Aspects of aperiodic order: Spectral theory via dynamical systems

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Daniel Lenz
geboren am 05. 10. 1970

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Dekan: Prof. Dr. Bernd Heinrich
Gutachter: Prof. Dr. Werner Kirsch, Ruhr-Universität Bochum, Prof. Dr. Robert V. Moody, University of Alberta, Prof. Dr. Peter Stollmann, TU Chemnitz
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Structure of the work

This work has two parts. The second part consists of the following original manuscripts:

[**Le3**] D. Lenz, *Uniform ergodic theorems on subshifts over a finite alphabet*, Ergodic Theory & Dynamical Systems **22** (2002), 245–255.

[**Le4**] D. Lenz, *Singular spectrum of Lebesgue measure zero for one-dimensional quasicrystals*, Communications in Mathematical Physics **227** (2002), 129–130.

[**Le5**] D. Lenz, *Existence of non-uniform cocycles on uniquely ergodic systems*, Ann. Inst. Henri Poincaré: Prob. & Stat. **40** (2004), 197–206.

[**DL8**] D. Damanik, D. Lenz, *A condition of Boshernitzan and uniform convergence in the multiplicative ergodic theorem*, preprint 2004.

[**LS3**] D. Lenz, P. Stollmann, *Algebras of random operators associated to Delone dynamical systems*, Mathematical Physics, Analysis and Geometry **6** (2003), 269–290.

[**LS4**] D. Lenz, P. Stollmann, *An ergodic theorem for Delone dynamical systems and existence of the integrated density of states*, to appear in: Journal d' Analyse Mathématique.

[**BL1**] M. Baake, D. Lenz, *Dynamical systems on translation bounded measures and pure point diffraction*, Ergodic Theory & Dynamical Systems **24** (2004), 1867–1893.

[**BL2**] M. Baake, D. Lenz, *Deformation of Delone dynamical systems and topological conjugacy*, Journal of Fourier Analysis and Applications **11** (2005), 125–150.

All these manuscripts are concerned with spectral consequences of aperiodic order. They can be divided in three groups:

- [**Le3, Le4, Le5, DL8**] : dealing with one-dimensional systems and operators.
- [**LS3, LS4**] : dealing with higher dimensional systems and operators.
- [**BL1, BL2**] : dealing with diffraction.

A common feature in the treatment of the three topics lies in the method: The approach is based on a study of the associated dynamical systems.

The first part of this work gives an introduction into aperiodic order in general and the lines of research pursued. More precisely, a brief outline and a summary can be found in Chapter 1. The three lines of research are then discussed in Chapter 2, Chapter 3 and Chapter 4 respectively. These chapters do not contain proofs. Apart from this they are completely self contained. They provide background, necessary definitions and precise statements of the results.

Part 1

Introduction

CHAPTER 1

Aperiodic order: Some introductory remarks

Long range aperiodic order or aperiodic order for short is a specific form of weak disorder. It may be considered to mark a borderline between order and disorder. This intermediate position in the regime of (dis)ordered systems is its distinctive feature. It is responsible for its properties. So far, no precise definition of aperiodic order is known. Instead various classes of examples have been considered. For further background and recent surveys we refer the reader to [**BM1, Ja, Mo1, Se**].

In order to be more concrete and to set a perspective let us illustrate this by considering the simplest one-dimensional examples in the range of functions from the integers $\mathbb{Z}$ to $\{0,1\}$. Here, a completely ordered situation corresponds to a periodic function. A highly disordered situation is given by a typical realization of a fair coin tossing experiment. In between, there are functions such as

$$V_{I,\vartheta} : \mathbb{Z} \longrightarrow \{0,1\}, \;\; V_{I,\vartheta}(n) := \chi_I(n\vartheta \mod 1), \tag{1}$$

where χ_I is the characteristic function of an nonempty subinterval $I = [a, b)$ of $[0, 1]$ and ϑ is an irrational number in $(0, 1)$. Such a function, known as a *circle map*, is not periodic, as ϑ is irrational. However, it has many regularity features. It gives an example of aperiodic order.

The most prominent example of aperiodic order in one-dimension belongs to this class. It is the so called Fibonacci model. In this case $\vartheta = \vartheta_{\rm gm}$ and $I = [1 - \vartheta_{gm}, 1]$ with $\vartheta_{\rm gm} :=$ golden mean. This example can be seen as a one-dimensional analogue of the well-known Penrose tiling. Note that in this case the parameter ϑ_{gm} appears twice, viz as rotation and an interval length. This can of course be generalized to other values of ϑ. The corresponding functions $V_{I,\vartheta}$ with $I = [1-\vartheta, 1)$ are known as *Sturmian*. They have been intensely studied (see Chapter 2 for references).

The recent interest in aperiodic order draws from mathematical and physical sources. There, of course, interesting features of special models play a key role. Still, two general aspects may be singled out as well. These are:

- The actual discovery of physical substances exhibiting aperiodic order.
- The conceptual interest in aperiodic order as an intermediate stage of disorder.

Let us discuss these points in more detail.

In 1984, Shechtman/Blech/Gratias/Cahn [**SBGC**] and independently one year later, Ishimasa/Nissen/Fukano [**INF**] reported the discovery of solids which showed pure point diffraction with 5-fold symmetry. This discovery set a new paradigm in crystallography for the following reason:

The diffraction pattern comes from interference of the various scattered parts of an incoming beam. Thus, pure point diffraction can only occur if “a lot” of interference takes place. This means that the positions of the scatterers are highly correlated. Put differently: These solid exhibit long range order.

Of course, there are well known solids with long range order, viz crystals. In their case the atoms form a lattice structure. However, basic discrete geometry shows that 5-fold symmetry is incompatible with a lattice structure: a solid with a 5-fold symmetry is aperiodic. Thus, the outcome of the diffraction experiment can be summarized as follows:

$$\begin{aligned} \text{pure point diffraction} &\simeq \text{long range order},\\ \text{5-fold symmetry} &\simeq \text{aperiodicity}. \end{aligned}$$

The solids discovered exhibit long range aperiodic order. They were soon called *quasicrystals* and mathematicians and physicists alike started their investigation. On the theoretical side, three aspects of aperiodic order received particular attention, viz modeling and complexity, diffraction properties, and electronic properties.

Modeling and diffraction have already been mentioned. Let us now turn to electronic properties of aperiodic order next.

Indeed, one starting point of the conceptional study of aperiodic order and its consequences is marked by the investigations of Kohmoto/Kadanoff/ Tang [**KKT**] and Ostlund/Pandit/Rand/Schellnhuber/Siggia [**OPRSS**]. These groups study electronic properties. More precisely, they consider the operators of the form

$$H_V : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z}), \ (H_V u)(n) = u(n+1) + u(n-1) + V(n)u(n), \tag{2}$$

where $V : \mathbb{Z} \longrightarrow \{0,1\}$ is the Fibonacci potential considered above, viz $V(n) = \chi_{(1-\vartheta_{gm},1]}(n\theta_{gm} \mod 1)$ with ϑ_{gm} = golden mean. Such operators can serve as quantum mechanical models for solids with intermediate disorder. The effects of this intermediate disorder are the prime motivation for [**KKT, OPRSS**].

Let us be more precise by discussing the properties of H_V depending on the degree of disorder and randomness captured by V. Of course, this topic (and corresponding higher dimensional and continuous models) have attracted immense attention over the last decades. We refer the reader to the books [**CL, CFKS, PF, St**] for further discussion and references.

For periodic V the spectrum of H_V is known to consist of bands with purely absolutely continuous spectrum. On the other extreme, in the high random case pure point spectrum is known to occur. This applies in particular to typical Bernoulli-type potentials, i.e. for V modeling the outcome of the coin tossing experiment discussed above [**CKM**].

In between these extreme cases one finds almost-periodic V, -the most prominent example being the almost-Mathieu-operator with $V(n) = \lambda \cos(\vartheta n + \beta)$ with $\lambda \neq 0$ and $\beta \in (0, 2\pi)$,- and V associated to aperiodic order. In these cases other interesting phenomena occur, which were earlier unexpected. This includes Cantor spectrum and purely singular continuous spectrum, i.e. absence of both point spectrum and absolutely continuous spectrum. As for the almost-Mathieu-operator we refer the reader to the surveys [**Ji1, La**] and to [**AJ, AK, Ji2, Pu**] for some recent developments.

Here we will now restrict attention to the case of aperiodic order. In this case, V only takes finitely many values and the combinatorial structure of finite pieces of V is the crucial ingredient. This combinatorial structure is independent of the actual values taken by V, as replacement of V by λV with $\lambda \neq 0$ does not change the combinatorics and leaves many spectral features unchanged. This is a key difference (and in some sense simplification) compared to the almost periodic case.

Now, as discussed in [**KKT**] and [**OPRSS**] the spectrum in the aperiodically ordered case seems to be a Cantor set of Lebesgue measure zero and the spectral type seems to be neither absolutely continuous nor pure point but rather purely singular continuous Both [**KKT**] and [**OPRSS**] are non-rigorous. It has been a particular focus of research to make the corresponding statements rigorous. By now this has been achieved for many examples (see Chapter 2 for details and references).

Of course, operators in higher dimensions have also attracted attention. Here the situation in the aperiodically ordered case is much less satisfactory as far as spectral theory goes. Precise information on spectral type and the spectrum as a set is essentially completely missing. Instead research has been focused on the so called integrated density of states see ([**BHZ, Be2, Ho1, Ho3, Ke, KP**] and references therein). The integrated density of states is the distribution function of a measure on the real line. It is an averaged quantity giving the mean number of electron states per unit volume. In particular, the spectrum can be shown to be the set of its points of non-constancy. Conversely, gaps of the spectrum correspond to intervals of constancy of the integrated density of states.

Here, the first task has been to establish existence of the corresponding average. More detailed investigations then show that it is related to a trace on a certain C^*-algebra. This is known as Shubin-Pastur trace formula. This C^*-algebraic approach allows one to determine the set of possible gaps in the spectrum via K-theory. For quasicrystals, these topics have been investigated starting with the work of Kellendonk [**Ke**], which in turn is strongly stimulated by the corresponding program due to Bellissard and his co-workers (see e.g. [**Be1, Be3**]).

To summarize, aperiodic order gives rise to various new and previously unexpected phenomena. Three of these phenomena are investigated in the present work. These are:

- Cantor spectrum of Lebesgue measure zero,
- uniform existence of certain averages,
- pure point diffraction.

Below, these topics and their respective context will be discussed separately and in more detail. Here, we continue our general discussion.

It is a crucial feature of disorder that various manifestations of a fixed kind of disorder exist. In the non-periodic examples discussed above they are given by all (typical) examples of a coin tossing experiment and by the functions $V_{I,\vartheta}^{\beta}$ with

$$V_{I,\vartheta}^{\beta}(n) := \chi_I(n\vartheta + \beta \mod 1)$$

for $\beta \in [0,1)$. These manifestations can be gathered to form a set Ω with certain regularity features. For example, Ω is invariant under translations

$$T : \Omega \longrightarrow \Omega$$

and therefore gives rise to a dynamical system (Ω, T). The quantities of interest are then suitable functions on Ω.

The key point is that properties of (Ω, T) reflect properties of the single $\omega \in \Omega$ and vice versa. In particular, rather intricate properties of single $\omega \in \Omega$ may lead to simple and easily accessible features of (Ω, T).

This is particularly interesting in the regime of aperiodic order. Namely, one may try to formulate the so far not completely understood order requirements for the $\omega's$ in terms of properties of Ω.

More generally, the use of (Ω, T) gives a tool to investigate properties of its elements.

The link between dynamical systems and the properties of its points is a key element in our considerations. Indeed, somewhat loosely our main results may be phrased as follows:

Result 1. [**Le3, Le4, Le5, DL8**] *A strong version of unique ergodicity implies Cantor spectrum of Lebesgue measure zero for one-dimensional quasicrystals. This strong version of unique ergodicity holds for many models.*

Result 2. [**LS3, LS4**] *The averages of almost additive Banach space valued functions exist in arbitrary dimension whenever the dynamical system is uniquely ergodic and of low complexity. This implies, in particular, strongly uniform existence of the integrated density of states.*

Result 3. [**BL1, BL2**] *Pure point diffraction is equivalent to pure point dynamical spectrum for rather general measure dynamical systems. In this context, pure point diffraction is stable under equivariant perturbations.*

Precise versions of these results and further details are discussed in the next chapters. Each of these chapters starts with a general introduction into its particular topic. In these introductions, we also discuss the particular contribution of the author, whenever the results are obtained in joint work.

CHAPTER 2

Uniform ergodic theorems and spectral theory of one-dimensional discrete Schrödinger operators

In this chapter we give an overview on the authors works [**Le3, Le4, Le5**] and the authors joint work with David Damanik [**DL8**]. The presentation is not uninfluenced by the authors survey type article [**Le6**].

The chapter is concerned with certain discrete random Schrödinger operators associated to compact topological dynamical systems. This means we are given a dynamical system (Ω, T) consisting of a compact space Ω and a homeomorphism T as well as a continuous function $f : \Omega \longrightarrow \mathbb{R}$. The associated selfadjoint operators $(H_\omega)_{\omega\in\Omega}$ are acting on $\ell^2(\mathbb{Z})$ by

$$(3) \qquad (H_\omega u)(n) \equiv u(n+1) + u(n-1) + f(T^n\omega)u(n),$$

This type of operator arises in the quantum mechanical treatment of disordered solids. The underlying discretization is known as tight binding approximation (see e.g. [**BHZ**] for further study of this approximation in the context of aperiodic order). The operator describe the behavior of a single electron which does not interact with other electrons. This is known as one particle approximation. The influence of the solid i.e. of its disorder, is completely absorbed into the choice of the effective potential $n \mapsto f(T^n\omega)$. The influence of disorder is therefore intimately related to features of the dynamical system (Ω, T).

We will assume that (Ω, T) is *strictly ergodic*, i.e.

(SE) (Ω, T) is minimal and uniquely ergodic.

As usual, the dynamical system (Ω, T) is called *minimal* if every orbit is dense and it is called *uniquely ergodic* if there exists only one T-invariant probability measure on Ω. For minimal (Ω, T), there exists a set $\Sigma \subset \mathbb{R}$ s.t.

$$\Sigma = \sigma(H_\omega) \quad \text{for all} \quad \omega \in \Omega,$$

where we denote the spectrum of the operator H by $\sigma(H)$ (see for example [**BIST, Le1**]). We will furthermore assume that (Ω, T) is *aperiodic*, i.e. satisfies

(AP) $T^n\omega \neq \omega$ for all $\omega \in \Omega$ and all $n \neq 0$.

The main focus of the chapter will be the case that (Ω, T) is a subshift over a finite set $S \subset \mathbb{R}$. Recall that (Ω, T) is called a subshift over S if Ω is a closed subset of $S^{\mathbb{Z}}$, invariant under the shift operator $T : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$ given by $(Ta)(n) \equiv a(n+1)$. The function f is then given by $f : \Omega \longrightarrow S \subset \mathbb{R}$, $f(\omega) \equiv \omega(0)$. Here, S carries the discrete topology and $S^{\mathbb{Z}}$ is given the product topology.

Subshifts satisfying (SE) and (AP) have attracted particular attention in recent years, as they can serve as simple models for quasicrystals: They are close to periodic structures by (SE) and not periodic by (AP). They exhibit special features and have been subject to intensive research since then (see Chapter 1). From the mathematical point of view, the associated operators have a tendency to have rather interesting properties such as:

- ($\mathcal{Z}$) Cantor spectrum of Lebesgue measure zero (i.e. Σ is a Cantor set of Lebesgue measure zero);
- ($\mathcal{SC}$) Purely singular continuous spectrum;
- ($\mathcal{AT}$) Anomalous transport.

These properties should be consequences of the underlying disorder which is random by (AP) but still in some sense close to the periodic case by (SE). Absence of point spectrum should then hold as it holds in the periodic case. Absence of absolutely continuous spectrum is expected due to the randomness. Finally, Cantor spectrum (i.e. occurrence of "many" gaps) can be understood by regarding (Ω, T) as periodic with period infinity.

While these considerations are rather convincing on the heuristic level, so far only absence of absolutely continuous spectrum has been established in the general case due to recent results of Last/Simon [**LS**] in combination with earlier results of Kotani [**Ko**]. More precisely, [**Ko**] gives almost sure absence of absolutely continuous spectrum for aperiodic systems and [**LS**] shows that the absolutely continuous spectrum is constant in the minimal case. The other points have rather been proven for large classes of examples. The main examples can be divided in two classes. These classes are given by

- primitive substitution operators as studied e.g. in [**Be2, BBG, BG, Da1, Su1, Su2**] and
- Sturmian operators respectively more generally circle map operators investigated e.g. in [**BIST, Da2, DKL, DL1, DP, HKS, JL1, Ka**] (see [**Da4**] for a recent survey).

The most prominent example is the Fibonacci model (golden mean). This model actually belongs to both classes.

The aim here is to discuss a method to investigate ($\mathcal{Z}$) which was developed by the author in [**Le4**]. It shows that ($\mathcal{Z}$) holds whenever a suitable uniform ergodic type theorem is valid. A quite strong version of such a theorem has been shown to hold for large classes of examples by the author in [**Le3**]. In fact, [**Le3**] even characterizes the subshifts allowing for this strong version theorem. Recently, a slightly weaker ergodic type theorem could be established in joint work with David Damanik [**DL8**]. This weak form is still sufficient to conclude ($\mathcal{Z}$). The results of [**Le4**] rely on work of Furman [**Fu**]. Alternative proofs and partial strengthening of this work are given by the author in [**Le5**]. For discussion of ($\mathcal{SC}$), ($\mathcal{AT}$) and further details we refer the reader to the cited literature.

The property ($\mathcal{Z}$) has been investigated for various models: The starting point are the non-rigorous works Kohmoto/Kadanoff/Tang [**KKT**] and Ostlund/Pandit/Rand/Schellnhuber/Siggia, [**OPRSS**] already discussed in the first chapter. First rigorous results were then obtained by Casdagli [**Ca**]. They show that a certain set called the pseudo-spectrum is a Cantor set of measure zero. While this is rigorously proven, the relation between pseudo-spectrum and spectrum remained open. These works concern the Fibonacci model.

Subsequently rigorous results on ($\mathcal{Z}$) for both Sturmian operators and primitive substitutions were obtained:

For Sturmian operators, $(\mathcal{Z})$ was first shown in the golden mean case by Sütő [**Su1, Su2**]. The general case was then treated by Bellissard/Iochum/Scoppola / Testard [**BIST**]. This has been extended to Quasi-Sturmian models by David Damanik and the author [**DL6**]. A different approach, which recovers some of these results, is given in [**Da3, DL7**]. Most of the cited works tackle not only $(\mathcal{Z})$ but also $(\mathcal{SC})$.

As for primitive substitutions, following work by Bellissard/Bovier/ Ghez [**BBG**] on the period doubling substitution, it was shown for several primitive substitutions by Bovier/Ghez [**BG**]. These works apply to a large class of substitutions given by an algorithmically accessible condition. The Rudin-Shapiro substitution does not belong to this class. In [**Le4**], the author then established $(\mathcal{Z})$ for all primitive substitutions (and in fact a larger class of subshifts). Independently, a proof of $(\mathcal{Z})$ for primitive substitutions was given by Liu/Tan/Wen/Wu in [**LTWW**].

The method presented in [**Le4**] has subsequently been used by other authors as well: It has been applied to show $(\mathcal{Z})$ for certain circle maps by Adamczewski/Damanik [**AD**]. Moreover, Lima/ de Oliveira used it in [**dOL**] to show $(\mathcal{Z})$ for certain non-primitive substitutions. In fact, it can be applied to a large class of non-primitive substitutions as discussed in [**DL7**].

As mentioned already, the range of [**Le4**] was extended in joint work with David Damanik. This extension reproduces all earlier results of this type. Moreover, it allows one to show $(\mathcal{Z})$ for almost all circle maps (i.e. potentials of the form (1)). Earlier results could only treat a set of circle maps of zero measure.

Let us point out that the method given below does not rely on a renormalization scheme, so-called trace maps, as do all other results cited above. Trace maps provide a very powerful tool in the study of random operators. In particular, they allow one to not only study $(\mathcal{Z})$ but also absence of eigenvalues. However, not all systems allow for trace maps and even if there are trace maps they may be hard to analyze. Thus, a main advantage of the approach below is its independence of trace maps.

The method is rather based on relating ergodic features of (Ω, T) to spectral features of the associated operators. The abstract cornerstone is Theorem 1.3 below. It is actually valid for arbitrary dynamical systems satisfying (SE). It gives a characterization of Σ in terms of uniform existence of the Lyapunov exponent γ (precise definition given below). As a consequence, we obtain a necessary and sufficient condition for validity of the equation

$$\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}. \tag{4}$$

in Theorem 1.5 in terms of uniform existence of the Lyapunov exponent. Now, trying to establish (4) is a canonical strategy in the proofs of $(\mathcal{Z})$, as by fundamental results of Kotani [**Ko**], the set $\{E \in \mathbb{R} : \gamma(E) = 0\}$ has Lebesgue measure zero if (Ω, T) is an aperiodic subshift.

Thus, Theorem 1.5 reduces the study of $(\mathcal{Z})$ to establishing validity of a uniform ergodic theorem for certain matrix valued functions over (Ω, T). This effectively, transforms the spectral theoretic problem into an ergodic problem.

This ergodic problem can be solved for a large class of examples including all primitive substitutions by the main result of [**Le3**]. More precisely, [**Le3**] characterizes the class of subshifts for which the averages for every subadditive function on the associated set of words exist by a combinatorial condition (PW). This class contains all primitive substitutions.

In order to apply [**Le4**] to establish $(\mathcal{Z})$ one does not need the full strength of [**Le3**] dealing with arbitrary subadditive functions. It suffices to know existence of averages for all subadditive functions coming from matrices. This idea is the starting point for the authors joint work with David Damanik in [**DL8**].

There existence of averages for certain matrix valued functions is shown to hold for all subshifts satisfying a condition (B). This condition is due to Boshernitzan in his study of unique ergodicity. One may think of (B) as saying that (PW) holds "on many scales". Accordingly, validity of the ergodic theorem is established in [**DL8**] in two steps:

In the first step, it is shown that (B) implies existence of averages along many scales. Then, in the next step one shows that existence of the averages along many scales implies existence of the averages. This step requires an "extrapolation" of scales. This is provided by the so called avalanche principle. The avalanche principle was introduced by Goldstein/Schlag in [**GS**]. We use it in the form given by Bourgain/Jitomirskaya in [**BJ**].

This part of the considerations of [**DL8**] is essentially due to the author.

In the second part of [**DL8**], validity of (B) is proven for large classes of examples. In particular, in joint work with David Damanik, it is shown to hold for in a suitable sense almost all circle maps potentials defined above in (1).

1. Uniformity of cocycles and Cantor spectrum of Lebesgue measure zero

In this section we introduce the necessary notation and give precise versions of our results.

For a continuous function $A : \Omega \longrightarrow GL(2,\mathbb{R})$, $\omega \in \Omega$, and $n \in \mathbb{Z}$, the cocycle $A(\omega, n)$ is defined by

$$A(\omega,n) \equiv \begin{cases} A(T^{n-1}\omega)\cdots A(\omega) & : \quad n>0 \\ Id & : \quad n=0 \\ A^{-1}(T^{n}\omega)\cdots A^{-1}(T^{-1}\omega) & : \quad n<0 \end{cases}$$

By Kingmans subadditive ergodic theorem, there exists $\Lambda(A) \in \mathbb{R}$ with

$$\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(\omega,n)\|$$

for μ almost every $\omega \in \Omega$ if (Ω, T) is uniquely ergodic with invariant probability measure μ. Following Furman [**Fu**], we introduce the following definition.

Definition 1.1. Let (Ω, T) be strictly ergodic. The continuous function $A : (\Omega, T) \longrightarrow GL(2,\mathbb{R})$ is called uniform if the limit $\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(\omega,n)\|$ exists for all $\omega \in \Omega$ and the convergence is uniform on Ω.

Remark 1.2. As shown by Furstenberg and Weiss [**FW**], uniform existence of the limit in the definition already implies uniform convergence. In fact, this is even true for a continuous subadditive cocycle $(f_n)_{n\in\mathbb{N}}$ on a minimal (Ω, T) (i.e. f_n are continuous real-valued functions on Ω with $f_{n+m}(\omega) \le f_n(\omega) + f_m(T^n\omega)$ for all $n, m \in \mathbb{N}$ and $\omega \in \Omega$).

For spectral theoretic investigations a special type of $SL(2,\mathbb{R})$-valued function is relevant. Namely, for $E\in\mathbb{R}$, let the continuous function $M^E:\Omega\longrightarrow SL(2,\mathbb{R})$ be given by

$$M^E(\omega)\equiv\begin{pmatrix} E-f(T\omega) & -1\\ 1 & 0\end{pmatrix}.$$

Straightforward calculations show that a sequence u is a solution of the difference equation

$$u(n+1)+u(n-1)+(f(T^n\omega)-E)u(n)=0 \tag{5}$$

if and only if

$$\begin{pmatrix} u(n+1)\\ u(n)\end{pmatrix}=M^E(\omega,n)\begin{pmatrix} u(1)\\ u(0)\end{pmatrix},\ \text{ for all } n\in\mathbb{Z}. \tag{6}$$

By the above considerations, M^E gives rise to the average $\gamma(E)\equiv\Lambda(M^E)$. This average is called the Lyapunov exponent for the energy E. It measures the rate of exponential growth of solutions of (5). Our abstract result now reads as follows.

Theorem 1.3. [**Le4**] *Let (Ω,T) be strictly ergodic. Then,*

$$\Sigma=\{E\in\mathbb{R}:\gamma(E)=0\}\cup\{E\in\mathbb{R}:\ M^E\ \textit{is not uniform}\},$$

where the union is disjoint.

Remark 1.4. This theorem is related to results of Johnson on so called exponential dichotomy [**Jo**]. More precisely, Johnson shows that the resolvent is exactly the set of energies with exponential dichotomy, i.e. strong exponential behaviour of the solutions. Now, the results of Furman [**Fu**] can be understood as saying that exponential dichotomy is equivalent to uniformity of M^E with $\Lambda(M^E)>0$. This line of thought can probably be used to provide a proof for Theorem 1.3. When [**Le4**] was written the author was not aware of Johnson's results. Therefore, [**Le4**] contains a different proof of the theorem.

The theorem has two consequences. The first says that uniform positivity of the Lyapunov exponent is equivalent to M^E being non-uniform for all $E\in\Sigma$. This is interesting when one tries to construct examples of non-uniform cocycles. This is further discussed below.

The other consequence is the following theorem of [**Le4**], which is crucial to our method of proving ($\mathcal{Z}$).

Theorem 1.5. [**Le4**] *Let (Ω,T) be strictly ergodic. Then the following are equivalent:*

(i) *The function M^E is uniform for every $E\in\mathbb{R}$.*
(ii) $\Sigma=\{E\in\mathbb{R}:\gamma(E)=0\}$.

In this case the Lyapunov exponent $\gamma:\mathbb{R}\longrightarrow[0,\infty)$ is continuous.

The theorem relates the validity of (4) to ergodic features of the underlying subshift. It turns out that uniformity of the transfer matrices and more generally of locally constant matrices can be shown for large classes of subshifts. Here, a function $A:\Omega\longrightarrow SL(2,\mathbb{R})$, where (Ω,T) is a subshift over S, is called locally constant if there exists an $N\in\mathbb{N}$ with $A(\omega)=A(\rho)$ whenever $\omega(-N)\ldots\omega(N)=\rho(-N)\ldots\rho(N)$.

To introduce these classes we need some more notation. We consider sequences over S as words and use standard concepts from the theory of words ([**Lo**]). In particular, Sub(w) denotes the set of subwords of w, the number of occurrences of v in w is denoted by $\sharp_v(w)$ and the length $|w|$ of the word $w = w(1)\ldots w(n)$ is given by n. To Ω we associate the set $\mathcal{W} = \mathcal{W}(\Omega)$ of finite words associated to Ω given by $\mathcal{W} \equiv \cup_{\omega\in\Omega}\mathrm{Sub}(\omega)$. For a finite set M, we define $\sharp M$ to be the number of elements in M.

We can now present the two classes of subshifts we will be dealing with. The first class consists of those satisfying the condition (PW) of uniform positive weights:

Definition 1.6. The subshift (Ω, T) satisfies (PW) if there exists a $C > 0$ with $\liminf_{|x|\to\infty} \frac{\sharp_v(x)}{|x|}|v| \geq C$ for every $v \in \mathcal{W}$.

As discussed in [**Le4, Le3**], this class contains all primitive substitution subshifts. It allows for a rather strong ergodic type theorem [**Le3, Le5**]. In fact, it can be characterized by validity of such a theorem. This is the content of the next result.

Before we state the result, let us recall that $F : \mathcal{W} \longrightarrow \mathbb{R}$ is called *subadditive* if $F(xy) \leq F(x) + F(y)$ whenever $xy \in \mathcal{W}$.

Theorem 1.7. [**Le3, Le5**] *Let (Ω, T) be minimal. Then the following assertions are equivalent:*

(i) *(Ω, T) satisfies (PW).*
(ii) *The limit $\lim_{|x|\to\infty} \frac{F(x)}{|x|}$ exists for every subadditive $F : \mathcal{W} \longrightarrow \mathbb{R}$.*

In this case, every locally constant $A : \Omega \longrightarrow SL(2, \mathbb{R})$ is uniform.

Remark 1.8. (a) This theorem underlines the importance of condition (PW). A further discussion of this condition and its relation to other conditions can be found in the next section.
(b) The theorem generalizes the corresponding results of [**DL5, Le2**]. In [**Le2**], the equivalence is shown to hold for special subshifts, viz Sturmian dynamical systems. Moreover, a variant of (PW) is shown to be necessary for (ii).

The previous ergodic theorem together with Theorem 1.5 implies Cantor spectrum of Lebesgue measure zero for the corresponding systems by the results of Kotani [**Ko**] discussed at the beginning of the chapter.

Theorem 1.9. [**Le4**] *Let (Ω, T) be an aperiodic subshift. If (Ω, T) satisfies (PW), then Σ is a Cantor set of Lebesgue measure zero.*

As discussed above, this result applies to a wide range of examples. These include all primitive substitutions [**Le4**] as well as certain circle maps [**AD**] and certain non-primitive substitutions [**dOL**]. As primitive substitutions have attracted a lot of attention, we explicitely state the following corollary.

Corollary 1.10. [**Le4**] *Let (Ω, T) be an aperiodic subshift associated to a primitive substitution, then Σ is a Cantor set of Lebesgue measure zero.*

While condition (PW) holds for many examples, it is not necessary for Σ being a Cantor set of measure zero, as can be seen by considering suitable Sturmian potentials [**Le6**]. This rises the question for generalizations of (PW). In this context the following condition is of interest.

Definition 1.11. Let (Ω, T) be a subshift over a finite alphabet. For $w \in \mathcal{W}$ define $V_w := \{\omega \in \Omega : w = \omega(1)\ldots\omega(|w|)\}$. Then, (Ω, T) is said to satisfy condition

(B) if there exists an ergodic probability measure ν on Ω, a sequence (l_n) in $\mathbb{N}$ with $l_n \to \infty$, $n \to \infty$, and $C > 0$ such that $|w|\nu(V_w) \geq C$, whenever $w \in \mathcal{W}$ satisfies $|w| = l_n$ for some $n \in \mathbb{N}$.

This condition was introduced by Boshernitzan in his study of unique ergodicity in [**Bo1**]. It may be considered as giving validity of (PW) on certain scales. Namely, as shown by Boshernitzan in [**Bo2**], this condition implies unique ergodicity. It is therefore [**Bo2, DL8**] equivalent with the requirement that there exists a $C > 0$ and a sequence (l_n) in $\mathbb{N}$ with $l_n \to \infty$, $n \to \infty$, with

$$(B') \qquad \liminf_{|x|\to\infty} \frac{\sharp_v(x)}{|x|}|v| \geq C,$$

whenever $|v| = l_n$ for some $n \in \mathbb{N}$.

The intuition that (B) means validity of (PW) on certain scales is supported by the following extension of Theorem 1.7.

In order to state this extension, we need one more piece of notation. Let (Ω, T) be a uniquely ergodic subshift. Every subadditive $F : \mathcal{W} \longrightarrow \mathbb{R}$ induces a subadditive cocycle (f_n) on Ω defined by $f_n : \Omega \longrightarrow \mathbb{R}$, $f_n(\omega) := F(\omega(1)\ldots\omega(n))$. In particular, by Kingmans subadditive ergodic theorem, we can associate to every subadditive F a number $\Lambda(F)$ with $\Lambda(F) = \lim_{n\to\infty} 1/n f_n(\omega)$ for almost every $\omega \in \Omega$.

Theorem 1.12. [**DL8**] *Let (Ω, T) be a minimal subshift over a finite alphabet. Then the following conditions are equivalent:*

(i) *(Ω, T) satisfies* (B).
(ii) *(Ω, T) is uniquely ergodic and there exists a sequence (l'_n) in $\mathbb{N}$ with $l'_n \to \infty$ for $n \to \infty$ such that $\lim_{n\to\infty} |w_n|^{-1}F(w_n) = \Lambda(F)$ for every subadditive F and every sequence (w_n) in $\mathcal{W}(\Omega)$ with $|w_n| = l'_n$ for every $n \in \mathbb{N}$.*

This result gives the existence of averages on many length scales. To obtain existence of averages on all lengths scales, one needs an "extrapolation procedure". Such a procedure is provided by the avalanche principle introduced in [**GS**] and later varied in [**BJ**]. Combined with the avalanche principle of [**BJ**], the previous theorem can be used to give the following theorem, which is the main abstract result of [**DL8**]. As discussed there, this result covers all earlier results of this form.

Theorem 1.13. [**DL8**] *Let (Ω, T) be a minimal subshift which satisfies* (B). *Let $A : \Omega \longrightarrow SL(2, \mathbb{R})$ be locally constant. Then, A is uniform.*

The previous theorem and our general method immediately give the following result.

Theorem 1.14. [**DL8**] *Let (Ω, T) be an aperiodic subshift. If (Ω, T) satisfies (B), then Σ is a Cantor set of measure zero.*

As an application we obtain the following result, where for $b, \theta, \beta \in (0, 1)$ arbitrary, the function $V^\beta_{[b,1),\vartheta}$ is defined as in Chapter 1 by

$$V^\beta_{[b,1),\vartheta} : \mathbb{Z} \longrightarrow \{0, 1\}, \quad \text{by } n \mapsto \chi_{[b,1)}(n\vartheta + \beta \mod 1).$$

THEOREM 1.15. [**DL8**] *Let $\vartheta \in (0,1)$ be irrational.*
(a) *For almost every $b \in (0,1)$, the spectrum $\sigma(H_{V^{\beta}_{[b,1),\vartheta}})$ is a Cantor set of Lebesgue measure zero for every $\beta \in (0,1)$.*
(b) *If ϑ has bounded continued fraction expansion, then $\sigma(H_{V^{\beta}_{[b,1),\vartheta}})$ is a Cantor set of Lebesgue measure zero for every $\beta \in (0,1)$ and every $b \in (0,1)$.*

REMARK 1.16. (a) The proof relies on Diophantine approximation.
(b) As discussed in [**DL8**], this result is particularly relevant as all earlier results on Cantor spectrum for circle maps [**AD, BIST, DL6, Su1, Su2**] only cover a set of parameters (ϑ, b) of Lebesgue measure zero in $(0,1)\times(0,1)$.
(c) Condition (B) does not hold for all circle maps. Thus, the method developed above can not be used to infer Cantor spectrum for all irrational circle maps. Still, Cantor spectrum may be true for all irrational circle maps (cf. discussion in Section 4).

2. The conditions (PW) and (B)

In this section, we would like to shortly discuss conditions (PW) and (B). We will use condition (B') introduced on the previous page instead of (B). As discussed there, it is equivalent to (B).

We start by giving a geometric interpretation for terms of the form

$$\frac{\sharp_v(x)}{|x|}|v|$$

for words v and x. If the copies of v in x are disjoint, then $\sharp_v(x)\cdot|v|$ is just the amount of "space" in x covered by v. The term $\frac{\sharp_v(x)}{|x|}|v|$ gives then the fraction of x covered by v. Thus, conditions like (PW) and (B') mean that in an averaged sense all words cover a certain minimal amount of space.

Let us now discuss the relation between the various conditions mentioned. Condition (B') implies that

$$\limsup_{|x|\to\infty}\frac{\sharp_v(x)}{|x|} > 0$$

for every $v \in \mathcal{W}$. By [**Qu**], (B') then implies minimality. Moreover, as shown by Boshernitzan [**Bo2**] (see [**DL8**] as well), (B') implies unique ergodicity. Now, obviously (B') is weaker than (PW). Condition (PW) was introduced by the author in [**Le3**]. It is related to linear repetitivity as studied by Durand in [**Du2**] for subshifts (see [**DHS**] as well) and, independently, by Lagarias/Pleasants for Delone sets in [**LP**]. Here, a subshift is called linearly repetitive if there exists a constant $C>0$, with $\sharp_v(x)\geq 1$ whenever $x, v\in\mathcal{W}$ satisfy $|x|\geq C|v|$. Thus, (PW) can be considered to be an averaged version of (LR).

These considerations provide the following chain of implications:

$$(LR) \Longrightarrow (PW) \Longrightarrow (B') \Longrightarrow (SE).$$

It is then natural to ask for the reverse implications: As discussed in [**DL8**], there exist circle map subshifts which do not satisfy (B'). As all circle maps are strictly ergodic, the rightmost arrow can not be reversed. Similarly, the middle implication can not be reversed as all Sturmian models satisfy (B') by [**DL8**] but not all of them satisfy (PW) [**Le2, Le6**].

The precise relation between (PW) and (LR) is still unclear. It is shown by Monteil that (LR) is equivalent to (PW) combined with a bound on the highest occurring power in $\mathcal{W}$ [**Mon**]. It is not known whether this bound is forced by (PW) already. Also, as shown by the author in [**Le3**], (LR) is equivalent to uniform validity of (PW) on all systems derived from the original ones by return words. Here, again, it is not known whether this really is an additional requirement.

3. Uniformity of certain cocycles: Ergodic theoretic background

This section gives a short study of ergodic theoretic background to the proof of Theorem 1.3 and a discussion of non-uniformity of certain cocycles. This partly summarizes the results of [**Le5**].

We start with our characterization of uniformity. To state it we need some further notation. The projective space over $\mathbb{R}^2$ consisting of all one-dimensional subspaces of $\mathbb{R}^2$ is denoted by $\mathcal{P}$. To $X \in \mathbb{R}^2 \setminus \{0\}$, we associate the element $[X] = \{\lambda X : \lambda \in \mathbb{R}\}$ of $\mathcal{P}$.

We have the following theorem.

THEOREM 3.1. *Let (Ω, T) be uniquely ergodic and $A : \Omega \longrightarrow SL(2,\mathbb{R})$ be continuous. Then the following are equivalent:*

(i) A is uniform with $\Lambda(A) > 0$.

(ii) There exist constants $\kappa, C > 0$ and continuous functions $u, v : \Omega \longrightarrow \mathcal{P}$ with

$$\|A(\omega,n)U\| \leq C\exp(-\kappa n)\|U\| \quad \text{and} \quad \|A(-n,\omega)V\| \leq C\exp(-\kappa n)\|V\|. \tag{7}$$

for arbitrary $\omega \in \Omega$, $n \in \mathbb{N}$, $U \in u(\omega)$ and $V \in v(\omega)$.

(iii) There exists $\delta > 0$ and $m \in \mathbb{N}$ with $0 < \delta \leq \frac{1}{n}\ln\|A(\omega,n)\| \leq \frac{3}{2}\delta$ for every $\omega \in \Omega$ and $n \geq m$.

In this case, $u(\omega) \neq v(\omega)$, $[A(\omega,n)U] = u(T^n\omega)$ and $[A(\omega,n)V] = v(T^n\omega)$ for arbitrary $\omega \in \Omega$, $n \in \mathbb{Z}$, $U \in u(\omega)$ and $V \in v(\omega)$ with $U, V \neq 0$.

REMARK 3.2. (a) The equivalence of (i) and (ii) essentially generalizes the corresponding results of Furman for strictly ergodic systems [**Fu**]. A connection between (i) and (ii) in certain examples had already been established by Herman in [**He1**].
(b) Condition (ii) is essentially the condition known as exponential dichotomy in [**Jo**].
(c) Our proof is based on results of Ruelle [**Ru**] as given by Last/Simon in [**LS**].

The proof of the theorem gives the following corollary [**Le5**].

COROLLARY 3.3. *Let (Ω, T) be strictly ergodic. Then, the continuous $A : \Omega \longrightarrow SL(2,\mathbb{R})$ is uniform with $\Lambda(A) > 0$ if and only if there exists $m \in \mathbb{N}$ and $\delta > 0$ such that $\delta \leq \frac{1}{n}\ln\|A(\omega,n)\|$ for every $\omega \in \Omega$ and $n \geq m$.*

REMARK 3.4. The corollary deals with uniform lower bounds on $\frac{1}{n}\ln\|A(\omega,n)\|$. Let us point out that for arbitrary continuous (not necessarily uniform) $A : \Omega \longrightarrow SL(2,\mathbb{R})$ a uniform upper bound holds whenever (Ω, T) is strictly ergodic. This is shown by Furman in Corollary 2 of [**Fu**].

We finish this section with a discussion of existence of non-uniform cocycles. The question of existence of non-uniform $SL(2,\mathbb{R})$-valued-cocycles has attracted attention in recent years [**Wa1, He1, Fu**]. In fact, in 1984 Walters asked the following question [**Wa1**]:

(Q) Does every uniquely ergodic dynamical system with non-atomic measure μ admit a non-uniform cocycle?

Using results of Veech [**Ve**], Walters presents one class of examples admitting non-uniform cocycles.

As shown by Herman [**He1**], further classes are given by suitable irrational rotations.

In this context, the result of the previous section, give examples of subshifts on which locally constant cocycles can not be non-uniform.

On the other hand, Theorem 1.3 has the following immediate consequence.

THEOREM 3.5. *Let (Ω, T) be strictly ergodic and (H_ω) the associated operators. Then the following are equivalent:*

(i) $\gamma(E) > 0$ *for every* $E \in \mathbb{R}$.
(ii) $\Sigma = \{E \in \mathbb{R} : M^E$ *is non-uniform*$\}$.

The theorem shows that examples of operators with uniform positive Lyapunov exponent give rise to non-uniform cocycles. There is a well-known class of random operators with uniform positive cocycles, viz the almost-Mathieu-operators already shortly discussed at the beginning of the first chapter. Let us be more precise:

Choose an irrational $\alpha \in (0,1)$ and an arbitrary $\lambda > 0$. Denote the irrational rotation by α on the unit circle, $\mathcal{S}$, by R_α (i.e. $R_\alpha z \equiv \exp(i\alpha)z$, where i is the square root of -1). Define $f^\lambda : \mathcal{S} \longrightarrow \mathbb{R}$ by $f^\lambda(z) \equiv \lambda(z + z^{-1})$ (i.e. $f^\lambda(\exp(i\theta)) = 2\lambda\cos(\theta)$). Denote the associated operators by (H_z^λ) and their spectrum by $\Sigma(\lambda)$. The operators (H_z^λ) are called almost-Mathieu-operators. They have attracted much attention (see discussion above for further references). Now, by [**AA, AS**] (see [**He2**] for an alternative proof as well), we have

$$\gamma(E) > 0 \text{ for all } E \in \mathbb{R} \text{ whenever } \lambda > 1.$$

Combining this result with the previous theorem, we infer the following theorem.

THEOREM 3.6. *For arbitrary irrational $\alpha \in (0,1)$ and $\lambda > 1$, the function M^E is non-uniform if and only if E belongs to $\Sigma(\lambda)$.*

By this result every irrational rotation allows for a non-uniform matrix. This generalizes the results of Herman [**He1**] mentioned above. Let us emphasize, however, that the results of Herman in [**He2**] combined with Theorem 4 of [**Fu**] (or Theorem 3.1 above) also give existence of non-uniform cocycles for every irrational rotation. Still, the above result is more explicit as the set of energies with non-uniform transfer matrices is identified as $\Sigma(\lambda)$.

4. Further remarks

The preceeding considerations establish a link between uniform existence of the Lyapunov exponent and its vanishing on the spectrum. This can be combined with results of Kotani to conclude Cantor spectrum of measure zero for many examples.

Two questions present themselves in this context:

Question: Can one establish $(\mathcal{Z})$ for further classes of subshifts? More generally: Is $(\mathcal{Z})$ valid for every strictly ergodic subshift?

Question: Can one use these considerations to exclude eigenvalues? In particular, is Cantor spectrum of measure zero sufficient for absence of eigenvalues?

While these are certainly interesting question, they might be out of reach of current research.

On a more concrete level, one may try to extend the results of the previous sections to continuum models and to models on strips. In fact, models on strips are currently being considered in joint work with David Damanik.

In this task, the ergodic theory considerations for functions with values in $SL(2,\mathbb{R})$ have to be be extended to functions with values in $\mathrm{SL}(2n,\mathbb{R})$. Here, the strategy is to follow [**Le3, Le5**] in adopting the relevant results from Ruelle's work [**Ru**]. On the operator theoretic level, one needs a suitable version of Kotani's theory for a strip. One version of this theory is given by Kotani/Simon in [**KS**]. The problem when dealing with these results is that some Lyapunov exponents may vanish while others do not.

It should also be noted that the preceeding results give a new perspective on Walter's question on existence of non-uniform cocycles:

Namely, let (Ω, T) be a dynamical system on which every cocycle is uniform. Then, by Theorem 1.3, the spectrum of any Schrödinger operator associated to it according to (3) agrees with the set of zeros of the corresponding Lyapunov exponents. This, of course, is a rather strong restriction for (Ω, T) and one may try to show that this actually implies periodicity. This may be seen as an analogue to Kotani's results on deterministic potentials.

Alternatively, by Theorem 1.5, to answer Walters question affirmatively, it suffices to find a Schrödinger operator whose Lyapunov exponent has a point of discontinuity.

CHAPTER 3

Operators of finite range: Uniform existence of the integrated density of states

In this chapter, we provide a discussion of the authors joint work with Peter Stollmann [**LS3, LS4**].

The chapter is concerned with aperiodic order in arbitrary dimensions. Thus, we consider the higher dimensional analogues to the sequences and subshifts over a finite alphabet, which were discussed in Chapter 2. These analogues are given by Delone sets (with suitable regularity features) and the corresponding Delone dynamical systems (Ω, T). The associated Hamiltonians H_ω act on $\ell^2(\omega)$, $\omega \in \Omega$. Thus, both the operators and the underlying Hilbert space depends on $\omega \in \Omega$. The overall goal is to study the order features of (Ω, T), the spectral theory of $(H_\omega)_{\omega\in\Omega}$ and the interplay between these two. More specifically, our aims are

- to carry over basic theory of random operators to our setting,
- to show that aperiodic order gives a very uniform existence result for the so called integrated density of states.

Here, the second point is actually a special case of a more general result:

- aperiodic order gives a very strong type of ergodic theorem for arbitrary Banach space valued functions.

Let us discuss this in more detail.

We use basic results from Connes' non-commutative integration theory to construct a von Neuman algebra $\mathcal{N}(\Omega, m)$. This von Neumann algebra contains the Hamiltonians and their spectral projections. It provides a natural setting to state and prove those features of the Hamiltonians which are "usual" for random operators. This includes almost sure constancy of the spectral properties of H_ω, $\omega \in \Omega$, and absence of discrete spectrum. Here, ergodicity is the essential assumption. Moreover, there is a canonical trace on the von Neumann algebra. This trace allows one to abstractly define the integrated density of states.

By its very nature aperiodic order is a topological concept rather than a measure theoretical one. Thus, we restrict attention to a suitable sub C^*-algebra $\mathcal{A}(\Omega)$ for our further considerations. This algebras comes from the operators which are "local" in a suitable sense and therefore rather directly reflect the underlying disorder. For these operators we can show that the integrated density defined above can be calculated via an averaging procedure. This is our version of the so called Pastur-Shubin trace formula. It generalizes in some sense the corresponding results of [**Ke, Ho1, Ho3**].

The results mentioned so far are all contained in [**LS3**]. They are not specific for the underlying form of disorder. They are in fact well known and standard for random operators and almost random operators [**KM**] (see e.g. [**CL, PF**] and references therein). In our context, the proofs require some additional care is as the Hilbert space depends on the randomness as well. Still the proofs are essentially straightforward variants of the "usual" ones.

We now come to a specific feature of the underlying disorder, which is discussed in [**LS4**]. Namely, assuming unique ergodicity and a suitable finite complexity condition, we can actually show that the convergence of the approximants to the integrated density of states takes place in a very uniform way. More precisely, the distribution functions of the converging measures converge uniformly. This seems to be the first result of its kind for random operators.

As mentioned already, our proof relies on a strong ergodic type theorem for almost additive Banach space valued functions, which may be of independent interest.

Such a result was first proven for systems associated to substitutions by Geerse / Hof in [**GH**]. Their proof uses two ingredients: unique ergodicity i.e. uniform existence of the frequencies and certain decompositions of large clusters into smaller ones. These decompositions are naturally present in their framework of substitution systems. In general, one has to work to produce them and that is a key issue in our study. We use partitions according to return words as introduced by Durand [**Du1**] in the case of subshifts and later studied by Priebe for tilings [**Pr**].

A one-dimensional variant of this ergodic theorem and the proof outlined above has been studied by the author in [**Le2, Le3**]. Accordingly, the basic line of argument of [**LS4**] as well as essential parts of its actual proofs are due to the author.

Most of the content of [**LS3**] is in one way or other suggested by Peter Stollmann. Still many of the actual details are supplied by the author. In this context, we would also like to mention the authors work with Norbert Peyerimhof and Ivan Veselić [**LPV**], which works out a very general setup for the treatment of random operators based on Connes non-commutative integration theory.

1. Uniform existence of averages on Delone dynamical systems

In arbitrary dimensions, long range order is usually modeled by tilings or, equivalently, Delone sets. Here, we will be concerned with Delone set (see [**LP, Sol1**] for further discussion). In order to be more precise, let $d \in \mathbb{N}$ be fixed and denote by $B_S(p)$ the closed ball centered at $p \in \mathbb{R}^d$ with radius $S > 0$.

DEFINITION 1.1. A subset $\omega \subset \mathbb{R}^d$ is called a *Delone set* if there exist $0 < r < R$ such that for any $p \in \mathbb{R}^d$ the ball $B_r(p)$ contains at most one and $B_R(p)$ contains at least one element of ω.

The points of a Delone set ω are thought to model the positions of the atoms of a quasicrystal. Apparently, every Delone set is closed.

The Hausdorff metric on the set of compact subsets of $\mathbb{R}^d$ induces the so called *natural topology* [**LP**] on the set of closed subsets of $\mathbb{R}^d$. This topology was introduced in [**LP**] and is studied in detail in [**LS2**].

We will not define this topology here (see [**LS2**]). We will rather note two of its crucial properties: Firstly, the set of all closed subsets of $\mathbb{R}^d$ is compact in the natural topology. Secondly, the natural action T of $\mathbb{R}^d$ on the closed sets given by $T_t C \equiv C + t$ is continuous.

Having introduced a topology and an action of $\mathbb{R}^d$, we can now define the dynamical systems of interest: (Ω, T) is called a *Delone dynamical system* and abbreviated as *DDS* if Ω is a set of Delone sets that is invariant under the shift T and closed under the natural topology.

These can be considered as analogues to subshifts. However, it should be noted that we do not have build in an analogue to the finiteness of the alphabet so far.

To do so and for our further investigations we will have to deal with finite parts of Delone sets. This topic will be considered next. The appropriate definition in our context is the following:

DEFINITION 1.2. (a) A pair (Λ, Q) consisting of a bounded subset Q of $\mathbb{R}^d$ and $\Lambda \subset Q$ finite is called pattern. The set Q is called the support of the pattern.
(b) A pattern (Λ, Q) is called ball pattern if $Q = B_s(x)$ with $x \in \Lambda$ for suitable $x \in \mathbb{R}^d$ and $s > 0$.

We will identify patterns which are equal up to translation. More precisely, on the set of patterns we introduce an equivalence relation via $(\Lambda_1, Q_1) \simeq (\Lambda_2, Q_2)$ if and only if there exists a $t \in \mathbb{R}^d$ with $\Lambda_1 = \Lambda_2 + t$ and $Q_1 = Q_2 + t$. The class of a pattern (Λ, Q) is denoted by $[(\Lambda, Q)]$.

Two sets of pattern classes are naturally associated to a Delone set ω. These are the set $\mathcal{P}(\omega)$ of all pattern classes occurring in ω

$$\mathcal{P}(\omega) = \{[Q \wedge \omega] : Q \subset \mathbb{R}^d \text{ bounded and measurable}\},$$

and the and set $\mathcal{P}_B(\omega))$ of ball pattern classes occurring in ω

$$\mathcal{P}_B(\omega)) = \{[B(p, s) \wedge \omega] : p \in \omega, s \in \mathbb{R}\}.$$

Here, we define

$$Q \wedge \omega = (\omega \cap Q, Q). \tag{8}$$

For $s \in (0, \infty)$, we denote by $\mathcal{P}_B^s(\omega)$ the set of ball patterns with radius s. A Delone set is said to be of finite local complexity if the set $\mathcal{P}_B^s(\omega)$ is finite for every $s > 0$. This type of finiteness is a strong assumption. It can be considered to be the analogue of the finiteness of the alphabet in the case of one-dimensional subshifts.

DEFINITION 1.3. (Ω, T) is called a Delone dynamical system of finite local complexity (DDSF) if Ω is a closed T-invariant set of Delone sets such that $\cup_{\omega \in \Omega} P_B^s(\omega)$ is finite for every $s > 0$.

We will show that suitable ergodic averages exist on (DDSF). Thus, we will have to take means of suitable functions along suitable sequences of patterns and pattern classes. These functions and sequences will be introduced next.

Here and in the sequel we will use the following notation: For $Q \subset \mathbb{R}^d$ and $h > 0$ we define

$$Q_h \equiv \{x \in Q : \text{dist}(x, \partial Q) \geq h\}, \;\; Q^h \equiv \{x \in \mathbb{R}^d : \text{dist}(x, Q) \leq h\},$$

where, of course, dist denotes the usual distance and ∂Q is the boundary of Q. Moreover, we denote the Lebesgue measure of a measurable subset $Q \subset \mathbb{R}^d$ by $|Q|$. Then, a sequence (Q_n) of subsets in $\mathbb{R}^d$ is called a *van Hove sequence* if the sequence $(|Q_n|^{-1}|Q_n^h \setminus Q_{n,h}|)$ tends to zero for every $h \in (0, \infty)$. Similarly, a sequence (P_n) of pattern classes, (i.e. $P_n = [(\Lambda_n, Q_n)]$ with suitable Q_n, Λ_n) is called a *van Hove sequence* if Q_n is a van Hove sequence. Obviously, this is well defined.

Moreover, for pattern classes P_i, $i = 1, \ldots, k$ and P, we write

$$P = \oplus_{i=1}^k P_i$$

if there exist $X_i = (\Lambda_i, Q_i)$, $i = 1, \ldots, k$, and $X = (\Lambda, Q)$, such that $[X_i] = P_i$, $i = 1, \ldots, k$, $[X] = P$, $\Lambda = \cup \Lambda_i$, $Q = \cup Q_i$ and the Q_i are disjoint up to their boundaries.

We can now introduce the class of functions we want to average.

DEFINITION 1.4. Let Ω be a DDS and $\mathcal{B}$ be a vector space with seminorm $\| \cdot \|$. A function $F : \mathcal{P}(\Omega) :\longrightarrow \mathcal{B}$ is called *almost additive (with respect to* $\| \cdot \|$*)* if there exists a function $b : \mathcal{P}(\Omega) \longrightarrow [0, \infty)$ (called *associated error function*) and a constant $D > 0$ such that

(A1) $\|F(\oplus_{i=1}^k P_i) - \sum_{i=1}^k F(P_i)\| \leq \sum_{i=1}^k b(P_i)$,
(A2) $\|F(P)\| \leq D|P| + b(P)$.
(A3) $b(P_1) \leq b(P) + b(P_2)$ whenever $P = P_1 \oplus P_2$,
(A4) $\lim_{n\to\infty} |P_n|^{-1} b(P_n) = 0$ for every van Hove sequence (P_n).

Apparently, condition (A4) says that b is a boundary or surface type term. Then, (A1) means that F is additive up to a boundary term. Condition (A2) gives an apriori bound. Condition (A3) may look surprising at first sight. However, it reflects exactly the behaviour of surfaces under taking disjoint unions: If $C = A \cup B$ with subsets A, B of $\mathbb{R}^d$ which are disjoint up to their boundary, then the boundary of A is contained in the union of the boundaries of C and B.

Now, our ergodic theorem reads as follows.

THEOREM 1.5. *For a minimal, aperiodic DDSF (Ω, T) the following are equivalent:*

(i) *(Ω, T) is uniquely ergodic.*
(ii) *The limit $lim_{k\to\infty} |P_k|^{-1} F(P_k)$ exists for every van Hove sequence (P_k) and every almost additive F on (Ω, T) with values in a Banach space.*

REMARK 1.6. (a) The proof uses methods of Geerse/Hof [**GH**] and ideas from Durand [**Du1, Du2**] and Priebe [**Pr**]. In fact, Geerse/Hof established a similar result for a tiling associated to a primitive substitution.
(b) In the one-dimensional case related results have been shown by the author in [**Le2, Le3**].
(c) Existence of averages for real-valued functions on linearly repetitive Delone dynamical systems have been established by Lagarias/Pleasansts [**LP**].

2. The integrated density of states

In this section we present an operator algebraic approach to operators associated with quasicrystals. This allows one to establish basic properties of these operators. Moreover, it can be combined with the results of the previous section to prove existence of the integrated density of states in a very uniform sense. Related material can be found in [**Ke, KP, BHZ**].

We shall use Delone dynamical systems as parameter spaces for operators associated with quasicrystals. As discussed in Chapter 1, Ω is then viewed as standing for a fixed type of aperiodic order and the elements $\omega \in \Omega$ are considered as specific realizations of this type of (dis)order.

This is very similar to the random models [**CL, PF, St**] and the almost random framework introduced in [**Be1, Be3**]. However, there is one important difference: in the situation at hand the Hamiltonian H_ω is naturally defined on $\ell^2(\omega)$ and the latter space varies with $\omega \in \Omega$. The family $(H_\omega)_{\omega\in\Omega}$ of random operators should satisfy the *covariance condition*

$$H_{\omega+t} = U_t H_\omega U_t^*,$$

where $U_t : \ell^2(\omega) \to \ell^2(\omega + t)$ is the unitary operator induced by translation.

Let a DDSF (Ω, T) be given and consider the bundle

$$\Xi := \{(\omega, x) | \omega \in \Omega, x \in \omega\} \subset \Omega \times \mathbb{R}^d,$$

that carries a natural measurable structure induced by $C(\Omega) \otimes C_c(\mathbb{R}^d)$. Since Ω is compact, there exist T–invariant measures on Ω. Let μ be such an invariant measure. Then, μ induces an measure m on Ξ according to

$$m = \int_\Omega \left(\sum_{x\in\omega} \delta_x \right) d\mu(\omega),$$

where δ_x denotes the unit mass at x. This yields a direct integral decomposition

$$L^2(\Xi, m) = \int_\Omega^\oplus \ell^2(\omega) d\mu(\omega).$$

For a measurable, essentially bounded $H = (H_\omega)_{\omega\in\Omega}$ let

$$\pi(H) = \int_\Omega^\oplus H_\omega d\mu(\omega) \in \mathcal{B}(L^2(\Xi, m)).$$

Using Connes' non-commutative integration theory, one can easily see that

$$\mathcal{N}(\Omega, \mu) := \{A = (A_\omega)_{\omega\in\Omega} | A \text{ covariant, measurable and essentially bounded}\}/\sim$$

is a von Neumann algebra. Here, $\sim$ is the equivalence relation which identifies random operators which agree up to a set of measure zero. This von Neumann algebra carries a canonical trace, viz

$$\tau(H) = \int_\Omega \mathrm{tr}(H_\omega M_f) d\mu(\omega).$$

Here, $f \in C_c(\mathbb{R}^d)$ is arbitrary with $f \geq 0$ and $\int f(x)dx = 1$, M_f acts on $\ell^2(\omega)$ by $(M_f h)(x) = f(x)h(x)$, and τ does not depend on the particular choice of $f \in C_c(\mathbb{R}^d)$ satisfying these requirements. Note that the operator $H_\omega M_f$ has finite rank, since only finitely many points of ω lie in the support of f by the Delone property.

This trace allows one to associated a canonical measure ρ_H to every selfadjoint $H \in \mathcal{N}(\Omega, m)$ by

$$\langle \rho_H, \varphi \rangle := \tau(\varphi(H)) \text{ for } \varphi \in C_b(\mathbb{R}),$$

The following holds.

THEOREM 2.1. [**LS3**] *Let (Ω, T) be an DDS, μ an invariant measure and $H \in \mathcal{N}(\Omega, \mu)$ selfadjoint.*

(a) ρ_H is a spectral measure of H and $\pi(H)$.

(b) Let furthermore μ be ergodic. Then for μ–a.e. $\omega \in \Omega$ the spectral properties of H_ω do not depend on $\omega \in \Omega$. In particular, we have supp$\rho_H = \sigma(H_\omega)$ for μalmost-every $\omega \in \Omega$ in this case. Moreover, the discrete spectrum of H_ω is void.

This result gives the already mentioned analog to the basic results for random operators, [**CL, PF, St**]. In different context it can be found in [**Le2, LPV**].

As mentioned already, aperiodic order is a topological topic and we therefore have to single out a C^*-subalgebra of random operators. This will be done next.

DEFINITION 2.2. Let Ω be a DDSF. A family $A = (A_\omega)$, $A_\omega \in \mathcal{B}(\ell^2(\omega))$ is said to be an *operator (family) of finite range* if there exists $R > 0$ such that

- $(A_\omega \delta_x | \delta_y) = 0$ if $x, y \in \omega$ and $|x - y| \geq R$.
- $(A_\omega \delta_{x+t} | \delta_{y+t}) = (A_{\tilde{\omega}} \delta_x | \delta_y)$ if $\omega \cap (B_R(x+t) \cup B_R(y+t)) = \tilde{\omega} \cap (B_R(x) \cup B_R(y)) + t$ and $x, y \in \tilde{\omega}$.

This definition means that the matrix elements $A_\omega(x,y) = (A_\omega \delta_x | \delta_y)$ of A_ω only depend on a sufficiently large patch around x and y and vanish if the distance between x and y is too large. Since there are only finitely many nonequivalent patches, an operator of finite range is bounded in the sense that $\|A\| = \sup_{\omega \in \Omega} \|A_\omega\| < \infty$. The completion of the space of all finite range operators with respect to this norm is a C^*–algebra. We denote it by $\mathcal{A}(\Omega)$. By definition of the norm, the representations $\pi_\omega : A \mapsto A_\omega$ can be uniquely extended to representations of $\mathcal{A}(\Omega)$ which are again denoted by $\pi_\omega : \mathcal{A}(\Omega) \to \mathcal{B}(\ell^2(\omega))$.

THEOREM 2.3. [**LS3**] *The following conditions on Ω are equivalent:*

(i) *(Ω, T) is minimal.*
(ii) *For any selfadjoint $A \in \mathcal{A}(\Omega)$ the spectrum $\sigma(A_\omega)$ is independent of $\omega \in \Omega$.*
(iii) *π_ω is faithful for every $\omega \in \Omega$.*

Next we relate the "abstract integrated density of states" ρ_H to the integrated density of states as considered in random or almost random models and defined by a volume limit over finite parts of the operator.

To do so note, that $\mathcal{A}(\Omega) \subset \mathcal{N}(\Omega, \mu)$ for every invariant measure μ.

Now, for a selfadjoint $A \in \mathcal{A}(\Omega)$ and bounded $Q \subset \mathbb{R}^d$ the restriction $A_\omega|_Q$ of A_ω on $\ell^2(Q \cap \omega)$ has finite rank. Thus, the spectral counting function

$$n(A_\omega, Q)(E) := \#\{ \text{ eigenvalues of } A_\omega|_Q \text{ below } E\}$$

is finite. Then, $\frac{1}{|Q|} n(A_\omega, Q)$ is the distribution function of the measure $\rho(A_\omega, Q)$, defined by

$$\langle \rho(A_\omega, Q), \varphi \rangle := \frac{1}{|Q|} \mathrm{tr}(\varphi(A_\omega|_Q)) \text{ for } \varphi \in C_b(\mathbb{R}).$$

One of the fundamentals of random operator theory is the existence of the infinite volume limit

$$N(E) = \lim_{Q \nearrow \mathbb{R}^d} \frac{1}{|Q|} n(A_\omega, Q)(E)$$

independently of ω a.s. This amounts to the convergence in distribution of the measures $\rho(A_\omega, Q)$ just defined. As a first result on weak convergence we get:

THEOREM 2.4. *Let (Ω, T) be a uniquely ergodic DDSF and $A \in \mathcal{A}(\Omega)$ selfadjoint. Then, for any van Hove sequence Q_n,*

$$\rho(A_\omega, Q_n) \to \rho_A \text{ weakly as } n \to \infty,$$

where μ is the unique ergodic probability measure.

This result can be regarded as abstract versions of the celebrated Shubin's trace formula [**Sh**]. See also the discussion [**AS, CL, PF, Be1, Be3**] for the almost periodic, random and almost random case. In some sense, it falls well within the standard theory. Following the strategy of [**Be3, Ke**] the proof proceeds in two steps. In the first step we show existence of the limit

$$\lim_{n\to\infty} \frac{1}{|Q_n|} \mathrm{tr}(\varphi(A_\omega)|_{Q_n}).$$

This uses the usual ergodic theorem. In the second step, we show that the difference

$$\frac{1}{|Q_n|} \mathrm{tr}(\varphi(A_\omega)|_{Q_n} - \varphi(A_\omega|_{Q_n}))$$

becomes small for large n. For the special case of primitive substitutions this is contained in [**Ke**]. For this case, a different proof of existence of the limit is contained in [**Ho1**]. A general treatment of the first step can also be infered from [**Ho2**].

Now, using Theorem 1.5 discussed in the previous section, we can actually do better than vague convergence. More precisely, we can show that the actual distribution functions $(E \mapsto \rho(A_\omega, Q_n)((-\infty, E])$ converge uniformly:

THEOREM 2.5. [**LS4**] *Let (Ω, T) be a minimal, uniquely ergodic, aperiodic DDSF and (A_ω) an selfadjoint operator of finite range. Then, for any van Hove sequence Q_n, and any $\omega \in \Omega$*

$$(E \mapsto \rho(A_\omega, Q_n)((-\infty, E])) \to \ (E \mapsto \rho_A((-\infty, E]))$$

uniformly for $n \to \infty$.

To infer this theorem from Theorem 1.5, we have to first provide a Banach space containing the functions $E \mapsto \rho(A_\omega, Q_n)((-\infty, E])$ and $E \mapsto \rho_A((-\infty, E])$ and then prove suitable almost additivity properties. Details are given in [**LS4**]. Here, we would like to shortly discuss the relevance of this uniform convergence statement.

If the limiting distribution is continuous, uniform convergence follows from weak convergence by abstract measure theory. In our cases the limiting distribution $E \mapsto \rho_A((-\infty, E])$ may have points of discontinuity (see e.g. [**KLS**]). Thus, uniformity is a remarkable feature. It is very much in line with the overall intuition that aperiodic order is close to a periodic structure.

3. Further remarks

The results presented above serve two purposes. They show that the order is reflected in very strong averaging properties of the underlying dynamical system and they provide a first modest step towards spectral theory of quasicrystal operators in higher dimensions.

Concerning this latter point the situation is much less than satisfactory. A key deficiency in the higher dimensional case (compared to the one-dimensional situation) is the missing of the transfer matrix formalism.

In one dimension the transfer matrix formalism allows one to rather directly "translate" combinatorial properties of the underlying system into properties of solutions of the corresponding eigenvalue equations. Properties of the solutions are then related to spectral theory. This strategy is particularly powerful as combinatorial properties are in some sense simple due to the very meaning of aperiodic order.

There is no equivalent to this strategy in the higher dimensional case. In this respect the higher dimensional case for aperiodic order is worse than the case of high disorder for the following reason: Due to the strong order in the models there are not enough parameters to wiggle. This leaves much space for new ideas ....

Let us mention that the results of this chapter and in particular the result on uniform convergence of the integrated density of states can be used to tackle a specific feature of aperiodic order, viz the occurrence of locally supported eigenfunctions. This has been investigated in joint work with Steffen Klassert and Peter Stollmann [**KLS**]. There, it is shown that points of discontinuity of the integrated density of states occur exactly at those energies for which compactly supported eigenfunctions exist.

Also, we would like to mention that statements on generic operators associated to Delone sets in the continuum are also available: Generically such an operator has purely singularly continuous spectrum in certain intervals. This has recently been proven in joint work with Peter Stollmann [**LS5**]. The proof relies on (a slight extension of) Simon's Wonderland theorem [**Si**].

CHAPTER 4

Pure point diffraction and measure dynamical systems

In this chapter we present and discuss the results of [**BL1, BL2**]. These results are obtained by the author in joint work with Michael Baake.

Since the discovery of quasicrystals twenty years ago [**SBGC**] the phenomenon of pure point diffraction has attracted a lot of attention. Next, we discuss a basic set up for diffraction and give a brief outline of our corresponding results. A precise version of these results is then given in the next section.

In a simplified manner diffraction can be modeled as follows [**Co**]: The positions of the atoms of a solid are described by a set $\varLambda \subset \mathbb{R}^d$. To $\varLambda \subset \mathbb{R}^d$ one associates the *Dirac comb*:

$$\delta_\varLambda := \sum_{x\in\varLambda} \delta_x,$$

where δ_z denotes the unit point measure at $z \in \mathbb{R}^d$. If $\varLambda$ is finite, one can take the Fourier transform of the Dirac comb and obtains $\mathcal{F}(\delta_\varLambda)(q) := \sum_{x\in\varLambda} \exp(ixq)$. The intensity $I_\varLambda$ of the diffraction is then given by

$$I_\varLambda(q) := |\mathcal{F}(\delta_\varLambda)(q)|^2 = \mathcal{F}(\delta_\varLambda * \widetilde{\delta_\varLambda})(q) = \sum_{x,y\in\varLambda} \exp(i(x-y)q).$$

It is this quantity that describes the outcome of a diffraction experiment.

In order to avoid the complicated surface effects, we assume that the solid take the whole space. This is an idealization. Thus, we consider $\varLambda$ which fills the space in not too irregular a manner. Again, there is a Dirac comb: $\delta_\varLambda := \sum_{x\in\varLambda} \delta_x$. However, in this case the formal Fourier transform $\mathcal{F}(\delta_\varLambda)(q) = \sum_{x\in\varLambda} \exp(ixq)$ does not make sense. In fact, $\mathcal{F}(\delta_\varLambda)$ is, in general, not even a measure.

This problem can be solved by averaging. Thus, one calculates the intensity per unit volume:

$$\begin{aligned} I_\varLambda &:= \lim_{n\to\infty} \frac{1}{|B_n|} I_{\varLambda\cap B_n} \\ &= \lim_{n\to\infty} \frac{1}{|B_n|} \mathcal{F}(\,\delta_{\varLambda\cap B_n} * \widetilde{\delta_{\varLambda\cap B_n}}\,) \\ &= \mathcal{F}(\lim_{n\to\infty} \frac{1}{|B_n|} \delta_{\varLambda\cap B_n} * \widetilde{\delta_{\varLambda\cap B_n}}\,) \\ &= \mathcal{F}(\gamma) \end{aligned}$$

with the *autocorrelation measure*

$$\gamma = \lim_{n\to\infty} \frac{1}{|B_n|}\, \delta_{\Lambda\cap B_n} * \widetilde{\delta_{\Lambda\cap B_n}}.$$

Here, the limits are taken in the vague topology and assumed to exist. By its very definition γ is an averaged quantity. It deals with differences of positions in the solid and describes their mean number of occurrences.

The Fourier transform $\mathcal{F}(\gamma) = \widehat{\gamma}$ of the autocorrelation measure is called the *diffraction measure*. It describes the intensity of a scattered beam. If $\widehat{\gamma}$ is a pure point measure, we speak about *pure point diffraction*. Note that this is only possible if "a lot" of interference occurs. This means the positions of Λ are correlated on long range. Therefore, as already discussed in Chapter 1, pure point diffraction means a high degree of order in the solid.

A special case of this situation is, of course, given by a periodic Λ. There, it is a direct consequence of the Poisson summation formula, that $\widehat{\gamma}$ is a pure point measure.

There are two questions we want to tackle. They are the following:

Question 1: How can one characterize pure point diffraction?

Question 2: How stable is pure point diffraction?

Let us start by discussing Question 1: There are three characterization of pure point diffraction. These are the characterizations by:

- Pure point spectrum of the associated dynamical system (see [**BL1, Dwo, vEM, Gou2, LMS, Schl, Qu**]).
- Almost periodicity of the autocorrelation measure γ (see [**BM2**]).
- Almost periodicity of the underlying set Λ (see [**Gou2, MS, Qu**]).

Here, we will be concerned with the first characterization.

In this case, one does not only consider a single Λ but rather the set Ω of all subsets of $\mathbb{R}^d$ with the same kind of (dis)order as Λ (see discussion in Chapter 1). This set is invariant under the translations α_t, $t \in \mathbb{R}^d$. Thus, (Ω, α) is a topological dynamical system. Now, assume that (Ω, α) is equipped with an α-invariant probability measure m. Then, there is a unitary representation

$$T : \mathbb{R}^d \longrightarrow \text{Unitary operators on } L^2(\Omega, m),\ (T^t f)(\omega) = f(\alpha_{-t}\omega).$$

If $L^2(\Omega, m)$ possess an orthonormal basis of eigenfunctions (i.e. functions $f \neq 0$ with $T^t f = \exp(iyt) f$ for all $t \in \mathbb{R}^d$), then T is said to have *pure point dynamical spectrum.*

Concerning the notation, a word of **warning** may be in order. The pieces of notation just discussed are chosen to fit with the notation of [**BL1, BL2**]. Thus, T does not denote the action of $\mathbb{R}^d$ on Ω as in the previous chapters, but rather denotes the unitary representation. The action of $\mathbb{R}^d$ is denoted by α.

The relation between pure point dynamical spectrum and pure point diffraction has been investigated by various people: Starting with the work of Dworkin [**Dwo**], it has been shown in increasing levels of generality that pure point dynamical spectrum implies pure point diffraction [**Ho2, Schl**] (see [**vEM**] as well).

These results have played a prominent role in establishing pure point diffraction for examples [**Ho2, Schl, Sol2, Rob**].

In a special one-dimensional situation (and a somewhat different context) equivalence of the two notions was shown by Queffélec [**Qu**]. Lee/Moody/Solomyak could then show equivalence in arbitrary dimensions provided the elements of Ω satisfied some regularity assumptions [**LMS**]. For quite general point sets, equivalence has been shown recently by Gouéré [**Gou2**]. He also provides a closed formula for γ in terms of the underlying dynamical system [**Gou1**]. His work relies on a connection to stochastic processes and Palm measures.

Now, both from the physical and from the mathematical point of view, the restriction to point sets (i.e. measures of the form $\delta_\Lambda = \sum_{x\in\Lambda}\delta_x$), is rather restrictive and somewhat arbitrary. In fact, weighted Dirac combs $\sum_{x\in\Lambda} a_x\delta_x$ with suitable $a_x \in \mathbb{C}$, $x \in \Lambda$, or density distributions of the form $\sum_{x\in\Lambda}\rho(\cdot - x)$ with a $\rho \in C_c(\mathbb{R}^d)$, have also been considered in the past [**BM2, BD, LMS, Ric**].

This suggests to work with measures instead of point sets. In fact, these situations can be unified and extended by considering **translation bounded measures** instead of δ_Λ.

It turns out that characterization of pure point diffraction by pure point dynamical spectrum can be proven in the context of such measures. Moreover, one can give a closed formula for the autocorrelation measure in this context as well. This is the content of of the authors work with Michael Baake in [**BL1**]. It generalizes the corresponding results of [**Dwo, Ho2, Gou2, LMS**].

This generalization from point sets to measures requires some care as essentially all information of geometric type is lost. Roughly speaking, geometric considerations have to be replaced by functional analytic ones.

We will illustrate this by discussing the three key steps behind our proof of the equivalence of pure point dynamical and pure point diffraction spectrum:

The first step is to prove that γ is actually a spectral measure for a sub-representation $T|_{\mathcal{U}}$ of T. This shows that pure point diffraction is equivalent to pure point spectrum of this sub-representation (see below).

The next step is to realize that this sub-representation is "large" in a suitable sense.

The third step then is to show that pure point spectrum of "large" sub-representations forces pure point spectrum of the whole representation. This is actually true in quite general a situation and it is not only the pure pointedness of the spectrum but various other properties that are determined by the sub-representation.

A word on the meaning of "large" in the context of sub-representations may be in order. In our context "large" is given in terms of some algebra of continuous functions separating points. This allows one to apply the Stone/Weierstraß Theorem at an appropriate point.

Both this strategy to prove the equivalence and the proofs of the mentioned key results are essentially due to the author.

Let us now discuss the question of stability of pure point diffraction. There, one starts with one set Λ with pure point diffraction and asks whether a suitable deformed set Λ' has pure point diffraction as well.

Stability of pure point diffraction under suitable deformation of the underlying set has been studied in [**BD, BSW, Ho4**] (see [**CS1, CS2**] for related results on deformations as well).

It turns out that the dynamical systems approach gives a very direct way to study it. This is carried out in [**BL2**] in joint work with Michael Baake. It generalizes the corresponding results of [**BD, Ho4**]. Here, the main idea is to proceed in the following three steps.

In the first step, one considers the associated dynamical systems $(\Omega(\Lambda), \alpha)$ and $(\Omega(\Lambda'), \alpha)$ and shows that $(\Omega(\Lambda'), \alpha)$ is a factor of $(\Omega(\Lambda), \alpha)$.

Now, in the second step, one shows that a factor actually inherits certain spectral properties.

In the third step stability of pure point diffraction can then be established by appealing to equivalence of pure point diffraction and pure point dynamical spectrum twice: As Λ has pure point diffraction, the dynamical system $(\Omega(\Lambda), \alpha)$ has pure point spectrum. This is inherited by $(\Omega(\Lambda'), \alpha)$ and we infer pure point diffraction of Λ'.

This strategy has been suggested by Michael Baake in the context of point dynamical systems. It has then been worked out for measure dynamical systems by the author in joint work with Michael Baake. Substantial parts of the actual arguments were supplied by the author.

To summarize, the aims of our work as studied in [**BL1, BL2**] are as follows:

- Set up a framework involving measures instead of point sets.
- Provide a closed formula for the autocorrelation measure in this context.
- Study relations between dynamical spectrum and diffraction spectrum in this setup.
- Study stability of pure point diffraction in this context.

It should be mentioned that [**BL1, BL2**] are not restricted to $\mathbb{R}^d$ but rather deal with arbitrary locally compact, σ-compact Abelian groups. However, the main physical motivation is clearly the case of $\mathbb{R}^d$. Thus, we restrict our attention to this case in this chapter.

1. Spectral properties forced by sub representations

The aim of this section is to present the abstract dynamical system part of the results of [**BL1**]. These results will show that a "lot of pure point spectrum" actually forces pure point spectrum.

Let Ω be a compact Hausdorff space and

$$\alpha\colon \mathbb{R}^d \times \Omega \longrightarrow \Omega \tag{9}$$

be a continuous action of $\mathbb{R}^d$ on Ω, where, of course, $\mathbb{R}^d \times \Omega$ carries the product topology Then, (Ω, α) is called a *topological dynamical system.* The set of continuous functions on Ω will be denoted by $C(\Omega)$.

Let m be a G-invariant probability measure on Ω and denote by $L^2(\Omega, m)$ the corresponding space of square integrable functions on Ω. As discussed above, the action α induces a unitary representation $T = T_m$ of $\mathbb{R}^d$ on $L^2(\Omega, m)$ in the obvious way, by

$$T : \mathbb{R}^d \longrightarrow \text{Unitary operators on } L^2(\Omega, m),\ (T^t f)(\omega) = f(\alpha_{-t}\omega).$$

A non-zero $f \in L^2(\Omega, m)$ is called an *eigenvector* (or eigenfunction) of T if there exists an $y \in \mathbb{R}^d$ with $T^t h = \exp(ity)h$ for every $t \in G$. The closure (in $L^2(\Omega, m)$) of the linear span of all eigenfunctions of T will be denoted by $\mathcal{H}_{pp}(T)$.

A crucial ingredient in our considerations is the following variant (and extension) of a result from [**LMS**].

Lemma 1.1. *Let* (Ω, α) *be a topological dynamical system with an invariant measure* m. *Then,* $\mathcal{H}_{pp}(T) \cap C(\Omega)$ *is a sub-algebra of* $C(\Omega)$ *which is closed under complex conjugation and contains all constant functions. Similarly,* $\mathcal{H}_{pp}(T) \cap L^\infty(\Omega, m)$ *is a sub-algebra of* $L^\infty(\Omega, m)$ *that is closed under complex conjugation and contains all constant functions.*

When combined with the theorem of Stone/Weierstraß, this lemma allows one to rather directly prove the following theorem.

Theorem 1.2. *Let* (Ω, α) *be a topological dynamical system with invariant probability measure* m. *Then, the following assertions are equivalent.*

(a) T *has pure point spectrum, i.e.,* $\mathcal{H}_{pp}(T) = L^2(\Omega, m)$.
(b) *There exists a subspace* $\mathcal{V} \subset \mathcal{H}_{pp}(T) \cap C(\Omega)$ *which separates points.*

This result is one of the two abstract cornerstones of our characterization of pure point diffraction by pure point dynamical spectrum. The other cornerstone is the result that diffraction measure is a spectral measure for a sub-representation (see below).

The previous result can be generalized and extended. To do so, we need a special concept of "density of a subspace with respect to multiplication". This is defined next.

Definition 1.3. A subspace $\mathcal{V}$ of $L^2(\Omega, m)$ is said to satisfy condition MD if the set of products $f_1 \cdot \ldots \cdot f_n$ with $n \in \mathbb{N}$, $f_i \in \mathcal{V} \cap L^\infty(\Omega, m)$ or $\overline{f_i} \in \mathcal{V} \cap L^\infty(\Omega, m)$, $1 \leq i \leq n$, is total in $L^2(\Omega, m)$.

Now, our result on spectral properties forced by sub-representations reads as follows.

Theorem 1.4. *Let* (Ω, α) *be a topological dynamical system over* G *with* α-*invariant measure* m. *Let* $\mathcal{V}$ *be a closed* T-*invariant subspace of* $L^2(\Omega, m)$ *satisfying* MD. *If* $T|_{\mathcal{V}}$ *has pure point spectrum, then the following assertions hold:*

(a) T *has pure point spectrum.*
(b) *The group of eigenvalues of* T *is generated by the set of eigenvalues of* $T|_{\mathcal{V}}$.
(c) *If* $\mathcal{V}$ *has a basis consisting of continuous eigenfunctions of* $T|_{\mathcal{V}}$, *then* $L^2(\Omega, m)$ *has a basis consisting of continuous eigenfunctions of* T, *provided the multiplicity of each eigenvalue of* T *is at most countably infinite.*

2. Characterization of pure point spectrum

In this section, we discuss those results of [**BL1**] which refer to measure dynamical systems.

Uniformly discrete sets will be replaced by translation bounded measures. These are defined as follows.

Definition 2.1. For $C > 0$ and $V \subset \mathbb{R}^d$ open and relatively compact, we define

$$\mathcal{M}_{C,V} := \{\mu \text{ measure} : |\mu|(x + V) \leq C \text{ for all } x \in \mathbb{R}^d\}.$$

The elements of $\mathcal{M}_{C,V}$ are called (C, V)-translation bounded measures.

Note: A uniformly discrete point sets $\varLambda$ in $\mathbb{R}^d$ can be identified with a translation bounded measure via $\delta_\varLambda := \sum_{x\in\varLambda}\delta_x$.

As usual the convolution $\mu * \nu$ of two finite measures μ and ν on $\mathbb{R}^d$ is defined by

$$\mu * \nu(\varphi) := \int\int \varphi(x+y)d\mu(x)d\nu(y)$$

for $\varphi \in C_c(\mathbb{R}^d)$. For an arbitrary measure μ on $\mathbb{R}^d$, we define $\widetilde{\mu}$ by

$$\widetilde{\mu}(\varphi) := \overline{\int \overline{\varphi(-x)} d\mu(x)}.$$

We are now heading towards introducing dynamical systems based on translation bounded measures. To do so, we need a topology and an action. This will be discussed next.

The space $\mathcal{M}_{C,V}$ of (C,V)-translation bounded measures on $\mathbb{R}^d$ is equipped with the vague topology i.e. the weakest topology which makes all functionals of the form

$$\mathcal{M}_{C,V} \longrightarrow \mathbb{C},\ \ \mu \mapsto \mu(\varphi),$$

with $\varphi \in C_c(\mathbb{R}^d)$ continuous. Equipped with this topology $\mathcal{M}_{C,V}$ is a very nice space. More precisely, the following holds.

Theorem 2.2. *Let $C>0$ and a relatively compact open set V in $\mathbb{R}^d$ be given. Then, $\mathcal{M}_{C,V}$ is a compact metrizable space.*

Remark 2.3. When restricted to uniformly discrete point sets, this topology agrees with the topologies introduced in [**BHZ**] and [**LS1**] and we recover their compactness results.

We discuss the action next. There is a canonical continuous action α of $\mathbb{R}^d$ on $\mathcal{M}_{C,V}$:

$$\alpha\colon \mathbb{R}^d \times \mathcal{M}_{C,V} \longrightarrow \mathcal{M}_{C,V}\,, \quad \alpha_t(\mu) := \delta_t * \mu.$$

Now, we can finally define the objects of interest.

Definition 2.4. $(\varOmega,\alpha)$ is called a dynamical system on the translation bounded measures on $\mathbb{R}^d$ (TMDS) if there exist $C>0$ and $V\subset\mathbb{R}^d$ open, relatively compact, such that $\varOmega$ is a closed, α-invariant subset of $\mathcal{M}_{C,V}$,

Having introduced measure dynamical systems, we can now discuss the autocorrelation measure. In the introduction to this chapter this measure has been defined by an averaging procedure on $\mathbb{R}^d$. To make this averaging procedure work one needs an ergodicity assumption. It turns out that a different type of averaging, viz an averaging over $\varOmega$, can be applied without an ergodicity assumption. This gives a closed formula for the autocorrelation measure. In the ergodic case, both ways to define this measure agree by an ergodic theorem. Details are discussed next.

We start with the averaging over the space $\varOmega$.

Lemma 2.5. *Let $(\varOmega,\alpha)$ be a measure dynamical system with α-invariant probability measure m on it. Let $\sigma \in C_c(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d}\sigma(t)\,\mathrm{d}t = 1$ be given. For $\varphi\in C_c(\mathbb{R}^d)$ define*

$$\gamma_{\sigma,m}(\varphi) \ := \ \int_\varOmega\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(t+s)\sigma(t)\,\mathrm{d}\widetilde{\omega}(s)\,\mathrm{d}\omega(t)\,\mathrm{d}m(\omega).$$

Then, $\gamma_{\sigma,m}$ is a positive definite measure on $\mathbb{R}^d$ and does not depend on σ.

The lemma suggests the following definition.

DEFINITION 2.6. The measure $\gamma_m := \gamma_{\sigma,m}$ is called the autocorrelation measure. Its Fourier transform $\widehat{\gamma_m}$ is called the diffraction measure.

It remains to show that the autocorrelation just defined agrees with the one obtained by a limiting procedure. This is the content of the next theorem.

THEOREM 2.7. *Let (B_n) be a van Hove sequence. If (Ω, α) is a (TMDS) with a unique α-invariant ergodic probability measure m, then*

$$\frac{1}{|B_n|}\,\omega_{B_n} * \widetilde{\omega_{B_n}} \longrightarrow \gamma_m, \;\; n \longrightarrow \infty,$$

in the vague topology for every $\omega \in \Omega$. Here, ω_{B_n} denotes the restriction of ω to B_n.

REMARK 2.8. (a) In the ergodic case ergodic case a similar result is true with convergence holding for almost every $\omega \in \Omega$ [**BL1**].

(b) The proof of convergence is similar to the corresponding arguments of [**Dwo, Ho4, Schl**]. These works deal with special point sets. Thus, the above result generalizes them.

(c) Related results have been obtained by Gouéré [**Gou1, Gou2**] for special dynamical systems coming from point sets. In fact, these papers provide the first proof of a closed formula for the autocorrelation measure. He uses point processes and Palm measures. While the closed formula above is inspired by his work, our approach is different and in some sense more general as we deal with measures instead of point sets. Let us mention, however, that he proves an analogue of Theorem 2.7 without an ergodicity assumption.

Our next goal is to discuss the role of the diffraction measure as a spectral measure of a certain sub-representation. To do so, we need some further notation.

As discussed already, when equipped with an invariant probability measure m, the (TMDS) (Ω, α) gives rise to a unitary representation T of $\mathbb{R}^d$ with $(T^t f)(\omega) := f(\alpha_{-t}\omega)$.

By definition, Ω is a subset of the dual of $C_c(\mathbb{R}^d)$. Thus, we can embed $C_c(\mathbb{R}^d)$ into $C(\Omega)$ in the canonical way via

$$f : C_c(\mathbb{R}^d) \longrightarrow C(\Omega) \;\text{ by }\; f_\varphi(\omega) := \omega(\varphi).$$

LEMMA 2.9. *The set of functions $\mathcal{U}_0 := \{f_\varphi : \varphi \in C_c(\mathbb{R}^d)\}$ is a T-invariant subspace of $L^2(\Omega, m)$, and so is its closure $\mathcal{U}$.*

REMARK 2.10. The definition of f given here differs from the one in [**BL1**] by a sign. This does not matter as only the space $\mathcal{U}$ is relevant. However, this space is not changed.

The lemma has the following important consequence: In the context of TMDS, the canonical representation T of $\mathbb{R}^d$ on $L^2(\Omega, m)$ has a canonical sub-representation, namely, the restriction $T|_{\mathcal{U}}$ of T to $\mathcal{U}$. Thus, in this context we have two natural representations of $\mathbb{R}^d$. It turns out that the sub representation $T|_{\mathcal{U}}$ completely determines the pure point spectrum in the sense discussed in the previous section. This is discussed next.

By Stone's Theorem, every unitary representation S of $\mathbb{R}^d$ on $L^2(\Omega, m)$ possesses a projection valued measure

$$E_S\colon \text{Borel sets on } \mathbb{R}^d \longrightarrow \text{Projections on } L^2(\Omega, m)$$

with

$$\langle f, S_x f\rangle = \int_{\mathbb{R}^d} \exp(ixs)\, \mathrm{d}\langle f, E_S(s) f\rangle.$$

In this sense S is the (inverse) Fourier transform of E_S.

Definition 2.11. A measure ρ on $\mathbb{R}^d$ is is called a spectral measure for the unitary representation S if $E_S(B) = 0$ if and only if $\rho(B) = 0$.

After these preparations, we can state our result on the role of the diffraction measure.

Theorem 2.12. *Let (Ω, α, m) be a TMDS and $\mathcal{U}$ as above. Then, the measure $\widehat{\gamma_m}$ is a spectral measure for the restriction $T|_{\mathcal{U}}$ of T to $\mathcal{U}$.*

Remark 2.13. This type of result is implicit in [**Dwo, vEM, Ho4, Schl, LMS**].

The previous theorem can be combined with the results of Section 1 to give:

Theorem 2.14. *(Ω, α) TMDS with invariant probability measure m, $\mathcal{U}$ as above. If $\widehat{\gamma_m}$ is a pure point measure, the following assertions hold:*

(a) *T has pure point spectrum.*
(b) *The group of eigenvalues of T is generated by the set of points in $\widehat{G}$ with non-vanishing $\widehat{\gamma_m}$ measure.*
(c) *If $\mathcal{U}$ has a basis consisting of continuous eigenfunctions of T, then so has $L^2(\Omega, m)$.*

As a corollary of the preceeding theorem, we obtain the following result.

Corollary 2.15. *(Ω, α) TMDS with invariant probability measure m, $\mathcal{U}$ as above. Then, $\widehat{\gamma_m}$ is a pure point measure if and only if T has pure point spectrum.*

Remark 2.16. This result generalizes the corresponding results of [**Dwo, Gou2, LMS**].

3. Stability of pure point diffraction

This section is devoted to a discussion of results from [**BL2**]. This work deals with stability of pure point diffraction under suitable deformations. By the results of the previous section (see [**LMS**] as well), stability of pure point diffraction is equivalent to stability of pure point dynamical spectrum. This in turn can be investigated by means of factors.

The main application of this setup is a perturbation result for deformed model sets on $\mathbb{R}^d$. This generalizes the corresponding results of [**BD, Ho4**] (see [**BSW, CS1, CS2, Gou2, Ho2**] for results on deformed Delone sets as well).

Let us also mention that the results of [**BL2**] are more general than discussed below in two ways: They are not only valid on $\mathbb{R}^d$ but also on arbitrary locally compact, σ-compact Abelian groups. Moreover, they apply to measure dynamical systems and not only to point sets. However, as our main application concerns point sets on $\mathbb{R}^d$, we restrict our attention to this case in the sequel.

Our approach is based on studying factors of dynamical systems. A factor is defined as follows:

DEFINITION 3.1. (Θ, β) is called a *factor* of (Ω, α), with factor map Φ, if $\Phi : \Omega \longrightarrow \Theta$ is continuous and onto with $\Phi(\alpha_x(\omega)) = \beta_x(\Phi(\omega))$ for all $\omega \in \Omega$ and $x \in \mathbb{R}^d$.

The key point is that a factor inherits many properties of the underlying dynamical system.

PROPOSITION 3.2. *Let (Θ, β) be a factor of (Ω, α). Then the following holds:*
(a) If (Ω, α) is uniquely ergodic, minimal or strictly ergodic, the analogous property holds for (Θ, β) as well.
(b) If (Ω, α) is uniquely ergodic with pure point dynamical spectrum, the same holds for (Θ, β).
(c) If (Ω, α) is uniquely ergodic and all of its eigenfunctions are continuous, the same holds for (Θ, β)

We can now present a first abstract perturbation result.

As usual, the dynamical system generated by a uniformly discrete set $\Gamma \subset \mathbb{R}^d$ is denoted by $(\Omega(\Gamma), \alpha)$, i.e.

$$\Omega(\Gamma) := \overline{\{\alpha_t(\Gamma) : t \in G\}}.$$

Let Λ be an r-discrete point set in $\mathbb{R}^d$ (i.e. $\|x - y\| \geq r$ whenever $x, y \in \Lambda$ with $x \neq y$) and let

$$q : \Xi(\Omega(\Lambda)) := \{\Gamma \in \Omega(\Lambda) : 0 \in \Gamma\} \longrightarrow B_{\frac{r}{3}}$$

be continuous. For $\Gamma \in \Omega(\Lambda)$ define

$$\Gamma_q := \{x + q(-x + \Gamma) : x \in \Gamma\}.$$

THEOREM 3.3. *Assume the setting above. Then, the following assertions hold:*

(a) *If $(\Omega(\Lambda), \alpha)$ is uniquely ergodic or minimal, so is $(\Omega(\Lambda_q), \alpha)$.*
(b) *If $(\Omega(\Lambda), \alpha)$ is uniquely ergodic with pure point spectrum, so is $(\Omega(\Lambda_q), \alpha)$.*
(c) *If $(\Omega(\Lambda), \alpha)$ is uniquely ergodic and all of its eigenfunctions are continuous, the same holds for $(\Omega(\Lambda_q), \alpha)$.*

This theorem follows from Proposition 3.2, once we show that the dynamical system $(\Omega(\Lambda_q), \alpha)$ is a factor of $(\Omega(\Lambda), \alpha)$ via $\Gamma \mapsto \Gamma_q$. This, however, is a rather direct computation.

We will apply our result to so called model sets. These sets play a most prominent role in the study of aperiodic order. For example, both circle maps considered in Chapter 1 and Penrose tilings can be considered to be examples of model sets. In various ways model sets can be considered to be the simplest non-periodic sets (see [**Mo2**] for detailed discussion of model sets). This becomes already clear from their definition. Namely, loosely speaking they are "shadows" of periodic structures as they arise as a projection to $\mathbb{R}^d$ of a periodic structure in a higher dimensional space. Here are the details:

A *cut and project scheme* is given by

$$\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times H & \xrightarrow{\;\pi_{\text{int}}\;} & H \\
\cup & & \cup & & \cup \text{ dense} \\
L & \xleftarrow{\;1-1\;} & \tilde{L} & \longrightarrow & L^\star \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^\star
\end{array}$$

where

- H is a locally compact, σ-compact group, called the *internal space*,
- $\tilde{L}$ is a *lattice* in $\mathbb{R}^d \times H$,
- π and π_{int} are the canonical projections and π restricted to $\tilde{L}$ is injective and the image $\pi_{\text{int}}(\tilde{L})$ is dense in H.

In this case $\star : L \longrightarrow L^\star$, $x \mapsto \pi_{\text{int}}(\pi^{-1}(x))$ is well defined. A *model set* is any translate of a set of the form

$$\curlywedge(W) \;:=\; \{x \in L : x^\star \in W\},$$

where the *window* W is a relatively compact subset of H with nonempty interior. A model set is called *regular* if ∂W has Haar measure 0 in H,

By their very definition, regular model sets are closely related to periodic structures. Periodic structures have pure point diffraction. A fundamental result states that this holds for model sets as well.

THEOREM 3.4. [**Ho2, Schl**] *Regular model sets are pure point diffractive. In fact,* $(\varOmega(\varLambda), \alpha)$ *is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions.*

We can now study deformed model sets. Thus, let $\varLambda := \curlywedge(W)$ be a regular r-discrete model set. Let

$$\vartheta \colon H \longrightarrow B_{\frac{r}{3}}$$

be a continuous function with compact support. Then, we define the *deformed model set*

$$\varLambda_\vartheta \;:=\; \{x + \vartheta(x^\star) : x \in \varLambda\}.$$

Our main result on deformed model sets now reads as follows.

THEOREM 3.5. *Assume the setting above. Then,* $\varLambda_\vartheta$ *is pure point diffractive. In fact, the dynamical system* $(\varOmega(\varLambda_\vartheta), \alpha)$ *is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions.*

A proof of this result can be outlined as follows: By the previous theorem, we know already that $(\varOmega(\varLambda), \alpha)$ has pure point spectrum. By Theorem 3.3, it suffices to show that

$$\theta \circ^\star : \Xi(\varOmega(\varLambda)) \longrightarrow B_{\frac{r}{3}}$$

is continuous. This requires some care, as $\star$ is not really defined on $\Xi(\varOmega(\varLambda))$. So, we first have to extend $\star$. This can be done by appealing to results of Schlottmann [**Schl**].

4. Further remarks

The results of the previous sections underline the relevance of (measure) dynamical systems in the study of aperiodic order. It is shown that pure point diffraction and its stability can be understood in terms of suitable dynamical systems.

This point of view of dynamical systems has actually been of use for further lines of research.

One such line is the study of model sets. As mentioned already, these form a most prominent class of aperiodically ordered sets. In this case, the associated dynamical systems is not only strictly ergodic with pure point dynamical spectrum, but has further regularity features: The eigenfunctions are continuous and separate almost all points. It turns out that this actually characterizes regular model sets (among so called Meyer sets). This is shown in joint work with Michael Baake and Robert V. Moody [**BLM**]. In this case, it is even possible to mark the border between periodicity and aperiodicity: The dynamical system is periodic if and only if the eigenfunctions separate all points [**BLM**].

Another line of research concerns dense Dirac combs. There one considers measures of the form

$$\sum_{x\in\varLambda} a_x\delta_x,$$

where $\varLambda$ is a dense subset of $\mathbb{R}^d$ and the coefficients a_x, $x\in\varLambda$, are suitably decaying. These models can be seen as suitable approximations to disordered systems. As such they have been studied by Christoph Richard in [**Ric**]. There, it is shown that they are pure point diffractive under suitable assumptions on the coefficients. It turns out that a rather direct dynamical systems approach to his results exists. Via this approach Richard's original results can be extended in various directions. This is currently being worked out by the author in joint work with Michael Baake and Christoph Richard [**BLR**].

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Part 2

Original manuscripts

CHAPTER 5

D. Lenz, *Uniform ergodic theorems on subshifts over a finite alphabet*, Ergodic Theory & Dynamical Systems 22 (2002), 245–255

UNIFORM ERGODIC THEOREMS ON SUBSHIFTS OVER A FINITE ALPHABET

DANIEL LENZ

Fachbereich Mathematik, Johann Wolfgang Goethe-Universität, 60054 Frankfurt, Germany

E-mail: dlenz@math.uni-frankfurt.de

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Abstract. We investigate uniform ergodic type theorems for almost additive and subadditive functions on a subshift over a finite alphabet. We show that every uniquely ergodic subshift admits a uniform ergodic theorem for Banach-space-valued almost additive functions. We then give a necessary and sufficient condition on a minimal subshift to allow for a uniform subadditive ergodic theorem. This provides in particular a sufficient condition for unique ergodicity.

1. Introduction

Ergodic theorems for almost additive and subadditive functions play a role in several branches of mathematics and physics. In particular, they are an important tool in statistical mechanics as well as in the theory of random operators (cf. [1, 9, 18, 19] and references therein).

During recent years lattice gas models and random operators on aperiodic tilings have received a lot of interest both in one dimension and in higher dimensions (cf. [2, 9, 10, 11, 12] and references therein). In these cases one rather expects uniform ergodic theorems to hold.

The aim of this paper is to provide a thorough study of the validity of such theorems in the one-dimensional case.

In particular, we show that every uniquely ergodic subshift over a finite alphabet admits a uniform additive ergodic theorem. Moreover, we give a necessary and sufficient condition for a minimal subshift to allow for a subadditive theorem. This gives in particular a sufficient condition for unique ergodicity.

The proofs are quite elementary and conceptual. Thus, it is to be expected that a considerable part of the material presented here, can be extended to higher dimensional tiling dynamical systems.

More precisely, we consider the following situation:
Let A be a finite set called the alphabet and equipped with the discrete topology. Let Ω be a subshift over A. This means that Ω is a closed subset of $A^{\mathbb{Z}}$, where $A^{\mathbb{Z}}$ is given the product topology and Ω is invariant under the shift operator $T : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$, $Ta(n) \equiv a(n+1)$.

We consider sequences over A as words and use standard concepts from the theory of words ([6, 16]). In particular, $\mathrm{Sub}(w)$ denotes the set of subwords of w, the empty word is denoted by ϵ, the number of occurrences of v in w is denoted by $\sharp_v(w)$ and the length $|w|$ of the word $w = w(1)\ldots w(n)$ is given by n. To Ω we associate the set $\mathcal{W} = \mathcal{W}(\Omega)$ of finite words associated to Ω given by $\mathcal{W} \equiv \cup_{\omega\in\Omega}\mathrm{Sub}(\omega)$. Similarly, we define $\mathcal{W}_n \equiv \{w \in \mathcal{W} : |w| = n\}$ for $n \in \mathbb{N}$. A word $w \in \mathcal{W}$ is called primitive if v can not be written as $v = w^l$, with $w \in \mathcal{W}$ and $l \geq 2$. For a finite set M, we define $\sharp M$ to be the number of elements in M.

To phrase our additive ergodic theorem, we need the following definition.

Definition 1.1. *Let $(B, \|\cdot\|)$ be a Banach space. A function $F : \mathcal{W} \longrightarrow B$ is called almost additive if there exists a constant $D > 0$ and a nonincreasing function $c : [0,\infty) \longrightarrow [0,\infty)$ with $\lim_{r\to\infty} c(r) = 0$ s.t. the following holds*

(A1) $\|F(v) - \sum_{j=1}^n F(v_j)\| \leq \sum_{j=1}^n c(|v_j|)|v_j|$ *for* $v = v_1 \ldots v_n \in \mathcal{W}$.
(A2) $\|F(v)\| \leq D|v|$ *for every* $v \in \mathcal{W}$.

Then the additive theorem can be stated as follows.

Theorem 1. *Let (Ω, T) be a subshift over the finite alphabet A with associated set of words $\mathcal{W}$. Then the following are equivalent:*

(i) *(Ω, T) is uniquely ergodic, i.e.* $\lim_{|w|\to\infty} \frac{\sharp_v(w)}{|w|}$ *exists for all* $v \in \mathcal{W}$.
(ii) *The limit* $\lim_{|w|\to\infty} \frac{F(w)}{|w|}$ *exists for every Banach-space-valued almost additive function F on $\mathcal{W}$.*
(iii) *The limit* $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n f(T^j\omega)$ *exists uniformly in $\omega \in \Omega$ for every continuous Banach-space-valued function f on Ω.*

Remark 1. (a) The equivalence of (iii) and (i) and the implication (ii) $\Longrightarrow$ (i) are standard. Thus, the main content of the theorem is the implication (i) $\Longrightarrow$ (ii).
(b) The proof of the theorem is the same for numerical functions as for Banach-space-valued functions. However, in applications (e.g. to random operators) one considers measure valued functions. For this reason we have stated it in the Banach space version.

To introduce the second result of this paper, recall that a function $F : \mathcal{W} \longrightarrow \mathbb{R}$ is called subadditive if it satisfies $F(ab) \leq F(a) + F(b)$. The dynamical system (Ω, T) is said to satisfy (SET), i.e. to admit a uniform subadditive ergodic theorem, if, for every subadditive function F, the limit $\lim_{|w|\to\infty} \frac{F(w)}{|w|}$ exists. As shown below (cf. Section 3), the limit is then given by $\inf_{n\in\mathbb{N}} F^{(n)}$, where $F^{(n)} \equiv \max\{\frac{F(v)}{|v|} : v \in \mathcal{W}_n\}$.

Define the functions $l_v : \mathcal{W} \longrightarrow \mathbb{R}$, for $v \in \mathcal{W}$, and $\nu : \mathcal{W} \longrightarrow \mathbb{R}$ by

$$l_v(w) \equiv (\text{Maximal number of disjoint copies of } v \text{ in } w) \cdot |v|$$

and

$$\nu(v) \equiv \liminf_{|w|\to\infty} \frac{l_v(w)}{|w|}.$$

Then a subshift (Ω, T) over A is said to satisfy uniform positivity of quasiweights (PQ) if the following condition holds:

(PQ) There exists a constant $C > 0$ with $\nu(v) \geq C$ for all $v \in \mathcal{W}$.

Our result on subadditive ergodic theorems then reads as follows:

Theorem 2. *Let* (Ω, T) *be a minimal subshift. Then the following are equivalent:*
(i) *The subshift* (Ω, T) *satisfies (SET).*
(ii) *The subshift* (Ω, T) *satisfies (PQ).*

Remark 2. (a) As the validity of a subadditive ergodic theorem immediately implies that the underlying subshift is uniquely ergodic, we see that (PQ) is a sufficient condition for unique ergodicity.
(b) Condition (PQ) might be set in prospective by comparing it with uniform positivity of weights (PW) as well as with a highest power condition (HP) given as follows:

(PW) There exists $E > 0$ with $\liminf_{|w|\to\infty} \frac{\sharp_v(w)}{|w|}|v| \geq E$ for all $v \in \mathcal{W}$.

(HP) There exists an $N > 0$ s.t. $v^k \in \mathcal{W}$ implies $k \leq N$ (or equivalently: there exists a $\kappa > 0$ s.t. v prefix of uv with u not empty implies $|u| \geq \frac{|v|}{\kappa}$).

It is not hard to show, that (PW) together with (HP) implies (PQ). In fact, as pointed out to the author by the referee (PW) alone implies (PQ). A proof is given in Proposition 4.2.
(c) A particular important class of systems satisfying (PW) (or (PQ)) are those satisfying condition (LR) given as follows:

(LR) There exists a $D > 0$ s.t. every $v \in \mathcal{W}$ is a factor of every $w \in \mathcal{W}$ with $|w| \geq D|v|$.

These systems were introduced by Durand and called linearly recurrent [7, 8]. A detailed study including a constructive characterization can be found in [7]. In the context of quasicrystals (i.e. arbitrary dimensional tilings) these systems were recently discussed under the name of linearly repetitive Delone sets in [13].
(d) It is not hard to see that (PQ) (or (PW)) implies in particular the minimality of the subshift (Ω, T). In fact, minimality of (Ω, T) is equivalent to $\nu(v) > 0$ for every $v \in \mathcal{W}$. From this point of view (PQ) (or (PW)) can be seen as strong type of minimality condition.

The organisation of the paper is as follows. The additive ergodic theorem is contained in Section 2. The subadditive ergodic theorem is contained in Section 3. Finally, in Section 4 we provide examples of subshifts satisfying (PQ), thereby giving a precise form to Remark 2 (b) and (c). A further discussion of the relationship between (LR) and (PW) is then contained in Section 5.

2. Uniform additive ergodic theorems

This section is devoted to a proof of theorem 1. The following lemma is the key to its proof. The proof given below has been suggested by the referee. It simplifies and extends (by not using a minimality condition) the proof given by the author.

Lemma 2.1. *Let* (Ω, T) *be a uniquely ergodic subshift over* A *and let* B *be a Banach space. Let* $F : \mathcal{W} \longrightarrow B$ *be almost additive. Then the limit* $\lim_{|w|\to\infty} \frac{F(w)}{|w|}$ *exists.*

Proof. By unique ergodicity, there exists, for $u \in \mathcal{W}$, the limit $lim_{|w|\to\infty} \frac{\sharp_u w}{|w|}$. Call it $\mu(u)$. Moreover, for $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $L = L(n, \varepsilon)$ with

$$(1) \qquad \sum_{u \in \mathcal{W}_n} \left| \frac{\sharp_u(w)}{|w|} - \mu(u) \right| < \varepsilon$$

for every $w \in \mathcal{W}$, with $|w| \geq L(n,\varepsilon)$. Now, choose $n \in \mathbb{N}$ arbitrary and $w \in \mathcal{W}$ with $|w| \geq 2n$. For $j = 0, 1, \ldots, n-1$ we can cut w in words of length n starting at position j. This means that we can write

$$w = a^{(j)} u_1^{(j)} \ldots u_{l(j)}^{(j)} b^{(j)}$$

with $|a^{(j)}| = j$, $|u_1^{(j)}| = \ldots = |u_{l(j)}^{(j)}| = n$, $|b^{(j)}| < n$. Almost additivity of F implies

$$\|F(w) - \sum_{k=1}^{l(j)} F(u_k^{(j)})\| \leq 2n(D + c(0)) + |w| c(n). \tag{2}$$

Moreover, we have

$$\sum_{j=0}^{n-1} \sum_{k=1}^{l(j)} F(u_k^{(j)}) = \sum_{u \in \mathcal{W}_n} \sharp_u(w) F(u), \tag{3}$$

as both sides of the equation contain the same terms. Using (2) and (3), we can calculate

$$\begin{aligned}
\left\| \frac{F(w)}{|w|} - \sum_{u \in \mathcal{W}_n} \frac{\sharp_u(w)}{|w|} \frac{F(u)}{|u|} \right\| &= \frac{1}{|w| n} \left\| n F(w) - \sum_{j=0}^{n-1} \sum_{k=1}^{l(j)} F(u_k^{(j)}) \right\| \\
&= \frac{1}{|w| n} \left\| \sum_{j=0}^{n-1} \left(F(w) - \sum_{k=1}^{l(j)} F(u_k^{(j)}) \right) \right\| \\
&\leq \frac{2n(D + c(0))}{|w|} + c(n).
\end{aligned}$$

Combining this with (1) we arrive at

$$\left\| \frac{F(w)}{|w|} - \sum_{u \in \mathcal{W}_n} \mu(u) \frac{F(u)}{n} \right\| \leq \frac{2n}{|w|} (D + c(0)) + c(n) + D\varepsilon$$

for w with $|w| \geq L(n, \varepsilon)$ which yields

$$\limsup_{|w| \to \infty} \left\| \frac{F(w)}{|w|} - \sum_{u \in \mathcal{W}_n} \mu(u) \frac{F(u)}{n} \right\| \leq c(n).$$

This shows that $\frac{F(w)}{|w|}$ is a Cauchy sequence and the proof is finished. □

Proof of Theorem 1. The equivalence (i)$\Longleftrightarrow$ (iii) is standard. We include a proof for completeness.

(i)$\Longrightarrow$ (iii). This is clear if f is locally constant, i.e. if there exists a $k \geq 0$ and $g : \mathcal{W}_{2k+1} \longrightarrow B$ with $f(\omega) = g(\omega(-k) \ldots g(0) \ldots g(k))$ for $\omega \in \Omega$. For arbitrary f the statement then follows by density arguments.

(iii)$\Longrightarrow$(i). Define for $v \in \mathcal{W}$ the cylinder function χ_v on Ω by $\chi_v(\omega) = 1$ if $\omega(0) \ldots \omega(|v| - 1) = v$ and $\chi_v(\omega) = 0$ otherwise. Applying (iii) to χ_v gives the existence of the frequency of v. As $v \in \mathcal{W}$ was arbitrary, statement (i) follows.

(ii) $\Longrightarrow$ (i). This is clear, as the function $w \mapsto \sharp_v(w)$ is almost additive for every $v \in \mathcal{W}$.

(i)$\Longrightarrow$ (ii). This is just the previous lemma. □

3. Uniform subadditive ergodic theorems

Here, we consider subadditive functions on $\mathcal{W}$. The key results are the following two lemmas. Their proofs use and considerably extend ideas from [14, 15]. We first introduce some notation and review some basic facts concerning subadditive functions. For a subadditive function $F : \mathcal{W} \longrightarrow \mathbb{R}$ and $n \in \mathbb{N}$ we set $\overline{F} \equiv \inf_{n\in\mathbb{N}} F^{(n)}$, where $F^{(n)} \equiv \max\{\frac{F(v)}{|v|} : v \in \mathcal{W}_n,\}$ as well as $\phi(n) \equiv \max\{F(v) : v \in \mathcal{W}_n\} = nF^{(n)}$. Subadditivity of F yields $\phi(n+m) \leq \phi(n) + \phi(m)$ i.e. ϕ is subadditive in the classical sense. This implies in particular

$$\lim_{n\to\infty} F^{(n)} = \lim_{n\to\infty} \frac{\phi(n)}{n} = \inf_{n\in\mathbb{N}} \frac{\phi(n)}{n} = \overline{F}. \tag{4}$$

Lemma 3.1. *Let (Ω, T) be a minimal subshift over A satisfying (PQ). Then, the subshift (Ω, T) satisfies (SET) as well.*

Proof. It is clearly enough to show:

$$(A)\ \limsup_{|w|\to\infty} \frac{F(w)}{|w|} \leq \overline{F} \quad \text{and } (B)\ \liminf_{|w|\to\infty} \frac{F(w)}{|w|} \geq \overline{F}.$$

Ad (A): This follows immediately from (4).
Ad (B): Assume the contrary. This implies, in particular, $\overline{F} > -\infty$ and that there exists a sequence (v_n) in $\mathcal{W}$ as well as a $\delta > 0$ with $|v_n|$ tending to ∞ for $n \to \infty$ and

$$\frac{F(v_n)}{|v_n|} \leq \overline{F} - \delta \tag{5}$$

for every $n \in \mathbb{N}$. Moreover, by (A), there exists an $L_0 \in \mathbb{R}$ with

$$\frac{F(w)}{|w|} \leq \overline{F} + \frac{C\delta}{8} \tag{6}$$

for all $w \in \mathcal{W}$, $|w| \geq L_0$, where C is the constant from (PQ).

Fix $m \in \mathbb{N}$ with $|v_m| \geq L_0$. Using (PQ), we can now find an $L_1 \in \mathbb{R}$ s.t. every $w \in \mathcal{W}$ with $|w| \geq L_1$ can be written as $w = x_1 v_m x_2 v_m \ldots x_l v_m x_{l+1}$ with

$$\frac{l-2}{2} \geq \frac{C}{4} \frac{|w|}{|v_m|}. \tag{7}$$

Now, considering only every other copy of v_m in w, we can write w as $w = y_1 v_m y_2 \ldots y_r v_m y_{r+1}$, with $|y_j| \geq |v_m| \geq L_0$, $j = 1, \ldots, r+1$, and by (7)

$$r \geq \frac{l-2}{2} \geq \frac{C}{4} \frac{|w|}{|v_m|}.$$

Using (5), (6) and this estimate, we can now calculate

$$\begin{aligned} \frac{F(w)}{|w|} &\leq \sum_{j=1}^{r+1} \frac{F(y_j)}{|y_j|} \frac{|y_j|}{|w|} + \frac{F(v_m)}{|v_m|} \frac{r|v_m|}{|w|} \leq \sum_{j=1}^{r+1} (\overline{F} + \frac{C}{8}\delta) \frac{|y_j|}{|w|} + (\overline{F} - \delta) \frac{r|v_m|}{|w|} \\ &\leq \overline{F} + \frac{C}{8}\delta - \frac{C}{4} \frac{|w|}{|v_m|} \frac{|v_m|}{|w|} \delta \leq \overline{F} - \frac{C}{8}\delta. \end{aligned}$$

As this holds for arbitrary $w \in \mathcal{W}$ with $|w| = L_1$, we arrive at the obvious contradiction $F^{(L_1)} \leq \overline{F} - \frac{C}{8}\delta < \inf_{n\in\mathbb{N}} F^{(n)}$. This finishes the proof. □

Lemma 3.2. *Let (Ω, T) be a minimal subshift over A satisfying (SET). Then, the subshift (Ω, T) satisfies (PQ) as well.*

Proof. Note that, for $v \in \mathcal{W}$, the function $(-l_v)$ is subadditive. Thus, the equation

$$\nu(v) \equiv \liminf_{|w|\to\infty} \frac{l_v(w)}{|w|} = \lim_{|w|\to\infty} \frac{l_v(w)}{|w|} \tag{8}$$

holds by (SET). The proof will now be given by contraposition. So, let us assume that the values $\nu(v)$, $v \in \mathcal{W}$, are not bounded away from zero. As the system is minimal, we have $\nu(w) > 0$ for every $w \in \mathcal{W}$. Thus, there exists a sequence (v_n) in $\mathcal{W}$ with

$$\nu(v_n) > 0, \text{ and } \nu(v_n) \longrightarrow 0, \quad n \to \infty. \tag{9}$$

As the alphabet A is finite, there are only finitely many words of a prescribed length. Thus, (9) implies

$$|v_n| \longrightarrow \infty, \quad n \to \infty. \tag{10}$$

Replacing (v_n) by a suitable subsequence, we can assume by (9) that the equation

$$\sum_{n=1}^{\infty} \nu(v_n) < \frac{1}{2} \tag{11}$$

holds. Set $l_n \equiv l_{v(n)}$ for $n \in \mathbb{N}$. By (8), (10) and (11), we can choose inductively for each $k \in \mathbb{N}$ a number $n(k)$, with

$$\sum_{j=1}^{k} \frac{l_{n(j)}(w)}{|w|} < \frac{1}{2} \tag{12}$$

for every $w \in \mathcal{W}$ with $|w| \geq \frac{|v_{n(k+1)}|}{2}$. Note that (12) implies

$$|v_{n(k)}| < \frac{|v_{n(k+1)}|}{2} \tag{13}$$

as $\frac{l_{n(k)}(v_{n(k)})}{|v_{n(k)}|} = 1$. Define the function $l : \mathcal{W} \longrightarrow \mathbb{R}$ by

$$l(w) \equiv \sum_{j=1}^{\infty} l_{n(j)}(w).$$

Note that the sum is actually finite for each $w \in \mathcal{W}$. Obviously, $(-l)$ is subadditive. Thus, by assumption, the limit $\lim_{|w|\to\infty} \frac{l(w)}{|w|}$ exists. On the other hand, we clearly have

$$\frac{l(v_{n(k)})}{|v_{n(k)}|} \geq \frac{l_{n(k)}(v_{n(k)})}{|v_{n(k)}|} \geq 1$$

as well as by the induction construction (12), (13)

$$\frac{l(w)}{|w|} = \sum_{j=1}^{k} \frac{l_{n(j)}(w)}{|w|} < \frac{1}{2}$$

for $w \in \mathcal{W}$ with $\frac{|v_{n(k+1)}|}{2} \leq |w| < |v_{n(k+1)}|$. This gives a contradiction proving the lemma. □

Proof of Theorem 2. This theorem follows immediately from the foregoing two lemmas. □

4. Examples

In this section we discuss certain classes of examples satisfying condition (PQ). We will need the following lemma.

Lemma 4.1. *Let (Ω, T) be a subshift over A with associated set of words $\mathcal{W}$. Let $a \in \mathcal{W}$ be primitive with $2 \leq n \equiv \max\{k \in \mathbb{N} : a^k \in \mathcal{W}\} < \infty$. Set $b \equiv a^n$. Then for every prefix v of b with $|v| \geq |a|$ the inequality*

$$l_v(w) \geq \frac{1}{8} \sharp_b(w)|b|$$

holds for every $w \in \mathcal{W}$.

Proof. We start with the following claim.

Claim. The distance between two distinct copies of $b = a^n$ in a word $w \in \mathcal{W}$ is larger than $(n-1)|a|$.

Proof of claim. Let d be the distance between two copies of a^n in w and assume $d \leq (n-1)|a|$. We write $d = m|a| + j$ with $0 \leq j < |a|$. Thus, a is a factor of aa starting at position j. By primitivity of a the word aa contains exactly two copies of a i.e. $j = 0$. This gives $d = m|a|$ with $0 < m \leq n-1$. Thus $a^m a^n = a^{m+n}$ is a factor of w yielding a contradiction to the maximality of n. The proof of the claim is finished.

Take only every other copy of b in w. By the claim, the distance between their starting points is larger than $2(n-1)|a| \geq |b|$ and they are therefore disjoint. This implies

$$l_b(w) \geq \frac{1}{2} \sharp_b(w)|b| \tag{14}$$

for every $w \in \mathcal{W}$. Moreover, if $|v|$ is a prefix of b with $|v| \geq |a|$, then each copy of b contains at least $\frac{|b|}{4|v|}$ disjoint copies of v. This implies

$$l_v(w) \geq \frac{l_b(w)}{|b|} \frac{|b|}{4|v|}|v| = \frac{1}{4} l_b(w) \geq \frac{1}{8} \sharp_b(w)|b|,$$

where we used (14) in the last inequality. The desired result follows. □

From this lemma, we can derive a sufficient (and necessary) condition for (PQ). Recall that a minimal subshift is called periodic if there exists an $n \neq 0$ with $T^n \omega = \omega$ for every $\omega \in \Omega$. A minimal subshift is called aperiodic if it is not periodic.

Proposition 4.2. *If (Ω, T) satisfies (PW), then it satisfies (PQ) as well and (SET) holds.*

Proof. By (PW) the subshift is minimal. If it is periodic, the statement is immediate. So, assume that (Ω, T) is aperiodic and satisfies (PW) with constant C. We will show that it satisfies (PQ) with constant $\frac{C}{8}$. Let $v \in \mathcal{W}$ be given. There are two cases.

Case 1. The distance between two copies of v is always at least $\frac{1}{8}|v|$: In this case, we have $l_v(w) \geq \frac{1}{8} \sharp_v(w)|v|$.

Case 2. There exist two copies of v in some $w \in \mathcal{W}$ with distance smaller than $\frac{1}{8}|v|$: Then, $v = z^m y$ with $m \geq 8$ a prefix y of z, and $z^{m+1} \in \mathcal{W}$. The word z can be written as $z = a^k$ for some primitive word a and $k \geq 1$. By minimality and aperiodicity, we have $n \equiv \max\{k \in \mathbb{N} : a^k \in \mathcal{W}\} < \infty$. By construction, the inequality $n \geq k(m+1) \geq 2$ holds and v is prefix of $b \equiv a^n$ with $|v| \geq |a|$. Thus, we can apply the previous lemma and infer $l_v(w) \geq \frac{1}{8}\sharp_b(w)|b|$.
Thus, by (PW) with constant C we have $\liminf_{|w|\to\infty} \frac{l_v(w)}{|w|} \geq \frac{C}{8}$ in both cases and (PQ) follows. As (PQ) implies minimality, (SET) is an immediate consequence of Theorem 2 □

We also have the following sufficient condition for (PQ).

Proposition 4.3. *If (Ω, T) satisfies (LR), then it satisfies (PQ) as well and a subadditive ergodic theorem holds.*

Proof. Straightforward arguments show that (LR) implies (PQ). Now, (SET) follows from Theorem 2. □

Remark 3. (a) In [7], it is shown that systems satisfying (LR) are uniquely ergodic. Proposition 4.3 shows that in fact a stronger statement holds viz these systems allow for a subadditive ergodic theorem.
(b) The validity of (LR) is known for primitive substitutions and for Sturmian systems whose rotation number has bounded continued fraction expansion [6, 7]. Thus, Proposition 4.3 applies in particular to these systems.
(c) As shown in [15], (LR) implies a subadditive ergodic theorem in tiling dynamical systems of arbitrary dimension (cf. [5] as well).

5. Dynamical Systems satisfying (LR)

In this section we further investigate the relationship between (LR) and (PW). Of course, (LR) is a "local" positivity condition whereas (PW) is a positivity condition on averages. Thus, (LR) is a much stronger condition. However, as shown below, (LR) is in fact equivalent to uniform validity of (PW) on all induced systems (s. below). For a detailed study of (LR) and a characterization in terms of primitive S-adic systems we refer the reader to recent work of Durand [7].
The notion of return word used in the sequel was introduced in [6]. The notion of derived sequence and induced system used below are (up to slight reformulation) taken from [8].

Let Ω be a subshift over the finite alphabet A with associated set $\mathcal{W}$ of finite words. A return word of $u \in \mathcal{W}$ is a word $x \in \mathcal{W}$ s.t.

$$xu \in \mathcal{W}, \quad \sharp_u(xu) = 2, \quad u \text{ is prefix of } xu.$$

The set of return words of $u \in \mathcal{W}$ will be denoted by $\mathcal{R}(u) \equiv \mathcal{R}(u, \mathcal{W})$. Recall that (Ω, T) is minimal if and only if for every $w \in \mathcal{W}$ there exists an $R(w) > 0$ s.t. w is a factor of every $v \in \mathcal{W}$ with $|v| \geq R(w)$ [17]. Thus, if (Ω, T) is minimal, the length of a return word of $u \in \mathcal{W}$ is bounded by $R(u)$ and the set $\mathcal{R}(u)$ is finite for every $u \in \mathcal{W}$. Let (Ω, T) be minimal and an arbitrary $u \in \mathcal{W}$ be given. Then, every $\omega \in \Omega$ can uniquely be writen as $\omega = \ldots x_{-2}x_{-1}x_0x_1x_2\ldots$ with $x_j \in R(u)$, $j \in \mathbb{Z}$, s.t.

(∗) $\omega(0)$ belongs to x_0,

($*$) every occurrence of u in ω begins begins at the position of one of the x_j, $j \in \mathbb{Z}$.

To $\omega \in \Omega$, we can thus associate the derived sequence $\widehat{\omega} \in \mathcal{R}(u)^{\mathbb{Z}}$ given by $\widehat{\omega}(n) \equiv x_n$. It is not hard to show that the induced system $\Omega(u) \equiv \{\widehat{\omega} : \omega \in \Omega\} \subset \mathcal{R}(u)^{\mathbb{Z}}$ is closed and invariant i.e. an subshift over the alphabet $\mathcal{R}(u)$. We set $\mathcal{W}(u) \equiv \mathcal{W}(\Omega(u))$. If u is the empty word ϵ we set $\Omega(\epsilon) = \Omega$ and $\widehat{\omega} = \omega$. The elements of $\mathcal{R}(u)$ can be considered as letters of the alphabet $\mathcal{R}(u)$ or as words over the alphabet A. To avoid confusion we therefore write $w = \widehat{x_1} \ldots \widehat{x_n}$ for $w \in \mathcal{R}(u)^n$ with $w(j) = x_j \in \mathcal{R}(u)$. Note that $w = \widehat{x_1} \ldots \widehat{x_n}$ belongs to $\mathcal{W}(u)$ if and only if $x_1 \ldots x_n u$ belongs to $\mathcal{W}$. Let $x_1, \ldots, x_n \in \mathcal{R}(u)$ be arbitrary with $x_1 \ldots x_n u \in \mathcal{W}$. If (Ω, T) is uniquely ergodic, we have by construction of $\Omega(u)$ the equality

$$\lim_{|w| \to \infty, w \in \mathcal{W}} \frac{\sharp_{x_1 \ldots x_n u}(w)}{\sharp_u(w)} = \lim_{|w| \to \infty, w \in \mathcal{W}(u)} \frac{\sharp_{\widehat{x_1} \ldots \widehat{x_n}}(w)}{|w|}. \tag{15}$$

Thus, unique ergodicity of (Ω, T) implies unique ergodicity of $(\Omega(u), T)$ for every $u \in \mathcal{W}$. Similarly, validity of (PW) for (Ω, T) implies validity of (PW) for $(\Omega(u), T)$ for $u \in \mathcal{W}$. However, the constant C_Ω in (PW) for (Ω, T) and the constant $C_{\Omega(u)}$ in (PW) for $(\Omega(u), T)$ might be different. This suggests the following definition of uniform positivity of weights

(UP) The system (Ω, T) is said to satisfy (UP) if there exists a constant $C > 0$ s.t. $(\Omega(u), T)$ satisfies (PW) with this C for every $u \in \mathcal{W}$.

Note that (UP) implies (PW) on (Ω, T) (for $u \equiv \epsilon$), which in turn implies unique ergodicity and (by Remark 2 (d)) minimality of (Ω, T).

We have the following result.

Theorem 3. *For a subshift (Ω, T) over A the following are equivalent:*
(i) (Ω, T) satisfies (LR).
(ii) (Ω, T) satisfies (UP).

To prove $(ii) \Longrightarrow (i)$ we need some preparation. We start with a lemma whose proof is similar to the proof of Lemma 4.1.

Lemma 5.1. *Let (Ω, T) be minimal and aperiodic. Let $a \in \mathcal{W}$ be primitive and $n \equiv \max\{k \in \mathbb{N} : a^k \in \mathcal{W}\}$. Then, there exists an $x \in \mathcal{R}(a)$ and $v \in \mathcal{R}(xa)$ s.t. xa^{n-1} is a prefix of v and xa^n is a prefix of vxa.*

Proof. Let an arbitrary $\omega \in \Omega$ be given and consider an occurrence of a^n in ω. Inspecting the occurrences of a to the left of a^n, we see that there exists an $x \in \mathcal{R}(a)$ with $xa^n \in \mathcal{W}$. Assume w.l.o.g. $xa^n = \omega(1) \ldots \omega(|xa^n|)$. Define $j > 1$ to be the smallest number in $\mathbb{N}$ with $\omega(j) \ldots \omega(j + |xa| - 1) = xa$ i.e. $\omega(1) \ldots \omega(j-1)$ is a return word to xa. There are three cases:
Case 1. $1 < j \leq |x|$: As a is a prefix of xa and x is a return word of a, this case cannot occur.
Case 2. $|x| < j \leq |xa^{n-1}|$: As a is a prefix of xa and as a is primitive, we infer that j must be of the form $j = |x| + k|a| + 1$ with $0 \leq |k| \leq n-2$. Thus, xa and aa have a common prefix of length $\min\{|xa|, |aa|\}$. By primitivity of a and $x \in \mathcal{R}(a)$, we see $x = a$. But this implies $a^{n+1} = xa^n \in \mathcal{W}$ yielding a contradiction to the maximality of n. Thus, this case cannot occur either.
Case 3. $|xa^{n-1}| < j$: In this case the statement of the lemma follows easily. □

Proposition 5.2. *Let (Ω, T) be minimal and aperiodic. If (Ω, T) satisfies (UP), it satisfies (HP) as well i.e. there exists an $N \in \mathbb{N}$ s.t. $a^n \in \mathcal{W}$ implies $n \leq N$.*

Proof. Let a be primitive and $n \in \mathbb{N}$ be maximal with $a^n \in \mathcal{W}$. Let x and v be given by the previous lemma. By (UP) and (15), we have $\lim_{|w|\to\infty} \frac{\sharp_{xa}(w)}{\sharp_a(w)} \geq C$ as well as $\lim_{|w|\to\infty} \frac{\sharp_{xa}(w)}{\sharp_{vxa}(w)} \leq \frac{1}{C}$. Moreover, we clearly have $\frac{\sharp_{vxa}(w)}{\sharp_a(w)} \leq \frac{\sharp_{a^n}(w)}{\sharp_a(w)} \leq \frac{1}{n-1}$. Here, we used in the last inequality that two copies of a^n must be disjoint up to one copy of a (cf. proof of Lemma 4.1). Combining these inequalities with the identity

$$\frac{\sharp_{xa}(w)}{\sharp_a(w)} = \frac{\sharp_{xa}(w)}{\sharp_{vxa}(w)} \frac{\sharp_{vxa}(w)}{\sharp_a(w)}$$

(valid vor $|w|$ large enough), we arrive at $n - 1 \leq \frac{1}{C^2}$. This proves the proposition. □

Proof of Theorem 3. We can assume that (Ω, T) is strictly ergodic, as this is implied by both (LR) and (UP). If (Ω, T) is periodic, it satisfies both (LR) and (UP). Thus, it is enough to consider aperiodic strictly ergodic (Ω, T).
(i)$\Longrightarrow$ (ii). By (LR) there exist constants c_1, c_2 with $c_1|u| \leq |x| \leq c_2|u|$ for every $u \in \mathcal{W}$ and every $x \in \mathcal{R}(u)$. Here, the second inequality is immediate from the definition of (LR). The first inequality is just (HP) which is valid for aperiodic systems satisfying (LR) (cf. [7]). Applying these inequalities to (15) gives

$$\lim_{|w|\to\infty, w\in\mathcal{W}(u)} \frac{\sharp_{\widehat{x_1}\ldots\widehat{x_n}}(w)}{|w|} n \geq \liminf_{|w|\to\infty, w\in\mathcal{W}} \frac{\frac{|w|n}{2c_2|x_1\ldots x_n u}}{\frac{|w|}{c_1|u|}} \geq \frac{c_1}{2c_2} \frac{n|u|}{|x_1 \ldots x_n u|} \geq \frac{c_1}{4c_2^2}.$$

This shows (ii).
(ii) $\Longrightarrow$ (i). Let $u \in \mathcal{W}$ be given. Let $x \in \mathcal{R}(u)$ be arbitrary. Apparently, for $|w|$ large enough, the following equation holds

$$\frac{\sharp_{xu}(w)}{\sharp_u(w)} = \frac{\sharp_{xu}(w)}{|w|} |xu| \frac{|w|}{|u|\sharp_u(w)} \frac{|u|}{|xu|}. \tag{16}$$

Next, we estimate the various factors appearing in this equation. By (UP) and (15) we have $C \leq \lim_{|w|\to\infty} \frac{\sharp_{xu}(w)}{\sharp_u(w)}$. As (UP) implies (PW) (with the same C) we have furthermore $\lim_{|w|\to\infty} \frac{|w|}{|u|\sharp_u(w)} \leq \frac{1}{C}$. As (UP) implies (HP) by the previous proposition, we see that two distinct copies of xu in w must have distance at least $\frac{1}{N}|xu|$. This can easily be seen to imply $\frac{\sharp_{xu}(w)}{|w|}|xu| \leq N+1$. We emphasize that the constants C and N appearing in these inequalities do not depend on x or u. Taking limits in (16) and using the above inequalities we arrive at

$$C \leq (N+1)\frac{1}{C}\frac{|u|}{|xu|}.$$

But, this yields immediately $|x| \leq (N+1)\frac{1}{C^2}|u|$. As u in $\mathcal{W}$ and $x \in \mathcal{R}(u)$ were arbitrary, this shows (LR) with constant $(N+1)\frac{1}{C^2} + 1$. □

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CHAPTER 6

D. Lenz, *Singular spectrum of Lebesgue measure zero for one-dimensional quasicrystals*, Communications in Mathematical Physics 227 (2002), 129–130

SINGULAR SPECTRUM OF LEBESGUE MEASURE ZERO FOR ONE-DIMENSIONAL QUASICRYSTALS

DANIEL LENZ [1,2,*]

[1] Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel

[2] Fachbereich Mathematik, Johann Wolfgang Goethe-Universität, 60054 Frankfurt, Germany

E-mail: dlenz@math.uni-frankfurt.de

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Abstract. The spectrum of one-dimensional discrete Schrödinger operators associated to strictly ergodic dynamical systems is shown to coincide with the set of zeros of the Lyapunov exponent if and only if the Lyapunov exponent exists uniformly. This is used to obtain Cantor spectrum of zero Lebesgue measure for all aperiodic subshifts with uniform positive weights. This covers, in particular, all aperiodic subshifts arising from primitive substitutions including new examples as e.g. the Rudin-Shapiro substitution.

Our investigation is not based on trace maps. Instead it relies on an Oseledec type theorem due to A. Furman and a uniform ergodic theorem due to the author.

1. Introduction

This article is concerned with discrete random Schrödinger operators associated to minimal topological dynamical systems. This means we consider a family $(H_\omega)_{\omega\in\Omega}$ of operators acting on $\ell^2(\mathbb{Z})$ by

$$(H_\omega u)(n) \equiv u(n+1) + u(n-1) + f(T^n\omega)u(n), \tag{1}$$

where Ω is a compact metric space, $T : \Omega \longrightarrow \Omega$ is a homeomorphism and $f : \Omega \longrightarrow \mathbb{R}$ is continuous. The dynamical system (Ω, T) is called minimal if every orbit is dense. For minimal (Ω, T), there exists a set $\Sigma \subset \mathbb{R}$ s.t.

$$\sigma(H_\omega) = \Sigma, \quad \text{for all} \quad \omega \in \Omega, \tag{2}$$

where we denote the spectrum of the operator H by $\sigma(H)$ (cf. [6, 36]).

We will be particularly interested in the case that (Ω, T) is a subshift over a finite alphabet $A \subset \mathbb{R}$. In this case Ω is a closed subset of $A^{\mathbb{Z}}$, invariant under the shift operator $T : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ given by $(Ta)(n) \equiv a(n+1)$ and f is given by

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$f : \Omega \longrightarrow A \subset \mathbb{R}$, $f(\omega) \equiv \omega(0)$. Here, A carries the discrete topology and $A^{\mathbb{Z}}$ is given the product topology.

Operators associated to subshifts arise in the quantum mechanical treatment of quasicrystals (cf. [3, 40] for background on quasicrystals). Various examples of such operators have been studied in recent years. The main examples can be divided in two classes. These classes are given by primitive substitution operators (cf. e.g. [4, 5, 7, 11, 41, 42]) and Sturmian operators respectively more generally circle map operators (cf. e.g. [6, 12, 15, 16, 26, 27, 30]). A recent survey can be found in [14].

For these classes and in fact for arbitrary operators associated to subshifts satisfying suitable ergodicity and aperiodicity conditions, one expects the following features:
($\mathcal{S}$) Purely singular spectrum; ($\mathcal{A}$) absence of eigenvalues; ($\mathcal{Z}$) Cantor spectrum of Lebesgue measure zero.

Note that ($\mathcal{S}$) combined with ($\mathcal{A}$) implies purely singular continuous spectrum and note also that ($\mathcal{S}$) is a consequence of ($\mathcal{Z}$). Let us mention that ($\mathcal{S}$) is by now completely established for all relevant subshifts due do recent results of Last/Simon [34] in combination with earlier results of Kotani [32]. For discussion of ($\mathcal{A}$) and further details we refer the reader to the cited literature.

The aim of this article is to investigate ($\mathcal{Z}$) and to relate it to ergodic properties of the underlying subshifts.

The property ($\mathcal{Z}$) has been investigated for several models by a number of authors: Following work by Bellissard/Bovier/Ghez [5], the most general result for primitive substitutions so far has been obtained by Bovier/Ghez [7]. They can treat a large class of substitutions which is given by an algorithmically accessible condition. The Rudin-Shapiro substition does not belong to this class. For arbitrary Sturmian operators, Bellissard/Iochum/Scoppola/Testard established ($\mathcal{Z}$) [6], thereby extending the work of Sütő in the golden mean case [41, 42]. A different approach, which recovers some of these results, is given in [13, 19].

A canonical starting point in the investigation of ($\mathcal{Z}$) for subshifts is the fundamental result of Kotani [32] that the set $\{E \in \mathbb{R} : \gamma(E) = 0\}$ has Lebesgue measure zero if (Ω, T) is an aperiodic subshift. Here, γ denotes the Lyapunov exponent (precise definition given below). This reduces the problem ($\mathcal{Z}$) to establishing the equality

$$\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}. \tag{3}$$

As do all other investigations of ($\mathcal{Z}$) so far, our approach starts from (3). Unlike the earlier treatments mentioned above our approach does not rely on the so called trace maps. Instead, we present a new method, the cornerstones of which are the following: (1) A strong type of Oseledec theorem by A. Furman [21]. (2) A uniform ergodic theorem for a large class of subshifts by the author [37]. This new setting allows us

- (∗) to characterize validity of (3) for arbitrary strictly ergodic dynamical systems by an essentially ergodic property viz by uniform existence of the Lyapunov exponent (Theorem 1),
- (∗∗) to present a large class of subshifts satisfying this property (Theorem 2).

Here, (∗) gives the new conceptual point of view of our treatment and (∗∗) gives a large class of examples. Put together (∗) and (∗∗) provide a soft argument for ($\mathcal{Z}$)

for a large class of examples which contains, among other examples, all primitive substitutions.

The paper is organized as follows. In Section 2 we present the subshifts we will be interested in, introduce some notation and state our results. In Section 3, we recall results of Furman [21] and of the author [37] and adopt them to our setting. Section 4 is devoted to a proof of our results. Finally, in Section 5 we provide some further comments and discuss a variant of our main result.

Note added. After this work was completed, we learned about the very recent preprint "Measure Zero Spectrum of a Class of Schrödinger Operators" by Liu/Tan/Wen/Wu (mp-arc 01-189). They present a detailed and thorough analysis of trace maps for primitive substitutions. Based on this analysis, they establish ($\mathcal{Z}$) for all primitive substitutions thereby extending the approach developed in [5, 7, 9, 41].

2. Notation and Results

In this section we discuss basic material concerning topological dynamical systems and the associated operators and state our results.

As usual a dynamical system is said to be strictly ergodic if it is uniquely ergodic (i.e. there exists only one invariant probability measure) and minimal. A minimal dynamical system is called aperiodic if there does not exist an $n \in \mathbb{Z}$, $n \neq 0$, and $\omega \in \Omega$ with $T^n \omega = \omega$.

As mentioned already, our main focus will be the case that (Ω, T) is a subshift over the finite alphabet $A \subset \mathbb{R}$. We will then consider the elements of (Ω, T) as double sided infinite words and use notation and concepts from the theory of words. In particular, we then associate to Ω the set $\mathcal{W}$ of words associated to Ω consisting of all finite subwords of elements of Ω. The length $|x|$ of a word $x \equiv x_1 \ldots x_n$ with $x_j \in A$, $j = 1, \ldots, n$, is defined by $|x| \equiv n$. The number of occurences of $v \in \mathcal{W}$ in $x \in \mathcal{W}$ is denoted by $\sharp_v(x)$.

We can now introduce the class of subshifts we will be dealing with. They are those satisfying uniform positivity of weights (PW) given as follows:

(PW) There exists a $C > 0$ with $\liminf_{|x|\to\infty} \frac{\sharp_v(x)}{|x|}|v| \geq C$ for every $v \in \mathcal{W}$.

One might think of (PW) as a strong type of minimality condition. Indeed, minimality can easily be seen to be equivalent to $\liminf_{|x|\to\infty} |x|^{-1}\sharp_v(x)|v| > 0$ for every $v \in \mathcal{W}$ [39]. The condition (PW) implies strict ergodicity [37]. The class of subshifts satisfying (PW) is rather large. By [37], it contains all linearly repetitive subshifts (see [20, 33] for definition and thorough study of linerarly repetitive systems). Thus, it contains, in particular, all subshifts arising from primitive substitutions as well as all those Sturmian dynamical systems whose rotation number has bounded continued fraction expansion [20, 33, 38].

In our setting the class of subshifts satisfying (PW) appears naturally as it is exactly the class of subshifts admitting a strong form of uniform ergodic theorem [37]. Such a theorem in turn is needed to apply Furmans results (s. below for details).

After this discussion of background from dynamical systems we are now heading towards introducing key tools in spectral theoretic considerations viz transfer matrices and Lyapunov exponents.

The operator norm $\|\cdot\|$ on the set of 2×2-matrices induces a topology on $GL(2,\mathbb{R})$ and $SL(2,\mathbb{R})$. For a continuous function $A:\Omega\longrightarrow GL(2,\mathbb{R})$, $\omega\in\Omega$, and $n\in\mathbb{Z}$, we define the cocycle $A(n,\omega)$ by

$$A(n,\omega)\equiv\begin{cases} A(T^{n-1}\omega)\cdots A(\omega) & : \quad n>0\\ Id & : \quad n=0\\ A^{-1}(T^{n}\omega)\cdots A^{-1}(T^{-1}\omega) & : \quad n<0\end{cases}$$

By Kingmans subadditive ergodic theorem (cf. e.g. [31]), there exists $\Lambda(A)\in\mathbb{R}$ with

$$\Lambda(A)=\lim_{n\to\infty}\frac{1}{n}\log\|A(n,\omega)\|$$

for μ a. e. $\omega\in\Omega$ if (Ω,T) is uniquely ergodic with invariant probability measure μ. Following [21], we introduce the following definition.

Definition 1. *Let (Ω,T) be strictly ergodic. The continuous function $A:(\Omega,T)\longrightarrow GL(2,\mathbb{R})$ is called uniform if the limit $\Lambda(A)=\lim_{n\to\infty}\frac{1}{n}\log\|A(n,\omega)\|$ exists for all $\omega\in\Omega$ and the convergence is uniform on Ω.*

Remark 1. It is possible to show that uniform existence of the limit in the definition already implies uniform convergence. The author learned this from Furstenberg and Weiss [22]. They actually have a more general result. Namely, they consider a continuous subadditive cocycle $(f_n)_{n\in\mathbb{N}}$ on a minimal (Ω,T) (i.e. f_n are continuous real-valued functions on Ω with $f_{n+m}(\omega)\le f_n(\omega)+f_m(T^n\omega)$ for all $n,m\in\mathbb{N}$ and $\omega\in\Omega$). Their result then gives that existence of $\phi(\omega)=\lim_{n\to\infty}n^{-1}f_n(\omega)$ for all $\omega\in\Omega$ implies constancy of ϕ as well as uniform convergence.

For spectral theoretic investigations a special type of $SL(2,\mathbb{R})$-valued function is relevant. Namely, for $E\in\mathbb{R}$, we define the continuous function $M^E:\Omega\longrightarrow SL(2,\mathbb{R})$ by

$$M^E(\omega)\equiv\begin{pmatrix} E-f(T\omega) & -1\\ 1 & 0\end{pmatrix}. \tag{4}$$

It is easy to see that a sequence u is a solution of the difference equation

$$u(n+1)+u(n-1)+(f(T^n\omega)-E)u(n)=0 \tag{5}$$

if and only if

$$\begin{pmatrix} u(n+1)\\ u(n)\end{pmatrix}=M^E(n,\omega)\begin{pmatrix} u(1)\\ u(0)\end{pmatrix},\ n\in\mathbb{Z}. \tag{6}$$

By the above considerations, M^E gives rise to the average $\gamma(E)\equiv\Lambda(M^E)$. This average is called the Lyapunov exponent for the energy E. It measures the rate of exponential growth of solutions of (5). Our main result now reads as follows.

Theorem 1. *Let (Ω,T) be strictly ergodic. Then the following are equivalent:*
(i) The function M^E is uniform for every $E\in\mathbb{R}$.
(ii) $\Sigma=\{E\in\mathbb{R}:\gamma(E)=0\}$.
In this case the Lyapunov exponent $\gamma:\mathbb{R}\longrightarrow[0,\infty)$ is continuous.

Remark 2. (a) As will be seen later on, M^E is always uniform for E with $\gamma(E)=0$ and for $E\in\mathbb{R}\setminus\Sigma$. From this point of view, the theorem essentially states that M^E can not be uniform for $E\in\Sigma$ with $\gamma(E)>0$.
(b) Continuity of the Lyapunov exponent can easily be infered from (ii) (though

this does not seem to be in the literature). More precisely, continuity of γ on $\{E \in \mathbb{R} : \gamma(E) = 0\}$ is a consequence of subharmonicity. Continuity of γ on $\mathbb{R} \setminus \Sigma$ follows from the Thouless formula (see e. g. [10] for discussion of subharmonicity and the Thouless formula). Below, we will show that continuity of γ follows from (i) and this will be crucial in our proof of (i) $\Longrightarrow$ (ii).

Having studied $(*)$ of the introduction in the above theorem, we will now state our result on $(**)$.

Theorem 2. *If (Ω, T) is a subshift satisfying (PW), then the function M^E is uniform for each $E \in \mathbb{R}$.*

Remark 3. (a) Uniformity of M^E is rather unusual. This is, of course, clear from Theorem 1. Alternatively, it is not hard to see directly that it already fails for discrete almost periodic operators. More precisely, the Almost-Mathieu-Operator with coupling bigger than 2 has uniform positive Lyapunov exponent [24]. By a deterministic version of the theorem of Oseledec (cf. Theorem 8.1 of [34] for example), this would force pure point spectrum for all these operators, if M^E were uniform on the spectrum. However, there are examples of such Almost-Mathieu Operators without point spectrum [2, 29].
(b) The above theorem generalizes [18, 35], which in turn unified the work of Hof [25] on primitive substitutions and of Damanik and the author [17] on certain Sturmian subshifts.
(c) The theorem is a rather direct consequence of the subadditive theorem of [37].

The two theorems yield some interesting conclusions. We start with the following consequence of Theorem 1 concerning $(\mathcal{Z})$. A proof is given in Section 4.

Corollary 2.1. *Let (Ω, T) be an aperiodic strictly ergodic subshift. If M^E is uniform for every $E \in \mathbb{R}$, then the spectrum Σ is a Cantor set of Lebesgue measure zero.*

As $\Sigma = \{E : \gamma(E) = 0\}$ holds for arbitrary Sturmian dynamical subshifts [6, 41] (cf. [19] as well), Theorem 1 immediately implies the following corollary.

Corollary 2.2. *Let $(\Omega(\alpha), T)$ be a Sturmian dynamical system with rotation number α. Then M^E is uniform for every $E \in \mathbb{R}$.*

Remark 4. So far uniformity of M^E for Sturmian systems could only be established for rotation numbers with bounded continued fraction expansion [17]. Moreover, the corollary is remarkable as a general type of uniform ergodic theorem actually fails as soon as the continued fraction expansion of α is unbounded [37, 38].

Theorem 1, Theorem 2 and Corollary 2.1 directly yield the following corollary.

Corollary 2.3. *Let (Ω, T) be a subshift sastisfying (PW). Then $\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}$. If (Ω, T) is furthermore aperiodic, then Σ is a Cantor set of Lebesgue measure zero.*

Remark 5. For aperiodic (Ω, T) satisfying (PW), this gives an alternative proof of $(\mathcal{S})$.

As discussed above primitive substitutions satisfy (PW). As validity of $(\mathcal{Z})$ for primitive substitutions has been a special focus of earlier investigations (cf. discussion in Section 1 and Section 5), we explicitely state the following consequence of the foregoing corollary.

Corollary 2.4. *Let (Ω, T) be aperiodic and associated to a primitive substitution, then Σ is a Cantor set of Lebesgue measure zero.*

3. Key results

In this section, we present (consequences of) results of Furman [21] and of the author [37].

We start with some simple facts concerning uniquely ergodic systems. Define for a continuous $b : \Omega \longrightarrow \mathbb{R}$ and $n \in \mathbb{Z}$ the averaged function $A_n(b) : \Omega \longrightarrow \mathbb{R}$ by

$$A_n(b)(\omega) \equiv \begin{cases} n^{-1} \sum_{k=0}^{n-1} b(T^k\omega) & : \quad n > 0 \\ 0 & : \quad n = 0 \\ |n|^{-1} \sum_{k=1}^{|n|} b(T^{-k}\omega) & : \quad n < 0 \end{cases} \tag{7}$$

Moreover, for a continuous b as above and a finite measure μ on Ω we set $\mu(b) \equiv \int_\Omega b(\omega)\, d\mu(\omega)$. The following proposition is well known see e.g. [43].

Proposition 3.1. *Let (Ω, T) be uniquely ergodic with invariant probability measure μ. Let b be a continuous function on Ω. Then the averaged functions $A_n(b)$ converge uniformly towards the constant function with value $\mu(b)$ for $|n|$ tending to infinity.*

The following consequence of a result by A. Furman is crucial to our approach.

Lemma 3.2. *Let (Ω, T) be strictly ergodic with invariant probability measure μ. Let $B : \Omega \longrightarrow SL(2, \mathbb{R})$ be uniform with $\Lambda(B) > 0$. Then, for arbitrary $U \in \mathbb{C}^2 \setminus \{0\}$ and $\omega \in \Omega$, there exist constants $D, \kappa > 0$ such that $\|B(n,\omega)U\| \geq D \exp(\kappa|n|)$ holds for all $n \geq 0$ or for all $n \leq 0$. Here, $\|\cdot\|$ denotes the standard norm on $\mathbb{C}^2$.*

Proof. Theorem 4 of [21] states that uniformity of B implies that (in the notation of [21]) either $\Lambda(B) = 0$ or B is continuously diagonalizable. As we have $\Lambda(B) > 0$, we infer that B is continuously diagonalizable. This means that there exist continuous functions $C : \Omega \longrightarrow GL(2, \mathbb{R})$ and $a, d : \Omega \longrightarrow \mathbb{R}$ with

$$B(1, \omega) = C(T\omega)^{-1} \begin{pmatrix} \exp(a(\omega)) & 0 \\ 0 & \exp(d(\omega)) \end{pmatrix} C(\omega).$$

By multiplication and inversion, this immediately gives

$$B(n, \omega) = C(T^n\omega)^{-1} \begin{pmatrix} \exp(nA_n(a)(\omega)) & 0 \\ 0 & \exp(nA_n(d)(\omega)) \end{pmatrix} C(\omega), \quad n \in \mathbb{Z}. \tag{8}$$

As $C : \Omega \longrightarrow GL(2, \mathbb{R})$ is continuous on the compact space Ω, there exists a constant $\rho > 0$ with

$$0 < \rho \leq \|C(\omega)\|, |\det C(\omega)|, \|C^{-1}(\omega)\|, |\det C^{-1}(\omega)| \leq \frac{1}{\rho} < \infty, \text{ for all } \omega \in \Omega. \tag{9}$$

In view of (8) and (9), exponential growth of terms as $\|B(n,\omega)U\|$ will follow from suitable upper and lower bounds on $A_n(a)(\omega)$ and $A_n(d)(\omega)$ for large $|n|$. To obtain these bounds we proceed as follows.

Assume without loss of generality $\mu(a) \geq \mu(d)$. By (9), (8) and Proposition 3.1, we then have

$$0 < \Lambda(B) = \Lambda(C(T\cdot)^{-1}BC) = \Lambda(\begin{pmatrix} \exp(a(\cdot)) & 0 \\ 0 & \exp(d(\cdot)) \end{pmatrix} = \mu(a). \tag{10}$$

Moreover, $\det B(\omega) = 1$ implies $\det B(n,\omega) = 1$ for all $n \in \mathbb{Z}$. Thus, taking determinants, logarithms and averaging with $\frac{1}{n}$ in (8), we infer

$$0 = A_n(a)(\omega) + A_n(d)(\omega) + \frac{1}{n}\log|\det(C(T^n\omega)^{-1}C(\omega))|.$$

Taking the limit $n \to \infty$ in this equation and invoking (9) as well as Proposition 3.1, we obtain $\mu(a) = -\mu(d)$. As $\mu(a) > 0$ by (10), Proposition 3.1 then shows that there exists $\kappa > 0$, e.g. $\kappa = \frac{1}{2}\mu(a)$, s.t. for large $|n|$, we have

$$A_n(a)(\omega) > \kappa, \quad \text{and} \quad A_n(d)(\omega) < -\kappa \quad \text{for all } \omega \in \Omega.$$

Now, the statement of the lemma is a direct consequence of (8) and (9). □

Lemma 3.3. *Let (Ω, T) be strictly ergodic. Let $A : \Omega \longrightarrow SL(2,\mathbb{R})$ be uniform. Let (A_n) be a sequence of continuous $SL(2,\mathbb{R})$-valued functions converging to A in the sense that $d(A_n, A) \equiv \sup_{\omega\in\Omega}\{\|A_n(\omega) - A(\omega)\|\} \longrightarrow 0$, $n \longrightarrow \infty$. Then, $\Lambda(A_n) \longrightarrow \Lambda(A)$, $n \longrightarrow \infty$.*

Proof. This is essentially a result of [21]. More precisely, Theorem 5 of [21] shows that $\Lambda(A_n)$ converges to $\Lambda(A)$ whenever the following holds: A is a uniform $GL(2,\mathbb{R})$-valued function and $d(A_n, A) \longrightarrow 0$ and $d(A_n^{-1}, A^{-1}) \longrightarrow 0$, $n \longrightarrow \infty$. Now, for functions A_n, A with values in $SL(2,\mathbb{R})$, it is easy to see that $d(A_n^{-1}, A^{-1}) \longrightarrow 0$, $n \longrightarrow \infty$ if $d(A_n, A) \longrightarrow 0, n \longrightarrow \infty$. The proof of the lemma is finished. □

Lemma 3.4. *Let (Ω, T) be uniquely ergodic. Let $A : \Omega \longrightarrow GL(2,\mathbb{R})$ be continuous. Then, the inequality $\limsup_{n\to\infty} n^{-1}\log\|A(n,\omega)\| \le \Lambda(A)$ holds uniformly on Ω.*

Proof. This follows from Corollary 2 of [21] (cf. Theorem 1 of [21] as well). □

Finally, we need the following lemma providing a large supply of uniform functions if (Ω, T) is a subshift satisfying (PW).

Lemma 3.5. *Let (Ω, T) be a subshift satisfying (PW). Let $F : \mathcal{W} \longrightarrow \mathbb{R}$ satisfy $F(xy) \le F(x) + F(y)$ (i.e. F is subadditive). Then, the limit $\lim_{|x|\to\infty}\frac{F(x)}{|x|}$ exists.*

Proof. This is just one half of Theorem 2 of [37]. □

4. Proofs of the main results

In this section, we use the results of the foregoing section to prove the theorems stated in Section 2.

We start with some lemmas needed for the proof of Theorem 1.

Lemma 4.1. *Let (Ω, T) be strictly ergodic. If M^E is uniform for every $E \in \mathbb{R}$ then $\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}$ and the Lyapunov exponent $\gamma : \mathbb{R} \longrightarrow [0,\infty)$ is continuous.*

Proof. We start by showing continuity of the Lyapunov exponent. Consider a sequence (E_n) in $\mathbb{R}$ converging to $E \in \mathbb{R}$. As the function M^E is uniform by assumption, by Lemma 3.3, it suffices to show that $d(M^{E_n}, M^E) \to 0, n \to \infty$. This is clear from the definition of M^E in (4).

Now, set $\Gamma \equiv \{E \in \mathbb{R} : \gamma(E) = 0\}$. The inclusion $\Gamma \subset \Sigma$ follows from general principles (cf. e.g. [10]). Thus, it suffices to show the opposite inclusion $\Sigma \subset \Gamma$. By (2), it suffices to show $\sigma(H_\omega) \subset \Gamma$ for a fixed $\omega \in \Omega$.

Assume the contrary. Then there exists spectrum of H_ω in the complement $\Gamma^c \equiv \mathbb{R} \setminus \Gamma$ of Γ in $\mathbb{R}$. As γ is continuous, the set Γ^c is open. Thus, spectrum of H_ω can only exist in Γ^c, if spectral measures of H_ω give actually weight to Γ^c. By standard results on generalized eigenfunction expansion [8], there exists then an $E \in \Gamma^c$ admitting a polynomially bounded solution $u \neq 0$ of (5). By (6), this solution satisfies $(u(n+1), u(n))^t = M^E(n,\omega)(u(1), u(0))^t$, $n \in \mathbb{Z}$, where v^t denotes the transpose of v. By $E \in \Gamma^c$, we have $\Lambda(M^E) \equiv \gamma(E) > 0$. As M^E is uniform by assumption, we can thus apply Lemma 3.2 to M^E to obtain that $\|(u(n+1), u(n))^t\|$ is, at least, exponentially growing for large values of n or large values of $-n$. This contradicts the fact that u is polynomially bounded and the proof is finished. □

Lemma 4.2. *If (Ω, T) is uniquely ergodic, M^E is uniform for each $E \in \mathbb{R}$ with $\gamma(E) = 0$.*

Proof. By $\det M^E(\omega) = 1$, we have $1 \leq \|M^E(n,\omega)\|$ and therefore $0 \leq \liminf_{n\to\infty} n^{-1} \log \|M^E(n,\omega)\| \leq \limsup_{n\to\infty} n^{-1} \log \|M^E(n,\omega)\|$. Now, the statement follows from Lemma 3.4. □

The following lemma is probably well known. However, as we could not find it in the literature, we include a proof.

Lemma 4.3. *If (Ω, T) is strictly ergodic, M^E is uniform with $\gamma(E) > 0$ for each $E \in \mathbb{R} \setminus \Sigma$.*

Proof. Let $E \in \mathbb{R} \setminus \Sigma$ be given. The proof will be split in four steps. Recall that Σ is the spectrum of H_ω for every $\omega \in \Omega$ by (2) and thus E belongs to the resolvent of H_ω for all $\omega \in \Omega$.

Step 1. For every $\omega \in \Omega$, there exist unique (up to a sign) normalized $U(\omega), V(\omega) \in \mathbb{R}^2$ such that $\|M^E(n,\omega)U(\omega)\|$ is exponentially decaying for $n \longrightarrow \infty$ and $\|M^E(n,\omega)V(\omega)\|$ is exponentially decaying for $n \longrightarrow -\infty$. The vectors $U(\omega), V(\omega)$ are linearly independent. For fixed $\omega \in \Omega$ they can be choosen to be continuous in a neighborhood of ω.

Step 2. Define the matrix $C(\omega)$ by $C(\omega) \equiv (U(\omega), V(\omega))$. Then $C(\omega)$ is invertible and there exist functions $a, b : \Omega \longrightarrow \mathbb{R} \setminus \{0\}$ such that

$$C(T\omega)^{-1} M^E(\omega) C(\omega) = \begin{pmatrix} a(\omega) & 0 \\ 0 & b(\omega) \end{pmatrix}. \tag{11}$$

Step 3. The functions $|a|, |b|, \|C\|, \|C^{-1}\| : \Omega \longrightarrow \mathbb{R}$ are continuous.

Step 4. M^E is uniform with $\gamma(E) > 0$.

Ad Step 1. This can be seen by standard arguments. Here is a sketch of the construction. Fix $\omega \in \Omega$ and set $u_0(n) \equiv (H_\omega - E)^{-1}\delta_0(n)$ and $u_{-1}(n) \equiv (H_\omega - E)^{-1}\delta_{-1}(n)$, where δ_k , $k \in \mathbb{Z}$, is given by $\delta_k(k) = 1$ and $\delta_k(n) = 0$, $k \neq n$. By Combes/Thomas arguments, see e.g. [10], the initial conditions $(u_0(0), u_0(1))$ and $(u_{-1}(0), u_{-1}(1))$ give rise to solutions of (5) which decay exponentially for $n \to \infty$. It is easy to see that not both of these solutions can vanish identically. Thus, after normalizing, we find a vector $U(\omega)$ with the desired properties. The continuity statement follows easily from continuity of $\omega \mapsto (H_\omega - E)^{-1}x$, for $x \in \ell^2(\mathbb{Z})$. The construction for $V(\omega)$ is similar. Uniqueness follows by standard arguments from constancy of the Wronskian. Linear independence is clear as E is not an eigenvalue of H_ω.

Ad Step 2. The matrix C is invertible by linear independence of U and V. The uniqueness statements of Step 1, show that there exist functions $a, b : \Omega \longrightarrow \mathbb{R}$ with $M^E(\omega)U(\omega) = a(\omega)U(T\omega)$ and $M^E(\omega)V(\omega) = b(\omega)V(T\omega)$. This easily yields (11). As the left hand side of this equation is invertible, the right hand side is invertible as well. This shows that a and b do not vanish anywhere.

Ad Step 3. Direct calculations show that the functions in question do not change if $U(\omega)$ or $V(\omega)$ or both are replaced by $-U(\omega)$ resp. $-V(\omega)$. By Step 1, such a replacement can be used to provide a version of V and U continuous arround an arbitrary $\omega \in \Omega$. This gives the desired continuity.

Ad Step 4. As $\|C\|$ and $\|C^{-1}\|$ are continuous by Step 3 and Ω is compact, there exists a constant $\kappa > 0$ with $\kappa \leq \|C(\omega)\|, \|C^{-1}(T\omega)\| \leq \kappa^{-1}$ for every $\omega \in \Omega$. Thus, uniformity of M^E will follow from uniformity of $\omega \mapsto C^{-1}(T\omega)M^E(\omega)C(\omega)$, which in turn will follow by Step 2 from uniformity of

$$\omega \mapsto D(\omega) \equiv \begin{pmatrix} |a|(\omega) & 0 \\ 0 & |b|(\omega) \end{pmatrix}.$$

As $|a|$ and $|b|$ are continuous by Step 3 and do not vanish by Step 2, the functions $\ln|a|, \ln|b| : \Omega \longrightarrow \mathbb{R}$ are continuous. The desired uniformity of D follows now by Proposition 3.1 (see proof of Lemma 3.2 for a similar reasoning). Positivity of $\gamma(E)$ is immediate from Step 1. □

A simple but crucial step in the proof of Theorem 2 is to relate the transfer matrices to subadditive functions. This will allow us to use Lemma 3.5 to show that the uniformity assumption of Lemma 3.2 and Lemma 3.3 holds for subshifts satisfying (PW). We proceed as follows. Let (Ω, T) be a strictly ergodic subshift and let $E \in \mathbb{R}$ be given. To the matrix valued function M^E we associate the function $F^E : \mathcal{W} \longrightarrow \mathbb{R}$ by setting

$$F^E(x) \equiv \log \|M^E(|x|, \omega)\|,$$

where $\omega \in \Omega$ is arbitrary with $\omega(1) \ldots \omega(|x|) = x$. It is not hard to see that this is well defined. Moreover, by submultiplicativity of the norm $\|\cdot\|$, we infer that F^E satisfies $F^E(xy) \leq F^E(x) + F^E(y)$.

Proposition 4.4. *M^E is uniform if and only if the limit $\lim_{|x|\to\infty} \frac{F^E(x)}{|x|}$ exists.*

Proof. This is straightforward. □

Now, we can prove the results stated in Section 2.

Proof of Theorem 1. The implication (i)$\Longrightarrow$(ii) is an immediate consequence of Lemma 4.1. This lemma also shows continuity of the Lyapunov exponent. The implication (ii)$\Longrightarrow$(i) follows from Lemma 4.2 and Lemma 4.3. □

Proof of Corollary 2.1 As Σ is closed and has no discrete points by general principles on random operators, the Cantor property will follow if Σ has measure zero. But this follows from the assumption and Theorem 1, as the set $\{E \in \mathbb{R} : \gamma(E) = 0\}$ has measure zero by the results of Kotani theory discussed in the introduction. □

Proof of Theorem 2. This is immediate from Lemma 3.5 and Proposition 4.4. □

5. Further discussion

In this section we will present some comments on the results proven in the previous sections.

As shown in the introduction and the proof of Theorem 1, the problem $(\mathcal{Z})$ for subshifts can essentially be reduced to establishing the inclusion $\Sigma \subset \{E \in \mathbb{R} : \gamma(E) = 0\}$. This has been investigated for various models by various authors [5, 6, 7, 13, 19, 42]. All these proofs rely on the same tool viz trace maps (see [1, 9] for study of trace maps as well). Trace maps are very powerful as they capture the underlying hierarchical structures. Besides beeing applicable in the investigation of $(\mathcal{Z})$, trace maps are extremely useful because

- trace map bounds are an important tool to prove absence of eigenvalues.

Actually, most of the cited literature studies both $(\mathcal{A})$ and $(\mathcal{Z})$. In fact, $(\mathcal{Z})$ can even be shown to follow from a strong version of $(\mathcal{A})$ [19] (cf. [13] as well). While this makes the trace map approach to $(\mathcal{Z})$ very attractive, it has two drawbacks:

- The analysis of the actual trace maps may be quite hard or even impossible.
- The trace map formalism only applies to substitution-like subshifts.

Thus, trace map methods can not be expected to establish zero-measure spectrum in a generality comparable to the validity of the underlying Kotani result.

Let us now compare this with the method presented above. Essentially, our method has a complementary profile: It does not seem to give information concerning absence of eigenvalues. But on the other hand it only requires a weak ergodic type condition. This condition is met by subshifts satisfying (PW) and this class of subshifts contains all primitive substitutions. In particular, it gives information on the Rudin-Shapiro substitution which so far had been unattainable. Moreover, quite likely, the condition (PW) will be satisfied for certain circle maps, where $(\mathcal{Z})$ could not be proven by other means.

All the same, it seems worthwhile pointing out that (PW) does not contain the class of Sturmian systems whose rotation number has unbounded continued fraction expansion. This is in fact the only class known to satisfy $(\mathcal{Z})$ (and much more [6, 12, 15, 16, 17, 27, 28, 41]) not covered by (PW). For this class, one can use the implication (ii) $\Longrightarrow$ (i) of Theorem 1, to conclude uniform existence of the Lyapunov exponent as done in Corollary 2.2. Still it seems desirable to give a direct proof of uniform existence of the Lyapunov exponent for these systems.

Finally, let us give the following strengthening of (the proof of) Theorem 1. It may be of interest whenever the strictly ergodic system is not a subshift.

Theorem 3. *Let (Ω, T) be strictly ergodic. Then,*

$$\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\} \cup \{E \in \mathbb{R} : M^E \text{ is not uniform}\},$$

where the union is disjoint.

Proof. The union is disjoint by Lemma 4.2. The inclusion "$\supset$" follows from Lemma 4.3.

To prove the inclusion "$\subset$", let $E \in \mathbb{R}$ with M^E uniform and $\gamma(E) > 0$ be given. By Lemma 3.3, we infer positivity of the Lyapunov exponent for all $F \in \mathbb{R}$ close to E. Moreover, by Theorem 4 of [21], for $F \in \mathbb{R}$ with $\gamma(F) > 0$, uniformity of M^F is equivalent to existence of an $n \in \mathbb{N}$ and a continuous $C : \Omega \longrightarrow GL(2, \mathbb{R})$ such that all entries of $C(T^n\omega)^{-1}M^F(n, \omega)C(\omega)$ are positive for all $\omega \in \Omega$. By uniformity of M^E this latter condition holds for M^E. By continuity of $(F, \omega) \mapsto C(T^n\omega)^{-1}M^F(n, \omega)C(\omega)$ and compactness of Ω, it must then hold for M^F as well whenever F is sufficiently close to E.

These considerations prove existence of an open interval $I \subset \mathbb{R}$ containing E on which uniformity of the transfer matrices and positivity of the Lyapunov exponent hold (cf. top of page 811 of [21] for related arguments). Now, replacing Γ^c with I, one can easily adopt the proof of Lemma 4.1 to obtain the desired inclusion. □

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CHAPTER 7

D. Lenz, *Existence of non-uniform cocycles on uniquely ergodic systems*, Ann. Inst. Henri Poincaré: Prob. & Stat. 40 (2004), 197–206.

EXISTENCE OF NON-UNIFORM COCYCLES ON UNIQUELY ERGODIC SYSTEMS

DANIEL LENZ [1,*]

[1] Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany
E-mail: dlenz@mathematik.tu-chemnitz.de

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Abstract. We study existence of non-uniform continuous $SL(2,\mathbb{R})$-valued cocycles over uniquely ergodic dynamical systems. We present a class of subshifts over finite alphabets on which every locally constant cocycle is uniform. On the other hand, we also show that every irrational rotation admits non-uniform cocycles. Finally, we discuss characterizations of uniformity.

1. Introduction

This paper is concerned with $SL(2,\mathbb{R})$-valued cocycles over dynamical systems. Throughout, (Ω, T) will be a uniquely ergodic dynamical system (i.e. Ω is a compact metric space, $T : \Omega \longrightarrow \Omega$ is a homeomorphism and there is only one T-invariant probability measure on Ω). The unique T-invariant probability measure on Ω will be denoted by μ. Let $SL(2,\mathbb{R})$ denote the group of real-valued 2×2-matrices with determinant equal to one. This is a topological group whose topology is induced by the standard metric on the 2×2-matrices. To a continuous function $A : \Omega \longrightarrow SL(2,\mathbb{R})$ we associate the cocycle

$$A(\cdot,\cdot) : \mathbb{Z} \times \Omega \longrightarrow SL(2,\mathbb{R})$$

defined by

$$A(n,\omega) \equiv \begin{cases} A(T^{n-1}\omega)\cdots A(\omega) & : \quad n > 0 \\ Id & : \quad n = 0 \\ A^{-1}(T^{n}\omega)\cdots A^{-1}(T^{-1}\omega) & : \quad n < 0. \end{cases}$$

By the multiplicative ergodic theorem, there exists a $\Lambda(A) \in \mathbb{R}$ with

$$\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$$

for μ-almost every $\omega \in \Omega$. Following [6] (cf. [21] as well), we introduce the following definition.

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Definition 1. *Let (Ω, T) be uniquely ergodic. The continuous function $A : \Omega \longrightarrow SL(2,\mathbb{R})$ is called uniform if the limit $\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$ exists for all $\omega \in \Omega$ and the convergence is uniform on Ω.*

Remark 1. For minimal (not necessarily uniquely ergodic) systems, uniform existence of the limit in the definition already implies uniform convergence, as proven by Furstenberg and Weiss [7]. Their result is actually even more general and applies to arbitrary real valued continuous cocycles.

Existence or non-existence of uniform $SL(2,\mathbb{R})$-valued functions has been studied by various people, e.g. in [21, 8, 6, 15]. In fact, Walters asked the following question [21]:

(Q) Does every uniquely ergodic dynamical system with non-atomic measure μ admit a non-uniform cocycle?

Using results of Veech [20], Walters presents a class of examples admitting non-uniform cocycles. He also discusses a further class of examples, namely suitable irrational rotations, for which non-uniformity was shown by Herman [8]. Recently, Furman carried out a careful study of uniformity of cocycles [6]. For strictly ergodic dynamical systems, he characterizes uniform cocycles with positive $\Lambda(A)$ in terms of uniform diagonalizability. Related results on positivity of cocycles can also be found in [12].

The aim of this article is to adress (Q) for certain examples and to study conditions for uniformity of cocycles. In order to be more precise recall that (Ω, T) is called a subshift over the compact S, if Ω is a closed subset of $S^{\mathbb{Z}}$ (with product topology) invariant under the shift $T : S^{\mathbb{Z}} \longrightarrow S^{\mathbb{Z}}$, $(Ts)(n) \equiv s(n+1)$. If S is finite, it is called the alphabet. A function f on a subshift over S is called locally constant if there exists an $N \in \mathbb{N}$ such that

$$f(\omega) = f(\rho), \;\text{ whenever }\; (\omega(-N),\ldots,\omega(N)) = (\rho(-N),\ldots,\rho(N)). \tag{1}$$

Our results will show the following:

- There exist subshifts over finite alphabets which do not admit locally constant non-uniform cocycles (Theorem 1).
- Every irrational rotation admits a non-uniform cocycle (Theorem 2).
- For strictly ergodic dynamical systems, uniformity of A with $\Lambda(A) > 0$ follows already from suitable lower bounds on $n^{-1} \ln \|A(n,\omega)\|$ (Theorem 3).
- For uniquely ergodic dynamical system, uniformity of A with $\Lambda(A) > 0$ can be characterized by a certain uniform hyperbolicity condition (Theorem 4).

As mentioned already, these results are closely related to results of Furman [6] and Herman [8, 9] respectively. This will be discussed in more detail at the corresponding places.

This paper is organised as follows. In Section 2, we prove Theorem 1. Section 3 is devoted to a proof of Theorem 2 and discussion of its background. Finally, we discuss Theorem 3 and Theorem 4 in Section 4.

2. Subshifts with only uniform locally constant functions

In this section we present a class of subshifts over finite alphabets on which every locally constant cocycle is uniform.

For a subshift (Ω, T) over the finite set S, let $\mathcal{W}$ be the associated set of finite words i.e.

$$\mathcal{W} \equiv \{\omega(n)\cdots\omega(n+k) : \omega \in \Omega, n \in \mathbb{Z}, k \in \mathbb{N}_0\}.$$

We will use standard concepts from combinatorics on words. In particular, we define the length $|w|$ of a word $w = w(1)\dots w(n)$ to be n and we denote the number of copies of v in w by $\sharp_v(w)$ for arbitrary $v, w \in \mathcal{W}$. The class of subshifts we are particularly interested in is presented in the next definition.

Definition 2. *A subshift (Ω, T) over the finite set S is said to satisfy uniform positivity of weights, (PW), if there exists a constant $C > 0$ with $\liminf_{|w|\to\infty} \frac{\sharp_v(w)}{|w|}|v| \geq C$ for all $v \in \mathcal{W}$.*

Remark 2. (a) Condition (PW) says roughly that the amount of "space" covered by a word $v \in \mathcal{W}$ in a long word $w \in \mathcal{W}$ is bounded below uniformly in $v \in \mathcal{W}$. In particular, (PW) implies minimality.
(b) The condition (PW) is in particular satisfied for subshifts associated to primitive substitutions and more generally for linearly recurrent subshifts [5, 16].
(c) It is not hard to see that (PW) implies that the subshift has linear complexity. More precisely, the number of different words in $\mathcal{W}$ of length n is bounded by $C^{-1}n$ (see e.g. [17]).

Theorem 1. *Let (Ω, T) be a subshift over the finite set S. If (Ω, T) satisfies (PW), then every locally constant function $G : \Omega \longrightarrow SL(2, \mathbb{R})$ is uniform.*

The theorem is a rather direct consequence of the following lemma. The lemma relates (PW) to existence of averages for subadditive functions on $\mathcal{W}$. Recall that $F : \mathcal{W} \longrightarrow \mathbb{R}$ is called subadditive if $F(xy) \leq F(x) + F(y)$ for arbitrary $x, y \in \mathcal{W}$ with $xy \in \mathcal{W}$.

Lemma 2.1. *Let (Ω, T) be a minimal subshift over the finite S. Then, the limit $\lim_{|x|\to\infty} |x|^{-1}F(x)$ exists for every subadditive $F : \mathcal{W} \longrightarrow \mathbb{R}$ if and only if (Ω, T) satisfies (PW).*

Proof. One implication follows from Theorem 2 of [16] and the other by Proposition 4.2. of [16]. □

Proof of Theorem 1. Define $F^G : \mathcal{W} \longrightarrow \mathbb{R}$ by

$$F^G(x) \equiv \sup\{\log \|G(|x|, \omega)\| : \omega(1)\dots\omega(|x|) = x\}.$$

Apparently, F^G is subadditive. Thus, by the preceeding lemma, the limit $\lim_{|x|\to\infty} |x|^{-1}F^G(x)$ exists. Therefore, it remains to show that

$$\Delta(n, \sigma, \rho) \equiv \left| \frac{1}{n} \log \|G(n, \sigma)\| - \frac{1}{n} \log \|G(n, \rho)\| \right| \tag{2}$$

is arbitrarily small for all $\sigma, \rho \in \Omega$ with

$$\sigma(1)\dots\sigma(n) = \rho(1)\dots\rho(n) \tag{3}$$

whenever $n \in \mathbb{N}$ is large enough. Let $N \in \mathbb{N}$ be the constant of (1) for the locally constant G. Consider an arbitrary $n \in \mathbb{N}$ with $n \geq 2N$.

From $G(n, \omega) = G(N, T^{n-N}\omega)G(n - 2N, T^N\omega)G(N, \omega)$ for arbitrary $n \geq 2N$, we infer

$$\log \|G(n, \omega)\| \leq \log \|G(n - 2N, T^N\omega)\| + \log \|G(N, T^{n-N}\omega)\| + \log \|G(N, \omega)\|$$

as well as $G(n-2N, T^N\omega) = G(N, T^{n-N}\omega)^{-1} G(n,\omega) G(N,\omega)^{-1}$ for arbitrary $\omega \in \Omega$. Combining this latter equality with the fact that $\|M\| = \|M^{-1}\|$ for all $M \in SL(2,\mathbb{R})$, we infer

$$\log \|G(n-2N, T^N\omega)\| - \log \|G(N, T^{n-N}\omega)\| - \log \|G(N,\omega)\| \leq \log \|G(n,\omega)\|$$

for all $\omega \in \Omega$. By local constancy, we have $G(n-2N, T^N\sigma) = G(n-2N, T^N\rho)$ whenever σ and ρ satisfy (3) with $n \geq 2N$. Thus, for such σ, ρ the above inequalities yield

$$|\log \|G(n,\sigma)\| - \log \|G(n,\rho)\|| \leq 4 \sup\{|\log \|G(N,\omega)\|| : \omega \in \Omega\}.$$

As the right hand side is independent of n, this easily gives the desired smallness of the $\Delta(n,\sigma,\rho)$ in (2) for large n. □

3. Non-uniform functions

In this section we will discuss certain examples of non-uniform cocycles. These examples will be based on recent results of the author [15] on spectral theory of certain Schrödinger operators and known results on positivity of Lyapunov exponents [1, 2, 9].

Let (Ω, T) be as above and let $f : \Omega \longrightarrow \mathbb{R}$ be a continuous function. To these data we can associate a family $(H_\omega)_{\omega\in\Omega}$ of operators $H_\omega : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$, $\omega \in \Omega$, given by

$$(H_\omega u)(n) \equiv u(n+1) + u(n-1) + f(T^{n-1}\omega)u(n). \tag{4}$$

Such families of operators arise in the study of disordered media. Depending on the underlying dynamical systems, they provide examples for a variety of interesting spectral features such as dense pure point spectrum, purely singularly continuous spectrum and Cantor spectrum of measure zero (see [3, 4] for details and further references.)

An important tool in the investigation of their spectral theory is the study of solutions u of the associated eigenvalue equation

$$u(n+1) + u(n-1) + (\omega(n) - E)u(n) = 0 \tag{5}$$

for $E \in \mathbb{R}$. It is not hard to see that u is a solution of this equation if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M^E(n,\omega) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix},\ n \in \mathbb{Z}, \tag{6}$$

where the continuous function $M^E : \Omega \longrightarrow SL(2,\mathbb{R})$ is defined by

$$M^E(\omega) \equiv \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}. \tag{7}$$

As discussed in the introduction M^E gives rise to the average $\gamma(E) \equiv \Lambda(M^E)$. This average is called the Lyapunov exponent for the energy E. It measures the rate of exponential growth of solutions of (5).

As is well known (see e.g. Proposition 1.2.2 in [18]), for minimal (Ω, T) the spectrum $\Sigma = \sigma(H_\omega)$ of the self-adjoint operator H_ω does not depend on the point $\omega \in \Omega$. Moreover, for strictly ergodic systems, it was shown by the author in Theorem 3 of [15] that

$$\Sigma = \{E : \gamma(E) = 0\} \cup \{E : M^E \ \text{ is not uniform}\}, \tag{8}$$

where the union is disjoint. This implies immediately the following result.

Lemma 3.1. *Let (Ω, T) be strictly ergodic and (H_ω) as above. Then $\Sigma = \{E : M^E$ is not uniform$\}$ if and only if $\gamma(E) > 0$ for every $E \in \mathbb{R}$.*

Thus, examples of operators of the form (H_ω) with positive Lyapunov exponent give rise to non-uniform matrices. Indeed, there are well known examples of operators with uniformly positive Lyapunov exponent and we will discuss one of them next.

Fix $\alpha \in (0,1)$ irrational and $\lambda > 0$. Denote the irrational rotation by α on the unit circle, $\mathcal{S}$, by R_α (i.e. $R_\alpha z \equiv \exp(2\pi i\alpha)z$, where i is the square root of -1). Define $f^\lambda : \mathcal{S} \longrightarrow \mathbb{R}$ by $f^\lambda(z) \equiv \lambda(z + z^{-1})$ (i.e. $f^\lambda(\exp(i\theta)) = 2\lambda\cos(\theta)$). Denote the associated operators by (H_z^λ) and their spectrum by $\Sigma(\lambda)$. The operators (H_z^λ) are called almost-Mathieu operators. They have attracted much attention (see e.g. [10, 11, 13] for further discussion and references). We have the following theorem.

Theorem 2. *For arbitrary irrational $\alpha \in (0,1)$ and $\lambda > 1$, the function M^E is non-uniform if and only if E belongs to $\Sigma(\lambda)$.*

Proof. By the foregoing lemma, it suffices to show positivity of $\gamma(E)$ for every $E \in \mathbb{R}$. This is well known [1, 2] (see [9] for an alternative proof as well). □

Remark 3. The result shows that every irrational rotation allows for a non-uniform matrix. This generalizes results of Herman [8], where this was only shown for certain rotation numbers. Note, however, that the results of [9] combined with Theorem 4 of [6] (or Theorem 4 below) also show existence of non-uniform cocycles for every irrational rotation. Still, the above result is more explicit in that the set of energies with non-uniform matrices is identified as $\Sigma(\lambda)$.

4. Characterizations of uniformity

In this section we study uniformity of cocycles for uniquely ergodic and strictly ergodic systems.

Let $\mathcal{P} = \mathcal{P}\mathbb{R}^2$ be the projective space over $\mathbb{R}^2$. Thus, $\mathcal{P}$ is the space of all one-dimensional subspaces of $\mathbb{R}^2$. To $X \in \mathbb{R}^2 \setminus \{0\}$, we associate the element $[X] = \{\lambda X : \lambda \in \mathbb{R}\} \in \mathcal{P}$. Obviously, every element in $\mathcal{P}$ can be written as $[(\cos(\theta), \sin(\theta))]$ with a suitable $\theta \in [0, \pi]$. The space $\mathcal{P}$ is a complete metric space, when equipped with the metric

$$d([(\cos(\theta), \sin(\theta))], [(\cos(\eta), \sin(\eta))]) = \min\{|\theta - \eta|, |\theta - \eta - \pi|, |\theta - \eta + \pi|\}$$

We start with a characterization of uniformity of cocycles for strictly ergodic systems.

Theorem 3. *Let (Ω, T) be strictly ergodic. Then, a continuous $A : \Omega \longrightarrow SL(2, \mathbb{R})$ is uniform with $\Lambda(A) > 0$ if and only if there exist $m \in \mathbb{N}$ and $\delta > 0$ such that $\delta \leq \frac{1}{n} \ln \|A(n, \omega)\|$ for all $\omega \in \Omega$ and $n \geq m$.*

Remark 4. The theorem deals with a uniform lower bound on $\frac{1}{n} \ln \|A(n, \omega)\|$. As for an upper bound, we mention Corollary 2 of [6] which shows $\limsup_{n\to\infty} n^{-1} \ln \|A(n, \omega)\| \leq \Lambda(A)$ uniformly in $\omega \in \Omega$ for arbitrary (not necessarily uniform) continuous $A : \Omega \longrightarrow SL(2, \mathbb{R})$.

The proof of this theorem and of further results will be based on some auxiliary propositions.

Proposition 4.1. *Let* (A_n) *be a sequence in* $SL(2,\mathbb{R})$*. Then, there exists at most one* $v \in \mathcal{P}$ *with* $\|A_n V\| \longrightarrow 0$*,* $n \longrightarrow \infty$*, for every* $V \in v$*.*

Proof. Assume the contrary. Then, there exist linearly independent vectors V_1 and V_2 in $\mathbb{R}^2$ with $\|A_n V_i\| \longrightarrow 0$, $n \longrightarrow \infty$, $i = 1, 2$. Thus, $\|A\| \longrightarrow 0$, $n \longrightarrow 0$ and this contradicts $\|A_n\| \geq 1$ (which is a direct consequence of $\det A_n = 1$). □

Part (a) of the following proposition contains the key to our considerations, viz the estimate (10) below. We take it from recent work of Last/Simon in [14] which in turn essentially abstracts a result of Ruelle [19]. As pointed out to the author by the referee it can also be understood as a consequence of the classical geometric Morse-Lemma by viewing $SL(2,\mathbb{R})$ as the group of isometries of the hyperbolic plane and then using that the orbit in question $V_n = A_n^{-1} V_0$ is quasi-geodesic (due to the assumptions).

While (a) of the proposition is clearly the main new input in our argument, we will mostly use the the variant of (a) given in part (b) of the proposition.

Proposition 4.2. *Let* (A_n) *be a sequence of matrices in* $SL(2,\mathbb{R})$ *with* $D \equiv \sup_{n\in\mathbb{N}} \|A_{n+1} A_n^{-1}\| < \infty$*. Define the selfadjoint operator* $|A_n|$ *by* $|A_n| \equiv (A_n^* A_n)^{\frac{1}{2}}$ *and let* u_n *be the eigenspace of* $|A_n|$ *associated to the eigenvalue* $a_n \equiv \||A_n|\|^{-1} = \|A_n\|^{-1}$*.*
(a) If there exist $\delta > 0$ *and* $m \in \mathbb{N}$ *with* $\delta \leq n^{-1} \ln \|A_n\|$ *for* $n \geq m$ *then* u_n *is one-dimensional for* $n \geq m$ *i.e.* $u_n \in \mathcal{P}$*, and there exists* $u \in \mathcal{P}$ *with* $d(u_n, u) \leq C \exp(-2\delta n)$ *for every* $n \geq m$*, where* $C = 2\pi D^2 (1 - \exp(-2\delta))^{-1}$*.*
(b) If there exist $\delta > 0$ *and* $m \in \mathbb{N}$ *with* $\delta \leq n^{-1} \ln \|A_n\| \leq \frac{3}{2}\delta$ *for* $n \geq m$*, then* $\|A_n U\| \leq (2C+1) \exp(-2^{-1}\delta n) \|U\|$ *for arbitrary* $n \geq m$ *and* $U \in u$*.*

Proof. (a) As $|A_n|$ is selfadjoint, $a_n^{-1} = \||A_n|\|$ is an eigenvalue of $|A_n|$. Thus, by $1 = \det A_n = \det |A_N|$, the selfadjoint $|A_n|$ has the eigenvalues a_n^{-1} and a_n. As by assumption

$$1 < \exp(\delta n) \leq \|A_n\| = a_n^{-1} \text{ for all } n \geq m. \tag{9}$$

the eigenspace u_n is then one-dimensional. By (8.5) of [14] (see [19] as well), the u_n converge to an element $u \in \mathcal{P}$ and

$$d(u_n, u) \leq \frac{\pi}{2} \sum_{k=n}^{\infty} \frac{D^2}{\|A_n\|^2}. \tag{10}$$

Combining (9) and (10), we infer

$$d(u_n, u) \leq C \exp(-2\delta n) \tag{11}$$

with C as above.

(b) Let $U \in u$ with $\|U\| = 1$ and $n \geq m$ be given. By (11), we can find $U_n \in u_n$ with $\|U_n\| = 1$ and

$$\|U - U_n\| \leq \sqrt{2} d([U],[U_n]) \leq C\sqrt{2} \exp(-2\delta n). \tag{12}$$

By (9) we have

$$\|A_n U_n\| = \||A_n| U_n\| = \|a_n U_n\| \leq \exp(-\delta n). \tag{13}$$

As, by assumption, $\ln \|A_n\| \leq \frac{3}{2}\delta n$, we obtain

$$\|A_n U\| \leq \|A_n(U - U_n)\| + \|A_n U_n\| \leq (2C+1)\exp(-\frac{1}{2}\delta n).$$

This implies (b) □

We also have the following "uniform version" of the foregoing Proposition.

Proposition 4.3. *Let $A : \Omega \longrightarrow SL(2,\mathbb{R})$ be continuous. For $n \in \mathbb{Z}$ and $\omega \in \Omega$, define the selfadjoint nonnegative operator $|A(n,\omega)|$ by $|A(n,\omega)| = (A(n,\omega)^* A(n,\omega))^{\frac{1}{2}}$ and let $u(n,\omega)$ be the eigenspace of $|A(n,\omega)|$ associated to the eigenvalue $a(n,\omega) = \|A(n,\omega)\|^{-1} = \||A(n,\omega)|\|^{-1}$.*
(a) If there exist $\delta > 0$ and $m \in \mathbb{N}$ with $\delta \leq n^{-1} \ln \|A(n,\omega)\|$ for every $n \geq m$ and every $\omega \in \Omega$, then $u(n,\omega)$ is one-dimensional, i.e. $u(n,\omega)$ belongs to $\mathcal{P}$, for $n \geq m$ and the functions $u(n,\cdot)$ converge uniformly to a continuous function $u : \Omega \longrightarrow \mathcal{P}$.
(b) If there exist $\delta > 0$ and $m \in \mathbb{N}$ with $\delta \leq n^{-1} \ln \|A(n,\omega)\| \leq \frac{3}{2}\delta$, for all $\omega \in \Omega$ and $n \geq m$, then there exists $\kappa > 0$ and $C > 0$ with $\|A(n,\omega)U\| \leq C\exp(-\kappa n)\|U\|$ for every $n \in \mathbb{N}$, $\omega \in \Omega$ and $U \in u(\omega)$.

Proof. To prove (a) and (b), we apply parts (a) and (b) respectively of the foregoing proposition simultanuously for all $\omega \in \Omega$. Note that all estimates in the foregoing proposition are rather explicit and are governed by constants not depending on $\omega \in \Omega$. In particular, the functions $u(n,\cdot)$ converge uniformly. As they are obviously continuous, their limit is also continuous. □

Proof of Theorem 3. The "only if" statement is clear. To show the other direction, we proceed as follows:

By assumption we can apply Proposition 4.3 (a) and obtain a continuous function $u : \Omega \longrightarrow \mathcal{P}$ (which is the limit of the function $u(n,\cdot)$). By the multiplicative ergodic theorem, there exists a T-invariant set $\Omega' \subset \Omega$ of full measure with

$$0 < \delta \leq \Lambda(A) = \liminf_{|n|\to\infty} \frac{1}{|n|} \ln \|A(n,\omega)\| = \limsup_{|n|\to\infty} \frac{1}{|n|} \ln \|A(n,\omega)\|$$

for every $\omega \in \Omega'$. This, of course, implies

$$0 < \Lambda(A) \leq \limsup_{n\to\infty} \frac{1}{n} \ln \|A(n,\omega)\| \leq \frac{4}{3} \liminf_{n\to\infty} \frac{1}{n} \ln \|A(n,\omega)\|$$

for every $\omega \in \Omega'$. By (b) of Proposition 4.2, we then infer exponential decay of $\|A(n,\omega)U\|$ for $n \to \infty$ for arbitrary but fixed $\omega \in \Omega'$ and $U \in u(\omega)$. As Ω' is invariant and the subspace of $\mathbb{R}^2$ with such exponential decay is unique by Proposition 4.1, we conclude, for $\omega \in \Omega'$,

$$[A(n,\omega)U] = u(T^n\omega) \tag{14}$$

for $n \in \mathbb{Z}$ and $U \in u(\omega) \setminus \{0\}$. Now, by continuity of $\omega \mapsto u(\omega)$ and minimality of (Ω, T), we infer validity of (14) for every $\omega \in \Omega$ and $n \in \mathbb{Z}$. Similarly, considering $n \to -\infty$, we infer existence of a continuous $v : \Omega \longrightarrow \mathcal{P}$, $\omega \mapsto v(\omega)$, such that $\|A(n,\omega)V\|$ is exponentially decaying for $n \to -\infty$ for every $\omega \in \Omega'$ and $V \in v(\omega)$ and

$$[A(n,\omega)V] = v(T^n\omega) \tag{15}$$

for arbitrary $\omega \in \Omega$, $n \in \mathbb{Z}$ and $V \in v(\omega) \setminus \{0\}$.

Now, choose, for each $\omega \in \Omega$, vectors $U(\omega) \in u(\omega)$ and $V(\omega) \in v(\omega)$ with $\|U(\omega)\| = \|V(\omega)\| = 1$. By (14) and (15), there exist $a, d : \Omega \longrightarrow \mathbb{R} \setminus \{0\}$, with $A(\omega)U(\omega) = a(\omega)U(T\omega)$ and $A(\omega)V(\omega) = d(\omega)V(T\omega)$. Define the matrix $C(\omega)$ by $C(\omega) = (U(\omega), V(\omega))$. By $\|U(\omega)\| = \|V(\omega)\| = 1$, $U(\omega)$ and $V(\omega)$ are unique up to a multiplication by -1. Moreover, for fixed $\omega_0 \in \Omega$, we can always find a neighbourhood of ω_0 on which U and V can be chosen continuously (as u and v are continuous). Therefore, the functions

$$\omega \mapsto \|C(\omega)\|,\ \omega \mapsto |a(\omega)|,\ \ \omega \mapsto |d(\omega)|$$

are continuous (as they are invariant under the replacement of $U(\omega)$ by $-U(\omega)$ or $V(\omega)$ by $-V(\omega)$.) A short calculation then gives

$$\ln \|A(n,\omega)U(\omega)\| = \begin{cases} \sum_{k=0}^{n-1} \ln |a(T^k\omega)| & : \quad n > 0 \\ 0 & : \quad n = 0 \\ -\sum_{k=n}^{-1} \ln |a(T^k\omega)| & : \quad n < 0. \end{cases}$$

Thus, the uniform ergodic theorem for continuous functions on uniquely ergodic systems, yields

$$\frac{1}{n} \ln \|A(n,\omega)U(\omega)\| \longrightarrow \int_\Omega \ln |a(\omega)|\, d\mu(\omega),\ |n| \longrightarrow \infty,$$

uniformly in $\omega \in \Omega$. As $\|A(n,\omega)U(\omega)\|$ is exponentially decaying for $n \to \infty$ and $\omega \in \Omega'$, we see $\int_\Omega \ln |a(\omega)|\, d\mu(\omega) < 0$. Putting this together, we infer that $\|A(n,\omega)U(\omega)\|$ is exponentially decaying for $n \to \infty$ and exponentially increasing for $n \to -\infty$ for every $\omega \in \Omega$. Similarly, $\|A(n,\omega)V(\omega)\|$ can be seen to be exponentially decaying for $n \to -\infty$ and exponentially increasing for $n \to \infty$ for every $\omega \in \Omega$. In particular, we have $u(\omega) \neq v(\omega)$ for every $\omega \in \Omega$. Thus, the matrix $C(\omega)$ is invertible and, by construction, we have

$$C(T\omega)^{-1} A(\omega) C(\omega) = \begin{pmatrix} a(\omega) & 0 \\ 0 & d(\omega) \end{pmatrix}. \tag{16}$$

Now, uniformity of A follows easily from continuity of $|a|$ and $|b|$, as the continuous functions $\omega \mapsto \|C(\omega)\|$ and $\omega \mapsto \|C^{-1}(\omega)\|$ are uniformly bounded on the compact Ω. □

Corollary 4.4. *Let (Ω, T) be strictly ergodic and $(H_\omega)_{\omega \in \Omega}$ as in Section 3. For $E \in \mathbb{R}$, define $\gamma_{min}(E)$ by $\gamma_{min}(E) \equiv \liminf_{n\to\infty} \min\{\frac{1}{n} \ln \|M^E(n,\omega)\| : \omega \in \Omega\}$. Then, $\Sigma = \{E \in \mathbb{R} : \gamma_{min}(E) = 0\}$.*

Proof. By Theorem 3, we have $\gamma_{min}(E) > 0$ if and only if M^E is uniform with $\gamma(E) > 0$. But this is equivalent to $E \notin \Sigma$ by (8). □

Remark 5. For the almost-Mathieu operators discussed in Section 3, it is possible to establish pure point spectrum (provided α, λ are suitable) (see references in Section 3). An important issue in the corresponding proofs is to obtain exponentially growing lower bounds on the modulus of the matrix elements of $M^E(n,\omega)$ for large $n \in \mathbb{N}$ (and suitable $\omega \in \Omega$ and $E \in \mathbb{R}$). The corollary shows that these bounds can not hold uniformly. This contrasts with the validity of uniform upper bounds discussed in Remark 4.

The methods developed above to treat strictly ergodic systems can be modified to characterize uniformity of cocylces for uniquely ergodic systems. This is the content of the next theorem.

Theorem 4. *Let (Ω, T) be uniquely ergodic and $A : \Omega \longrightarrow SL(2,\mathbb{R})$ be continuous. Then the following are equivalent:*

(i) A is uniform with $\Lambda(A) > 0$.

(ii) There exist constants $\kappa, C > 0$ and continuous functions $u, v : \Omega \longrightarrow \mathcal{P}$ with

$$\|A(n,\omega)U\| \le C\exp(-\kappa n)\|U\| \quad \text{and} \quad \|A(-n,\omega)V\| \le C\exp(-\kappa n)\|V\|. \tag{17}$$

for arbitrary $\omega \in \Omega$, $n \in \mathbb{N}$, $U \in u(\omega)$ and $V \in v(\omega)$.

(iii) There exists $\delta > 0$ and $m \in \mathbb{N}$ with $0 < \delta \le \frac{1}{n}\ln\|A(n,\omega)\| \le \frac{3}{2}\delta$ for every $\omega \in \Omega$ and $n \ge m$.

In this case, $u(\omega) \neq v(\omega)$, $[A(n,\omega)U] = u(T^n\omega)$ and $[A(n,\omega)V] = v(T^n\omega)$ for arbitrary $\omega \in \Omega$, $n \in \mathbb{Z}$, $U \in u(\omega)$ and $V \in v(\omega)$ with $U, V \neq 0$.

Remark 6. The equivalence of (i) and (ii) in some sense extends the corresponding result of Furman for strictly ergodic systems [6]. Namely, Theorem 4 of [6] shows that uniformity of A combined with $\Lambda(A) > 0$ holds if and only if A is continuously cohomologous to a diagonal matrix. Our extension to uniquely ergodic systems is made possible through the use of Proposition 4.2 (see discussion before this proposition). Let us also mention that the concept of hyperbolic structure studied in [9] essentially amounts to (ii) in our context (see [8] for connection to uniformity as well). Part (iii) of Theorem 4 is new. It is inspired by arguments in [14]. It provides an analogue of Theorem 3 for uniquely ergodic systems.

Proof of Theorem 4. (i) $\Longrightarrow$ (iii): This is clear.

(iii) $\Longrightarrow$ (ii): The construction of u is immediate from Proposition 4.3. The construction of v is similar by applying Proposition 4.3 to the function $\widetilde{A} : \widetilde{\Omega} \longrightarrow SL(2,\mathbb{R})$, where $\widetilde{\Omega} = \Omega$, $\widetilde{A}(\omega) = A(T^{-1}\omega)^{-1}$ and the action on $\widetilde{\Omega}$ is given by $\widetilde{T} = T^{-1}$.

(ii) $\Longrightarrow$ (i): Proposition 4.1 and assumption (ii) imply

$$[A(n,\omega)U] = u(T^n\omega) \ \text{ and } \ [A(n,\omega)V] = v(T^n\omega) \tag{18}$$

for arbitrary $\omega \in \Omega$, $n \in \mathbb{Z}$, $U \in u(\omega)$ and $V \in v(\omega)$ with $U, V \neq 0$. Let arbitrary $U \in u(\omega)$ and $n \in \mathbb{N}$ be given. By (18) and (ii), we then have $\|U\| = \|A(n, T^{-n}\omega)A(-n,\omega)U\| \le C\exp(-\kappa n)\|A(-n,\omega)U\|$ which implies $\|A(-n,\omega)U\| \ge C^{-1}\exp(\kappa n)\|U\|$. As this holds for all $n \in \mathbb{N}$, we infer $u(\omega) \neq v(\omega)$ from (ii). Now, (i) follows by mimicking the last part of the proof of Theorem 3.

Note that the last statement of the theorem has been shown in (ii)$\Longrightarrow$(i). $\square$

To formulate our last result, we recall that the set $C(\Omega, SL(2,\mathbb{R}))$ of continuous functions $A : \Omega \longrightarrow SL(2,\mathbb{R})$ is a complete metric space when equiped with the metric

$$d(A_1, A_2) \equiv \sup_{\omega\in\Omega} \|A_1(\omega) - A_2(\omega)\|.$$

Let $\mathcal{U}(\Omega)$ be the set of uniform $A \in C(\Omega, SL(2,\mathbb{R}))$ and $\mathcal{U}(\Omega)_+$ be the set of those $A \in \mathcal{U}(\Omega)$ with $\Lambda(A) > 0$. Then the following holds (see Theorem 5 of [6] as well).

Theorem 5. *Let (Ω, T) be uniquely ergodic. Then, $\mathcal{U}(\Omega)_+$ is open in $C(\Omega, SL(2,\mathbb{R}))$ and $\Lambda : \mathcal{U}(\Omega) \longrightarrow \mathbb{R}$ is continuous.*

This is essentially contained in Theorem 5 of [6] and its proof. Note, however, that there is a slight gap in the proof of that theorem in [6]: Its statement refers to arbitrary uniquely ergodic systems. But its proof makes crucial use of Theorem 4 of [6], which assumes not only unique ergodicity but also minimality. As far as the continuity statement goes, this gap can be bridged by restricting attention to a T-minimal subset Ω_0 of Ω. However, it does not seem to be clear that this yields the openess statement as well. Therefore, we conclude this section by noting that, given the methods provided in [6], one can base a *Proof of Theorem 5* on Theorem 4 above, similarly as the proof of Theorem 5 in [6] is based on Theorem 4 of [6].

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CHAPTER 8

D. Damanik, D. Lenz, *A condition of Boshernitzan and uniform convergence in the multiplicative ergodic theorem*, preprint 2004.

A CONDITION OF BOSHERNITZAN AND UNIFORM CONVERGENCE IN THE MULTIPLICATIVE ERGODIC THEOREM

DAVID DAMANIK [1], DANIEL LENZ [2]

[1] Department of Mathematics 253–37, California Institute of Technology, Pasadena, CA 91125, U.S.A., E-Mail: damanik@its.caltech.edu

[2] Fakultät für Mathematik, TU Chemnitz, D-09107 Chemnitz, Germany, E-Mail: dlenz@mathematik.tu-chemnitz.de

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Abstract. This paper is concerned with uniform convergence in the multiplicative ergodic theorem on aperiodic subshifts. If such a subshift satisfies a certain condition, originally introduced by Boshernitzan, every locally constant SL(2,$\mathbb{R}$)-valued cocycle is uniform. As a consequence, the corresponding Schrödinger operators exhibit Cantor spectrum of Lebesgue measure zero.

An investigation of Boshernitzan's condition then shows that these results cover all earlier results of this type and, moreover, provide various new ones. In particular, Boshernitzan's condition is shown to hold for almost all circle maps and almost all Arnoux-Rauzy subshifts.

1. Introduction

This paper is concerned with uniform convergence in the multiplicative ergodic theorem.

More precisely, let (Ω, T) be a topological dynamical system Thus, Ω is a compact metric space and $T : \Omega \longrightarrow \Omega$ is a homeomorphism. Assume furthermore that (Ω, T) is uniquely ergodic, that is, there exists a unique T-invariant probability measure μ on Ω.

As usual the dynamical system (Ω, T) is called minimal if every orbit $\{T^n\omega : n \in \mathbb{Z}\}$ is dense in Ω. It is called aperiodic if $T^n\omega \neq \omega$ for all $\omega \in \Omega$ and $n \neq 0$.

Let SL(2,$\mathbb{R}$) be the group of real valued 2×2-matrices with determinant equal to one equipped with the topology induced by the standard metric on 2×2 matrices.

To a continuous function $A : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ we associate the cocycle

$$A(\cdot, \cdot) : \mathbb{Z} \times \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$$

defined by

$$A(n, \omega) \equiv \begin{cases} A(T^{n-1}\omega) \cdots A(\omega) & : \quad n > 0 \\ Id & : \quad n = 0 \\ A^{-1}(T^n\omega) \cdots A^{-1}(T^{-1}\omega) & : \quad n < 0. \end{cases}$$

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By the multiplicative ergodic theorem, there exists a $\Lambda(A) \in \mathbb{R}$ with

$$\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\| \tag{1}$$

for μ-almost every $\omega \in \Omega$. Now, it is well known that unique ergodicity of (Ω, T) is equivalent to uniform convergence in the Birkhoff additive ergodic theorem when applied to continuous functions. Therefore, it is natural to investigate uniform convergence in (1). This motivates the following definition.

Definition 1.1. [39, 90]. *Let (Ω, T) be uniquely ergodic. The continuous function $A : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ is called uniform if the limit $\Lambda(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$ exists for all $\omega \in \Omega$ and the convergence is uniform on Ω.*

Remark 1. For minimal topological dynamical systems, uniform existence of the limit in the definition implies uniform convergence. This was proven by Furstenberg and Weiss [40]. In fact, their result is even more general and applies to arbitrary real-valued continuous cocycles.

Various aspects of uniformity of cocycles have been considered in the past:

A first topic has been to provide examples of non-uniform cocycles. In fact, in [90] Walters asks the question whether every uniquely ergodic dynamical system with non-atomic measure μ admits a non-uniform cocycle. He presents a class of examples admitting non-uniform cocycles based on results of Veech [86]. He also gives another class of examples, namely suitable irrational rotations, for which non-uniformity was shown by Herman [45]. In general, however, Walters' question is still open.

A different line of study has been pursued by Furman in [39]. He characterizes uniformity of A on a given uniquely ergodic minimal (Ω, T) by a suitable hyperbolicity condition. The results of Furman can essentially be extended to uniquely ergodic systems (and, in fact, a strengthening of some sort can be obtained for minimal uniquely ergodic systems), as shown by Lenz in [65]. They also give that the corresponding results of [46] provide examples of non-uniform cocycles as discussed in [65].

Finally, somewhat complementary to Walters' original question, it is possible to study conditions on subshifts over finite alphabets which imply uniformity of locally constant cocycles. This topic and variants of it have been discussed at various places [23, 47, 62, 63, 64, 65]. It is the main focus of the present article. It is not only of intrinsic interest but also relevant in the study of spectral theory of certain Schrödinger operators, as recently shown by Lenz [63] (see below for details).

To elaborate on this and state our main results, we recall some further notions.

(Ω, T) is called a subshift over $\mathcal{A}$ if $\mathcal{A}$ is finite with discrete topology and Ω is a closed T-invariant subset of $\mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}^{\mathbb{Z}}$ carries the product topology and $T : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ is given by $(Ts)(n) := s(n+1)$. A function F on Ω is called locally constant if there exists an $N \in \mathbb{N}$ with

$$F(\omega) = F(\rho) \ \text{ whenever } \ (\omega(-N), \ldots, \omega(N)) = (\rho(-N), \ldots, \rho(N)). \tag{2}$$

We will freely use notions from combinatorics on words (see, e.g., [67, 68]). In particular, the elements of $\mathcal{A}$ are called letters and the elements of the free monoid

$\mathcal{A}^*$ over $\mathcal{A}$ are called words. The length $|w|$ of a word w is the number of its letters. The number of occurrences of a word w in a word x is denoted by $\#_w(x)$.

Each subshift (Ω, T) over $\mathcal{A}$ gives rise to the associated set of words

$$\mathcal{W}(\Omega) := \{\omega(k)\cdots\omega(k+n-1) : k \in \mathbb{Z}, n \in \mathbb{N}, \omega \in \Omega\}. \tag{3}$$

For $w \in \mathcal{W}$, we define

$$V_w := \{\omega \in \Omega : \omega(1)\cdots\omega(|w|) = w\}.$$

Finally, if ν is a T-invariant probability measure on (Ω, T) and $n \in \mathbb{N}$, we set

$$\eta_\nu(n) := \min\{\nu(V_w) : w \in \mathcal{W}, |w| = n\}. \tag{4}$$

If (Ω, T) is uniquely ergodic with invariant probability measure μ , we set $\eta(n) := \eta_\mu(n)$.

Definition 1.2. *Let (Ω, T) be a subshift over $\mathcal{A}$. Then, (Ω, T) is said to satisfy condition* (B) *if there exists an ergodic probability measure ν on Ω with*

$$\limsup_{n\to\infty} n\,\eta_\nu(n) > 0.$$

Thus, (Ω, T) satisfies (B) *if and only if there exists an ergodic probability measure ν on Ω, a constant $C > 0$ and a sequence (l_n) in $\mathbb{N}$ with $l_n \to \infty$ for $n \to \infty$ such that $|w|\nu(V_w) \geq C$ whenever $w \in \mathcal{W}(\Omega)$ with $|w| = l_n$ for some $n \in \mathbb{N}$.*

This condition was introduced by Boshernitzan in [11] (also see [12] for related material). For minimal interval exchange transformations, it was shown to imply unique ergodicity by Veech in [89]. Finally, in [14], Boshernitzan showed that it implies unique ergodicity for arbitrary minimal subshifts.

Our main result is:

Theorem 1. *Let (Ω, T) be a minimal subshift which satisfies* (B). *Let $A : \Omega \longrightarrow$* SL(2,$\mathbb{R}$) *be locally constant. Then, A is uniform.*

As discussed below, this result covers all earlier results of this form as given in [23, 47, 64, 65]. Moreover, as we will show below, it also applies to various new examples, including many circle maps and Arnoux-Rauzy subshifts. This point is worth emphasizing, as most circle maps and Arnoux-Rauzy subshifts seem to have been rather out of reach of earlier methods.

The proof of the main result is based on two steps. In the first step, we give various equivalent characterizations of condition (B). This is made precise in Theorem 5. This result may be of independent interest. In our context it shows that (B) implies uniform convergence on "many scales." In the second step, we use the so-called Avalanche Principle introduced by Goldstein and Schlag in [41] and extended by Bourgain and Jitomirskaya in [15] to conclude uniform convergence from uniform convergence on "many scales."

As a by-product of our proof, we obtain a simple combinatorial argument for unique ergodicity for subshifts satisfying (B). Unlike the proof given in [14], we do not need any apriori estimates on the number of invariant measures.

As mentioned already, our results are particularly relevant in the study of certain Schrödinger operators. This is discussed next:

To each bounded $V : \mathbb{Z} \longrightarrow \mathbb{R}$, we can associate the Schrödinger operator $H_V : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ acting by

$$(H_V u)(n) \equiv u(n+1) + u(n-1) + V(n)u(n).$$

The spectrum of H_V is denoted by $\sigma(H_V)$.

Now, let (Ω, T) be a subshift over $\mathcal{A}$ and assume without loss of generality that $\mathcal{A} \subset \mathbb{R}$. Then, (Ω, T) gives rise to the family $(H_\omega)_{\omega\in\Omega}$ of selfadjoint operators. These operators arise in the study of aperiodically ordered solids, so-called quasicrystals. They exhibit interesting spectral features such as Cantor spectrum of Lebesgue measure zero, purely singularly continuous spectrum and anomalous transport. They have attracted a lot of attention in recent years (see, e.g., the surveys [21, 85] and discussion below for details). Recently, Lenz has shown that uniformity of certain locally constant cocycles is intimately related to Cantor spectrum of Lebesgue measure zero for these operators [63]. This can be combined with our main result to give the following theorem (see below for details).

Theorem 2. *Let (Ω, T) be a minimal subshift which satisfies* (B). *If (Ω, T) is aperiodic, then there exists a Cantor set $\Sigma \subset \mathbb{R}$ of Lebesgue measure zero with $\sigma(H_\omega) = \Sigma$ for every $\omega \in \Omega$.*

This result covers all earlier results on Cantor spectrum of measure zero [1, 7, 8, 16, 24, 25, 63, 66, 73, 83, 84] as discussed below. More importantly, it gives various new ones. In particular, it covers almost all circle maps and Arnoux-Rauzy subshifts.

To give a flavor of these new examples, we mention the following theorem. Define for $\alpha, \theta, \beta \in (0,1)$ arbitrary, the function

$$V_{\alpha,\beta,\theta} : \mathbb{Z} \longrightarrow \{0,1\}, \quad \text{by } V_{\alpha,\beta,\theta}(n) := \chi_{[1-\beta,1)}(n\alpha + \theta \mod 1),$$

where χ_M denotes the characteristic function of the set M. These functions are called circle maps.

Theorem 3. *Let $\alpha \in (0,1)$ be irrational. Then, we have the following:*

(a) *For almost every $\beta \in (0,1)$, the spectrum $\sigma(H_{V_{\alpha,\beta,\theta}})$ is a Cantor set of Lebesgue measure zero for every $\theta \in (0,1)$.*

(b) *If α has bounded continued fraction expansion, then $\sigma(H_{V_{\alpha,\beta,\theta}})$ is a Cantor set of Lebesgue measure zero for every $\beta \in (0,1)$ and every $\theta \in (0,1)$.*

Remark 2. This result is particularly relevant as all earlier results on Cantor spectrum for circle maps [1, 8, 24, 83, 84] only cover a set of parameters (α, β) of Lebesgue measure zero in $(0,1) \times (0,1)$ (cf. Appendix A).

Finally, we mention the following by-product of our investigation. Details (and precise definitions) will be discussed in Section 8.

Theorem 4. *Let (Ω, T) be a minimal subshift which satisfies* (B) *and $(H_\omega)_{\omega\in\Omega}$ the associated family of operators. Then the Lyapunov exponent $\gamma : \mathbb{R} \longrightarrow [0, \infty)$ is continuous.*

The paper is organized as follows: In Section 2 we study condition (B) and show its equivalence to various other conditions. As a by-product this shows unique ergodicity of subshifts satisfying (B). Moreover, it is used in Section 3 to give a proof of our main result. Stability of the results under certain operations on the subshift is discussed in Section 4. In Section 5 we discuss examples for which (B) is known to hold. New examples, viz certain circle maps and Arnoux-Rauzy subshifts, are given in Sections 6 and 7. Finally, the application to Schrödinger operators is discussed in Section 8.

2. Boshernitzan's Condition (B)

In this section, we give various equivalent characterizations of (B). This is made precise in Theorem 5. Then, we provide a new proof of unique ergodicity for systems satisfying (B) in Theorem 6. Theorem 5 in some sense generalizes the main results of [62] and its proof heavily uses and extends ideas from there.

To state our result, we need some preparation. We start by introducing a variant of Boshernitzan's condition (B). Namely, if (Ω, T) is a subshift, we define for $w \in \mathcal{W}(\Omega)$ the set U_w by

$$U_w := \{\omega \in \Omega : \exists n \in \{0, 1, \ldots, |w| - 1\} \text{ such that } \omega(-n+1)\ldots\omega(-n+|w|) = w\}.$$

If ω belongs to U_w, we say that w occurs in ω around one.

Definition 2.1. *Let (Ω, T) be a subshift over $\mathcal{A}$. Then, (Ω, T) is said to satisfy condition* (B') *if there exists an ergodic probability measure ν on Ω, a constant $C' > 0$, and a sequence (l'_n) in $\mathbb{N}$ with $l'_n \to \infty$ for $n \to \infty$ such that $\nu(U_w) \geq C'$ whenever $w \in \mathcal{W}(\Omega)$ with $|w| = l'_n$ for some $n \in \mathbb{N}$.*

Next, we discuss a consequence of Kingman's ergodic theorem. Recall that $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ is called subadditive if it satisfies $F(xy) \leq F(x) + F(y)$ whenever $x, y, xy \in \mathcal{W}(\Omega)$, where (Ω, T) is an arbitrary subshift.

Proposition 2.2. *Let (Ω, T) be a uniquely ergodic subshift with invariant probability measure μ. Let $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ be subadditive, then there exists a number $\Lambda(F) \in \mathbb{R} \cup \{-\infty\}$ with*

$$\Lambda(F) = \lim_{n\to\infty} n^{-1} F(\omega(1)\cdots\omega(n))$$

for μ-almost every ω in Ω.

Proof. For $n \in \mathbb{N}$, we define the continuous function $f_n : \Omega \longrightarrow \mathbb{R}$, by

$$f_n(\omega) := F(\omega(1)\ldots\omega(n)).$$

As F is subadditive, (f_n) is a subadditive cocycle. Thus Kingman's subadditive theorem applies. This proves the statement. □

Theorem 5. *Let (Ω, T) be a minimal subshift over $\mathcal{A}$. Then the following conditions are equivalent:*

- (i) *(Ω, T) satisfies* (B).
- (ii) *(Ω, T) satisfies* (B').
- (iii) *(Ω, T) is uniquely ergodic and there exists a sequence (l'_n) in $\mathbb{N}$ with $l'_n \to \infty$ for $n \to \infty$ such that $\lim_{n\to\infty} |w_n|^{-1} F(w_n) = \Lambda(F)$ for every subadditive F and every sequence (w_n) in $\mathcal{W}(\Omega)$ with $|w_n| = l'_n$ for every $n \in \mathbb{N}$.*

The remainder of this section is devoted to a proof of this theorem. The proof will be split into several parts.

Lemma 2.3. *Let (Ω, T) be a minimal subshift. Then, (Ω, T) satisfies* (B) *if and only if it satisfies* (B').

Proof. If (Ω, T) is periodic, validity of (B) and (B') is immediate. Thus, we can restrict our attention to aperiodic (Ω, T).

Apparently, $\nu(U_w) \leq |w|\nu(V_w)$ for all $w \in \mathcal{W}(\Omega)$ and all ergodic probability measures ν on Ω. Thus, (B') implies (B) (with the same ν, l_n, and C).

Conversely, assume that (Ω, T) satisfies (B). We will show that it satisfies (B') with $l'_n = [2l_n/3] + 1$, $n \in \mathbb{N}$, and $C' = C/9$. Here, for arbitrary $a \in \mathbb{R}$, we set $[a] := \sup\{n \in Z : n \leq a\}$.

Consider $v \in \mathcal{W}(\Omega)$ with $|v| = l'_n$ for some $n \in \mathbb{N}$. Choose $w \in \mathcal{W}(\Omega)$ with $|w| = l_n$ such that v is a prefix of w. There are two cases:

Case 1. There exists a primitive $x \in \mathcal{W}(\Omega)$ and a prefix $\widetilde{x}$ of x such that $w = x^k \widetilde{x}$ for some $k \geq 6$.

As (Ω, T) is minimal and aperiodic, the word x does not occur with arbitrarily high powers. Thus, we can find $y \in \mathcal{W}(\Omega)$ such that

$$\widetilde{w} := x^{k-1} y \in \mathcal{W}(\Omega)$$

satisfies $|\widetilde{w}| = l_n$ but x^k is not a prefix of $\widetilde{w}$. Now, as x is primitive, it does not appear non-trivially in x^{k-1}. Therefore, different copies of $\widetilde{w}$ have distance at least $(k-2)|x|$. This gives

$$\nu(U_{\widetilde{w}}) \geq (k-2)|x|\nu(V_{\widetilde{w}}) \geq \frac{(k-2)|x|}{(k+2)|x|}|\widetilde{w}|\nu(V_{\widetilde{w}}) \geq \frac{1}{2}C.$$

Moreover, by construction, v is a subword of $\widetilde{w}$ (and even of x^{k-1}) with

$$\frac{|v|}{|\widetilde{w}|} \geq \frac{1}{2}.$$

Putting these estimates together, we infer

$$\nu(U_v) \geq \frac{1}{2}\nu(U_{\widetilde{w}}) \geq \frac{1}{2} \cdot \frac{1}{2} \cdot C = \frac{C}{4}.$$

Case 2. There does not exist a primitive x in $\mathcal{W}$ and a prefix $\widetilde{x}$ of x with $w = x^k \widetilde{x}$ for some $k \geq 6$.

In this case, different copies of w have distance at least $\frac{1}{6}|w|$. Therefore, we have

$$\nu(U_w) \geq \frac{1}{6}|w|\nu(V_w)$$

and this gives

$$\nu(U_v) \geq \frac{2}{3}\nu(U_w) \geq \frac{2}{3} \cdot \frac{1}{6} \cdot |w|\nu(V_w) \geq \frac{1}{9}C.$$

In both cases the desired estimates hold and the proof of the lemma is finished. □

We next give our proof of unique ergodicity for systems satisfying (B'). The proof proceeds in two steps. In the first step, we use (B') to show existence of the frequencies along certain sequences. In the second step, we show existence of the frequencies along all sequences. Let us emphasize that it is exactly this two-step procedure which is underlying the proof of our main result on locally constant matrices. However, in that case the details are more involved.

We need the following proposition.

Proposition 2.4. *Let (Ω, T) be a subshift with ergodic probability measure ν. Let $f : \Omega \longrightarrow \mathbb{R}$ be a bounded measurable function. Then,*

$$\lim_{n,m \geq 0, n+m \to \infty} \frac{1}{n+m} \sum_{k=-m}^{n} f(T^k \omega) = \nu(f)$$

for ν-almost every $\omega \in \Omega$.

Proof. By Birkhoff's ergodic theorem, we have both

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k\omega)=\nu(f) \text{ and } \lim_{m\to\infty}\frac{1}{m}\sum_{k=-m}^{0} f(T^k\omega)=\nu(f)$$

for ν-almost every $\omega\in\Omega$. Now, for every sequence $(a_k)_{k\in\mathbb{Z}}$ with

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} a_k=\lim_{m\to\infty}\frac{1}{m}\sum_{k=-m}^{0} a_k=a,$$

one easily infers

$$\lim_{n,m\geq 0,n+m\to\infty}\frac{1}{n+m}\sum_{k=-m}^{n} a_k=a.$$

The statement follows immediately. □

Theorem 6. *If the subshift* (Ω,T) *satisfies* (B'), *it is uniquely ergodic and minimal.*

Proof. It suffices to show that the frequencies $\lim_{|x|\to\infty}\frac{\#_w(x)}{|x|}$ exist for every $w\in\mathcal{W}$. Then, the system is uniquely ergodic by standard reasoning. Moreover, in this case, the system is minimal as well as all frequencies are positive by (B').

Thus, let an arbitrary $w\in\mathcal{W}(\Omega)$ be given. We proceed in two steps.

Step 1. For all $\varepsilon>0$, there exists an $n_0=n_0(\varepsilon)$ with $\left|\frac{\#_w(x)}{|x|}-\nu(V_w)\right|\leq\varepsilon$ whenever $|x|=l'_n$ with $n\geq n_0$.

Step 2. For $\varepsilon>0$, there exists an $N_0=N_0(\varepsilon)$ with $\left|\frac{\#_w(x)}{|x|}-\nu(V_w)\right|\leq\varepsilon$ whenever $|x|\geq N_0$.

Here, Step 2 follows easily from Step 1 by partitioning long words x into pieces of length l'_n with sufficiently large $n\in\mathbb{N}$.

Thus, we are left with the task of proving Step 1. To do so, assume the contrary. Then, there exist $\delta>0$, (x_n) in $\mathcal{W}$ and $(l'_{k(n)})$ in $\mathbb{N}$ with $|x_n|=l'_{k(n)}$, $k(n)\longrightarrow\infty$ and

$$\left|\frac{\#_w(x_n)}{|x_n|}-\nu(V_w)\right|\geq\delta \tag{5}$$

for every $n\in\mathbb{N}$. Consider

$$E:=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}U_{x_k}.$$

By (B'), we have

$$\nu(E)=\lim_{n\to\infty}\nu(\cup_{k=n}^{\infty}U_{x_k})\geq C'>0.$$

Thus, by Proposition 2.4, we can find an ω in E with

$$\lim_{n,m\geq 0,n+m\to\infty}\frac{\#_w(\omega(-m)\dots\omega(n))}{n+m}=\nu(V_w). \tag{6}$$

As ω belongs to E, there are infinitely many x_n occurring around one in ω. Now, if we calculate the occurrences of w along this sequence of x_n, we stay away from $\nu(V_w)$ by at least δ according to (5). On the other hand, by (6), we come arbitrarily close to $\nu(V_w)$ when calculating the frequency of w along any sequence of words occurring in ω around one. This contradiction proves Step 1 and therefore finishes the proof of the theorem by the discussion above. □

Our next task is to relate (B') and convergence in subadditive ergodic theorems. We need two auxiliary results.

Proposition 2.5. *Let (Ω, T) be a uniquely ergodic subshift and $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ be subadditive. Then,* $\limsup_{|x|\to\infty} |x|^{-1} F(x) \leq \Lambda(F)$.

Proof. Define f_n as in the proof of Proposition 2.2. Then, the statement is a direct consequence of Corollary 2 in [39]. □

Proposition 2.6. *Let (Ω, T) be a uniquely ergodic subshift with invariant probability measure μ. Let $w \in \mathcal{W}(\Omega)$ be arbitrary and denote by χ_{U_w} the characteristic function of U_w. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{U_w}(T^k \omega) = \mu(U_w)$$

uniformly in $\omega \in \Omega$.

Proof. As U_w is both closed and open, the characteristic function χ_{U_w} is continuous. Thus, the statement follows from unique ergodicity. □

Now, our result on subadditive ergodic theorems and (B') reads as follows.

Lemma 2.7. *Let (Ω, T) be a uniquely ergodic and minimal subshift. Let (w_n) be a sequence in $\mathcal{W}(\Omega)$ with $|w_n| \longrightarrow \infty$, $n \to \infty$. Then, the following assertions are equivalent:*

(i) $\lim_{n\to\infty} |w_n|^{-1} F(w_n) = \Lambda(F)$ *for every subadditive* $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$.
(ii) *There exists a* $C' > 0$ *with* $\mu(U_{w_n}) \geq C'$ *for every* $n \in \mathbb{N}$.

Proof. The proof can be thought of as an adaptation and extension of the proofs of Lemma 3.1 and Lemma 3.2 in [62] to our setting.

(i) $\Longrightarrow$ (ii). Assume the contrary. Then, the sequence $(\mu(U_{w_n}))$ is not bounded away from zero. By passing to a subsequence, we may then assume without loss of generality that

$$\sum_{n=1}^{\infty} \mu(U_{w_n}) < \frac{1}{2}. \tag{7}$$

As (Ω, T) is minimal, we have $\mu(U_{w_n}) > 0$ for every $n \in \mathbb{N}$. Moreover, by assumption, we have

$$|w_n| \longrightarrow \infty, n \longrightarrow \infty. \tag{8}$$

For $w, x \in \mathcal{W}(\Omega)$, we say that w occurs in x around $j \in \{1, \ldots, |x|\}$ if there exists $l \in \mathbb{N}$ with $l \leq j < l + |w| - 1$ and $x(l) \ldots x(l + |w| - 1) = w$.

Now, define for $n \in \mathbb{N}$, the function $F_n : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ by

$$F_n(x) := \#\{j \in \{1, \ldots, |x|\} : w_n \text{ occurs in } x \text{ around } j\}.$$

Here, $\#M$ denotes the cardinality of M. Thus $F_n(x)$ measures the amount of "space" covered in x by copies of w_n. Obviously, $-F_n$ is subadditive for every $n \in \mathbb{N}$.

The definition of F_n shows

$$F_n(\omega(1)\ldots\omega(m)) = \sum_{k=0}^{m-|w_n|-1} \chi_{U_{w_n}}(T^k\omega)$$

for arbitrary $\omega \in \Omega$ and $m \in \mathbb{N}$. Thus, by Proposition 2.6, we have

$$\lim_{|x|\to\infty} |x|^{-1} F_n(x) = \mu(U_{w_n})$$

for arbitrary but fixed $n \in \mathbb{N}$.

Invoking this equality and (7) and (8), we can choose inductively for every $k \in \mathbb{N}$ a number $n(k) \in \mathbb{N}$ with

$$\frac{|w_{n(k+1)}|}{2} > |w_{n(k)}|$$

and

$$\sum_{j=1}^{k} \frac{F_{n(j)}(x)}{|x|} < \frac{1}{2},$$

whenever $|x| \geq |w_{n(k+1)}|$. It is not hard to see that

$$F(x) := \sum_{j=1}^{\infty} F_{n(2j)}(x)$$

is finite for every $x \in \mathcal{W}(\Omega)$ and $-F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$, $x \mapsto -F(x)$, is subadditive. Therefore, by our assumption (i) the limit

$$-\Lambda(-F) = \lim_{n\to\infty} \frac{F(w_n)}{|w_n|}$$

exists. On the other hand, for every $k \in \mathbb{N}$, we have

$$\frac{F(w_{n(2k)})}{|w_{n(2k)}|} \geq \frac{F_{n(2k)}(w_{n(2k)})}{|w_{n(2k)}|} = 1$$

as well as

$$\frac{F(w_{n(2k+1)})}{|w_{n(2k+1)}|} = \frac{1}{|w_{n(2k+1)}|} \sum_{j=1}^{k} F_{n(2j)}(w_{n(2k+1)}) \leq \frac{1}{|w_{n(2k+1)}|} \sum_{j=1}^{2k} F_{n(j)}(w_{n(2k+1)}) < \frac{1}{2}.$$

This is a contradiction and the proof of this part of the lemma is finished.

(ii) $\Longrightarrow$ (i). Let $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ be subadditive. By Proposition 2.5, we have

$$\limsup_{|x|\to\infty} \frac{F(x)}{|x|} \leq \Lambda(F). \tag{9}$$

Thus, it remains to show

$$\Lambda(F) \leq \liminf_{n\to\infty} \frac{F(w_n)}{|w_n|}.$$

Assume the contrary. Then, $\Lambda(F) > -\infty$ and there exists a subsequence $(w_{n(k)})$ of (w_n) and $\delta > 0$ with

$$\frac{F(w_{n(k)})}{|w_{n(k)}|} \leq \Lambda(F) - \delta \tag{10}$$

for every $k \in \mathbb{N}$. For $w, x \in \mathcal{W}(\Omega)$, we define $\#_w^*(x)$ to be the maximal number of disjoint copies of w in x.

It is not hard to see that

$$|w| \cdot \#_w^*(\omega(1)\ldots\omega(m)) \geq \frac{1}{2} \sum_{k=0}^{m-|w|-1} \chi_{U_w}(T^k\omega)$$

for all $\omega \in \Omega$ and $m \in \mathbb{N}$. By Proposition 2.6, this implies

$$\liminf_{|x|\to\infty} \frac{\#_w^*(x)}{|x|}|w| \geq \frac{1}{2}\mu(U_w).$$

Combining this with our assumption (ii), we infer

$$\liminf_{|x|\to\infty} \frac{\#_{w_{n(k)}}^*(x)}{|x|}|w_{n(k)}| \geq \frac{C'}{2} \tag{11}$$

for every $k \in \mathbb{N}$. By (9), we can choose L_0 such that

$$\frac{F(x)}{|x|} \leq \Lambda(F) + \frac{C'}{16}\delta, \tag{12}$$

whenever $|x| \geq L_0$. Fix $k \in \mathbb{N}$ with $|w_{n(k)}| \geq L_0$. Using (11), we can now find an $L_1 \in \mathbb{R}$ such that every $x \in \mathcal{W}(\Omega)$ with $|x| \geq L_1$ can be written as $x = x_1 w_{n(k)} x_2 w_{n(k)} \ldots x_l w_{n(k)} x_{l+1}$ with

$$\frac{l-2}{2} \geq \frac{C'}{8}\frac{|x|}{|w_{n(k)}|}. \tag{13}$$

Now, considering only every other copy of $w_{n(k)}$ in x, we can write x as $x = y_1 w_{n(k)} y_2 \ldots y_r w_{n(k)} y_{r+1}$, with $|y_j| \geq |w_{n(k)}| \geq L_0$, $j = 1, \ldots, r+1$, and by (13)

$$r \geq \frac{l-2}{2} \geq \frac{C'}{8}\frac{|x|}{|w_{n(k)}|}.$$

Using (12), (10) and this estimate, we can now calculate

$$\begin{aligned}
\frac{F(x)}{|x|} &\leq \sum_{j=1}^{r+1} \frac{F(y_j)}{|y_j|}\frac{|y_j|}{|x|} + \frac{F(w_{n(k)})}{|w_{n(k)}|}\frac{r|w_{n(k)}|}{|x|} \\
&\leq \sum_{j=1}^{r+1}(\Lambda(F) + \frac{C'}{16}\delta)\frac{|y_j|}{|x|} + (\Lambda(F) - \delta)\frac{r|w_{n(k)}|}{|x|} \\
&\leq \Lambda(F) + \frac{C'}{16}\delta - \frac{C'}{8}\frac{|x|}{|w_{n(k)}|}\frac{|w_{n(k)}|}{|x|}\delta \\
&\leq \Lambda(F) - \frac{C'}{16}\delta.
\end{aligned}$$

As this holds for arbitrary $x \in \mathcal{W}(\Omega)$ with $|x| \geq L_1$, we arrive at the obvious contradiction $\Lambda(F) \leq \Lambda(F) - \frac{C'}{16}\delta$. This finishes the proof. □

Proof of Theorem 5. Given the previous results, the proof is simple: The equivalence of (i) and (ii) is shown in Lemma 2.3. The implication (ii) $\Longrightarrow$ (iii) follows from Theorem 6 combined with Lemma 2.7. The implication (iii) $\Longrightarrow$ (ii) is immediate from Lemma 2.7. This finishes the proof of Theorem 5. □

3. Uniformity of Locally Constant Cocycles

In this section we provide a proof of our main result, Theorem 1. As mentioned already, the cornerstones of the proof are Theorem 5 and the so-called Avalanche Principle, introduced in [41] and later extended in [15].

We use the Avalanche Principle in the following form given in Lemma 5 of [15].

Lemma 3.1. *There exist constants $\lambda_0 > 0$ and $\kappa > 0$ such that*

$$\left| \log \|A_N \dots A_1\| + \sum_{j=2}^{N-1} \log \|A_j\| - \sum_{j=1}^{N-1} \log \|A_{j+1} A_j\| \right| \le \frac{\kappa \cdot N}{\exp(\lambda)},$$

whenever $N = 3^P$ with $P \in \mathbb{N}$ and $A_1, \dots, A_N$ are elements of SL(2,$\mathbb{R}$) *such that*

- $\log \|A_j\| \ge \lambda \ge \lambda_0$ *for every $j = 1, \dots, N$;*
- $|\log \|A_j\| + \log \|A_{j+1}\| - \log \|A_j A_{j+1}\|| < \frac{1}{2}\lambda$ *for every $j = 1, \dots, N$.*

Remark 3. Actually, Lemma 5 in [15] is more general in that more general N are allowed.

Before we can give the proof of Theorem 1, we need one more auxiliary result.

Proposition 3.2. *Let (Ω, T) be an arbitrary subshift and $A : \Omega \longrightarrow$ SL(2,$\mathbb{R}$) a locally constant function. Then,*

$$0 = \lim_{n \to \infty} \sup \left\{ \frac{1}{n} |\log \|A(n,\omega)\| - \log \|A(n,\rho)\|| : \omega(1) \dots \omega(n) = \rho(1) \dots \rho(n) \right\}.$$

Proof. As A is locally constant, there exists an $N \in \mathbb{N}$ such that $A(\omega) = A(\rho)$, whenever $\omega(-N) \dots \omega(N) = \rho(-N) \dots\ \rho(N)$. Thus,

$$A(n - 2N, T^N \omega) = A(n - 2N, T^N \rho),$$

whenever $n \ge 2N$ and $\omega(1) \dots \omega(n) = \rho(1) \dots \rho(n)$. Moreover, for arbitrary matrices X, Y, Z in SL(2,$\mathbb{R}$), we have

$$\log \|Y\| - \log \|X\| - \log \|Z\| \le \log \|XYZ\| \le \log \|X\| + \log \|Y\| + \log \|Z\|,$$

where we used the triangle inequality as well as $\|M\| = \|M^{-1}\|$ for $M \in$ SL(2,$\mathbb{R}$). Finally, we have

$$A(n, \sigma) = A(N, T^{n-N} \sigma) A(n - 2N, T^N \sigma) A(N, \sigma).$$

Putting these three equations together, we arrive at the desired conclusion. □

Remark 4. Let us point out that the previous proposition is the only point in our considerations where local constancy of A enters. In particular, our main result holds for all A for which the conclusion of the proposition holds.

Proof of Theorem 1. Let (Ω, T) be a subshift satisfying (B) and let $A : \Omega \longrightarrow$ SL(2,$\mathbb{R}$) be locally constant. We have to show that A is uniform.

Case 1. $\Lambda(A) = 0$: As A takes values in SL(2,$\mathbb{R}$), we have $\|A(n,\omega)\| \ge 1$ and the estimate

$$0 \le \liminf_{n \to \infty} \frac{1}{n} \log \|A(n,\omega)\|$$

holds uniformly in $\omega \in \Omega$. On the other hand, by Corollary 2 of [39], we have

$$\limsup_{n \to \infty} \frac{1}{n} \log \|A(n,\omega)\| \le \Lambda(A)$$

uniformly in $\omega \in \Omega$. This shows the desired uniformity in this case.

Case 2. $\Lambda(A) > 0$: Define $F : \mathcal{W}(\Omega) \longrightarrow \mathbb{R}$ by

$$F(x) := \sup\{\log \|A(n,\omega)\| : \omega(1)\ldots\omega(n) = x\}.$$

Apparently, F is subadditive. As discussed above, there exists then $\Lambda(F)$ with

$$\Lambda(F) = \lim_{n\to\infty} \frac{F(\omega(1)\ldots\omega(n))}{n}$$

for μ-almost every $\omega \in \Omega$. On the other hand, by the multiplicative ergodic theorem, there also exists $\Lambda(A)$ with

$$\Lambda(A) = \lim_{n\to\infty} \frac{\log \|A(n,\omega)\|}{n}$$

for μ-almost every $\omega \in \Omega$. By Proposition 3.2, we infer that $\Lambda(A) = \Lambda(F)$. Summarizing, we have

$$\Lambda(A) = \Lambda(F) > 0. \tag{14}$$

Combining this equation with Theorem 5, we infer

$$\lim_{n\to\infty} \frac{F(w_n)}{|w_n|} = \Lambda(A),$$

whenever (w_n) is a sequence with $|w_n| = l'_n$. Also, combining (14) with Proposition 2.5, we infer

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\| \le \limsup_{|x|\to\infty} \frac{F(x)}{|x|} \le \Lambda(A)$$

uniformly in $\omega \in \Omega$. It remains to show

$$\Lambda(A) \le \liminf_{n\to\infty} \frac{1}{n} \log \|A(n,\omega)\|$$

uniformly in $\omega \in \Omega$. To do so, let $\varepsilon > 0$ with $\varepsilon \le 1/12$ be given.

The preceding considerations and Proposition 3.2 give existence of $n_0 \in \mathbb{N}$ such that with

$$l := \frac{l'_{n_0}}{2},$$

the following holds:

(I) $\log \|A(n,\omega)\| \le \Lambda(A)(1+\varepsilon)n$ for all $\omega \in \Omega$ whenever $n \ge l$.

(II) $\log \|A(2l,\omega)\| \ge \Lambda(A)(1-\varepsilon)2l$ for all $\omega \in \Omega$.

(III) $\Lambda(A)(1-3\varepsilon)l \ge \lambda_0$.

(IV) $\frac{2\kappa}{l\exp(\lambda_0)} < \varepsilon\Lambda(A)$.

Here, λ_0 and κ are the constants from Lemma 3.1. Using (II), subadditivity and (I), we can calculate

$$\begin{aligned} \Lambda(A)(1-\varepsilon)2l &\le \log \|A(2l,\omega)\| \\ &\le \log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| \\ &\le \log \|A(l,\omega)\| + \Lambda(A)(1+\varepsilon)l. \end{aligned}$$

This implies $\Lambda(A)(1-3\varepsilon)l \le \log \|A(l,\omega)\|$ and therefore by (III),

$$\lambda_0 \le \Lambda(A)(1-3\varepsilon)l \le \log \|A(l,\omega)\| \tag{15}$$

for every $\omega \in \Omega$. Moreover, by subadditivity, (I) and (II), we have

$$\begin{aligned} & \left|\log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| - \log \|A(2l,\omega)\|\right| \\ = \ & \log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| - \log \|A(2l,\omega)\| \\ \leq \ & \Lambda(A)2l(1+\varepsilon) - \log \|A(2l,\omega)\| \\ \leq \ & \Lambda(A)2l(1+\varepsilon) - \Lambda(A)2l(1-\varepsilon) \\ = \ & \Lambda(A)4l\varepsilon \end{aligned}$$

for arbitrary $\omega \in \Omega$. Using the assumption $\varepsilon \leq 1/12$, we infer

$$\left|\log \|A(l,\omega)\| + \log \|A(l,T^l\omega)\| - \log \|A(l,\omega)\|\right| \leq \frac{1}{2}\Lambda(A)(1-3\varepsilon)l. \tag{16}$$

Equations (15) and (16) and (III) show that the Avalanche Principle, Lemma 3.1, with

$$\lambda = \Lambda(A)(1-3\varepsilon)l$$

can be applied to the matrices $A_1, \ldots, A_N$, where $N = 3^P$ with $P \in \mathbb{N}$ arbitrary and

$$A_j = A(l, T^{(j-1)l}\omega), \ \ j = 1, \ldots, N$$

with $\omega \in \Omega$ arbitrary. This gives

$$\left|\log \|A_N \ldots A_1\| + \sum_{j=2}^{N-1} \log \|A_j\| - \sum_{j=1}^{N-1} \log \|A_{j+1}A_j\|\right| \leq \frac{\kappa N}{\exp(\lambda)}.$$

This yields

$$\begin{aligned} \log \|A_N \ldots A_1\| \ & \geq \ \sum_{j=1}^{N-1} \log \|A_{j+1}A_j\| - \sum_{j=2}^{N-1} \log \|A_j\| - \frac{\kappa \cdot N}{\exp(\lambda)} \\ & \geq \ (N-1)\Lambda(A)(1-\varepsilon)2l - (N-2)\Lambda(A)(1+\varepsilon)l - \frac{\kappa \cdot N}{\exp(\lambda)} \\ & = \ \Lambda(A)Nl(1-3\varepsilon) + \Lambda(A)4\varepsilon l - \frac{\kappa \cdot N}{\exp(\lambda)} \\ & \geq \ \Lambda(A)Nl(1-3\varepsilon) - \frac{\kappa \cdot N}{\exp(\lambda)}. \end{aligned}$$

Here, we used (I) and (II) in the second step and positivity of $\Lambda(A)4\varepsilon l$ in the last step. Dividing by by $n := Nl$, and invoking (IV), we obtain

$$\Lambda(A)(1-4\varepsilon) \leq \frac{1}{n} \log \|A(n,\omega)\| \tag{17}$$

for all $\omega \in \Omega$ and all $n = 3^P \cdot l$ with $P \in \mathbb{N}$.

We finish the proof by showing that

$$\Lambda(A)(1-44\varepsilon) \leq \frac{1}{n} \log \|A(n,\omega)\| \tag{18}$$

for all $n \geq l$ and all $\omega \in \Omega$. As ε was arbitrary, this gives the desired statement.

To show (18), choose $\omega \in \Omega$ and and $n \geq l$. Let $P \in \mathbb{N} \cup \{0\}$ be such that

$$3^P \cdot l \leq n < 3^{P+1} \cdot l.$$

Then, by (17) and subadditivity we have

$$\begin{aligned}\Lambda(A)(1-4\varepsilon) &\le \frac{1}{3^{P+2}l}\log\|A(3^{P+2}l,\omega)\| \\ &\le \frac{1}{3^{P+2}l}\log\|A(n,\omega)\| + \frac{1}{3^{P+2}l}\ \log\|A(3^{P+2}l-n,T^n\omega)\| \\ &\le \frac{1}{n}\log\|A(n,\omega)\|\cdot\frac{n}{3^{P+2}l} + \Lambda(A)(1+\varepsilon)(1-\frac{n}{3^{P+2}l}),\end{aligned}$$

where we could use (I) in the last estimate as, by assumption on n, $3^{P+2}l - n \ge 3^{P+1}2l > l$. Now, a direct calculation gives

$$\Lambda(A)\left(1+\varepsilon-5\varepsilon\frac{3^{P+2}l}{n}\right) \le \frac{1}{n}\log\|A(n,\omega)\|.$$

As $3^{P+2}l/n \le 9$ by the very choice of P, the desired equation (18) follows easily. This finishes the proof of our main theorem. □

4. Stability of Uniform Convergence Under Substitutions

In the last section, we studied sufficient conditions on (Ω,T) to ensure property

(P) : Every locally constant $A:\Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ is uniform.

In this section, we consider "perturbations" $(\Omega(S),T)$ of (Ω,T) by substitutions S and study how validity of (P) for (Ω,T) is related to validity of (P) for $(\Omega(S),T)$.

We start with the necessary notation. Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets. A map $S:\mathcal{A}\longrightarrow\mathcal{B}^*$ is called a substitution. Obviously, S can be extended to $\mathcal{A}^*$ in the obvious way. Moreover, for a two-sided infinite word $(\omega(n))_{n\in\mathbb{Z}}$ over $\mathcal{A}$, we can define $S(\omega)$ by

$$S(\omega) := \cdots S(\omega(-2))S(\omega(-1))|S(\omega(0))S(\omega(1))S(\omega(2))\cdots,$$

where $|$ denotes the position of zero. If (Ω,T) is a subshift over A and $S:\mathcal{A}\longrightarrow\mathcal{B}^*$ is a substitution, we define $\Omega(S)$ by

$$\Omega(S) := \{T^k S(\omega) : \omega\in\Omega, k\in\mathbb{Z}\}.$$

Then, $(\Omega(S),T)$ is a subshift over $\mathcal{B}$. It is not hard to see that $(\Omega(S),T)$ is minimal (uniquely ergodic) if Ω is minimal (uniquely ergodic).

Theorem 7. *Let (Ω,T) be a minimal uniquely ergodic subshift over $\mathcal{A}$ that satisfies* (P). *Let S be a substitution over $\mathcal{A}$. Then, $(\Omega(S),T)$ satisfies* (P) *as well.*

Proof. Let $B:\Omega(S)\longrightarrow \mathrm{SL}(2,\mathbb{R})$ be locally constant. Define

$$A:\Omega\longrightarrow \mathrm{SL}(2,\mathbb{R}) \text{ by } A(\omega) := B(|S(\omega(0))|, S(\omega)).$$

Then, A is locally constant as well and

$$A(n,\omega) = B(|S(\omega(0)\ldots\omega(n-1))|, S(\omega)).$$

In particular, we have

$$\frac{\log\|B(|S(\omega(0)\ldots\omega(n))|,S(\omega))\|}{|S(\omega(0)\ldots\omega(n))|} = \frac{n+1}{|S(\omega(0)\ldots\omega(n))|}\cdot\frac{\log\|A(n,\omega)\|}{n+1}. \tag{19}$$

By

$$|S(\omega(0)\ldots\omega(n))| = \sum_{a\in\mathcal{A}} |S(a)|\#_a(\omega(0)\ldots\omega(n))$$

and unique ergodicity of (Ω, T), the quotients

$$\frac{n+1}{|S(\omega(0)\dots\omega(n))|}$$

converge uniformly in $\omega \in \Omega$ towards a number ρ. From (19) and validity of (P) for (Ω, T) we infer that

$$\lim_{n\to\infty} \frac{\log \|B(|S(\omega(0)\dots\omega(n))|, S(\omega))\|}{|S(\omega(0)\dots\omega(n))|} = \rho \cdot \Lambda(A)$$

uniformly on Ω. As every $\sigma \in \Omega(S)$ has the form $\sigma = T^k S(\omega)$ with $|k| \leq \max\{|S(a)| : a \in \mathcal{A}\}$, uniform convergence of $\frac{1}{n} \log \|B(n,\sigma)\|$ follows. □

In certain cases, a converse of this theorem holds. To be more precise, let (Ω, T) be a subshift over $\mathcal{A}$ and S a substitution on $\mathcal{A}$. Then, S is called *recognizable* (with respect to (Ω, T)) if there exists a locally constant map

$$\widetilde{S} : \Omega(S) \longrightarrow \Omega \times \mathbb{Z}$$

with $\widetilde{S}(T^k S(\omega)) = (\omega, k)$, whenever $0 \leq k \leq |S(\omega(0))|$. Recognizability is known for various classes of substitutions that generate aperiodic subshifts, including all primitive substitutions [72] and all substitutions of constant length that are one-to-one [3] (cf. the discussion in [38]).

Theorem 8. *Let* (Ω, T) *be a uniquely ergodic minimal subshift over* $\mathcal{A}$. *Let* S *be a recognizable substitution over* $\mathcal{A}$. *If* $(\Omega(S), T)$ *satisfies* (P), *then* (Ω, T) *satisfies* (P) *as well.*

Proof. Let $B : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ be locally constant. For $\sigma \in \Omega(S)$ define

$$A(\sigma) \equiv \begin{cases} B(\omega) & : \quad \sigma = S(\omega) \\ id & : \quad \text{otherwise.} \end{cases}$$

Note that $\sigma = S(\omega)$ if and only if the second component of $\widetilde{S}(\sigma)$ is 0. As $\widetilde{S}$ is locally constant, this shows that A is locally constant as well.

Moreover, by definition of A and recognizability of S, we have

$$A(|S(\omega(0)\dots\omega(n-1))|, S(\omega)) = B(n, \omega).$$

Now, the proof can be finished similarly to the proof of the previous theorem. □

There is an instance of the previous theorem that deserves special attention, viz subshifts derived by return words. Return words and the derived subshifts have been discussed by various authors since they were first introduced by Durand in [32]. We recall the necessary details next.

Let (Ω, T) be a minimal subshift and $w \in \mathcal{W}(\Omega)$ arbitrary. Then, $x \in \mathcal{W}(\Omega)$ is called a return word of w if xw satisfies the following three properties: it belongs to $\mathcal{W}(\Omega)$, it starts with w and it contains exactly two copies of w. We then introduce a new alphabet $\mathcal{A}_w$ consisting of the return words of w. Obviously, there is a natural map

$$S_w : \mathcal{A}_w \longrightarrow \mathcal{A}^*$$

which maps the return word x of w (which is a letter of $\mathcal{A}_w$) to the word x over $\mathcal{A}$. Partitioning every word $\omega \in \Omega$ according to occurrences of w, we obtain a unique two-sided infinite word ω_w over $\mathcal{A}_w$ with

$$T^{-k} S_w(\omega_w) = \omega$$

for $k \leq 0$ maximal with $\omega(k) \dots \omega(k + |w| - 1) = w$. We define

$$\Omega_w := \{\omega_w : \omega \in \Omega\}.$$

Then, (Ω_w, T) is a subshift, called the subshift derived from (Ω, T) with respect to w. It is not hard to see that (Ω_w, T) is minimal. Moreover, (Ω_w, T) is uniquely ergodic if (Ω, T) is uniquely ergodic. Clearly, S_w is recognizable and $(\Omega, T) = (\Omega_w(S_w), T)$ since the whole construction only depends on the (local) information of occurrences of w. Thus, we obtain the following corollary from the previous theorem.

Corollary 1. *Let (Ω, T) be a minimal uniquely ergodic subshift that satisfies* (P). *Let $w \in \mathcal{W}(\Omega)$ be arbitrary. Then, (Ω_w, T) satisfies* (P) *as well.*

The aim of this paper is to study (P). Given that (B) is a sufficient condition for (P), it is then natural to ask for stability properties of (B) as well. It turns out that (B) shares the stability features of (P).

Theorem 9. *Let (Ω, T) be a minimal uniquely ergodic subshift over $\mathcal{A}$. Let S be a substitution on $\mathcal{A}$ and $(\Omega(S), T)$ the corresponding subshift.*

(a) *If (Ω, T) satisfies* (B)*, so does $(\Omega(S), T)$.*

(b) *If $(\Omega(S), T)$ satisfies* (B) *and S is recognizable, then (Ω, T) satisfies* (B) *as well.*

Before we can give a proof, we note the following simple observation.

Proposition 4.1. *Let (Ω, T) be a minimal uniquely ergodic subshift satisfying* (B) *with length scales (l_n) and constant $C > 0$. Then,*

$$|w|\mu(V_w) \geq \frac{C}{N},$$

whenever $w \in \mathcal{W}(\Omega)$ satisfies $l_n/N \leq |w| \leq l_n$ for some $n \in \mathbb{N}$ and $N \in N$.

Proof. Every $w \in \mathcal{W}(\Omega)$ with $l_n/N \leq |w| \leq l_n$ is a prefix of a $v \in \mathcal{W}$ with $|v| = l_n$. Then, $V_v \subset V_w$ holds and (B) implies

$$|w|\mu(V_w) \geq \frac{|v|}{N}\mu(V_w) \geq \frac{|v|}{N}\mu(V_v) \geq \frac{C}{N}.$$

This finishes the proof of the proposition. □

Proof of Theorem 9. Define $M := \{|S(a)| : a \in \mathcal{A}\}$ and denote the unique T-invariant probability measure on Ω (resp., $\Omega(S)$) by μ (resp., μ_S).

(a) We assume that (Ω, T) satisfies (B) with length scales (l_n) and constant $C > 0$. Let $w \in \mathcal{W}(\Omega(S))$ with $|w| = l_n$ for some $n \in \mathbb{N}$ be given. Then, there exists a word $v \in \mathcal{W}(\Omega)$ such that w is a subword of $S(v)$ and satisfies the estimate

$$\frac{|w|}{M} \leq |v| \leq |w|. \tag{20}$$

Choose $\omega \in \Omega$ arbitrary. Obviously,

$$\#_w(S(\omega(1) \dots \omega(k))) \geq \#_v(\omega(1) \dots \omega(k)).$$

Thus, counting occurrences of $w \in S(\omega)$ and occurrences of v in ω, we obtain by unique ergodicity

$$\begin{aligned} |w|\mu_S(V_w) &= |w| \lim_{n\to\infty} \frac{\#_w(S(\omega)(1) \dots S(\omega)(n))}{n} \geq |w| \lim_{k\to\infty} \frac{\#_v(\omega(1) \dots \omega(k))}{kM} \\ &= \frac{|w|}{M}\mu(V_v) \geq \frac{1}{M}|v|\mu(V_v) \geq \frac{1}{M^2}C, \end{aligned}$$

where we used (20) in the second-to-last step and Proposition 4.1 combined with (20) in the last step. This shows (B) for $(\Omega(S), T)$ along the same length scales (l_n) with new constant C/M^2.

(b) We assume that $(\Omega(S), T)$ satisfies (B) with constant $C > 0$ and length scales (l_n). By recognizability, there exists a map $\widetilde{S} : \Omega(S) \longrightarrow \Omega \times \mathbb{Z}$ and an $N \in \mathbb{N}$ with $\widetilde{S}(T^k S(\omega)) = (\omega, k)$, whenever $0 \le k \le |S(\omega(0))|$, and $\widetilde{S}(\omega) = \widetilde{S}(\rho)$, whenever $\omega(-N) \dots \omega(N) = \rho(-N) \dots \rho(N)$. Let n_0 be chosen such that

$$\left[\frac{l_n}{3M}\right] \ge N,$$

for all $n \ge n_0$.

Choose an arbitrary $v \in \mathcal{W}(\Omega)$ with $|v| = \left[\frac{l_n}{3M}\right]$ for some $n \ge n_0$.

Let $x, y \in \mathcal{W}(\Omega)$ be given with $|x| = |y| = |v|$ and $xvy \in \mathcal{W}(\Omega)$. By recognizability and our choice of the lengths of x, y and v, occurrences of $S(xvy)$ in $S(\omega)$ correspond to occurrences of v in ω for any $\omega \in \Omega$. Thus, we obtain

$$\#_v(\omega(1) \dots \omega(n)) \ge \#_{S(xvy)}(S(\omega(1) \dots \omega(n)))$$

for every $n \in \mathbb{N}$ and $\omega \in \Omega$. Therefore, a short calculation invoking unique ergodicity gives

$$\begin{aligned} |v|\mu(V_v) &= |v| \lim_{n\to\infty} \frac{\#_v(\omega(1)\dots\omega(n))}{n} \ge |v| \lim_{n\to\infty} \frac{\#_{S(xvy)}(S(\omega(1)\cdots\omega(n)))}{n} \\ &\ge \frac{|v|}{|S(xvy)|} \lim_{n\to\infty} \frac{|S(\omega(1)\cdots\omega(n))|}{n} |S(xvy)| \frac{\#_{S(xvy)}(S(\omega(1)\cdots\omega(n)))}{|S(\omega(1)\cdots\omega(n))|} \\ &\ge \frac{1}{3M} |S(xvy)| \mu_S(V_{S(xvy)}), \end{aligned}$$

where we used the trivial bound $|S(x)|/|x| \ge 1$ in the second-to-last step. By construction, we have

$$\frac{l_n}{2M} \le |xvy| \le |S(xvy)| \le l_n.$$

Thus, we can apply Proposition 4.1, and the assumption (B) on $\Omega(S)$, to our estimate on $|v|\mu(V_v)$ to obtain $|v|\mu(V_v) \ge \frac{C}{6M^2}$. As $v \in \mathcal{W}$ with $|v| = \left[\frac{l_n}{3M}\right]$ was arbitrary, we infer (B) with the new length scales $[l_n/3]$ for $n \ge n_0$ and new constant $C/(6M^2)$. □

5. Examples Known to Satisfy (B)

In this section we discuss the classes of subshifts for which the Boshernitzan condition is either known or a simple consequence of known results. In our discussion of the occurrence of zero-measure Cantor spectrum for Schrödinger operators in Section 8, this will be relevant since all the models for which this spectral property was previously known will be shown to satisfy (B). Hence we present a unified approach to all these results.

5.1. **Examples Satisfying (PW): Linearly Recurrent Subshifts and Subshifts Generated by Primitive Substitutions.** A subshift (Ω, T) over $\mathcal{A}$ satisfies the condition (PW) (for *positive weights*) if there exists a constant $C > 0$ such that

$$\liminf_{|x|\to\infty} \frac{\#_v(x)}{|x|} |v| \ge C \text{ for every } v \in \mathcal{W}(\Omega).$$

This condition was introduced by Lenz in [62]. There, it was shown that the class of subshifts satisfying (PW) is exactly the class of subshifts for which a uniform subadditive ergodic theorem holds. Moreover, (PW) implies minimality and unique ergodicity.

The following is obvious:

Proposition 5.1. *If the subshift* (Ω, T) *satisfies* (PW)*, then it satisfies* (B).

The condition (PW) holds in many cases of interest. For example, it is easily seen to be satisfied for all linearly recurrent subshifts. Here, a subshift (Ω, T) is called *linearly recurrent* (or *linearly repetitive*) if there exists a constant K such that if $v, w \in \mathcal{W}(\Omega)$ with $|w| \geq K|v|$, then v is a subword of w.

We note:

Proposition 5.2. *If the subshift* (Ω, T) *is linearly recurrent, then it satisfies* (PW).

The class of linearly recurrent subshifts was studied, for example, in [33, 34].

A popular way to generate linearly recurrent subshifts is via primitive substitutions. A substitution $S : \mathcal{A} \to \mathcal{A}^*$ is called *primitive* if there exists $k \in \mathbb{N}$ such that for every $a, b \in \mathcal{A}$, $S^k(a)$ contains b. Such a substitution generates a subshift (Ω, T) as follows. It is easy to see that there are $m \in \mathbb{N}$ and $a \in \mathcal{A}$ such that $S^m(a)$ begins with a. If we iterate S^m on the symbol a, we obtain a one-sided infinite limit, u, called a *substitution sequence*. Ω then consists of all two-sided sequences for which all subwords are also subwords of u. One can verify that this construction is in fact independent of the choice of u, and hence Ω is uniquely determined by S. Prominent examples are given by

$a \mapsto ab,\ b \mapsto a$	Fibonacci
$a \mapsto ab,\ b \mapsto ba$	Thue-Morse
$a \mapsto ab,\ b \mapsto aa$	Period Doubling
$a \mapsto ab,\ b \mapsto ac,\ c \mapsto db,\ d \mapsto dc$	Rudin-Shapiro

The following was shown in [34]:

Proposition 5.3. *If the subshift* (Ω, T) *is generated by a primitive substitution, then it is linearly recurrent.*

It may happen that a non-primitive substitution generates a linearly recurrent subshift. An example is given by $a \mapsto aaba$, $b \mapsto b$. In fact, the class of linearly recurrent subshifts generated by substitutions was characterized in [25]. In particular, it turns out that a subshift generated by a substitution is linearly recurrent if and only if it is minimal.

5.2. **Sturmian and Quasi-Sturmian Subshifts.** Consider a minimal subshift (Ω, T) over $\mathcal{A}$. Recall that the associated set of words is given by

$$\mathcal{W}(\Omega) := \{\omega(k) \cdots \omega(k+n-1) : k \in \mathbb{Z}, n \in \mathbb{N}, \omega \in \Omega\}.$$

The (factor) complexity function $p : \mathbb{N} \to \mathbb{N}$ is then defined by

$$p(n) = \#\mathcal{W}_n(\Omega), \tag{21}$$

where $\mathcal{W}_n(\Omega) = \mathcal{W}(\Omega) \cap \mathcal{A}^n$ and $\#$ denotes cardinality.

It is a fundamental result of Hedlund and Morse that periodicity can be characterized in terms of the complexity function [44]:

Theorem 10 (Hedlund-Morse)**.** *(Ω, T) is aperiodic if and only if $p(n) \geq n+1$ for every $n \in \mathbb{N}$.*

Aperiodic subshifts of minimal complexity, $p(n) = n+1$ for every $n \in \mathbb{N}$, exist and they are called Sturmian. If the complexity function satisfies $p(n) = n+k$ for $n \geq n_0$, $k, n_0 \in \mathbb{N}$, the subshift is called quasi-Sturmian. It is known that quasi-Sturmian subshifts are exactly those subshifts that are a morphic image of a Sturmian subshift; compare [18, 19, 74].

There are a large number of equivalent characterizations of Sturmian subshifts; compare [9]. We are mainly interested in their geometric description in terms of an irrational rotation. Let $\alpha \in (0,1)$ be irrational and consider the rotation by α on the circle,

$$R_\alpha : [0,1) \to [0,1), \quad R_\alpha \theta = \{\theta + \alpha\},$$

where $\{x\}$ denotes the fractional part of x, $\{x\} = x \mod 1$. The coding of the rotation R_α according to a partition of the circle into two half-open intervals of length α and $1-\alpha$, respectively, is given by the sequences

$$v_n(\alpha, \theta) = \chi_{[0,\alpha)}(R_\alpha^n \theta).$$

We obtain a subshift

$$\begin{aligned} \Omega_\alpha &= \overline{\{v(\alpha,\theta) : \theta \in [0,1)\}} \\ &= \{v(\alpha,\theta) : \theta \in [0,1)\} \cup \{\tilde{v}^{(k)}(\alpha) : k \in \mathbb{Z}\} \subset \{0,1\}^{\mathbb{Z}} \end{aligned}$$

which can be shown to be Sturmian. Here, $\tilde{v}_n^{(k)}(\alpha) = \chi_{(0,\alpha]}(R_\alpha^{n+k} 0)$. Conversely, every Sturmian subshift is essentially of this form, that is, if Ω is minimal and has complexity function $p(n) = n+1$, then up to a one-to-one morphism, $\Omega = \Omega_\alpha$ for some irrational $\alpha \in (0,1)$.

By uniform distribution, the frequencies of factors of Ω are given by the Lebesgue measure of certain subsets of the torus. Explicitly, if we write $I_0 = [0,\alpha)$ and $I_1 = [\alpha, 1)$, then the word $w = w_1 \ldots w_n \in \{0,1\}^n$ occurs in $v(\alpha,\theta)$ at site $k+1$ if and only if

$$\{k\alpha + \theta\} \in I(w_1, \ldots, w_n) := \bigcap_{j=1}^{n} R_\alpha^{-j}(I_{w_j}).$$

This shows that the frequency of w is θ-independent and equal to the Lebesgue measure of $I(w_1, \ldots, w_n)$. It is not hard to see that $I(w_1, \ldots, w_n)$ is an interval whose boundary points are elements of the set

$$P_n(\alpha) := \{\{-j\alpha\} : 0 \leq j \leq n\}.$$

The $n+1$ points of $P_n(\alpha)$ partition the torus into $n+1$ subintervals and hence the length $h_n(\alpha)$ of the smallest of these intervals bounds the frequency of a factor of length n from below. It is therefore of interest to study $\limsup n h_n(\alpha)$.

To this end we recall the notion of a continued fraction expansion; compare [56, 79]. For every irrational $\alpha \in (0,1)$, there are uniquely determined $a_k \in \mathbb{N}$ such that

$$\alpha = [a_1, a_2, a_3, \ldots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}. \tag{22}$$

The associated rational approximants $\frac{p_k}{q_k}$ are defined by

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_k = a_k p_{k-1} + p_{k-2}, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_k = a_k q_{k-1} + q_{k-2}. \end{aligned}$$

These rational numbers are best approximants to α in the following sense,

$$\min_{\substack{p,q\in\mathbb{N} \\ 0<q<q_{k+1}}} |q\alpha - p| = |q_k\alpha - p_k|\,, \tag{23}$$

and the quality of the approximation can be estimated according to

$$\frac{1}{q_k + q_{k+1}} < |q_k\alpha - p_k| < \frac{1}{q_{k+1}}. \tag{24}$$

By definition, we have

$$h_n(\alpha) = \min_{0<|q|\leq n} \{q\alpha\}.$$

Notice that for $0 < q \leq n$, we have $\min\{\{q\alpha\}, \{-q\alpha\}\} = \|q\alpha\|$, where we denote $\|x\| = \min_{p\in\mathbb{Z}} |x - p|$.

In particular,

$$h_n(\alpha) = \min_{0<q\leq n} \|q\alpha\|. \tag{25}$$

As noted by Hartman [43], this shows that $h_n(\alpha)$ can be expressed in terms of the continued fraction approximants. Indeed, if we combine (23) and (25), we obtain:

Lemma 5.4 (Hartman)**.** *If $q_k \leq n < q_{k+1}$, then*

$$h_n(\alpha) = |q_k\alpha - p_k|.$$

This allows us to show the following:

Theorem 11. *Every Sturmian subshift obeys the Boshernitzan condition* (B).

Proof. We only need to show that

$$\limsup_{n\to\infty} n h_n(\alpha) \geq C > 0. \tag{26}$$

We shall verify this on the subsequence $n_k = q_{k+1} - 1$. Hartman's lemma together with (24) shows that

$$n_k h_{n_k}(\alpha) = (q_{k+1} - 1)|q_k\alpha - p_k| \geq \frac{q_{k+1} - 1}{q_k + q_{k+1}} = \frac{1 - q_{k+1}^{-1}}{1 + q_k q_{k+1}^{-1}}.$$

Thus (26) holds (with $C = 1/3$, say). ☐

Corollary 2. *Every quasi-Sturmian subshift obeys* (B)*.*

Proof. This follows from Theorem 11 along with the stability result, Theorem 9. ☐

5.3. **Interval Exchange Transformations.** Subshifts generated by interval exchange transformations (IET's) are natural generalizations of Sturmian subshifts. They were studied, for example, in [13, 35, 36, 37, 53, 54, 55, 69, 77, 87, 88, 89].

IET's are defined as follows. Given a probability vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_i > 0$ for $1 \le i \le m$, we let $\mu_0 = 0$, $\mu_i = \sum_{j=1}^{i} \lambda_j$, and $I_i = [\mu_{i-1}, \mu_i)$. Let τ be a permutation of $\mathcal{A}_m = \{1, \ldots, m\}$, that is, $\tau \in S_m$, the symmetric group. Then $\lambda^\tau = (\lambda_{\tau^{-1}(1)}, \ldots, \lambda_{\tau^{-1}(m)})$ is also a probability vector and we can form the corresponding μ_i^τ and I_i^τ. Denote the unit interval $[0, 1)$ by I. The (λ, τ) interval exchange transformation is then defined by

$$T : I \to I, \;\; T(x) = x - \mu_{i-1} + \mu^\tau_{\tau(i)-1} \text{ for } x \in I_i, \; 1 \le i \le m.$$

It exchanges the intervals I_i according to the permutation τ.

The transformation T is invertible and its inverse is given by the $(\lambda^\tau, \tau^{-1})$ interval exchange transformation.

The symbolic coding of $x \in I$ is $\omega_n(x) = i$ if $T^n(x) \in I_i$. This induces a subshift over the alphabet $\mathcal{A}_m$: $\Omega_{\lambda,\tau} = \overline{\{\omega(x) : x \in I\}}$.

Sturmian subshifts correspond to the case of two intervals as a first return map construction shows.

Keane [53] proved that if the orbits of the discontinuities μ_i of T are all infinite and pairwise distinct, then T is minimal. In this case, the coding is one-to-one and the subshift is minimal and aperiodic. This holds in particular if τ is irreducible and λ is irrational. Here, τ is called irreducible if $\tau(\{1, \ldots, k\}) \neq (\{1, \ldots, k\})$ for every $k < m$ and λ is called irrational if the λ_i are rationally independent.

Regarding property (B), Boshernitzan has proved two results. First, in [12] the following is shown:

Theorem 12 (Boshernitzan)**.** *For every irreducible $\tau \in S_m$ and for Lebesgue almost every λ, the subshift $\Omega_{\lambda,\tau}$ satisfies* (B)*.*

In fact, Boshernitzan shows that for every irreducible $\tau \in S_m$ and for Lebesgue almost every λ, the subshift $\Omega_{\lambda,\tau}$ satisfies a stronger condition where the sequence of n's for which $\eta(n)$ is large cannot be too sparse. This condition is easily seen to imply (B), and hence the theorem above.

Note that when combined with Keane's minimality result, Theorem 12 implies that almost every subshift arising from an interval exchange transformation is uniquely ergodic. The latter statement confirms a conjecture of Keane [53] and had earlier been proven by different methods by Masur [69] and Veech [88]. Keane had in fact conjectured that all minimal interval exchange transformations would give rise to a uniquely ergodic system. This was disproved by Keynes and Newton [55] using five intervals, and then by Keane [54] using four intervals (the smallest possible number). The conjecture was therefore modified in [54] and then ultimately proven by Masur and Veech.

In a different paper, [13], Boshernitzan singles out an explicit class of subshifts arising from interval exchange transformations that satisfy (B). The transformation T is said to be of (rational) rank k if the μ_i span a k-dimensional space over $\mathbb{Q}$ (the field of rational numbers).

Theorem 13 (Boshernitzan)**.** *If T has rank* 2*, the subshift $\Omega_{\lambda,\tau}$ satisfies* (B)*.*

6. Circle Maps

Let $\alpha \in (0,1)$ be irrational and $\beta \in (0,1)$ arbitrary. The coding of the rotation R_α according to a partition into two half-open intervals of length β and $1-\beta$, respectively, is given by the sequences

$$v_n(\alpha,\beta,\theta) = \chi_{[0,\beta)}(R_\alpha^n \theta).$$

We obtain a subshift

$$\Omega_{\alpha,\beta} = \overline{\{v(\alpha,\beta,\theta) : \theta \in [0,1)\}} \subset \{0,1\}^{\mathbb{Z}}. \tag{27}$$

Subshifts generated this way are usually called circle map subshifts or subshifts generated by the coding of a rotation. These natural generalizations of Sturmian subshifts were studied, for example, in [1, 2, 10, 28, 29, 30, 48, 52, 80].

To the best of our knowledge, the Boshernitzan condition for this class of subshifts has not been studied explicitly. It is, however, intimately related to classical results on inhomogeneous diophantine approximation problems. In this section we make this connection explicit and study the condition (B) for circle map subshifts.

To describe the relation of frequencies of finite words occurring in a subshift to the length of intervals on the circle, let us write, in analogy to the Sturmian case, $I_0 = [0,\beta)$ and $I_1 = [\beta,1)$. The word $w = w_1 \dots w_n \in \{0,1\}^n$ occurs in $v(\alpha,\beta,\theta)$ at site $k+1$ if and only if

$$R_\alpha^k(\theta) \in I(w_1,\dots,w_n) := \bigcap_{j=1}^n R_\alpha^{-j}(I_{w_j}).$$

Thus the frequency of w is θ-independent and equal to the Lebesgue measure of $I(w_1,\dots,w_n)$. Moreover, $I(w_1,\dots,w_n)$ is an interval whose boundary points are elements of the set

$$P_n(\alpha,\beta) := \{\{-j\alpha + k\beta\} : 1 \le j \le n,\ 0 \le k \le 1\}.$$

This shows in particular that $\Omega_{\alpha,\beta}$ is quasi-Sturmian when $\beta \in \mathbb{Z} + \alpha\mathbb{Z}$ as in this case $P_n(\alpha,\beta)$ splits the unit interval into $n+k$ subintervals for large n. On the other hand, when $\beta \notin \mathbb{Z} + \alpha\mathbb{Z}$, $P_n(\alpha,\beta)$ contains $2n$ elements and the complexity of $\Omega_{\alpha,\beta}$ is $p(n) = 2n$ for n large enough.

Again, the points of $P_n(\alpha,\beta)$ partition the torus into $2n$ (resp., $n+k$) subintervals and hence the length $h_n(\alpha,\beta)$ of the smallest of these intervals bounds the frequency of a factor of length n from below. Explicitly, we have

$$h_n(\alpha,\beta) = \min\left\{\|q\alpha + r\beta\| : 0 \le |q| \le n,\ 0 \le r \le 1,\ (q,r) \neq (0,0)\right\}.$$

Let us also define

$$\tilde{h}_n(\alpha,\beta) = \min\left\{\|q\alpha + \beta\| : 0 \le |q| \le n\right\}.$$

Then $h_n(\alpha,\beta) \le \tilde{h}_n(\alpha,\beta)$ and therefore

$$\limsup_{n\to\infty} n\tilde{h}_n(\alpha,\beta) = 0 \Rightarrow \limsup_{n\to\infty} n h_n(\alpha,\beta) = 0. \tag{28}$$

Since we saw in Theorem 11 above that the points of $P_n(\alpha)$ are nicely spaced for many values of n, the Boshernitzan condition can only fail for a circle map subshift $\Omega_{\alpha,\beta}$ if the orbit of the α-rotation comes too close to β. In other words, to prove such a negative result for a circle map subshift, it should be sufficient to study $\tilde{h}_n(\alpha,\beta)$, followed by an application of (28).

Motivated by Hardy and Littlewood [42], Morimoto [70, 71] carried out an in-depth analysis of the asymptotic behavior of the numbers $\tilde{h}_n(\alpha,\beta)$. Morimoto's results and related ones were summarized in [57]. While it is possible to deduce consequences regarding the Boshernitzan condition from these papers, we choose to give direct and elementary proofs of our positive results below and make reference to a specific theorem of Morimoto only for a complementary negative result.

Our first result shows that the Boshernitzan condition holds in almost all cases.

Theorem 14. *Let $\alpha \in (0,1)$ be irrational. Then the subshift $\Omega_{\alpha,\beta}$ satisfies* (B) *for Lebesgue almost every $\beta \in (0,1)$.*

Proof. Denote the set of β's for which the Boshernitzan condition fails by $N(\alpha)$,

$$N(\alpha) = \{\beta \in (0,1) : \Omega_{\alpha,\beta} \text{ does not satisfy (B)}\}.$$

By (26) and Theorem 11, there exists a sequence $n_k \to \infty$ such that

$$\liminf_{k\to\infty} n_k h_{n_k}(\alpha) = C > 0.$$

Let $\varepsilon > 0$ with $\varepsilon < C$ be given and denote the $\frac{\varepsilon}{2n}$-neighborhood of the set $\{\{q\alpha\} : 0 < |q| \le n\}$ by $U(\varepsilon, n)$. Clearly, every $\beta \in N(\alpha)$ belongs to $U(\varepsilon, n_k)$ for $k \ge k_0(\beta)$. Therefore,

$$N(\alpha) \subseteq \liminf_{k\to\infty} U(\varepsilon, n_k) = \bigcup_{m=1}^{\infty} \bigcap_{k \ge m} U(\varepsilon, n_k). \tag{29}$$

The sets

$$S_m = \bigcap_{k\ge m} U(\varepsilon, n_k)$$

obey $S_m \subseteq S_{m+1}$ and $|S_m| \le \varepsilon$ for every m; $|\cdot|$ denoting Lebesgue measure. Hence,

$$\left| \liminf_{k\to\infty} U(\varepsilon, n_k) \right| \le \varepsilon.$$

It follows that $N(\alpha)$ has zero Lebesgue measure. □

The next result concerns a subclass of α's for which the Boshernitzan condition holds for all β's.

Theorem 15. *Let $\alpha \in (0,1)$ be irrational with bounded continued fraction coefficients, that is, $a_n \le C$. Then, $\Omega_{\alpha,\beta}$ satisfies* (B) *for every $\beta \in (0,1)$.*

Proof. By Lemma 5.4 and (24), we have

$$h_n(\alpha) > \frac{1}{q_k + q_{k+1}},$$

where k is chosen such that $q_k \le n < q_{k+1}$. Thus, for every n, we have

$$n h_n(\alpha) > \frac{n}{q_k + q_{k+1}} \ge \frac{q_k}{(a_{k+1}+2)q_k} \ge \frac{1}{C+2}. \tag{30}$$

Now assume there exists $\beta \in (0,1)$ such that $\Omega_{\alpha,\beta}$ does not satisfy (B). Let $\varepsilon = (7C+14)^{-1}$. As $\limsup_{n\to\infty} n\varepsilon(n) = 0$, we have $n\varepsilon(n) < \epsilon$ for every sufficiently large n. Thus, for each such n we can find a word of length n with frequency less than ε/n. Now, each such word corresponds to an interval with length less than ε/n with boundary points in $P_n(\alpha,\beta)$. Moreover, invoking (30) and the fact that $\epsilon < 1/(C+2)$, we infer that the length of the interval has the form $|m_n\alpha - \beta - k_n|$

with $|m_n| \le n$. To summarize, we see that for every n large enough there exist k_n, m_n with $|m_n| \le n$ such that

$$|m_n\alpha - \beta - k_n| \le \frac{\varepsilon}{n}.$$

Clearly, the mapping $n \mapsto m_n$ can take on each value only finitely many times. Therefore, there exists a sequence $n_j \to \infty$ such that $m_{n_j} \neq m_{n_j+1}$. This implies

$$\begin{aligned}\left|\left(m_{n_j+1} - m_{n_j}\right)\alpha - \left(k_{n_j+1} - k_{n_j}\right)\right| &\le \left|m_{n_j+1}\alpha - \beta - k_{n_j+1}\right| + \left|m_{n_j}\alpha - \beta - k_{n_j}\right| \\ &\le \frac{\varepsilon}{n_j+1} + \frac{\varepsilon}{n_j} \\ &\le \frac{2\varepsilon}{n_j}.\end{aligned}$$

Since $0 < \left|m_{n_j+1} - m_{n_j}\right| \le 2(n_j+1) \le 3n_j =: \tilde{n}_j$, we obtain $\tilde{n}_j h_{\tilde{n}_j}(\alpha) \le 6\varepsilon < (C+2)^{-1}$, which contradicts (30). ☐

This raises the question whether $\Omega_{\alpha,\beta}$ satisfies (B) for every β also in the case where α has unbounded coefficients a_n. It is a consequence of a result of Morimoto [71] that this is not the case.

Theorem 16 (Morimoto)**.** *Let $\alpha \in (0,1)$ be irrational with unbounded continued fraction coefficients. Then, there exists $\beta \in (0,1)$ such that*

$$\limsup_{n\to\infty} n\tilde{h}_n(\alpha,\beta) = 0.$$

Corollary 3. *Let $\alpha \in (0,1)$ be irrational with unbounded continued fraction coefficients. Then, there exists $\beta \in (0,1)$ such that $\Omega_{\alpha,\beta}$ does not satisfy* (B).

Proof. This is an immediate consequence of Theorem 16 and (28). ☐

We close this section with a brief discussion of the case where the circle is partitioned into a finite number of half-open intervals. To be specific, let $0 < \beta_1 < \cdots < \beta_{p-1} < 1$ and associate the intervals of the induced partition with p symbols: Let $\beta_p = \beta_0 = 0$ and

$$v_n(\theta) = k \Leftrightarrow R_\alpha^n(\theta) \in [\beta_k, \beta_{k+1}).$$

We obtain a subshift over the alphabet $\{0, 1, \ldots, p-1\}$,

$$\Omega_\beta = \overline{\{v(\theta) : \theta \in [0,1)\}}.$$

Again, the word $w = w_1 \ldots w_n \in \{0,1\}^n$ occurs in $v(\theta)$ at site $k+1$ if and only if

$$R_\alpha^k \theta \in I(w_1, \ldots, w_n) := \bigcap_{j=1}^n R_\alpha^{-j}(I_{w_j})$$

and the connected components of the sets $I(w_1, \ldots, w_n)$ are bounded by the points

$$\{-j\alpha + \beta_k : 1 \le j \le n, 0 \le k \le p-1\}. \tag{31}$$

Recall that $\limsup_{n\to\infty} n\tilde{h}_n(\alpha,\beta)$ is an important quantity in the case of a partition of the circle into two intervals. In fact, we showed that this quantity being positive is a necessary condition for (B) to hold. When there are three or more intervals, however, we will need to require a much stronger condition as the β_i's may now "take turns" in being well approximated by the α-orbit. Indeed, we shall now be interested in studying $\liminf_{n\to\infty} n\tilde{h}_n(\alpha,\gamma)$ (for certain values of γ, associated with the β_i's). More precisely, define the following quantity:

$$M(\alpha,\gamma) = \liminf_{|n|\to\infty} |n| \cdot \|n\alpha - \gamma\|.$$

Let

$$P(\alpha) = \{\gamma : M(\alpha,\gamma) > 0\}.$$

Then, we have the following result:

Theorem 17. *Let $\alpha \in (0,1)$ be irrational. Suppose that $0 = \beta_0 < \beta_1 < \cdots < \beta_{p-1} < \beta_p = 1$ are such that*

$$\beta_k - \beta_l \in P(\alpha) \text{ for } 0 \le k \ne l \le p-1.$$

Then the subshift (Ω_β, T) satisfies the Boshernitzan condition (B).

Remarks. 1. This gives a finite number of conditions whose combination is a sufficient condition for (B) to hold.

2. The set $P(\alpha)$ is non-empty for every irrational α. In fact, for every irrational α there exists a suitable γ such that $M(\alpha,\gamma) > 1/32$; compare [79, Theorem IV.9.3].

3. We discuss in Appendix B how $M(\alpha,\gamma)$ can be computed with the help of the so-called *negative continued fraction expansion* of α and the α*-expansion* of γ.

Proof. By (31), all frequencies of words of length n are bounded from below by

$$\hat{h}_n(\alpha,\beta) = \min\left\{\|q\alpha + \beta_k - \beta_l\| : 0 \le |q| \le n,\ 0 \le k,l \le p-1,\ (q,k-l) \ne (0,0)\right\}.$$

As in our considerations above, we choose a sequence $n_k \to \infty$ such that

$$\liminf_{k\to\infty} n_k h_{n_k}(\alpha) = C > 0.$$

By assumption, we have

$$D = \min\{M(\alpha,\beta_k - \beta_l) : 0 \le k \ne l \le p-1\} > 0.$$

Notice that with these choices of C and D, frequencies of words of length n_k are bounded from below by

$$\hat{h}_{n_k}(\alpha,\beta) \ge \min\left\{\frac{C - o(1)}{n_k}, \frac{D - o(1)}{n_k}\right\}.$$

Putting everything together, we obtain

$$\limsup_{n\to\infty} n \cdot \eta(n) \ge \liminf_{k\to\infty} n_k \cdot \eta(n_k) \ge \min\{C,D\} > 0,$$

and hence (B) is satisfied. □

7. Arnoux-Rauzy Subshifts and Episturmian Subshifts

In this section we consider another natural generalization of Sturmian subshifts, namely, Arnoux-Rauzy subshifts and, more generally, episturmian subshifts. These subshifts were studied, for example, in [4, 27, 31, 50, 51, 78, 91]. They share with Sturmian subshifts the fact that, for each n, there is a unique subword of length n that has multiple extensions to the right. Our main results will show that, similarly to the circle map case, the Boshernitzan condition is almost always satisfied, but not always.

Let us consider a minimal subshift (Ω, T) over the alphabet $\mathcal{A}_m = \{1, 2, \ldots, m\}$, where $m \ge 2$. Recall that the set of subwords of length n occurring in elements of Ω is denoted by $\mathcal{W}_n(\Omega)$ (cf. (3)) and that the complexity function p is defined by $p(n) = \sharp\mathcal{W}_n(\Omega)$ (cf. (21)). A word $w \in \mathcal{W}(\Omega)$ is called *right-special* (resp., *left-special*) if there are distinct symbols $a, b \in \mathcal{A}_m$ such that $wa, wb \in \mathcal{W}(\Omega)$ (resp., $aw, bw \in \mathcal{W}(\Omega)$). A word that is both right-special and left-special is called *bispecial*.

For later use, let us recall the *Rauzy graphs* that are associated with $\mathcal{W}(\Omega)$. For each n, we consider the directed graph $\mathcal{R}_n = (V_n, A_n)$, where the vertex set is given by $V_n = \mathcal{W}_n(\Omega)$, and A_n contains the arc from aw to wb, $a, b \in \mathcal{A}_m$, $|w| = n-1$, if and only if $awb \in \mathcal{W}_{n+1}(\Omega)$. That is, $|V_n| = p(n)$ and $|A_n| = p(n+1)$. Moreover, a word is right-special (resp., left-special) if and only if its out-degree (resp., in-degree) is ≥ 2.

Note that the complexity function of a Sturmian subshift obeys $p(n+1)-p(n) = 1$ for every n and hence for every length, there is a unique right-special factor and a unique left-special factor, each having exactly two extensions. This property is clearly characteristic for a Sturmian subshift.

Arnoux-Rauzy subshifts and episturmian subshifts relax this restriction on the possible extensions somewhat, and they are defined as follows: Ω is called an *Arnoux-Rauzy subshift* if for every n, there is a unique right-special word $r_n \in \mathcal{W}(\Omega)$ and a unique left-special word $l_n \in \mathcal{W}(\Omega)$, both having exactly m extensions. This implies in particular that $p(1) = m$ and hence

$$p(n) = (m-1)n + 1.$$

Arnoux-Rauzy subshifts over $\mathcal{A}_2$ are exactly the Sturmian subshifts.

On the other hand, Ω is called *episturmian* if $\mathcal{W}(\Omega)$ is closed under reversal (i.e., for every $w = w_1 \dots w_n \in \mathcal{W}(\Omega)$, we have $w^R = w_n \dots w_1 \in \mathcal{W}(\Omega)$) and for every n, there is exactly one right-special word $r_n \in \mathcal{W}(\Omega)$.

It is easy to see that every Arnoux-Rauzy subshift is episturmian. On the other hand, every episturmian subshift turns out to be a morphic image of some Arnoux-Rauzy subshift. We shall explain this connection below. Since we are interested in studying the Boshernitzan condition, this fact is important and allows us to limit our attention to the Arnoux-Rauzy case.

Risley and Zamboni [78] found two useful descriptions of a given Arnoux-Rauzy subshift, namely, in terms of the recursive structure of the bispecial words and in terms of an S-adic system.

Let ϵ be the empty word and let $\{\epsilon = w_1, w_2, \dots\}$ be the set of all bispecial words in $\mathcal{W}(\Omega)$, ordered so that $0 = |w_1| < |w_2| < \cdots$. Let $I = \{i_n\}$ be the sequence of elements i_n of $\mathcal{A}_m$ so that $w_n i_n$ is left-special. The sequence I is called the *index sequence* associated with Ω. Risley and Zamboni prove that, for every n, w_{n+1} is the *palindromic closure* $(w_n i_n)^+$ of $w_n i_n$, that is, the shortest palindrome that has $w_n i_n$ as a prefix. Conversely, given any sequence I, one can associate a subshift Ω as follows: Start with $w_1 = \epsilon$ and define w_n inductively by $w_{n+1} = (w_n i_n)^+$. The sequence of words $\{w_n\}$ has a unique one-sided infinite limit $w_\infty \in \mathcal{A}_m^{\mathbb{N}}$, called the *characteristic sequence*, which then gives rise to the subshift $(\Omega(I), T)$ in the standard way; $\Omega(I)$ consists of all two-sided infinite sequences whose subwords occur in w. Risley and Zamboni prove the following characterization.

Proposition 7.1 (Risley-Zamboni)**.** *For every Arnoux-Rauzy subshift (Ω, T) over $\mathcal{A}_m$, every $a \in \mathcal{A}_m$ occurs in the index sequence $\{i_n\}$ infinitely many times and $\Omega = \Omega(I)$. Conversely, for every sequence $\{i_n\} \in \mathcal{A}_m^{\mathbb{N}}$ such that every $a \in \mathcal{A}_m$*

occurs in $\{i_n\}$ *infinitely many times,* $(\Omega(I), T)$ *is an Arnoux-Rauzy subshift and* $\{i_n\}$ *is its index sequence.*

The S-adic description of an Arnoux-Rauzy subshift, that is, involving iterated morphisms chosen from a finite set, found in [78] reads as follows.

Proposition 7.2 (Risley-Zamboni)**.** *Let* (Ω, T) *be an Arnoux-Rauzy subshift over* $\mathcal{A}_m$ *and* $\{i_n\}$ *the associated index sequence. For each* $a \in \mathcal{A}_m$*, define the morphism* τ_a *by*

$$\tau_a(a) = a \text{ and } \tau_a(b) = ab \text{ for } b \in \mathcal{A}_m \setminus \{a\}.$$

Then for every $a \in \mathcal{A}_m$*, the characteristic sequence is given by*

$$\lim_{m\to\infty} \tau_{i_1} \circ \cdots \circ \tau_{i_m}(a).$$

We can now state our positive result regarding the Boshernitzan condition for Arnoux-Rauzy subshifts.

Theorem 18. *Let* (Ω, T) *be an Arnoux-Rauzy subshift over* $\mathcal{A}_m$ *and* $\{i_n\}$ *the associated index sequence. Suppose there is* $N \in \mathbb{N}$ *such that for a sequence* $k_j \to \infty$*, each of the words* $i_{k_j} \dots i_{k_j+N-1}$ *contains all symbols from* $\mathcal{A}_m$*. Then the Boshernitzan condition* (B) *holds.*

This result is similar to Theorem 14 in the sense that if we put any probability measure ν on $\mathcal{A}_m$ assigning positive weight to each symbol, then almost all sequences $\{i_n\}$ with respect to the product measure $\nu^{\mathbb{N}}$ correspond to Arnoux-Rauzy subshifts that satisfy the assumption of Theorem 18.

Before proving this theorem, we state our negative result, which is an analog of Corollary 3.

Theorem 19. *For every* $m \geq 3$*, there exists an Arnoux-Rauzy subshift over* $\mathcal{A}_m$ *that does not satisfy the Boshernitzan condition* (B).

Remark 5. The assumption $m \geq 3$ is of course necessary since the case $m = 1$ is trivial and the case $m = 2$ corresponds to the Sturmian case, where the Boshernitzan condition always holds; compare Theorem 11.

The Arnoux-Rauzy subshifts are uniquely ergodic and we set

$$d(w) \equiv \mu(V_w),\ w \in \mathcal{W}(\Omega),$$

where, as usual, the unique invariant probability measure is denoted by μ.

Proof of Theorem 18. This proof employs the description of the subshift in terms of the bispecial words; compare Proposition 7.1.

Observe that there is some k_0 such that $|w_k| \leq 2|w_{k-1}|$ for every $k \geq k_0$. Essentially, we need that $i_1, \dots, i_{k_0-1}$ contains all symbols from $\mathcal{A}_m$.

Now consider a value of $k \geq k_0$ such that $i_k \dots i_{k+N-1}$ contains all symbols from $\mathcal{A}_m$. We claim that

$$|w_k| \cdot \eta(|w_k|) \geq 2^{-N}. \tag{32}$$

By the assumption, this implies

$$\limsup_{n\to\infty} n \cdot \eta(n) \geq 2^{-N}$$

and hence the Boshernitzan condition (B).

The Rauzy graph $\mathcal{R}_{|w_k|}$ has one vertex (namely, w_k) with in-degree and out-degree m, while all other vertices have in-degree and out-degree 1. Thus, the graph splits up into m loops that all contain w_k and are pairwise disjoint otherwise. These loops can be indexed in an obvious way by the elements of the alphabet $\mathcal{A}_m$.

Since $w_{k+1} = (w_k i_k)^+$, w_{k+1} begins and ends with w_k and, moreover, w_{k+1} contains all words that correspond to the loop in $\mathcal{R}_{|w_k|}$ indexed by i_k. Iterating this argument, we see that w_{k+N} contains the words from all loops and hence all words from $\mathcal{W}_{|w_k|}(\Omega)$. This implies

$$\min_{w\in\mathcal{W}_{|w_k|}(\Omega)} d(w) \geq d(w_{k+N}) \geq \frac{1}{|w_{k+N}|} \geq \frac{1}{2^N|w_k|}$$

and hence (32), finishing the proof. □

Proof of Theorem 19. This proof employs the description of the subshift in terms of an S-adic structure; compare Proposition 7.2.

We shall construct an index sequence $\{i_n\}$ over three symbols (i.e., over the alphabet $\mathcal{A}_3$) such that the corresponding Arnoux-Rauzy subshift does not satisfy the Boshernitzan condition (B). It is easy to verify that the same idea can be used to construct such a subshift over $\mathcal{A}_m$ for any $m \geq 3$.

The index sequence will have the form

$$i_1 i_2 i_3 \ldots = 1^{a_1} 2^{a_2} 3^{a_3} 1^{a_4} 2^{a_5} 3^{a_6} 1^{a_7} \ldots \tag{33}$$

with a rapidly increasing sequence of integers, $\{a_n\}$.

By the special form of the Rauzy graph, the words $w_k a$ label all the frequencies of words in $\mathcal{W}_{|w_k|+1}(\Omega)$ since words corresponding to arcs on a given loop in $\mathcal{R}_{|w_k|}$ must have the same frequency. Put differently,

$$\eta(|w_k|+1) = \min_{a\in\mathcal{A}_3} d_{w_\infty}(w_k a). \tag{34}$$

Here, we make the dependence of the frequency on w_∞ explicit.

Moreover, it is sufficient to control $\eta(n)$ for these special values of n since every subword u that is not bispecial has a unique extension to either the left or the right, and this extension must have the same frequency. This shows

$$\eta(|w_k|+1) \geq \eta(n) \text{ for } |w_k|+1 \leq n \leq |w_{k+1}|. \tag{35}$$

Now write $\mu_{k,m} = \tau_{i_k} \circ \cdots \circ \tau_{i_{k+m-1}}$. Proposition 7.2 says that the characteristic sequence is given by the limit

$$w = \lim_{m\to\infty} \mu_{1,m}(a) \text{ for every } a \in \mathcal{A}_3.$$

We also define

$$w^{(k)} = \lim_{m\to\infty} \mu_{k,m}(a) = (\mu_{1,k-1})^{-1}(w).$$

By [50], $w^{(k)}$ is the derived sequence labeling the return words of w_k in w_∞. In particular, $w^{(k)}$ labels the occurrences of $w_k a$, $a \in \mathcal{A}_3$, in w. Moreover,

$$d_{w_\infty}(w_k a) = \frac{d_{w^{(k)}}(a)}{\sum_{b\in\mathcal{A}_3} d_{w^{(k)}}(b)|\mu_{1,k-1}(b)|} \leq \frac{d_{w^{(k)}}(a)}{\min_{b\in\mathcal{A}_3}|\mu_{1,k-1}(b)|}. \tag{36}$$

Combining (34) and (36), we obtain

$$(|w_k|+1)\cdot\eta(|w_k|+1) \leq \frac{|w_k|+1}{\min_{b\in\mathcal{A}_3}|\mu_{1,k-1}(b)|} \cdot \min_{a\in\mathcal{A}_3} d_{w^{(k)}}(a). \tag{37}$$

Notice that $(|w_k|+1)(\min_{b\in\mathcal{A}_3}|\mu_{1,k-1}(b)|)^{-1}$ only depends on $i_1,\dots,i_{k-1}$ and $\min_{a\in\mathcal{A}_3} d_{w^{(k)}}(a)$ only depends on $i_k, i_{k+1},\dots$. Thus, if we choose a rapidly increasing sequence $\{a_n\}$ in (33), we can arrange for

$$\lim_{k\to\infty}(|w_k|+1)\cdot\eta(|w_k|+1)=0. \tag{38}$$

This together with (35) implies

$$\lim_{n\to\infty} n\cdot\eta(n)=0,$$

proving the theorem.

Let us briefly comment on (38). Choose a monotonically decreasing sequence $e_k\to 0$. Assign any value ≥ 1 to a_1. Then, a_2 should be chosen large enough so that for $1\le k\le a_1$, (37) yields

$$(|w_k|+1)\cdot\eta(|w_k|+1)\le e_k. \tag{39}$$

Here we use that between consecutive 3's in $w^{(k)}$, there must be at least a_2 2's. Next, we choose a_3 so large that (39) holds for $a_1+1\le k\le a_2$. Here we use that between consecutive 1's in $w^{(k)}$, there must be at least a_3 3's. We can continue in this fashion, thereby generating a sequence $\{a_n\}$ such that (39) holds for all k. This shows in particular that $(|w_k|+1)\cdot\eta(|w_k|+1)$ can go to zero arbitrarily fast. □

One may wonder what sequences are generated by the procedures described before Propositions 7.1 and 7.2 if one starts with an index sequence that does not necessarily satisfy the assumption above, namely, that all symbols occur infinitely often. It was shown by Droubay, Justin and Pirillo [31, 50] that one obtains episturmian subshifts and, conversely, every episturmian subshift can be generated in this way.

Proposition 7.3 (Droubay, Justin, Pirillo)**.** *For every episturmian subshift (Ω,T) over $\mathcal{A}_m$, there exists an index sequence $\{i_n\}$ such that $\Omega=\Omega(I)$. Conversely, for every sequence $\{i_n\}\in\mathcal{A}_m^{\mathbb{N}}$, $(\Omega(I),T)$ is an episturmian subshift and $\{i_n\}$ is its index sequence. For every $a\in\mathcal{A}_m$, the characteristic sequence is given by*

$$\lim_{m\to\infty}\tau_{i_1}\circ\cdots\circ\tau_{i_m}(a).$$

We can now quickly deduce results concerning (B) (and hence (P)) for episturmian subshifts. If (Ω,T) is an episturmian subshift over $\mathcal{A}_m$, denote by $\mathcal{A}\subseteq\mathcal{A}_m$ the set of all symbols that occur in its index sequence infinitely many times. Fix k such that $i_k,i_{k+1},\dots$ only contains symbols from $\mathcal{A}$. Thus this tail sequence corresponds to an Arnoux-Rauzy subshift over $|\mathcal{A}|$ symbols and the given episturmian subshift is a morphic image (under $\mu_{1,k-1}$) of it. (Note that $|\mathcal{A}|\ge 2$ since (Ω,T) is aperiodic.) If the associated Arnoux-Rauzy subshift satisfies (B) (if, e.g., Theorem 18 applies), then (Ω,T) satisfies (B) by Theorem 9. On the other hand, since every Arnoux-Rauzy subshift is episturmian, Theorem 19 shows that not all episturmian subshifts satisfy (B). In this context, it is interesting to note that Justin and Pirillo showed that all episturmian subshifts are uniquely ergodic [50].

8. Application to Schrödinger Operators

In this section we discuss applications of our previous study to spectral theory of Schrödinger operators. This is based on methods introduced in [63] by Lenz.

Let (Ω, T) be a minimal uniquely ergodic subshift over the finite set $\mathcal{A}$ and assume $\mathcal{A} \subset \mathbb{R}$. As discussed in the introduction, (Ω, T) gives rise to the family $(H_\omega)_{\omega\in\Omega}$ of selfadjoint operators $H_\omega : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ acting by

$$(H_V u)(n) \equiv u(n+1) + u(n-1) + \omega(n)u(n).$$

As (Ω, T) is minimal, there exists a set $\Sigma \equiv \Sigma((\Omega, T)) \subset \mathbb{R}$ with

$$\sigma(H_\omega) = \Sigma \text{ for all } \omega \in \Omega$$

(see, e.g., [8]). We will assume furthermore that (Ω, T) is aperiodic. Such subshifts have attracted a lot of attention in recent years for both physical and mathematical reasons:

These subshifts can serve as models for a special class of solids discovered in 1984 by Shechtman et al. [81]. These solids, later called quasicrystals, have very special mechanical, electrical, and diffraction properties [49, 82]. In the quantum mechanical description of electrical (i.e., conductance) properties of these solids, one is led to the operators (H_ω) above. These operators in turn have a tendency to display intriguing mathematical features. These features include:

($\mathcal{Z}$) Cantor spectrum of Lebesgue measure zero, that is, Σ is a Cantor set of Lebesgue measure zero.
($\mathcal{SC}$) Purely singular continuous spectrum, that is, absence of both point spectrum and absolutely continuous spectrum.
($\mathcal{AT}$) Anomalous transport.

By now, absence of absolutely continuous spectrum is completely established for all relevant subshifts due to results of Last and Simon [61] in combination with earlier results of Kotani [59]. The other spectral features have been investigated for large, but special, classes of examples. Here, our focus is on ($\mathcal{Z}$). As for the other properties, we refer the reader to the survey articles [21, 83].

The property ($\mathcal{Z}$) has been investigated for several models: For the period-doubling substitution and the Thue-Morse substitution, it was shown to hold by Bellissard et al. in [7] (cf. earlier work of Bellissard [6] as well). A more general result for primitive substitutions has then been obtained by Bovier and Ghez [16]. Recently, proofs of ($\mathcal{Z}$) for all primitive substitutions were obtained by Liu et al. [66] and, independently, by Lenz [63]. For special examples of on-primitive substitutions, ($\mathcal{Z}$) has recently been investigated by de Oliveira and Lima [73]. Their results were extended by Damanik and Lenz [25].

For Sturmian operators, ($\mathcal{Z}$) has been proven by Sütő in the golden mean case (= Fibonacci substitution) [83, 84]. The general case was then treated by Bellissard et al. [8]. A different approach to ($\mathcal{Z}$) in the Sturmian case has been developed in [26] by Damanik and Lenz. A suitably modified version of this approach can also be used to study ($\mathcal{Z}$) for a certain class of substitutions as shown by Damanik [22].

For quasi-Sturmian operators, ($\mathcal{Z}$) was shown in [24]. Later a different proof was given in [64].

All approaches to ($\mathcal{Z}$) are based on a fundamental result of Kotani [59]. To discuss this result, we need some preparation.

Spectral properties of the operators (H_ω) are intimately linked to behavior of solutions of the difference equation

$$u(n+1) + u(n-1) + (\omega(n) - E)u(n) = 0 \tag{40}$$

for $E \in \mathbb{R}$. To study this behavior, we define, for $E \in \mathbb{R}$, the locally constant function $M^E : \Omega \longrightarrow \mathrm{SL}(2,\mathbb{R})$ by

$$M^E(\omega) \equiv \begin{pmatrix} E - \omega(1) & -1 \\ 1 & 0 \end{pmatrix}. \tag{41}$$

Then, it is easy to see that a sequence u is a solution of the difference equation (40) if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M^E(n,\omega) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix},\ n \in \mathbb{Z}. \tag{42}$$

The rate of exponential growth of solutions of (40) is then measured by the so-called Lyapunov exponent $\gamma(E) \equiv \Lambda(M^E)$. The fundamental result of Kotani, mentioned above, says that (due to aperiodicity)

$$|\{E \in \mathbb{R} : \gamma(E) = 0\}| = 0, \tag{43}$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}$. By general principles, it is clear that $\{E \in \mathbb{R} : \gamma(E) = 0\} \subset \Sigma$ [17]. The overall strategy to prove $(\mathcal{Z})$ is then to show

$$\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}. \tag{44}$$

Given (44), Σ can not contain an interval by (43). Moreover, Σ is a closed set, as the spectrum of an operator always is. Finally, Σ does not contain isolated points, again by general principles on random operators [17]. Hence, Σ is a Cantor set of measure zero if (44) holds.

The standard approach to (44) used to rely on trace maps. Trace maps are a powerful tool in the study of spectral properties. In particular, they can be used not only to study $(\mathcal{Z})$, but also to investigate $(\mathcal{SC})$ and $(\mathcal{AT})$. However, trace maps do not seem to be available as soon as the dynamical systems get more complicated. This difficulty is avoided in a new approach to $(\mathcal{Z})$ introduced in [63]. There, validity of (44) is related to certain ergodic properties of the underlying dynamical system. More precisely, the abstract cornerstone of this new approach is the following result.

Theorem 20. [63] *Let (Ω, T) be a minimal uniquely ergodic subshift over $\mathcal{A} \subset \mathbb{R}$. Then, $\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}$ if and only if M^E is uniform for every $E \in \mathbb{R}$. In this case, the map $\gamma : \mathbb{R} \longrightarrow [0, \infty)$ is continuous.*

Given this theorem, it becomes possible to show (44) by studying uniformity of the functions M^E. In fact, as shown in [63], uniformity of M^E holds for all systems satisfying (PW) and, in particular, for all linearly repetitive systems (see [62] as well). In [63], this was used to prove $(\mathcal{Z})$ for all primitive substitutions. Later $(\mathcal{Z})$ has been established for various further systems by showing linear repetitivity [1, 25, 73].

Proof of Theorem 2. Given Theorem 20 and Kotanis result (43), the assertion follows easily from our main result: By (B) and Theorem 1, the function M^E is uniform for every $E \in \mathbb{R}$. By Theorem 20, this implies $\Sigma = \{E : \gamma(E) = 0\}$. By (43), this gives that Σ is a Cantor set of Lebesgue measure zero, as discussed above. □

Proof of Theorem 3. The result follows from Theorem 2 and the results regarding the validity of (B) for circle map subshifts of Section 6. □

Proof of Theorem 4. The assertion is immediate from Theorem 20 and our main result, Theorem 1. □

As becomes clear from the discussion in Section 5, Theorem 2 generalizes all earlier results on $(\mathcal{Z})$. Moreover, it gives various new ones. One of these new results on $(\mathcal{Z})$ is Theorem 3. Similarly, combining Theorem 2 and the results of Subsection 5.3, we obtain another new result on $(\mathcal{Z})$ for subshifts associated with interval exchange transformations. To the best of our knowledge this is the first result on $(\mathcal{Z})$ for operators associated to interval exchange transformations (not counting those which are Sturmian or linearly repetitive).

Theorem 21. *Let $\tau \in S_m$ be irreducible. Then, for Lebesgue almost every λ, $\Sigma = \Sigma(\Omega_{\lambda,\tau})$ is a Cantor set of Lebesgue measure zero.*

Proof. As discussed in Subsection 5.3, if $\tau \in S_m$ is irreducible, $\Omega_{\lambda,\tau}$ is minimal, aperiodic, and satisfies (B) for almost every λ. This, combined with Theorem 2, yields the assertion. □

Finally, we also mention the following result for Arnoux-Rauzy subshifts, which follows from Theorem 2 and the discussion in Section 7

Theorem 22. *Let (Ω, T) be an aperiodic Arnoux-Rauzy subshift over $\mathcal{A}_m$ and $\{i_n\}$ the associated index sequence. Suppose there is $N \in \mathbb{N}$ such that for a sequence $k_j \to \infty$, each of the words $i_{k_j} \dots i_{k_j+N-1}$ contains all symbols from $\mathcal{A}_m$. Then, Σ is a Cantor set of measure zero.*

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Appendix A. Almost Every Circle Map Subshift Has Infinite Index

In this section, we show that the previous results on zero-measure Cantor spectrum for Schrödinger operators associated with circle map subshifts only cover a zero-measure set in parameter space. This should be seen in connection with Theorem 3, where this spectral result is established for almost all parameter values.

Recall that for every $(\alpha, \beta) \in (0,1) \times (0,1)$, we may define a subshift $\Omega_{\alpha,\beta}$ as in (27).

Proofs of zero-measure spectrum for the associated operators based on trace map dynamics were given in [8, 24, 83, 84]. They cover the case of arbitrary irrational $\alpha \in (0,1)$ and β's in $(0,1)$ of the form $\beta = m\alpha + n$. This is clearly a zero-measure set in $(0,1) \times (0,1)$.

The paper [1] applies the results of [63] and shows zero-measure spectrum for a class of circle map subshifts that is characterized by means of a generalized continued fraction algorithm. Essentially, [1] characterizes the pairs (α, β) for which the associated subshifts are linearly recurrent. We want to show that these, too, form a set of measure zero.

To this end, we note that every aperiodic linearly recurrent subshift Ω has finite index in the sense that there is $N < \infty$ such that its set of finite subwords, $\mathcal{W}(\Omega)$, contains no word of the form w^N. (This is immediate from the definition.)

We say that a subshift Ω has infinite index if for every $n \geq 1$, there is a word w such that $w^n \in \mathcal{W}(\Omega)$ and prove the following:

Proposition A.1. *For almost every* $(\alpha, \beta) \in (0,1) \times (0,1)$*, the subshift* $\Omega_{\alpha,\beta}$ *has infinite index.*

Remarks. (a) This implies that for almost every $(\alpha, \beta) \in (0,1) \times (0,1)$, $\Omega_{\alpha,\beta}$ is not linearly recurrent.

(b) Our proof is an extension of arguments from [28, 52].

Proof. It suffices to show that for each fixed $\beta \in (0,1)$, $\Omega_{\alpha,\beta}$ has infinite index for almost every $\alpha \in (0,1)$.

For a sequence $l_k \to \infty$ with

$$\sum_{k=1}^{\infty} l_k^{-1} = \infty \tag{45}$$

(e.g., $l_k = k$), we define the sets $G_{\alpha,\beta}(k) \subseteq [0,1)$ by

$$G_{\alpha,\beta}(k) = \{\theta \in [0,1) : V_\theta(mq_k + j) = V_\theta(j),\ -2l_k + 1 \leq m \leq 2l_k - 1, 1 \leq j \leq q_k\},$$

where

$$V_\theta(n) = \chi_{[0,\beta)}(R_\alpha^n \theta).$$

It is clearly sufficient to show that for each $\beta \in (0,1)$ fixed (and $|\cdot|$ denoting Lebesgue measure),

$$\left| \limsup_{k\to\infty} G_{\alpha,\beta}(k) \right| > 0 \text{ for almost every } \alpha.$$

Since $|\limsup G_{\alpha,\beta}(k)| \geq \limsup |G_{\alpha,\beta}(k)|$, this will follow from

$$\limsup_{k\to\infty} |G_{\alpha,\beta}(k)| > 0 \text{ for almost every } \alpha. \tag{46}$$

Define

$$G^{(1)}_{\alpha,\beta}(k) = \left\{ \theta : \min_{(-2l_k+2)q_k+1 \leq m \leq (2l_k-1)q_k} \|m\alpha + \theta\| > |q_k\alpha - p_k| \right\},$$

$$G^{(2)}_{\alpha,\beta}(k) = \left\{ \theta : \min_{(-2l_k+2)q_k+1 \leq m \leq (2l_k-1)q_k} \|m\alpha + \theta - \beta\| > |q_k\alpha - p_k| \right\}.$$

It follows from (23) that

$$\|((m \pm q_k)\alpha + \theta) - (m\alpha + \theta)\| = |q_k\alpha - p_k|.$$

This in turn implies

$$G_{\alpha,\beta}(k) \subseteq G^{(1)}_{\alpha,\beta}(k) \cap G^{(2)}_{\alpha,\beta}(k) \tag{47}$$

On the other hand, we have

$$G^{(1)}_{\alpha,\beta}(k)^c = \bigcup_{m=(-2l_k+2)q_k+1}^{(2l_k-1)q_k} \{\theta : \|m\alpha + \theta\| \leq |q_k\alpha - p_k|\},$$

$$G^{(2)}_{\alpha,\beta}(k)^c = \bigcup_{m=(-2l_k+2)q_k+1}^{(2l_k-1)q_k} \{\theta : \|m\alpha + \theta - \beta\| \leq |q_k\alpha - p_k|\},$$

which, by (24), gives for $i = 1, 2$,

$$|G^{(i)}_{\alpha,\beta}(k)^c| \le 2q_k(4l_k - 3)|q_k\alpha - p_k| \le (8l_k - 6)\frac{q_k}{q_{k+1}} \le \frac{8l_k}{a_{k+1}}. \tag{48}$$

Combining (47) and (48), we get

$$\limsup_{k\to\infty} |G_{\alpha,\beta}(k)| \ge 1 - \liminf_{k\to\infty} \frac{16l_k}{a_{k+1}}.$$

By our assumption (45), we have that $\liminf_{k\to\infty} \frac{16l_k}{a_{k+1}}$ is less than $1/2$, say, for almost every α [56, Theorem 30]. This shows (46) and hence concludes the proof. □

Appendix B. Some Remarks on Inhomogeneous Diophantine Approximation

Let $\alpha \in (0,1)$ be irrational and let $\gamma \in [0,1)$. The *two-sided inhomogeneous approximation constant* $M(\alpha,\gamma)$ is given by

$$M(\alpha,\gamma) = \liminf_{|n|\to\infty} |n| \cdot \|n\alpha - \gamma\|,$$

where $\|\cdot\|$ denotes the distance from the closest integer. The number $M(\alpha,\gamma)$ turned out to be important in our study of the Boshernitzan condition for circle map subshifts corresponding to partitions of the unit circle into at least three intervals; compare Section 6. In this appendix we sketch a way to compute $M(\alpha,\gamma)$ which was proposed by Pinner. For background information, we refer the reader to the excellent texts by Khinchin [56] and Rockett and Szüsz [79]. We shall present results from [75]. Related work can be found in Cusick et al. [20] and Komatsu [58].

The *negative continued fraction expansion* of α is given by

$$\alpha = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots}}} =: [0; a_1, a_2, a_3, \ldots]^-,$$

where the integers $a_i \ge 2$ are generated as follows:

$$\alpha_0 := \{\alpha\},\ a_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil,\ \alpha_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil - \frac{1}{\alpha_n}.$$

The corresponding *convergents* $p_n/q_n = [0; a_1, a_2, \ldots, a_n]^-$ are given by

$$\begin{aligned} p_{-1} &= -1, \quad p_0 = 0, \quad p_{n+1} = a_{n+1}p_n - p_{n-1}, \\ q_{-1} &= 0, \quad\ \ q_0 = 1, \quad q_{n+1} = a_{n+1}q_n - q_{n-1}. \end{aligned}$$

There is a simple way to switch back and forth between regular and negative continued fraction expansion; see [75].

Write

$$\overline{\alpha}_i := [0; a_i, a_{i-1}, \ldots, a_1]^-,\ \ \alpha_i := [0; a_{i+1}, a_{i+2}, \ldots]^-.$$

Then

$$D_i := q_i\alpha - p_i = \alpha_0 \cdots \alpha_i,\ \ q_i = (\overline{\alpha}_1 \cdots \overline{\alpha}_i)^{-1}.$$

The α-*expansion* of γ is now obtained as follows: Let

$$\gamma_0 := \{\gamma\}, \quad b_{n+1} := \left\lfloor \frac{\gamma_n}{\alpha_n} \right\rfloor, \quad \gamma_{n+1} := \left\{ \frac{\gamma_n}{\alpha_n} \right\},$$

so that

$$\{\gamma\} = \sum_{i=1}^{\infty} b_i D_{i-1}.$$

Finally, with $t_k := 2b_k - a_k + 2$, let

$$d_k^- := \sum_{j=1}^{k} \frac{t_j \, q_{j-1}}{q_k}, \quad d_k^+ := \sum_{j=k+1}^{\infty} \frac{t_j \, D_{j-1}}{D_{k-1}}$$

and

$$\begin{aligned} s_1(k) &:= \tfrac{1}{4}(1 - \overline{\alpha}_k + d_k^-)(1 - \alpha_k + d_k^+) q_k D_{k-1}, \\ s_2(k) &:= \tfrac{1}{4}(1 + \overline{\alpha}_k + d_k^-)(1 + \alpha_k + d_k^+) q_k D_{k-1}, \\ s_3(k) &:= \tfrac{1}{4}|1 - \overline{\alpha}_k - d_k^-| \, |1 - \alpha_k - d_k^+| q_k D_{k-1}, \\ s_4(k) &:= \tfrac{1}{4}(1 + \overline{\alpha}_k - d_k^-)(1 + \alpha_k + d_k^+) q_k D_{k-1}. \end{aligned}$$

We have the following result [75]:

Theorem 23 (Pinner)**.** *Suppose that $\gamma \notin \mathbb{Z}\alpha + \mathbb{Z}$ and that its α-expansion has $b_i = a_i - 1$ at most finitely many times. Then*

$$M(\alpha, \gamma) = \liminf_{k \to \infty} \min\{s_1(k), s_2(k), s_3(k), s_4(k)\}.$$

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CHAPTER 9

D. Lenz, P. Stollmann, *Algebras of random operators associated to Delone dynamical systems*, Mathematical Physics, Analysis and Geometry 6 (2003), 269–290.

ALGEBRAS OF RANDOM OPERATORS ASSOCIATED TO DELONE DYNAMICAL SYSTEMS

DANIEL LENZ [2], PETER STOLLMANN [3]

Abstract. We carry out a careful study of operator algebras associated with Delone dynamical systems. A von Neumann algebra is defined using noncommutative integration theory. Features of these algebras and the operators they contain are discussed. We restrict our attention to a certain C^*-subalgebra to discuss a Shubin trace formula.

[1,2] Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany; E-mail: D.Lenz@mathematik.tu-chemnitz.de,

[1,3] Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany; E-mail: P.Stollmann@mathematik.tu-chemnitz.de

Introduction

The present paper is part of a study of Hamiltonians for aperiodic solids. Among them, special emphasis is laid on models for quasicrystals. To describe aperiodic order, we use Delone (Delaunay) sets. Here we construct and study certain operator algebras which can be naturally associated with Delone sets and reflect the aperiodic order present in a Delone dynamical system. In particular, we use Connes noncommutative integration theory to build a von Neumann algebra. This is achieved in Section 2 after some preparatory definitions and results gathered in Section 1. Let us stress the following facts: it is not too hard to write down explicitely the von Neumann algebra $\mathcal{N}(\Omega, T, \mu)$ of observables, starting from a Delone dynamical system (Ω, T) with an invariant measure μ. As in the case of random operators, the observables are families of operators, indexed by a set Ω of Delone sets. This set represents a type of (aperiodic) order and the ergodic properties of (Ω, T) can often be expressed by combinatorial properties of its elements ω. The latter are thought of as realizations of the type of disorder described by (Ω, T). The algebra $\mathcal{N}(\Omega, T, \mu)$ incorporates this disorder and playes the role of a noncommutative space underlying the algebra of observables. To see that this algebra is in fact a von Neumann algebra is by no means clear. At that point the analysis of Connes [9] enters the picture.

In order to verify the necessary regularity properties we rely on work done in [29], where we studied topological properties of a groupoid that naturally comes with (Ω, T). Using this, we can construct a measurable (even topological) groupoid. Any invariant measure μ on the dynamical system gives rise to a transversal measure Λ and the points of the Delone sets are used to define a random Hilbert space $\mathcal{H}$. This latter step uses specifically the fact that we are dealing with a dynamical system

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consisting of point sets and leads to a noncommutative random variable that has no analogue in the general framework of dynamical systems. We are then able to identify $\mathcal{N}(\Omega, T, \mu)$ as $End_\Lambda(\mathcal{H})$. While in our approach we use noncommutative integration theory to verify that a certain algebra is a von Neumann algebra we should also like to point out that at the same time we provide interesting examples for the theory. Of course, tilings have been considered in this connection quite from the start as seen on the cover of [10]. However, we emphasize the point of view of concrete operators and thus are led to a somewhat different setup.

The study of traces on this algebra is started in Section 3. Traces are intimately linked to transversal functions on the groupoid. These can also be used to study certain spectral properties of the operator families constituting the von Neumann algebra. For instance, spectral properties are almost surely constant for the members of any such family. This type of results is typical for random operators. In fact, we regard the families studied here in this random context. An additional feature that is met here is the dependence of the Hilbert space on the random parameter $\omega \in \Omega$.

In Section 4 we introduce a C^*-algebra that had already been encountered in a different form in [6, 17]. Our presentation here is geared towards using the elements of the C^*-algebra as tight binding hamiltonians in a quantum mechanical description of disordered solids (see [6] for related material as well). We relate certain spectral properties of the members of such operator families to ergodic features of the underlying dynamical system. Moreover, we show that the eigenvalue counting functions of these operators are convergent. The limit, known as the integrated density of states, is an object of fundamental importance from the solid state physics point of view. Apart from proving its existence, we also relate it to the canonical trace on the von Neumann algebra $\mathcal{N}(\Omega, T, \mu)$ in case that the Delone dynamical system (Ω, T) is uniquely ergodic. Results of this genre are known as Shubin's trace formula due to the celebrated results from [36].

We conclude this section with two further remarks.

Firstly, let us mention that starting with the work of Kellendonk [17], C^*-algebras associated to tilings have been subject to intense research within the framework of K-theory (see e.g. [18, 19, 32]). This can be seen as part of a program originally initiated by Bellissard and his co-workers in the study of so called gap-labelling for almost periodic operators [3, 4, 5]. While the C^*-algebras we encounter are essentially the same, our motivation, aims and results are quite different.

Secondly, let us remark that some of the results below have been announced in [28, 29]. A stronger ergodic theorem will be found in [30] and a spectral theoretic application is given in [20].

1. Delone dynamical systems and coloured Delone dynamical systems

In this section we recall standard concepts from the theory of Delone sets and introduce a suitable topology on the closed sets in euclidian space. A slight extension concerns the discussion of *coloured (decorated) Delone sets*.

A subset ω of $\mathbb{R}^d$ is called a *Delone set* if there exist $0 < r, R < \infty$ such that $2r \leq \|x - y\|$ whenever $x, y \in \omega$ with $x \neq y$, and $B_R(x) \cap \omega \neq \emptyset$ for all $x \in \mathbb{R}^d$. Here, the Euclidean norm on $\mathbb{R}^d$ is denoted by $\| \cdot \|$ and $B_s(x)$ denotes the (closed) ball in $\mathbb{R}^d$ around x with radius s. The set ω is then also called an (r, R)-set. We will sometimes be interested in the restrictions of Delone sets to bounded sets. In order to treat these restrictions, we introduce the following definition.

Definition 1.1. (a) *A pair* (Λ, Q) *consisting of a bounded subset* Q *of* $\mathbb{R}^d$ *and* $\Lambda \subset Q$ *finite is called a* pattern. *The set* Q *is called the* support of the pattern.
(b) *A pattern* (Λ, Q) *is called a* ball pattern *if* $Q = B_s(x)$ *with* $x \in \Lambda$ *for suitable* $x \in \mathbb{R}^d$ *and* $s \in (0, \infty)$.

The pattern (Λ_1, Q_1) is contained in the pattern (Λ_2, Q_2) written as $(\Lambda_1, Q_1) \subset (\Lambda_2, Q_2)$ if $Q_1 \subset Q_2$ and $\Lambda_1 = Q_1 \cap \Lambda_2$. Diameter, volume etc. of a pattern are defined to be the diameter, volume etc of its support. For patterns $X_1 = (\Lambda_1, Q_1)$ and $X_2 = (\Lambda_2, Q_2)$, we define $\sharp_{X_1} X_2$, *the number of occurences of* X_1 *in* X_2, to be the number of elements in $\{t \in \mathbb{R}^d : \Lambda_1 + t \subset \Lambda_2, Q_1 + t \subset Q_2\}$.

For further investigation we will have to identify patterns that are equal up to translation. Thus, on the set of patterns we introduce an equivalence relation by setting $(\Lambda_1, Q_1) \sim (\Lambda_2, Q_2)$ if and only if there exists a $t \in \mathbb{R}^d$ with $\Lambda_1 = \Lambda_2 + t$ and $Q_1 = Q_2 + t$. In this latter case we write $(\Lambda_1, Q_1) = (\Lambda_2, Q_2) + t$. The class of a pattern (Λ, Q) is denoted by $[(\Lambda, Q)]$. The notions of diameter, volume, occurence etc. can easily be carried over from patterns to pattern classes.

Every Delone set ω gives rise to a set of pattern classes, $\mathcal{P}(\omega)$ viz $\mathcal{P}(\omega) = \{[Q \wedge \omega] : Q \subset \mathbb{R}^d \text{ bounded and measurable}\}$, and to a set of ball pattern classes $\mathcal{P}_B(\omega)) = \{[B_s(x) \wedge \omega] : x \in \omega, s > 0\}$. Here we set $Q \wedge \omega = (\omega \cap Q, Q)$.

For $s \in (0, \infty)$, we denote by $\mathcal{P}_B^s(\omega)$ the set of ball patterns with radius s; note the relation with s-patches as considered in [21]. A Delone set is said to be of *finite local complexity* if for every radius s the set $\mathcal{P}_B^s(\omega)$ is finite. We refer the reader to [21] for a detailed discussion of Delone sets of finite type.

Let us now extend this framework a little, allowing for coloured Delone sets. The alphabet $\mathbb{A}$ is the set of possible *colours* or *decorations*. An $\mathbb{A}$-*coloured Delone set* is a subset $\omega \subset \mathbb{R}^d \times \mathbb{A}$ such that the projection $pr_1(\omega) \subset \mathbb{R}^d$ onto the first coordinate is a Delone set. The set of all $\mathbb{A}$-coloured Delone sets is denoted by $\mathcal{D}_{\mathbb{A}}$.

Of course, we speak of an (r, R)-*set* if $pr_1(\omega)$ is an (r, R)-set. The notions of *pattern, diameter, volume* of pattern etc. easily extend to coloured Delone sets. E.g.

Definition 1.2. *A pair* (Λ, Q) *consisting of a bounded subset* Q *of* $\mathbb{R}^d$ *and* $\Lambda \subset Q \times \mathbb{A}$ *finite is called an* $\mathbb{A}$- decorated pattern. *The set* Q *is called the* support of the pattern.

A coloured Delone set ω is thus viewed as a Delone set $pr_1(\omega)$ whose points $x \in pr_1(\omega)$ are labelled by colours $a \in \mathbb{A}$. Accordingly, the translate $T_t\omega$ of a coloured Delone set $\omega \subset \mathbb{R}^d \times \mathbb{A}$ is given by

$$T_t\omega = \{(x + t, a) : (x, a) \in \omega\}.$$

From [29] we infer the notion of the *natural topology*, defined on the set $\mathcal{F}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$. Since in our subsequent study in [30] the alphabet is supposed to be a finite set, the following construction will provide a suitable topology for coloured Delone sets. Define, for $a \in \mathbb{A}$,

$$p_a : \mathcal{D}_{\mathbb{A}} \to \mathcal{F}(\mathbb{R}^d), p_a(\omega) = \{x \in \mathbb{R}^d : (x, a) \in \omega\}.$$

The initial topolgy on $\mathcal{D}_{\mathbb{A}}$ with respect to the family $(p_a)_{a \in \mathbb{A}}$ is called the *natural topology* on the set of $\mathbb{A}$- decorated Delone sets. It is obvious that metrizability and compactness properties carry over from the natural topology without decorations to the decorated case.

Finally, the notions of Delone dynamical system and Delone dynamical system of finite local complexity carry over to the coloured case in the obvious manner.

Definition 1.3. *Let $\mathbb{A}$ be a finite set.* (a) *Let Ω be a set of Delone sets. The pair (Ω, T) is called a* Delone dynamical system *(DDS) if Ω is invariant under the shift T and closed in the natural topology.*
(a') *Let Ω be a set of $\mathbb{A}$-coloured Delone sets. The pair (Ω, T) is called an* $\mathbb{A}$-coloured Delone dynamical system *($\mathbb{A}$-DDS) if Ω is invariant under the shift T and closed in the natural topology.*
(b) *A DDS (Ω, T) is said to be of* finite local complexity *if $\cup_{\omega\in\Omega} P_B^s(\omega)$ is finite for every $s > 0$.*
(b') *An $\mathbb{A}$-DDS (Ω, T) is said to be of* finite local complexity *if $\cup_{\omega\in\Omega} P_B^s(\omega)$ is finite for every $s > 0$.*
(c) *Let $0 < r, R < \infty$ be given. A DDS (Ω, T) is said to be an* (r, R)-system *if every $\omega \in \Omega$ is an (r, R)-set.*
(c') *Let $0 < r, R < \infty$ be given. An $\mathbb{A}$-DDS (Ω, T) is said to be an* (r, R)-system *if every $\omega \in \Omega$ is an (r, R)-set.*
(d) *The set $\mathcal{P}(\Omega)$ of* pattern classes associated to a DDS Ω *is defined by $\mathcal{P}(\Omega) = \cup_{\omega\in\Omega}\mathcal{P}(\omega)$.*

In view of the compactness properties known for Delone sets, [29], we get that Ω is compact whenever (Ω, T) is a DDS or an $\mathbb{A}$-DDS.

2. Groupoids and non commutative random variables

In this section we use concepts from Connes non-commutative integration theory [9] to associate a natural von Neumann algebra with a given DDS (Ω, T). To do so, we introduce

- a suitable *groupoid* $\mathcal{G}(\Omega, T)$,
- a *transversal measure* $\Lambda = \Lambda_\mu$ for a given invariant measure μ on (Ω, T)
- and a Λ-*random Hilbert space* $\mathcal{H} = (\mathcal{H}_\omega)_{\omega\in\Omega}$

leading to the von Neumann algebra

$$\mathcal{N}(\Omega, T, \mu) := \mathrm{End}_\Lambda(\mathcal{H})$$

of *random operators*, all in the terminology of [9]. Of course, all these objects will now be properly defined and some crucial properties have to be checked. Part of the topological prerequisites have already been worked out in [29]. Note that comparing the latter with the present paper, we put more emphasis on the relation with noncommutative integration theory.

The definition of the groupoid structure is straightforward see also [6], Sect. 2.5. A set $\mathcal{G}$ together with a partially defined associative multiplication $\cdot : \mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$, and an inversion ${}^{-1} : \mathcal{G} \longrightarrow \mathcal{G}$ is called a groupoid if the following holds:

- $(g^{-1})^{-1} = g$ for all $g \in \mathcal{G}$,
- If $g_1 \cdot g_2$ and $g_2 \cdot g_3$ exist, then $g_1 \cdot g_2 \cdot g_3$ exists as well,
- $g^{-1} \cdot g$ exists always and $g^{-1} \cdot g \cdot h = h$, whenever $g \cdot h$ exists,
- $h \cdot h^{-1}$ exists always and $g \cdot h \cdot h^{-1} = g$, whenever $g \cdot h$ exists.

A groupoid is called topological groupoid if it carries a topology making inversion and multiplication continuous. Here, of course, $\mathcal{G} \times \mathcal{G}$ carries the product topology and $\mathcal{G}^2 \subset \mathcal{G} \times \mathcal{G}$ is equipped with the induced topology.

A given groupoid $\mathcal{G}$ gives rise to some standard objects: The subset $\mathcal{G}^0 = \{g \cdot g^{-1} \mid g \in \mathcal{G}\}$ is called the set of *units*. For $g \in \mathcal{G}$ we define its *range* $r(g)$ by $r(g) = g \cdot g^{-1}$ and its *source* by $s(g) = g^{-1} \cdot g$. Moreover, we set $\mathcal{G}^\omega = r^{-1}(\{\omega\})$ for any unit $\omega \in \mathcal{G}^0$. One easily checks that $g \cdot h$ exists if and only if $r(h) = s(g)$.

By a standard construction we can assign a groupoid $\mathcal{G}(\Omega, T)$ to a Delone dynamical system. As a set $\mathcal{G}(\Omega, T)$ is just $\Omega \times \mathbb{R}^d$. The multiplication is given by $(\omega, x)(\omega - x, y) = (\omega, x + y)$ and the inversion is given by $(\omega, x)^{-1} = (\omega - x, -x)$. The groupoid operations can be visualized by considering an element (ω, x) as an arrow $\omega - x \xrightarrow{x} \omega$. Multiplication then corresponds to concatenation of arrows; inversion corresponds to reversing arrows and the set of units $\mathcal{G}(\Omega, T)^0$ can be identified with Ω.

Apparently this groupoid $\mathcal{G}(\Omega, T)$ is a topological groupoid when Ω is equipped with the topology of the previous section and $\mathbb{R}^d$ carries the usual topology.

The groupoid $\mathcal{G}(\Omega, T)$ acts naturally on a certain topological space $\mathcal{X}$. This space and the action of $\mathcal{G}$ on it are of crucial importance in the sequel. The space $\mathcal{X}$ is given by

$$\mathcal{X} = \{(\omega, x) \in \mathcal{G} : x \in \omega\} \subset \mathcal{G}(\Omega, T).$$

In particular, it inherits a topology form $\mathcal{G}(\Omega, T)$. This $\mathcal{X}$ can be used to define a *random variable* or *measurable functor* in the sense of [9]. Following the latter reference, p. 50f, this means that we are given a functor F from $\mathcal{G}$ to the category of measurable spaces with the following properties:

- For every $\omega \in \mathcal{G}^0$ we are given a measure space $F(\omega) = (\mathcal{Y}^\omega, \beta^\omega)$.
- For every $g \in \mathcal{G}$ we have an isomorphism $F(g)$ of measure spaces, $F(g) : \mathcal{Y}^{s(g)} \to \mathcal{Y}^{r(g)}$ such that $F(g_1 g_2) = F(g_1) F(g_2)$, whenever $g_1 g_2$ is defined, i.e., whenever $s(g_1) = r(g_2)$.
- A measurable structure on the disjoint union

  $$\mathcal{Y} = \cup_{\omega \in \Omega} \mathcal{Y}^\omega$$

  such that the projection $\pi : \mathcal{Y} \to \Omega$ is measurable as well as the natural bijection of $\pi^{-1}(\omega)$ to $\mathcal{Y}^\omega$.
- The mapping $\omega \mapsto \beta^\omega$ is measurable.

We will use the notation $F : \mathcal{G} \rightsquigarrow \mathcal{Y}$ to abbreviate the above.

Let us now turn to the groupoid $\mathcal{G}(\Omega, T)$ and the bundle $\mathcal{X}$ defined above. Since $\mathcal{X}$ is closed ([29], Prop.2.1), it carries a reasonable Borel structure. The projection $\pi : \mathcal{X} \to \Omega$ is continuous, in particular measurable. Now, we can discuss the action of $\mathcal{G}$ on $\mathcal{X}$. Every $g = (\omega, x)$ gives rise to a map $J(g) : \mathcal{X}^{s(g)} \longrightarrow \mathcal{X}^{r(g)}$, $J(g)(\omega - x, p) = (\omega, p + x)$. A simple calculation shows that $J(g_1 g_2) = J(g_1) J(g_2)$ and $J(g^{-1}) = J(g)^{-1}$, whenever $s(g_1) = r(g_2)$. Thus, $\mathcal{X}$ is an $\mathcal{G}$-space in the sense of [27]. It can be used as the target space of a measurable functor $F : \mathcal{G} \rightsquigarrow \mathcal{X}$. What we still need is a *positive random variable* in the sense of the following definition, taken from [29]. First some notation:

Given a locally compact space Z, we denote the set of continuous functions on Z with compact support by $C_c(Z)$. The support of a function in $C_c(Z)$ is denoted by supp(f). The topology gives rise to the Borel-σ-algebra. The measurable non-negative functions with respect to this σ-algebra will be denoted by $\mathcal{F}^+(Z)$. The measures on Z will be denoted by $\mathcal{M}(Z)$.

Definition 2.1. *Let* (Ω, T) *be an* (r, R)*-system.*

(a) *A choice of measures* $\beta : \Omega \to \mathcal{M}(\mathcal{X})$ *is called* a positive random variable with values in $\mathcal{X}$ *if the map* $\omega \mapsto \beta^\omega(f)$ *is measurable for every* $f \in \mathcal{F}^+(\mathcal{X})$, β^ω *is supported on* $\mathcal{X}^\omega$, *i.e.,* $\beta^\omega(\mathcal{X} - \mathcal{X}^\omega) = 0$, $\omega \in \Omega$, *and* β *satisfies the following invariance condition*

$$\int_{\mathcal{X}^{s(g)}} f(J(g)p)d\beta^{s(g)}(p) = \int_{\mathcal{X}^{r(g)}} f(q)d\beta^{r(g)}(q)$$

for all $g \in \mathcal{G}$ *and* $f \in \mathcal{F}^+(\mathcal{X}^{r(g)})$.

(b) *A map* $\Omega \times C_c(\mathcal{X}) \longrightarrow \mathbb{C}$ *is called a* complex random variable *if there exist an* $n \in \mathbb{N}$, *positive random variables* β_i, $i = 1, \ldots, n$ *and* $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, n$ *with* $\beta^\omega(f) = \sum_{i=1}^n \beta_i^\omega(f)$.

We are now heading towards introducing and studying a special random variable. This variable is quite important as it gives rise to the ℓ^2-spaces on which the Hamiltonians act. Later we will see that these Hamiltonians also induce random variables.

Proposition 2.2. *Let* (Ω, T) *be an* (r, R)*-system. Then the map* $\alpha : \Omega \longrightarrow \mathcal{M}(\mathcal{X})$, $\alpha^\omega(f) = \sum_{p\in\omega} f(p)$ *is a random variable with values in* $\mathcal{X}$. *Thus the functor* F_α *given by* $F_\alpha(\omega) = (\mathcal{X}^\omega, \alpha^\omega)$ *and* $F_\alpha(g) = J(g)$ *is measurable.*

Proof. See [29], Corollary 2.6. □

Clearly, the condition that (Ω, T) is an (r, R)-system is used to verify the measurability conditions needed for a random variable. We should like to stress the fact that the above functor given by $\mathcal{X}$ and $\alpha^\bullet$ differs from the canonical choice, possible for any dynamical system. In the special case at hand this canonical choice reads as follows:

Proposition 2.3. *Let* (Ω, T) *be a DDS. Then the map* $\nu : \Omega \longrightarrow \mathcal{M}(\mathcal{G})$, $\nu^\omega(f) = \int_{\mathbb{R}^d} f(\omega, t)dt$ *is a* transversal function, *i.e., a random variable with values in* $\mathcal{G}$.

Actually, one should possibly define transversal functions before introducing random variables. Our choice to do otherwise is to underline the specific functor used in our discussion of Delone sets. As already mentioned above, the analogue of the transversal function ν from Proposition 2.3 can be defined for any dynamical system. In fact this structure has been considered by Bellissard and coworkers in a C^*-context. The notion *almost random operators* has been coined for that; see [3] and the literature quoted there.

After having encountered functors from $\mathcal{G}$ to the category of measurable spaces under the header random variable or measurable functor, we will now meet *random Hilbert spaces*. By that one designates, according to [9], a *representation* of $\mathcal{G}$ in the category of Hilbert spaces, given by the following data:

- A measurable family $\mathcal{H} = (\mathcal{H}_\omega)_{\omega \in \mathcal{G}^0}$ of Hilbert spaces.
- For every $g \in \mathcal{G}$ a unitary $U_g : \mathcal{H}_{s(g)} \to \mathcal{H}_{r(g)}$ such that

$$U(g_1 g_2) = U(g_1)U(g_2)$$

whenever $s(g_1) = r(g_2)$. Moreover, we assume that for every pair (ξ, η) of measurable sections of $\mathcal{H}$ the function

$$\mathcal{G} \to \mathbb{C}, g \mapsto (\xi|\eta)(g) := (\xi_{r(g)}|U(g)\eta_{s(g)})$$

is measurable.

Given a measurable functor $F : \mathcal{G} \rightsquigarrow \mathcal{Y}$ there is a natural representation $L^2 \circ F$, where

$$\mathcal{H}_\omega = L^2(\mathcal{Y}^\omega, \beta^\omega)$$

and $U(g)$ is induced by the isomorphism $F(g)$ of measure spaces.

Let us assume that (Ω, T) is an (r, R)-system. We are especially interested in the representation of $\mathcal{G}(\Omega, T)$ on $\mathcal{H} = (\ell^2(\mathcal{X}^\omega, \alpha^\omega))_{\omega\in\Omega}$ induced by the measurable functor $F_\alpha : \mathcal{G}(\Omega, T) \rightsquigarrow \mathcal{X}$ defined above. The necessary measurable structure is provided by [29], Proposition 2.8. It is the measurable structure generated by $C_c(\mathcal{X})$.

The last item we have to define is a *transversal measure*. We denote the set of nonnegative transversal functions on a groupoid $\mathcal{G}$ by $\mathcal{E}^+(\mathcal{G})$ and consider the unimodular case ($\delta \equiv 1$) only. Following [9], p. 41f, a *transversal measure* Λ is a linear mapping

$$\Lambda : \mathcal{E}^+(\mathcal{G}) \to [0, \infty]$$

satisfying

- Λ is *normal*, i.e., $\Lambda(\sup \nu_n) = \sup \Lambda(\nu_n)$ for every increasing sequence (ν_n) in $\mathcal{E}^+(\mathcal{G})$.
- Λ is *invariant*, i.e., for every $\nu \in \mathcal{E}^+(\mathcal{G})$ and every kernel λ with $\lambda^\omega(1) = 1$ we get

$$\Lambda(\nu * \lambda) = \Lambda(\nu).$$

Given a fixed transversal function ν on $\mathcal{G}$ and an invariant measure μ on $\mathcal{G}^0$ there is a unique transversal measure $\Lambda = \Lambda_\nu$ such that

$$\Lambda(\nu * \lambda) = \mu(\lambda^\bullet(1)),$$

see [9], Theoreme 3, p.43. In the next Section we will discuss that in a little more detail in the case of DDS groupoids.

We can now put these constructions together.

Definition 2.4. *Let (Ω, T) be an (r, R)-system and let μ be an invariant measure on Ω. Denote by $\mathcal{V}_1$ the set of all $f : \mathcal{X} \longrightarrow \mathbb{C}$ which are measurable and satisfy $f(\omega, \cdot) \in \ell^2(\mathcal{X}^\omega, \alpha^\omega)$ for every $\omega \in \Omega$.*

A family $(A_\omega)_{\omega\in\Omega}$ of bounded operators $A_\omega : \ell^2(\omega, \alpha^\omega) \longrightarrow \ell^2(\omega, \alpha^\omega)$ is called measurable *if $\omega \mapsto \langle f(\omega), (A_\omega g)(\omega)\rangle_\omega$ is* measurable *for all $f, g \in \mathcal{V}_1$. It is called* bounded *if the norms of the A_ω are uniformly bounded. It is called* covariant *if it satisfies the covariance condition*

$$H_{\omega+t} = U_t H_\omega U_t^*, \ \omega \in \Omega, t \in \mathbb{R}^d,$$

where $U_t : \ell^2(\omega) \longrightarrow \ell^2(\omega + t)$ is the unitary operator induced by translation. Now, we can define

$$\mathcal{N}(\Omega, T, \mu) := \{A = (A_\omega)_{\omega\in\Omega} | A \text{ covariant, measurable and bounded}\}/\sim,$$

where $\sim$ means that we identify families which agree μ almost everywhere.

As is clear from the definition, the elements of $\mathcal{N}(\Omega, T, \mu)$ are classes of families of operators. However, we will not distinguish too pedantically between classes and their representatives in the sequel.

Remark 2.5. It is possible to define $\mathcal{N}(\Omega, T, \mu)$ by requiring seemingly weaker conditions. Namely, one can consider families (A_ω) that are essentially bounded and satisfy the covariance condition almost everywhere. However, by standard procedures (see [9, 25]), it is possible to show that each of these families agrees almost everywhere with a family satisfying the stronger conditions discussed above.

Obviously, $\mathcal{N}(\Omega, T, \mu)$ depends on the measure class of μ only. Hence, for uniquely ergodic (Ω, T), $\mathcal{N}(\Omega, T, \mu) =: \mathcal{N}(\Omega, T)$ gives a canonical algebra. This case has been considered in [28, 29].

Apparently, $\mathcal{N}(\Omega, T, \mu)$ is an involutive algebra under the obvious operations. Moreover, it can be related to the algebra $\mathrm{End}_\Lambda(\mathcal{H})$ defined in [9] as follows.

Theorem 2.6. *Let (Ω, T) be an (r, R)-system and let μ be an invariant measure on Ω. Then $\mathcal{N}(\Omega, T, \mu)$ is a weak-*-algebra. More precisely,*

$$\mathcal{N}(\Omega, T, \mu) = \mathrm{End}_\Lambda(\mathcal{H}),$$

where $\Lambda = \Lambda_\nu$ and $\mathcal{H} = (\ell^2(\mathcal{X}^\omega, \alpha^\omega))_{\omega\in\Omega}$ are defined as above.

Proof. The asserted equation follows by plugging in the respective definitions. The only thing that remains to be checked is that $\mathcal{H}$ is a square integrable representation in the sense of [9], Definition, p. 80. In order to see this it suffices to show that the functor F_α giving rise to $\mathcal{H}$ is proper. See [9], Proposition 12, p. 81.

This in turn follows by considering the transversal function ν defined in Proposition 2.3 above. In fact, any $u \in C_c(\mathbb{R}^d)^+$ gives rise to the function $f \in \mathcal{F}^+(\mathcal{X})$ by $f(\omega, p) := u(p)$. It follows that

$$(\nu * f)(\omega, p) = \int_{\mathbb{R}^d} u(p + t)dt = \int_{\mathbb{R}^d} u(t)dt,$$

so that $\nu * f \equiv 1$ if the latter integral equals 1 as required by [9], Definition 3, p. 55. ☐

We can use the measurable structure to identify $L^2(\mathcal{X}, m)$, where $m = \int_\Omega \alpha^\omega \mu(\omega)$ with $\int_\Omega^\oplus \ell^2(\mathcal{X}^\omega, \alpha^\omega)\, d\mu(\omega)$. This gives the faithful representation

$$\pi : \mathcal{N}(\Omega, T, \mu) \longrightarrow B(L^2(\mathcal{X}, m)), \pi(A)f((\omega, x)) = (A_\omega f_\omega)((\omega, x))$$

and the following immediate consequence.

Corollary 2.7. *$\pi(\mathcal{N}(\Omega, T, \mu)) \subset B(L^2(\mathcal{X}, m))$ is a von Neumann algebra.*

Next we want to identify conditions under which $\pi(\mathcal{N}(\Omega, T, \mu))$ is a factor. Recall that a Delone set ω is said to be *non-periodic* if $\omega + t = \omega$ implies that $t = 0$.

Theorem 2.8. *Let (Ω, T) be an (r, R)-system and let μ be an ergodic invariant measure on Ω. If ω is non-periodic for μ-a.e. $\omega \in \Omega$ then $\mathcal{N}(\Omega, T, \mu)$ is a factor.*

Proof. We want to use [9], Corollaire 7, p. 90. In our case $\mathcal{G} = \mathcal{G}(\Omega, T)$, $\mathcal{G}^0 = \Omega$ and

$$\mathcal{G}_\omega^\omega = \{(\omega, t) : \omega + t = \omega\}.$$

Obviously, the latter is trivial, i.e., equals $\{(\omega, 0)\}$ iff ω is non-periodic. By our assumption this is valid μ-a.s. so that we can apply [9], Corollaire 7, p. 90. Therefore the centre of $\mathcal{N}(\Omega, T, \mu)$ consists of families

$$f = (f(\omega)1_{\mathcal{H}_\omega})_{\omega\in\Omega},$$

where $f : \Omega \to \mathbb{C}$ is bounded, measurable and invariant. Since μ is assumed to be ergodic this implies that $f(\omega)$ is a.s. constant so that the centre of $\mathcal{N}(\Omega, T, \mu)$ is trivial. □

Remark 2.9. Since μ is ergodic, the assumption of non-periodicity in the theorem can be replaced by assuming that there is a set of positive measure consisting of non-periodic ω.

Note that the latter result gives an extension of part of what has been announced in [28], Theorem 2.1 and [29], Theorem 3.8. The remaining assertions of [29] will be proven in the following Section, again in greater generality.

3. Transversal functions, traces and deterministic spectral properties.

In the preceding section we have defined the von Neumann algebra $\mathcal{N}(\Omega, T, \mu)$ starting from an (r, R)-system (Ω, T) and an invariant measure μ on (Ω, T). In the present section we will study traces on this algebra. Interestingly, this rather abstract and algebraic enterprise will lead to interesting spectral consequences. We will see that the operators involved share some fundamental properties with "usual random operators".

Let us first draw the connection of our families to "usual random operators", referring to [7, 31, 39] for a systematic account. Generally speaking one is concerned with families $(A_\omega)_{\omega\in\Omega}$ of operators indexed by some probability space and acting on $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$ typically. The probability space Ω encodes some statistical properties, a certain kind of disorder that is inspired by physics in many situations. One can view the set Ω as the set of all possible realization of a fixed disordered model and each single ω as a possible realization of the disorder described by Ω. Of course, the information is mostly encoded in a measure on Ω that describes the probability with which a certain realization is picked.

We are faced with a similar situation, one difference being that in any family $A = (A_\omega)_{\omega\in\Omega} \in \mathcal{N}(\Omega, T, \mu)$, the operators A_ω act on the possibly different spaces $\ell^2(\omega)$. Apart from that we have the same ingredients as in the usual random business, where, of course, Delone dynamical systems still bear quite some order. That is, we are in the realm of weakly disordered systems. For a first idea what this might have to do with aperiodically ordered solids, quasicrystals, assume that the points $p \in \omega$ are the atomic positions of a quasicrystal. In a tight binding approach (see [6] Section 4 for why this is reasonable), the Hamiltonian H_ω describing the respective solid would naturally be defined on $\ell^2(\omega)$, its matrix elements $H_\omega(p, q), p, q \in \omega$ describing the diagonal and hopping terms for an electron that undergoes the influence of the atomic constellation given by ω. The definite choice of these matrix elements has to be done on physical grounds. In the following subsection we will propose a C^*-subalgebra that contains what we consider the most reasonable candidates; see also [6, 17]. It is clear, however, that $\mathcal{N}(\Omega, T, \mu)$ is a reasonable framework, since translations should not matter. Put in other words, every reasonable Hamiltonian family $(H_\omega)_{\omega\in\Omega}$ should be covariant.

The remarkable property that follows from this "algebraic" fact is that certain spectral properties of the H_ω are *deterministic*, i.e., do not depend on the choice of the realization ω μ-a.s.

Let us next introduce the necessary algebraic concepts, taking a second look at transversal functions and random variables with values in $\mathcal{X}$. In fact, random variables can be integrated with respect to transversal measures by [9], i.e for a given non-negative random variable β with values in $\mathcal{X}$ and a transversal measure Λ, the expression $\int F_\beta d\Lambda$ is well defined. More precisely, the following holds:

Lemma 3.1. *Let (Ω, T) be an (r,R)-system and μ be T-invariant.*

(a) *Let β be a nonnegative random variable with values in $\mathcal{X}$. Then $\int_\Omega \beta^\omega(f(\omega,\cdot))\,d\mu(\omega)$ does not depend on $f \in \mathcal{F}^+(\mathcal{X})$ provided f satisfies $\int f((\omega + t, x+t)\,dt = 1$ for every $(\omega, x) \in \mathcal{X}$ and*

$$\int_\Omega \beta^\omega(f(\omega,\cdot))\,d\mu(\omega) = \int F_\beta d\Lambda,$$

where $F_\beta : \mathcal{G} \rightsquigarrow \mathcal{X}$ is the measurable functor induced by $F_\beta(\omega) = (\mathcal{X}^\omega, \beta^\omega)$ and $\Lambda = \Lambda_\nu$ the transversal measure defined in the previous section.

(b) *An analogous statement remains true for a complex random variable $\beta = \sum_k \lambda_k \beta_k$, when we define*

$$\int F_\beta d\Lambda = \sum_k \lambda_k \int F_{\beta_k} d\Lambda$$

and restrict to $f \in \mathcal{F}^+(\mathcal{X})$ with $\mathrm{supp} f$ *compact.*

Proof. Part (a) is a direct consequence of the definitions and results in [9]. Part (b), then easily follows from (a) by linearity. □

A special instance of the foregoing lemma is given in the following proposition.

Proposition 3.2. *Let (Ω, T) be an (r,R)-system and let μ be T-invariant. If λ is a transversal function on $G(\Omega, T)$ then*

$$\varphi \mapsto \int_\Omega \langle \lambda^\omega, \varphi \rangle d\mu(\omega)$$

defines an invariant functional on $C_c(\mathbb{R}^d)$, i.e., a multiple of the Lebesgue measure. In particular, if μ is an ergodic measure, then either $\lambda^\omega(1) = 0$ a.s. or $\lambda^\omega(1) = \infty$ a.s.

Proof. Invariance of the functional follows by direct checking. By uniqueness of the Haar measure, this functional must then be a multiple of Lebesgue measure. If μ is ergodic, the map $\omega \mapsto \lambda^\omega(1)$ is almost surely constant (as it is obviously invariant). This easily implies the last statement. □

Each random operator gives rise to a random variable as seen in the following proposition whose simple proof we omit.

Proposition 3.3. *Let (Ω, T) be an (r,R)-system and μ be T-invariant. Let $(A_\omega) \in \mathcal{N}(\Omega, T, \mu)$ be given. Then the map $\beta_A : \Omega \longrightarrow \mathcal{M}(\mathcal{X})$, $\beta_A^\omega(f) = \mathrm{tr}(A_\omega M_f)$ is a complex random variable with values in $\mathcal{X}$.*

Now, choose a nonnegative measurable u on $\mathbb{R}^d$ with compact support and $\int_{\mathbb{R}^d} u(x)dx = 1$. Combining the previous proposition with Lemma 3.1, $f(\omega, p) := u(p)$, we infer that the map

$$\tau : \mathcal{N}(\Omega, T, \mu) \longrightarrow \mathbb{C}, \;\; \tau(A) = \int_\Omega \operatorname{tr}(A_\omega M_u)\, d\mu(\omega)$$

does not depend on the choice of f viz u as long as the integral is one. Important features of τ are given in the following lemma.

Lemma 3.4. *Let (Ω, T) be an (r, R)-system and μ be T-invariant. Then the map $\tau : \mathcal{N}(\Omega, T, \mu) \longrightarrow \mathbb{C}$ is continuous, faithful, nonegative on $\mathcal{N}(\Omega, T, \mu)^+$ and satisfies $\tau(A) = \tau(U^*AU)$ for every unitary $U \in \mathcal{N}(\Omega, T, \mu)$ and arbitrary $A \in \mathcal{N}(\Omega, T, \mu)$, i.e., τ is a trace.*

We include the elementary proof, stressing the fact that we needn't rely on the noncommutative framework; see also [27] for the respective statement in a different setting.

Proof. Choosing a continuous u with compact support we see that $|\tau(A) - \tau(B)| \leq \int \|A_\omega - B_\omega\| \operatorname{tr} M_u d\mu(\omega) \leq \|A - B\| C$, where $C > 0$ only depends on u and Ω. On the other hand, choosing u with arbitrary large support we easily infer that τ is faithful. It remains to show the last statement.

According to [12], I.6.1, Cor.1 it suffices to show $\tau(K^*K) = \tau(KK^*)$ for every $K = (K_\omega)_{\omega\in\Omega} \in \mathcal{N}(\Omega, T, \mu)$. We write $k_\omega(p, q) := (K_\omega \delta_q | \delta_p)$ for the associated kernel and calculate

$$\begin{aligned}
\tau(K^*K) &= \int_\Omega \operatorname{tr}(K_\omega^* K_\omega M_u) d\mu(\omega) \\
&= \int_\Omega \operatorname{tr}(M_{u^{\frac{1}{2}}} K_\omega^* K_\omega M_{u^{\frac{1}{2}}}) d\mu(\omega) \\
&= \int_\Omega \sum_{m\in\omega} \|K_\omega M_{u^{\frac{1}{2}}} \delta_m\|^2 \mu(\omega) \\
&= \int_\Omega \sum_{l,m\in\omega} |k_\omega(l, m)|^2 u(m) \int_{\mathbb{R}^d} u(l - t) dt d\mu(\omega)
\end{aligned}$$

where we used that $\int_{\mathbb{R}^d} u(l - t)dt = 1$ for all $l \in \omega$. By covariance and Fubinis theorem we get

$$\cdots = \int_{\mathbb{R}^d} \int_\Omega \sum_{l,m\in\omega} |k_{\omega-t}(l - t, m - t)|^2 u(m)u(l - t) d\mu(\omega)dt.$$

As μ is T-invariant, we can replace $\omega - t$ by ω and obtain

$$\begin{aligned}
&= \int_{\mathbb{R}^d} \int_\Omega \sum_{l,m\in\omega+t} |k_\omega(l - t, m - t)|^2 u(m)u(l - t) dt d\mu(\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} \sum_{l,m\in\omega} |k_\omega(l, m)|^2 u(m + t)u(l) dt d\mu(\omega) \\
&= \int_\Omega \operatorname{tr}(K_\omega K_\omega^* M_u) d\mu(\omega)
\end{aligned}$$

by reversing the first steps. □

Having defined τ, we can now associate a canonial measure ρ_A to every selfadjoint $A \in \mathcal{N}(\Omega, T, \mu)$.

Definition 3.5. *For $A \in \mathcal{N}(\Omega, T, \mu)$ selfadjoint, and $B \subset \mathbb{R}$ Borel measurable, we set $\rho_A(B) \equiv \tau(\chi_B(A))$, where χ_B is the characteristic function of B.*

For the next two results we refer to [27] where the context is somewhat different.

Lemma 3.6. *Let (Ω, T) be an (r, R)-system and μ be T-invariant. Let $A \in \mathcal{N}(\Omega, T, \mu)$ selfadjoint be given. Then ρ_A is a spectral measure for A. In particular, the support of ρ_A agrees with the spectrum Σ of A and the equality $\rho_A(F) = \tau(F(A))$ holds for every bounded measurable F on $\mathbb{R}$.*

Lemma 3.7. *Let (Ω, T) be an (r, R)-system and μ be T-invariant. Let μ be ergodic and $A = (A_\omega) \in \mathcal{N}(\Omega, T, \mu)$ be selfadjoint. Then there exists $\Sigma, \Sigma_{ac}, \Sigma_{sc}, \Sigma_{pp}, \Sigma_{ess} \subset \mathbb{R}$ and a subset $\widetilde{\Omega}$ of Ω of full measure such that $\Sigma = \sigma(A_\omega)$ and $\sigma_\bullet(A_\omega) = \Sigma_\bullet$ for $\bullet = ac, sc, pp, ess$ and $\sigma_{disc}(A_\omega) = \emptyset$ for every $\omega \in \widetilde{\Omega}$. In this case, the spectrum of A is given by Σ.*

We now head towards evaluating the trace τ.

Definition 3.8. *The number*

$$\int F_\alpha d\Lambda =: D_{\Omega,\mu}$$

is called the mean density *of Ω with respect to μ.*

Theorem 3.9. *Let (Ω, T) be an (r, R)-system and μ be ergodic. If ω is non-periodic for μ-a.e. $\omega \in \Omega$ then $\mathcal{N}(\Omega, T, \mu)$ is a factor of type II_D, where $D = D_{\Omega,\mu}$, i.e., a finite factor of type II and the canonical trace τ satisfies $\tau(1) = D$.*

Proof. We already know that $\mathcal{N}(\Omega, T, \mu)$ is a factor. Using Proposition 3.2 and [9], Cor. 9, p. 51 we see that $\mathcal{N}(\Omega, T, \mu)$ is not of type I. Since it admits a finite faithful trace, $\mathcal{N}(\Omega, T, \mu)$ has to be a finite factor of type II.

Note that Lemma 3.1, the definition of τ and α give the asserted value for $\tau(1)$. □

Remark 3.10. It is a simple consequence of Proposition 4.6 below that

$$D_\omega = \lim_{R\to\infty} \frac{\#(\omega \cap B_R(0))}{|B_R(0)|}$$

exists and equals $D_{\Omega,\mu}$ for almost every $\omega \in \Omega$. Therefore, the preceding result is a more general version of the results announced as [28], Theorem 2.1 and [29], Theorem 3.8, respectively. Of course, existence of the limit is not new. It can already be found e.g. in [6].

4. The C*-algebra associated to finite range operators and the integrated density of states

In this section we study a C*-subalgebra of $\mathcal{N}(\Omega, T, \mu)$ that contains those operators that might be used as hamiltonians for quasicrystals. The approach is direct and does not rely upon the framework introduced in the preceding sections.

We define

$$\mathcal{X} \times_\Omega \mathcal{X} := \{(p, \omega, q) \in \mathbb{R}^d \times \Omega \times \mathbb{R}^d : p, q \in \omega\},$$

which is a closed subspace of $\mathbb{R}^d \times \Omega \times \mathbb{R}^d$ for any DDS Ω.

Definition 4.1. *A* kernel of finite range *is a function $k \in C(\mathcal{X} \times_\Omega \mathcal{X})$ that satisfies the following properties:*

(i) *k is bounded.*
(ii) *k has finite range, i.e., there exists $R_k > 0$ such that $k(p, \omega, q) = 0$, whenever $|p - q| \geq R_k$.*
(iii) *k is invariant, i.e.,*

$$k(p + t, \omega + t, q + t) = k(p, \omega, q),$$

for $(p, \omega, q) \in \mathcal{X} \times_\Omega \mathcal{X}$ and $t \in \mathbb{R}^d$.

The set of these kernels is denoted by $\mathcal{K}^{fin}(\Omega, T)$.

We record a few quite elementary observations. For any kernel $k \in \mathcal{K}^{fin}(\Omega, T)$ denote by $\pi_\omega k := K_\omega$ the operator $K_\omega \in \mathcal{B}(\ell^2(\omega))$, induced by

$$(K_\omega \delta_q | \delta_p) := k(p, \omega, q) \text{ for } p, q \in \omega.$$

Clearly, the family $K := \pi k$, $K = (K_\omega)_{\omega \in \Omega}$, is bounded in the product (equipped with the supremum norm) $\Pi_{\omega \in \Omega} \mathcal{B}(\ell^2(\omega))$. Now, pointwise sum, the convolution (matrix) product

$$(a \cdot b)(p, \omega, q) := \sum_{x \in \omega} a(p, \omega, x) b(x, \omega, q)$$

and the involution $k^*(p, \omega, q) := \overline{k}(q, \omega, p)$ make $\mathcal{K}^{fin}(\Omega, T)$ into a $*$-algebra. Then, the mapping $\pi : \mathcal{K}^{fin}(\Omega, T) \to \Pi_{\omega \in \Omega} \mathcal{B}(\ell^2(\omega))$ is a faithful $*$-representation. We denote $\mathcal{A}^{fin}(\Omega, T) := \pi(\mathcal{K}^{fin}(\Omega, T))$ and call it the *operators of finite range*. The completion of $\mathcal{A}^{fin}(\Omega, T)$ with respect to the norm $\|A\| := \sup_{\omega \in \Omega} \|A_\omega\|$ is denoted by $\mathcal{A}(\Omega, T)$. It is not hard to see that the mapping $\pi_\omega : \mathcal{A}^{fin}(\Omega, T) \to \mathcal{B}(\ell^2(\omega)), K \mapsto K_\omega$ is a representation that extends by continuity to a representation of $\mathcal{A}(\Omega, T)$ that we denote by the same symbol.

Proposition 4.2. *Let $A \in \mathcal{A}(\Omega, T)$ be given. Then the following holds:*
(a) $\pi_{\omega+t}(A) = U_t \pi_\omega(A) U_t^*$ for arbitrary $\omega \in \Omega$ and $t \in \mathbb{R}^d$.
(b) For $F \in C_c(\mathcal{X})$, the map $\omega \mapsto \langle \pi_\omega(A) F_\omega, F_\omega \rangle_\omega$ is continuous.

Proof. Both statements are immediate for $A \in \mathcal{A}^{fin}(\Omega, T)$ and then can be extended to $\mathcal{A}(\Omega, T)$ by density and the definition of the norm. □

We get the following result that relates ergodicity properties of (Ω, T), spectral properties of the operator families from $\mathcal{A}(\Omega, T)$ and properties of the representations π_ω.

Theorem 4.3. *The following conditions on a DDS (Ω, T) are equivalent:*

(i) *(Ω, T) is minimal.*
(ii) *For any selfadjoint $A \in \mathcal{A}(\Omega, T)$ the spectrum $\sigma(A_\omega)$ is independent of $\omega \in \Omega$.*
(iii) *π_ω is faithful for every $\omega \in \Omega$.*

Proof. (i)$\Longrightarrow$(ii):
Choose $\phi \in C(\mathbb{R})$. We then get $\pi_\omega(\phi(A)) = \phi(\pi_\omega(A))$ since π_ω is a continuous algebra homomorphism. Set $\Omega_0 = \{\omega \in \Omega : \pi_\omega(\phi(A)) = 0\}$. By Proposition 4.2 (a), Ω_0 is invariant under translations. Moreover, by Proposition 4.2 (b) it is closed. Thus, $\Omega_0 = \emptyset$ or $\Omega_0 = \Omega$ by minimality. As ϕ is arbitrary, this gives the desired equality of spectra by spectral calculus.

(ii)$\Longrightarrow$(iii):
By (ii) we get that $\|\pi_\omega(A)\|^2 = \|\pi_\omega(A^*A)\|$ does not depend on $\omega \in \Omega$. Thus $\pi_\omega(A) = 0$ for some A implies that $\pi_\omega(A) = 0$ for all $\omega \in \Omega$ whence $A = 0$.
(iii)$\Longrightarrow$(i):
Assume that Ω is not minimal. Then we find ω_0 and ω_1 such that $\omega_1 \notin \overline{(\omega_0 + \mathbb{R}^d)}$.
Consequently , there is $r > 0, p \in \omega, \delta > 0$ such that

$$d_H((\omega_0 - p) \cap B_r(0), (\omega_1 - q) \cap B_r(0)) > 2\delta$$

for all $q \in \omega_1$. Let $\rho \in C(\mathbb{R})$ such that $\rho(t) = 0$ if $t \geq \frac{1}{2}$ and $\rho(0) = 1$. Moreover, let $\psi \in C_c(\mathbb{R}^d)$ such that supp$\psi \subset B_\delta(0)$ and $\phi \in C_c(\mathbb{R}^d)$ and $\phi = 1$ on $B_{2r}(0)$.
Finally, let

$$\begin{aligned} a(x,\omega,y) &:= \rho\Bigg(\| \left(\sum_{p\in\omega} T_p\psi\right) T_x\phi - \left(\sum_{q\in\omega_0} T_q\psi\right) T_y\phi\|_\infty \\ &+ \| \left(\sum_{p\in\omega_0} T_p\psi\right) T_x\phi - \left(\sum_{q\in\omega} T_q\psi\right) T_y\phi\|_\infty \Bigg) \end{aligned}$$

It is clear that a is a symmetric kernel of finite range and by construction the corresponding operator family satisfies $A_{\omega_1} = 0$ but $A_{\omega_0} \neq 0$, which implies (iii). □

Let us now comment on the relation between the algebra $\mathcal{A}(\Omega, T)$ defined above and the C*-algebra introduced in [6, 17] for a different purpose and in a different setting. Using the notation from [6] we let

$$\mathcal{Y} = \{\omega \in \Omega : 0 \in \omega\}$$

and

$$G_\mathcal{Y} = \{(\omega, t) \in \mathcal{Y} \times \mathbb{R}^d : t \in \omega\} \subset \mathcal{X}.$$

In [6] the authors introduce the algebra $C^*(G_\mathcal{Y})$, the completion of $C_c(G_\mathcal{Y})$ with respect to the convolution

$$fg(\omega, q) = \sum_{t\in\omega} f(\omega, t) g(\omega - t, q - t)$$

and the norm induced by the representations

$$\Pi_\omega : C_c(G_\mathcal{Y}) \to \mathcal{B}(\ell^2(\omega)), \Pi_\omega(f)\xi(q) = \sum_{t\in\omega} f(\omega - t, t - q)\xi(q), q \in \omega.$$

The following result can be checked readily, using the definitions.

Proposition 4.4. *For a kernel $k \in \mathcal{K}^{fin}(\Omega, T)$ denote $f_k(\omega, t) := k(0, \omega, t)$. Then*

$$J : \mathcal{K}^{fin}(\Omega, T) \to C_c(G_\mathcal{Y}), k \mapsto f_k$$

is a bijective algebra isomorphism and $\pi_\omega = \Pi_\omega \circ J$ for all ω. Consequently, $\mathcal{A}(\Omega, T)$ and $C^*(G_\mathcal{Y})$ are isomorphic.

Note that the setting in [6] and here are somewhat different. In the tiling framework, the analogue of these algebras have been considered in [17].

We now come to relate the abstract trace τ defined in the last section with the mean trace per unit volume. The latter object is quite often considered by physicists and bears the name *integrated density of states*. Its proper definition rests on ergodicity. We start with the following preparatory result for which we need the notion of a van Hove sequence of sets.

For $s > 0$ and $Q \subset \mathbb{R}^d$, we denote by $\partial_s Q$ the set of points in $\mathbb{R}^d$ whose distance to the boundary of Q is less than s. A sequence (Q_n) of bounded subsets of $\mathbb{R}^d$ is called a *van Hove sequence* if $|Q_n|^{-1}|\partial_s Q_n| \longrightarrow 0, n \longrightarrow 0$ for every $s > 0$.

Proposition 4.5. *Assume that (Ω, T) is a uniquely ergodic (r, R)-system with invariant probability measure μ and $A \in \mathcal{A}(\Omega, T)$. Then, for any van Hove sequence (Q_n) it follows that*

$$\lim_{n\in\mathbb{N}} \frac{1}{|Q_n|}\mathrm{tr}(A_\omega|_{Q_n}) = \tau(A)$$

for every $\omega \in \Omega$.

Clearly, $A_\omega|_Q$ denotes the restriction of A_ω to the subspace $\ell^2(\omega \cap Q)$ of $\ell^2(\omega)$. Note that this subspace is finite-dimensional, whenever $Q \subset \mathbb{R}^d$ is bounded.

We will use here the shorthand $A_\omega(p, q)$ for the kernel associated with A_ω.

Proof. Fix a nonnegative $u \in C_c(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} u(x)dx = 1$ and support contained in $B_r(0)$ and let $f(\omega, p) := u(p)$. Then

$$\begin{aligned} \tau(A) &= \int_\Omega \mathrm{tr}(A_\omega M_u)\, d\mu(\omega) \\ &= \int_\Omega \left(\sum_{p\in\omega} A_\omega(p,p)u(p)\right) d\mu(\omega) \\ &= \int_\Omega F(\omega)\, d\mu(\omega), \end{aligned}$$

where

$$F(\omega) := \sum_{p\in\omega} A_\omega(p,p)u(p)$$

is continuous by virtue of [29], Proposition 2.5 (a). Therefore, the ergodic theorem for uniquely ergodic systems implies that for every $\omega \in \Omega$:

$$\frac{1}{|Q_n|}\int_{Q_n} F(\omega + t)dt \to \int_\Omega F(\omega)\, d\mu(\omega).$$

On the other hand,

$$\begin{aligned} \frac{1}{|Q_n|}\int_{Q_n} F(\omega + t)dt &= \frac{1}{|Q_n|}\int_{Q_n}\left(\sum_{p\in\omega+t} A_{\omega+t}(p,p)u(p)\right) dt \\ &= \frac{1}{|Q_n|}\underbrace{\int_{Q_n}\left(\sum_{q\in\omega} A_\omega(q,q)u(q+t)\right) dt}_{I_n} \end{aligned}$$

by covariance of A_ω. Since supp$u \subset B_r(0)$ and the integral over u equals 1, every $q \in \omega$ such that $q + B_r(0) \subset Q_n$ contributes $A_\omega(q,q)\cdot 1$ in the sum under the integral I_n. For those $q \in \omega$ such that $q + B_r(0) \cap Q_n = \emptyset$, the corresponding summand gives 0. Hence

$$\begin{aligned} \left|\frac{1}{|Q_n|}\left(\sum_{q\in\omega\cap Q_n} A_\omega(q,q) - I_n\right)\right| &\leq \frac{1}{|Q_n|}\cdot \#\{q \in \partial_{2r}Q_n\}\cdot \|A_\omega\| \\ &\leq C\cdot\frac{|\partial_{2r}Q_n|}{|Q_n|} \to 0 \end{aligned}$$

since (Q_n) is a van Hove sequence. □

A variant of this proposition also holds in the measurable situation.

Proposition 4.6. *Let μ be an ergodic measure on (Ω, T). Let $A \in \mathcal{N}(\Omega, T, \mu)$ and an increasing van Hove sequence (Q_n) of compact sets in $\mathbb{R}^d$ with $\mathbb{R}^d = \cup Q_n$, $0 \in Q_1$ and $|Q_n - Q_n| \leq C|Q_n|$ for some $C > 0$ and all $n \in \mathbb{N}$ be given. Then,*

$$\lim_{n\in\mathbb{N}} \frac{1}{|Q_n|} \operatorname{tr}(A_\omega|_{Q_n}) = \tau(A)$$

for μ-almost every $\omega \in \Omega$.

Proof. The proof follows along similar lines as the proof of the preceeding proposition after replacing the ergodic theorem for uniquely ergodic systems by the Birkhoff ergodic theorem. Note that for $A \in \mathcal{N}(\Omega, T, \mu)$, the function F defined there is bounded and measurable. □

In the proof we used ideas of Hof [14]. The following result finally establishes an identity that one might call an abstract Shubin's trace formula. It says that the abstractly defined trace τ is determined by the integrated density of states. The latter is the limit of the following eigenvalue counting measures. Let, for selfadjoint $A \in \mathcal{A}(\Omega, T)$ and $Q \subset \mathbb{R}^d$:

$$\langle \rho[A_\omega, Q], \varphi \rangle := \frac{1}{|Q|} \operatorname{tr}(\varphi(A_\omega|_Q)), \varphi \in C(\mathbb{R}).$$

Its distribution function is denoted by $n[A_\omega, Q]$, i.e. $n[A_\omega, Q](E)$ gives the number of eigenvalues below E per volume (counting multiplicities).

Theorem 4.7. *Let (Ω, T) be a uniquely ergodic (r, R)-system and μ its ergodic probability measure. Then, for selfadjoint $A \in \mathcal{A}(\Omega, T)$ and any van Hove sequence (Q_n),*

$$\langle \rho[A_\omega, Q_n], \varphi \rangle \to \tau(\varphi(A)) \text{ as } n \to \infty$$

for every $\varphi \in C(\mathbb{R})$ and every $\omega \in \Omega$. Consequently, the measures $\rho_\omega^{Q_n}$ converge weakly to the measure ρ_A defined above by $\langle \rho_A, \varphi \rangle := \tau(\varphi(A))$, for every $\omega \in \Omega$.

Proof. Let $\varphi \in C(\mathbb{R})$ and (Q_n) be a van Hove sequence. From Proposition 4.5, applied to $\varphi(A) = (\varphi(A_\omega))_{\omega\in\Omega}$, we already know that

$$\lim_{n\in\mathbb{N}} \frac{1}{|Q_n|} \operatorname{tr}(\varphi(A_\omega)|_{Q_n}) = \tau(\varphi(A))$$

for arbitrary $\omega \in \Omega$. Therefore, it remains to show that

$$\lim_{n\in\mathbb{N}} \frac{1}{|Q_n|} \left(\operatorname{tr}(\varphi(A_\omega)|_{Q_n}) - \operatorname{tr}(\varphi(A_\omega|_{Q_n}))\right) = 0 \qquad (*).$$

This latter property is stable under uniform limits of functions φ, since both $\varphi(A_\omega|_{Q_n})$ and $\varphi(A_\omega)|_{Q_n}$ are operators of rank dominated by $c \cdot |Q_n|$.

It thus suffices to consider a polynomial φ.

Now, for a fixed polynomial φ with degree N, there exists a constant $C = C(\varphi)$ such that

$$\|\varphi(A) - \varphi(B)\| \leq C\|A - B\|(\|A\| + \|B\|)^N$$

for any A, B on an arbitrary Hilbert space. In particular,

$$\frac{1}{|Q_n|} |\operatorname{tr}(\varphi(A_\omega)|_{Q_n}) - \operatorname{tr}(\varphi(B_\omega)|_{Q_n})| \leq C\|A_\omega - B_\omega\|(\|A_\omega\| + \|B_\omega\|)^N$$

and

$$\frac{1}{|Q_n|}\,|\mathrm{tr}(\varphi(A_\omega|_{Q_n})) - \mathrm{tr}(\varphi(B_\omega|_{Q_n}))| \le C\|A_\omega - B_\omega\|(\|A_\omega\| + \|B_\omega\|)^N$$

for all A_ω and B_ω.

Thus, it suffices to show $(*)$ for a polynomial φ and $A \in \mathcal{A}^{fin}(\Omega, T)$, as this algebra is dense in $\mathcal{A}(\Omega, T)$. Let such A and φ be given.

Let R_a the range of the kernel $a \in C(\mathcal{X} \times_\Omega \mathcal{X})$ corresponding to A. Since the kernel of A^k is the k-fold convolution product $b := a \cdots a$ one can easily verify that the range of A^k is bounded by $N \cdot R_a$. Thus, for all $p, q \in \omega \cap Q_n$ such that the distance of p, q to the complement of Q_n is larger than $N \cdot R_a$, the kernels of $A^k_\omega|_{Q_n}$ and $(A|_{Q_n})^k$ agree for $k \le N$. We get:

$$((\varphi(A_\omega)|_{Q_n})\delta_q|\delta_p) = b(p, \omega, q) = (\varphi(A_\omega|_{Q_n})\delta_q|\delta_p).$$

Since this is true outside $\{q \in \omega \cap Q_n : \mathrm{dist}(q, Q_n^c) > N \cdot R_a\} \subset \partial_{N \cdot R_a} Q_n$ the matrix elements of $(\varphi(A_\omega)|_{Q_n})$ and $\varphi(A_\omega|_{Q_n})$ differ at at most $c \cdot |\partial_{N \cdot R_a} Q_n|$ sites, so that

$$|\mathrm{tr}(\varphi(A_\omega)|_{Q_n}) - \mathrm{tr}(\varphi(A_\omega|_{Q_n}))| \le C \cdot |\partial_{N \cdot R_a} Q_n|.$$

Since (Q_n) is a van Hove sequence, this gives the desired convergence.

□

The above statement has many precursors: [2, 3, 4, 31, 36] in the context of almost periodic, random or almost random operators on $\ell^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$. It generalizes results by Kellendonk [17] on tilings associated with primitive substitutions. Its proof relies on ideas from [2, 3, 4, 17] and [14]. Nevertheless, it is new in the present context.

For completeness reasons, we also state the following result.

Theorem 4.8. *Let (Ω, T) be an (r, R)-system with an ergodic probabiltiy measure μ . Let $A \in \mathcal{A}(\Omega, T)$ be selfadjoint (Q_n) be an increasing van Hove sequence (Q_n) of compact sets in $\mathbb{R}^d$ with $\cup Q_n = \mathbb{R}^d$, $0 \in Q_1$ and $|Q_n - Q_n| \le C|Q_n|$ for some $C > 0$ and all $n \in \mathbb{N}$. Then,*

$$\langle \rho[A_\omega, Q_n], \varphi\rangle \to \tau(\varphi(A)) \text{ as } n \to \infty$$

for μ-almost every $\omega \in \Omega$. Consequently, the measures $\rho^{Q_n}_\omega$ converge weakly to the measure ρ_A defined above by $\langle \rho_A, \varphi\rangle := \tau(\varphi(A))$, for μ-almost every $\omega \in \Omega$.

The *Proof* follows along similar lines as the proof of the previous theorem with two modifications: Instead of Proposition 4.5, we use Proposition 4.6; and instead of dealing with arbitrary polynomials we choose a countable set of polynomials which is dense in $C_c([-\|A\| - 2, \|A\| + 2])$.

The primary object from the physicists point of view is the finite volume limit:

$$N[A](E) := \lim_{n\to\infty} n[A_\omega, Q_n](E)$$

known as the integrated density of states. It has a striking relevance as the number of energy levels below E per unit volume, once its existence and independence of ω are settled.

The last two theorems provide the mathematically rigorous version. Namely, the distribution function $N_A(E) := \rho_A(-\infty, E]$ of ρ_A is the right choice. It gives a limit of finite volume counting measures since

$$\rho[A_\omega, Q_n] \to \rho_A \text{ weakly as } n \to \infty.$$

Therefore, the desired independence of ω is also clear. Moreover, by standard arguments we get that the distribution functions of the finite volume counting functions converge to N_A at points of continuity of the latter.

In [30] we present a much stronger result for uniquely ergodic minimal DDS that extends results for onedimensional models by the first named author, [26]. Namely we prove that the distribution functions converge uniformly, uniform in ω. The above result can then be used to identify the limit as given by the tace τ. Let us stress the fact that unlike in usual random models, the function N_A does exhibit discontinuities in general, as explained in [20].

Let us end by emphasizing that the assumptions we posed are met by all the models that are usually considered in connection with quasicrystals. In particular, included are those Delone sets that are constructed by the cut-and-project method as well as models that come from primitive substitution tilings.

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CHAPTER 10

D. Lenz, P. Stollmann, *An ergodic theorem for Delone dynamical systems and existence of the integrated density of states*, to appear in: Journal d' Analyse Mathématique

AN ERGODIC THEOREM FOR DELONE DYNAMICAL SYSTEMS AND EXISTENCE OF THE INTEGRATED DENSITY OF STATES

DANIEL LENZ [1,*], PETER STOLLMANN [2,*]

Abstract. We study strictly ergodic Delone dynamical systems and prove an ergodic theorem for Banach space valued functions on the associated set of pattern classes. As an application, we prove existence of the integrated density of states in the sense of uniform convergence in distribution for the associated random operators.

Dedicated to J. Voigt on the occasion of his 60th birthday

[1] Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany; E-mail: D.Lenz@mathematik.tu-chemnitz.de,

[2] Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany; E-mail: P.Stollmann@mathematik.tu-chemnitz.de

1. Introduction

This paper is concerned with Delone dynamical systems and the associated random operators.

Delone dynamical systems can be seen as the higher dimensional analogues of subshifts over finite alphabets. They have attracted particular attention as they can serve as models for so called quasicrystals. These are substances, discovered in 1984 by Shechtman, Blech, Gratias and Cahn [38] (see the report [18] of Ishimasa et al. as well), which exhibit features similar to crystals but are non-periodic. Thus, they belong to the reign of disordered solids and their distinctive feature is their special form of weak disorder.

This form of disorder and its effects have been immensely studied in recent years, both from the theoretical and the experimental point of view (see [2, 19, 34, 37] and references therein). On the theoretical side, there does not exist an axiomatic framework (yet) to describe quasicrystals. However, they are commonly modeled by either Delone dynamical systems or tiling dynamical systems [37] (see [25, 26] for recent study of Delone sets as well). In fact, these two descriptions are essentially equivalent (see e.g. [31]). The main focus of the theoretical study lies then on diffraction properties, ergodic and combinatorial features and the associated random operators (see [2, 34, 37]).

Here, we will deal with ergodic features of Delone dynamical systems and the associated random operators. The associated random operators (Hamiltonians)

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describe basic quantum mechanical features of the models (e.g. conductance properties). In the one dimensional case, starting with [5, 36], various specific features of these Hamiltonians have been rigorously studied. They include purely singular continuous spectrum, Cantor spectrum and anomalous transport (see [8] for a recent review and an extended bibliography). In the higher dimensional case our understanding is much more restricted. In fact, information on spectral types is completely missing. However, there is K-Theory providing some overall type information on possible gaps in the spectra. This topic was initiated by Bellissard [3] for almost periodic operators. It has then been investigated for tilings starting with the work of Kellendonk [20] (see [4, 21] for recent reviews).

Now, our aim in this paper is to study the integrated density of states. This is a key quantity in the study of random operators. It gives some average type of information on the involved operators.

We will show uniform existence of the density of states in the sense of uniform convergence in distribution of the underlying measures. This result is considerably stronger than the corresponding earlier results of Kellendonk [20], and Hof [15], which only gave weak convergence. It fits well within the general point of view that quasicrystals should behave very uniformly due to their proximity to crystals.

These results are particularly relevant as the limiting distribution may well have points of discontinuity. In fact, points of discontinuity are an immediate consequence of existence of locally supported eigenfunctions. Such eigenfunctions had already been observed in certain models [1, 13, 23, 24]. In fact, as discussed by the authors and Steffen Klassert in [22], they can easily be "introduced" without essentially changing the underlying Delone dynamical system. Moreover, based on the methods presented here, it is possible to show that points of discontinuity of the integrated density of states are exactly those energies for which locally supported eigenfunctions exist (see [22] again).

Let us emphasize that the limiting distribution is known to be continuous for models on lattices [10] (and, in fact, even stronger continuity properties hold [6]). In these cases uniform convergence of the distributions is an immediate consequence of general measure theory.

To prove our result on uniform convergence (Theorem 3) we introduce a new method. It relies on studying convergence of averages in suitable Banach spaces. Namely, the integrated density of states turns out to be given by an almost additive function with values in a certain Banach space (Theorem 2). To apply our method we prove an ergodic theorem (Theorem 1), for Banach space valued functions on the associated set of pattern classes.

This ergodic theorem may be of independent interest. It is an analog of a result of Geerse/Hof [14] for tilings associated to primitive substitutions. For real valued almost additive functions on linearly repetitive Delone sets related results have been obtained by Lagarias and Pleasants [26]. The one-dimensional case has been studied by one of the authors in [28, 29].

The proof of our ergodic theorem uses ideas from the cited work of Geerse and Hof. Their work relies on suitable decompositions. These decompositions are naturally present in the framework of primitive substitutions. However, we need to construct them separately in the case we are dealing with. To do so, we use techniques of "partitioning according to return words" as introduced by Durand in

[11, 12] for symbolic dynamics and later studied for tilings by Priebe [35]. Note, however, that we need quite some extra effort, as we do not assume aperiodicity.

The paper is organized as follows. In Section 2 we introduce the notation and present our results. Section 3 is devoted to a discussion of the relevant decomposition. The ergodic theorem is proved in Section 4. Uniform convergence of the integrated density of states is proven in Section 5 after proving the necessary almost additivity.

2. Setting and results

The aim of this section is to introduce some notation and to present our results, which cover part of what has been announced in [30]. In a companion paper [31] more emphasis has been laid on the topological background and the basics of the groupoid construction and the noncommutative point of view.

For the remainder of the paper an integer $d \geq 1$ will be fixed and all Delone sets, patterns etc. will be subsets of $\mathbb{R}^d$. The Euclidean norm on $\mathbb{R}^d$ will be denoted by $\|\cdot\|$ as will the norms on various other normed spaces. For $s > 0$ and $p \in \mathbb{R}^d$, we let $B(p, s)$ be the closed ball in $\mathbb{R}^d$ around p with radius s. A subset ω of $\mathbb{R}^d$ is called Delone set if there exist $r > 0$ and $R > 0$ such that

- $2r \leq \|x - y\|$ whenever $x, y \in \omega$ with $x \neq y$,
- $B(x, R) \cap \omega \neq \emptyset$ for all $x \in \mathbb{R}^d$,

and the limiting values of r and R are called *packing radius* and *covering radius*, respectively. Such an ω will also be denoted as (r, R)-set. Of particular interest will be the restrictions of ω to bounded subsets of $\mathbb{R}^d$. In order to treat these restrictions, we introduce the following definition.

Definition 2.1. *(a) A pair (Λ, Q) consisting of a bounded subset Q of $\mathbb{R}^d$ and $\Lambda \subset Q$ finite is called pattern. The set Q is called the support of the pattern.*
(b) A pattern (Λ, Q) is called ball pattern if $Q = B(x, s)$ with $x \in \Lambda$ for suitable $x \in \mathbb{R}^d$ and $s > 0$.

The diameter and the volume of a pattern are defined to be the diameter and the volume of its support respectively. For patterns $X_1 = (\Lambda_1, Q_1)$ and $X_2 = (\Lambda_2, Q_2)$, we define $\sharp_{X_1} X_2$, the number of occurrences of X_1 in X_2, to be the number of elements in $\{t \in \mathbb{R}^d : \Lambda_1 + t = \Lambda_2 \cap (Q_1 + t), Q_1 + t = Q_2\}$. Moreover, for patterns $X_i = (\Lambda_i, Q_i)$, $i = 1, \ldots, k$, and $X = (\Lambda, Q)$, we write $X = \oplus_{i=1}^k X_i$ if $\Lambda = \cup \Lambda_i$, $Q = \cup Q_i$ and the Q_i are disjoint up to their boundaries.

For further investigation we will have to identify patterns which are equal up to translation. Thus, on the set of patterns we introduce an equivalence relation by setting $(\Lambda_1, Q_1) \simeq (\Lambda_2, Q_2)$ if and only if there exists a $t \in \mathbb{R}^d$ with $\Lambda_1 = \Lambda_2 + t$ and $Q_1 = Q_2 + t$. The class of a pattern (Λ, Q) is denoted by $[(\Lambda, Q)]$. The notions of diameter, volume occurrence etc. can easily be carried over from patterns to pattern classes.

Every Delone set ω gives rise to a set of pattern classes, $\mathcal{P}(\omega)$ viz $\mathcal{P}(\omega) = \{Q \wedge \omega : Q \subset \mathbb{R}^d \text{ bounded and measurable}\}$, and to a set of ball pattern classes $\mathcal{P}_B(\omega)) = \{[B(p, s) \wedge \omega] : p \in \omega, s \in \mathbb{R}\}$. Here we set

$$Q \wedge \omega = (\omega \cap Q, Q). \tag{1}$$

Furthermore, for arbitrary ball patterns P, we define $s(P)$ to be the radius of the underlying ball, i.e.

$$s(P) = s \ \text{ for } P = [(\Lambda, B(p,s))]. \tag{2}$$

For $s \in (0,\infty)$, we denote by $\mathcal{P}_B^s(\omega)$ the set of ball patterns with radius s. A Delone set is said to be of finite type if for every radius s the set $\mathcal{P}_B^s(\omega)$ is finite.

The Hausdorff metric on the set of compact subsets of $\mathbb{R}^d$ induces the so called *natural topology* on the set of closed subsets of $\mathbb{R}^d$. It is described in detail in [31] and shares some nice properties: Firstly, the set of all closed subsets of $\mathbb{R}^d$ is compact in the natural topology. Secondly, the natural action T of $\mathbb{R}^d$ on the closed sets given by $T_t C \equiv C + t$ is continuous.

Definition 2.2. *(a) If Ω is a set of Delone sets that is invariant under the shift T and closed under the natural topology, then (Ω, T) is called a* Delone dynamical system *and abbreviated as* DDS.
(b) A DDS (Ω, T) is said to be of finite local complexity *if $\cup_{\omega\in\Omega} P_B^s(\omega)$ is finite for every $s > 0$.*
*(c) A DDS (Ω, T) is called an (r,R)-*system *if every $\omega \in \Omega$ is an (r,R)-set.*
(d) The set $\mathcal{P}(\Omega)$ of patterns classes associated to a DDS Ω is defined by $\mathcal{P}(\Omega) = \cup_{\omega\in\Omega}\mathcal{P}(\omega)$.

Due to compactness of the set of all closed sets in the natural topology a DDS Ω is compact.

Let us record the following notions of ergodic theory along with an equivalent "combinatorial" characterization available for Delone dynamical systems (see e.g. [26, 39] for further discussion and references) : (Ω, T) is called *aperiodic* if $T_t\omega \neq \omega$ whenever $\omega \in \Omega$ and $t \in \mathbb{R}^d$ with $t \neq 0$. Is is called *minimal* if every orbit is dense. This is equivalent to $\mathcal{P}(\Omega) = \mathcal{P}(\omega)$ for every $\omega \in \Omega$. This latter property is called *local isomorphism property* in the tiling framework [39]. It is also referred to as *repetitivity*. Namely, it is equivalent to the existence of an $R(P) > 0$ for every $P \in \mathcal{P}(\Omega)$ such that $B(p, R(P)) \wedge \omega$ contains a copy of P for every $p \in \mathbb{R}^d$ and every $\omega \in \Omega$. Note also that every minimal DDS is an (r,R)-system.

We are interested in ergodic averages. More precisely, we will take means of suitable functions along suitable sequences of patterns and pattern classes. These functions and sequences will be introduced next. Here and in the sequel we will use the following notation: For $Q \subset \mathbb{R}^d$and $h > 0$ we define

$$Q_h \equiv \{x \in Q : \text{dist}(x, \partial Q) \geq h\}, \ \ Q^h \equiv \{x \in \mathbb{R}^d : \text{dist}(x, Q) \leq h\},$$

where, of course, dist denotes the usual distance and ∂Q is the boundary of Q. Moreover, we denote the Lebesgue measure of a measurable subset $Q \subset \mathbb{R}^d$ by $|Q|$. Then, a sequence (Q_n) of subsets in $\mathbb{R}^d$ is called a *van Hove sequence* if the sequence $(|Q_n|^{-1}|Q_n^h \setminus Q_{n,h}|)$ tends to zero for every $h \in (0,\infty)$. Similarly, a sequence (P_n) of pattern classes, (i.e. $P_n = [(\Lambda_n, Q_n)]$ with suitable Q_n, Λ_n) is called a *van Hove sequence* if Q_n is a van Hove sequence. (This is obviously well defined.) We can now discuss unique ergodicity. A dynamical system (Ω, T) is called *uniquely ergodic* if it admits only one T-invariant measure (up to normalization). For a Delone dynamical system, this is equivalent to the fact that for every pattern class P the frequency

$$f(P) \equiv \lim_{n\to\infty} |Q_n|^{-1} \sharp_P(\omega \wedge Q_n), \tag{3}$$

exists uniformly in $\omega \in \Omega$ for every van Hove sequence (Q_n). This equivalence was shown in Theorem 1.6 in [31] (see [27] as well). It goes back to [39], Theorem 3.3, in the tiling setting.

Definition 2.3. *Let Ω be a DDS and $\mathcal{B}$ be a vector space with seminorm $\|\cdot\|$. A function $F : \mathcal{P}(\Omega) :\longrightarrow \mathcal{B}$ is called* almost additive (with respect to $\|\cdot\|$) *if there exists a function $b : \mathcal{P}(\Omega) \longrightarrow [0,\infty)$ (called* associated error function*) and a constant $D > 0$ such that*

(A1) $\|F(\oplus_{i=1}^k P_i) - \sum_{i=1}^k F(P_i)\| \leq \sum_{i=1}^k b(P_i)$,
(A2) $\|F(P)\| \leq D|P| + b(P)$.
(A3) $b(P_1) \leq b(P) + b(P_2)$ *whenever* $P = P_1 \oplus P_2$,
(A4) $\lim_{n\to\infty} |P_n|^{-1} b(P_n) = 0$ *for every van Hove sequence* (P_n).

Now, our first result reads as follows.

Theorem 1. *For a minimal, aperiodic DDSF (Ω, T) the following are equivalent:*
(i) (Ω, T) is uniquely ergodic.
(ii) The limit $lim_{k\to\infty}|P_k|^{-1}F(P_k)$ exists for every van Hove sequence (P_k) and every almost additive F on (Ω, T) with values in a Banach space.

The proof of the the theorem makes use of completeness of the Banach space in crucial manner. However, it does not use the nondegeneracy of the norm. Thus, we get the following corollary (of its proof).

Corollary 2.4. *Let (Ω, T) be aperiodic and strictly ergodic. Let the vector space $\mathcal{B}$ be complete with respect to the topology induced by the seminorms $\|\cdot\|_\iota$, $\iota \in \mathcal{I}$. If $F : \mathcal{P} \longrightarrow \mathcal{B}$ is almost additive with respect to every $\|\cdot\|_\iota$. $\iota \in \mathcal{I}$, then* $\lim_{k\to\infty} |P_k|^{-1}F(P_k)$ *exists for every van Hove sequence (P_k) in $\mathcal{P}(\Omega)$.*

The main theorem may also be rephrased as a result on additive functions on Borel sets. As this may also be of interest we include a short discussion

Definition 2.5. *Let (Ω, T) be a DDS and $\mathcal{B}$ be a Banach space. Let $\mathcal{S}$ be the family of bounded measurable sets on $\mathbb{R}^d$. A function $F : \mathcal{S} \times \Omega \longrightarrow \mathcal{B}$ is called almost additive if there exists a function $b : \mathcal{S} \longrightarrow [0,\infty)$ and $D > 0$ such that*

(A0) $b(Q) = b(Q + t)$ *for arbitrary* $Q \in \mathcal{S}$ *and* $t \in \mathbb{R}^d$ *and* $\|F_\omega(Q) - F_\omega(Q')\| \leq b(Q)$ *whenever* $\omega \wedge Q = \omega \wedge Q'$.
(A1) $\|F_\omega(\cup_{j=1}^n Q_j) - \sum_{j=1}^n F_\omega(Q_j)\| \leq \sum_{j=1}^n b(Q_j)$ *for arbitrary* $\omega \in \Omega$ *and* $Q_j \in \mathcal{S}$ *which are disjoint up to their boundaries,*
(A2) $\|F_\omega(Q)\| \leq D|Q| + b(Q)$.
(A3) $b(Q_1) \leq b(Q) + b(Q_2)$ *whenever* $Q = Q_1 \cup Q_2$ *with* Q_1 *and* Q_2 *disjoint up to their boundaries.*
(A4) $\lim_{k\to\infty} |Q_k|^{-1} b(Q_k)$ *for every van Hove sequence* (Q_k) .

Corollary 2.6. *Let (Ω, T) be a strictly ergodic DDS and $F : \mathcal{S} \times \Omega$ be almost additive. Then $\lim_{k\to\infty} |Q_k|^{-1} F_\omega(Q_k)$ exists for arbitrary $\omega \in \Omega$ and every van Hove sequence (Q_k) in $\mathbb{R}^d$ and the convergence is uniform on Ω.*

Our further results concern selfadjoint operators in a certain C^* algebra associated to (Ω, T). The construction of this C^* algebra has been given in our earlier work [30, 31]. We recall the necessary details next.

Definition 2.7. *Let (Ω, T) be a DDSF. A family (A_ω) of bounded operators $A_\omega : \ell^2(\omega) \longrightarrow \ell^2(\omega)$ is called a* random operator of finite range *if there exists a constant $s > 0$ with*

- $A_\omega(x,y) = 0$ *whenever* $\|x-y\| \geq s$.
- $A_\omega(x,y)$ *only depends on the pattern class of* $((K(x,s) \cup K(y,s)) \wedge \omega)$.

The smallest such s *will be denoted by* R^A.

The operators of finite range form a $*$-algebra under the obvious operations. There is a natural C^*-norm on this algebra and its completion is a C^*-algebra denoted as $\mathcal{A}(\Omega, T)$ (see [4, 30, 31] for details). It consists again of families $(A_\omega)_{\omega\in\Omega}$ of operators $A_\omega : \ell^2(\omega) \longrightarrow \ell^2(\omega)$.

Note that for selfadjoint $A \in \mathcal{A}(\Omega, T)$ and bounded $Q \subset \mathbb{R}^d$ the restriction $A_\omega|_Q$ defined on $\ell^2(Q \cap \omega)$ has finite rank. Therefore, the spectral counting function

$$n(A_\omega, Q)(E) := \#\{ \text{ eigenvalues of } A_\omega|_Q \text{ below } E\}$$

is finite and $\frac{1}{|Q|}n(A_\omega, Q)$ is the distribution function of the measure $\rho(A_\omega, Q)$, defined by

$$\langle \rho(A_\omega, Q), \varphi\rangle := \frac{1}{|Q|}\mathrm{tr}(\varphi(A_\omega|_Q)) \text{ for } \varphi \in C_b(\mathbb{R}).$$

These spectral counting functions are obviously elements of the vector space $\mathcal{D}$ consisting of all bounded right continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ for which $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x)$ exists. Equipped with the supremum norm $\|f\|_\infty \equiv \sup_{x\in\mathbb{R}} |f(x)|$ this vector space is a Banach space. It turns out that the spectral counting function is essentially an almost additive function. More precisely the following holds.

Theorem 2. *Let* (Ω, T) *be a DDS. Let* A *be an operator of finite range. Then* $F^A : \mathcal{P}(\Omega) \longrightarrow \mathcal{D}$, *defined by* $F^A(P) \equiv n(A_\omega, Q_{R^A})$ *for* $P = [(\omega \wedge Q)]$ *is a well defined almost additive function.*

Remark 1. The theorem seems to be new even in the one-dimensional case. (There, of course, it is very easy to prove.)

Based on the foregoing two theorems it is rather clear how to show existence of the limit $\lim_{k\to\infty} |Q_k|^{-1}n(A_\omega, Q_n)$ for van Hove sequences (Q_k). This limit is called the spectral density of A. It is possible to express this limit in closed form using a certain trace on a von Neumann algebra [30, 31]. We will not discuss this trace here, but rather directly give a closed expression. This will be done next. To each selfadjoint element $A \in \mathcal{A}(\Omega, T)$, we associate the measure ρ^A defined on $\mathbb{R}$ by

$$\rho^A(F) \equiv \int_\Omega \mathrm{tr}_\omega(M_f(\omega)\pi_\omega(F(A)))d\mu(\omega).$$

Here, tr_ω is the standard trace on the bounded operators on $\ell^2(\omega)$, f is an arbitrary nonnegative continuous function with compact support on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} f(t)dt = 1$ and $M_f(\omega)$ denotes the operator of multiplication with f in $\ell^2(\omega)$ (see [30, 31] for details). It turns out that ρ^A is a spectral measure for A [31]. Our result on convergence of the integrated density of states is the following.

Theorem 3. *Let* (Ω, T) *be a strictly ergodic DDSF. Let* A *be a selfadjoint operator of finite range and* (Q_k) *be an arbitrary van Hove sequence. Then the distributions* $E \mapsto \rho^{A_\omega}_{Q_k}((-\infty, E])$ *converge to the distribution* $E \mapsto \rho^A((-\infty, E])$ *with respect to* $\|\cdot\|_\infty$ *and this convergence is uniform in* $\omega \in \Omega$.

Remark 2. (a) The usual proofs of existence of the integrated density of states only yield weak convergence of the measures.
(b) The proof of the theorem uses the fact, already established in [31, 32], that the measures $\rho(A_\omega, Q_n)$ converge weakly towards the measure ρ^A for every $\omega \in \Omega$ and $A \in \mathcal{A}(\Omega, T)$.

As mentioned in the preceeding remark, the usual proofs of existence of the integrated density of states only give weak convergence of the measures $\rho_{Q_n}^{A_\omega}$. Weak convergence of measures does in general not imply convergence in distribution. Convergence in distribution will follow, however, from weak convergence if the limiting distribution is continuous. Thus, the theorem is particularly interesting in view of the fact that the limiting distribution can have points of discontinuity.

Existence of such discontinuities is rather remarkable as it is completely different from the behaviour of random operators associated to models with higher disorder. It turns out that a very precise understanding of this phenomenon can be obtained invoking the results presented above. Details of this will be given separately [22]. Here, we only mention the following theorem.

Theorem 4. *Let (Ω, T) be a strictly ergodic DDSF and A an operator of finite range on (Ω, T). Then, E is a point of discontinuity of ρ^A if and only if there exists a locally supported eigenfunction of $A_\omega - E$ for one (every) $\omega \in \Omega$.*

3. Decomposing Delone sets

This section provides the main geometric ideas underlying the proof of our ergodic theorem, Theorem 1. We first discuss how to decompose a given Delone set in finite pieces, called cells, in a natural manner, Proposition 3.2. This is based on the Voronoi construction, as given in (4) and Lemma 3.1, together with a certain way to obtain Delone sets from a given Delone set and a pattern. This decomposition will be done on an increasing sequence of scales. As mentioned already, here we use ideas from [11, 35]. Having described these decompositions, our main concern is to study van Hove type properties of the induced sequences of cells. This study will be undertaken in a series of lemmas yielding as main results Proposition 3.12 and Proposition 3.14. Here, the proof of Proposition 3.14 requires quite some extra effort (compared with the proof of Proposition 3.12) as we have to cope with periods.

We start with a discussion of the well known Voronoi construction. Let ω be an (r, R)-set. To an arbitrary $x \in \omega$ we associate the Voronoi cell $V(x, \omega) \subset \mathbb{R}^d$ defined by

$$V(x,\omega) \equiv \{p \in \mathbb{R}^d : \|p - x\| \leq \|p - y\| \text{ for all } y \in \omega \text{ with } y \neq x\} \tag{4}$$

$$= \cap_{y\in\omega, y\neq x}\{p \in \mathbb{R}^d : \|p - x\| \leq \|p - y\|\}. \tag{5}$$

Note that $\{p \in \mathbb{R}^d : \|p-x\| \leq \|p-y\|\}$ is a half-space. Thus, $V(x, \omega)$ is a convex set. Moreover, it is obviously closed and bounded and therefore compact. It turns out that $V(x, \omega)$ is already determined by the elements of ω close to x. More precisely, the following is valid.

Lemma 3.1. *Let ω be an (r, R)-set. Then, $V(x, \omega)$ is determined by $B(x, 2R) \wedge \omega$, viz $V(x, \omega) \equiv \cap_{y\in B(x,2R)}\{p \in \mathbb{R}^d : \|p - x\| \leq \|p - y\|\}$. Moreover, $V(x, \omega)$ is contained in $B(x, R)$.*

Proof. The first statement follows from Corollary 5.2 in [37] and the second one is a consequence of Proposition 5.2 in [37]. □

Next we describe our notion of derived Delone sets. Let ω be an (r,R)-set and P be a ball pattern class with $P \in \mathcal{P}(\omega)$. Then, we define the Delone set derived from ω by P, denoted as ω_P, to be the set of all occurrences of P in $\mathbb{R}^d$, i.e.

$$\omega_P \equiv \{t \in \mathbb{R}^d : [B(t, s(P)) \wedge \omega] = P\}.$$

Now, let (Ω, T) be minimal. Choose $\omega \in \Omega$ and $P \in \mathcal{P}_B(\Omega)$. Then, the Voronoi construction applied to ω_P yields a decomposition of ω into cells

$$C(x, \omega, P) \equiv V(x, \omega_P) \wedge \omega, \quad x \in \omega_P.$$

More precisely,

$$\mathbb{R}^d = \cup_{x \in \omega_P} V(x, \omega_P), \text{ and } int(V(x, \omega_P)) \cap int(V(y, \omega_P)) = \emptyset,$$

whenever $x \neq y$. Here, $int(V)$ denotes the interior of V. This way of decomposing ω will be called the P-decomposition of ω. It is a crucial fact that each $C(x, \omega, P)$ is already determined by

$$B(x, 2R(P)) \wedge \omega,$$

as can be seen by Lemma 3.1, where $R(P)$ denotes the covering radius of ω_P. Thus, in particular the following holds.

Proposition 3.2. *Let (Ω, T) be a minimal DDS and $P \in \mathcal{P}_B(\Omega)$ be fixed. Let $\omega \in \Omega$ with $0 \in \omega_P$ and set $Q = B(0, 2R(P)) \wedge \omega$. Then $C(Q) \equiv [V(0, \omega_P) \wedge \omega]$ depends only on $[Q]$ (and not on ω). Moreover, if $\widetilde{C}$ is a cell occurring in the P-decomposition of some $\omega_1 \in \Omega$, then $[\widetilde{C}] = C(Q)$ for a suitable $\omega \in \Omega$ with $0 \in \omega_P$.*

The proposition says that the occurrences of certain cells in the P-decompositions are determined by the occurrences of the larger $Q \in \{[B(x, 2R(P)) \wedge \omega] : x \in \omega_P, \omega \in \Omega\}$. The proposition does not say that different Q induce different $C(Q)$ (and this will in fact not be true in general).

The main aim is now to study the decompositions associated to an increasing sequence of ball pattern classes (P_n). We begin by studying minimal and maximal distances between occurrences of a ball pattern class P. We need the following definition.

Definition 3.3. *Let (Ω, T) be minimal and $P \in \mathcal{P}_B$ be arbitrary. Define $r(P)$ as the packing radius of ω_P, i.e., by $r(P) \equiv \frac{1}{2} \inf\{\|x - y\| : x \neq y, x, y \in \omega_P, \omega \in \Omega\}$ and the occurrence radius $R(P)$ by $R(P) \equiv \inf\{R > 0 : \sharp_P([B(p, R) \wedge \omega]) \geq 1$ for every $p \in \mathbb{R}^d$ and $\omega \in \Omega\}$.*

Lemma 3.4. *Let (Ω, T) be minimal. Then $R(P) \equiv \min\{R > 0 : \sharp_P([B(p, R) \wedge \omega]) \geq 1$ for every $p \in \mathbb{R}^d$ and $\omega \in \Omega\}$. Moreover, ω_P is an $(r(P), R(P))$-set for every $\omega \in \Omega$.*

Proof. We show that the infimum is a minimum. Assume the contrary and set $R' := R(P)$. Then there exist $p \in \mathbb{R}^d$ and $\omega \in \Omega$ such that $B(p, R') \wedge \omega$ does not contain a copy of P. However, by definition of R', $B(p, R' + \epsilon) \wedge \omega$ contains a copy of P for every $\epsilon > 0$. As ω is a Delone set, $B(p, R' + 1) \wedge \omega$ contains only finitely many copies of P and a contradiction follows. The last statement of the lemma is immediate. □

We will have to deal with Delone sets which are not aperiodic. To do so the following notions will be useful. For a minimal DDS (Ω, T) let $\mathcal{L} \equiv \mathcal{L}(\Omega)$ be the periodicity lattice of (Ω, T), i.e.

$$\mathcal{L} \equiv \mathcal{L}(\Omega) \equiv \{t \in \mathbb{R}^d : T_t\omega = \omega \text{ for all } \omega \in \Omega\}.$$

Clearly, $\mathcal{L}$ is a subgroup of $\mathbb{R}^d$; it is discrete, since every ω is discrete. Thus, (see e.g. Proposition 2.3 in [37]), $\mathcal{L}$ is a lattice in $\mathbb{R}^d$, i.e. there exists $D(\mathcal{L}) \in \mathbb{N}$ and vectors $e_1, \ldots, e_{D(\mathcal{L})} \in \mathbb{R}^d$ which are linearly independent (in $\mathbb{R}^d$) such that

$$\mathcal{L} = \mathrm{Lin}_{\mathbb{Z}}\{e_j : j = 1, \ldots D(\mathcal{L})\} \equiv \{\sum_{j=1}^{D(\mathcal{L})} a_j e_j : a_j \in \mathbb{Z}, j = 1, \ldots, D(\mathcal{L})\}.$$

We define $r(\mathcal{L})$ by

$$r(\mathcal{L}) \equiv \begin{cases} \infty & : \; ; \text{if } \mathcal{L} = \{0\} \\ \frac{1}{2}\min\{\|t\| : t \in \mathcal{L} \setminus \{0\} & : \; ; \text{otherwise.} \end{cases}$$

Next, we provide a result on minimal distances, viz. Lemma 3.6. Variants of this result have been given in the literature on tilings [35] and on symbolic dynamics [11]. To prove it in our context we recall the following lemma concerning the natural topology from [33]:

Lemma 3.5. *A sequence (ω_n) of Delone sets converges to $\omega \in \mathcal{D}$ in the natural topology if and only if there exists for any $l > 0$ an $L > l$ such that the $\omega_n \cap B(0, L)$ converge to $\omega \cap B(0, L)$ with respect to the Hausdorff distance as $n \to \infty$.*

Lemma 3.6. *Let (Ω, T) be minimal. Let (P_n) be a sequence of ball pattern classes with $s(P_n) \longrightarrow \infty$, $n \longrightarrow \infty$. Then,*

$$\liminf_{n\to\infty} r(P_n) \geq r(\mathcal{L}).$$

Proof. As (Ω, T) is minimal, it is an (r, R)-system. Assume that the claim is false. Thus, there exists a sequence (P_n) in $\mathcal{P}_B(\Omega)$ with $s(P_n) \to \infty$, $n \to \infty$, but $r(P_n) \leq C$ with a suitable constant $C > 0$ with $C < r(\mathcal{L})$. Then, there exist $\omega_n \in \Omega$ and $t_n \in \mathbb{R}^d$ with $\|t_n\| \leq \frac{1}{2}C$ (and, of course, $\|t_n\| \geq \frac{1}{2}r$) with

$$B(0, s(P_n)) \wedge \omega_n = B(0, s(P_n)) \wedge (\omega_n - t_n). \tag{6}$$

By compactness of Ω and $B(0, \frac{1}{2}C)$, we can assume without loss of generality that $\omega_n \to \omega$ and $t_n \to t$, with $t \in B(0, C)$, $n \to \infty$. Thus, (6) implies

$$\omega = \omega - t. \tag{7}$$

In fact, let $p \in \omega$. Fix $R > 0$ such that $p \in \omega \cap B(0, R)$. By Lemma 3.5 we find $p_n \in \omega_n \cap B(0, R)$, for n sufficiently large, such that $p_n \to p$ for $n \to \infty$. Assuming $R < s(P_n)$ and utilizing (6) we find $q_n \in \omega_n$ such that $p_n = q_n - t_n$. Since $q_n \to p + t$ and $\omega_n \to \omega$ we see that $q = p + t \in \omega$ leaving us with

$$\omega \cap B(0, R) \subset (\omega - t) \cap B(0, R).$$

By symmetry and since R was arbitrary, this gives (7). Minimality yields that (7) extends to all $\omega \in \Omega$. As $0 < r \leq \|t\| \leq C < r(\mathcal{L})$, this gives a contradiction. □

Definition 3.7. *For a compact convex set $C \subset \mathbb{R}^d$ denote by $s(C) > 0$ the* inradius *of C, i.e. the largest s such that C contains a ball of radius s.*

In the sequel we write $\omega_{n,P_n} := (\omega_n)_{P_n}$ to shorten notation

Lemma 3.8. *Let (Ω_n, T), $n \in \mathbb{N}$ be a family of minimal DDS. Let a pattern class $P_n \in \mathcal{P}(\Omega_n)$, an $\omega_n \in \Omega_n$ and $x_n \in \omega$ be given for any $n \in \mathbb{N}$. If $r(P_n) \longrightarrow \infty$, $n \longrightarrow \infty$, then $s(V(x_n, \omega_{n,P_n})) \to \infty$, for $n \to \infty$.*

Proof. Without loss of generality we can assume that $x_n = 0$ for every $n \in \mathbb{N}$. By construction of $V_n \equiv V(x_n, \omega_{n,P_n})$, we have

$$\operatorname{dist}(0, \partial V_n) \geq \frac{r(P_n)}{2} = s(V_n),\ n \in \mathbb{N}.$$

This implies $s(V_n) \longrightarrow \infty$, $n \longrightarrow \infty$. □

Our next aim is to show that a sequence of convex sets with increasing inradii must be van Hove. We need the following two lemmas.

Lemma 3.9. *For every $d \in \mathbb{N}$, there exists a constant $c = c(d)$ with*

$$(1+s)^d - (1-s)^d \leq cs$$

for $|s| \leq 1$.

Proof. This follows by a direct computation. □

For $C \subset \mathbb{R}^d$ and $\lambda \geq 0$ we set

$$\lambda C \equiv \{\lambda x : x \in C\}.$$

Lemma 3.10. *Let C be a compact convex set in $\mathbb{R}^d$ with $B(0,s) \subset C$, then the inclusion*

$$C^h \setminus C_h \subset (1 + \frac{h}{s})C \setminus (1 - \frac{h}{s})|C|$$

holds, where we set $(1 - hs^{-1})C = \emptyset$ if $h > s$. In particular,

$$|C^h \setminus C_h| \leq \kappa \max\{\frac{h}{s}, \frac{h^d}{s^d}\}|C|,$$

with a suitable constant $\kappa = \kappa(d)$.

Proof. The first statement follows by convexity of C. The second is then an immediate consequence of the change of variable formula combined with the foregoing lemma. □

Lemma 3.11. *Let (C_n) be a sequence of convex sets in $\mathbb{R}^d$ with $s(C_n) \longrightarrow \infty$, $n \longrightarrow \infty$. Then (C_n) is a van Hove sequence.*

Proof. Let $h > 0$ be given and assume without loss of generality that $B(0, s(C_n)) \subset C_n$. The lemma follows from the foregoing lemma. □

The following consequence of the foregoing results is a key ingredient of our proof of Theorem 1.

Proposition 3.12. *Let (Ω, T) be minimal and aperiodic. Let (P_n) be a sequence in $\mathcal{P}_B(\Omega)$ with $s(P_n) \to \infty$, $n \to \infty$. Let $(\omega_n) \subset \Omega$ and $x_n \in \omega_{n,P_n}$ be arbitrary. Then, $V(x_n, \omega_{n,P_n})$ is a van Hove sequence.*

Proof. By Lemma 3.6, aperiodicity of (Ω,T) together with $s(P_n)\to\infty$, $n\to\infty$ yields $r(P_n)\to\infty$, $n\to\infty$. Therefore, by Lemma 3.8, we have $s(V(x_n,\omega_{n,P_n}))\to\infty$, for $n\to\infty$. Now, the statement is immediate from Lemma 3.11. □

We will also need an analogue of this proposition for arbitrary (i.e. not necessarily aperiodic) DDS. To obtain this analogue we will need some extra effort.

Let a minimal DDS (Ω,T) with periodicity lattice $\mathcal{L}$ be given. Let $U=U(\mathcal{L})$ be the subspace of $\mathbb{R}^d$ spanned by the e_j, $j=1,\ldots,D(\mathcal{L})$ and let $P_U:\mathbb{R}^d\longrightarrow U$ be the orthogonal projection onto U. The lattice $\mathcal{L}$ induces a grid on $\mathbb{R}^d$. Namely, we can set

$$G_0\equiv\{x\in\mathbb{R}^d : P_U x=\sum_{j=1}^{D(\mathcal{L})}\lambda_j e_j\ ;\text{with } 0\le\lambda_j<1, j=1,\ldots,D(\mathcal{L})\}$$

and

$$G_{(n_1,\ldots,n_{D(\mathcal{L})})}\equiv n_1e_1+\ldots+n_{D(\mathcal{L})}e_{D(\mathcal{L})}+G_0,$$

for $(n_1,\ldots,n_{D(\mathcal{L})})\in\mathbb{Z}^{D(\mathcal{L})}$.

We will now use coloring of Delone sets to obtain new DDS from (Ω,T). These new systems will essentially be the same sets but equipped with a coloring which "broadens" the periodicity lattice. Coloring has been discussed e.g. in [31].

Let C be a finite set. A Delone set with colorings in C is a subset of $\mathbb{R}^d\times C$ such that $p_1(\omega)$ is a Delone set, where $p_1:\mathbb{R}^d\times C$ is the canonical projection $p_1(x,c)=x$. When referring to an element (x,c) of a colored Delone set we also say that x is colored with c. Notions as patterns, pattern classes, occurrences, diameter etc. can easily be carried over to colored Delone sets.

Fix $\omega\in\Omega$ with $0\in\Omega$. For every $l\in\mathbb{N}$, we define a DDS as follows: Let $\omega^{(l)}$ be a Delone set with coloring in $\{0,1\}$ introduced by the following rule: $x\in\omega$ is colored with 1 if and only if there exists $(n_1,\ldots,n_{D(\mathcal{L})})\in\mathbb{Z}^{D(\mathcal{L})}$ with

$$x\in G_{(ln_1,\ldots,ln_{D(\mathcal{L})})},$$

in all other cases $x\in\omega$ is colored with 0. Set $\Omega^{(l)}\equiv\Omega(\omega^{(l)})\equiv\overline{\{T_t\omega^{(l)}:t\in\mathbb{R}^d\}}$, where the bar denotes the closure in the in the canonical topology associated to colored Delone sets [31]. Moreover, the DDS $(\Omega^{(l)},T)$ is minimal, as can easily be seen considering repetitions of patterns in $\omega^{(l)}$. Also, $(\Omega^{(l)},T)$ is uniquely ergodic if (Ω,T) is uniquely ergodic, as follows by considering existence of frequencies in $\omega^{(l)}$. The important point about $(\Omega^{(l)},T)$ is the following lemma.

Lemma 3.13. *Let (Ω,T) be a minimal DDS and $(\Omega^{(l)},T)$ for $l\in\mathbb{N}$ be constructed as above, then $r(\mathcal{L}(\Omega^{(l)}))=l\cdot r(\mathcal{L}(\Omega))$.*

Proof. This is immediate from the construction. □

Now, we can state the following analog of Proposition 3.12.

Proposition 3.14. *Let (Ω,T) be minimal and $(\Omega^{(n)},T)$, $n\in\mathbb{N}$ constructed as above. For each $n\in\mathbb{N}$, choose a pattern class $P_n\in\mathcal{P}(\Omega^{(n)})$. Let $(\omega_n)\subset\Omega^{(n)}$ and $x_n\in\omega_{n,P_n}$ be arbitrary. If $s(P_n)\to\infty$, $n\to\infty$, then, $V(x_n,\omega_{n,P_n})$ is a van Hove sequence.*

Proof. By the foregoing lemma and Lemma 3.6, we infer

$$r(P_n)\longrightarrow\infty, n\longrightarrow\infty.$$

The statement then follows as in the proof of Proposition 3.12. □

4. The ergodic Theorem

In this section we prove Theorem 1. The main idea of the proof is to combine the geometric decompositions studied in the last section with the almost additivity of F to reduce the study of F on large patterns to the study o F on smaller patterns.

Proof of Theorem 1. (ii) $\Longrightarrow$(i). For every $Q \in \mathcal{P}$ the function $P \mapsto \sharp_Q(P)$ is almost additive on $\mathcal{P}$. Thus, its average $\lim_{n\to\infty} |P_n|^{-1}\sharp_Q(P_n)$ exists along arbitrary van Hove sequences (P_n) in $\mathcal{P}$. But this easily implies (3) which in turn implies unique ergodicity, as discussed in Sectiuon 2.

(i) $\Longrightarrow$ (ii). Let $F : \mathcal{P}(\Omega) \longrightarrow \mathcal{B}$ be almost additive with error function b. Let (P_n) be a van Hove sequence in $\mathcal{P}(\Omega)$. We have to show that $\lim_{n\to\infty} |P_n|^{-1}F(P_n)$ exists. As $\mathcal{B}$ is a Banach space, it is clearly sufficient to show that $(|P_n|^{-1}F(P_n))$ is a Cauchy sequence. To do so we will provide $F^{(k)}$ in $\mathcal{B}$ such that

$$\||P_n|^{-1}F(P_n) - F^{(k)}\| \text{ is arbitrarily small for } n \text{ large and } k \text{ large.}$$

To introduce $F^{(k)}$ we proceed as follows: Fix $\omega \in \Omega$ with $0 \in \omega$. We will now first consider the case that (Ω, T) is aperiodic. The other case can be dealt with similarly. We will comment on this at the end of the proof. Let $B^{(k)}$ be the ball pattern class occurring in ω around zero with radius k i.e.

$$B^{(k)} \equiv [\omega \wedge B(0,k)]. \tag{8}$$

Thus, $(B^{(k)})$ is a sequence in $\mathcal{P}_B(\Omega)$ with $k = s(B^{(k)}) \longrightarrow \infty$ for$k \to \infty$ and the assumptions of Proposition 3.12 are satisfied.

As (Ω, T) is of finite local complexity, the set $\{[B(x, 2R(B^{(k)})) \wedge \omega] : x \in \omega, \omega \in \Omega \text{ with } [B(x,k) \wedge \omega] = B^{(k)}\}$ is finite. We can thus enumerate its elements by $B_j^{(k)}$, $j = 1, \ldots, N(k)$ with suitable $N(k) \in \mathbb{N}$ and $B_j^{(k)} \in \mathcal{P}(\Omega)$. Let $C_j^{(k)} \equiv C(B_j^{(k)})$ be the cells associated to $B_j^{(k)}$ according to Proposition 3.2. By Proposition 3.12,

$$(*) \quad (C_{j_k}^{(l_k)}) \text{ is a van Hove sequence}$$

for arbitrary $(l_k) \subset \mathbb{N}$ with $l_k \to \infty$, $k \to \infty$, and $j_k \in \{1, \ldots, N(l_k)\}$. This will be crucial. Denote the frequencies of the $B_j^{(k)}$ by $f(B_j^{(k)})$, i.e.

$$f(B_j^{(k)}) = \lim_{n\to\infty} |P_n|^{-1}\sharp_{B_j^{(k)}} P_n. \tag{9}$$

Define

$$F^{(k)} \equiv \sum_{j=1}^{N(k)} f(B_j^{(k)})F(C_j^{(k)}).$$

Choose $\epsilon > 0$. We have to show that

$$\||P_n|^{-1}F(P_n) - F^{(k)}\| < \epsilon, \text{ for } n \text{ and } k \text{ large}$$

(as this will imply that $|P_n|^{-1}F(P_n)$ is a Cauchy sequence). By $(*)$, there exists $k(\epsilon) > 0$ with

$$|C_j^{(k)}|^{-1}b(C_j^{(k)}) < \frac{\epsilon}{3} \text{ for every } j = 1, \ldots, N(k) \tag{10}$$

whenever $k \geq k(\epsilon)$. (Otherwise, we could find (l_k) in $\mathbb{N}$ and $j_k \in \{1, \ldots, N(l_k)\}$ with $l_k \to \infty, k \to \infty$ such that

$$|C_{j_k}^{(l_k)}|^{-1}b(C_{j_k}^{(l_k)}) \geq \frac{\epsilon}{3}.$$

Since $(C_{j_k}^{(k)})$ is a van Hoove sequence by $(*)$, this contradicts property (A4) from Definition 2.3.)

Let $P \in \mathcal{P}$ be an arbitrary pattern class. By minimality of (Ω, T), we can choose $Q = Q(P) \subset \mathbb{R}^d$ with $[Q \wedge \omega] = P$.

The idea is now to consider the decomposition of $\omega \wedge Q$ induced by the $B^{(k)}$-decomposition of ω. This decomposition of $\omega \wedge Q$ will (up to a boundary term) consist of representatives of $C_j^{(k)}$, $j = 1, \ldots, N(k)$. For $P = P_n$ with $n \in \mathbb{N}$ large, the number of representatives of a $C_j^{(k)}$ for j fixed occurring in $Q \wedge \omega$ will essentially be given by $f(C_j^{(k)})|P_n|$. Together with the almost-additivity of F, this will allow us to relate $F(P_n)$ to $F^{(k)}$ in the desired way. Here are the details:

Let $I(P,k) \equiv \{x \in \omega_{B^{(k)}} : B(x, 2R(B^{(k)})) \subset Q\}$. Then, by Lemma 3.1 and Proposition 3.2

$$Q \wedge \omega = S \wedge \omega \oplus \bigoplus_{x \in I(P,k)} C(x, \omega, B^{(k)}) \tag{11}$$

with a suitable surface type set $S \subset \mathbb{R}^d$ with

$$S \subset Q \setminus Q_{4R(B^{(k)})}. \tag{12}$$

The triangle inequality implies

$$\begin{aligned} \|\frac{F(P)}{|P|} - F^{(k)}\| \quad &\leq \quad \|\frac{F(P) - F([S \wedge \omega]) - \sum_{x \in I(P,k)} F([C(x, \omega, B^{(k)})])\|}{|P|} \\ &\quad + \|\frac{F([S \wedge \omega]) + \sum_{x \in I(P,k)} F([C(x, \omega, B^{(k)})])}{|P|} - F^{(k)}\| \\ &\equiv D_1(P,k) + D_2(P,k). \end{aligned}$$

The terms $D_1(P,k)$ and $D_2(P,k)$ can be estimated as follows.

By almost additivity of F, we have

$$\begin{aligned} D_1(P,k) \quad &\leq \quad \frac{b([S \wedge \omega])}{|P|} + \sum_{x \in I(P,k)} \frac{b([C(x, \omega, B^{(k)})])}{|C(x, \omega, B^{(k)})|} \frac{|C(x, \omega, B^{(k)})|}{|P|} \\ &\leq \quad \frac{b(P) + b([\oplus_{x \in I(P,k)} C(x, \omega, B^{(k)})])}{|P|} \\ &\qquad + \sup\left\{ \frac{b([C(x, \omega, B^{(k)})])}{|C(x, \omega, B^{(k)})|} \, : \, x \in I(P,k) \right\}. \end{aligned}$$

In the last inequality we used (A3).

Fix $k = k(\epsilon)$ from (10) and consider the above estimate for $P = P_n$. Then,

$$D_1(P_n, k) \leq \frac{b(P_n) + b([\oplus_{x \in I(P,k)} C(x, \omega, B^{(k)})])}{|P_n|} + \frac{\epsilon}{3}.$$

As (P_n) is a van Hove sequence, it is clear from (12) that $([\oplus_{x \in I(P,k)} C(x, \omega, B^{(k)})])$ is a van Hove sequence as well. Thus

$$\frac{b(P_n) + b([\oplus_{x \in I(P_n,k)} C(x, \omega, B^{(k)})]}{|P_n|} = \frac{b(P_n)}{|P_n|}$$

$$+\frac{b([\oplus_{x\in I(P_n,k)}C(x,\omega,B^{(k)})])}{|[\oplus_{x\in I(P_n,k)}C(x,\omega,B^{(k)})]|}\frac{|[\oplus_{x\in I(P_n,k)}C(x,\omega,B^{(k)})]|}{|P_n|}$$

tends to zero for n tending to infinity by the definition of b. Putting this together, we infer

$$D_1(P_n,k) < \frac{\epsilon}{2}$$

for large enough $n \in \mathbb{N}$.

Consider now D_2. Invoking the definition of $F^{(k)}$, we clearly have

$$D_2(P,k) \leq \frac{\|F([S\wedge\omega])\|}{|P|}$$

$$+\sum_{j=1}^{N(k)}\left|\frac{\sharp\{x\in I(P,k) : [B(x,2R(B^k))\wedge\omega] = B_j^{(k)}\}}{|P|} - f(B_j^{(k)})\right| \|F(C_j^{(k)})\|.$$

Choose k as above and consider $P = P_n$. By (12) and the almost additivity of F (property (A2)), we infer that the first term tends to zero for n tending to infinity. Again by (12) and the definition of the frequency, we infer that the second term tends to zero as well. Thus,

$$D_2(P_n,k) < \frac{\epsilon}{2}$$

for n large. Putting these estimates together, we infer

$$\||P_n|^{-1}F(P_n) - F^{(k)}\| \leq D_1(n,k) + D_2(n,k) < \epsilon$$

for large n and the proof is finished for aperiodic DDS.

For arbitrary strictly ergodic DDS, we replace the definition of $B^{(k)}$ in (8), by

$$B^{(k)} \equiv [B(0,k)\wedge\omega^{(k)}],$$

where $\omega^{(k)} \in \Omega^{(k)}$ is defined via colouring; see the paragraphs preceding Lemma 3.13 in Section 3. Then $B^{(k)}$ belongs to $\mathcal{P}_B(\Omega^{(k)})$ for every $k \in \mathbb{N}$ and

$$s^k \equiv s(B^{(k)}) \longrightarrow \infty, k \longrightarrow \infty.$$

Thus, Proposition 3.14 applies. The proof then proceeds along the same lines as above, with Ω replaced by $\Omega^{(k)}$ and Proposition 3.12 replaced by Proposition 3.14 at the corresponding places. □

Remark 3. *Using what could be called the* k-cells, $C_j^{(k)}$, $k \in \mathbb{N}, j = 1,\dots,N(k)$ *from the preceding proof we have actually proven that*

$$\lim_{k\to\infty}\sum_{j=1}^{N(k)} f(B_j^{(k)})F(C_j^{(k)}) = \lim_{n\to\infty}\frac{F(P_n)}{|P_n|}.$$

Proof of Corollary 2.4. We use the notation of the corollary. Apparently, the reasoning yielding (i) $\Longrightarrow$ (ii) in the foregoing proof remains valid for arbitrary seminorms $\|\cdot\|$. Thus, if F is almost-additive with respect to seminorms $\|\cdot\|_\iota$, $\iota \in \mathcal{I}$, then $(|P_n|^{-1}F(P_n))$, is a Cauchy sequence with respect to $\|\cdot\|_\iota$ for every $\iota \in \mathcal{I}$. The corollary now follows from completeness. □

Proof of Corollary 2.6. This can be shown by mimicking the arguments in the above proof. Alternatively, one can define the function $\widetilde{F} : \mathcal{P} \longrightarrow \mathcal{B}$ by setting $\widetilde{F}(P) := F(Q,\omega)$, where (Q,ω) is arbitrary with $P = [\omega \wedge Q]$. This definition may seem very arbitrary. However, by (A0), it is not hard to see that $\widetilde{F}(P)$ is (up to a boundary term) actually independent of the actual choice of Q and ω. By the same kind of reasoning, one infers that $\widetilde{F}$ is almost-additive. Now, existence of the limits $|P_n|^{-1}\widetilde{F}(P_n)$ follows for arbitrary van Hove sequnences (P_n). Invoking (A0) once more the corollary follows. □

5. Uniform convergence of the integrated density of states

This section is devoted to a proof of Theorem 2 and Theorem 3. We need some preparation.

Lemma 5.1. *Let B and C be selfadjoint operators in a finite dimensional Hilbert space. Then, $|n(B)(E) - n(B+C)(E)| \leq rank(C)$ for every $E \in \mathbb{R}$, where $n(D)$ denotes the eigenvalue counting function of D, i.e. $n(D)(E) \equiv \sharp\{$Eigenvalues of D not exceeding $E\}$.*

Proof. This is a consequence of the minmax principle, see e.g. Theorem 4.3.6 in [17] for details. □

From this lemma we infer the following proposition.

Proposition 5.2. *Let U be a subspace of the finite dimensional Hilbert space X with inclusion $j : U \longrightarrow X$ and orthogonal projection $p : X \longrightarrow U$. Then, $|n(A)(E) - n(pAj)(E)| \leq 4 \cdot rank(1 - j \circ p)$ for every selfadjoint operator A on X.*

Proof. Let $P : X \longrightarrow X$ be the orthogonal projection onto U, i.e. $P = j \circ p$. Set $P^\perp \equiv 1 - P$ and denote the range of $P^\perp$ by $U^\perp$. By

$$A - PAP = P^\perp AP + PAP^\perp + P^\perp AP^\perp,$$

and the foregoing lemma, we have $|n(A)(E) - n(PAP)(E)| \leq 3rank(P^\perp)$. As obviously,

$$PAP = pAj \oplus 0_{U^\perp},$$

with the zero operator $0_{U^\perp} : U^\perp \longrightarrow U^\perp$, $f \mapsto 0$, we also have

$$|n(PAP)(E) - n(pAj)(E)| \leq dim(U^\perp).$$

As $\dim U^\perp = rank(P^\perp)$, we are done. □

Lemma 5.3. *Let (Ω, T) be an (r, R)-system and $\omega \in \Omega$ and Q a bounded subset of $\mathbb{R}^d$. Then,*

$$\sharp Q \cap \omega \leq \frac{1}{|B(0,r)|} |Q^r|.$$

Proof. As (Ω, T) is an (r, R)-system, balls with radius r around different points in ω are disjoint and the lemma follows. □

Our main tool will be the following consequence of the foregoing two results.

Proposition 5.4. *Let (Ω, T) be an (r, R)-system. Let $Q, Q_j \subset \mathbb{R}^d$, $j = 1, \ldots, n$ be given with $Q = \cup_{j=1}^n Q_j$ and the Q_j pairwise disjoint up to their boundaries. Set $\delta(\omega, s) \equiv |\dim \ell^2(Q_s \cap \omega) - \dim \ell^2(\cup_{j=1}^n (Q_{j,s} \cap \omega))|$ for $\omega \in \Omega$ and $s > 0$ arbitrary. Then,*

$$\begin{aligned} \delta(\omega, s) &\leq |\dim \ell^2(Q \cap \omega) - \dim \ell^2(\cup_{j=1}^n (Q_{j,s} \cap \omega))| \\ &\leq \frac{1}{|B(0,r)|} \sum_{j=1}^n |Q_j^r \setminus Q_{j,s+r}|. \end{aligned}$$

Proof. Apparently

$$\dim \ell^2(\cup_{j=1}^n (Q_{j,s}\cap\omega)) \leq \dim \ell^2(Q_s\cap\omega) \leq \dim \ell^2(Q\cap\omega).$$

Now the first inequality is clear and the second follows by

$$\begin{aligned}\dim \ell^2(Q\cap\omega) - \dim \ell^2(\cup_{j=1}^n Q_{j,s}\cap\omega) &\leq \sum_{j=1}^n \sharp((Q_j\setminus Q_{j,s})\cap\omega)\\ &\leq \frac{1}{|B(0,r)|}\sum_{j=1}^n |Q_j^r\setminus Q_{j,s+r}|.\end{aligned}$$

Here, the last inequality follows by the foregoing lemma. □

We are now able to prove Theorem 2.

Proof of Theorem 2. We have to provide $b:\mathcal{P}(\Omega)\longrightarrow(0,\infty)$, and $D>0$ such that (A1) , (A2) and (A3) of Definition 2.3 are satisfied. Set

$$D \equiv \frac{2}{|B(0,r)|}$$

and define b by

$$b(P) \equiv \frac{8}{|B(0,r)|}|Q^r\setminus Q_{R^A+r}|$$

whenever $P\in\mathcal{P}(\Omega)$ with $P=[(Q,\Lambda)]$. Apparently, b is well defined. Moreover, (A4) follows by the very definition of b and the van Hove property.

Now, (A2) is satisfied as

$$\|F^A(P)\| = \|n(A_\omega, Q_{R^A})\|_\infty \leq \sharp(Q_{R^A}\cap\omega) \leq \frac{1}{B(0,r)}|Q^r| \leq D|P| + b(P),$$

for $P=[Q\wedge\omega]$. (A3) can be shown by a similar argument. It remains to show (A1). Let $P=\oplus_{j=1}^n P_j$. Then, there exists $\omega\in\Omega$ and bounded measurable sets $Q, Q_j,\ j=1,\ldots,n$ in $\mathbb{R}^d$ with Q_j pairwise disjoint up to their boundaries and $Q=\cup_{j=1}^n Q_j$ such that

$$P=[Q\wedge\omega] \text{ and } P_j=[Q_j\wedge\omega], j=1,\ldots,n.$$

As A is an operator of finite range, it follows from the definition of R^A that

$$A_\omega|_{\cup_{j=1}^n Q_{j,R^A}} = \oplus_{j=1}^n A_\omega|_{Q_{j,R^A}}$$

and in particular,

$$\sum_{j=1}^n n(A_\omega, Q_{j,R^A}) = n(A_\omega, \cup_{j=1}^n Q_{j,R^A}). \tag{13}$$

Thus, we can calculate as follows

$$\begin{aligned} \|F^A(P) - \sum_{j=1}^n F(P_j)\| &= \|n(A_\omega, Q_{R^A}) - \sum_{j=1}^n n(A_\omega, Q_{j,R^A})\|_\infty \\ &= \|n(A_\omega, Q_{R^A}) - n(A_\omega, \cup_{j=1}^n Q_{j,R^A})\|_\infty \\ \text{(Prop 5.2)} \quad &\leq 4(\dim \ell^2(Q_{R^A}) - \dim \ell^2(\cup_{j=1}^n Q_{j,R^A}) \\ \text{(Prop 5.4)} \quad &\leq \frac{4}{|B(0,r)|} \sum_{j=1}^n |Q_j^r \setminus Q_{j,R^A+r}| \\ &\leq \sum_{j=1}^n b(P_j). \end{aligned}$$

This finishes the proof. □

We can now proceed to show Theorem 3. The theorem will be an immediate consequence of Theorem 1 and Theorem 2, once we have proven the following lemma.

Lemma 5.5. *Let (Ω, T) be a strictly ergodic (r, R)-system. Let A be a finite range operator with range R^A. Then, $\|n(A_\omega, Q) - F^A([\omega \wedge Q])\|_\infty \leq 4|B(0,r)|^{-1}|Q^r \setminus Q_{R^A+r}|$ for all $\omega \in \Omega$ and all bounded subsets Q in $\mathbb{R}^d$.*

Proof By definition of F^A, we have

$$\|n(A_\omega, Q) - F^A([\omega \wedge Q])\|_\infty = \|n(A_\omega, Q) - n(A_\omega, Q_{R^A})\|_\infty.$$

Invoking Proposition 5.2, we see that the difference is bounded by $4\sharp(Q \setminus Q_{R^A}) \wedge \omega$. The statement of the lemma now follows by Lemma 5.3. □.

Proof of Theorem 3. Let (Q_n) be a van Hove sequence. Then $([Q_n \wedge \omega])$ is a van Hove sequence in $\mathcal{P}(\Omega)$ independent of ω. Thus, $|Q_n|^{-1} F^A([Q_n \wedge \omega])$ converges uniformly in $\omega \in \Omega$ by Theorem 1 and Theorem 2. The proof follows from the foregoing lemma. □

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CHAPTER 11

M. Baake, D. Lenz, *Dynamical systems on translation bounded measures and pure point diffraction*, Ergodic Theory & Dynamical Systems 24 (2004), 1867–1893.

DYNAMICAL SYSTEMS ON TRANSLATION BOUNDED MEASURES: PURE POINT DYNAMICAL AND DIFFRACTION SPECTRA

MICHAEL BAAKE AND DANIEL LENZ

Abstract. Certain topological dynamical systems are considered that arise from actions of σ-compact locally compact Abelian groups on compact spaces of translation bounded measures. Such a measure dynamical system is shown to have pure point dynamical spectrum if and only if its diffraction spectrum is pure point.

1. Introduction

This paper deals with certain dynamical systems build from measures on σ-compact locally compact Abelian groups. These dynamical systems give rise to two spectra: the dynamical spectrum and the diffraction spectrum. After introducing the dynamical systems and discussing their basic topological features, we will focus on studying the relationship between these two spectra. Particular attention will be paid to the case where one of the spectra is pure point. This will be shown to happen if and only if the other is pure point as well (read on for details and a discussion of related results.)

The motivation for our study comes from physics and, more precisely, from the study of solids with long range aperiodic order and crystal-like diffraction spectrum. Such solids are known as genuine quasicrystals. The existence of quasicrystals is now a well-established and widely accepted experimental fact. Even if discussions about the precise structures will still be going on for a while, the common feature of aperiodicity has opened a new chapter of crystallography and solid state research.

The original discovery of quasicrystals [35, 19] was somewhat accidental and only possible through one of their most striking features, namely their sharp Bragg diffraction with point symmetries that are not possible for 3-dimensional crystals (such as n-fold rotation axes with $n = 8, 10, 12$, or icosahedral symmetry); see [1] for a summary and [2] for a guide to the literature. These experimental results called for a mathematical explanation and created a subject now often referred to as mathematical diffraction theory.

Mathematical diffraction theory deals with the Fourier transform of the autocorrelation measure (or Patterson measure) of a given translation bounded (possibly complex) measure ω. Here, ω is the mathematical idealisation of the physical structure of a solid or, more generally, of any state of matter. In its simplest form, ω is just a Dirac comb, i.e., a countable collection of (possibly weighted) point measures which mimic the positions of the atoms (and their scattering strengths). The autocorrelation measure γ_ω of ω (see below for a precise definition) is then a positive definite measure. Its Fourier transform $\widehat{\gamma_\omega}$ is a positive measure, called the diffraction measure, which models the outcome of a diffraction experiment; see [10] for background material and details on the physical justification of this approach.

Now, given this setting, one of the most obvious questions to address is that for (the characterisation of all) examples of measures ω with a diffraction measure $\widehat{\gamma_\omega}$ that is pure point, i.e., consists of point measures only.

This was addressed by Bombieri and Taylor in [8]. However, no rigorous answer could be given at that time. Soon after, Hof [17] showed that structures obtained from the cut-and-project formalism [21] possess a pure point diffraction spectrum under rather decent assumptions, and Solomyak started a rather systematic study of substitution dynamical systems with pure point spectrum in [37]. By now, large classes of examples are known [5, 22], also beyond the class of ordinary projection sets [6]. Moreover, Schlottmann was able to free the cut-and-project formalism from basically all specific properties of Euclidean space [33] and established that all regular model sets are pure point diffractive [34], see also [26] for a summary on model sets.

A cornerstone in many of these considerations is the use of ergodic theory and the so-called Dworkin argument [11] (see [18, 34] as well). This argument links the diffraction spectrum to the dynamical spectrum. It can be used to infer pure point diffraction spectrum from pure point dynamical spectrum. These investigations heavily depend on the underlying point sets being point sets of finite local complexity (FLC). This, however, is not necessary, as becomes clear from two alternative approaches, one for general Dirac combs and measures on the basis of almost periodicity by Moody and one of the authors [5], and the other for so-called deformed model sets by Bernuau and Duneau [9].

Thus, at the moment, there is a considerable gap between the cases that can be treated by the method of almost periodicity of measures [14, 5] and those using ergodic theory and requiring FLC together with unique ergodicity. It is the primary aim of this article to narrow this gap. This will be achieved by thoroughly analysing the link between the diffraction spectrum and the dynamical spectrum given by the Dworkin argument.

The analysis carried out below will also be a crucial ingredient in a forthcoming paper of ours [4] which investigates the stability of pure point diffraction. Namely, we will set up a perturbation theory for pure point diffraction by studying deformations of dynamical systems with pure point dynamical spectrum. Particular emphasis will be put on deformed model sets and isospectral deformations of Delone sets. Note that the deformation of model sets almost immediately leads to point sets which violate FLC.

Let us now discuss our results in more detail. The first step in our approach is to choose a setting of measures rather than point sets. Defining appropriate dynamical systems with measures on a locally compact Abelian group will free us from essentially all restrictions mentioned. This setting is presented in Section 3, where also the relevant topological questions are discussed. The relationship between our measure dynamical systems and point dynamical systems is investigated in Section 4. It is shown that the measure dynamical systems enclose the usual Delone dynamical systems. This section introduces a topology on Delone sets (and actually all closed subsets of the group) with very nice compactness properties. The results generalise and strengthen the corresponding considerations of [34, 23] and may be of independent interest in further studies of point sets not satisfying FLC. The extension of diffraction theory and the Dworkin argument (as developed for point sets in, e.g., [11, 34]) to our setting is achieved in Section 5.

The main result of our paper is summarised in Theorem 7 in Section 7. It states that, under some rather mild assumptions,

- pure point dynamical spectrum is equivalent to pure point diffraction spectrum.

This generalises the main results of Lee, Moody and Solomyak [22] in at least three ways: It is not restricted to dynamical systems arising from Delone sets, and, in fact, not even to dynamical systems arising from point sets. It does not need any condition of finite local complexity. It does not need an ergodicity assumption for the invariant measure involved.

Let us mention that a generalisation of the main result of [22] sharing the last two features had already been announced by Gouéré [15, 16], within the framework of point processes and Palm measures, see his recent work [15] for a study of this framework as well. His result applies to the point dynamical systems studied below in Section 4. Thus, there is some overlap between his result an ours. However, in general, our setting, methods and results are quite different from his, as we leave the scenario of point sets. In fact, the measure theoretical setting seems very adequate and natural in view of the physical applications, where point sets are only a somewhat crude approximation of the arrangements of scatterers.

Let us also mention that, in general, the diffraction spectrum and the dynamical spectrum can be of different type, as has been investigated by van Enter and Miękisz in [12].

Our proof of the equivalence of the two notions of pure pointedness relies on two results which are of interest in their own right. These results are

- an abstract characterisation of pure point dynamical spectrum for arbitrary topological dynamical systems,
- a precise interpretation of the Dworkin argument.

Here, the abstract characterisation is achieved in Theorem 1 in Section 2. Roughly speaking, it states that a system has pure point spectrum once it has a lot of point spectrum. The precise interpretation of the Dworkin argument is given in Theorem 6 in Section 6. It says that the diffraction measure is a spectral measure for a suitable subrepresentation of the translation action at hand.

The relationship between suitable subrepresentation with pure point spectrum and the original representation can actually be analysed in more detail. To do so, we take a second look at the abstract theory in Section 8. Namely, we discuss how the group of all eigenvalues and the continuity of eigenfunctions is already determined by the set of eigenvalues and continuity of eigenfunctions associated to a suitable subrepresentation with pure point spectrum.

This material is rather general and may be of independent interest. Here, we apply it to our topological measure dynamical systems. This gives a criterion for the continuity of the eigenfunctions. More importantly, it shows that the group of all eigenvalues is generated by the support of the diffraction spectrum. The validity of such a result was brought to our attention by R. V. Moody for the case of point dynamical systems satisfying FLC [27].

The material presented above, and the abstract strategy to prove our main result, can be adopted to study a measurable framework (as opposed to a topological one). This will be analysed in the future.

2. An abstract criterion

In this section, we introduce some notation and provide a simple result which lies at the heart of our considerations. It is rather general and might also be useful elsewhere.

Let Ω be a compact topological space (by which we mean to include the Hausdorff property) and G be a locally compact Abelian (LCA) group which is σ-compact. Let

$$\alpha\colon\, G \times \Omega \;\longrightarrow\; \Omega \tag{1}$$

be a continuous action of G on Ω, where, of course, $G \times \Omega$ carries the product topology (later on, we will specify it via $\alpha_t(P) = t + P$ for $P \subset G$ and $t \in G$). Then, (Ω, α) is called a *topological dynamical system*. The set of continuous functions on Ω will be denoted by $C(\Omega)$. Let m be a G-invariant probability measure on Ω and denote the corresponding set of square integrable functions on Ω by $L^2(\Omega, m)$. This space is equipped with the inner product $\langle f, g\rangle := \int \overline{f(\omega)} g(\omega) \,\mathrm{d}m(\omega)$. The action α induces a unitary representation T of G on $L^2(\Omega, m)$ in the obvious way, where $T^t h$ is defined by $(T^t h)(\omega) := h(\alpha_{-t}(\omega))$. Whenever we want to emphasise the dependence of the inner product and the unitary representation on the chosen invariant measure m, we write $\langle f, g\rangle_m$ and T_m instead of $\langle f, g\rangle$ and T.

The dual group of G is denoted by $\widehat{G}$, and the pairing between a character $\hat{s} \in \widehat{G}$ and an element $t \in G$ is written as $(\hat{s}, t)$, which, of course, is a number on the unit circle, compare [31, Ch. 4] for background material.

A non-zero $h \in L^2(\Omega, m)$ is called an *eigenvector* (or eigenfunction) of T if there exists an $\hat{s} \in \widehat{G}$ with $T^t h = (\hat{s}, t) h$ for every $t \in G$. The closure (in $L^2(\Omega, m)$) of the linear span of all eigenfunctions of T will be denoted by $\mathcal{H}_{pp}(T)$.

The following is a variant (and an extension) of a result from [22].

Lemma 1. *Let (Ω, α) be a topological dynamical system with an invariant measure m. Then, $\mathcal{H}_{pp}(T) \cap C(\Omega)$ is a subalgebra of $C(\Omega)$ which is closed under complex conjugation and contains all constant functions. Similarly, $\mathcal{H}_{pp}(T) \cap L^\infty(\Omega, m)$ is a subalgebra of $L^\infty(\Omega, m)$ that is closed under complex conjugation and contains all constant functions.*

Proof. We only show the statement about $\mathcal{H}_{pp}(T) \cap C(\Omega)$. The other result can be shown in the same way.

The set $\mathcal{H}_{pp}(T) \cap C(\Omega)$ is a vector space because it is the intersection of two vector spaces. Moreover, every constant (non-vanishing) function is obviously continuous and an eigenvector of T (with eigenvalue 1, i.e., with the trivial character $(\hat{s}, t) \equiv 1$).

It remains to be shown that $\mathcal{H}_{pp}(T) \cap C(\Omega)$ is closed under complex conjugation and under forming products.

Closedness under complex conjugation: Let f be an eigenfunction of T to, say, $\hat{s}$. Then, $\overline{f}$ is an eigenfunction of T to the character $\hat{s}^{-1}$. Here, of course, the inverse $\hat{s}^{-1}$ of $\hat{s} \in \widehat{G}$ is given by $t \mapsto \overline{(\hat{s}, t)}$, where the bar denotes complex conjugation. Using this, it is not hard to see that $\mathcal{H}_{pp}(T)$ is closed under complex conjugation. As this is true of $C(\Omega)$ as well, we see that the intersection $\mathcal{H}_{pp}(T) \cap C(\Omega)$ is closed under complex conjugation.

Closedness under products: This is shown in Lemma 3.7 in [22] in the case that m is not only translation invariant but also ergodic. To adopt their argument to the case at hand, we note that every eigenfunction can be approximated arbitrarily well (in $L^2(\Omega, m)$) by bounded eigenfunctions via a simple cut-off procedure. More precisely, if f is an eigenfunction, then $|f|$ is an α-invariant function. Therefore, for an arbitrary $N > 0$, the function

$$f^N(\omega) \;:=\; \begin{cases} f(\omega), & |f(\omega)| \le N \\ 0\,, & \text{otherwise} \end{cases} \tag{2}$$

is again an eigenfunction (with the same $\hat{s}$ as f). Apparently, the f^N converge to f in $L^2(\Omega, m)$ as $N \to \infty$.

After this preliminary consideration, we can conclude the proof following [22]: Let two functions $f, g \in \mathcal{H}_{pp}(T) \cap C(\Omega)$ be given. Then, fg belongs to $C(\Omega)$. It remains to be shown that it belongs to $\mathcal{H}_{pp}(T)$ as well.

Choose $\varepsilon > 0$ arbitrarily. Observe that $\|g\|_\infty < \infty$, as $g \in C(\Omega)$ with Ω compact. Since f is in $\mathcal{H}_{pp}(T)$, there exists a finite linear combination $f' = \sum a_i f_i$ of eigenfunctions of T with

$$\|f - \sum a_i f_i\|_2 \;\le\; \frac{\varepsilon}{\|g\|_\infty}\,.$$

By the preliminary consideration around Eq. (2), we can assume that all f_i are bounded functions. Thus, in particular, $\|f'\|_\infty < \infty$.

Similarly, choose another finite linear combination $g' = \sum b_j g_j$ of bounded functions g_j in $\mathcal{H}_{pp}(T)$ with

$$\|g - \sum b_j g_j\|_2 \;\le\; \frac{\varepsilon}{\|f'\|_\infty}\,.$$

Then,

$$\|fg - f'g'\|_2 \;\le\; \|f'\|_\infty \, \|g - g'\|_2 + \|g\|_\infty \, \|f - f'\|_2 \;\le\; 2\varepsilon.$$

The proof is complete by observing that $f'g'$ is in $\mathcal{H}_{pp}(T)$ because the product of bounded eigenfunctions is again a bounded eigenfunction. □

With Lemma 1, the following result is a rather direct consequence of the Stone-Weierstraß Theorem.

Theorem 1. *Let (Ω, α) be a topological dynamical system with invariant probability measure m. Then, the following assertions are equivalent.*

(a) *T has pure point spectrum, i.e., $\mathcal{H}_{pp}(T) = L^2(\Omega, m)$.*
(b) *There exists a subspace $\mathcal{V} \subset \mathcal{H}_{pp}(T) \cap C(\Omega)$ which separates points.*

Proof. (a) $\Longrightarrow$ (b): This is clear, as one can take $\mathcal{V} = \mathcal{H}_{pp}(T) \cap C(\Omega) = C(\Omega)$.
(b) $\Longrightarrow$ (a): By Lemma 1, $\mathcal{H}_{pp}(T) \cap C(\Omega)$ is an algebra which is closed under complex conjugation and contains the constant functions. It also separates points as it contains a subspace, $\mathcal{V}$, with this property by (b). Thus, we can apply the Stone-Weierstraß Theorem (compare [28, Thm. 4.3.4]), to conclude that $\mathcal{H}_{pp}(T) \cap C(\Omega)$ is dense in $C(\Omega)$. By standard measure theory, see [28, Prop. 6.4.11], $\mathcal{H}_{pp}(T)$ is then dense in $L^2(\Omega, m)$ as well. As $\mathcal{H}_{pp}(T)$ is closed, statement (a) follows. □

3. Measure dynamical systems

For the remainder of the paper, let G be a fixed σ-compact LCA group with identity 0. Integration with respect to Haar measure is denoted by $\int_G \ldots \,\mathrm{d}t$, and the measure of a subset D of G is denoted by $|D|$. The vector space of complex valued continuous functions on G with compact support is denoted by $C_c(G)$. It is made into a locally convex space by the inductive limit topology, as induced by the canonical embeddings

$$C_K(G) \;\hookrightarrow\; C_c(G)\,, \quad K \subset G \text{ compact}.$$

Here, $C_K(G)$ is the space of complex valued continuous functions on G with support in K, which is equipped with the usual supremum norm $\|.\|_\infty$. The support of $\varphi \in C_c(G)$ is denoted by $\mathrm{supp}(\varphi)$.

The dual $C_c(G)^*$ of the locally convex space $C_c(G)$ is denoted by $\mathcal{M}(G)$. The space $\mathcal{M}(G)$ carries the vague topology. This topology equals the weak-$*$ topology of $C_c(G)^*$, i.e., it is the weakest topology which makes all functionals $\mu \mapsto \mu(\varphi)$, $\varphi \in C_c(G)$, continuous. As is well known (see e.g. [28, Thm. 6.5.6] together with its proof), every $\mu \in \mathcal{M}(G)$ gives rise to a unique $|\mu| \in \mathcal{M}(G)$, called the *total variation* of μ, which satisfies

$$|\mu|(\varphi) \;=\; \sup\{|\mu(\psi)| : \psi \in C_c(G,\mathbb{R}) \text{ with } |\psi| \leq \varphi\}$$

for every $\varphi \in C_c(G)_+$. Apparently, the total variation $|\mu|$ is positive, i.e., $|\mu|(\varphi) \geq 0$ for all $\varphi \in C_c(G)_+$. In particular, it can be identified with a measure on the σ-algebra of Borel sets of G that satisfies

- $|\mu|(K) < \infty$ for every compact set $K \subset G$,
- $|\mu|(A) = \sup\{|\mu|(K) : K \subset A, K \text{ compact}\}$ for every Borel set $A \subset G$

(see, e.g., [28, Thm. 6.3.4]). As G is σ-compact, we furthermore have

- $|\mu|(A) = \inf\{|\mu|(B) : A \subset B, B \text{ open}\}$ for every Borel set $A \subset G$

by [28, Prop. 6.3.6], i.e., $|\mu|$ is an (unbounded) regular Borel measure, in line with the Riesz-Markov Theorem [30, Thm. IV.18]. Finally, we note that there exists, by [28, Thm. 6.5.6], a measurable function $u\colon G \longrightarrow \mathbb{C}$ with $|u(t)| = 1$ for $|\mu|$-almost every $t \in G$ such that

$$\mu(\varphi) \;=\; \int_G \varphi\, u \,\mathrm{d}|\mu| \qquad \text{for all } \varphi \in C_c(G). \tag{3}$$

This polar decomposition permits us to identify the elements of $\mathcal{M}(G)$ with the regular complex Borel measures on G, which is the Riesz-Markov Theorem for this situation.

For later use, we also introduce some notation concerning Fourier transforms and convolutions, compare [32, 7] for details. The Fourier transform of a quantity q will always be denoted by $\widehat{q}$. For $\varphi, \psi \in C_c(G)$, we define the convolution $\varphi * \psi$ by $\big(\varphi * \psi\big)(t) := \int_G \varphi(s)\psi(t-s)\,\mathrm{d}s$ and the function $\widetilde{\varphi} \in C_c(G)$ by $\widetilde{\varphi}(t) := \overline{\varphi(-t)}$. For $\mu \in \mathcal{M}(G)$ and $\varphi \in C_c(G)$, the convolution $\varphi * \mu$ is the function given by $\big(\varphi * \mu\big)(t) := \int_G \varphi(t-s)\,\mathrm{d}\mu(s)$. For two convolvable measures $\mu, \nu \in \mathcal{M}(G)$, the convolution $\mu * \nu$ is the element of $\mathcal{M}(G)$ given by $\big(\mu * \nu\big)(\varphi) := \int_G \int_G \varphi(s+t)\,\mathrm{d}\mu(s)\,\mathrm{d}\nu(t)$ for $\varphi \in C_c(G)$; the measures $\widetilde{\mu}$ and $\overline{\mu}$ are defined by $\widetilde{\mu}(\varphi) := \overline{\mu(\widetilde{\varphi})}$ and $\overline{\mu}(\varphi) := \overline{\mu(\overline{\varphi})}$, respectively. For $\mu \in \mathcal{M}(G)$ and a measurable set $B \subset G$, we denote the restriction of μ to B by μ_B. Finally, for $x \in G$, we define the measure δ_x to be the normalised point measure at x.

We will consider actions of G on spaces consisting of measures on G. The relevant set of measures will be defined next.

Definition 1. *Let $C > 0$ and a relatively compact open set V in G be given. A measure $\mu \in \mathcal{M}(G)$ is called* (C,V)-translation bounded *if $|\mu|(t+V) \leq C$ for all $t \in G$. It is simply called* translation bounded *if there exist C, V such that it is (C,V)-translation bounded. The set of all (C,V)-translation bounded measures is denoted by $\mathcal{M}_{C,V}(G)$ and the set of all translation bounded measures by $\mathcal{M}^\infty(G)$.*

The vague topology on $\mathcal{M}(G)$ has very nice features when restricted to the translation bounded measures.

Theorem 2. *Let $C > 0$ and a relatively compact open set V in G be given. Then, $\mathcal{M}_{C,V}(G)$ is a compact Hausdorff space. If G is second countable, $\mathcal{M}_{C,V}(G)$ is metrisable.*

To prove the theorem, we start with the following simple result from measure theory.

Proposition 1. *Let $\mu \in \mathcal{M}(G)$ and a relatively compact open set V in G be given. Then, $|\mu|(V) = \sup\{|\mu(\varphi)| : \varphi \in C_c(G)$ with $\operatorname{supp}(\varphi) \subset V$ and $\|\varphi\|_\infty \leq 1\}$.*

Proof. Denote the supremum in the statement by S. Recall the polar decomposition and the definition of u in Eq. (3). As u is $|\mu|$ almost surely equal to 1, a direct calculation easily gives $S \leq |\mu|(V)$. Conversely, as $C_c(V)$ is dense in $L^1(V, |\mu|_V)$ (see, e.g., [28, Prop. 6.4.11]), there exists a sequence (φ_n) in $C_c(V)$ with $\varphi_n \xrightarrow{n\to\infty} \overline{u} \cdot 1_V$ in $L^1(V, |\mu|_V)$. By the $|\mu|$-almost sure boundedness of u, this implies $u\varphi_n \xrightarrow{n\to\infty} u\overline{u} \cdot 1_V = 1_V$ in $L^1(V, |\mu|_V)$. Then, a short calculation, invoking (3) again, shows $S \geq \lim_{n\to\infty} |\mu(\varphi_n)| = |\mu|(V)$. This proves the proposition. □

Proof of Theorem 2. By definition of $\mathcal{M}_{C,V}(G)$, for each $\varphi \in C_c(G)$, there exists a radius $R(\varphi) > 0$ such that $\mu(\varphi) \in \overline{B_{R(\varphi)}}$ for every $\mu \in \mathcal{M}_{C,V}(G)$, where B_r is the (open) ball of radius r around 0 and $\overline{B_r}$ its closure. Thus, we can consider $\mathcal{M}_{C,V}(G)$ as a subspace of $\mathit{\Pi} := \prod_{\varphi \in C_c(G)} \overline{B_{R(\varphi)}}$ equipped with the product topology via the embedding

$$j \colon \mathcal{M}_{C,V}(G) \hookrightarrow \mathit{\Pi}\,, \quad \big(j(\mu)\big)(\varphi) := \mu(\varphi).$$

As $\mathit{\Pi}$ is obviously a compact Hausdorff space, this shows immediately that $\mathcal{M}_{C,V}(G)$ is relatively compact and Hausdorff. It remains to be shown that $j(\mathcal{M}_{C,V}(G))$ is closed. This is a direct consequence of Definition 1 together with Proposition 1.

The statement about metrisability is standard: if G is second countable, there exists a countable dense subset $\{\varphi_n : n \in \mathbb{N}\}$ in $C_c(G)$. Then,

$$d \colon \mathcal{M}_{C,V}(G) \times \mathcal{M}_{C,V}(G) \longrightarrow \mathbb{R}\,, \quad d(\mu,\nu) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{|\mu(\varphi_n) - \nu(\varphi_n)|}{1 + |\mu(\varphi_n) - \nu(\varphi_n)|}$$

gives a metric on $\mathcal{M}_{C,V}(G)$ which generates the topology. □

Having discussed the topology of $\mathcal{M}(G)$, we can now introduce the topological dynamical systems associated to subsets of $\mathcal{M}(G)$. To do so, we will use the obvious action α of G on $\mathcal{M}(G)$ given by

$$\alpha \colon G \times \mathcal{M}(G) \longrightarrow \mathcal{M}(G)\,, \quad (t,\mu) \mapsto \alpha_t(\mu) := \delta_t * \mu,$$

or, more explicitly, $\big(\alpha_t(\mu)\big)(\varphi) = \int_G \varphi(t+s)\,\mathrm{d}\mu(s)$. We use the same symbol for the action as in Eq. (1), since misunderstandings are unlikely, and we will usually write $\alpha_t\mu$ for $\alpha_t(\mu)$.

This action is compatible with the topological structure of G and $\mathcal{M}(G)$.

Proposition 2. *Let $C > 0$ and a relatively compact open set $V \subset G$ be given. Then, the action $\alpha \colon G \times \mathcal{M}_{C,V}(G) \longrightarrow \mathcal{M}_{C,V}(G)$ is continuous, where $G \times \mathcal{M}_{C,V}(G)$ carries the product topology.*

Proof. Let (t_ι, μ_ι) be a net in $G \times \mathcal{M}_{C,V}(G)$ converging to (t,μ). We have to show that the net $(\alpha_{t_\iota}(\mu_\iota))$ converges to $\alpha_t(\mu)$, i.e., we have to check that

$$\int \varphi(s + t_\iota)\,\mathrm{d}\mu_\iota(s) \;\longrightarrow\; \int \varphi(s+t)\,\mathrm{d}\mu(s)\,, \quad \text{for all } \varphi \in C_c(G).$$

By $t_\iota \longrightarrow t$, there exists an index ι_0 and a compact set K such that the support of φ and the supports of all φ_ι with $\iota \geq \iota_0$ are contained in K. Moreover, as $\mu_\iota \in \mathcal{M}_{C,V}(G)$ for every ι, there exists a constant C' with $|\mu|(K) \leq C'$ as well as $|\mu_\iota|(K) \leq C'$ for all ι. Now, the desired statement follows easily from

$$\begin{aligned} |\mu_\iota(\varphi(.+t_\iota)) - \mu(\varphi(.+t))| &\leq |\mu_\iota(\varphi(.+t_\iota)) - \mu_\iota(\varphi(.+t))| + |\mu_\iota(\varphi(.+t)) - \mu(\varphi(.+t))| \\ &\leq |\mu_\iota|(C)\|\varphi(.+t_\iota) - \varphi(.+t)\|_\infty + |\mu_\iota(\varphi(.+t)) - \mu(\varphi(.+t))|, \end{aligned}$$

as both terms on the right hand side tend to zero for $t_\iota \longrightarrow t$ and $\mu_\iota \longrightarrow \mu$. □

The dynamical systems we are interested in are defined as follows.

Definition 2. (Ω, α) *is called a* dynamical system *on the translation bounded measures on* G (TMDS) *if there exist a constant* $C > 0$ *and a relatively compact open* $V \subset G$ *such that* Ω *is a closed subset of* $\mathcal{M}_{C,V}(G)$ *that is invariant under the* G*-action* α.

Remarks. (a) The α invariant subsets of $\mathcal{M}(G)$ are called *translation stable sets* in [14]. Thus, a TDMS is just a closed translation stable subset of $\mathcal{M}_{C,V}(G)$.
(b) The space Ω of a TMDS is always compact by Theorem 2 and the action α is continuous by Proposition 2. Thus, a TMDS is a topological dynamical system in the sense of Section 2.
(c) The considerations of this section (and those of the next) do not use commutativity of the underlying group G. They immediately extend to arbitrary σ-compact locally compact groups. But since we need harmonic analysis later on, we stick to Abelian groups here.

4. Point dynamical systems

This section has two aims. Firstly, we present an abstract topological framework which allows us to treat point dynamical systems which are not of finite local complexity. Secondly, we show how these systems fit into our setting of measure dynamical systems. As for the first aim, we actually introduce a topology on the set of all closed subsets of G. For the case of $\mathbb{R}^d$, this topology has already been studied by Stollmann and one of the authors in [23]. Our extension to arbitrary locally compact groups is strongly influenced by the investigation of Schlottmann [34] (which, however, is restricted to FLC systems).

As pointed out by the referee, the topology we introduce can also be obtained as a special case of a topology introduced by Fell in [13]. This is further discussed in the appendix. Given this connection, Theorem 3 below is a corollary of Theorem 1 in [13]. For this reason, we only give an outline of how it can be established in our setting.

We start by defining the relevant sets of points.

Definition 3. *Let* G *be a* σ*-compact* LCA *group, and* V *an open neighbourhood of* 0 *in* G.

(a) *A subset* $\varLambda$ *of* G *is called* V-discrete *if every translate of* V *contains at most one point of* $\varLambda$. *The set of all* V*-discrete subsets of* G *is denoted by* $\mathcal{D}_V(G)$.
(b) *A subset of* G *is called* uniformly discrete *if it is* V*-discrete for some* V. *The set of all uniformly discrete subsets of* G *is denoted as* $\mathcal{UD}(G)$.
(c) *The set of all discrete and closed subsets of* G *will be denoted by* $\mathcal{D}(G)$.
(d) *The set of all closed subsets of* G *is denoted as* $\mathcal{C}(G)$.
(e) *A subset* $\varLambda$ *of* G *is called* relatively dense *if there is a compact* K *with* $\varLambda + K = G$.

Note that a uniformly discrete subset of G is closed. As it presents no extra difficulty, we will actually topologise not only $\mathcal{D}(G)$ but rather the larger set $\mathcal{C}(G)$. This will be done by providing a suitable uniformity (see [20, Ch. 6] for details on uniformities). Namely, for $K \subset G$ compact and V a neighbourhood of 0 in G, we set

$$U_{K,V} \;:=\; \{(P_1, P_2) \in \mathcal{C}(G) \times \mathcal{C}(G) : P_1 \cap K \subset P_2 + V \text{ and } P_2 \cap K \subset P_1 + V\}.$$

It is not hard to check that

$$(P,P) \in U_{K,V}\,,\;\; U_{K,V} = U_{K,V}^{-1}\,,\;\; U_{K_1\cup K_2, V_1\cap V_2} \subset U_{K_1,V_1} \cap U_{K_2,V_2}\,,\;\; U_{K-W,W} \circ U_{K-W,W} \subset U_{K,V}$$

for V a neighbourhood of 0, W a compact neighbourhood of 0 with $W + W \subset V$, K in G compact, and P any closed subset of G. Here, on sets U, U_1, U_2 consisting of ordered pairs, we define $U^{-1} := \{(y,x) : (x,y) \in U\}$ and

$$U_1 \circ U_2 \;:=\; \{(x,z) : \exists\, y \in G \text{ with } (x,y) \in U_1 \text{ and } (y,z) \in U_2\}.$$

This guarantees that $\{U_{K,V} : K \text{ compact}, V \text{ open with } 0 \in V\}$ generates a uniformity, and hence a topology on $\mathcal{C}(G)$ via the neighbourhoods

$$U_{K,V}(P) \;:=\; \{Q : (Q,P) \in U_{K,V}\}\,, \quad P \in \mathcal{C}(G)\,.$$

Note that we could equally well generate the same uniformity with V running through compact neighbourhoods of $0 \in G$.

Definition 4. *The topology defined this way is called the* local rubber topology (LRT).

This topology essentially means that two sets P_1, P_2 are close if they "almost" agree on large compact sets.

Fundamental properties of the LRT are given in the following result (see [23] for an earlier result on $\mathbb{R}^d$).

Theorem 3. *With the* LRT*, the set $\mathcal{C}(G)$ of closed subsets of G is a compact Hausdorff space. If the topology of G has a countable base, then $\mathcal{C}(G)$ is metrisable.*

Proof. The set $\mathcal{C}(G)$ is Hausdorff, as the intersection of all $U_{K,V}$ contains only the diagonal set $\{(P,P) : P \in \mathcal{C}(G)\}$, see [20, Ch. 6].

We next show completeness: Let $(P_\iota)_{\iota \in I}$ be a Cauchy net in $\mathcal{C}(G)$, where I is an index set directed by $\leq$, compare [20, Ch. 2].

We have to show that the Cauchy net converges to a closed subset of G, hence an element of $\mathcal{C}(G)$. To this end, we introduce the set P of those $x \in G$ such that, for every neighbourhood V of 0, there exists $\iota_{x,V} \in I$ with

$$(x + V) \cap P_\iota \;\neq\; \varnothing \quad \text{for all } \iota \geq \iota_{x,V}. \tag{4}$$

It is not hard to see that P is closed. P will turn out to be the limit of our Cauchy net.

To show this, let a compact K in G and a neighbourhood V of 0 be given. We have to provide an $\iota_{K,V}$ with

$$P \cap K \;\subset\; P_\iota + V \quad \text{and} \quad P_\iota \cap K \;\subset\; P + V$$

for every $\iota \geq \iota_{K,V}$.

Rather direct arguments show existence of $\iota_{K,V}$ with $P \cap K \subset P_\iota + V$ for all $\iota \geq \iota_{K,V}$. We next establish the other inclusion. By a compactness argument,

$$P \cap C \;\neq\; \varnothing, \tag{5}$$

whenever $C \subset G$ is compact and $P_\kappa \cap C \neq \varnothing$ for all $\kappa \geq \kappa_0$ and some $\kappa_0 \in I$. Assume, without loss of generality, that V is compact and symmetric. As (P_ι) is a Cauchy net, there exists a $\iota_{K,V}$ with $(P_\iota, P_\kappa) \in U_{K,V}$ for all $\iota, \kappa \geq \iota_{K,V}$. Consider $\iota \geq \iota_{K,V}$ and choose an arbitrary $q \in P_\iota \cap K$. Then,

$$(q + V) \cap P_\kappa \;\neq\; \varnothing$$

for every $\kappa \geq \iota_{K,V}$, as V is symmetric and $P_\iota \cap K \subset P_\kappa + V$ by $(P_\iota, P_\kappa) \in U_{K,V}$. Thus, (5) gives existence of a $p \in P$ with $p \in q + V$. In particular, invoking the symmetry of V once more, we have $q \in p + V \subset P + V$. As $q \in P_\iota \cap K$ was arbitrary, we infer the desired inclusion $P_\iota \cap K \;\subset\; P + V$.

These considerations prove the desired completeness statement.

Finally, we show compactness of $\mathcal{C}(G)$. As $\mathcal{C}(G)$ is complete, it suffices to prove it is precompact. Thus, for any given K compact and V an open neighbourhood of 0 in G, we have to provide a natural number n and $P_i \in \mathcal{C}(G)$, $1 \leq i \leq n$, such that $\mathcal{C}(G) \;\subset\; \bigcup_{i=1}^{n} U_{K,V}(P_i)$. Since $U_{K,V}(P) \supset U_{K,V\cap(-V)}(P)$ for all $P \in \mathcal{C}(G)$ and $V \cap (-V)$ is symmetric, we can assume, without loss of generality, that V is symmetric (i.e., $V = -V$). As K is compact, there exists a finite set $D \subset K$ with $K \subset D + V$. Direct calculations then give

$$\mathcal{C}(G) \;\subset\; \bigcup_{i=1}^{n} U_{K,V}(D_i),$$

where D_i, $1 \leq i \leq n$, is an enumeration of the power set of D.

The statement about metrisability is a direct consequence of [20, Thm. 6.13] and the remark thereafter. □

Having topologised $\mathcal{C}(G)$, and thus $\mathcal{UD}(G)$ as well, we can now introduce our point dynamical systems. The natural action of G on $\mathcal{C}(G)$ by translation will also be denoted by α. Explicitly, we define $\alpha_t(P) = t + P$ for $P \in \mathcal{C}(G)$, where $t + P = \{t + x : x \in P\}$ as usual.

Definition 5. *Let $\varOmega$ be a subset of G and α the translation action just defined.*

(a) *The pair $(\varOmega, \alpha)$ is called a* set dynamical system *if $\varOmega$ is a closed subset of $\mathcal{C}(G)$ which is invariant under α.*

(b) *A set dynamical system $(\varOmega, \alpha)$ is called a* point dynamical system *if $\varOmega$ is a subset of $\mathcal{D}_V(G)$ for some open neighbourhood V of 0.*

(c) *A point dynamical system $(\varOmega, \alpha)$ is called a* Delone dynamical system*, if every element of $\varOmega$ is a relatively dense subset of G.*

It follows from Theorem 3 that a set dynamical system is indeed a topological dynamical system in the sense of Section 2.

A special way of obtaining set dynamical systems is the following: Choose $P \in \mathcal{C}(G)$. Then, the LRT-closure $X(P)$ of the orbit $\{\alpha_t(P) : t \in G\}$ of P in $\mathcal{C}(G)$ is a closed α-invariant subset of $\mathcal{C}(G)$, hence compact. Thus, $(X(P), \alpha)$ is a set dynamical system. We say that P and P' are *locally indistinguishable* if $P \subset X(P')$ and $P' \subset X(P)$, hence if $X(P) = X(P')$. The set of all P' which are locally indistinguishable form P is called the RLI class of P, written as RLI(P). Here, as before, the letter R stands for "rubber".

Given these notions, we can characterise minimality of a set dynamical system in the following way, which extends [34, Prop. 3.1] to our setting.

Proposition 3. *Let $P \in \mathcal{C}(G)$ be given. Then, the following assertions are equivalent.*

(a) *The set dynamical system $(X(P), \alpha)$ is minimal.*
(b) $X(P) = \mathrm{RLI}(P)$.
(c) *The set $\{t \in G : (t + P, P) \in U_{K,V}\}$ is relatively dense in G for all compact $K \subset G$ and all open neighbourhoods V of 0.*
(d) *The set P is repetitive, i.e., for every K compact and every open neighbourhood V of 0, there is a compact set $C = C(K, V) \subset G$ such that, for $t_1, t_2 \in G$, there is an $s \in C$ with $(t_1 + P, s + t_2 + P) \in U_{K,V}$.*

Proof. The equivalence of (a) and (b) is clear by the definition of minimality and the local indistinguishability class. This is also known as Gottschalk's Theorem, compare [29, Thm. 4.1.2]. The remaining assertions can be shown by mimicking and slightly extending the proof of [34, Prop. 3.1]. Since we do not use these results later on, we skip further details. □

Having discussed point dynamical systems, we can now relate them to special dynamical systems on measures. The connection relies on the map

$$\delta\colon \mathcal{UD}(G) \longrightarrow \mathcal{M}^\infty(G)\,, \quad \delta(\Lambda) := \sum_{x\in\Lambda} \delta_x\,,$$

Note that δ is indeed a map into $\mathcal{M}^\infty(G)$, as every $\Lambda \in \mathcal{UD}(G)$ is uniformly discrete.

Lemma 2. *The map $\delta\colon \mathcal{UD}(G) \longrightarrow \mathcal{M}^\infty(G)$ is injective, continuous and compatible with the action of G. The inverse $\delta^{-1}\colon \delta(\mathcal{UD}(G)) \longrightarrow \mathcal{UD}(G)$ is continuous as well.*

Proof. Injectivity and compatibility with the action of G are immediate. In particular, using $\delta_{t+x} = \delta_t * \delta_x$, one can check that

$$\delta(\alpha_t(\Lambda)) \,=\, \delta(t+\Lambda) \,=\, \delta_t * \delta(\Lambda) \,=\, \alpha_t(\delta(\Lambda))\,.$$

To show continuity of δ, we have to show $\delta(\Lambda_\iota)(\varphi) \longrightarrow \delta(\Lambda)(\varphi)$ for all $\varphi \in C_c(G)$, whenever $\Lambda_\iota \longrightarrow \Lambda$. Let V be an open neighbourhood of 0 in G such that $\Lambda \in \mathcal{D}_V(G)$. Let $\varphi \in C_c(G)$ be given, so $\mathrm{supp}(\varphi)$ is compact. By a simple partition of unity argument, we can now write $\varphi = \sum_{j=1}^n \varphi_j$, where each φ_j, $j = 1, \dots, n$, has its support in a set of the form $W + t_j$, with $W + W \subset V$. Now, for such φ_j, the convergence $\delta(\Lambda_\iota)(\varphi_j) \longrightarrow \delta(\Lambda)(\varphi_j)$ follows easily. This yields the desired convergence for φ. The continuity of δ^{-1} can be shown similarly. □

As can be seen from simple examples, $\delta(\mathcal{UD}(G))$ is, in general, not closed in $\mathcal{M}^\infty(G)$ and the LRT is not the same as the vague topology on $\delta(\mathcal{UD}(G))$. For example, if (x_n) is a sequence in G with $x_n \longrightarrow 0$ and $x_n \neq 0$, then, $\Lambda_n := \{0, x_n\} \in \mathcal{UD}(G)$ with $\Lambda_n \longrightarrow \{0\}$ in the LRT. However, $\delta(\Lambda_n) \longrightarrow 2\delta_0$ in the vague topology, and $2\delta_0$ does not belong to $\delta(\mathcal{UD}(G))$. Nevertheless, the following still holds.

Proposition 4. *Let V be an open neighbourhood of 0 in G. Then, $\mathcal{D}_V(G)$ is compact in the* LRT *and $\delta(\mathcal{D}_V(G))$ is compact in the vague topology.*

Proof. By compactness of $\mathcal{C}(G)$ and continuity of δ, it suffices to show that $\mathcal{D}_V(G)$ is closed. Since V is open, this is easy. □

Our main result on the relationship of point dynamical systems and the framework of measure dynamical systems reads as follows.

Theorem 4. *If (Ω,α) is a point dynamical system, the restriction $\delta|_{\Omega}\colon \Omega \longrightarrow \delta(\Omega)$ of δ to Ω establishes a topological conjugacy between the dynamical systems (Ω,α) and $(\delta(\Omega),\alpha)$.*

Proof. By Lemma 2, $\delta|_{\Omega}$ is an injective and continuous map which is compatible with the group action. As Ω is compact, so is $\delta(\Omega)$. By a standard argument [28, Prop. 1.6.8], $\delta|_{\Omega}$ is then a homeomorphism between (Ω,α) and $(\delta(\Omega),\alpha)$. Together, this establishes the topological conjugacy. □

We finish this section by briefly discussing the relationship of the LRT and the topology usually considered for Delone dynamical systems with the FLC property. A thorough discussion of the latter topology has been given in [34]. This discussion actually gives a topology on the closed subsets of G (though this is not explicitly noted in [34]). This topology will be called the local matching topology (LMT).

The definition of the LMT in [34] shows immediately that the LRT is coarser than the LMT. Thus, the identity

$$id\colon (\mathcal{C}(G),\mathrm{LMT}) \longrightarrow (\mathcal{C}(G),\mathrm{LRT})\,, \quad P \mapsto P$$

is continuous. This yields the following result, which essentially shows that our way of topologising the uniformly discrete sets coincides with the usual topology when restricted to sets of finite local complexity.

Proposition 5. *Let Ω be a subset of $\mathcal{C}(G)$. If Ω is compact in the* LMT*, then Ω is compact in the* LRT *as well, and the two topologies agree on Ω.*

Proof. The restriction $id_{\Omega}\colon (\Omega,\mathrm{LMT}) \longrightarrow (\Omega,\mathrm{LRT})$ of the identity to Ω is continuous. Thus, as (Ω,LMT) is compact, so is its image (Ω,LRT). Now, continuity of the inverse is standard, cf. [28, Prop. 1.6.8]. Thus, the two topologies agree. □

5. The diffraction spectrum

The basic concepts in the mathematical treatment of diffraction experiments are the *autocorrelation measure* and the *diffraction measure*. In the context of Delone dynamical systems, these concepts have been developed and investigated in a series of articles by theoretical physicists and mathematicians [3, 5, 11, 17, 18, 33, 34]. This will now be generalized and extended to our measure dynamical systems. More precisely, we show the existence of the autocorrelation measure by a limiting procedure, provided certain ergodicity assumptions hold.

Let us mention that we will provide an alternative approach to these quantities later on. It will be more general in that it does not need an ergodicity assumption.

To phrase our results, we need two more pieces of notation. Firstly, recall from [34] that a sequence (B_n) of compact subsets of G is called a *van Hove sequence* if

$$\lim_{n\to\infty} \frac{|\partial^K B_n|}{|B_n|} \;=\; 0$$

for all compact $K \subset G$. Here, for compact B, K, the "K-boundary" $\partial^K B$ of B is defined as

$$\partial^K B \;:=\; \overline{((B+K)\setminus B)} \cup (\overline{G\setminus B} - K) \cap B,$$

where the bar denotes the closure. The existence of van Hove sequences for all σ-compact LCA groups is shown in [34, p. 249], see also Section 3.3 and Theorem (3.L) of [38, Appendix]. Moreover, every van Hove sequence is a Følner sequence, i.e., $|B_n \triangle (B_n+K)|/|B_n| \xrightarrow{n\to\infty} 0$, for every compact set $K \subset G$, where $\triangle$ denotes the operator for the symmetric difference of two sets, compare Section 3.2 of [38, Appendix]; for a partial converse, consult Theorem (3.K) of the same reference.

Secondly, for $\varphi \in C_c(G)$ and $\mu \in \mathcal{M}(G)$, we define

$$f_\varphi(\mu) \;:=\; \big(\varphi * \mu\big)(0) \;=\; \int_G \varphi(-s)\,\mathrm{d}\mu(s).$$

This gives a way to "push" functions from $C_c(G)$ to $C(\Omega)$. In fact, up to the sign, f_φ is just the canonical embedding of $C_c(G)$ into its bidual $\mathcal{M}(G)^*$. Basic features of the map $\varphi \mapsto f_\varphi$ are gathered in the following lemma.

Lemma 3. *Recall the definition $\alpha_t\mu = \delta_t * \mu$ for measures μ, and set $\beta_t(\varphi) = \delta_t * \varphi$ for functions φ. Then one has:*

(a) *The function $f_\varphi\colon \mathcal{M}(G) \longrightarrow \mathbb{C}$, $\mu \mapsto f_\varphi(\mu)$ is continuous, for all $\varphi \in C_c(G)$.*
(b) *If (Ω, α) is a* TMDS*, the map $f : C_c(G) \longrightarrow C(\Omega)$, $\varphi \mapsto f_\varphi$, is linear, continuous and compatible with the action of G in that $f_\varphi(\alpha_t\mu) = f_{\beta_t(\varphi)}(\mu)$.*

Proof. (a) For $\varphi \in C_c(G)$, we have $f_\varphi(\mu) = \mu(\varphi_-)$, where $\varphi_-(t) = \varphi(-t)$. Thus, continuity of f_φ is immediate from the definition of the topology on $\mathcal{M}(G)$.

(b) Linearity of the map f is obvious. To show continuity of f, recall that $C_c(G)$ is equipped with the inductive limit topology induced from the embeddings $C_K(G) \hookrightarrow C_c(G)$ with $K \subset G$ compact. Thus, it suffices to show the continuity of the map

$$f_K\colon\, C_K(G) \;\hookrightarrow\; C_c(G)\,, \quad \varphi \mapsto f_\varphi$$

for every compact K in G. So, let (φ_ι) be a net in $C_K(G)$ converging to $\varphi \in C_K(G)$. Then, $\|\varphi_\iota - \varphi\|_\infty \longrightarrow 0$, and $\mathrm{supp}(\varphi), \mathrm{supp}(\varphi_\iota) \subset K$ for all ι. As $\Omega \subset \mathcal{M}_{C,V}(G)$ with suitable C, V, this easily implies $f_{\varphi_\iota} \longrightarrow f_\varphi$. Finally, a direct calculation shows $f_\varphi(\alpha_t\mu) = f_{\beta_t(\varphi)}(\mu)$. □

Remark. The lemma is particularly interesting as there does not seem to exist any canonical map from Ω to G or from G to Ω in our setting (let alone a map which is compatible with the corresponding group actions). However, if one views a function φ as the Radon-Nikodym derivative of a measure that is absolutely continuous with respect to the Haar measure of G, the action α_t induces β_t as defined.

Now, our result on existence of the autocorrelation function reads as follows.

Theorem 5. *Let α be the translation action of G on $\mathcal{M}^\infty(G)$ as introduced above.*

(a) *If (Ω, α) is a uniquely ergodic* TMDS*, there exists a translation bounded measure γ on G such that the sequence $(\frac{1}{|B_n|}\,\widetilde{\omega_{B_n}} * \omega_{B_n})$ converges, in the vague topology, to γ for every van Hove sequence (B_n) and every $\omega \in \Omega$. Moreover, the equation $\big(\widetilde{\varphi} * \psi * \gamma\big)(t) = \langle f_\varphi, T^t f_\psi\rangle$ holds for arbitrary $\varphi, \psi \in C_c(G)$ and $t \in G$.*
(b) *Let G have a topology with countable base. Let (Ω, α) be a* TMDS *with ergodic probability measure m. Then, there exists a translation bounded measure γ on G such that the sequence $(\frac{1}{|B_n|}\,\widetilde{\omega_{B_n}} * \omega_{B_n})$ converges, in the vague topology, to γ for m-almost every $\omega \in \Omega$, whenever (B_n) is a van Hove sequence along which the Birkhoff ergodic theorem holds. Moreover, the equation $\big(\widetilde{\varphi} * \psi * \gamma\big)(t) = \langle f_\varphi, T^t f_\psi\rangle$ holds for arbitrary $\varphi, \psi \in C_c(G)$ and $t \in G$.*

Remarks. (a) Every LCA group with a countable base of the topology admits a van Hove sequence along which the Birkhoff ergodic theorem holds, as follows from recent results of Lindenstrauss [24], see also Tempelman's monograph [38], in particular its Appendix, for background material. More precisely, every van Hove sequence is a Følner sequence, and thus contains a so-called tempered subsequence with the desired property, compare [24]. Note also that G second countable implies σ-compactness as well as metrisability of G.
(b) The theorem generalises the corresponding results of [11, 34, 18].

To prove Theorem 5, we need some preparation in form of the following results.

Lemma 4. *Let D be a dense subset of $C_c(G)$. Let $C > 0$ and a relatively compact open V in G be given. If (μ_ι) is a net of measures in $\mathcal{M}_{C,V}(G)$ such that $\mu_\iota(\varphi)$ converges for every $\varphi \in D$, then there exists a translation bounded measure $\mu \in \mathcal{M}(G)$ such that (μ_ι) converges vaguely to μ.*

Proof. As D is dense, every μ in $\mathcal{M}(G)$ is uniquely determined by its values on D. Thus, all converging subnets of (μ_ι) have the same limit. As $\mathcal{M}_{C,V}(G)$ is compact by Theorem 2, there exist converging subnets. Putting this together, we arrive at the desired statement. □

Lemma 5. [34, Lemma 1.2] *Let μ, ν be translation bounded measures on G and (B_n) a van Hove sequence. Then, in the vague topology, $\lim_{n\to\infty} \frac{1}{|B_n|}(\mu_{B_n} * \nu_{B_n} - \mu * \nu_{B_n}) = 0$.* □

Lemma 6. [34, Lemma 1.1 (2)] *Let (B_n) be a van Hove sequence in G and μ a translation bounded measure. Then, the sequence $\big(|\mu|(B_n)/|B_n|\big)$ is bounded.* □

Proof of Theorem 5. (a) As $\omega \in \mathcal{M}_{C,V}(G)$, Lemma 6 and a short calculation give a constant $C' > 0$ and and a relatively compact open $V' \subset G$ such that the sequence $\big((\widetilde{\omega_{B_n}} * \omega_{B_n})/|B_n|\big)$ is contained in $\mathcal{M}_{C',V'}$. Moreover, the set $\{\varphi * \psi : \varphi, \psi \in C_c(G)\}$ is dense in $C_c(G)$ by standard arguments involving approximate units [32]. Thus, by Lemma 4 and Lemma 5, it suffices to show $\lim_{n\to\infty} \frac{1}{|B_n|}\big(\widetilde{\varphi} * \psi * \widetilde{\omega_{B_n}} * \omega_{B_n}\big)(t) = \langle f_\varphi, T^t f_\psi\rangle$ for arbitrary $\varphi, \psi \in C_c(G)$ and $t \in G$. By Lemma 5, it suffices to show

$$\lim_{n\to\infty} \frac{1}{|B_n|}\big(\widetilde{\varphi} * \psi * \widetilde{\omega} * \omega_{B_n}\big)(t) \; = \; \langle f_\varphi, T^t f_\psi\rangle.$$

This follows by unique ergodicity and a Dworkin type calculation [11, 34, 22]. As the details are somewhat more involved than in the case of Delone sets, we include a sketch for the convenience of the reader. We define $Z_n := \big(\widetilde{\varphi} * \psi * \widetilde{\omega} * \omega_{B_n}\big)(t)$. Then,

$$Z_n \; = \; \int_G \big(\widetilde{\varphi} * \psi\big)(t-u)\,\mathrm{d}(\widetilde{\omega} * \omega_{B_n})(u) \; = \; \int_G \int_G \int_G \overline{\varphi}(v-t+r)\psi(v-s)1_{B_n}(s)\,\mathrm{d}v\,\mathrm{d}\widetilde{\omega}(r)\,\mathrm{d}\omega(s),$$

where 1_{B_n} denotes the characteristic function of B_n. Using Fubini's Theorem and sorting the terms, we arrive at

$$Z_n \; = \; \int_G \left(\int_G \overline{\varphi}(v-t+r)\,\mathrm{d}\widetilde{\omega}(r)\right)\left(\int_G \psi(v-s)1_{B_n}(s)\,\mathrm{d}\omega(s)\right)\mathrm{d}v\,.$$

We will now study the two terms in brackets. A short calculation shows $\int_G \overline{\varphi}(v-t+r)\,\mathrm{d}\widetilde{\omega}(r) = \overline{f_\varphi(\alpha_{t-v}\omega)}$. As for the other term, we consider the difference function

$$D(v) \; := \; \int_G \psi(v-s)\,1_{B_n}(s)\,\mathrm{d}\omega(s) - \int_G \psi(v-s)\,\mathrm{d}\omega(s)\,1_{B_n}(v)\,.$$

Let K be a compact set with $K = -K$, $0 \in K$ and $\mathrm{supp}(\psi) \subset K$. Then, it is not hard to see that the difference $D(v)$ (and in fact each of its terms alone) vanishes for $v \notin B_n + K$. Similarly, one can show that $D(v)$ is supported in $\partial^K B_n$. Apparently, $|D(v)|$ is bounded above by $C := 2\|\psi\|_\infty \sup\{|\omega|(t + \mathrm{supp}(\psi)) : t \in G\} < \infty$. As (B_n) is a van Hove sequence, we conclude that

$$0 \;\leq\; \frac{1}{|B_n|} \int_G \Big| D(v) \overline{f_\varphi(\alpha_{t-v}\omega)} \Big| \,\mathrm{d}v \;\leq\; C \|f_\varphi\|_\infty \frac{|\partial^K B_n|}{|B_n|} \;\xrightarrow{n\to\infty}\; 0\,.$$

Noting that $\int_G \psi(v - s)\,\mathrm{d}\omega(s) = f_\psi(\alpha_{-v}\omega)$, and putting these considerations together, we arrive at

$$\lim_{n\to\infty} \frac{Z_n}{|B_n|} \;=\; \lim_{n\to\infty} \frac{1}{|B_n|} \int_G \overline{f_\varphi(\alpha_{t-v}\omega)} f_\psi(\alpha_{-v}\omega)\, 1_{B_n}(v)\,\mathrm{d}v\,.$$

This yields

$$\begin{aligned} \lim_{n\to\infty} \frac{Z_n}{|B_n|} \;&=\; \int_\Omega \overline{f_\varphi(\alpha_t\omega)} f_\psi(\omega)\,\mathrm{d}m(\omega) \;=\; \int_\Omega \overline{f_\varphi(\omega)} f_\psi(\alpha_{-t}\omega)\,\mathrm{d}m(\omega) \\ &=\; \int_\Omega \overline{f_\varphi(\omega)} (T^t f_\psi)(\omega)\,\mathrm{d}m(\omega) \;=\; \langle f_\varphi, T^t f_\psi \rangle\,, \end{aligned}$$

where we used the pointwise ergodic theorem for continuous functions on a uniquely ergodic system in the first step and α-invariance of m in the second step. (Note that this ergodic theorem only relies on compactness of the underlying space Ω and does not require separability of G. This can easily be seen by going through a proof of this theorem as presented, e.g., in [39, Thm. 6.19].)

(b) This can be seen similarly: After replacing the pointwise ergodic theorem for uniquely ergodic systems by the Birkhoff ergodic theorem, the considerations of (a) can be carried through to show that, for each function $\widetilde{\varphi} * \psi$, there exists a set $\Omega_{\varphi,\psi} \subset \Omega$ of full measure such that

$$\frac{1}{|B_n|} \big(\widetilde{\varphi} * \psi * \widetilde{\omega_{B_n}} * \omega_{B_n}\big)(0) \;\xrightarrow{n\to\infty}\; \langle f_\varphi, f_\psi \rangle$$

for every $\omega \in \Omega_{\varphi,\psi}$. As G is second countable, there exists a countable set D in $C_c(G)$ such that D and $\{\widetilde{\varphi} * \psi : \varphi, \psi \in D\}$ are dense in G. Thus, there is a set $\Omega_0 \subset \Omega$ of full measure such that

$$\frac{1}{|B_n|} \big(\widetilde{\varphi} * \psi * \widetilde{\omega_{B_n}} * \omega_{B_n}\big)(0) \;\xrightarrow{n\to\infty}\; \langle f_\varphi, f_\psi \rangle$$

for all $\varphi, \psi \in D$ and all $\omega \in \Omega_0$. By the density of $\{\widetilde{\varphi} * \psi : \varphi, \psi \in D\}$ in $C_c(G)$ and Lemma 4, the vague convergence of $\big(\widetilde{\omega_{B_n}} * \omega_{B_n}\big)/|B_n|$ towards a translation bounded measure γ with $\big(\widetilde{\varphi} * \psi * \gamma\big)(0) = \langle f_\varphi, f_\psi \rangle$ for all $\varphi, \psi \in D$ follows. This gives the desired vague convergence.

It remains to show the last part of the statement: As D is dense in $C_c(G)$ and f is a continuous map, the formula

$$\big(\widetilde{\varphi} * \psi * \gamma\big)(0) \;=\; \langle f_\varphi, f_\psi \rangle$$

does not only hold for $\varphi, \psi \in D$, but for arbitrary $\varphi, \psi \in C_c(G)$. For $t \in G$, this implies

$$\big(\widetilde{\varphi} * \psi * \gamma\big)(t) \;=\; \big(\delta_{-t} * \widetilde{\varphi} * \psi * \gamma\big)(0) \;=\; \big(\widetilde{\varphi} * (\delta_{-t} * \psi) * \gamma\big)(0) \;=\; \langle f_\varphi, f_{\delta_{-t} * \psi} \rangle\,.$$

By Lemma 3, we have

$$f_{\delta_{-t}*\psi}(\omega) \;=\; f_{\beta_{-t}(\psi)}(\omega) \;=\; f_\psi(\alpha_{-t}\omega) \;=\; T^t f_\psi(\omega)\,.$$

Thus, we can conclude

$$\big(\widetilde{\varphi} * \psi * \gamma\big)(t) \;=\; \langle f_\varphi, f_{\delta_{-t}*\psi}\rangle \;=\; \langle f_\varphi, T^t f_\psi\rangle\,.$$

This finishes the proof. □

Remark. By Lemma 5, the convergence of $|B_n|^{-1}\widetilde{\omega_{B_n}} * \omega_{B_n}$ towards γ discussed in the previous theorem implies convergence of $|B_n|^{-1}\widetilde{\omega} * \omega_{B_n}$ towards γ as well.

The measure

$$\gamma \;=\; \gamma_\omega \;=\; \lim_{n\to\infty} \frac{1}{|B_n|}\,\widetilde{\omega_{B_n}} * \omega_{B_n}$$

appearing in the theorem is called the *autocorrelation measure* (or *autocorrelation* for short) of $\omega \in \varOmega$. It is obviously positive definite, and hence transformable. By Bochner's Theorem, compare [7, Ch. I.4], its Fourier transform is then a positive measure on $\widehat{G}$, called the *diffraction measure* of $\omega \in \varOmega$. We will have to say more about autocorrelation and diffraction measures in the next section.

6. Relating diffraction and dynamical spectrum

In this section, we show that the diffraction spectrum is equivalent to the spectrum of a certain subrepresentation of T. This type of result is implicit in essentially every work using the so-called Dworkin argument [11, 17, 34, 36]. However, it seems worthwhile to make this connection explicit. In fact, this is one of the two cornerstones of our approach to the characterisation of pure pointedness, the other being Theorem 1. A key ingredient in our considerations will be Proposition 7 below.

We start by giving a closed formula for the autocorrelation measure. This closed formula does not rely on any ergodicity assumptions. Thus, via this formula, an autocorrelation measure can be attached to any TMDS with an invariant probability measure m. We should like to mention that this is inspired by recent work of Gouéré [15], who gives a closed formula in the context of Palm measures and point processes.

Proposition 6. *Let $(\varOmega,\alpha)$ be a* TMDS *with invariant probability measure m. Let a function $\sigma\in C_c(G)$ be given with $\int_G \sigma(t)\,\mathrm{d}t = 1$. For $\varphi \in C_c(G)$, define*

$$\gamma_{\sigma,m}(\varphi) \;:=\; \int_\varOmega \int_G f_\varphi(\alpha_{-t}\overline{\omega})\sigma(t)\,\mathrm{d}\omega(t)\,\mathrm{d}m(\omega).$$

This leads to the following assertions.

(a) *The map $\gamma_{\sigma,m}\colon C_c(G) \longrightarrow \mathbb{C}$ is continuous, i.e., $\gamma_{\sigma,m}\in\mathcal{M}(G)$.*
(b) *For $\varphi,\psi\in C_c(G)$, the equation $\big(\widetilde{\varphi} * \psi * \gamma_{\sigma,m}\big)(t) = \langle f_\varphi, T^t f_\psi\rangle$ holds.*
(c) *The measure $\gamma_{\sigma,m}$ does not depend on $\sigma\in C_c(G)$, provided $\int_G \sigma\,\mathrm{d}t = 1$.*
(d) *The measure $\gamma_{\sigma,m}$ is positive definite.*

Proof. Note that $f_\varphi(\alpha_s\overline{\omega}) = \overline{f_{\overline{\varphi}}(\alpha_s(\omega))}$.
(a) Obviously, $|f_\varphi(\overline{\omega})| \le \|\varphi\|_\infty \sup\{|\omega|(t+\mathrm{supp}(\varphi)) : t\in G\}$. As σ has compact support, $\gamma_{\sigma,m}(\varphi)$ is then finite. Moreover, $\varphi_\iota \longrightarrow \varphi$ implies $\overline{\varphi_\iota} \longrightarrow \overline{\varphi}$ which, in turn, yields $\overline{f_{\overline{\varphi_\iota}}} \longrightarrow \overline{f_{\overline{\varphi}}}$ by continuity of f. As σ has compact support and $\varOmega\subset\mathcal{M}_{C,V}(G)$, this gives

$$\int_G \overline{f_{\overline{\varphi_\iota}}}(\alpha_{-t}(\omega))\sigma(t)\,\mathrm{d}\omega(t) \;\longrightarrow\; \int_G \overline{f_{\overline{\varphi}}(\alpha_{-t}(\omega))}\sigma(t)\,\mathrm{d}\omega(t)\,,$$

uniformly on Ω. The desired continuity statement follows.
(b) We first show the statement for $t = 0$. To this aim, we define

$$Z \;:=\; \big(\widetilde{\varphi} * \psi * \gamma_{\sigma,m}\big)(0) \;=\; \int_G \big(\widetilde{\varphi} * \psi\big)(-s)\,\mathrm{d}\gamma_{\sigma,m}(s) \;=\; \int_\Omega \int_G \overline{f_{\widetilde{\widetilde{\varphi}*\psi}}}(\alpha_{-t}\omega)\sigma(t)\,\mathrm{d}\omega(t)\,\mathrm{d}m(\omega).$$

Then, we can calculate

$$\begin{aligned} Z &= \int_\Omega \int_G \overline{\int_G \overline{(\widetilde{\varphi} * \psi)(s-t)}\,\mathrm{d}\omega(s)}\,\sigma(t)\,\mathrm{d}\omega(t)\,\mathrm{d}m(\omega) \\ &= \int_\Omega \int_G \int_G \int_G \widetilde{\varphi}(u)\psi(s-t-u)\,\mathrm{d}u\;\mathrm{d}\overline{\omega}(s)\sigma(t)\,\mathrm{d}\omega(t)\,\mathrm{d}m(\omega) \\ &= \int_\Omega \int_G \int_G \int_G \widetilde{\varphi}(u+s)\psi(-t-u)\,\mathrm{d}u\;\mathrm{d}\overline{\omega}(s)\sigma(t)\,\mathrm{d}\omega(t)\,\mathrm{d}m(\omega) \\ &= \int_G \int_\Omega \int_G \overline{f_\varphi(\alpha_u\omega)}\psi(-t-u)\sigma(t)\,\mathrm{d}\omega(t)\,\mathrm{d}m(\omega)\,\mathrm{d}u, \end{aligned}$$

where we used the translation invariance of the Haar measure in the second last step, and Fubini's Theorem and $\int_G \widetilde{\varphi}(u+s)\,\mathrm{d}\overline{\omega}(s) = \overline{f_\varphi(\alpha_u\omega)}$ in the last step. By the invariance of m, and Fubini's Theorem together with $\int \sigma(t)\,\mathrm{d}t = 1$, this gives:

$$\begin{aligned} Z &= \int_G \int_\Omega \int_G \overline{f_\varphi(\omega)}\psi(-t-u)\sigma(t)\,\mathrm{d}(\alpha_{-u}\omega)(t)\,\mathrm{d}m(\omega)\,\mathrm{d}u \\ &= \int_G \int_\Omega \int_G \overline{f_\varphi(\omega)}\psi(-t)\sigma(t-u)\,\mathrm{d}\omega(t)\,\mathrm{d}m(\omega)\,\mathrm{d}u \\ &= \int_\Omega \overline{f_\varphi(\omega)} f_\psi(\omega)\,\mathrm{d}m(\omega) \;\;=\;\; \langle f_\varphi, f_\psi\rangle\,. \end{aligned}$$

The case of arbitrary $t \in G$ can now be treated by mimicking the last part of the proof of part (b) of Theorem 5.
(c) This is immediate from (b) and (a) as $\{\widetilde{\varphi} * \psi : \varphi, \psi \in C_c(G)\}$ is dense in $C_c(G)$.
(d) This is a direct consequence of (b). □

Part (b) of the Lemma shows, in particular, that the measure $\gamma_{\sigma,m}$ equals the autocorrelation measure introduced in the last section if m is ergodic. Part (d) shows that $\gamma_{\sigma,m}$ is positive definite. Thus, by Bochner's Theorem, see [7], its Fourier transform is a positive measure on the dual group $\widehat{G}$.

Definition 6. *Let (Ω, α) be a* TMDS *with invariant probability measure m. The measure $\gamma_m := \gamma_{\sigma,m}$, where $\sigma \in C_c(G)$ with $\int_G \sigma\,\mathrm{d}t = 1$, is called the* autocorrelation measure *of the dynamical system (Ω, α) with invariant measure m. Its Fourier transform $\widehat{\gamma_m}$ is called the* diffraction measure *of the dynamical system (Ω, α) with invariant measure m.*

We summarise the preceding considerations in the following lemma, where we use $\psi_{\text{-}}$ for the function defined by $\psi_{\text{-}}(t) = \psi(-t)$.

Lemma 7. *Let (Ω, α) be a* TMDS *with invariant probability measure m. Then, there exists a unique measure on G assigning the value $\langle f_\varphi, f_\psi\rangle$ to the function $\overline{\varphi} * \psi_{\text{-}}$, for $\varphi, \psi \in C_c(G)$. This measure is the autocorrelation measure γ_m of (Ω, α).*

Proof. Uniqueness is clear as the set $\{\overline{\varphi} * \psi_{\_} : \varphi, \psi \in C_c(G)\}$ is dense in $C_c(G)$. Existence follows from Proposition 6, as

$$\gamma_m(\overline{\varphi} * \psi_{\_}) \;=\; \big(\widetilde{\varphi} * \psi * \gamma_m\big)(0) \;=\; \langle f_\varphi, f_\psi \rangle.$$

This proves the lemma. □

Remark. This definition of the autocorrelation and the diffraction of a dynamical system is to be compared with the corresponding objects of a single measure (namely an element in Ω) studied in the last section. In the latter case, one faces the problem of its dependence of the measure m, or of the averaging sequence (B_n). It is reasonable, both mathematically and physically, to replace this by the objects defined in Definition 6, at least for most aspects of the spectral theory connected with it.

Having cast the diffraction measure in an abstract context, we will now briefly discuss the basic quantities in the spectral theory of dynamical systems: Let (Ω, α) be a TMDS. By Stone's Theorem (compare [25, Sec. 36D]), there exists a projection valued measure

$$E_T \colon \text{Borel sets on } \widehat{G} \;\longrightarrow\; \text{Projections on } L^2(\Omega, m)$$

with

$$\langle f, T^t f\rangle \;=\; \int_{\widehat{G}} (\widehat{s}, t)\, \mathrm{d}\langle f, E_T(\widehat{s}) f\rangle \;=\; \int_{\widehat{G}} (\widehat{s}, t)\, \mathrm{d}\rho_f(\widehat{s})\,,$$

where ρ_f is the measure on $\widehat{G}$ defined by $\rho_f(B) := \langle f, E_T(B) f\rangle$. It is then not hard to see that T has pure point spectrum (in the sense defined in Section 2) if and only if all the measures ρ_f, with $f \in L^2(\Omega, m)$, are pure point measures.

To $\varphi \in C_c(G)$, we have associated the function $f_\varphi \in L^2(\Omega, m)$ in the last section. It turns out that the measure ρ_{f_φ} can be calculated in terms of the diffraction measure. While this connection is not hard to prove, it is underlying the main result of this section. Therefore, we isolate it in the following proposition (compare [7, 14]).

Proposition 7. *Let (Ω, α) be a* TMDS *with invariant probability measure m. Then, the equation $\rho_{f_\varphi} = |\widehat{\varphi}|^2\, \widehat{\gamma_m}$ holds for every $\varphi \in C_c(G)$.*

Proof. By the very definition of ρ_{f_φ} above, the (inverse) Fourier transform (on $\widehat{G}$) of ρ_{f_φ} is $t \mapsto \langle f_\varphi, T^t f_\varphi\rangle$. By Lemma 6, we have $\langle f_\varphi, T^t f_\varphi\rangle = \big(\widetilde{\varphi} * \varphi * \gamma_m\big)(t)$. Thus, taking the Fourier transform (on G), we infer $\rho_{f_\varphi} = |\widehat{\varphi}|^2\, \widehat{\gamma_m}$. □

Note that every closed T-invariant subspace $\mathcal{V}$ of $L^2(\Omega, m)$ gives rise to a representation $T|_{\mathcal{V}}$ of G on $\mathcal{V}$ by restricting the representation T to $\mathcal{V}$. The spectral family of $T|_{\mathcal{V}}$ will be denoted by $E_{T|_{\mathcal{V}}}$. With the canonical inclusion $i_{\mathcal{V}} : \mathcal{V} \longrightarrow L^2(\Omega, m)$ and projection $P_{\mathcal{V}} \colon L^2(\Omega, m) \longrightarrow \mathcal{V}$, we obviously have

$$T|_{\mathcal{V}} \;=\; P_{\mathcal{V}}\, T\, i_{\mathcal{V}} \quad \text{and} \quad E_{T|_{\mathcal{V}}} = P_{\mathcal{V}}\, E_T\, i_{\mathcal{V}}.$$

In our setting, a translation invariant subspace appears naturally. This is discussed next.

Lemma 8. *Let (Ω, α) be a* TMDS *with invariant probability measure m. The set of functions $\mathcal{U}_0 := \{f_\varphi : \varphi \in C_c(G)\}$ is a translation invariant subspace of $L^2(\Omega, m)$, and so is its closure.*

Proof. The first part of the statement follows from Lemma 3 (b). The second part of the statement is then immediate. □

Definition 7. *Let $\mathcal{U}$ be the closure of the space $\mathcal{U}_0$ from Lemma* 8 *in $L^2(\Omega, m)$.*

Before we can give a precise version of the relationship between $\widehat{\gamma_m}$ and T we need one more definition.

Definition 8. *Let ρ be a measure on $\widehat{G}$ and S be an arbitrary unitary representation of G on $L^2(\Omega, m)$. Then, ρ is called a* spectral measure *for S if the following holds for all Borel sets B: $E_S(B) = 0$ if and only if $\rho(B) = 0$.*

Now, the relationship between $\widehat{\gamma_m}$ and $T = T_m$ can be phrased as follows.

Theorem 6. *Let (Ω, α) be a* TMDS *with invariant probability measure m. Then, the measure $\widehat{\gamma_m}$ is a spectral measure for the restriction $T|_{\mathcal{U}}$ of T to $\mathcal{U}$.*

Proof. Let B be a Borel set in $\widehat{G}$. Then, obviously, $E_{T|_{\mathcal{U}}}(B) = 0$ if and only if we have $0 = \langle f_\varphi, E_{T|_{\mathcal{U}}}(B) f_\varphi \rangle$ for every $\varphi \in C_c(G)$. As $\langle f_\varphi, E_T(B) f_\varphi \rangle = \langle f_\varphi, E_{T|_{\mathcal{U}}}(B) f_\varphi \rangle$ for every $\varphi \in C_c(G)$, by the very definition of $\mathcal{U}$ and $E_{T|_{\mathcal{U}}}$, we infer that $E_{T|_{\mathcal{U}}}(B) = 0$ if and only if

$$\langle f_\varphi, E_T(B) f_\varphi \rangle \; = \; 0 \,, \quad \text{for every } \varphi \in C_c(G) \,.$$

By Proposition 7, we have $\rho_{f_\varphi} = |\widehat{\varphi}|^2 \widehat{\gamma_m}$ and, in particular,

$$\langle f_\varphi, E_T(B) f_\varphi \rangle \; = \rho_{f_\varphi}(B) \; = \int_B |\widehat{\varphi}|^2 \, \mathrm{d}\widehat{\gamma_m} \,.$$

These considerations show that $E_{T|_{\mathcal{U}}}(B) = 0$ if and only if $0 = \int_B |\widehat{\varphi}|^2 \, \mathrm{d}\widehat{\gamma_m}$ for every function $\varphi \in C_c(G)$. Thus, it remains to be shown that $\widehat{\gamma_m}(B) = 0$ if and only if $0 = \int_B |\widehat{\varphi}|^2 \, \mathrm{d}\widehat{\gamma_m}$ for every $\varphi \in C_c(G)$. The only if part is clear. As for the converse, recall that the image of $L^1(G, \mathrm{d}t)$ under the Fourier transform separates points in $\widehat{G}$, see [32]. As $C_c(G)$ is dense in in $L^1(G, \mathrm{d}t)$, the same holds for the image of $C_c(G)$ under the Fourier transform. Therefore, for every $\widehat{s} \in \widehat{G}$, there exists a $\varphi \in C_c(G)$ with $\widehat{\varphi}(\widehat{s}) \neq 0$. Thus, $\widehat{\gamma_m}(B \cap K) = 0$ for every compact K is a direct consequence of $0 = \int_B |\widehat{\varphi}|^2 \, \mathrm{d}\widehat{\gamma_m}$ for every $\varphi \in C_c(G)$. As $\widehat{\gamma_m}$ is regular, $\widehat{\gamma_m}(B) = 0$ follows. □

We finish this section with a brief discussion of some consequences of the above results for the definition of γ_m. We thank the referee for useful comments on this point.

The following is a consequence of Proposition 7 (compare [5, Prop. 3] and discussion preceding it for similar considerations).

Corollary 1. *Let (φ_ι) be an approximate unit in $C_c(G)$ with respect to convolution, i.e., $\varphi_\iota * \psi \longrightarrow \psi$ in $C_c(G)$ for every $\psi \in C_c(G)$. Then, the measures $\rho_{f_{\varphi_\iota}}$ converge vaguely to $\widehat{\gamma_m}$.*

Proof. As (φ_ι) is an approximate unit in $C_c(G)$, the continuous functions $\varphi_\iota * \widetilde{\varphi}_\iota$, viewed as absolutely continuous measures with respect to the Haar measure on G, converge vaguely towards δ_0, the unit point measure at $0 \in G$. Thus, Levy's continuity theorem [7, Thm 3.13] gives us compact convergence of $|\widehat{\varphi}_\iota|^2$ towards $\widehat{\delta_0} \equiv 1$. This easily implies vague convergence of the measures $|\widehat{\varphi}_\iota|^2 \, \widehat{\gamma_m}$ towards $\widehat{\gamma_m}$. As $\rho_{f_\varphi} = |\widehat{\varphi}|^2 \, \widehat{\gamma_m}$ by Proposition 7, we infer the statement of the corollary. □

Note that Corollary 1 gives another way to define the diffraction measure $\widehat{\gamma_m}$. Namely, we can define $\widehat{\gamma_m}$ to be any accumulation point of the net $(\rho_{f_{\varphi_\iota}})$ whenever (φ_ι) is an approximate unit in $C_c(G)$. The result is unique, once m is chosen.

7. The main result

In this section, we state and prove our main result. It shows equivalence of pure point diffraction and pure point dynamical spectrum for rather general measure theoretic dynamical systems.

Theorem 7. *Let (Ω, α) be a* TMDS *with invariant probability measure m. Let T_m be the associated unitary representation of G by translation operators and $\widehat{\gamma_m}$ the associated diffraction measure. The following assertions are now equivalent.*

(a) *The diffraction measure $\widehat{\gamma_m}$ is pure point.*
(b) *The representation T_m has pure point spectrum.*

Proof of Theorem 7. (a) $\Longrightarrow$ (b). This is a consequence of Theorem 1. More precisely, we will show that the vector space

$$\mathcal{V} := \{f_\varphi : \varphi \in C_c(G)\}$$

satisfies assertion (b) of this theorem:

As $\widehat{\gamma_m}$ has pure point spectrum by (a), Theorem 6 gives that $T|_\mathcal{U}$ has pure point spectrum, where $\mathcal{U}$ is the closure of $\mathcal{V}$. Thus, in particular, f_φ belongs to $\mathcal{H}_{pp}(T)$ for every $\varphi \in C_c(G)$. As every element of the form f_φ is continuous by Lemma 3, we see that $\mathcal{V}$ is indeed a subspace of $\mathcal{H}_{pp}(T) \cap C(\Omega)$.

It remains to be shown that $\mathcal{V}$ separates points. Let ω_1 and ω_2 be two different points of Ω. Then, ω_1 and ω_2 are different measures on G. Therefore, there exists a $\varphi \in C_c(G)$ with $\omega_1(\varphi) \neq \omega_2(\varphi)$. This implies $f_{\varphi_-}(\omega_1) \neq f_{\varphi_-}(\omega_2)$ with $\varphi_-(t) := \varphi(-t)$.

(b) $\Longrightarrow$ (a). This is immediate from Theorem 6. □

8. Spectral properties determined by subrepresentations

The ideas of the preceding sections can be refined to give some further information on how spectral properties of T are determined by spectral properties of $T_\mathcal{U}$. This concerns the continuity of the eigenfunctions, and the set of eigenvalues. While the TMDS are the application we have in mind here, the underlying result can be phrased rather abstractly.

We need a special concept on "density of a subspace with respect to multiplication". This is defined next.

Definition 9. *A subspace $\mathcal{V}$ of $L^2(\Omega, m)$ is said to satisfy condition* MD *if the set of products $f_1 \cdot \ldots \cdot f_n$ with $n \in \mathbb{N}$, $f_i \in \mathcal{V} \cap L^\infty(\Omega, m)$ or $\overline{f_i} \in \mathcal{V} \cap L^\infty(\Omega, m)$, $1 \leq i \leq n$, is total in $L^2(\Omega, m)$.*

Theorem 8. *Let (Ω, α) be a topological dynamical system over G with α-invariant measure m. Let $\mathcal{V}$ be a closed T-invariant subspace of $L^2(\Omega, m)$ satisfying* MD. *If $T|_\mathcal{V}$ has pure point spectrum, then the following assertions hold:*

(a) *T has pure point spectrum.*
(b) *The group of eigenvalues of T is generated by the set of eigenvalues of $T|_\mathcal{V}$.*
(c) *If $\mathcal{V}$ has a basis consisting of continuous eigenfunctions of $T|_\mathcal{V}$, then $L^2(\Omega, m)$ has a basis consisting of continuous eigenfunctions of T, provided the multiplicity of each eigenvalue of T is at most countably infinite.*

Remarks. (a) The countability assumption in (c) is trivially satisfied if the Hilbert space $L^2(\Omega, m)$ is separable, (which holds, e.g., if Ω is metrisable). It is also satisfied if α is ergodic with respect to m. In this case, the multiplicity of each eigenvalue is one.
(b) While we have stated the theorem for topological dynamical systems, its proof does not use the topology on Ω. It can therefore be carried over without changes to give the corresponding result for measurable actions of G on a measure space Ω.

Proof. (a)/(b) Let $\mathcal{S}_1$ be an orthonormal basis of $\mathcal{V}$ consisting of eigenfunctions of $T|_{\mathcal{V}}$. Set

$$\mathcal{S}_2 \;:=\; \{f^N : f \in \mathcal{S}_1 \;\text{ or }\; \overline{f} \in \mathcal{S}_1, N \in \mathbb{N}\},$$

where, for $N \in \mathbb{N}$ and a function f, the function f^N is defined in (2). As mentioned there, f^N is again an eigenfunction of T. However, f^N need not belong to $\mathcal{V}$. Let $\mathcal{S}_3$ be the set of finite products of elements of $\mathcal{S}_2$. In particular, all elements of $\mathcal{S}_3$ are bounded functions, and the same is true of all finite linear combinations of elements of $\mathcal{S}_3$.

Claim. *Every finite product $f_1 \cdot \ldots \cdot f_n$ with $n \in \mathbb{N}$, and f_i or $\overline{f_i}$ in $\mathcal{V} \cap L^\infty(\Omega, m)$, can be approximated arbitrarily well (in $L^2(\Omega, m)$) by finite linear combinations of elements of $\mathcal{S}_3$.*

Proof of the Claim. This is shown by induction. The case $n = 1$ is simple, as $\mathcal{S}_1$ is an orthonormal basis of $\mathcal{V}$. Assume that the claim holds for fixed $n \in \mathbb{N}$. As in Lemma 1, we use again a variant of Lee, Moody and Solomyak [22]. Let $\varepsilon > 0$ be given. By the induction assumption, there exists a finite linear combination g of elements of $\mathcal{S}_3$ with

$$\|f_1 \cdot \ldots \cdot f_n - g\|_2 \;\leq\; \frac{\varepsilon}{\|f_{n+1}\|}\,.$$

Here, g is a bounded function (as all functions in $\mathcal{S}_3$ are bounded). Thus, there exists a finite linear combination h of elements in $\mathcal{S}_3$ with

$$\|f_{n+1} - h\|_2 \;\leq\; \frac{\varepsilon}{\|g\|_\infty}.$$

The proof of the claim can now be finished as in Lemma 1. □

The claim shows that $\mathcal{S}_3$ is total in $L^2(\Omega, m)$, as the products appearing in its statement are total in $L^2(\Omega, m)$ by the density assumption MD. Now, obviously, the elements of $\mathcal{S}_3$ are eigenfunctions of T and the corresponding eigenvalues are just the group generated by the eigenvalues of $T|_{\mathcal{V}}$. This proves (a) and (b).

To prove (c), we consider a basis $\mathcal{S}_1$ of $\mathcal{V}$ consisting of continuous eigenfunctions of $T|_{\mathcal{V}}$ and define

$$\mathcal{S}_4 \;:=\; \{f_1 \cdot \ldots \cdot f_n : n \in \mathbb{N},\; f_i \in \mathcal{S}_1 \;\text{ or }\; \overline{f_i} \in \mathcal{S}_1\}\,.$$

As above, one can show that $\mathcal{S}_4$ is total in $L^2(\Omega, m)$. Apparently, the elements in $\mathcal{S}_4$ are continuous eigenfunctions of T. Moreover, by general principles, eigenfunctions belonging to different eigenvalues are orthogonal. We now apply the Gram-Schmidt orthogonalisation procedure in each eigenspace, compare [28, Sec. 3.1.13]. This is possible because the multiplicity of each eigenspace is at most countably infinite by (c). As a result, we obtain a basis of eigenfunctions which are continuous. (Note that Gram-Schmidt deals only with finite sums in each step and therefore does not destroy continuity.) □

The preceding considerations can be applied to any TMDS. This will briefly be discussed next. To apply Theorem 8, we need the following reformulation of previous results.

Proposition 8. *Let* (Ω, α) *be a* TMDS *with invariant probability measure* m*, let* $\mathcal{U}$ *be the space introduced in Definition* 7 *and denote the characteristic function of* Ω *by* 1_Ω*. Then, the subspace* $\mathcal{V} := \mathcal{U} + \{c\, 1_\Omega : c \in \mathbb{C}\}$ *is closed, invariant and satisfies assumption* MD*.*

Proof. The subspace $\mathcal{S} := \{c\, 1_\Omega : c \in \mathbb{C}\}$ is one-dimensional. So, as $\mathcal{U}$ is closed, $\mathcal{V} = \mathcal{U} + \mathcal{S}$ is closed as well. As $\mathcal{U}$ and $\mathcal{S}$ are T-invariant, so is $\mathcal{V}$. It remains to be shown that MD is satisfied. This is a consequence of the Stone-Weierstraß Theorem as $\{f_\varphi : \varphi \in C_c(G)\}$ separates points (see the proof of Theorem 7). □

It is possible to base the proof of our main result, Theorem 7, on Theorem 8 and Proposition 8. Here, our focus is in a somewhat different direction.

Theorem 9. *Let* (Ω, α) *be a* TMDS *with invariant probability measure* m*,* T_m *be the corresponding unitary representation of* G *by translation operators, and* $\widehat{\gamma_m}$ *the associated diffraction measure. Let* $\mathcal{U}$ *be the space of functions defined in Definition* 7*. If* $\widehat{\gamma_m}$ *is a pure point measure, the following assertions hold.*

(a) *The group of eigenvalues of* T_m *is generated by the set of points in* $\widehat{G}$ *of positive* $\widehat{\gamma_m}$ *measure, i.e., the points* $\hat{s}$ *with* $\widehat{\gamma_m}(\{\hat{s}\}) > 0$*.*
(b) *If* $\mathcal{U}$ *has a basis of continuous eigenfunctions of* T_m*, then so has* $L^2(\Omega, m)$*, provided the multiplicity of each eigenvalue is at most countably infinite.*

Proof. As $\widehat{\gamma_m}$ is a pure point measure, $T|_{\mathcal{U}}$ has pure point spectrum by Theorem 6. Set $\mathcal{V} := \mathcal{U} + \{c\, 1_\Omega : c \in \mathbb{C}\}$. As 1_Ω is obviously a (continuous) eigenfunction of T (to the eigenvalue 1), $T_{\mathcal{V}}$ has pure point spectrum as well. Moreover, by Proposition 8, $\mathcal{V}$ is invariant, closed and satisfies assumption MD. Thus, the conditions of Theorem 8 are satisfied, and our assertions follow. □

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Appendix A. The local rubber topology is a Fell topology

The aim of this appendix is to show that the topology introduced in Section 4 is a special case of a topology introduced by Fell in [13] on the closed subsets of an arbitrary locally compact space.

We start by recalling the definition of Fell's topology: The locally compact space in question is G. For a compact set C in G and a finite family $\mathcal{F}$ of open sets in G, we define $\mathcal{U}(C, \mathcal{F})$ by

$$\mathcal{U}(C, \mathcal{F}) : \{C \in \mathcal{C}(G) : \varLambda \cap C = \varnothing \text{ and } \varLambda \cap A \neq \varnothing \text{ for every } A \in \mathcal{F}\}.$$

The family of all $\mathcal{U}(C, \mathcal{F})$ with C compact in G and $\mathcal{F}$ a finite family of open sets in G is a basis of the Fell topology. This is a typical example of a so-called "hit and miss" topology, where $\mathcal{U}(C, \mathcal{F})$ consists of all closed sets which hit the sets of $\mathcal{F}$ and miss the set C.

This topology agrees with the one introduced in Section 4, as follows from the next lemma.

Lemma 9. (a) *Let $C \subset G$ compact and $\mathcal{F}$ a finite family of open subsets of G be given with $\mathcal{U}(C,\mathcal{F}) \neq \varnothing$. Then, there exists a closed set H in G, a compact $K \subset G$ and an open neighbourhood V of $0 \in G$ with*

$$U_{K,V}(H) \subset \mathcal{U}(C,\mathcal{F}).$$

(b) *Let a closed set H in G, a compact subset $K \subset G$ and an open neighbourhood V of $0 \in G$ be given. Then, there exists a compact $C \subset G$ and a finite family of open sets $\mathcal{F}$ in G with $\mathcal{U}(C,\mathcal{F}) \neq \varnothing$ and*

$$\mathcal{U}(C,\mathcal{F}) \subset U_{K,V}(H).$$

Proof. (a) By $\mathcal{U}(C,\mathcal{F}) \neq \varnothing$, we have $A \setminus C \neq \varnothing$ for every $A \in \mathcal{F}$. Thus, in every $A \in \mathcal{F}$ there exists an $x_A \in A \setminus C$. As $A \setminus C$ is open and $\mathcal{F}$ is a finite family, we can find a neighbourhood V of 0 in G with $V = -V$ such that

$$x_A + \overline{V} \subset A \setminus C \quad \text{for every } A \in \mathcal{F}. \tag{6}$$

Define $H := \{x_A : A \in \mathcal{F}\}$ and $K := C \cup (H + \overline{V})$. Then, K is the disjoint union of C and $H + \overline{V}$ by the very construction of H.

Now, let an arbitrary $L \in U_{K,V}(H)$ be given. We have to show that $L \in \mathcal{U}(C,\mathcal{F})$:
By $L \in U_{K,V}(H)$, we have $L \cap K \subset H + V \subset H + \overline{V} \subset K \setminus C$ and, as $C \subset K$, this implies

$$L \cap C = L \cap C \cap K = \varnothing.$$

Moreover, for every $A \in \mathcal{F}$, we have $x_A \in H = H \cap K \subset L + V$ and therefore $(x_A - V) \cap L \neq \varnothing$. By $V = -V$ and (6), this implies

$$\varnothing \neq (x_A + V) \cap L \subset (A \setminus C) \cap L \subset A \cap L$$

for every $A \in \mathcal{F}$. These considerations show $L \in \mathcal{U}(C,\mathcal{F})$. As $L \in U_{K,V}(H)$ was arbitrary, we infer $U_{K,V}(H) \subset \mathcal{U}(C,\mathcal{F})$.

(b) Let W be an open neighbourhood of 0 in G with $W = -W$ and $W + W \subset V$. Define $C := K \setminus (H + W)$, where H and K are given by assumption. As $K \cap H$ is compact, there exist $t_1, \dots, t_n \in G$ with

$$K \cap H \subset \bigcup_{i=1}^{n} (t_i + W) \quad \text{and} \quad (t_j + W) \cap (K \cap H) \neq \varnothing \quad \text{for } 1 \leq j \leq n. \tag{7}$$

Set $\mathcal{F} := \{t_j + W : 1 \leq j \leq n\}$.

Let now $L \in \mathcal{U}(C,\mathcal{F})$ be arbitrary. We have to show that $L \in U_{K,V}(H)$:
By $L \in \mathcal{U}(C,\mathcal{F})$ and the definition of C, we have $\varnothing = L \cap C = L \cap (K \setminus (H + W))$, so

$$L \cap K = L \cap (K \cap (H + W)) \cup L \cap (K \setminus (H + W)) = L \cap (K \cap (H + W)) \subset H + W \subset H + V.$$

By $L \cap (t_j + W) \neq \varnothing$, $1 \leq j \leq n$, and $W - W \subset V$, we also have $t_j + W \subset L + V$. Combined with (7), this implies

$$H \cap K \subset \bigcup_{j=1}^{n} (t_j + W) \subset L + V$$

and we infer $L \in U_{K,V}(H)$. As $L \in \mathcal{U}(C,\mathcal{F})$ was arbitrary, the inclusion $\mathcal{U}(C,\mathcal{F}) \subset U_{K,V}(H)$ is established. Moreover, $\mathcal{U}(C,\mathcal{F})$ is not empty, as it obviously contains H. □

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany
E-mail address: mbaake@mathematik.uni-bielefeld.de

Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany
E-mail address: dlenz@mathematik.tu-chemnitz.de

CHAPTER 12

M. Baake, D. Lenz, *Deformation of Delone dynamical systems and topological conjugacy*, Journal of Fourier Analysis and Applications 11 (2005), 125–150.

DEFORMATION OF DELONE DYNAMICAL SYSTEMS AND PURE POINT DIFFRACTION

MICHAEL BAAKE AND DANIEL LENZ

Abstract. This paper deals with certain dynamical systems built from point sets and, more generally, measures on locally compact Abelian groups. These systems arise in the study of quasicrystals and aperiodic order, and important subclasses of them exhibit pure point diffraction spectra. We discuss the relevant framework and recall fundamental results and examples. In particular, we show that pure point diffraction is stable under "equivariant" local perturbations and discuss various examples, including deformed model sets. A key step in the proof of stability consists in transforming the problem into a question on factors of dynamical systems.

1. Introduction

Aperiodic order has become a topic of intense research over the last two decades [34, 38, 7, 47, 49]. While the term is not rigorously defined (yet), it roughly refers to forms of order at the very verge between periodic and non-periodic structures. As such, it has attracted attention in various branches of mathematics including geometry, combinatorics, ergodic theory, operator theory and harmonic analysis.

An important trigger in these developments has been the actual discovery of physical substances with strong aperiodic order [42], which are now called quasicrystals. They owe their discovery to their remarkable diffraction patterns: These patterns imply a high degree of order as they are pure point spectra (or Bragg spectra), while, at the same time, they exclude periodicity by their non-crystallographic symmetries. Accordingly, the study of pure point diffraction has been an important topic in this context ever since.

This paper is concerned with pure point diffraction. More precisely, we study the stability of pure point diffraction under certain deformations. This issue is a very natural one, both from the physical and the mathematical point of view. In order to study stability under deformations, we need to review the undeformed case first. To make the paper essentially self-contained, this discussion is carried out at some length, including relevant concepts and examples. Moreover, we hope that the paper can serve as an introductory survey over the treatment of diffraction via dynamical systems for the reader unfamiliar with the field.

Delone sets provide an important model class for the description of aperiodic order. In particular, they can be viewed as a mathematical abstraction of the set of atomic positions of a physical quasicrystal (at zero temperature, or at a given instant of time). Many of the rather intriguing spectral properties of quasicrystals can be formulated, in a simplified manner, on the basis of Delone sets. This is also a rather common class of structures in the mathematical theory of aperiodic order [29]. It is attractive because it admits a direct geometric interpretation with two Delone sets being close to one another if large patches (around some fixed point of reference, say) coincide, possibly after a tiny local rearrangement of the individual points.

However, from a more physical point of view, other scenarios are also very important. In particular, the description of an aperiodic distribution of matter by means of (continuous) quasi- or almost periodic density functions has been emphasized right from the very beginning of quasicrystal theory [9]. Here, closeness of two structures is more adequately described by means of the supremum norm, as in the theory of almost periodic functions.

As is apparent, these two pictures are not compatible — unless they are embedded into a larger class of structures that admit both the Delone (or tiling) picture and the continuous density description as special cases. One possibility is the use of translation bounded (complex) measures, equipped with the vague topology. Here, two structures (i.e., measures) are close if their evaluations with continuous functions supported on a large compact set K are close. This entails both situations mentioned above, one being described by pure point measures, the other by absolutely continuous measures with continuous Radon-Nikodym densities.

In view of the fact that the original distinction between the discrete and the continuous approach led to rather hefty disputes on the justification and appropriateness of the two approaches, we believe that the systematic development of a unified frame is overdue. We take this as our main motivation for a dynamical systems approach based on *measures*, though we will also spell out the details for the more conventional (and perhaps more intuitive) approach via Delone sets.

As mentioned already, one important issue in this context is that of the *stability* of certain features, e.g., stability under slight modifications or deformations. The question of stability of pure point diffractivity is addressed in this paper.

Our main abstract result shows that pure point diffraction is stable under local "equivariant" perturbations. The proof relies on two steps: We use a recent result of ours [3] (see [30, 22] for related material) which establishes when pure point dynamical spectrum is equivalent to pure point diffraction spectrum. This effectively transforms the stability problem into a question on dynamical systems. This question is then solved by studying certain factors of the original dynamical system.

To give the reader a flavour of this procedure, we include the following rather informal statement of our main result, when restricted to Delone sets.

Result. *The hull of an admissibly deformed Delone set is a topological factor of the hull of the original Delone set. In particular, if a Delone set has pure point diffraction spectrum, its deformation has pure point diffraction spectrum as well.*

A precise version of this result is given in Theorem 3. As mentioned already, our setting is general enough to treat not only the case of Delone sets but rather the case of arbitrary measure dynamical systems. This is made precise in Theorem 4.

The abstract result is applied to various examples. In particular, we study perturbations of model sets in the context of cut and project schemes. This generalizes the corresponding considerations of Hof [24] and Bernuau and Duneau [11]. It also shows that related results of Clark and Sadun [12] fall well within our framework.

Our results should be compared to complementary results of Hof [26]. They show that *random* perturbations do not leave a pure point spectrum unchanged, but rather introduce an absolutely continuous component, see also [2, 5] for further examples.

We are well aware of the fact that considerable parts of the following investigation dealing with topological dynamical systems can be generalized to measurable dynamical systems.

However, by its very nature, the subject of aperiodic order seems to be a topological one. For this reason, we stick to the topological category.

The paper is organized as follows. In Section 2, we introduce some basic notation concerning topological dynamical systems. In Section 3, we recall and establish various facts on factors. These considerations are the abstract core behind our deformation procedure. Section 4 is devoted to a discussion of diffraction in the context of dynamical systems of Delone sets and measures. The abstract deformation procedure and the stability of pure point diffraction under this type of deformation is discussed in Section 5. Applications to model sets are studied in Section 6, which also contains a brief summary of their general definition. The various concepts and results will then be illustrated with a concrete example, the silver mean chain, in Section 7. Further aspects of the deformation procedure, in particular concerning topological conjugacy, are discussed in Section 8.

2. Generalities on dynamical systems

Our considerations are set in the framework of topological dynamical systems. We are dealing with σ-compact locally compact topological groups and compact spaces. Thus, we start with some basic notation and facts concerning locally compact topological spaces used throughout the paper.

Whenever X is a σ-compact locally compact space (by which we mean to include the Hausdorff property), we denote the space of continuous functions on X by $C(X)$ and the subspace of continuous functions with compact support by $C_c(X)$. This space is equipped with the locally convex limit topology induced by the canonical embeddings $C_K(X) \hookrightarrow C_c(X)$, where $C_K(X)$ is the space of complex continuous functions with support in a given compact set $K \subset X$. Here, each $C_K(X)$ is equipped with the topology induced by the standard supremum norm.

As X is a topological space, it carries a natural σ-algebra, namely the Borel σ-algebra generated by all closed subsets of X. The set $\mathcal{M}(X)$ of all complex regular Borel measures on G can then be identified with the space $C_c(X)^*$ of complex valued, continuous linear functionals on $C_c(G)$. This is justified by the Riesz-Markov representation theorem, compare [39, Ch. 6.5] for details. In particular, we can write $\int_X f \, \mathrm{d}\mu = \mu(f)$ for $f \in C_c(X)$ and simplify the notation this way. The space $\mathcal{M}(X)$ carries the vague topology, i.e., the weakest topology that makes all functionals $\mu \mapsto \mu(\varphi)$, $\varphi \in C_c(X)$, continuous. The total variation of a measure $\mu \in \mathcal{M}(X)$ is denoted by $|\mu|$.

We now fix a σ-compact locally compact Abelian (LCA) group G for the remainder of the paper. The dual group of G is denoted by $\widehat{G}$, and the pairing between a character $\hat{s} \in \widehat{G}$ and $t \in G$ is written as $(\hat{s}, t)$. Whenever G acts on the compact space $\varOmega$ (which is then also Hausdorff by our convention) by a continuous action

$$\alpha \colon \; G \times \varOmega \;\longrightarrow\; \varOmega \, , \quad (t, \omega) \;\mapsto\; \alpha_t(\omega) \, ,$$

where $G \times \varOmega$ carries the product topology, the pair $(\varOmega, \alpha)$ is called a *topological dynamical system* over G. We will often write $\alpha_t \omega$ for $\alpha_t(\omega)$. If $\omega \in \varOmega$ satisfies $\alpha_t \omega = \omega$, t is called a *period* of ω. If all $t \in G$ are periods, ω is called G-invariant, or α-*invariant* to refer to the action involved.

The set of all Borel probability measures on $\varOmega$ is denoted by $\mathcal{P}(\varOmega)$, and the subset of α-invariant probability measures by $\mathcal{P}_G(\varOmega)$. As $\varOmega$ is compact, $C_c(\varOmega)$ equipped with the supremum norm is a Banach space. The vague topology on $\mathcal{M}(\varOmega)$ is then just the weak-$*$

topology. By Alaoglu's theorem on weak-$*$ compactness of the unit sphere (compare [39, Thm. 2.5.2]), we easily conclude that $\mathcal{P}(\Omega)$ is compact. As $\mathcal{P}_G(\Omega)$ is obviously closed in $\mathcal{P}(\Omega)$, it is then compact as well. Apparently, $\mathcal{P}_G(\Omega)$ is convex. More importantly, it is always non-empty. For $G = \mathbb{Z}$, this is standard, compare Section 6.2 in [50]. This proof only uses the existence of a van Hove sequence and the compactness of $\mathcal{P}(\Omega)$. Thus, it can be carried over to our setting (for the existence of van Hove sequences, we refer the reader to [44, p. 145] and [48, Appendix, Sec. 3.3]).

An α-invariant probability measure is called *ergodic* if every (measurable) invariant subset of Ω has either measure zero or measure one. The ergodic measures are exactly the extremal points of the convex set $\mathcal{P}_G(\Omega)$. The dynamical system (Ω, α) is called *uniquely ergodic* if $\mathcal{P}_G(\Omega)$ is a singleton set, i.e., if it consists of exactly one element. As usual, (Ω, α) is called *minimal* if, for all $\omega \in \Omega$, the G-orbit $\{\alpha_t \omega : t \in G\}$ is dense in Ω. If (Ω, α) is both uniquely ergodic and minimal, it is called *strictly ergodic*.

Given an $m \in \mathcal{P}_G(\Omega)$, we can form the Hilbert space $L^2(\Omega, m)$ of square integrable measurable functions on Ω. This space is equipped with the inner product

$$\langle f, g \rangle \;=\; \langle f, g \rangle_\Omega \;:=\; \int_\Omega \overline{f(\omega)}\, g(\omega)\, \mathrm{d}m(\omega).$$

The action α gives rise to a unitary representation $T = T^\Omega := T^{(\Omega,\alpha,m)}$ of G on $L^2(\Omega, m)$ by

$$T_t\colon\; L^2(\Omega, m) \;\longrightarrow\; L^2(\Omega, m)\,, \quad (T_t f)(\omega) \;:=\; f(\alpha_{-t}\omega)\,,$$

for every $f \in L^2(\Omega, m)$ and arbitrary $t \in G$.

An $f \in L^2(\Omega, m)$ is called an *eigenfunction* of T with *eigenvalue* $\hat{s} \in \widehat{G}$ if $T_t f = (\hat{s}, t) f$ for every $t \in G$. An eigenfunction (to $\hat{s}$, say) is called *continuous* if it has a continuous representative f with $f(\alpha_{-t}\omega) = (\hat{s}, t) f(\omega)$, for all $\omega \in \Omega$ and $t \in G$. The representation T is said to have *pure point spectrum* if the set of its eigenfunctions is total in $L^2(\Omega, m)$. One then also says that the dynamical system (Ω, α) has *pure point dynamical spectrum*.

By Stone's theorem, compare [32, Sec. 36D], there exists a projection valued measure

$$E_T\colon \text{ Borel sets of } \widehat{G} \;\longrightarrow\; \text{projections on } L^2(\Omega, m)$$

with

$$\langle f, T_t f \rangle \;=\; \int_{\widehat{G}} (\hat{s}, t)\, \mathrm{d}\langle f, E_T(\hat{s}) f \rangle \;:=\; \int_{\widehat{G}} (\hat{s}, t)\, \mathrm{d}\rho_f(\hat{s})\,,$$

where $\rho_f = \rho_f^\Omega := \rho_f^{(\Omega,\alpha,m)}$ is the measure on $\widehat{G}$ defined by $\rho_f(B) := \langle f, E_T(B) f \rangle$. In fact, by Bochner's theorem [41], ρ_f is the unique measure on $\widehat{G}$ with $\langle f, T_t f \rangle = \int_{\widehat{G}} (\hat{s}, t)\, \mathrm{d}\rho_f(\hat{s})$ for every $t \in G$.

3. Factors

Factors of dynamical systems and the corresponding subrepresentations will be an important tool in our study of deformation. In this section, we recall their basic theory, most of which is well known. Since details are somewhat scattered in the literature, we sketch some of the proofs for the sake of completeness, or give precise references. Readers who are familiar with it, or are more interested to first learn about diffraction, may skip this section at first reading.

Let $(\mathit{\Omega},\alpha)$ and $(\mathit{\Theta},\beta)$ be two topological dynamical systems under the action of G, with a mapping $\mathit{\Phi}\colon \mathit{\Omega} \longrightarrow \mathit{\Theta}$ that gives rise to the following diagram:

$$\begin{array}{ccc} \mathit{\Omega} & \xrightarrow{\alpha} & \mathit{\Omega} \\ {\scriptstyle \mathit{\Phi}}\big\downarrow & & \big\downarrow{\scriptstyle \mathit{\Phi}} \\ \mathit{\Theta} & \xrightarrow{\beta} & \mathit{\Theta} \end{array} \tag{1}$$

Definition 1. *Let two topological dynamical systems $(\mathit{\Omega},\alpha)$ and $(\mathit{\Theta},\beta)$ under the action of G and a mapping $\mathit{\Phi} : \mathit{\Omega} \longrightarrow \mathit{\Theta}$ be given. Then, $(\mathit{\Theta},\beta)$ is called a* factor *of $(\mathit{\Omega},\alpha)$, with factor map $\mathit{\Phi}$, if $\mathit{\Phi}$ is a continuous surjection that makes the diagram* (1) *commutative, i.e., $\mathit{\Phi}(\alpha_t(\omega)) = \beta_t(\mathit{\Phi}(\omega))$ for all $\omega \in \mathit{\Omega}$ and $t \in G$.*

Factors inherit many features from the underlying dynamical system. Due to the commutativity of diagram (1), a period $t \in G$ of ω is also a period of $\mathit{\Phi}(\omega)$. Clearly, the converse need not be true, as we will see in an example later on. Let us next recall three other properties of dynamical systems which are inherited by factors.

Fact 1. *Let $(\mathit{\Theta},\beta)$ be a factor of $(\mathit{\Omega},\alpha)$ with factor map $\mathit{\Phi}\colon \mathit{\Omega} \longrightarrow \mathit{\Theta}$. Then, $U \subset \mathit{\Theta}$ is open if and only if $\mathit{\Phi}^{-1}(U)$ is open in $\mathit{\Omega}$.*

Proof. As $\mathit{\Phi}$ is continuous, the only if part is clear. So, assume that $U \subset \mathit{\Theta}$ is given with $\mathit{\Phi}^{-1}(U)$ open. Then, $\mathit{\Phi}^{-1}(\mathit{\Theta} \setminus U) = \mathit{\Omega} \setminus \mathit{\Phi}^{-1}(U)$ is closed and thus compact, as $\mathit{\Omega}$ is compact. Thus, by continuity and surjectivity of $\mathit{\Phi}$, the set $\mathit{\Theta} \setminus U = \mathit{\Phi}(\mathit{\Phi}^{-1}(\mathit{\Theta} \setminus U))$ is compact and, in particular, closed. Thus, U is open. □

Clearly, $\mathit{\Phi}$ induces a mapping $\mathit{\Phi}_*\colon \mathcal{M}(\mathit{\Omega}) \longrightarrow \mathcal{M}(\mathit{\Theta})$, $\mu \mapsto \mathit{\Phi}_*(\mu)$, via $\big(\mathit{\Phi}_*(\mu)\big)(g) := \mu(g \circ \mathit{\Phi})$ for all $g \in C(\mathit{\Theta})$. If μ is a probability measure on $\mathit{\Omega}$, its image, $\mathit{\Phi}_*(\mu)$, is a probability measure on $\mathit{\Theta}$. Moreover, if $\mathit{\Phi}$ is a factor map, invariance under the group action is preserved. So, in this case, we obtain the mapping

$$\mathit{\Phi}_*\colon \mathcal{P}_G(\mathit{\Omega}) \longrightarrow \mathcal{P}_G(\mathit{\Theta}), \quad \mu \mapsto \mathit{\Phi}_*(\mu), \tag{2}$$

where we stick to the same symbol, $\mathit{\Phi}_*$, for simplicity.

Fact 2. *Let $(\mathit{\Theta},\beta)$ be a factor of $(\mathit{\Omega},\alpha)$ with factor map $\mathit{\Phi}\colon \mathit{\Omega} \longrightarrow \mathit{\Theta}$. Then, $\mathit{\Phi}_*$ of* (2) *is a continuous surjection. Moreover, it satisfies $\mathit{\Phi}_*\big(\sum_i c_i\mu_i\big) = \sum_i c_i\,\mathit{\Phi}_*(\mu_i)$, whenever $\sum_i c_i\mu_i$ is a finite convex combination of measures $\mu_i \in \mathcal{P}_G(\mathit{\Omega})$. Finally, $\mathit{\Phi}_*$ maps ergodic measures to ergodic measures, and thus extremal points of $\mathcal{P}_G(\mathit{\Omega})$ to extremal points of $\mathcal{P}_G(\mathit{\Theta})$.*

Proof. By [15, Prop. 3.2], the mapping $\mathit{\Phi}_*$ is continuous, and by [15, Prop. 3.11], it is onto. Direct calculations show $\mathit{\Phi}_*\big(\sum_i c_i\mu_i\big) = \sum_i c_i\,\mathit{\Phi}_*(\mu_i)$ for every finite convex combination $\sum_i c_i\mu_i$ of measures in $\mathcal{P}_G(\mathit{\Omega})$.

Let $\mu \in \mathcal{P}_G(\mathit{\Omega})$ be ergodic, i.e., any α-invariant measurable subset A of $\mathit{\Omega}$ satisfies either $\mu(A) = 0$ or $\mu(A) = 1$. Consider $\nu := \mathit{\Phi}_*(\mu) \in \mathcal{P}_G(\mathit{\Theta})$, and let B be a β-invariant measurable subset of $\mathit{\Theta}$, i.e., $\beta_t(B) = B$ for all $t \in G$. Clearly, one has $\nu(B) = \mu(\mathit{\Phi}^{-1}(B))$, where $A := \mathit{\Phi}^{-1}(B) = \{\omega \in \mathit{\Omega} : \mathit{\Phi}(\omega) \in B\}$ is α-invariant, as a consequence of (1). So, $\nu(B) = \mu(A)$ is either 0 or 1, and ν is also ergodic. The final claim about the extremal points is then standard, compare [15, Prop. 5.6]. □

Fact 3. *Let $(\mathit{\Theta},\beta)$ be a factor of $(\mathit{\Omega},\alpha)$ with factor map $\mathit{\Phi}\colon \mathit{\Omega} \longrightarrow \mathit{\Theta}$. If $(\mathit{\Omega},\alpha)$ is uniquely ergodic, minimal or strictly ergodic, the analogous property holds for $(\mathit{\Theta},\beta)$ as well.*

Proof. If (Ω,α) is uniquely ergodic, $\mathcal{P}_G(\Omega)$ is a singleton set, and $\mathcal{P}_G(\Theta) = \Phi_*(\mathcal{P}_G(\Omega))$ must then also be a singleton set, by Fact 2. So, also (Θ,β) is uniquely ergodic. Apparently, every G-orbit in Θ is the image of a G-orbit in Ω, under the factor map Φ. Continuity of Φ implies $\Phi(C) \subset \Phi\big(\overline{C}\big) \subset \overline{\Phi(C)}$ for arbitrary $C \subset \Omega$. If C is dense, $\overline{C} = \Omega$, and $\overline{\Phi(C)} = \Phi(\overline{C}) = \Theta$ because Φ is onto. This shows that minimality is properly inherited, and the last claim on strict ergodicity is then obvious. □

Now, let (Θ,β) be a factor of (Ω,α) with factor map $\Phi\colon \Omega \longrightarrow \Theta$ and let $m \in \mathcal{P}_G(\Omega)$ be fixed. For the remainder of this section, we denote the induced measure on Θ by $n = \Phi_*(m)$. Consider the mapping

$$i^{\Phi}\colon\; L^2(\Theta,n) \longrightarrow L^2(\Omega,m)\,, \quad f \mapsto f\circ\Phi\,, \tag{3}$$

and let $p_{\Phi} : L^2(\Omega,m) \longrightarrow L^2(\Theta,n)$ be the adjoint of i^{Φ}. The maps i^{Φ} and p_{Φ} are partial isometries. More precisely, i^{Φ} is even an isometric embedding because

$$\langle i^{\Phi}(g), i^{\Phi}(f)\rangle_{\Omega} \;=\; \int_{\Omega} \overline{(g\circ\Phi)}\,(f\circ\Phi)\,\mathrm{d}m \;=\; \big(\Phi_*(m)\big)(\overline{g}f) \;=\; \langle g,f\rangle_{\Theta}$$

for arbitrary $f,g \in L^2(\Theta,n)$. As i^{Φ} is an isometry from $L^2(\Theta,n)$ with range $i^{\Phi}(L^2(\Theta,n))$, standard theory of partial isometries (compare [51, Thm. 4.34]) implies

$$p_{\Phi}\circ i^{\Phi} \;=\; \mathrm{id}_{L^2(\Theta,n)} \quad\text{and}\quad i^{\Phi}\circ p_{\Phi} \;=\; P_{i^{\Phi}(L^2(\Theta,n))}\,,$$

where $\mathrm{id}_{L^2(\Theta,n)}$ is the identity on $L^2(\Theta,n)$ and $P_{i^{\Phi}(L^2(\Theta,n))}$ is the orthogonal projection of $L^2(\Omega,m)$ onto $\mathcal{V} := i^{\Phi}(L^2(\Theta,n))$.

Given these maps, we can discuss the relation between the spectral theory of T^{Ω} and T^{Θ}.

Theorem 1. *Let $L^2(\Omega,m)$ and $L^2(\Theta,n)$ be the canonical Hilbert spaces attached to the dynamical systems (Ω,α) and (Θ,β), with factor map Φ and $n=\Phi_*(m)$. Then, the partial isometries i^{Φ} and p_{Φ} are compatible with the unitary representations T^{Ω} and T^{Θ} of G on $L^2(\Omega,m)$ and $L^2(\Theta,n)$, i.e.,*

$$i^{\Phi}\circ T^{\Theta}_t \;=\; T^{\Omega}_t\circ i^{\Phi} \quad\textit{and}\quad T^{\Theta}_t\circ p_{\Phi} \;=\; p_{\Phi}\circ T^{\Omega}_t\,,$$

for all $t\in G$. Similarly, the spectral families $E_{T^{\Theta}}$ and $E_{T^{\Omega}}$ satisfy

$$i^{\Phi}\circ E_{T^{\Theta}}(\cdot) \;=\; E_{T^{\Omega}}(\cdot)\circ i^{\Phi} \quad\textit{and}\quad E_{T^{\Theta}}(\cdot)\circ p_{\Phi} \;=\; p_{\Phi}\circ E_{T^{\Omega}}(\cdot).$$

The corresponding measures satisfy $\rho^{\Theta}_g = \rho^{\Omega}_{i^{\Phi}(g)}$ for every $g\in L^2(\Theta,n)$.

Proof. Let $g \in L^2(\Theta,n)$ be given. As Φ is a factor map, a short calculation gives

$$\big(T^{\Omega}_t(i^{\Phi}(g))\big)(\omega) \;=\; g(\Phi(\alpha_{-t}\omega)) \;=\; g(\beta_{-t}\Phi(\omega)) \;=\; \big((i^{\Phi}T^{\Theta}_t)(g)\big)(\omega)$$

and the first of the equations stated above follows. The second follows by taking adjoints.

Choose $g\in L^2(\Theta,n)$. As discussed above, ρ^{Θ}_g is the unique measure on $\widehat{G}$ with

$$\langle g, T^{\Theta}_t g\rangle_{\Theta} \;=\; \int_{\widehat{G}} (\hat{s},t)\,\mathrm{d}\rho^{\Theta}_g(\hat{s})\,, \quad \text{for all } t\in G.$$

Similarly, $\rho^{\Omega}_{i^{\Phi}(g)}$ is the unique measure on $\widehat{G}$ with

$$\langle i^{\Phi}(g), T^{\Omega}_t i^{\Phi}(g)\rangle_{\Omega} \;=\; \int_{\widehat{G}} (\hat{s},t)\,\mathrm{d}\rho^{\Omega}_{i^{\Phi}(g)}(\hat{s})\,, \quad \text{for all } t\in G.$$

Moreover, as i^Φ is an isometry, we obtain from the statements proved so far that

$$\langle g, T^\Theta_t g\rangle_\Theta \;=\; \langle i^\Phi(g), i^\Phi(T^\Theta_t g)\rangle_\Omega \;=\; \langle i^\Phi(g), T^\Omega_t i^\Phi(g)\rangle_\Omega \,.$$

Putting the last three equations together, we obtain

$$\int_{\widehat{G}} (\hat{s}, t)\, \mathrm{d}\rho^\Theta_g(\hat{s}) = \int_{\widehat{G}} (\hat{s}, t)\, \mathrm{d}\rho^\Omega_{i^\Phi(g)}(\hat{s})$$

for every $t \in G$. By the mentioned uniqueness of the involved measures, this gives

$$\rho^\Theta_g = \rho^\Omega_{i^\Phi(g)}.$$

This, in turn, implies

$$\langle g, E_{T^\Theta}(B) g\rangle_\Theta \;=\; \rho^\Theta_g(B) \;=\; \rho^\Omega_{i^\Phi(g)}(B) \;=\; \langle i^\Phi(g), E_{T^\Omega}(B) i^\Phi(g)\rangle_\Omega \;=\; \langle g, p_\Phi E_{T^\Omega}(B) i^\Phi(g)\rangle_\Omega$$

for all Borel measurable $B \subset \widehat{G}$ and every $g \in L^2(\Theta, n)$. As $g \in L^2(\Theta, n)$ is arbitrary, we infer $E_{T^\Theta}(\cdot) = p_\Phi E_{T^\Omega}(\cdot) i^\Phi$. □

One succinct way to summarize the core of Theorem 1 is to say that the following diagram is commutative, with the map i^Φ (resp. p_Φ) being injective (resp. surjective).

$$\begin{array}{ccccc} L^2(\Theta,n) & \xrightarrow{\;i^\Phi\;} & L^2(\Omega,m) & \xrightarrow{\;p_\Phi\;} & L^2(\Theta,n) \\ {\scriptstyle T^\Theta}\big\downarrow & & {\scriptstyle T^\Omega}\big\downarrow & & {\scriptstyle T^\Theta}\big\downarrow \\ L^2(\Theta,n) & \xrightarrow{\;i^\Phi\;} & L^2(\Omega,m) & \xrightarrow{\;p_\Phi\;} & L^2(\Theta,n) \end{array} \tag{4}$$

Corollary 1. *Assume the situation of Theorem 1 and define $\mathcal{V} = i^\Phi(L^2(\Theta, n))$. Then, $U\colon L^2(\Theta, n) \longrightarrow \mathcal{V}$, $f \mapsto i^\Phi(f)$, is a unitary map, the subspace $\mathcal{V}$ of $L^2(\Omega, m)$ is invariant under T^Ω, and the restriction $T^\Omega|_\mathcal{V}$ of T^Ω to $\mathcal{V}$ is unitarily equivalent to T^Θ via U.*

Proof. As i^Φ is an isometric embedding, the map $U\colon L^2(\Theta, n) \longrightarrow i^\Phi(L^2(\Theta, n))$ is unitary. By Theorem 1, we have $i^\Phi \circ T^\Theta_t = T^\Omega_t \circ i^\Phi$. Consequently, the space $\mathcal{V} = i^\Phi(L^2(\Theta, n))$ is invariant under T^Ω, with $T^\Omega|_\mathcal{V} U g = U T^\Theta g$ for every $g \in L^2(\Theta, n)$. □

The foregoing results describe the relationship between T^Θ and T^Ω in the general case. In the special case of pure point spectrum, we can be more explicit as follows.

Proposition 1. *Let (Θ, β) be a factor of the dynamical system (Ω, α), with factor map $\Phi\colon \Omega \longrightarrow \Theta$. Let $m \in \mathcal{P}_G(\Omega)$ be given, $n = \Phi_*(m)$, and let $L^2(\Theta, n)$ and $L^2(\Omega, m)$ be the corresponding Hilbert spaces. Then, the following assertions hold.*

(a) *If g is an eigenfunction of T^Θ to the eigenvalue $\hat{s}$, $i^\Phi(g) = g \circ \Phi$ is an eigenfunction of T^Ω to the eigenvalue $\hat{s}$.*

(b) *If T^Ω has pure point dynamical spectrum, the same is true of T^Θ.*

Proof. (a): Let g be an eigenfunction of T^Θ. Then, $i^\Phi(g) = g \circ \Phi$ is an eigenfunction of T^Ω, as $i^\Phi \circ T^\Theta_t = T^\Omega_t \circ i^\Phi$ by Theorem 1.
(b): If T^Ω has pure point dynamical spectrum, there exists an orthonormal basis of $L^2(\Omega, m)$ which entirely consists of eigenfunctions of T^Ω. Now, by Theorem 1, we have $T^\Theta_t p_\Phi = p_\Phi T^\Omega_t$. Therefore, $p_\Phi f$ is an eigenfunction of T^Θ (or zero) if f is an eigenfunctions of T^Ω. As p_Φ is onto, the statement follows. □

Let us now discuss the *continuity* of eigenfunctions. Recall that a sequence $(B_n)_{n\in\mathbb{N}}$ of compact sets in G with non-empty interior is called *van Hove*, if it exhausts G and if

$$\lim_{n\to\infty} \frac{|\partial^K B_n|}{|B_n|} \;=\; 0$$

for every compact K in G, where $\partial^K B := \overline{((B+K)\setminus B)} \cup ((\overline{G\setminus B} - K)\cap B)$.

For $G=\mathbb{R}^d$ and $G=\mathbb{Z}^d$, the following lemma (and much more) was shown by Robinson in [40]. His proof carries over easily to our situation. For the convenience of the reader, we include a brief discussion.

Lemma 1. *Let $(\varOmega,\alpha)$ be a uniquely ergodic dynamical system. Denote the unique invariant probability measure on $\varOmega$ by m. Let $\hat{s}$ be an eigenvalue of $T=T^{(\varOmega,\alpha,m)}$. Then, the following assertions are equivalent.*

(i) *There exists a continuous eigenfunction f to $\hat{s}$ (i.e., f is continuous with $f(\alpha_{-t}(\omega)) = (\hat{s},t)f(\omega)$ for all $t\in G$ and $\omega\in\varOmega$).*

(ii) *The sequence $A_{B_n}(h)$ of continuous functions on $\varOmega$, defined by*

$$A_{B_n}(h)(\omega) \;:=\; \frac{1}{|B_n|}\int_{B_n} \overline{(\hat{s},t)}\, h(\alpha_{-t}(\omega))\,\mathrm{d}t,$$

converges uniformly, for every van Hove sequence (B_n) and every $h\in C(\varOmega)$.

Proof. (i) $\Longrightarrow$ (ii) (cf. [40]). If f is the continuous eigenfunction, $|f|$ is invariant and continuous. As $(\varOmega,\alpha)$ is uniquely ergodic, we may assume, without loss of generality, that $|f(\omega)|=1$ for every $\omega\in\varOmega$. Let $h\in C(\varOmega)$ be given. Apparently, the function $g=h\overline{f}$ is continuous. Therefore, by unique ergodicity, the functions

$$\frac{1}{|B_n|}\int_{B_n} g(\alpha_{-t}(\omega))\,\mathrm{d}t \;=\; \frac{1}{|B_n|}\int_{B_n} h(\alpha_{-t}(\omega))\overline{f(\alpha_{-t}(\omega))}\,\mathrm{d}t \;=\; \frac{\overline{f(\omega)}}{|B_n|}\int_{B_n} \overline{(\hat{s},t)}\, h(\alpha_{-t}(\omega))\,\mathrm{d}t$$

converge uniformly in $\omega\in\varOmega$. Multiplying by f and using $f\overline{f}=1$, we infer (ii).

(ii) $\Longrightarrow$ (i). As $\hat{s}$ is an eigenvalue of T, the projection $E(\{\hat{s}\})$ onto the eigenspace of $\hat{s}$ is not zero. Since $C(\varOmega)$ is dense in $L^2(\varOmega,m)$, there exists an $h\in C(\varOmega)$ with $E(\{\hat{s}\})h\neq 0$. Now, by the von Neumann ergodic theorem, see [28, Thm. 6.4.1] for a formulation that allows its derivation in the generality we need it here, the sequence $A_{B_n}(h)$ converges in $L^2(\varOmega,m)$ to $E(\{\hat{s}\})h$. By assumption (ii), this sequence converges uniformly to a function g. Thus, $g=E(\{\hat{s}\})h$ in $L^2(\varOmega,m)$. Moreover, by uniform convergence, g is continuous and satisfies $g(\alpha_{-t}(\omega)) = (\hat{s},t)g(\omega)$ for every $\omega\in\varOmega$ and $t\in G$. This gives (i). □

Lemma 1 has the following interesting consequence.

Proposition 2. *Let $(\varOmega,\alpha)$ be a uniquely ergodic dynamical system, all eigenfunctions of which are continuous. If $(\varTheta,\beta)$ is a factor of $(\varOmega,\alpha)$ with factor map $\varPhi$, it is a uniquely ergodic dynamical system, all eigenfunctions of which are continuous as well.*

Proof. Fact 3 gives that $(\varTheta,\beta)$ is uniquely ergodic. Let $\hat{s}$ be an eigenvalue of $T^{\varTheta}$. Then, $\hat{s}$ is an eigenvalue of $T^{\varOmega}$ by Proposition 1. We now apply Lemma 1 to infer continuity. To that end, choose an arbitrary $g\in C(\varTheta)$, wherefore $h=g\circ\varPhi$ belongs to $C(\varOmega)$. By (i) $\Longrightarrow$ (ii) of Lemma 1, the sequence $(A_{B_n}(h))$ converges uniformly for every van Hove sequence (B_n). A short calculation then gives

$$A_{B_n}(h)(\omega) \;=\; \frac{1}{|B_n|}\int_{B_n} \big(g\circ\Phi\big)(\alpha_{-t}(\omega))\overline{(\hat{s},t)}\,\mathrm{d}t \;=\; \frac{1}{|B_n|}\int_{B_n} g(\beta_{-t}(\Phi(\omega)))\overline{(\hat{s},t)}\,\mathrm{d}t.$$

As Φ is onto, this shows uniform convergence of $\theta\mapsto \frac{1}{|B_n|}\int_{B_n} g(\beta_{-t}(\theta))\overline{(\hat{s},t)}\,\mathrm{d}t$. As $g\in C(\Theta)$ was arbitrary, this gives the desired continuity statement, by (ii) $\Longrightarrow$ (i) of Lemma 1. □

Although we have not made use of it so far, it is possible to express p_Φ via a disintegration. Since it is instructive and also useful in applications, we finish this section by giving the details for the case when Ω and Θ are metrizable. We are in the somewhat simpler situation that a continuous map $\Phi\colon\, \Omega\longrightarrow\Theta$ exists. By standard theory, compare [19, Thm. 5.8] and [37, Thm. 4.5], there exists a measurable map

$$k\colon\, \Theta \;\longrightarrow\; \mathcal{M}(\Omega)\,, \quad \vartheta\;\mapsto\; k^{\vartheta}$$

that satisfies the following three properties.

(1) For n-almost every $\vartheta\in\Theta$, k^{ϑ} is a probability measure on Ω supported in $\Phi^{-1}(\vartheta)$.
(2) For all $f\in L^1(\Omega,m)$, the function $f^{\{k\}}:\Theta\longrightarrow\mathbb{C}$, $\vartheta\mapsto k^{\vartheta}(f)$, is integrable with respect to $n=\Phi_*(m)$.
(3) For all $f\in L^1(\Omega,m)$, one has $n(f^{\{k\}})=m(f)$.

In terms of integrals, the last property reads

$$\int_{\Theta} f^{\{k\}}\,\mathrm{d}n \;=\; \int_{\Theta}\int_{\Phi^{-1}(\vartheta)} f(\omega)\,\mathrm{d}k^{\vartheta}(\omega)\,\mathrm{d}n(\vartheta) \;=\; \int_{\Omega} f\,\mathrm{d}m.$$

Remarks. (1) Note that [37, Thm. 4.5] only deals with bounded functions f. However, using standard monotone class arguments, it is not hard to extend the statements given there to functions $f\in L^1(\Omega,m)$. This yields (2) and (3).
(2) The function f^k can also be considered as a conditional expectation of f (see part (i) of [19, Thm. 5.8] or part (b) of [37, Thm. 4.5]).

Given this disintegration, one can now describe the action of p_Φ on $f\in L^2(\Omega,m)$ explicitly, namely in terms of partial averages over the fibres $\Phi^{-1}(\vartheta)$.

Proposition 3. *Assume that Ω and Θ are compact metric spaces, and let $n=\Phi_*(m)$ as before. Then, the equation $\big(p_\Phi(f)\big)(\vartheta)=k^{\vartheta}(f)$ holds for all $f\in L^2(\Omega,m)$ and n-almost every $\vartheta\in\Theta$.*

Proof. Fix $f\in L^2(\Omega,m)$, and let $g\in L^2(\Theta,n)$ be arbitrary. Then, f belongs to $L^1(\Omega,m)$, since Ω is compact and $\overline{g\circ\Phi}\cdot f$ belongs to $L^1(\Omega,m)$, as it is the product of two L^2 functions. Using the properties of k, we can then calculate

$$\begin{aligned}
\langle g, f^{\{k\}}\rangle_{\Theta} \;&=\; \int_{\Theta}\overline{g}\,f^{\{k\}}\,\mathrm{d}n \;=\; \int_{\Theta}\overline{g(\vartheta)}\int_{\Phi^{-1}(\vartheta)} f(\omega)\,\mathrm{d}k^{\vartheta}(\omega)\,\mathrm{d}n(\vartheta)\\
&=\; \int_{\Theta}\int_{\Phi^{-1}(\vartheta)}\overline{g(\Phi(\omega))}\,f(\omega)\,\mathrm{d}k^{\vartheta}(\omega)\,\mathrm{d}n(\vartheta) \;=\; n\Big(\big(\overline{i^{\Phi}(g)}f\big)^{\{k\}}\Big)\\
&=\; m\big(\overline{i^{\Phi}(g)}f\big) \;=\; \int_{\Omega}\overline{g(\Phi(\omega))}f(\omega)\,\mathrm{d}m(\omega) \;=\; \langle i^{\Phi}(g), f\rangle_{\Omega}\\
&=\; \langle g, p_{\Phi}(f)\rangle_{\Theta}\,.
\end{aligned}$$

As $g\in L^2(\Theta,n)$ is arbitrary, this gives $f^{\{k\}}=p_\Phi(f)$ in $L^2(\Theta,n)$, and our claim follows. □

4. Diffraction theory of measure and Delone dynamical systems

In this section, we specify the dynamical systems we are dealing with and discuss the necessary background from diffraction theory. The material is taken from [3], where the proofs and further details can be found. For related material dealing with point dynamical systems, we refer the reader to [16, 23, 30, 44, 45, 46].

As discussed in the introduction, our main focus is on measure dynamical systems which includes the case of point dynamical systems. For the convenience of the reader, however, we start this section with a short discussion of point dynamical systems and discuss the general case of measures only afterwards.

Let V be an open neighbourhood of 0 in G. A subset $\varLambda$ of G is called *V-discrete* if every translate of V contains at most one point of $\varLambda$. Such sets are necessarily closed. A set is *uniformly discrete* if it is V-discrete for some open neighbourhood V of 0. The set of V-discrete point sets in G is abbreviated as $\mathcal{D}_V(G)$, while the set of all uniformly discrete subsets of G is denoted by $\mathcal{UD}(G)$. The set $\mathcal{UD}(G)$ (and actually even the set $\mathcal{C}(G)$ of all closed subsets of G) can be topologized by a uniformity as follows. For $K \subset G$ compact and V an open neighborhood of 0 in G, we set

$$U_{K,V} \;:=\; \{(P_1,P_2)\in \mathcal{UD}(G)\times\mathcal{UD}(G) : P_1\cap K \subset P_2+V \text{ and } P_2\cap K\subset P_1+V\}.$$

It is not hard to check that $\{U_{K,V} : K \text{ compact}, V \text{ open with } 0\in V\}$ generates a uniformity (see [27, Ch. 6] for basics about uniformities), and hence, via the neighbourhoods

$$U_{K,V}(P) \;:=\; \{Q : (Q,P)\in U_{K,V}\}\,, \quad P\in\mathcal{UD}(G),$$

a topology on $\mathcal{UD}(G)$. This topology is called the *local rubber topology* (LRT). For each open neighbourhood V of 0 in G, the set $\mathcal{D}_V(G)$ is compact in LRT. Apparently, G acts on $\mathcal{UD}(G)$ by translation. By slight abuse of notation, this action is again called α, i.e., we define

$$\alpha_t(\varLambda) \;:=\; \{t+x : x\in\varLambda\} \;=\; t+\varLambda.$$

To distinguish (compact) sets of measures ω from sets of point sets $\varLambda$, we will use the suggestive notation $\varOmega$ and $\varOmega_{\mathrm{p}}$ from now on.

Definition 2. *The pair $(\varOmega_{\mathrm{p}},\alpha)$ is called a* point dynamical system *if $\varOmega_{\mathrm{p}}$ is a closed α-invariant subset of $\mathcal{D}_V(G)$ for a suitable neighbourhood V of 0 in G.*

Apparently, every $\varLambda\in\mathcal{UD}(G)$ gives rise to a point dynamical system $(\varOmega(\varLambda),\alpha)$, where $\varOmega(\varLambda)$ is the closure of $\{\alpha_t(\varLambda) : t\in G\}$ in LRT and α is the action induced from the natural action of G on $\mathcal{UD}(G)$.

After this short look at point dynamical systems, we now introduce our main object of interest: measure dynamical systems. As mentioned already, they generalize point dynamical systems (see below for details).

Let $C>0$ and a relatively compact open set V in G be given. A measure $\mu\in\mathcal{M}(G)$ is called *(C,V)-translation bounded* if $|\mu|(t+V)\le C$ for all $t\in G$. It is called *translation bounded* if there exists such a pair C,V so that μ is (C,V)-translation bounded. The set of all (C,V)-translation bounded measures is denoted by $\mathcal{M}_{C,V}(G)$, the set of all translation bounded measures by $\mathcal{M}^\infty(G)$. In the vague topology, the set $\mathcal{M}_{C,V}(G)$ is a compact Hausdorff space. There is an obvious action of G on $\mathcal{M}^\infty(G)$, again denoted by α, given by

$$\alpha\colon\, G\times\mathcal{M}^\infty(G) \;\longrightarrow\; \mathcal{M}^\infty(G)\,, \quad (t,\mu)\mapsto\alpha_t\mu \quad\text{with}\quad (\alpha_t\mu) := \delta_t*\mu.$$

Restricted to $\mathcal{M}_{C,V}(G)$, this action is continuous.

Here, the convolution of two convolvable measures μ, ν is defined by

$$(\mu * \nu)(\varphi) \; = \; \int_G \varphi(r+s)\,\mathrm{d}\mu(r)\,\mathrm{d}\nu(s).$$

Definition 3. (Ω, α) *is called a dynamical system on the translation bounded measures on* G (TMDS *for short*) *if there exist a constant* $C > 0$ *and a relatively compact open set* $V \subset G$ *such that* Ω *is a closed* α*-invariant subset of* $\mathcal{M}_{C,V}(G)$.

It is possible to consider a point dynamical system as a TDMS. Namely, define

$$\delta\colon\, \mathcal{UD}(G) \;\longrightarrow\; \mathcal{M}^{\infty}(G)\,, \quad \delta(\varLambda) \;:=\; \textstyle\sum_{x\in\varLambda} \delta_x\,,$$

where δ_x is the unit point (or Dirac) measure at x. The mapping δ is continuous and injective.

Lemma 2. *If* $(\Omega_{\mathrm{p}}, \alpha)$ *is a point dynamical system, the mapping* $\delta : \Omega_{\mathrm{p}} \longrightarrow \Omega := \delta(\Omega_{\mathrm{p}})$ *establishes a topological conjugacy between the point dynamical system* $(\Omega_{\mathrm{p}}, \alpha)$ *and its image, the* TMDS (Ω, α).

Proof. By [3, Lemma 2], $\delta : \Omega_{\mathrm{p}} \longrightarrow \delta(\Omega_{\mathrm{p}})$ is a homeomorphism that is compatible with the G-action α, i.e., $\delta\big(\alpha_t(\varLambda)\big) = \alpha_t\big(\delta(\varLambda)\big)$ for all $\varLambda \in \Omega_{\mathrm{p}}$ and all $t \in G$. So, δ provides a topological conjugacy as claimed. □

Having introduced our models, we can now discuss some key issues of diffraction theory. Let (Ω, α) be a TMDS, equipped with an α-invariant measure $m \in \mathcal{P}_G(\Omega)$. We will need the mapping

$$f\colon\, C_c(G) \longrightarrow C(\Omega)\,, \quad f_{\varphi}(\omega) \;:=\; \int_G \varphi(-s)\,\mathrm{d}\omega(s).$$

Then, there exists a unique measure $\gamma = \gamma_m$ on G, called the *autocorrelation* (often called Patterson function in crystallography [14], though it is a measure in our setting) with

$$\gamma(\overline{\varphi} * \psi_{\text{-}}) \;=\; \langle f_{\varphi}, f_{\psi} \rangle$$

for all $\varphi, \psi \in C_c(G)$, where $\psi_{\text{-}}(s) := \psi(-s)$. The convolution $\varphi * \psi$ is defined by $(\varphi * \psi)(t) = \int \varphi(t-s)\psi(s)\,\mathrm{d}s$. For a more explicit formulation in terms of a weighted average, see [3, Prop. 6].

The measure γ is positive definite. Therefore, its Fourier transform is a positive measure $\widehat{\gamma}$; it is called the *diffraction measure*. This measure describes the outcome of a diffraction experiment, see [14] for background material.

Remark. This concept of an autocorrelation is defined via the entire dynamical system, which implicitly involves a local averaging procedure. The conventional approach uses a limit of a sequence of finite measures along a van Hove averaging sequence in G. If the dynamical system is (uniquely) ergodic, the two notions coincide [3]. In general, the definition we use here has the advantage of removing the dependence of the averaging sequence and automatically deals with the *typical* autocorrelation, at least with reference to the measure m.

In view of the fact that, in reality, one always faces *finite* structures, one can give a justification along the following lines. Among all elements of the full system that are compatible with a given finite part, "typical" ones are those to be considered, if no other piece of information is available. This means to take into account all structures which, after a small translation and/or up to some tiny local deformation, coincide with a fixed finite patch. One way to do so is to take an average over all these possibilities (on the level of their autocorrelations), which is essentially what our γ does. In the situation of unique ergodicity, compare [3], the precise method for forming the average is irrelevant – the result is independent of it.

Theorem 2 (Theorem 7 in [3])**.** *Let* $(\varOmega, \alpha)$ *be a* TMDS *with invariant measure* m*. Then, the following assertions are equivalent.*

(i) *The measure* $\widehat{\gamma}$ *is a pure point measure.*
(ii) $T^{\varOmega}$ *has pure point dynamical spectrum.* □

Theorem 2 links pure point diffraction spectrum to pure point dynamical spectrum. This is of particular relevance for our considerations. It will allow us to set up a perturbation and stability theory for pure point diffraction spectrum by studying (perturbations of) dynamical systems. This is the abstract core of our investigation, to be analyzed next.

5. Deforming measure and Delone dynamical systems: Abstract setting

In this section, we introduce a deformation procedure for dynamical systems that, under certain conditions, is *isospectral*, i.e., the deformation does not change the dynamical spectrum. In particular, we will later consider deformations of regular model sets and show that a relevant class of deformations preserves their pure point diffraction property. As discussed in the introduction, these considerations are motivated by questions from the mathematical theory of quasicrystals. They generalize the corresponding results in [24, 11].

For pedagogic reasons, we start with a short discussion of deformations of Delone dynamical systems. This results in Theorem 3. The general case of measure dynamical systems is treated afterwards.

Let $(\varOmega_{\mathrm{p}}, \alpha)$ be a Delone dynamical system with $\varOmega_{\mathrm{p}}$ contained in $\mathcal{D}_V(G)$ and consider a continuous mapping $q\colon \varOmega_{\mathrm{p}} \longrightarrow G$ whose image then is a compact set. In fact, let us assume that $q(\varOmega_{\mathrm{p}}) - q(\varOmega_{\mathrm{p}}) \subset V$ for some neighbourhood V of $0 \in G$. Note that there exists an open neighbourhood V' of 0 in G with

$$V' + q(\varOmega_{\mathrm{p}}) - q(\varOmega_{\mathrm{p}}) \;\subset\; V.$$

In particular, for arbitrary $\varLambda \in \varOmega_{\mathrm{p}}$ and $y, z \in \varLambda$ with $y \neq z$, we have $y + q(\varLambda - y) \neq z + q(\varLambda - z)$ as well as

$$\varLambda_q \;:=\; \{x + q(\varLambda - x) : x \in \varLambda\} \;\subset\; \mathcal{D}_{V'}(G). \tag{5}$$

$\varLambda_q$ can be viewed as a "deformed" version of $\varLambda$, which exlains the terminology. Moreover, $\varOmega_{\mathrm{p}}^q := \{\varLambda_q : \varLambda \in \varOmega_{\mathrm{p}}\}$ can rather directly be seen to be α-invariant and closed in $\mathcal{D}_{V'}(G)$. Thus, $(\varOmega_{\mathrm{p}}^q, \alpha)$ is a point dynamical system, and we have a mapping $\varPhi^q\colon \varOmega_{\mathrm{p}} \longrightarrow \varOmega_{\mathrm{p}}^q$ given by $\varPhi^q(\varLambda) = \varLambda_q$. This map can easily be seen to be a factor map.

In fact, it turns out that we do not need q to be defined on the whole of $\varOmega_p$ to obtain a factor map. It suffices to have it defined on a "transversal". To be more precise here, we introduce the following subset of $\varOmega_{\mathrm{p}}$,

$$\varXi \;:=\; \{\varLambda \in \varOmega_{\mathrm{p}} : 0 \in \varLambda\}. \tag{6}$$

Since the elements of $\varOmega_{\mathrm{p}}$ are non-empty point sets of G, it is clear that each G-orbit in $\varOmega_{\mathrm{p}}$ contains at least one element of $\varXi$. Moreover, the following holds.

Lemma 3. *If* $\varOmega_{\mathrm{p}}$ *is a point dynamical system under the action of the* LCA *group* G*, the subset* $\varXi$ *of* (6) *is compact.*

Proof. By definition, $\varOmega_{\mathrm{p}}$ is a closed subset of $\mathcal{D}_V(G)$ for a suitable neighbourhood V of 0 in G. As $\mathcal{D}_V(G)$ is compact in LRT, $\varOmega_{\mathrm{p}}$ is compact in LRT as well. So, we need to show that $\varXi \subset \varOmega_{\mathrm{p}}$ is a closed set.

Let $(\mathit{\Gamma}_\iota)$ be a net in $\mathit{\Xi}$ (so, $0 \in \mathit{\Gamma}_\iota$ for all ι) which converges to some $\mathit{\Lambda}$, where the latter must then lie in $\mathit{\Omega}_{\mathrm{p}}$. Assume that $0 \notin \mathit{\Lambda}$. Since $\mathit{\Lambda}$ is itself a closed subset of G, we know that $G \setminus \mathit{\Lambda}$ is an open set. By assumption, this open set would contain 0, and hence also an entire open neighbourhood of 0. This, however, contradicts the convergence $\mathit{\Gamma}_\iota \longrightarrow \mathit{\Lambda}$ in the LRT. □

As $\mathit{\Xi}$ is compact, every continuous function q on $\mathit{\Xi}$ can be extended to a continuous function $\tilde{q}$ on $\mathit{\Omega}_{\mathrm{p}}$ (the latter being compact and hence normal) by Tietze's extension theorem, compare [39, Prop. 1.5.8]. The very definition of $\mathit{\Omega}_p^{\tilde{q}}$, compare (5), shows that it only depends on q (and not on the extension chosen). In this situation, we can thus consistently define

$$\mathit{\Omega}_p^q \;:=\; \mathit{\Omega}_p^{\tilde{q}}. \tag{7}$$

Now, we can state our result on Delone dynamical systems.

Theorem 3. *Let $(\mathit{\Omega}_{\mathrm{p}}, \alpha)$ be a point dynamical system under the action of the LCA group G with $\mathit{\Omega}_{\mathrm{p}} \subset \mathcal{D}_V(G)$ for a suitable neighbourhood V of 0 in G. Let $q\colon \mathit{\Xi} \longrightarrow G$ be continuous with $q(\mathit{\Omega}_{\mathrm{p}}) - q(\mathit{\Omega}_{\mathrm{p}}) \subset V$. Then, the following assertions hold.*

(a) *If $(\mathit{\Omega}_{\mathrm{p}}, \alpha)$ has pure point diffraction spectrum (w.r.t. an invariant probability measure m), so does $(\mathit{\Omega}_{\mathrm{p}}^q, \alpha)$ (w.r.t. the measure $\mathit{\Phi}_*^q(m)$).*
(b) *If $(\mathit{\Omega}_{\mathrm{p}}, \alpha)$ is minimal or uniquely ergodic, then so is $(\mathit{\Omega}_{\mathrm{p}}^q, \alpha)$.*
(c) *If $(\mathit{\Omega}_{\mathrm{p}}, \alpha)$ is uniquely ergodic with pure point diffraction spectrum and all of its eigenfunctions are continuous, the same holds for $(\mathit{\Omega}_{\mathrm{p}}^q, \alpha)$.*

Proof. Let $\tilde{q}$ be a continuous extension of q from $\mathit{\Xi}$ to $\mathit{\Omega}_{\mathrm{p}}$. As discussed above in (7), we then have a factor map $\mathit{\Phi}^q : \mathit{\Omega}_{\mathrm{p}} \longrightarrow \mathit{\Omega}_{\mathrm{p}}^q$. Now, we can prove the assertions.

(a): If $(\mathit{\Omega}_{\mathrm{p}}, \alpha)$ has pure point diffraction spectrum, it has pure point dynamical spectrum, by Theorem 2. As $(\mathit{\Omega}_{\mathrm{p}}^q, \alpha)$ is a factor of $(\mathit{\Omega}, \alpha)$, it has pure point dynamical spectrum as well, by Proposition 1. Now, another application of Theorem 2 shows that $(\mathit{\Omega}_{\mathrm{p}}^q, \alpha)$ has pure point diffraction spectrum.

(b): This follows from Fact 3.

(c): The statement about continuity of the eigenfunctions is immediate from Proposition 2. The other statements follow by (a) and (b). □

Having discussed the special case of point dynamical systems, we now treat the general case. Let $(\mathit{\Omega}, \alpha)$ be a TMDS. We will deform $(\mathit{\Omega}, \alpha)$ by means of a measure-valued mapping

$$\lambda :\; \mathit{\Omega} \;\longrightarrow\; \mathcal{M}(G)\,, \quad \omega \;\mapsto\; \lambda^\omega,$$

which satisfies the following two properties.

(D1) The mapping $\mathit{\Omega} \times C_c(G) \longrightarrow \mathbb{C}$, $(\omega, \varphi) \mapsto \lambda^\omega(\varphi)$, is continuous.
(D2) There exists a compact $K \subset G$ such that $\mathrm{supp}(\lambda^\omega) \subset K$ for all $\omega \in \mathit{\Omega}$.

Such a deformation map λ will be called *admissible.* This definition entails the case that $\lambda^\omega \equiv \delta_0$, which we will call the trivial deformation map.

Proposition 4. *Let $(\mathit{\Omega}, \alpha)$ be a* TMDS *and let $\lambda\colon \mathit{\Omega} \longrightarrow \mathcal{M}(G)$ be an admissible deformation map. Then, $\omega \mapsto |\lambda^\omega|(1)$ is bounded.*

Proof. Let K be given according to (D2), and let $V \subset G$ be open and relatively compact. Since $\overline{K+V}$ is compact, one has

$$|\lambda^{\omega}|(1) \;=\; |\lambda^{\omega}|(K+V) \;=\; \sup\{|\lambda^{\omega}(\varphi)| : \operatorname{supp}(\varphi) \subset \overline{K+V},\, \|\varphi\|_{\infty} \leq 1\},$$

where we used [3, Prop. 1] in the last step. Due to compactness of $\varOmega$, the statement now follows from (D1) and the uniform boundedness principle (see [39, Thm. 2.2.9]). □

For $\omega \in \varOmega$ and $\varphi \in C_c(G)$, we define the actual *deformation* of ω into $\varPhi^{\lambda}(\omega)$ via

$$\big(\varPhi^{\lambda}(\omega)\big)(\varphi) \;:=\; \int_G \int_G \varphi(r+s)\, \mathrm{d}\lambda^{\alpha_{-r}(\omega)}(s)\, \mathrm{d}\omega(r),$$

where the double integral exists by (D1) and (D2). The constant deformation map $\lambda^{\omega} \equiv \delta_t$, with $t \in G$, results in a translation, i.e., $\varPhi^{\lambda}(\omega) = \delta_t * \omega$ in this case, for all $\omega \in \varOmega$. The trivial deformation map thus induces the identity. In general, the following is true.

Proposition 5. *Let a TMDS $(\varOmega, \alpha)$ be given and let λ be an admissible deformation map. Then, the following assertions hold.*

(a) *For every $\omega \in \varOmega$, the map $\varPhi^{\lambda}(\omega)\colon\, C_c(G) \longrightarrow \mathbb{C}$, $\varphi \mapsto \big(\varPhi^{\lambda}(\omega)\big)(\varphi)$, is continuous, i.e., $\varPhi^{\lambda}(\omega)$ belongs to $\mathcal{M}(G)$. Moreover, the map $\varPhi^{\lambda}\colon\, \varOmega \longrightarrow \mathcal{M}(G)$, $\omega \mapsto \varPhi^{\lambda}(\omega)$, is continuous as well.*

(b) *There exists a constant $C > 0$ and an open neighbourhood V of 0 in G such that $\varPhi^{\lambda}(\omega)$ belongs to $\mathcal{M}_{C,V}(G)$, for all $\omega \in \varOmega$.*

(c) *For all $t \in G$ and $\omega \in \varOmega$, one has $\varPhi^{\lambda}(\alpha_t(\omega)) = \alpha_t(\varPhi^{\lambda}(\omega))$.*

Proof. (a): Let K be compact according to (D2). For fixed $\omega \in \varOmega$ and $\varphi \in C_c(G)$ with support in the compact set L, the function

$$r \;\mapsto\; \int_G \varphi(r+s)\, \mathrm{d}\lambda^{\alpha_{-r}(\omega)}(s)$$

has support contained in $L - K$. Moreover, this function is continuous, since it can easily be expressed as a composition of continuous functions. In fact, extending this type of reasoning, one can show that

$$F\colon\, \varOmega \times C_c(G) \;\longrightarrow\; C_c(G)\,, \quad F(\omega,\varphi)(r) \;:=\; \int_G \varphi(r+s)\, \mathrm{d}\lambda^{\alpha_{-r}(\omega)}(s),$$

is continuous. In particular, $C_c(G) \longrightarrow \mathbb{C}$, $\varphi \mapsto \omega(F(\omega,\varphi))$, is continuous for fixed $\omega \in \varOmega$ and $\varOmega \longrightarrow \mathbb{C}$, $\omega \mapsto \omega(F(\omega,\varphi))$, is continuous for $\varphi \in C_c(G)$. As

$$\varPhi^{\lambda}(\omega)(\varphi) = \omega((F(\omega,\varphi))),$$

we infer (a).

(b): Let L be an arbitrary non-empty open set with compact closure. Let $1_{\overline{L}-K}$ be the characteristic function of $\overline{L} - K$, where K is taken from (D1). Then, for every $\varphi \in C_L(G)$, we have

$$|\varPhi^{\lambda}(\omega)(\varphi)| \;\leq\; \int_G |\lambda|^{\alpha_{-r}\omega}(1)\, \|\varphi\|_{\infty}\, 1_{\overline{L}-K}(r)\, \mathrm{d}|\omega|(r) \;\leq\; C(\lambda)\|\varphi\|_{\infty}\, |\omega|\,(\overline{L}-K),$$

where $C(\lambda)$ is the bound on $\omega \mapsto |\lambda^{\omega}|(1)$ obtained in Proposition 4. Thus,

$$\begin{aligned}
|\varPhi^{\lambda}(\omega)|(L+t) \;&=\; \sup\{|\varPhi^{\lambda}(\omega)(\varphi)| : \varphi \in C_{L+t}(G), \|\varphi\|_{\infty} \leq 1\} \\
&\leq\; C(\lambda)\|\varphi\|_{\infty}\, |\omega|\,(t + \overline{L} - K)
\end{aligned}$$

is uniformly bounded in $t \in G$, as ω is translation bounded, and (b) follows.
(c): This is immediate from

$$\begin{aligned}\big(\varPhi^{\lambda}(\alpha_t\omega)\big)(\varphi) &= \int_G \int_G \varphi(r+s)\, \mathrm{d}\lambda^{\alpha_{-r+t}(\omega)}(s)\, \mathrm{d}(\alpha_t\omega)(r) \\ &= \int_G \int_G \varphi(r+s+t)\, \mathrm{d}\lambda^{\alpha_{-r}(\omega)}(s)\, \mathrm{d}\omega(r) \\ &= \big(\alpha_t(\varPhi^{\lambda}(\omega))\big)(\varphi),\end{aligned}$$

which is valid for every $\varphi \in C_c(G)$. □

Define the set of periods of a measure ω as

$$\mathrm{Per}(\omega) \; := \; \{t \in G : \alpha_t\omega = \omega\}. \tag{8}$$

We then have the following consequence.

Corollary 2. *Let $(\varOmega, \alpha)$ be given and let λ be an admissible deformation map. For any $\omega \in \varOmega$, with resulting deformation $\varPhi^{\lambda}(\omega)$, one has*

$$\mathrm{Per}(\omega) \; \subset \; \mathrm{Per}(\varPhi^{\lambda}(\omega)).$$

Moreover, if any $\omega \in \varOmega$ exists where $\mathrm{Per}(\varPhi^{\lambda}(\omega))$ *is a true superset of* $\mathrm{Per}(\omega)$*, the mapping $\varPhi^{\lambda} \colon \varOmega \longrightarrow \mathcal{M}(G)$ fails to be injective.*

Proof. The first claim follows at once from part (c) of Proposition 5. For the second claim, let t be a period of $\varPhi^{\lambda}(\omega)$ that is *not* a period of ω. Then, $\omega \neq \alpha_t\omega$, but their images under $\varPhi^{\lambda}$ are equal. □

Part (a) of Proposition 5 implies that, for a given TMDS $(\varOmega, \alpha)$, the set

$$\varOmega^{\lambda} \; := \; \{\varPhi^{\lambda}(\omega) : \omega \in \varOmega\}$$

is compact, as it is the image of a compact set under a continuous map. Furthermore, by part (c) of the same proposition, $\varOmega^{\lambda}$ is invariant under α. In fact, by part (b) of Proposition 5, $\varOmega^{\lambda}$ is a subset of $\mathcal{M}_{C,V}(G)$ for suitable C, V. Putting this together, we have proved the following result.

Lemma 4. *Let $(\varOmega, \alpha)$ be a* TMDS *and let $\lambda \colon \varOmega \longrightarrow \mathcal{M}(G)$ be an admissible deformation map. Then, $(\varOmega^{\lambda}, \alpha)$ is a* TDMS*. Moreover, $(\varOmega^{\lambda}, \alpha)$ is a factor of $(\varOmega, \alpha)$, with factor map $\varPhi^{\lambda} \colon \varOmega \longrightarrow \varOmega^{\lambda}$.* □

If the situation of Lemma 4 applies, we call $(\varOmega^{\lambda}, \alpha)$ an *admissible* deformation of $(\varOmega, \alpha)$, with deformation map λ. The main abstract result of this paper now reads as follows.

Theorem 4. *Let $(\varOmega, \alpha)$ be a* TMDS *and let $\lambda \colon \varOmega \longrightarrow \mathcal{M}(G)$ be an admissible deformation map. Then, the following assertions hold.*

(a) *If $(\varOmega, \alpha)$ has pure point diffraction spectrum (w.r.t. some invariant probability measure m), so does $(\varOmega^{\lambda}, \alpha)$ (w.r.t. the corresponding induced measure).*
(b) *If $(\varOmega, \alpha)$ is minimal or uniquely ergodic, then so is $(\varOmega^{\lambda}, \alpha)$.*
(c) *If $(\varOmega, \alpha)$ is uniquely ergodic with pure point diffraction spectrum and all of its eigenfunctions are continuous, the same holds for $(\varOmega^{\lambda}, \alpha)$.*

Proof. The proof is essentially the same as the proof of Theorem 3.

(a): If (Ω, α) has pure point diffraction spectrum, it has pure point dynamical spectrum, by Theorem 2. As $(\Omega^{\lambda}, \alpha)$ is a factor of (Ω, α) by Lemma 4, it has pure point dynamical spectrum as well, by Proposition 1. Now, another application of Theorem 2 shows that $(\Omega^{\lambda}, \alpha)$ has pure point diffraction spectrum.

(b): This follows from Fact 3.

(c): The statement about continuity of the eigenfunctions is immediate from Proposition 2. The other statements follow by (a) and (b). □

Remarks. (1) Of course, the previous discussion of TMDS includes the case of Delone dynamical systems treated at the beginning of the section. To see this, one has to apply the mapping $\delta\colon \Omega_p \longrightarrow \Omega$ introduced in the previous section.
(2) The discussion of point dynamical systems given above requires a non-overlapping condition under deformation, here written as $q(\Omega_{\rm p}) - q(\Omega_{\rm p}) \subset V$ for a suitable open set V. In the TMDS setting, such a restriction is not necessary, which shows once more the greater flexibility of the approach via measures.

6. Model sets and their deformation

Model sets probably form the most important class of examples of aperiodic order. In their case, one starts with a periodic structure in a high dimensional space and considers a partial "image" in a lower dimensional space. This image will not be periodic any more but still preserve many regularity features due to the periodicity of the underlying high dimensional structure. For a survey and further references, we refer the reader to [33, 35].

Let us start with a brief recapitulation of the setting of a cut and project scheme and the definition of a model set. We need two locally compact Abelian groups, G and H, where G is also assumed to be σ-compact, see [44] for the reasons why this is needed. As usual, neutral elements will be denoted by 0 (or by $0_G, 0_H$, if necessary). A *cut and project scheme* emerges out of the following collection of groups and mappings:

$$
\begin{array}{ccccc}
G & \xleftarrow{\;\pi\;} & G \times H & \xrightarrow{\;\pi_{\rm int}\;} & H \\
\cup & & \cup & & \cup \text{ dense} \\
L & \xleftarrow{\;1-1\;} & \tilde{L} & \longrightarrow & L^{\star} \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^{\star}
\end{array}
\qquad (9)
$$

Here, $\tilde{L}$ is a *lattice* in $G \times H$, i.e., a cocompact discrete subgroup. The canonical projection π is one-to-one between $\tilde{L}$ and L (in other words, $\tilde{L} \cap \{0_G\} \times H = \{0\}$), and the image $L^{\star} = \pi_{\rm int}(\tilde{L})$ is dense in H, which is often called the internal space. In view of these properties of the projections π and $\pi_{\rm int}$, one usually defines the $\star$-map as $(.)^{\star} : L \longrightarrow H$ via $x^{\star} := \bigl(\pi_{\rm int} \circ (\pi|_L)^{-1}\bigr)(x)$, where $(\pi|_L)^{-1}(x) = \pi^{-1}(x) \cap \tilde{L}$, for all $x \in L$.

A *model set* is now any translate of a set of the form

$$\curlywedge(W) \;:=\; \{x \in L : x^{\star} \in W\} \qquad (10)$$

where the *window* W is a relatively compact subset of H with non-empty interior. Without loss of generality, we may assume that the stabilizer of the window,

$$H_W \;:=\; \{c \in H : c + W = W\}, \qquad (11)$$

is the trivial subgroup of H, i.e., $H_W = \{0\}$. If this were not the case (which could happen in compact groups H for instance), one could factor by H_W and reduce the cut and project scheme accordingly [44, 4]. Furthermore, we may assume that $\langle W - W\rangle$, the subgroup of H that is algebraically generated by the subset $W - W$, is the entire group, i.e., $\langle W - W\rangle = H$, again by reducing the cut and project scheme to this situation, compare [43] for details.

There are variations on the precise requirement to W which depend on the fine properties of the model sets one is interested in, compare [35, 44]. In particular, a model set is called *regular* if ∂W has Haar measure 0 in H, and *generic* if, in addition, $\partial W \cap L^\star = \varnothing$.

As discussed immediately after Definition 2, every model set $\varLambda$ gives rise to the dynamical system $(\varOmega(\varLambda), \alpha)$. It is one of the central results of this area, compare [35, 44] and references given there, that model sets provide a very natural generalization of the concept of a lattice.

Theorem 5. [44] *Regular model sets are pure point diffractive. In fact, $(\varOmega(\varLambda), \alpha)$ is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions.* □

For our purposes, it is sufficient to restrict our attention to regular model sets where W is a compact subset of H with $\overline{W^\circ} = W$ (in particular, W then has non-empty interior and, due to regularity, a boundary of Haar measure 0). This is motivated by the fact that diffraction cannot distinguish two model sets $\curlywedge(W)$ and $\curlywedge(W')$ if the symmetric difference $W \triangle W'$ of the windows has Haar measure 0 in H.

A regular model set with compact window W can be deformed as follows [23, 11]. Let $\vartheta \colon H \longrightarrow G$ be a continuous function with compact support, which, in view of the discussion around (7), we may assume to include W if necessary. If $\varLambda = \curlywedge(W)$, one defines

$$\varLambda_\vartheta \;:=\; \{x + \vartheta(x^\star) : x \in \varLambda\} \;=\; \{x + \vartheta(x^\star) : x \in L \text{ and } x^\star \in W\}. \tag{12}$$

To make sure that $\varLambda_\vartheta$ is still a Delone set, one usually requires that the compact set $K := \vartheta(H) - \vartheta(H)$ satisfies $K \subset V$ where V is an open neighbourhood of $0 \in G$ so that $\varLambda \in \mathcal{D}_V(G)$.

Note that $\varLambda_\vartheta$ (if it is Delone) has a well defined density, and one obtains

$$\mathrm{dens}(\varLambda_\vartheta) \;=\; \mathrm{dens}(\varLambda). \tag{13}$$

In other words, an admissible deformation does not change the density.

Our aim is now to show that the continuous mapping ϑ induces a deformation map q on $\varXi$. To do so, we will need the following lemma. It essentially says that the $\star$-map on $\varLambda$ can be extended to a unique continuous map on $\varXi$.

Lemma 5. *Let $\varLambda = \curlywedge(W)$, with $W = \overline{W^\circ}$ compact, be a regular model set and assume that $H_W = \{0\}$. Then, the set $\{\varLambda - x : x \in \varLambda\}$ is dense in the compact set $\varXi$ and there is precisely one continuous mapping $\sigma \colon \varXi \longrightarrow W$ with $\sigma(\varLambda - x) = x^\star$ for every $x \in \varLambda$.*

Proof. First, let us show that $\{\varLambda - x : x \in \varLambda\}$ is dense in $\varXi$, the latter being compact by Lemma 3.

To this end, let $\varGamma \in \varXi$ be given and consider an arbitrary neighbourhood $U_{K,V}(\varGamma)$ of $\varGamma$, where $K \subset G$ is compact and V is an open neighbourhood of 0 in G. Replacing K by $K \cup \{0\}$ if necessary, we can assume $0 \in K$ without loss of generality. We have to provide an element of the form $\varLambda - p$ with $p \in \varLambda$ which belongs to $U_{K,V}(\varGamma)$.

To do so, choose a compact neighbourhood V' of $0 \in G$ with

$$V' + V' \subset V \quad \text{and} \quad V' = -V'.$$

As Ξ is a subset of $\Omega_p(\Lambda)$, which is the orbit closure of $\{t+\Lambda : t \in G\}$, there exists a $t \in G$ with

$$t + \Lambda \in U_{K+V',V'}(\Gamma).$$

As 0 belongs to both Γ and K, we infer that

$$0 \in \Gamma \cap K \subset \Gamma \cap (K + V') \subset (t+\Lambda) + V'.$$

Therefore, $0 = t + p + v'$ with $p \in \Lambda$ and $v' \in V'$, or, put differently, $p = -t - v' \in \Lambda$. This gives

$$\Lambda - p = \Lambda + t + v' \in U_{K+V',V'}(\Gamma) + v' \subset U_{K,V}(\Gamma),$$

where the last inclusion follows by our choice of V'. As discussed above, this proves the density statement.

It remains to show the existence and uniqueness of a continuous map $\sigma \colon \Xi \longrightarrow H$ with $\sigma(\Lambda - x) = x^\star$ for every $x \in \Lambda$, where the uniqueness will be an immediate consequence of the continuity of σ and the already established denseness of $\{\Lambda - x : x \in \Lambda\}$ in Ξ.

Existence: By [44, Lemma 4.1], for every $\Gamma \in \Xi$, the set

$$\sigma(\Gamma) = \bigcap_{y\in\Gamma} (W - y^\star) \tag{14}$$

is a singleton set in H (note that the sign change in our formulation does not affect this statement). In the sequel, we will tacitly identify the singleton set $\sigma(\Gamma)$ with its unique element. Then, σ can be considered as a map on Ξ with values in H.

By (14), $\overline{\Gamma^\star} \subset W - \sigma(\Gamma)$. As $0 \in \overline{\Gamma^\star}$, we infer $0 = w - \sigma(\Gamma)$ for some $w \in W$, and hence $\sigma(\Gamma) \in W$. If $\Gamma = \Lambda - x$ for some $x \in \Lambda$, then we claim that $x^\star \in \sigma(\Lambda - x) = \bigcap_{y\in\Lambda-x}(W - y^\star)$. This is so because $y \in \Lambda - x$ implies $y = \ell - x$ for some $\ell \in \Lambda$, hence $W - y^\star = W - (\ell^\star - x^\star) = (W - \ell^\star) + x^\star$. Clearly, $\ell^\star \in W$, so $0 \in W - \ell^\star$, and this gives $x^\star \in W - y^\star$. With $y \in \Lambda - x$ arbitrary, we obtain $\sigma(\Lambda - x) = \{x^\star\}$, as $\sigma(\Gamma)$ is a singleton set.

Next, following [44, Prop. 4.3], we can show continuity of the mapping σ. Let $\Gamma \in \Xi$, and let $V = V(\sigma(\Gamma))$ be an open neighbourhood of $\sigma(\Gamma)$ in H. Since $\sigma(\Gamma) = \bigcap_{y\in\Gamma}(W - y^\star)$ is a singleton set, one has

$$\bigcap_{y\in\Gamma} (W - y^\star) \setminus V = \varnothing.$$

As V is open, each $(W - y^\star) \setminus V$ is closed, hence also compact. So, there must be a *finite* set $F \subset \Gamma$ such that we already have $\bigcap_{y\in F}(W - y^\star) \setminus V = \varnothing$. This implies that a compact set K exists such that $\bigcap_{y\in\Gamma\cap K}(W - y^\star) \setminus V = \varnothing$, so

$$\bigcap_{y\in\Gamma\cap K} (W - y^\star) \subset V.$$

This inclusion means that $\Gamma' \cap K = \Gamma \cap K$, for any $\Gamma' \in \Xi$, implies $\sigma(\Gamma') \subset V$. By a standard argument, this can now be turned into the claimed continuity of σ. □

We can now show how ϑ induces a deformation q.

Proposition 6. *Let $\Lambda = \curlywedge(W)$, with $W = \overline{W^\circ}$ compact, be a regular model set and assume that $H_W = \{0\}$. Let $\vartheta : W \longrightarrow G$ be continuous. Then, there is precisely one continuous mapping $q \colon \Xi \longrightarrow G$ with $q(\Lambda - x) = \vartheta(x^\star)$ for all $x \in \Lambda$.*

Proof. This follows directly from Lemma 5: Uniqueness follows because $\{\Lambda - x : x \in \Lambda\}$ is dense in Ξ. Existence follows as we can simply define $q := \vartheta \circ \sigma$ with the σ of Lemma 5. □

REMARK. Let us point out that continuity of ϑ is not necessary to obtain continuity of $\vartheta \circ \sigma$. In fact, it is easy to construct examples where ϑ may even have countably many points of discontinuity (at points of $L^\star$, in fact).

Based on Proposition 6, we can now directly prove our result on deformed model sets.

Theorem 6. *Let $\varLambda$ be a regular model set and $\vartheta : H \longrightarrow G$ a continuous map. Let $\varLambda_\vartheta$ be defined according to* (12)*, with the restriction that it is still a Delone set. Then, $\varLambda_\vartheta$ is pure point diffractive. In fact, the dynamical system $(\varOmega(\varLambda_\vartheta), \alpha)$ is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions.*

Proof. Consider the map $q \colon \varXi \longrightarrow G$ constructed in Proposition 6. Plugging in the definitions, we easily find $\varLambda_q = \varLambda_\vartheta$. This, in turn, gives

$$(\varOmega(\varLambda))^q \; = \; \varOmega(\varLambda_q) \; = \; \varOmega(\varLambda_\vartheta).$$

Thus, it suffices to show that $\big((\varOmega(\varLambda))^q, \alpha\big)$ is uniquely ergodic with pure point dynamical spectrum and continuous eigenfunctions. This, however, is immediate from Theorem 3. □

REMARK. Let us mention that the abstract result of Theorem 6 has a very concrete extension in that it is possible to calculate the diffraction of $\varLambda_\vartheta$ explicitly. For the Euclidean setting, this is explained in [24, 11], and we illustrate it below in a concrete example.

7. EXAMPLE: THE SILVER MEAN CHAIN

Let us explain the various notions with a simple example in one dimension, compare [6, Sec. 8.1]. To this end, consider the two letter substitution rule

$$\sigma : \begin{array}{ccc} a & \mapsto & aba \\ b & \mapsto & a \end{array} \tag{15}$$

which allows the construction of a bi-infinite (and reflection symmetric) fixed point as follows. Starting from the (admissible) seed $w_1 = a|a$, where $|$ denotes the reference point, and defining $w_{n+1} = \sigma(w_n)$, one obtains the iteration sequence

$$a|a \stackrel{\sigma}{\longmapsto} aba|aba \stackrel{\sigma}{\longmapsto} abaaaba|abaaaba \stackrel{\sigma}{\longmapsto} \ldots \xrightarrow{n\to\infty} w = \sigma(w)$$

where w is a bi-infinite word in the alphabet $\{a, b\}$ and convergence is in the obvious product topology as generated from the alphabet together with the discrete topology.

The corresponding substitution matrix reads

$$M_\sigma \; = \; \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

where $M_{k\ell}$ is the number of symbols of type ℓ in the word $\sigma(k)$, for $k, \ell \in \{a, b\}$. This matrix is primitive, with Perron-Frobenius eigenvalue $s = 1 + \sqrt{2}$, which happens to be a Pisot-Vijayaraghavan number. It is often called the silver mean, due to its continued fraction expansion ($s = [2; 2, 2, 2, \ldots]$, in contrast to $[1; 1, 1, 1, \ldots] = (1 + \sqrt{5}\,)/2$ for the golden mean). The corresponding eigenvectors (left and right) code the frequencies of the letters a and b in w, and also the information for a proper geometric representation of w as a point set in $\mathbb{R}$, such that the substitution turns into a geometric inflation rule. One convenient choice here is to represent a by an interval of length $1 + \sqrt{2}$, and b by one of length 1. Their frequencies are $\frac{1}{2}\sqrt{2}$ and $\frac{1}{2}\big(2 - \sqrt{2}\big)$, respectively.

This is an example of a so-called Pisot substitution with two symbols, and the derived point set is known to be a regular model set (with the projection scheme yet to be derived).

Also, the fixed point is a non-singular (or generic) member of the LI-class defined by it. At the same time, it is a Sturmian sequence, and we could have started with a concrete cut and project scheme (then with the compatibility with the inflation to be established). We prefer the former possibility here, as there is a rather elegant number theoretic formulation which we will now use.

Let $\varLambda_a$ and $\varLambda_b$ denote the left endpoints of the intervals of type a and b, with our reference point (formerly marked by $|$) being mapped to 0 in this process. Both point sets are subsets of the $\mathbb{Z}$-module

$$\mathbb{Z}[\sqrt{2}] \;:=\; \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$$

which happens to be the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{2})$. There is one non-trivial algebraic conjugation in this field, defined by ${}^\star : \sqrt{2} \mapsto -\sqrt{2}$, which maps $\mathbb{Z}[\sqrt{2}]$ onto itself. This will take the rôle of the $\star$-map in the cut and project scheme, which looks as follows.

$$\begin{array}{ccccccc} & \mathbb{R} & \xleftarrow{\;\pi\;} & \mathbb{R}\times\mathbb{R} & \xrightarrow{\;\pi_{\rm int}\;} & \mathbb{R} & \\ \text{dense} & \cup & & \cup & & \cup & \text{dense} \\ & \mathbb{Z}[\sqrt{2}] & \xleftarrow{\;1-1\;} & \tilde{L} & \xrightarrow{\;1-1\;} & \mathbb{Z}[\sqrt{2}] & \end{array}$$

where $\tilde{L} = \{(x, x^\star) : x \in \mathbb{Z}[\sqrt{2}]\}$ is a (rectangular) lattice in $\mathbb{R}^2$. In comparison to the standard situation of model sets, compare [35], this cut and project scheme is self-dual, see also [33, p. 418]. In particular, the $\star$-map is then one-to-one on $\mathbb{Z}[\sqrt{2}]$.

An explicit geometric realization of $\tilde{L}$ with basis vectors is

$$\tilde{L} \;=\; \Big\langle \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Big\rangle_{\mathbb{Z}} \tag{16}$$

which has the nice property that we can directly work with the standard Euclidean scalar product for our further analysis (rather than with the quadratic form defined by the lattice).

In particular, we will later also need the *dual lattice*

$$\tilde{L}^* \;=\; \{y \in \mathbb{R}^2 : xy \in \mathbb{Z} \text{ for all } x \in \tilde{L}\} \;=\; \Big\langle \frac{1}{4}\begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Big\rangle_{\mathbb{Z}} \tag{17}$$

(note the different star symbol), which has the projections

$$L^\circ \;=\; \pi(\tilde{L}^*) \;=\; \Big\{ \frac{1}{2}\big(m + \frac{n}{\sqrt{2}}\big) : m, n \in \mathbb{Z} \Big\} \;=\; \pi_{\rm int}(\tilde{L}^*) \;=\; (L^\circ)^\star .$$

Note that the $\star$-map is well defined on the rational span of L which includes L°.

Let us continue with the construction of our model set. By standard theory for the fixed point of a primitive substitution, the sets $\varLambda_a$ and $\varLambda_b$ satisfy the equations

$$\begin{array}{rcl} \varLambda_a & = & s\varLambda_a \;\dot\cup\; \big(s\varLambda_a + (1+s)\big) \;\dot\cup\; s\varLambda_b \\ \varLambda_b & = & s\varLambda_a + s \end{array}$$

with $s = 1 + \sqrt{2}$ from above, and $\dot\cup$ denoting the disjoint union of sets. Under the $\star$-map followed by taking the closure, one obtains a new set of equations for the windows $W_a = \overline{\varLambda_a^\star}$ and $W_b = \overline{\varLambda_b^\star}$,

$$\begin{array}{rcl} W_a & = & s^\star W_a \;\cup\; \big(s^\star W_a + (1 + s^\star)\big) \;\cup\; s^\star W_b \\ W_b & = & s^\star W_a + s^\star \end{array}$$

where $s^\star = 1 - \sqrt{2}$ is less than 1 in absolute value. This new set of equations constitutes a coupled iterated functions system that is a contraction. By standard Hutchinson theory, there is a unique pair of compact sets W_a and W_b that solves this system, compare [6, Thm. 1.1 and Sec. 4] for details. It is easy to check that this solution is given by

$$W_a \;=\; \left[\tfrac{\sqrt{2}-2}{2}, \tfrac{\sqrt{2}}{2}\right], \quad W_b \;=\; \left[-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}-2}{2}\right]. \tag{18}$$

From here, one can also see that $W = W_a \cup W_b = \left[-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right]$ is the window for the full set $\varLambda = \varLambda_a \dot\cup \varLambda_b$, with $W = \overline{W^\circ}$. Moreover, since $\pm 1/\sqrt{2}$ are not elements of $\mathbb{Z}[\sqrt{2}]$, we see that $\varLambda = \curlywedge(W) = \curlywedge(W^\circ)$, so that $\varLambda$ (and also $\varLambda_a$ and $\varLambda_b$) are regular, generic (or non-singular) model sets. The density of $\varLambda$ is $\mathrm{dens}(\varLambda) = 1/2$.

The deformation is now achieved by a suitable function $\vartheta\colon \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous on W and vanishes on its complement. This is consistent with (7) because the deformation rule (12) does not require the knowledge of ϑ for any value outside of W. A simple but interesting candidate is

$$\vartheta(y) \;=\; \begin{cases} \alpha y + \beta, & y \in W \\ 0, & y \notin W \end{cases} \tag{19}$$

with some constants $\alpha, \beta \in \mathbb{R}$. For admissible values of α, the affine nature of ϑ on W has the effect of changing the relative length ratio of the a and b intervals, with β being a global translation. It is easy to check that the admissible values of α include

$$-1 \;<\; \alpha \;<\; 3 + \sqrt{2}$$

which results in the ratio

$$\varrho \;=\; \frac{\mathrm{length}(a_\vartheta)}{\mathrm{length}(b_\vartheta)} \;=\; 1 + \frac{1-\alpha}{1+\alpha}\sqrt{2}. \tag{20}$$

Here, we use a_ϑ and b_ϑ for the intervals that result from the deformation (19). For a given ratio, the parameter α is given by $\alpha = (\sqrt{2} + 1 - \varrho)/(\sqrt{2} - 1 + \varrho)$. We will come back to this discussion in the next section.

Of particular interest is the fact that one does not only get the theoretical result of pure point diffraction, but also an explicit formula for the diffraction measure. A detailed account for its calculation can be found in [11], which can also be derived explicitly via Weyl's lemma on uniform distibution, compare [43, 36] for a formulation of the latter in the context of model sets. The result is

$$\widehat{\gamma}_{\varLambda_\vartheta} \;=\; \sum_{k \in L^\circ} |A_\vartheta(k)|^2 \, \delta_k \tag{21}$$

where the so-called Fourier-Bohr coefficients (or diffraction amplitudes) are given by

$$A_\vartheta(k) \;=\; \frac{1}{2\sqrt{2}} \int_W e^{2\pi i (k^\star y - k \vartheta(y))} \, \mathrm{d}y \tag{22}$$

for all $k \in L^\circ$, and $A_\vartheta(k) = 0$ otherwise. Note that $A_\vartheta(0) \equiv 1/2 = \mathrm{dens}(\varLambda)$ in agreement with a previous remark.

To arrive at (21) and (22), one first shows that $A_\vartheta(k)$ must vanish for all $k \notin L^\circ$, which is part of [11, Thm. 2.6]. Then, let $k \in L^\circ$, and consider the points of $\varLambda$ in a (large) finite patch, e.g., in the ball $B_r(0)$ of radius r around 0. We denote such a patch by $\varLambda^{(r)}$ and set

$$\varLambda_\vartheta^{(r)} \;=\; \{x + \vartheta(x^\star) : x \in \varLambda^{(r)}\}.$$

If we place unit point measures at the points of $\mathit{\Lambda}_{\vartheta}^{(r)}$, we obtain a finite measure whose Fourier transform exists and reads

$$\sum_{x'\in \mathit{\Lambda}_{\vartheta}^{(r)}} e^{-2\pi i k x'} \;=\; \sum_{x\in \mathit{\Lambda}^{(r)}} e^{-2\pi i (kx + k\vartheta(x^{\star}))} \;=\; \sum_{x\in \mathit{\Lambda}^{(r)}} e^{2\pi i (k^{\star}x^{\star} - k\vartheta(x^{\star}))}$$

where the last step used the fact that $e^{-2\pi i(kx+k^{\star}x^{\star})} = 1$ for $k \in L^{\circ}$ and $x \in L$. Now, after dividing by the volume of $B_r(0)$, one obtains the coefficient $A_{\vartheta}(k)$ by taking the limit as $r \to \infty$, which exists and gives (22) by Weyl's lemma.

Let us also mention that, if we use the formulation via measures, the diffraction formula (21) remains valid for *all* (continuous) functions ϑ, not just for those which preserve the Delone property.

For our special choice (19), one obtains

$$A_{\alpha,\beta}(k) \;=\; e^{-2\pi i \beta k}\, \frac{\sin(z)}{2z}\Big|_{z=\pi(\alpha k - k^{\star})\sqrt{2}} \tag{23}$$

for all $k \in L^{\circ}$.

8. Topological conjugacy and further aspects

In this section, we briefly comment on the question whether $(\mathit{\Omega}^{\lambda}, \alpha)$ is topologically conjugate to $(\mathit{\Omega}, \alpha)$. A deformed model set need not be topologically conjugate to the undeformed system. In our silver mean example, with the deformation function ϑ of (19), we can find values of the scaling parameter α where the factor becomes periodic, while $\mathit{\Lambda}$ itself (which corresponds to $\alpha = \beta = 0$) is aperiodic. In such a case, in view of Corollary 2, we cannot have topological conjugacy. Note that, in contrast to [12], we do *not* keep track of the type of the intervals here. If we did that (e.g., by giving different weights to the points of a and b intervals), topological conjugacy would always be preserved under the deformation.

In particular, $\alpha = 1$ (which gives $\varrho = 1$) results in $\mathit{\Lambda}_{\vartheta} = 2\mathbb{Z} + \beta$. Eq. (21) then reduces to $\widehat{\gamma}_{\mathit{\Lambda}_{\vartheta}} = \frac{1}{4}\delta_{\mathbb{Z}/2}$, as it has to. This is a concrete example of the phenomenon of an extinction rule, which can often be used to detect situations where topological conjugacy fails. Here, by analyzing (23) in detail, one finds that the Fourier-Bohr spectrum

$$\mathit{\Sigma}_{\alpha,\beta} \;:=\; \{k \in \mathbb{R} : A_{\alpha,\beta}(k) \neq 0\}$$

is independent of β, but depends on α. Concretely, one has

$$\langle \mathit{\Sigma}_{\alpha,\beta} \rangle_{\mathbb{Z}} \;=\; \begin{cases} \frac{1}{2}\mathbb{Z}, & \alpha = 1 \\ L^{\circ}, & \text{otherwise.} \end{cases}$$

Here, the $\mathbb{Z}$-span is needed because one can have systematic extinctions also for $\alpha \neq 1$. This happens for $\alpha \in \mathbb{Q}$ and for $\alpha = 1 + r\sqrt{2}$ with $r \in \mathbb{Q}$, through solutions of $\sin(z) = 0$ in (23). Such an extinction phenomenon is usually linked to the existence of symmetries. In our case, for these special values of α, the point set $\mathit{\Lambda}_{\vartheta}$ admits an inflation symmetry, and the extinctions can be understood from that [17], see [18] for a general discussion.

Whenever $\alpha \neq 1$, the deformed model $\mathit{\Lambda}_{\vartheta}$ set is actually topologically conjugate to the original model set $\mathit{\Lambda}$, though in general not via a local derivation rule, compare [13] for a recent clarification of the relation between these concepts.

Another interesting phenomenon is the appearance of periodic diffraction, even if the underlying structure is non-periodic. For simplicity, let us concentrate on the case $\beta = 0$.

Whenever ϱ of (20) is a rational number, $\varrho = p/q$ say with p, q coprime, the set of positions of $\varLambda_\vartheta$ is a subset of a lattice in $\mathbb{R}$ (of period $\lambda = \text{length}(a_\vartheta)/p = \text{length}(b_\vartheta)/q$). Consequently, by [1, Thm. 1], the diffraction measure of the corresponding Dirac comb is periodic, with period $1/\lambda$. As the diffraction is also pure point, by our Theorem 6, it is of the form $\mu * \delta_{\mathbb{Z}/\lambda}$, where μ is a finite positive pure point measure on $[0, 1/\lambda)$. Unless $\alpha = 1$, the Fourier-Bohr spectrum is dense in $\mathbb{R}$, and the underlying Dirac comb based on $\varLambda_\vartheta$ is not periodic. So, in our example, failure of topological conjugacy coincides with the existence of periods for $\varLambda_\vartheta$.

In the example, and also in our general discussion, we started from a model set and constructed a deformation scheme. In general, a deformation will not result in another model set, though its Fourier-Bohr spectrum remains unchanged. The latter is of central importance for the actual structure determination in crystallography, e.g., from a diffraction experiment. It is often implicitly assumed that the underlying structure is a model set, but our above analysis shows that this need not be the case. An important open question is thus how to effectively characterize model sets versus deformed model sets by means of intrinsic properties, preferably by easily accessible ones. Some first results can be infered from [4], but more has to be done in this direction.

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany
E-mail address: mbaake@math.uni-bielefeld.de

Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany
E-mail address: dlenz@mathematik.tu-chemnitz.de

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Erklärung

Hiermit erkläre ich, daß diese Arbeit bisher von mir weder der Fakultät für Mathematik der TU Chemnitz noch einer anderen wissenschaftlichen Einrichtung zum Zwecke der Habilitation eingereicht wurde.
Ferner erkläre ich, daß ich diese Arbeit selbständig verfaßt und keine anderen als die angegebenen Hilfsmittel benutzt habe.

(Dr. Daniel Lenz)