y} \to Y$ of singularities,*
(O) *$\widetilde{Y}$ is not universally CH_0-trivial, e.g., the function field $k(Y)$ admits a nontrivial étale unramified invariant such as $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Q}/\mathbb{Z}(1))$.*

Then, for a very general point $b \in B(k)$, the fiber $\mathcal{X}_b$ is not stably rational.

Here, Y is a particular instance of a "reference variety," a convenient notion that we adopt in this article, see Definition 1. In our applications, we will also make use of reference varieties that are reducible, see Proposition 2.

Recall, from [2], [16], that a proper variety X over k is *universally* CH_0*-trivial* if for every field extension k'/k, the degree homomorphism on the Chow group of 0-cycles $\mathrm{CH}_0(X_{k'}) \to \mathbb{Z}$ is an isomorphism. A proper morphism $f : \widetilde{X} \to X$ of k-varieties is *universally* CH_0*-trivial* if for every field extension k'/k, the push-forward homomorphism $f_* : \mathrm{CH}_0(\widetilde{X}_{k'}) \to \mathrm{CH}_0(X_{k'})$ is an isomorphism. Then a *universally CH_0-trivial resolution* of X is a proper birational universally CH_0-trivial morphism $f : \widetilde{X} \to X$ with $\widetilde{X}$ smooth.

In [24], the specialization method was applied to show that a very general hypersurface of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ is not stably rational over $\mathbb{C}$, utilizing the following reference variety:

$$(1.1) \qquad Y \ : \ yzs^2 + xzt^2 + xyu^2 + (x^2 + y^2 + z^2 - 2xy - 2xz - 2yz)v^2 = 0.$$

Such hypersurfaces have the structure of a quadric surface bundle over $\mathbb{P}^2$, by projection to the first factor, which inform the shape of the equation for the reference variety Y as in [34]. In [24], it was also shown that the locus, in the Hilbert scheme of all hypersurfaces of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$, where the quadric bundle admits a rational section, is dense (for the complex topology). This provided a first example showing that rationality is not a deformation invariant for smooth projective complex varieties. A detailed analysis of this same reference variety, from the point of view of the conic bundle structure obtained by projection onto the second factor, is made in [3].

Recently, Schreieder [37] developed a refinement of the specialization method, relaxing the condition that the reference variety admits a universally CH_0-trivial resolution. This helped establish the failure of stable rationality for many families of quadric bundles [37], and in particular a large class of quadric surface bundles over $\mathbb{P}^2$ of graded free type [36].

In two different directions, recent results of Ahmadinezhad and Okada [1] and of Krylov and Okada [29] imply the failure of stable rationality for many families of conic and del Pezzo surface bundles over projective space, including a very general hypersurface of bidegree $(2,d)$ and $(e,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ for $d \geq 4$ and $e \geq 3$. In this work, the specialization method is also used, where the reference varieties constructed have global differential forms in characteristic p, following the method of Kollár [28] developed by Totaro [39].

The goal of this note is threefold. First, we complete the stable rationality analysis for hypersurfaces of bidegree $(2,d)$ and $(e,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$.

Theorem 1. *The very general hypersurface of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ over $\mathbb{C}$ is not stably rational.*

We also produce loci of bidegree $(2,3)$ hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^3$ that are rational, giving another example of a family of smooth rationally connected fourfolds with some fibers rational but most not stably rational, see Remark 4.

We provide two different proofs of Theorem 1. Our first proof, in §2, uses a method, going back to Totaro [39] for hypersurfaces in projective space, for reducing the case of hypersurfaces of bidegree $(2, d+1)$ in $\mathbb{P}^n \times \mathbb{P}^m$ to those of bidegree $(2,d)$, by constructing reference varieties that are reducible, see also [7]. This method only works in general when $m = 2$, but with some additional geometric construction relying heavily on the analysis in [3], we are able to handle the case $m = 3$. We remark that hypersurfaces of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ have the structure of a conic bundle over $\mathbb{P}^3$ by projection onto the second factor, and a cubic surface bundle over $\mathbb{P}^2$ by projection onto the first factor. For our second proof, in §5, we construct a new reference hypersurface of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ such that the associated cubic surface bundle is generically smooth, as opposed to our first construction.

Our second goal, in §3, is to more generally study the Brauer groups of cubic surface bundles over a rational surface (see §3 for a definition), with an aim toward investigating properties of the discriminant curve (which arises from the work of Salmon [35], Clebsch [11], [10], and clarified by Edge [19]) and the ramification profile, in the spirit of the quadric surface bundle case in [34]. This analysis provides an example of a smooth cubic surface X over $K = \mathbb{C}(x, y)$ such that the cokernel of the natural map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ contains a nontrivial 2-torsion Brauer class that is unramified on any fourfold model of X, see Remark 14.

Finally, our third goal, in §4, is to provide a new family of reference fourfolds $X_n \to \mathbb{P}^2$, whose generic fiber is a diagonal quadric, and whose discriminant curve of degree $2n \geq 8$ consists of an $(n-1)$-gon of double lines with an inscribed conic. These are a generalization of (1.1) (which occurs as X_4). The varieties in this family have nontrivial unramified Brauer group by an application of the general formula in [34]. Just as for the family of reference varieties constructed by Schreieder [36], the morphism $X_n \to \mathbb{P}^2$ need not be flat; on the other hand, we prove that each reference variety in our family admits a CH_0-universally trivial resolution. As a corollary, we obtain particular cases of [36, Corollary 2] using the specialization method, see §4.2.

Acknowledgements. The authors would like to thank the organizers of the Simons Foundation conference on Birational Geometry held in New York City, August 21–25, 2017, as well as the Laboratory of Mirror Symmetry and Automorphic forms, HSE, Moscow, where some of this work was accomplished. The first author was partially supported by NSA Young Investigator Grant H98230-16-1-0321. The third author was partially supported by NSF grant DMS-1601680 and by the Laboratory of Mirror Symmetry NRU HSE, RF Government grant, ag. no. 14.641.31.0001. The authors would like to thank Jean-Louis Colliot-Thélène, Bjorn Poonen, Stefan Schreieder, Yuri Tschinkel, and the anonymous referee for helpful comments.

2. Hypersurfaces of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$

In this section we study hypersurfaces X of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$, which are Fano fourfolds of index 1 and Picard rank 2. Via projection onto the two factors, $X \to \mathbb{P}^2$ is a cubic surface bundle (its fibers are cubic surfaces in $\mathbb{P}^3$) and $X \to \mathbb{P}^3$ is a conic bundle. We point out that of the recent results [1], [7], [29], [36], [37] on stable rationality relevant to this case (e.g., for conic bundles and del Pezzo bundles), none actually cover the particular case of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$.

Definition 1. We call a proper variety Y over an algebraically closed field k a *reference variety* if for any discrete valuation ring A, with residue field k and fraction field K with choice of algebraic closure $\overline{K}$, and for any scheme $\mathcal{X}$ flat and proper over A with special fiber Y and smooth connected geometric generic fiber $\overline{X}$, the $\overline{K}$-scheme $\overline{X}$ is not universally CH_0-trivial.

For example, if Y is integral and satisfies conditions (R) and (O) from §1, then Y is a reference variety by [16]. More generally, examples of reducible reference varieties were utilized in [39]. We recall sufficient conditions for a reducible variety to be a reference variety.

Proposition 2. *Let Y be a proper scheme with two irreducible components Y_1 and Y_2 over an algebraically closed field k such that $Y_1 \cap Y_2$ is irreducible. If $Y_1 \cap Y_2$ is universally CH_0-trivial and Y_1 admits a universally CH_0-trivial resolution $f : Y_1' \to Y_1$ such that Y_1' is not universally CH_0-trivial, then Y is a reference variety.*

Proof. Let $U_i = Y_i \smallsetminus (Y_1 \cap Y_2)$. Since $Y_1 \cap Y_2$ is universally CH_0-trivial and Y_1 is not universally CH_0-trivial, there exists a field extension l/k such that in the localization exact sequence over l

$$\mathrm{CH}_0((Y_1 \cap Y_2)_l) \to \mathrm{CH}_0(Y_{1,l}) \xrightarrow{\iota} \mathrm{CH}_0(U_{1,l}) \to 0$$

the first map is not surjective. Hence, there is a non-trivial 0-cycle $\xi \in \mathrm{CH}_0(U_{1,l})$. Since k is algebraically closed, $U_1(k) \neq \varnothing$, thus we can also assume ξ has degree 0.

We can furthermore assume that ξ is supported on the smooth locus of U_1. Indeed, let $V \subset U_1$ be a smooth open subscheme such that f induces an isomorphism $f^{-1}(V) \xrightarrow{\sim} V$, and consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_0(Y'_{1,l}) \arrow[r] \arrow[d, "f_*"] & \mathrm{CH}_0(f^{-1}(U_{1,l})) \arrow[d, "f_*"] \\
\mathrm{CH}_0(Y_{1,l}) \arrow[r] & \mathrm{CH}_0(U_{1,l}).
\end{tikzcd}$$

The left vertical map is an isomorphism and the horizontal maps are surjective. We deduce that ξ comes from a non-trivial element $\xi' \in \mathrm{CH}_0(f^{-1}(U_{1,l}))$. Since $f^{-1}(U_{1,l})$ is smooth, by a moving lemma for 0-cycles [13, p. 599], we can assume that ξ' is supported on the open subscheme $f^{-1}(V_l)$ and, hence, that ξ is supported on V_l.

By construction, the image of ξ in $\mathrm{CH}_0(Y_l)$ is nonzero. In fact, this follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_0((Y_1 \cap Y_2)_l) \arrow[r] \arrow[d] & \mathrm{CH}_0(Y_{1,l}) \arrow[r] \arrow[d] & \mathrm{CH}_0(U_{1,l}) \arrow[r] \arrow[d] & 0 \\
\mathrm{CH}_0((Y_1 \cap Y_2)_l) \arrow[r] & \mathrm{CH}_0(Y_l) \arrow[r] & \mathrm{CH}_0(U_{1,l}) \oplus \mathrm{CH}_0(U_{2,l}) \arrow[r] & 0
\end{tikzcd}$$

where the horizontal sequences are the localization exact sequences, the leftmost vertical map is an isomorphism, the middle vertical map is induced by the inclusion $Y_1 \to Y$, and the rightmost vertical map is injective.

By the argument above, we can also assume that ξ is supported on the smooth locus of Y_l. Then [39, Lemma 2.4] shows that Y is a reference variety. □

We remark that there is a version of Proposition 2 with Y having multiple irreducible components whose intersections have multiple components. In what follows, though, we only use the case $Y_1 \cap Y_2$ irreducible.

We briefly mention two conditions ensuring that a proper variety Y is universally CH_0-trivial. First, Y is universally CH_0-trivial if there exists a proper surjective morphism $Y' \to Y$ with Y' universally CH_0-trivial and such that for any field extension k'/k and any scheme theoretic point $y \in Y_{k'}$, the fiber Y'_y has a 0-cycle of degree 1. Second, Y is universally CH_0-trivial if it admits a universally CH_0-trivial resolution by a universally CH_0-trivial variety.

For hypersurfaces, a lemma implied by Proposition 2 was utilized by Totaro to arrive at an inductive procedure for investigating the stable rationality of a smooth hypersurface W of degree $2n+1$ in projective space by degenerating W to the union of a smooth hypersurface of degree $2n$ and a hyperplane. In [7, §4], a similar inductive procedure is used, degenerating a hypersurface of bidegree $(2, n+1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ to the union of hypersurfaces of bidegree $(2, n)$ and $(0, 1)$. In this later case, the intersection of hypersurfaces of bidegree $(2, n)$ and $(0, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ can be chosen to be smooth and has the structure of a conic bundle over $\mathbb{P}^1$, hence is rational and thus universally CH_0-trivial. However, attempting this for a hypersurface of bidegree $(2, n+1)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ is more subtle. One can still degenerate to the union of hypersurfaces of bidegree $(2, n)$ and $(0, 1)$, but now the intersection of these components has the structure of a conic bundle over $\mathbb{P}^2$, which can certainly fail to be universally CH_0-trivial, cf. [23], [7]. We shall overcome this problem for hypersurfaces of bidegree $(2, 3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ by using the explicit geometry of the conic bundle structure on the reference variety (1.1) studied in [3].

In our construction, we start with the reference variety (1.1), a singular hypersurface Y_1 of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$. We recall from [3, §3] that projection to $\mathbb{P}^3$ gives the structure of a conic bundle $Y_1 \to \mathbb{P}^3$ defined by the 3×3 matrix

$$\begin{pmatrix} v^2 & u^2 - v^2 & t^2 - v^2 \\ u^2 - v^2 & v^2 & s^2 - v^2 \\ t^2 - v^2 & s^2 - v^2 & v^2 \end{pmatrix} \tag{2.1}$$

of homogeneous quadratic forms on $\mathbb{P}^3$. The discriminant of this conic bundle is the sextic surface $D \subset \mathbb{P}^3$ defined by

$$4v^6 - 4(s^2 + t^2 + u^2)v^4 + (s^2 + t^2 + u^2)^2 v^2 - 2s^2 t^2 u^2 = 0$$

which has two irreducible cubic surface components $D_\pm$, defined by

$$2v^3 - v(s^2 + t^2 + u^2) \pm \sqrt{2} stu = 0.$$

Each component $D_\pm$ is a tetrahedral Goursat surface [22], hence up to projective equivalence, is isomorphic to the Cayley nodal cubic surface. The intersection $D_+ \cap D_-$ is an arrangement of a triangle of lines and three conics, see [3, Figure 1]. A Cayley cubic surface contains 4 ordinary double points and 9 lines: 6 of the lines form the edges of a tetrahedron whose vertices are the 4 singular points, while the remaining 3 lines form a triangle not meeting the singular points. In our case, $D_+ \cap D_-$ does not contain the ordinary double points of either component, hence contains this later triangle of lines, which is thus common to both Cayley cubic surfaces D_+ and D_-.

Our aim is then to choose an appropriate hyperplane $H \subset \mathbb{P}^3$ such that the configuration $Y_1 \cup (\mathbb{P}^2 \times H)$, thought of as a union of hypersurfaces of bidegree $(2,2)$ and $(0,1)$, is a reference variety for hypersurfaces of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$. Since Y_1 admits a universally CH_0-trivial resolution that is not universally CH_0-trivial by [24] or [3], then by Proposition 2 we only need to verify, for our choice of $H \subset \mathbb{P}^3$, that $Y_1 \cap (\mathbb{P}^2 \times H)$ admits a universally CH_0-trivial resolution by a smooth variety that is universally CH_0-trivial.

We remark that $Y_1 \cap (\mathbb{P}^2 \times H) = Y_1|_H \to H$ is simply the restriction of the conic bundle $Y_1 \to \mathbb{P}^3$ to $\mathbb{P}^2 = H \subset \mathbb{P}^3$, whose discriminant divisor is now $D \cap H \subset H$. In particular, $Y_1|_H \subset \mathbb{P}^2 \times H = \mathbb{P}^2 \times \mathbb{P}^2$ is a hypersurface of bidegree $(2,2)$. By [7], the very general such hypersurface is not universally CH_0-trivial, so we would expect that for a very general choice of $H \subset \mathbb{P}^3$, the variety $Y_1|_H$ would *not* be universally CH_0-trivial. Indeed, in our case, we can verify this explicitly. Let $\alpha \in H^2(\mathbb{C}(\mathbb{P}^3), \mathbb{Z}/2)$ be the Brauer class corresponding to the generic fiber of the conic bundle $Y_1 \to \mathbb{P}^3$ and let $\gamma_\pm \in H^1(\mathbb{C}(D_\pm), \mathbb{Z}/2)$ be its residue class on the component $D_\pm$ of the discriminant. By the analysis in [3, §3], we know that $\gamma_\pm$ is étale away from the 4 singular points of $D_\pm$. The residues of the conic bundle $Y_1|_H \to H$ on the components of its discriminant $D \cap H = (D_+ \cap H) \cup (D_- \cap H)$ are simply the restrictions of $\gamma_\pm$. A general hypersurface $H \subset \mathbb{P}^3$ will cut D in the union of two smooth elliptic curves $E_+ \cup E_-$ meeting transversally and the restriction of $\gamma_\pm$ to $E_\pm$ are étale and would be nontrivial by a suitable version of the Lefschetz hyperplane theorem. Hence, by a formula due to Colliot-Thélène (cf. [34]), the unramified Brauer group of $Y_1|_H$ would have nontrivial 2-torsion, and since it only has isolated nodes, it admits a universally CH_0-trivial resolution by a smooth projective variety that would *not* be universally CH_0-trivial. In conclusion, we need to choose the hyperplane $H \subset \mathbb{P}^3$ in a special way.

We choose the plane $H \subset \mathbb{P}^3$ to be the unique plane spanned by the triangle of lines contained in $D_+ \cap D_-$. Then from the explicit representation (2.1) of Y_1 as a conic bundle, $H = \{v = 0\}$ and we have that $Y_0 = Y_1|_H \to H$ is the hypersurface of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$xyu^2 + xzt^2 + yzs^2 = 0 \tag{2.2}$$

where $(x : y : z)$ and $(u : t : s)$ are sets of homogeneous coordinates on $\mathbb{P}^2$. Given the above discussion, Theorem 1 will follow from the following.

Proposition 3. *The hypersurface Y_0 of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ defined by (2.2) is a singular projective rational variety admitting a universally CH_0-trivial resolution.*

We remark that since Y_0 is rational, any smooth proper variety birational to Y_0 will be CH_0-universally trivial, cf. [2, §1.2]. Hence to apply Proposition 2, we only need to verify that Y_0 admits a universally CH_0-trivial resolution.

Proof of Proposition 3. The singular locus Y_0^{sing} of Y_0 is the union of three curves:

$$\begin{aligned} C_x &: \ x = s = 0,\ u^2y + t^2z = 0 \\ C_y &: \ y = t = 0,\ u^2x + s^2z = 0 \\ C_z &: \ z = u = 0,\ t^2x + s^2y = 0. \end{aligned} \tag{2.3}$$

The intersection of Y_0^{sing} with the chart $z \neq 0$ in $\mathbb{P}^2 \times \mathbb{P}^2$ is contained in the affine chart $\mathbb{A}^4 \subset \mathbb{P}^2 \times \mathbb{P}^2$ with coordinates (x, y, s, t), defined by $z = 1, u = 1$. Hence it is enough to construct a resolution for the restriction U of Y_0 to this affine chart:

$$U \ : \ xy + xt^2 + ys^2 = 0$$

By making the change of variables $y_1 = y + t^2$ and $x_1 = x + s^2$ we obtain the equation:

$$U \ : \ x_1y_1 - s^2t^2 = 0$$

The singular locus of U is thus the union of two curves:

$$\begin{aligned} B_s &: \ x_1 = y_1 = s = 0 \\ B_t &: \ x_1 = y_1 = t = 0 \end{aligned}$$

Let $\widetilde{U} \to U$ be the composition of the blow up along B_s and then the blow up along the strict transform of B_t. We claim that $\widetilde{U}$ is smooth and that the map $\widetilde{U} \to U$ is universally CH_0-trivial. To check this, we use the local blow up calculations below. Note that we need to discuss only two charts in each case, using symmetry between x_1 and y_1, and x_2 and y_2, with the notations below.

1) First blow up U along B_s:
- In the chart defined by $y_1 = x_1y_2, s = x_1s_2$, the equation of the blow up is $y_2 - t^2s_2^2 = 0$, and the exceptional divisor is $x_1 = 0, y_2 - t^2s_2^2 = 0$.
- In the chart defined by $x_1 = sx_2, y_1 = sy_2$, the equation of the blow up is $x_2y_2 - t^2 = 0$ and the the exceptional divisor is $s = 0, x_2y_2 - t^2 = 0$.

2) Second blow up the proper transform B_t' of B_t:

$$B_t' \ : \ x_2 = y_2 = t = 0.$$

- In the chart defined by $y_2 = x_2y_3, t = x_2t_3$, the equation of the blow up is $y_3 - t_3^2 = 0$, the exceptional divisor is $x_2 = 0, y_3 - t_3^2 = 0$.
- In the chart defined by $x_2 = tx_3, y_2 = ty_3$, the equation of the blow up is $x_3y_3 - 1 = 0$, the exceptional divisor is $t = 0, x_3y_3 - 1 = 0$.

We see immediately that $\widetilde{U}$ is smooth, and that the resolution $\widetilde{U} \to U$ has scheme-theoretic fibers that are either smooth rational conics or chains of lines, hence these fibers are universally CH_0-trivial. We conclude, using [16, Proposition 1.8], that $\widetilde{U} \to U$ is a universally CH_0-trivial resolution.

Now we verify the rationality of Y_0. As a divisor of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, the variety Y_0 admits a conic bundle structure for each of the projections to $\mathbb{P}^2$. From the explicit representation (2.1), we see that under the projection to the $\mathbb{P}^2$ with homogeneous coordinates $(u : t : s)$, the conic bundle $Y_0 \to \mathbb{P}^2$ is defined by the matrix of homogeneous forms:

$$\begin{pmatrix} 0 & u^2 & t^2 \\ u^2 & 0 & s^2 \\ t^2 & s^2 & 0 \end{pmatrix} \tag{2.4}$$

Since the quadratic form is clearly isotropic over the generic point of the base $\mathbb{P}^2$ (as its symmetric matrix has a diagonal zero entry), the generic fiber of Y_0 is rational over a rational base, hence Y_0 is rational. □

We can also analyze the singularities of Y_0 from the point of view of the conic bundle structure $Y_0 \to \mathbb{P}^2$ considered in (2.4). Note that the discriminant is the double triangle $(uts)^2$ in $\mathbb{P}^2$ and the conic fibers have constant rank 2 along the discriminant. The singular locus of Y_0 consists of a rational curve above each line of the double triangle forming the discriminant. We now describe the analytic local normal forms for such a conic bundle.

Let k be an algebraically closed field of characteristic $\neq 2$. Considering a point on only one of the double lines, we have the following. Any conic over a complete 2-dimensional regular local ring $k[[u,t]]$ degenerating to conics of rank 2 over u^2 can be brought to the normal form

$$x^2 + y^2 + u^2 z^2 = 0.$$

In this case, the singular locus is the rational curve $x = y = u = 0$ lying over $u = 0$ in the base. Simply blowing up this curve is a universally CH_0-trivial resolution.

Now considering a point on the intersection of two of the double lines, we have the following. Any conic over a complete 2-dimensional regular local ring $k[[u,t]]$ degenerating to conics of rank 2 over $(uv)^2 = 0$ can be brought to the normal form

$$x^2 + y^2 + u^2 v^2 z^2 = 0.$$

In this case, the singular locus is the union of rational curves $x = y = u = 0$ and $x = y = v = 0$ lying over $uv = 0$ in the base. Blowing up one of these rational curves, and then blowing up the strict transform of the other yields a universally CH_0-trivial resolution whose fiber above the generic point of either $u = 0$ or $v = 0$ is a $\mathbb{P}^1$ (over the function field of a curve over k) and above the intersection point is a chain of three smooth rational curves.

Finally, we combine all this together to give our first proof of Theorem 1.

Proof of Theorem 1. Let Y_1 be the reference variety (1.1) for hypersurfaces of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^3$. The discriminant of the conic bundle $Y_1 \to \mathbb{P}^3$ defined by projection to the second factor is the union of two Cayley cubic surfaces meeting along a triangle of lines and a configuration of three smooth conics. Let $H \subset \mathbb{P}^3$ be the unique hyperplane through this triangle of lines. Then $Y_2 = \mathbb{P}^2 \times H$ is a hypersurface of bidegree $(0,1)$ in $\mathbb{P}^2 \times \mathbb{P}^3$. Consider the reducible projective variety $Y = Y_1 \cup Y_2$, which is a hypersurface of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$. Then $Y_0 = Y_1 \cap Y_2$ is the irreducible projective variety with equation (2.2), which by Proposition 3, admits a universally CH_0-trivial resolution $\widetilde{Y}_0 \to Y_0$ with $\widetilde{Y}_0$ smooth and rational, hence in particular, Y_0 is universally CH_0-trivial. Thus by an application of Proposition 3, we have that Y is a reference variety for hypersurfaces of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$. By the specialization method [40], [16], the very general hypersurface of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ is not universally CH_0-trivial, and in particular, is not stably rational. □

Remark 4. We remark that in the Hilbert scheme of all hypersurfaces of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$, there is a nonempty closed subvariety whose general element Y is rational. Indeed, any bidegree $(2,3)$ hypersurface Y in $\mathbb{P}^2 \times \mathbb{P}^3$ of the form

$$Asv + Bsu + Ctv + Dtu = 0,$$

where A, B, C, D are general homogeneous forms of bidegree $(2,1)$ on $\mathbb{P}^2 \times \mathbb{P}^3$, is smooth and the generic fiber of the associated cubic surface bundle $Y \to \mathbb{P}^2$ contains the disjoint lines $\{s = t = 0\}$ and $\{u = v = 0\}$, hence is a rational cubic surface over $k(\mathbb{P}^2)$, hence Y is a rational variety. This provides a simple and geometrically appealing family of smooth projective rationally connected fourfolds whose very general fiber is not stably rational but where some fibers are rational, cf. [24].

3. Remarks on the Brauer group of cubic surface bundles

In this section, we study the unramified Brauer group of a cubic surface bundle (defined below) over a rational surface. Cubic surface bundles naturally arise in the study of hypersurfaces of bidegree $(e,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$, via projection onto the first factor.

Let S be a smooth projective rational surface over an algebraically closed field k of characteristic not dividing 6 and let $K = k(S)$. By a *cubic surface bundle* over S, we mean a flat projective morphism $\pi : X \to S$ arising from a relative cubic hypersurface $X \subset \mathbb{P}(\mathscr{E})$ defined by the vanishing of a global section of $S^3(\mathscr{E}^\vee) \otimes \mathscr{L}$ for some vector bundle $\mathscr{E}$ of rank 4 and some line bundle $\mathscr{L}$ on S. We will usually assume that the generic fiber X_K is a smooth cubic surface over K.

The locus in S over which the fibers are singular is a divisor that carries a canonical scheme structure called the *discriminant* divisor $\Delta \subset S$ of the cubic surface bundle. This divisor can be constructed using invariant theory as follows.

Consider the Hilbert scheme

$$\mathcal{H} = \mathbb{P}(H^0(\mathbb{P}^3, \mathscr{O}(3))) = \mathbb{P}^{19}$$

of cubic surfaces in $\mathbb{P}^3$ over k and the action of SL_4 on $\mathcal{H}$ by change of variables. Salmon [35], and independently Clebsch [11], [10], found that the ring of invariants of cubic forms in four variables is generated by *fundamental invariants* A, B, C, D, E in degrees 8, 16, 24, 32, 40, and 100 where the square of the invariant of degree 100 is a polynomial in the remaining invariants. This implies that the associated GIT quotient is isomorphic to a weighted projective space

$$\mathbb{P}(H^0(\mathbb{P}^3, \mathscr{O}(3))) /\!\!/ \mathrm{SL}_4 \cong \mathbb{P}(8, 16, 24, 32, 40) \cong \mathbb{P}(1, 2, 3, 4, 5).$$

Salmon found a formula for the discriminant in terms of the fundamental invariants, though his formula contained an error. This error had been repeated throughout the 19th and 20th century until it was corrected by Edge [19]

$$\Delta = (A^2 - 64B)^2 - 2^{11}(8D + AC)$$

For further modern study of the fundamental invariants and the discriminant of a cubic surface, see [5], [18, §10.4], [20, §2].

The formula for the discriminant Δ in terms of the monomials of a generic cubic surface defines an integral hypersurface of degree 32 in $\mathcal{H}$. Given a quadric surface bundle $\pi : X \to S$, with $X \subset \mathbb{P}(\mathscr{E})$ for a rank 4 vector bundle $\mathscr{E}$ on S, and a Zariski open $U \subset S$ over which $\mathscr{E}$ is trivial, the restricted cubic surface bundle $\pi|_U : X|_U \to U$ is defined by the restriction of the universal cubic hypersurface on $\mathcal{H}$ via a classifying map $U \to \mathcal{H}$. Then the locus of points in U over which the fibers of $\pi|_U$ are singular is contained in the divisor $\Delta|_U \subset U$. By choosing a Zariski open cover of S trivializing $\mathscr{E}$, we can glue these divisors to yield the discriminant divisor $\Delta \subset S$ of the cubic surface bundle $\pi : X \to S$.

In the rest of the section, we make a series of remarks about the second unramified cohomology group $H^2_{\mathrm{nr}}(k(X)/k, \mathbb{Q}/\mathbb{Z}(1))$ of the total space of a cubic surface bundle $\pi : X \to S$, or what is the same, the Brauer group of a smooth proper model of X. We will often call this group the *unramified Brauer group* $\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$.

We have $\mathrm{Br}_{\mathrm{nr}}(k(X)/k) \subset \mathrm{Br}_{\mathrm{nr}}(k(X)/K) = \mathrm{Br}(X_K)$, where the second equality holds by purity (cf. [12, §2.2.2]) because the generic fiber X_K is a smooth projective cubic surface over K. Part of the sequence of low degree terms of the Leray spectral sequence for the étale sheaf $\mathbb{G}_{\mathrm{m}}$ associated to the structural morphism $X_K \to \mathrm{Spec}(K)$ is

(3.1)

$$\mathrm{Pic}(X) \to H^0(K, \mathrm{Pic}(X_{K^s})) \to \mathrm{Br}(K) \to \mathrm{Br}(X_K) \to H^1(K, \mathrm{Pic}(X_{K^s})) \to H^3(K, \mathbb{G}_{\mathrm{m}})$$

where here K^s is a separable closure of K and $\mathrm{Br}(X_K) = \ker(\mathrm{Br}(X_K) \to \mathrm{Br}(X_{K^s}))$, because X_K is rational over K^s, see [17]. Hence the cokernel of $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ is isomorphic to a subgroup of $H^1(K, \mathrm{Pic}(X_{K^s}))$. In fact, Swinnerton-Dyer [38] has computed the possible nontrivial values for $H^1(K, \mathrm{Pic}(X_{K^s}))$ when X_K is a smooth cubic surface: they are $\mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/3$, and $\mathbb{Z}/3 \times \mathbb{Z}/3$. As a corollary, we arrive at the following.

Proposition 5. *Let X_K be a smooth cubic surface over a field K. Then the cokernel of the map $\mathrm{Br}(K) \to \mathrm{Br}(X_K)$ is killed by 6.*

We remark that there is a general method to obtain bounds on the torsion exponent of the Brauer group of a smooth proper variety V over a field K from torsion exponent bounds on the Chow group $A_0(V)$ of 0-cycles of degree 0. By a generalization of an argument of Merkurjev, see [2, Theorem 1.4], if $A_0(V)$ is universally N-torsion (this is related to the *torsion order* of V, cf. [9]) then the cokernel of the map $\mathrm{Br}(K) \to \mathrm{Br}(V)$ is killed by $N \cdot I(V)$, where $I(V)$ is the index of V, the minimal degree of a 0-cycle. If V is a smooth projective cubic surface over a field K, the Chow group $A_0(V)$ of 0-cycles of degree 0 is universally 6 torsion by an argument going back to Roitman, see [9, Proposition 4.1]. The index of a cubic hypersurface always divides 3, as one can see by cutting with a line. Thus, in the case of cubic surfaces, the Galois cohomology computations of Swinnerton-Dyer achieve a better bound than that which can be obtained by the method involving 0-cycles.

Given a smooth cubic surface V over a field K, we can also bound the kernel of the map $\operatorname{Br}(K) \to \operatorname{Br}(V)$, otherwise known as the *relative Brauer group* $\operatorname{Br}(V/K)$. The following is well known.

Proposition 6. *Let V be a smooth proper K-variety. Then the kernel of the map $\operatorname{Br}(K) \to \operatorname{Br}(V)$ is killed by the index $I(V)$.*

Proof. Let $x \in V$ be a closed point with residue field L/K. We recall the existence of a corestriction homomorphism $\operatorname{cor}_{L/K} : \operatorname{Br}(L) \to \operatorname{Br}(K)$ with the property that the composition $\operatorname{Br}(K) \to \operatorname{Br}(L) \to \operatorname{Br}(K)$ is multiplication by the degree $[L : K]$. The algebra-theoretic norm (determinant of the left-regular representation) yields a norm homomorphism $n_{L/K} : R_{L/K}\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ of group schemes over K. The corestriction homomorphism is then the composition

$$\operatorname{Br}(L) = H^2(L, \mathbb{G}_{\mathrm{m}}) = H^2(K, R_{L/K}\mathbb{G}_{\mathrm{m}}) \xrightarrow{H^2(n_{L/K})} H^2(K, \mathbb{G}_{\mathrm{m}}) = \operatorname{Br}(K).$$

When L/K is separable, this coincides with the classical corestriction map on Galois cohomology; when L/K is purely inseparable, this is simply multiplication by the degree. Then define a map $\operatorname{Br}(V) \to \operatorname{Br}(L) \to \operatorname{Br}(K)$ by restriction to the point $x \in V$ followed by corestriction. This map has the property that the composition $\operatorname{Br}(K) \to \operatorname{Br}(V) \to \operatorname{Br}(K)$ is multiplication by the degree of the point x. In particular, the kernel of the map $\operatorname{Br}(K) \to \operatorname{Br}(V)$ is killed by the degree of x. Since the index $I(V)$ agrees with the greatest common divisor of the degrees of closed points, we arrive at the claim. □

We remark that given a 0-cycle $z = \sum_i a_i z_i$ of minimal degree $I(V)$, we can define (in analogy with the proof of Proposition 6) a map $\operatorname{Br}(V) \to \operatorname{Br}(K)$ by the formula $\alpha \mapsto \sum_i a_i \operatorname{cor}_{K(z_i)/K}(\alpha|_{z_i})$, where $\alpha|_{z_i}$ is the restriction of the Brauer class to the closed point z_i. This map has the property that the composition of maps $\operatorname{Br}(K) \to \operatorname{Br}(V) \to \operatorname{Br}(K)$ is multiplication by the index $I(V)$. Finally, we remark that in the proof of Proposition 6, we could have used the classical corestriction from Galois cohomology; indeed, answering a question of Lang and Tate from the late 1960s, it is a result of Gabber, Liu, and Lorenzini [21] that there always exists a 0-cycle of minimal degree on a smooth proper variety V whose support is on points with separable residue fields.

Note that since a smooth cubic surface always has index dividing 3 (by cutting with a line), we see that the relative Brauer group $\operatorname{Br}(V/K)$ is always killed by 3.

We remark that the relative Brauer group of a smooth cubic surface can be nontrivial. Indeed, if V admits a Galois invariant set of 6 non-intersecting exceptional curves, then V is the blow up of a Severi–Brauer surface along an effective 0-cycle of degree 6. If this Severi–Brauer surface is associated to a nontrivial Brauer class $\alpha \in \operatorname{Br}(K)[3]$ then the relative Brauer group $\operatorname{Br}(V/K)$ is generated by α. In general, for a smooth cubic surface V over K, by (3.1), the relative Brauer group $\operatorname{Br}(V/K)$ is isomorphic to $H^0(K, \operatorname{Pic}(V_{K^s}))/\operatorname{Pic}(V)$, hence is a finite abelian group killed by 3.

We also remark that the bound in Proposition 6 is not sharp in general. Indeed, let Q be a smooth projective quadric surface with nontrivial discriminant (i.e., Picard rank 1) and no rational point over a field K of characteristic not 2. Then $I(Q) = 2$ yet the map $\mathrm{Br}(K) \to \mathrm{Br}(Q)$ is injective, cf. [2, Theorem 3.1].

Now we will study the subgroup $\mathrm{im}(\mathrm{Br}(K) \to \mathrm{Br}(k(X)) \cap \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$, where $\pi : X \to S$ is the total space of a cubic surface bundle over k, using a method going back to Colliot-Thélène and Ojanguren [14].

Let $H^2_{\mathrm{nr}}(k(X)/X, \mu_\ell)$ denote all those classes in $H^2(k(X), \mu_\ell)$ that are unramified with respect to divisorial valuations corresponding to prime divisors (integral threefolds) on X. Let $H^2_{\mathrm{nr}}(k(X)/K, \mu_\ell)$ be the subset of those classes in $H^2(k(X), \mu_\ell)$ that are unramified with respect to divisorial valuations that are trivial on K, hence correspond to prime divisors of X dominating the base S. Let ℓ be a prime different from the characteristic.

Lemma 7. *Let $X \to S$ be a flat projective morphism of integral varieties over an algebraically closed field k with S a smooth projective rational surface with function field K. Let ℓ be a prime number different from the characteristic of k. Then the following diagram with exact rows is commutative:*

$$\begin{tikzcd}
0 \arrow[r] & H^2_{\mathrm{nr}}(k(X)/X, \mu_\ell) \arrow[r] & H^2_{\mathrm{nr}}(k(X)/K, \mu_\ell) \arrow[r, "\oplus\partial^2_T"] & \bigoplus_{T \in X^{(1)}_S} H^1(k(T), \mathbb{Z}/\ell) \\
& 0 = H^2_{\mathrm{nr}}(K/k, \mu_\ell) \arrow[r] & H^2(K, \mu_\ell) \arrow[r, "\oplus\partial^2_C"] \arrow[u, "\iota"] & \bigoplus_{C \in S^{(1)}} H^1(k(C), \mathbb{Z}/\ell) \arrow[u, "\tau"]
\end{tikzcd} \tag{3.2}$$

where $X^{(1)}_S$ is the set of all prime divisors in X that do not dominate the base S, and $S^{(1)}$ is the set of all prime divisors (i.e., integral curves) in S. The maps ∂^2_T and ∂^2_C are the residue maps. The map ι is the usual restriction map in Galois cohomology associated to the field extension $k(X)/K$. When $X|_C$ is generically reduced over C and $T \subset X|_C$ is a component (resp. when T is the reduced subscheme of a component of $X|_C$), the map τ is the usual restriction map in Galois cohomology for the field extensions $k(T)/k(C)$ (resp. multiplied by the multiplicity).

Proof. The upper row is exact by definition. The bottom row coincides with the usual Bloch–Ogus complex for degree 2 étale cohomology associated to S. The ith cohomology group of this complex is computed by the Zariski cohomology $H^i(S, \mathscr{H}^2)$ of the sheaf $\mathscr{H}^i$, which is the sheafification of the Zariski presheaf $U \mapsto H^2_{\text{ét}}(U, \mu_\ell)$, see [6, Theorem 6.1]. In particular, the lower row is exact in the first two places because $H^0(S, \mathscr{H}^2) = \mathrm{Br}(S)[\ell] = 0$ and $H^1(S, \mathscr{H}^2) \subset H^3_{\text{ét}}(S, \mathbb{Z}/\ell) = 0$ since S is a smooth projective rational surface, where the later inclusion arises from the sequence of low terms associated to the Bloch–Ogus spectral sequence $H^i(S, \mathscr{H}^j) \implies H^{i+j}_{\text{ét}}(S, \mu_\ell)$. For the commutativity of the right square see [14, p. 143]. □

If we want to understand classes $\xi \in H^2(K, \mu_\ell)$ such that $\iota(\xi) \in H^2_{\mathrm{nr}}(k(X)/X, \mu_\ell)$, then $\partial^2_T \iota(\xi) = 0$ for all T as in the diagram. By the commutativity of the central square in the diagram (3.2), we see that then $\tau(\partial^2_C \xi) = 0$ for all integral curves C

in S. Thus we are interested in the kernel $\ker(\tau)$, which we call $\mathcal{K}_\ell$. Clearly, $\mathcal{K}_\ell$ is the direct sum of the kernels of all restriction maps $\tau : H^1(k(C), \mathbb{Z}/\ell) \to H^1(k(T), \mathbb{Z}/\ell)$. The following well known lemma becomes relevant (see [30, §3.2.2, Corollary 2.14(d)]).

Lemma 8. *Let κ be a field and let ℓ be a prime number. Let Z be an integral κ-variety with function field F. Let l be the separable closure of κ inside F, i.e., the field extension of κ determined by the finite Galois set of irreducible components of $Z \times_\kappa \kappa^s$. Then the kernel of the restriction map $H^1(\kappa, \mathbb{Z}/\ell) \to H^1(F, \mathbb{Z}/\ell)$ coincides with the kernel of the restriction map $H^1(\kappa, \mathbb{Z}/\ell) \to H^1(l, \mathbb{Z}/\ell)$. In particular, if Z is geometrically integral, then the restriction map $H^1(\kappa, \mathbb{Z}/\ell) \to H^1(F, \mathbb{Z}/\ell)$ is injective.*

Recall that since the generic fiber X_K of the cubic surface fibration $\pi : X \to S$ is smooth, the locus in S over which the fibers become singular is the support of a divisor, the discriminant divisor $\Delta \subset S$. Thus if C is not contained in the discriminant divisor, then $T = X|_C \to C$ is a generically smooth cubic surface fibration (in particular, there is a unique codimension 1 point of X above the generic point of C), and Lemma 8 implies that τ is injective on the part of the direct sum corresponding to $T \to C$. Hence, we are only interested in components of the discriminant divisor $\Delta = \cup \Delta_i$. However, for a general cubic surface fibration, we expect that the generic fiber above a reduced component of the discriminant has only isolated singularities, in particular is integral.

Let C be a component of the discriminant. When T is the reduced subscheme of a component of $X|_C$ with multiplicity $e > 1$, then the map τ in diagram (3.2) will be zero if ℓ divides e. Now assume that the generic fiber of $X|_C \to C$ is reduced and admits an irreducible component that is not geometrically integral. Considering the possible degenerations of a cubic surface in $\mathbb{P}^3$, we see that the only possible cases are when the geometric generic fiber above C is a union of three planes or a union of a plane and an irreducible quadric. In the first case, the finite Galois set of connected components of the geometric generic fiber of $X|_C \to C$ is isomorphic to the spectrum of an étale $k(C)$-algebra of degree 3.

Lemma 9. *Let κ be a field and l/κ a finite separable extension of degree $n = 2, 3$. Let ℓ be a prime number and write $\mathcal{K}_\ell = \ker(H^1(\kappa, \mathbb{Z}/\ell) \to H^1(l, \mathbb{Z}/\ell))$. Then $\mathcal{K}_\ell = 0$ whenever $\ell \neq n$ or when $\ell = n = 3$ and l/κ is not Galois. Otherwise, when $n = \ell$ and l/κ is cyclic, then $\mathcal{K}_\ell$ is generated by the class $[l/\kappa] \in H^1(\kappa, \mathbb{Z}/\ell)$.*

Proof. If l/κ is a G-Galois extension of fields, then the exact sequence of low degree terms of the Hochschild–Serre spectral sequence implies that the kernel of the restriction map $H^1(\kappa, \mathbb{Z}/\ell) \to H^1(l, \mathbb{Z}/\ell)$ is isomorphic to the group cohomology $H^1(G, \mathbb{Z}/\ell)$. When $n = \ell$ and l/κ is cyclic, this shows that the kernel of restriction is cyclic of order ℓ, and since we already know that the class $[l/\kappa]$ becomes trivial, it is a generator.

Now consider the case when l/κ is cubic and not Galois, in which case the normal closure N/κ of l/κ is an S_3-Galois extension. Hence the kernel of the composition of restriction maps $H^1(\kappa, \mathbb{Z}/\ell) \to H^1(l, \mathbb{Z}/\ell) \to H^1(N, \mathbb{Z}/\ell)$ is isomorphic to the group cohomology $H^1(S_3, \mathbb{Z}/\ell)$, which has order 2 when $\ell = 2$ and is trivial for all primes $\ell \neq 2$. In the case $\ell = 2$, we know that N/l is degree 2 so that the kernel of the restriction map $H^1(l, \mathbb{Z}/2) \to H^1(N, \mathbb{Z}/2)$ already has order 2. Hence the map $H^1(\kappa, \mathbb{Z}/2) \to H^1(l, \mathbb{Z}/2)$ is injective. $\square$

We combine these considerations to arrive at a sufficient condition for the triviality of $\mathrm{im}(\mathrm{Br}(K) \to \mathrm{Br}(k(X))) \cap \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$.

Theorem 10. *Let $\pi : X \to S$ be a flat cubic surface bundle over a smooth projective connected rational surface S over an algebraically closed field k of characteristic not dividing 6. Assume that the fibers of π over all generic points of irreducible curves $C \subset S$ are reduced. Then $\mathrm{im}(\mathrm{Br}(K)[2] \to \mathrm{Br}(k(X)) \cap \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ is trivial.*

If we furthermore assume that the fibers of π over all generic points of irreducible curves $C \subset S$ are never geometrically the union of three planes permuted cyclically by Galois, then $\mathrm{im}(\mathrm{Br}(K)[3] \to \mathrm{Br}(k(X)) \cap \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ is trivial.

Proof. Let $\alpha \in \mathrm{Br}(K)$ be a nontrivial element, then there is an irreducible curve $C \subset S$ such that the residue of α at C is nontrivial. We argue that there is always an irreducible component $T \subset X|_C$ of the fiber of π above C such that the map $\tau : H^1(k(C), \mathbb{Z}/\ell) \to H^1(k(T), \mathbb{Z}/\ell)$, for either $\ell = 2$ in the first case or $\ell = 3$ in the second case, is injective. Hence $\iota(\alpha)$ will have a nontrivial residue at a valuation of $k(X)$ centered on $k(T)$, so $\iota(\alpha)$ is not in $\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$. Indeed, if the generic fiber of $X|_C \to C$ is geometrically irreducible, the injectivity follows from Lemma 8; if this generic fiber is not geometrically irreducible, then by the discussion above, there are only the following possible cases. Either the generic fiber of $X|_C \to C$ has an irreducible component that is a plane; in this case, we take T equal to this component and again apply Lemma 8. Or, the generic fiber of $X|_C \to C$ is irreducible and is geometrically the union of three planes; then by assumption, we are in the situation of Lemmas 8 and 9, and we can take $T = X|_C$. $\square$

For an application of Theorem 10, see Remark 14.

4. Quadric surface bundles with polygonal discriminant

In this section, we construct new reference quadric surface bundle fourfolds.

4.1. **Polygonal discriminant.** Let $C \subset \mathbb{P}^2$ be a conic and for $m \geq 3$ let P_m be a regular m-gon on lines in $\mathbb{P}^2$, such that C is inscribed in P_m, all defined over an algebraically closed field k of characteristic zero. Let $L_1, \ldots, L_m \subset \mathbb{P}^2$ be the lines corresponding to the sides of the m-gon and let $\ell_1, \ldots, \ell_m$ be linear forms defining these lines. To make the notations consistent with [24, 25], we assume that the conic C is defined by the vanishing of

$$F(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

where $(x : y : z)$ are homogeneous coordinates on $\mathbb{P}^2$. Let a, b, c, d be arbitrary products of some of the linear forms $\ell_1, \ldots, \ell_m$, or constant, satisfying the following conditions:

$$
\begin{aligned}
& abcd = (\ell_1 \cdots \ell_m)^2, \\
& \text{the degrees of } a, b, c, d \text{ are either all even or all odd,} \\
& a, b, c, d \text{ are square free and mutually distinct,} \\
& \text{at most one of } a, b, c \text{ is constant.}
\end{aligned}
\tag{4.1}
$$

Write d_0, d_1, d_2, d_3 for the degrees of a, b, c, dF, respectively, and write $d_i = 2d_i' + e$ where e is either 0 or 1. Letting

$$\mathscr{E} = \mathscr{O}_{\mathbb{P}^2}(-d_0') \oplus \mathscr{O}_{\mathbb{P}^2}(-d_1') \oplus \mathscr{O}_{\mathbb{P}^2}(-d_2') \oplus \mathscr{O}_{\mathbb{P}^2}(-d_3')$$

we define a quadratic form $q : \mathscr{E} \to \mathscr{O}_{\mathbb{P}^2}(e)$ by

$$q = as^2 + bt^2 + cu^2 + dFv^2 \tag{4.2}$$

on a local section (s, t, u, v) of $\mathscr{E}$. Viewing q as a global section of $S^2(\mathscr{E}^\vee) \otimes \mathscr{O}_{\mathbb{P}^2}(e)$, its vanishing defines a subvariety $X_n \subset \mathbb{P}(\mathscr{E})$, where we write $n = m + 1$. The restriction $X_n \to \mathbb{P}^2$ of the projective bundle projection is then a quadric surface bundle over $\mathbb{P}^2$ in a weak sense (i.e., no flatness condition is assumed, see [36, 37]). The discriminant of $X_n \to \mathbb{P}^2$ is then a curve of degree $2n$, the union of a double regular $(n-1)$-gon and a smooth inscribed conic.

While the quadric surface bundle fourfolds X_n are not smooth, in §6 we verify that they admit universally CH_0-trivial resolutions.

4.2. **Second unramified cohomology group.** In this section, we show that the quadric bundle fourfolds constructed in §4.1 have nontrivial unramified Brauer group, which together with the fact that they admit universally CH_0-trivial resolutions (see Theorem 16), completes the proof that they are reference varieties.

Theorem 11. *Let X_n be defined as in (4.2). Then*

$$H^2_{\mathrm{nr}}(k(X_n)/k, \mathbb{Z}/2) \neq 0.$$

Proof. We apply [34, Theorem 3.17]. The generic fiber of the quadric bundle X_n is given by a diagonal quadratic form $q \simeq \langle a, b, c, dF \rangle$. The Clifford invariant of q is given by (a dehomogenization)

$$\alpha = c(q) = (a, b) + (c, dF) + (ab, cdF)$$

over $k(\mathbb{P}^2)$. From the construction, since the conic C is tangent to the lines $L_1, \ldots, L_m$, we deduce that the residue $\partial^2_C(\alpha)$ at the generic point of C is trivial. Hence the ramification divisor $ram\, \alpha$ is supported on the union of lines $L_1 \cup \cdots \cup L_m$, which is a simple normal crossings divisor. Hence *loc. cit.* applies and we deduce that $H^2_{\mathrm{nr}}(k(X_n)/k, \mathbb{Z}/2) \neq 0$. More precisely, the image α' of α in $H^2(k(X_n), \mathbb{Z}/2)$ is a nontrivial element of the group $H^2_{\mathrm{nr}}(k(X_n)/k, \mathbb{Z}/2)$: with the notations in [34, Theorem 3.17], we have $T = ram\, \alpha$, and the diagonal class $(1, \ldots, 1)$, corresponding to the set of residues of α' gives an element of the group H. ∎

Finally, we mention that the reference varieties X_n can be used to obtain the following special case of a result of Schreieder [36, Corollary 2] via the specialization method. Consider a tuple (d_0, d_1, d_2, d_3) of degrees, all of the same parity, and a symmetric 4×4 matrix $A = (a_{ij})$ consisting of homogeneous forms a_{ij} on $\mathbb{P}^2$ of degree $(d_i + d_j)/2$. Fixing $e \in \mathbb{Z}$ of the same parity as the degrees, write $d_i = 2d_i' + e$, and let $\mathscr{E} = \bigoplus_{i=0}^{3} \mathscr{O}_{\mathbb{P}^2}(-d_i')$. Then as before, A defines a quadratic form $q : \mathscr{E} \to \mathscr{O}_{\mathbb{P}^2}(e)$ and thus a (weak) quadric surface bundle $X \to \mathbb{P}^2$, which is called of *graded-free type* (d_0, d_1, d_2, d_3). As explained in [37, Section 3.5], if X is smooth, its deformation type only depends on the tuple (d_0, d_1, d_2, d_3). For a fixed $n \geq 4$, the construction in §4.1 produces a reference variety that is a weak quadric surface bundle $X_n \to \mathbb{P}^2$ of any graded-free type (d_0, d_1, d_2, d_3) with $d_0 + d_1 + d_2 + d_3 = 2n$, at most one of the d_i allowed to be zero, and the following additional restrictions: $d_i \leq n - 1$ for $i = 0, 1, 2$, and $2 \leq d_3 \leq n + 1$. The specialization method then shows that the very general quadric surface bundle fourfold of any such graded-free type is not stably rational. Schreieder [36, Corollary 2] obtains this result with the additional restrictions on the individual degrees removed. On the other hand, the reference varieties constructed here may be of independent interest in other applications.

5. A cubic surface bundle with nontrivial unramified Brauer group

In this section, we construct an irreducible reference hypersurface Y of bidegree $(2, 3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ over an algebraically closed field k of characteristic zero, thereby giving another proof of Theorem 1. As a consequence, we also arrive at an explicit example of a smooth cubic surface X over $K = k(\mathbb{P}^2)$ such that the cokernel of the map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ contains a nontrivial 2-torsion class that is unramified on a fourfold model of X over k.

Consider the following varieties:

- The weak quadric surface bundle $V \to \mathbb{P}^2$ of graded-free type $(3, 3, 1, 3)$ with polygonal discriminant defined by

$$u(u-v)(t-v)x^2 + u(u-v)ty^2 + tw^2 + (t-v)F(u,t,v)z^2 = 0 \tag{5.1}$$

  where $\mathbb{P}^2$ has homogeneous coordinates $(u : v : t)$. Here, the polygon is the square whose sides are defined by the four linear forms $\ell_1 = u$, $\ell_2 = u - v$, $\ell_3 = t$, and $\ell_4 = t - v$ and where the inscribed conic is defined by the form

$$F(u, t, v) = 4u^2 + 4t^2 + v^2 - 4uv - 4tv.$$

- The hypersurface Y of bidegree $(2, 3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ defined by

$$u(u-v)(t-v)x^2 + u(u-v)ty^2 + tw^2z^2 + (t-v)F(u,t,v)z^2 = 0. \tag{5.2}$$

  where here, $\mathbb{P}^2$ has homogeneous coordinates $(x : y : z)$ and $\mathbb{P}^3$ has homogeneous coordinates $(u : t : v : w)$.

Note that V is birational to Y since the open subset of V defined by $z = 1, u = 1$ and the open subset of Y defined by $z = 1, u = 1$ are given by the same affine equation in the variables v, t, w, x, y.

We need the following technical result, which will be proved in §5.1.

Proposition 12. *Let V and Y be defined as in (5.1) and (5.2), respectively. Then there is a proper birational morphism $\pi : Y' \to Y$ such that:*

a) the rational map $Y \dashrightarrow V$ extends to a morphism $f : Y' \to V$;
b) the maps π and f are universally CH_0-trivial.

With this result, we are ready to give our second proof of Theorem 1.

Proof of Theorem 1. By the specialization method [40, 16], it is enough to insure that Y is a reference variety, i.e., that the conditions (O) and (R) as in the introduction are satisfied for Y. Since Y is birational to V, we have by Theorem 11 that $H^2_{\mathrm{nr}}(k(Y)/k, \mathbb{Z}/2) = H^2_{\mathrm{nr}}(k(V)/k, \mathbb{Z}/2) \neq 0$. Let Y' be as in Proposition 12. By Theorem 16 below, there exists a CH_0-trivial resolution $\widetilde{V} \to V$. Let $\widetilde{Y} \to Y'$ be a resolution such that the rational map $Y' \dashrightarrow \widetilde{V}$ extends to a morphism $\widetilde{Y} \to \widetilde{V}$. Since $\widetilde{Y}$ and $\widetilde{V}$ are smooth, the map $\widetilde{Y} \to \widetilde{V}$ is universally CH_0-trivial. This follows from weak factorization and the fact that the blow up of a smooth variety along a smooth center is a universally CH_0-trivial morphism. We then have the following diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[d] \arrow[dr] & \\ Y' \arrow[d] \arrow[dr] & \widetilde{V} \arrow[d] \\ Y & V, \end{tikzcd}$$

where we know that all the maps, except possibly $\widetilde{Y} \to Y'$, are universally CH_0-trivial. From this diagram, we see that the map $\widetilde{Y} \to Y'$ is universally CH_0-trivial, hence the composition $\widetilde{Y} \to Y' \to Y$ is a universally CH_0-trivial resolution. Finally, Y is a reference variety. $\square$

Remark 13. Note that if ϕ is the composition $\phi : Y \dashrightarrow V \to \mathbb{P}^2$ and $U \subset Y$ is an open where ϕ is defined, then the generic fiber of the map $\phi : U \to \mathbb{P}^2$ is not proper, so that we could not directly apply [36, Theorem 9].

Remark 14. A nice feature of the reference variety Y is that the cubic surface bundle $Y \to \mathbb{P}^2$ has as generic fiber $X = Y_K$ an explicit example of a smooth cubic surface over $K = k(\mathbb{P}^2)$ with a nontrivial element $\alpha \in \mathrm{Br}(X)[2] \neq 0$ that is not contained in the image of the map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ and $\alpha \in \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ is globally unramified on the function field of the fourfold Y. The fact that α is unramified follows from Theorem 11. To prove that α is nonconstant, we explicitly compute that the discriminant of the cubic surface fibration $Y \to \mathbb{P}^2$ is the union of the line $\{x = 0\}$ with multiplicity 4, the line $\{y = 0\}$ with multiplicity 4, the line $\{z = 0\}$ with multiplicity 30, the pair of conjugate lines $\{x^2 + z^2 = 0\}$ with multiplicity 6, the smooth conic $\{x^2 - y^2 + z^2 = 0\}$ with multiplicity 4, and the integral sextic defined by

$$x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + x^4z^2 - 8y^4z^2 + 16y^2z^4 + 20x^2y^2z^2 = 0.$$

For example, one could use Magma's `ClebschSalmonInvariants` command [8]. Then, over each component besides $\{z = 0\}$, one can check that the generic fiber is geometrically integral, while over the component $\{z = 0\}$, the fiber is the union of three planes defined over the residue field of the line $\{z = 0\}$, hence Theorem 10 applies and no constant Brauer class is unramified on X.

5.1. **Birational transformation.** In this section, we prove Proposition 12. We have the following natural rational map $Y \dashrightarrow V$

$$(u : v : t : w, x : y : z) \mapsto (u : v : t, x : y : wz : z). \tag{5.3}$$

This map is not defined on the locus $u = v = t = 0$; on the complement of this locus it is not an isomorphism along $z = 0$.

We construct Y' as a composition of two blow ups:

a) we consider the blow up $Y_1 \to Y$ of the locus $u = v = t = z = 0$,

b) we consider the blow up $Y' \to Y_1$ of the locus $u_1 = v_1 = t_1 = 0$ in the chart corresponding to the exceptional divisor defined by $z = 0$ (see below).

5.2. **First blow up.** We give equations of the blowup in each chart. Note that around the exceptional divisor we always have $w = 1$, and $x = 1$ or $y = 1$.

a) In the chart $v = uv_1, t = ut_1, z = uz_1$, the exceptional divisor is $u = 0$, and the blowup is given by

$$(1 - v_1)(t_1 - v_1)x^2 + (1 - v_1)t_1y^2 + t_1w^2z_1^2 + (t_1 - v_1)F(1, t_1, v_1)u^2z_1^2 = 0.$$

We extend the map (5.3) to the map $Y_1 \dashrightarrow V$ defined by

$$(u, v_1, t_1, w\ ;\ x : y, z_1) \mapsto (1 : v_1 : t_1, x : y : wz_1 : uz_1), \tag{5.4}$$

which is everywhere defined on this chart. Here, we mean that around the exceptional divisor, we have $w = 1$ and $x = 1$ or $y = 1$. The fibers of this map are either points or an $\mathbb{A}^1$ (if $uz_1 = 0$). The fiber of the map $Y_1 \to Y$ over the point $(0 : 0 : 0 : 1, x_0 : y_0 : 0)$ is given by

$$(1 - v_1)(t_1 - v_1)x_0^2 + (1 - v_1)t_1y_0^2 + t_1z_1^2 = 0. \tag{5.5}$$

This defines a (singular) cubic surface. By Lemma 15 below, this cubic surface is universally CH_0-trivial. We deduce that the map $Y_1 \to Y$ is universally CH_0-trivial over this chart.

b) In the chart $u = tu_1, v = tv_1, z = tz_1$, the exceptional divisor is $t = 0$, and the blowup is given by

$$u_1(u_1 - v_1)(1 - v_1)x^2 + u_1(u_1 - v_1)y^2 + w^2z_1^2 + (1 - v_1)F(u_1, 1, v_1)t^2z_1^2 = 0.$$

Similar as in the previous case, we define the map $Y_1 \dashrightarrow V$ by

$$(u_1, v_1, t, w\ ;\ x : y, z_1) \mapsto (u_1 : v_1 : 1, x : y : wz_1 : tz_1), \tag{5.6}$$

which is everywhere defined on this chart. The fibers of this map are either points or an $\mathbb{A}^1$. On the intersection of the charts, the maps (5.4) and (5.6) coincide. The universal CH_0-triviality of the fibers of the map $Y_1 \to Y$ follows from Lemma 15.

c) The chart with exceptional divisor $v = 0$ is similar.

d) In the chart $u = zu_1, v = zv_1, t = zt_1$, the exceptional divisor is $z = 0$, and the blowup is given by

$$u_1(u_1 - v_1)(t_1 - v_1)x^2 + u_1(u_1 - v_1)t_1y^2 + t_1w^2 + (t_1 - v_1)F(u_1, t_1, v_1)z^2 = 0. \tag{5.7}$$

We define the map $Y_1 \dashrightarrow V$ by

$$(u_1, v_1, t_1, w\ ;\ x : y, z) \mapsto (u_1 : v_1 : t_1, x : y : w : z), \tag{5.8}$$

which is defined everywhere on this chart, except at the locus $u_1 = v_1 = t_1 = 0$. On the domain of definition, the fibers are points or lines. The universal CH_0-triviality of the fibers of the map $Y_1 \to Y$ follows from Lemma 15.

Lemma 15. *Let k be a field, $a, b \in k$, and $S \subset \mathbb{P}^3_k$ be the cubic surface defined by*

$$au(u - v)(t - v) + bu(u - v)t + tz^2 = 0. \tag{5.9}$$

If $a \neq 0$, then S is rational.

- *(i) If $a \neq 0$ and $(a + b) \neq 0$, then S has three isolated singular points. The blowup $\widetilde{S} \to S$ at the singular points is smooth and the exceptional divisors are smooth and rational.*
- *(ii) If $a = 1, b = -1$, the cubic surface (5.9) has a unique singular point and it has a universally CH_0-trivial resolution $\widetilde{S} \to S$, given by successive blow ups over this point.*
- *(iii) If $a = 0$, $b = 1$, the cubic surface (5.9) is a union of a plane $t = 0$ and a rational quadric surface $u(u - v) + z^2 = 0$.*

Note that in the cases *(i)* and *(ii)*, since $\widetilde{S}$ is smooth and rational, it is universally CH_0-trivial. The lemma implies that the map $\widetilde{S} \to S$ is universally CH_0-trivial, hence S is universally CH_0-trivial as well. In part *(iii)*, we have that S is universally CH_0-trivial as well from its description.

Proof. The rational parameterization of S is given by projection from the point $z = u = v = 0$.

Part *(iii)* is straightforward. For part *(i)*, by direct computation we obtain the following description of the singular locus:

- $z = u = v = 0$;
- $z = 0, u = 0, av - (a + b)t = 0$;
- $z = 0, u - v = 0, av - (a + b)t = 0$.

We consider the exceptional divisor of the blowup of the first point. The other cases are similar, up to a linear change of variables. Put $c = a + b$. We write the equation of S, in the open chart $t = 1$, as

$$u(u - v)(c - av) + z^2 = 0.$$

We then have the following charts for the blowup of $u = v = z = 0$:

- In the chart $u = zu_1, v = zv_1$, the blowup $u_1(u_1 - v_1)(c - azv_1) + 1 = 0$ is smooth, and the exceptional divisor $u_1(u_1 - v_1)c + 1 = 0$ is smooth and rational.

- In the chart $z = uz_1, v = uv_1$, the blowup $(1-v_1)(c-auv_1)+z_1^2 = 0$ is smooth, and the exceptional divisor $(1-v_1)c+z_1^2 = 0$ is smooth and rational.
- In the chart $z = vz_1, u = vu_1$, the blowup $u_1(u_1-1)(c-av)+z_1^2 = 0$ is smooth, and the exceptional divisor $(1-u_1)c+z_1^2 = 0$ is smooth and rational.

In the part *(ii)*, the unique singular point of S is given by $z = u = v = 0$. We have the same equations as above, with $c = 0, a = 1$. We have additional double point singularities, in the second and the third chart (for example, the point $z_1 = u = v_1 = 0$ in the second chart $z_1^2 - uv_1(1-v_1) = 0$) which are resolved by one single blowup, and then the exceptional divisor is smooth and rational. □

5.3. **Second blowup.** We consider the chart (5.7) and we blow up the locus $u_1 = v_1 = t_1 = 0$:

- In the chart $v_1 = u_1v_2, t_1 = u_1t_2$, the blowup is given by

$$u_1^2(1-v_2)(t_2-v_2)x^2 + u_1^2(1-v_2)t_2y^2 + t_2w^2 + (t_2-v_2)F(1,v_2,t_2)u_1^2z_1^2 = 0.$$

We consider the map $Y' \to V$ defined by

$$(u_1, v_2, t_2, w\,;\, x:y, z_1) \mapsto (1:v_2:t_2, u_1x:u_1y:w:u_1z_1), \tag{5.10}$$

which is everywhere defined ($w \neq 0$). The fibers of the map are points or affine spaces. The fibers of the blow up $Y' \to Y_1$ on this chart are also either points or affine spaces.
- The analysis of the other two charts is similar.

6. Appendix: Analysis of singularities

The main goal of this section is to prove the following result.

Theorem 16. *Let X_n be as defined in (4.2). Then X_n admits a universally CH_0-trivial resolution $\widetilde{X}_n \to X_n$.*

The following lemma is known:

Lemma 17. *Let X be a proper variety over a field k of characteristic zero. If X admits a universally CH_0-trivial resolution $\widetilde{X} \to X$, then any resolution $X' \to X$ is universally CH_0-trivial.*

Proof. We note that by resolution of singularities, there is a smooth projective variety $\widetilde{X}'$ together with birational morphisms $\widetilde{X}' \to X'$ and $\widetilde{X}' \to \widetilde{X}$. Then, it is enough to observe that any birational morphism of smooth projective varieties over k is universally CH_0-trivial. For this, we use weak factorization, the fact that a blowup with smooth center is a universally CH_0-trivial morphism, and the fact that, by definition, a composition of two universally CH_0-trivial maps is universally CH_0-trivial. □

By [16, Proposition 1.8] we have the following criterion: a proper morphism $\widetilde{X} \to X$ is universally CH_0-trivial if for any scheme-theoretic point $P \in X$, the fiber $\widetilde{X}_P$ is a universally CH_0-trivial variety over the residue field $\kappa(P)$. Using this criterion, in order to prove that a sequence of blowups of a variety X provides a universally CH_0-trivial resolution $f : \widetilde{X} \to X$, it is enough to work formally locally on X. Indeed, if $\widehat{\mathscr{O}}_{X,P}$ is the completion of the local ring $\mathscr{O}_{X,P}$, the fiber of the induced map $\widetilde{X} \times_X \operatorname{Spec} \widehat{\mathscr{O}}_{X,P} \to \operatorname{Spec} \widehat{\mathscr{O}}_{X,P}$ at the closed point of $\operatorname{Spec} \widehat{\mathscr{O}}_{X,P}$ is $\widetilde{X}_P$.

We now analyze different types of singularities that could appear for X_n.

6.1. **Equations defining singular locus.** By symmetry, we may work over an open chart $z \neq 0$ of $\mathbb{P}^2$. Then we have the following equations defining the singular locus:

$$
\begin{aligned}
&as = bt = cu = dFv = 0, \\
&\frac{\partial a}{\partial x}s^2 + \frac{\partial b}{\partial x}t^2 + \frac{\partial c}{\partial x}u^2 + \frac{\partial dF}{\partial x}v^2 = 0,\ \frac{\partial a}{\partial y}s^2 + \frac{\partial b}{\partial y}t^2 + \frac{\partial c}{\partial y}u^2 + \frac{\partial dF}{\partial y}v^2 = 0.
\end{aligned}
\tag{6.1}
$$

Note that if $P = ((x, y), [s : t : u : v]) \in X_n$ satisfies that $(x, y) \notin D := L_1 \cup \cdots \cup L_m \cup C$, then P is a smooth point. Indeed, we then have $abcdF(P) \neq 0$ and the conditions above imply that

$$s = t = u = v = 0,$$

which is not possible since s, t, u, v are projective coordinates.

Also, if $P \in C$, but not on any line $L_1, \dots, L_m$, the conditions (6.1) imply that

$$s = t = u = 0, \quad v = 1, \quad F = 0, \quad \frac{\partial dF}{\partial x}(P) = \frac{\partial dF}{\partial y}(P) = 0,$$

which is impossible since the conic C is smooth.

Hence we need to analyze the following four types of singularities:

- over the generic point of lines L_i;
- over the intersection points $L_i \cap L_j$;
- over the tangency point of C and L_i;
- over closed points of lines L_i, that are not on other lines or on the conic C.

Note that by [24, 25], a universally CH_0-trivial resolution exists in the following cases:

$$
\begin{aligned}
&a = yz, b = xy, c = xz, d = 1; \\
&a = 1, b = xy, c = xz, d = yz.
\end{aligned}
\tag{6.2}
$$

In the arguments below, up to a linear change of variables x and y, we may assume that $\ell_i = x$ and $\ell_j = y$.

The analysis below provide the following global description of singularities:

a) the curves C_i (6.4), (6.6) over the lines L_i, some of these curves are singular at a point P_i (6.12) over the tangency point of C and L_i;
b) singular lines (6.7) and (6.10), over the intersection points of L_i and L_j;
c) the curves D_i (6.14) over the tangency points of C et L_i.

We consider the map

$$X'_n \to X_n \tag{6.3}$$

given by successive blow ups of the singular locus in the following order:

- we blow up lines (6.7);

- we blow up the points P_i and then we blow up the exceptional divisors over P_i,
- we blow up of the proper transform of C_1, the proper transform of $C_2, \dots$ and then proper transform of C_m,
- we blow up successively the proper transforms of lines (6.10),
- finally, we blow up the proper transforms of the curves D_i.

We claim that the only singularities of the variety X_n' are over some intersection points of L_i and L_j, the blowup $\widetilde{X}_n$ of these singularities is smooth, and the resulting map $\widetilde{X}_n \to X_n' \to X_n$ is a universally CH_0-trivial resolution.

6.2. **Singularities over lines L_i.** We have two cases to consider:

a) $\ell_i = x$ divides precisely two among the coefficients a, b, c, by symmetry we may assume that $x \,|\, a, b$, we then write $a = xa_1, b = xb_1$;

b) x divides d, by symmetry, we may assume that $x \,|\, a, d$ and we write $d = xd_1, a = xa_1$.

Then the analysis of the singular locus in each case is as follows:

a) The equations (6.1) imply $u = v = 0, a_1 s^2 + b_1 t^2 = 0$. Let λ be the common factor of a_1 and b_1: λ is the product of (some of) lines $\ell_1, \dots, \ell_m$ and $a_1 = \lambda a_2, b_1 = \lambda b_2$. We obtain a curve $C_i \subset X_n^{sing}$ in the singular locus X_n^{sing} of X_n:

$$x = 0, u = v = 0, a_2 s^2 + b_2 t^2 = 0. \tag{6.4}$$

We claim that the blowup of the curve C_i is smooth at any point of the exceptional divisor that is not over a point of C or a point on another line, and that the corresponding fibers are universally CH_0-trivial.

For the fibers over a point $P \in X_n$ lying over a closed point $Q \in L_i$, we work over the local ring $\widehat{\mathscr{O}}_{X_n,P}$. Since the residue field of Q is the field of complex numbers, any element of $\widehat{\mathscr{O}}_{\mathbb{P}^2,Q}$, which does not vanish at Q, is a square in $\widehat{\mathscr{O}}_{\mathbb{P}^2,Q}$, and hence in $\widehat{\mathscr{O}}_{X_n,P}$. We then obtain the following formal equation:

$$xs^2 + xt^2 + u^2 + v^2 = 0, \tag{6.5}$$

the singularity is defined by $u = v = 0, x = 0, s^2 + t^2 = 0$. This type of singularity has been already treated in [24] (compare with equations (6.2)), it is resolved with one blow up, and the corresponding fiber is universally CH_0-trivial.

For the fiber over the generic point of C_i, it is enough to consider the following charts of the blowup:

(a) $a_2 s^2 + b_2 t^2 = xw, u = xu_1, v = xv_1$.

The blowup is given by the conditions

$$\lambda w + cu_1^2 + dFv_1^2 = 0, a_2 s^2 + b_2 t^2 = xw$$

and the exceptional divisor is defined by

$$x = 0, \lambda w + cu_1^2 + Fdv_1^2 = 0, a_2 s^2 + b_2 t^2 = 0$$

This variety is smooth and rational over the generic point of C_i since $\frac{b_2}{a_2}$ is a square at the generic point of C_i.

(b) $a_2s^2 + b_2t^2 = uw, x = ux_1, v = uv_1$.
The blowup is given by the conditions

$$\lambda x_1 w + c + dFv_1^2 = 0, a_2s^2 + b_2t^2 = uw,$$

it is smooth since a_2, b_2, λ and c do not vanish at the generic point of L_i and $s \neq 0$ or $t \neq 0$. The exceptional divisor is also smooth and rational, defined by $u = 0$.
The chart with the exceptional divisor defined by $v = 0$ is similar.

(c) $x = (a_2s^2 + b_2t^2)x_1, u = (a_2s^2 + b_2t^2)u_1, v = (a_2s^2 + b_2t^2)v_1$.
The blowup is given by the condition

$$\lambda x_1 + cu_1^2 + dFv_1^2 = 0,$$

it is smooth. The exceptional divisor is smooth and rational, defined by

$$\lambda x_1 + cu^2 + dFv^2 = 0, a_2s^2 + b_2t^2 = 0.$$

b) Similarly as in the previous case, let λ be the common factor of a_1 and d_1 and $a_1 = \lambda a_2, d_1 = \lambda d_2$. We obtain the equation of the singular locus:

$$x = 0, t = u = 0, a_2s^2 + d_2Fv^2 = 0. \tag{6.6}$$

The formal equation at closed points are the same as the equations in the previous case (6.5). For the generic fiber, we consider the following chart of the blowup

$$a_2s^2 + d_2Fv^2 = xw, t = xt_1, u = xu_1.$$

The blowup is given by the conditions

$$\lambda w + bt_1^2 + cu_1^2 = 0, a_2s^2 + d_2Fv^2 = xw$$

and the exceptional divisor is defined by

$$x = 0, \lambda w + bt_1^2 + cu_1^2 = 0, a_2s^2 + d_2Fv^2 = 0$$

This variety is smooth and rational over the generic point of C_i since $\frac{d_2F}{a_2}$ is a square at the generic point of C_i.

The analysis of the other charts is similar to the previous case.

6.3. **Singularities over intersection points** $L_i \cap L_j$**.** Let Q be the intersection point of L_i and L_j and let P be a singular point of X_n over Q. We have the following cases to consider:

a) Only two coefficients among a, b, c, d vanish at Q. We then have the following possibilities (up to a symmetry): $xy|a, b$ or $xy|a, d$ and the corresponding singular lines are given by the (global) conditions

$$M_{ij} : x = y = u = v = 0 \text{ or } x = y = t = u = 0. \tag{6.7}$$

Again, working formally locally, we may assume that any function that does not vanish at Q is a square. Hence, in all cases, up to a symmetry, we have the following type of local equation:

$$xys^2 + xyt^2 + u^2 + v^2 = 0 \tag{6.8}$$

and the singularity is given by $x = y = u = v = 0$. Also we can change variables $s^2 + t^2 = s_1 t_1$ and consider the chart $t_1 = 1$, so that we have the following local equation:

$$xys_1 + u^2 + v^2 = 0.$$

The map $X_n' \to X_n$, restricted to $\widehat{\mathscr{O}}_{X_n,P}$ is the following composition:

- blow up of the line $x = y = u = v = 0$;
- blow up of the proper transform of $x = s_1 = u = v = 0$;
- blow up of the proper transform of $y = s_1 = u = v = 0$.

By symmetry between u and v we consider the following charts of the first blow up:

(a) $x = yx_1, u = yu_1, v = yv_1$, the equation of the blowup is $x_1 s_1 + u_1^2 + v_1^2 = 0$ and the exceptional divisor is given by $y = 0$. This blow up map is universally CH_0-trivial over this chart. Next we blow up the locus $x_1 = s_1 = u_1 = v_1 = 0$, corresponding to the product of a line y and the ordinary double point singularity, hence the second blowup is smooth, the fibers are universally CH_0-trivial. By smoothness, the third blow up is universally CH_0-trivial over this chart.

(b) $y = xy_1, u = xu_1, v = xv_1$, the equation of the blowup is $y_1 s_1 + u_1^2 + v_1^2 = 0$ and the exceptional divisor is given by $x = 0$. Next we blow up the locus $x = s_1 = u_1 = v_1 = 0$. We consider the following charts (again, using symmetry between u_1 and v_1):

   (i) $s_1 = xs_2, u_1 = xu_2, v_2 = xv_2$, the equation of the blowup is $y_1 s_2 + x(u_2^2 + v_2^2) = 0$, the exceptional divisor is $x = 0$. The fibers of the second blow up are universally CH_0-trivial. Next we blow up the locus $y_1 = s_2 = u_2 = v_2 = 0$. The charts corresponding to the exceptional divisors $y_1 = 0$ and $s_2 = 0$ are smooth, and the fibers are universally CH_0-trivial:

   - $s_2 = y_1 s_3, u_2 = y_1 u_3, v_2 = y_1 v_3$, the blowup is given by $s_3 + x(u_3^2 + v_3^2) = 0$;
   - $y_1 = s_2 y_3, u_2 = s_2 u_3, v_2 = s_2 v_3$, the blowup is given by $y_3 + x(u_3^2 + v_3^2) = 0$;

   In the chart $y_1 = u_2 y_3, s_2 = u_2 y_3, v_2 = u_2 v_3$ we have the following equation: $y_3 s_3 + x(1 + v_3^2) = 0$, the ordinary double singularities $v_3 = \pm i, x = y_3 = s_3 = 0$ are resolved after one blowup, and the exceptional divisor is rational.

   (ii) $x = s_1 x_2, u_1 = s_1 u_2, v_2 = s_1 v_2$, the equation of the blowup is $y_1 + s_1(u_2^2 + v_2^2) = 0, x = s_1 x_2$, the exceptional divisor is $s_1 = 0$, the fibers are universally CH_0-trivial. The blowup is smooth, hence the third blow up is universally CH_0-trivial over this chart.

(iii) $x = u_1x_2, s = u_1s_2, v_2 = u_1v_2$, the equation of the blowup is $y_1s_2 + u_1 + u_1v_2^2 = 0, x = u_1x_2$, the exceptional divisor is given by $u_1 = 0$. Next we blow up $y_1 = s_2 = u_1 = v_2 = 0$. Note that this chart is smooth along this locus, hence the third blow up is universally CH_0-trivial in this chart. The remaining singularity $y_1 = s_2 = u_1 = 0, v_2 = \pm i$ is resolved as at the end of the case (i) above.

(c) $x = ux_1, y = uy_1, v = uv_1$, the equation of the blow up is $x_1y_1s_1+1+v_1^2 = 0$, it is smooth and the exceptional divisor is given by $u = 0$. This blow up map is universally CH_0-trivial in this chart. Since the first blow up is smooth over this chart, the second and third blow ups are universally CH_0-trivial.

b) Only three coefficients among a, b, c, d vanish at Q. Then, we may assume that $xy\,|a, x\,|b, y\,|c$. The case when x or y divides d is similar. The singular locus over $x = y = 0$ is given by $t = u = v = 0$, hence, it is the point of the intersection of the curves C_i and C_j. The restriction of the map $X'_n \to X_n$ to $\widehat{\mathscr{O}}_{X_n,P}$ is the composition of the blow up of C_i and the proper transform of C_j. Note that over the point P we have the following formal equation:

$$xys^2 + xt^2 + yu^2 + v^2 = 0 \tag{6.9}$$

The same type of formal equation correspond to the case considered in [24] (compare with equations (6.2)), where it is showed that the two blow ups as above provide a universally CH_0-trivial resolution.

c) All coefficients a, b, c, d vanish at Q. We then assume $a = xa_1, b = xb_1, c = yc_1, d = yd_1$ and we have the following equation for X_n

$$xa_1s^2 + xb_1t^2 + yc_1u^2 + yd_1Fv^2 = 0$$

and the expression of the singular locus

$$x = y = 0, a_1s^2 + b_1t^2 = 0, c_1u^2 + d_1Fv^2 = 0 \tag{6.10}$$

Since a_1, b_1, c_1, d_1F evaluated at Q are nontrivial constants, this singular locus is the union of four lines, and the points with coordinates $s = t = 0$ or $u = v = 0$ correspond to the intersection points of these lines with the curves C_i and C_j.

We now describe the the formal equation. Changing variables, we may replace $a_1s^2 + b_1t^2$ by s_1t_1 and $c_1u^2 + d_1Ft^2$ by u_1v_1, and, by symmetry, we may consider the affine chart $t_1 = 1$. We then have the following formal equation

$$xs_1 + yu_1v_1 = 0 \tag{6.11}$$

The singular locus is the union of two lines $x = y = s_1 = u_1 = 0$ et $x = y = s_1 = v_1 = 0$. The restriction of the map $\widetilde{X}_n \to X_n$ is as follows:

(a) blow up of the locus $x = s_1 = u_1 = v_1 = 0$ (we separate two lines);

(b) blow up of the strict transform of the lines $x = y = s_1 = u_1 = 0$ and $x = y = s_1 = v_1 = 0$.

(Note that only one of the blowups of C_i and C_j is not an isomorphism for the case $t_1 = 1$ we consider). By symmetry, we have the following charts of the first blowup to consider:

(a) $s = xs_2, u = xu_2, v = xv_2$, the equation of the blowup is $s_2 + yu_2v_2 = 0$, which is smooth, the exceptional divisor is $x = 0$, it is smooth and rational; by smoothness, the second blowup is universally CH_0-trivial over this chart;

(b) $s = u_1s_2, x = u_1x_2, v = u_1v_2$, the equation of the blowup is $x_2s_2 + yv_2 = 0$, the exceptional divisor is rational, given by $u_1 = 0$; the singular locus $x_2 = s_2 = y = v_2 = 0$ is resolved by the second blowup, and the exceptional divisor is rational.

From the equations above, the fibers of the map $X'_n \to X_n$ at P are universally CH_0-trivial over this chart.

6.4. **Singularities over the tangency point of C and L_i.** Assume C is tangent to L_i at the point Q. We have the following cases:

a) d vanishes at Q. By symmetry, we assume $x|a$ and write $a = xa_1, d = xd_1$. Then the conditions (6.1) imply

$$t = u = 0,\ a_1s^2 + d_1F(Q)v^2 = 0,$$

hence we obtain a point

$$P_i : s = t = u = x = y - 1 = 0 \tag{6.12}$$

on the curve C_i in the case (6.6). The local form is as follows:

$$xs^2 + t^2 + u^2 + xFv^2 = 0,\ \text{singular locus: } s = t = u = x = y - 1 = 0 \tag{6.13}$$

This is the same type of formal equation as for the quadric bundle we considered in [25] (see equations (6.2)), by *loc. cit.*, the map (6.3) provide a universally CH_0-trivial resolution: we blow up successively the point P_i, then the exceptional divisor of the blowup and the proper transform of C_i.

b) d does not vanish at Q. We may then assume $a = xa_1, b = xb_1$, so that the conditions (6.1) imply

$$D_i : u = 0, a_1s^2 + b_1t^2 + d\frac{\partial F}{\partial x}(Q)v^2 = 0. \tag{6.14}$$

Arguing as in the previous cases, we obtain the following formal local form of the singular locus:

$$xs^2 + xt^2 + u^2 + Fv^2 = 0,\ \text{singular locus: } u = 0, s^2 + t^2 + \frac{\partial F}{\partial x}(Q)v^2 = 0. \tag{6.15}$$

This is the same type of formal equation as for the quadric bundle we considered in [24] (see equations (6.2)), by *loc. cit.*, the map (6.3) provide a universally CH_0-trivial resolution.

This finishes the proof of Theorem 16.

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