

Optimal portfolios with bounded shortfall risks

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Abstract

This paper considers dynamic optimal portfolio strategies of utility maximizing investors in the presence of risk constraints. In particular, we investigate the optimization problem with an additional constraint modeling bounded shortfall risk measured by Value at Risk or Expected Loss. Using the Black-Scholes model of a complete financial market and applying martingale methods we give analytic expressions for the optimal terminal wealth and the optimal portfolio strategies and present some numerical results.

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1 Introduction

One of the basic problems in applied stochastic finance deals with optimal strategies for portfolios consisting of risky stocks and riskless bonds. Giving a finite planning horizon $[0, T]$ and starting with some initial endowment the aim is to maximize the expected utility of the terminal wealth of the portfolio by optimal selection of the proportions of the portfolio wealth invested in stocks and bond, respectively. Assuming a continuous-time market allowing for permanent trading and rebalancing the portfolio these proportions have to be found for every time t up to T .

This problem has been solved in the context of the Black-Scholes model of a complete financial market, see e.g. Cox and Huang [4], Karatzas, Lehoczky and Shreve [6, 7]. Here the portfolio can contain shares of a riskless bond and of stocks whose prices follow a geometric Brownian motion.

Following the optimal portfolio strategy leads (by definition) to the maximum expected utility of the terminal wealth. Nevertheless, the terminal wealth is a random variable with a distribution which is often extremely skew and shows considerable probability in regions of small values of the terminal wealth. This means that the optimal terminal wealth may exhibit large so-called shortfall risks. By the term shortfall risk we denote the event, that the terminal wealth falls below some threshold value, e.g. the initial capital or the result of an investment in a pure bond portfolio.

In Germany companies offering some kind of private pension insurances (Riester-Rente) are obliged by law to pay at least the invested capital without any interest to the insured person. So the company is confronted with the risk of a terminal wealth of the portfolio (created with the deposits of the insured person) below the value the non-interest-bearing deposits.

In order to incorporate such shortfall risks into the optimization it is necessary to quantify them by using appropriate risk measures. Lets denote the terminal wealth of the portfolio at time $t = T$ by X_T . Further, let $q > 0$ be some threshold or shortfall level. Then the shortfall risk consists in the random event $\{X_T < q\}$ or $\{Z := X_T - q < 0\}$. Next we assign risk measures to the random variable (risk) Z and denote them by $\varrho(Z)$. Using these measures constraints of the type $\varrho(Z) \leq \varepsilon$ for some $\varepsilon \geq 0$ can be added to the formulation of the portfolio optimization problem.

A natural idea is to restrict the probability of a shortfall, i.e.

$$\varrho(Z) = \mathbf{P}(Z < 0) = \mathbf{P}(X_T < q) \leq \varepsilon.$$

Here $\varepsilon \in [0, 1]$ is the maximum shortfall probability which is accepted by the investor. This approach corresponds to the widely used concept of Value at Risk (VaR) which is defined as

$$\text{VaR}_\varepsilon(Z) = -\zeta_\varepsilon(Z)$$

where $\zeta_\varepsilon(Z)$ denotes the ε -quantile of the random variable Z . VaR can be interpreted as the threshold value for the risk Z such that Z falls short below this value with some

given probability ε . It holds

$$\mathbf{P}(Z < 0) \leq \varepsilon \Leftrightarrow \text{VaR}_\varepsilon(Z) \leq 0 \Leftrightarrow \text{VaR}_\varepsilon(X_T) \leq -q.$$

Another risk measure is the Expected Loss defined by

$$\text{EL}(Z) = \mathbf{E}Z^- = \mathbf{E}[(X_T - q)^-].$$

In the example of the pension insurance this is a measure for the average additional capital the company is obliged to pay as compensation for the shortfall. The constraint $\mathbf{E}Z^- \leq \varepsilon$ bounds this average additional capital by $\varepsilon \geq 0$.

Further risk measures can be found in the class of coherent measures introduced by Artzner, Delbaen, Eber and Heath [1] and Delbaen [5]. These are measures possessing the properties of monotonicity, subadditivity, positive homogeneity and the translation property. The above two risk measures do not belong to this class, since VaR is not subadditive and EL violates the translation invariance property.

The paper is organized as follows. Section 2 introduces basic notation for the considered Black-Scholes model of the financial market and formulates the portfolio optimization problem. Thereby we restrict to the case of a financial market with only one risky asset. The derivations of the paper can easily be generalized to the multi-dimensional Black-Scholes model with $d > 1$ risky assets.

Before the optimization with constrained risk measures is considered Section 3 deals with the unconstrained problem. Here the basic ideas of martingale methods are explained and an application to the case of the so-called CRRA utility function is given. Based on these results in Section 4 risk measure constraints are added to the optimization. First in Subsection 4.1 the shortfall probability or equivalently the Value at Risk is bounded. We follow the paper of Basak and Shapiro [2], but give slightly different solutions. In Subsection 4.2 the expected loss is bounded. This case is not considered explicitly in [2] and we give the detailed solution for the case of a CRRA utility function. Finally Section 5 presents some numerical results.

2 The portfolio optimization problem

We consider a continuous-time economy with finite horizon $[0, T]$ which is built on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$, on which is defined a 1-dimensional Brownian motion W . We assume that all stochastic processes are adapted to (\mathcal{F}_t) , the augmented filtration generated by W . It is assumed through this paper that all inequalities as well as equalities hold \mathbf{P} -almost surely. Moreover, it is assumed that all stated processes are well defined without giving any regularity conditions ensuring this, since our focus is a characterization problem.

Financial investment opportunities are given by an instantaneously riskless money market account and a risky stock as in the Black-Scholes model [3]. We suppose the money

market provides an interest rate r . The stock price S is represented by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad (2.1)$$

where the interest rate r , the stock instantaneous mean return μ and the volatility σ are assumed to be constants.

The dynamic market completeness implies the existence of a unique state price density process H_t , given by

$$dH_t = -H_t(rdt + \kappa dW_t), \quad H_0 = 1, \quad (2.2)$$

where $\kappa := (\mu - r)/\sigma$ is the market price of risk in the economy, which can be regarded as the driving economic parameter in an agent's dynamic investment problem.

We assume that an agent in this economy is endowed at time zero with an initial wealth of x . The agent chooses an investment policy θ , where θ_t denotes the fraction of wealth invested in the stock at time t . The portfolio process θ_t is assumed to be self-financing so that the agent's wealth process X follows

$$dX_t = [r + \theta_t(\mu - r)]X_t dt + \theta_t \sigma X_t dW_t, \quad X_0 = x. \quad (2.3)$$

At time $t = T$ the agent reaches the terminal wealth X_T . The portfolio process is assumed to be admissible in the following sense.

Definition 2.1

Given $x > 0$, we say that a portfolio process θ is admissible at x , and write $\theta \in \mathcal{A}(x)$, if the wealth process X_t^θ starting at $X_0^\theta = x$ satisfies

$$X_t^\theta \geq 0, \quad 0 \leq t \leq T.$$

In this economy, the agent is assumed to derive from the terminal wealth X_T a utility $u(X_T)$ and he is looking to maximize the expected utility by choosing an optimal strategy from the set of admissible strategies.

The dynamic problem

Find an admissible strategy θ^* in $\mathcal{A}(x)$ that solves

$$\max_{\theta \in \mathcal{A}} \mathbf{E}[u(X_T^\theta)]. \quad (2.4)$$

Thereby, the utility function $u(\cdot)$ satisfies the following conditions

- $u(\cdot)$ is twice continuously differentiable,
- $u(\cdot)$ is strictly increasing and strictly concave,
- $\lim_{x \rightarrow 0} u'(x) = \infty$ and $\lim_{x \rightarrow \infty} u'(x) = 0$.

3 Martingale methods for the unconstrained problem

With no additional restrictions such as risk management, the maximization problem (2.4) was solved in the case of a complete market, by Cox and Huang [4] and independently by Karatzas, Lehocky and Shreve [6] using martingale and duality approaches. The method consists of converting the dynamic optimization problem of finding an admissible strategy that maximizes the expected utility from terminal wealth, into a static optimization problem consisting of finding an optimal terminal wealth, and via a representation problem one gets the optimal strategy associated with this optimal terminal wealth.

Itô's Formula implies that the process $H_t X_t^\theta$ is a supermartingale which implies that the so called *budget constraint*

$$\mathbf{E}[H_T X_T^\theta] \leq x$$

is satisfied for every $\theta \in \mathcal{A}(x)$. This means that the expected discounted terminal wealth can not exceed the initial wealth. Here the state price density H_t serves as a discounting process.

In the present case of a complete market, the following theorem is a basic tool in martingale methods.[7]

Theorem 3.1

Let $x > 0$ be given and let ξ be a nonnegative, \mathcal{F}_T -measurable random variable such that

$$\mathbf{E}[H_T \xi] = x.$$

Then there exists a portfolio process $\theta(\cdot)$ in $\mathcal{A}(x)$ such that $\xi = X_T^\theta$.

Define

$$\mathcal{B}(x) := \{\xi \geq 0 : \xi \text{ is } \mathcal{F}_T\text{-measurable and } \mathbf{E}[H_T \xi] \leq x\}.$$

In contrast to the dynamic problem, where the investor is required to maximize expected utility from terminal wealth over a set of processes, in a first step the static problem is considered. Here, the investor has the advantage to maximize only over a set of random variables.

The static problem

Find an \mathcal{F}_T -measurable random variable ξ^* in $\mathcal{B}(x)$ that solves

$$\max_{\xi \in \mathcal{B}} \mathbf{E}[u(\xi)]. \quad (3.1)$$

In a second step the optimal strategy is found as the solution of the representation problem.

The representation problem

Given $\xi^* \in \mathcal{B}$ that solves (3.1), find an admissible strategy $\theta^* \in \mathcal{A}(x)$ such that $X_T^{\theta^*} = \xi^*$.

For $y > 0$, we denote by $I(y) := (u')^{-1}(y)$ the inverse function of the derivative of the utility function. Define $\chi(y) := \mathbf{E}[H_T I(yH_T)]$.

The following lemma provides some properties of the function $\chi(\cdot)$ and $I(\cdot)$, see [8].

Lemma 3.2

The function $\chi(\cdot)$ satisfies

$$\chi(y) < \infty \quad \text{for all } y > 0.$$

Moreover, the function $\chi(\cdot)$ is continuous, strictly decreasing on $(0, \infty)$ with

$$\chi(0) := \lim_{y \downarrow 0} \chi(y) = \infty, \quad \chi(\infty) := \lim_{y \rightarrow \infty} \chi(y) = 0.$$

For $0 < x, y < \infty$, we have

$$u(I(y)) \geq u(x) + y(I(y) - x).$$

The next theorem stated in [8] solves the static optimization problem (3.1).

Theorem 3.3

Consider the portfolio problem (2.4). Let $x > 0$ and set $y := \chi^{-1}(x)$, i.e. y solves $x = \mathbf{E}[H_T I(yH_T)]$. Then there exists for $\xi^* := I(yH_T)$ a self-financing portfolio process $\theta_t^*, t \in [0, T]$, such that

$$\theta_t^* \in \mathcal{A}(x), \quad X_T^{\theta^*} = \xi^*,$$

and the portfolio process solves the dynamic problem (2.4).

The representation problem can be solved using the fact that the process $H_t X_t^\theta$ is a martingale. Markov property of solution of stochastic differential equation allows the optimal wealth process before the horizon $X_t^{\theta^*}$ to be written as a function of H_t for which we apply Itô's Formula. By equating coefficients with the wealth process (2.3) one gets the optimal portfolio.

Example 3.4 The problem of the so called benchmark manager or non-risk managing agent was studied by Cox and Huang [4], where the agent has a constant relative risk aversion (CRRA) γ which is contained as a parameter of the utility function

$$u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma \in (0, \infty) \setminus \{1\}, \\ \ln x, & \gamma = 1. \end{cases} \quad (3.2)$$

According to Theorem 3.3, the static problem (3.1) has the optimal solution

$$\xi^B = I(yH_T),$$

with $I(x) = x^{\frac{-1}{\gamma}}$ is the inverse function of the derivative of the utility function $u(\cdot)$ and $y := \frac{1}{x^\gamma} e^{(1-\gamma)(r + \frac{\kappa^2}{2\gamma})T}$.

Let X_t^B be the optimal solution before the horizon. Itô's lemma applied to Equations (2.2) and (2.3) implies that the process $H_t X_t^B$ is \mathcal{F}_t -martingale: $X_t^B = \mathbf{E}\left[\frac{H_T}{H_t} X_T^B \mid \mathcal{F}_t\right]$. Here the optimal terminal wealth X_T^* is given by Theorem 3.3 as $X_T^* := \xi^B = I(yH_T)$. We apply Markov Property of the solution H_t of Equation (2.2) to compute this conditional expectation using the fact that $\ln H_T$ is normally distributed with mean $\ln H_t - (r + \frac{\kappa^2}{2})(T - t)$ and variance $\kappa^2(T - t)$. We get for the optimal terminal wealth before the horizon the following form

$$X_t^B = \frac{e^{\Gamma(t)}}{(yH_t)^{\frac{1}{\gamma}}} \quad \text{with} \quad \Gamma(t) := \frac{1 - \gamma}{\gamma} \left(r + \frac{\kappa^2}{2}\right)(T - t).$$

The optimal strategy is obtained by a representation approach. In this case we have $X_t^* = f(H_t)$ with $f(x) = \frac{e^{\Gamma(t)}}{(yx)^{\frac{1}{\gamma}}}$ for which we apply Itô's lemma to get

$$X_t^B = f(H_0) + \int_0^t \left(-rH_s f'(H_s) + \frac{\kappa^2}{2} H_s^2 f''(H_s)\right) ds + \int_0^t \left(-\kappa H_s f'(H_s)\right) dW_s.$$

If we equate the volatility coefficient of this equation with the volatility coefficient of Equation (2.3), we get in the absence of risk-constraint the following constant optimal strategy

$$\theta_t^B = \theta^B = \frac{\kappa}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2} = \text{const.}$$

4 Optimization with constraints

4.1 Value at Risk constraint

In this section we present the portfolio maximization problem constrained by the Value at Risk. More precisely, we consider an investor who wishes in addition to maximize his expected utility from terminal wealth, to control the probability for a shortfall. Given a probability $\varepsilon \in [0, 1]$ this constraint can be written as

$$\mathbf{P}(Z < 0) = \mathbf{P}(X_T < q) \leq \varepsilon. \quad (4.1)$$

From the definition of VaR given in the Introduction this is equivalent to

$$\text{VaR}_\varepsilon(Z) \leq 0 \Leftrightarrow \text{VaR}_\varepsilon(X_T) \leq -q.$$

With constraint (4.1) the agent bounds the probability of negative values of the risk $Z = X_T - q$ by ε . We will denote this strategy as VaR-strategy.

We give an alternative solution of the dynamic optimization problem of the VaR agent studied by Basak and Shapiro in [2]. The problem is solved using the martingale representation approach which consists of formulating the problem as the following static variational problem:

$$\begin{aligned} & \max_{\xi \in \mathcal{B}} \mathbf{E}[u(\xi)] \\ \text{subject to} & \quad \mathbf{E}[H_T \xi] \leq x \\ & \quad \mathbf{P}(\xi < q) \leq \varepsilon. \end{aligned}$$

The VaR constraint leads to nonconcavity with which the maximization is more complicated. The optimal terminal wealth is characterized by Basak and Shapiro [2], Proposition 1, where the authors assumed that the solution exists.

Proposition 4.1

The VaR-optimal terminal wealth is

$$\xi^{VaR} = \begin{cases} I(yH_T) & \text{if } H_T < \underline{h}, \\ q & \text{if } \underline{h} \leq H_T < \bar{h}, \\ I(yH_T) & \text{if } \bar{h} \leq H_T, \end{cases}$$

where $I(\cdot)$ is the inverse function of $u'(\cdot)$, $\underline{h} = \frac{u'(q)}{y}$, \bar{h} is such that $\mathbf{P}(H_T > \bar{h}) = \varepsilon$ and $y \geq 0$ solves $\mathbf{E}[H_T \xi^{VaR}(y)] = x$.

The VaR-constraint (4.1) is binding if, and only if, $\underline{h} < \bar{h}$.

In the following proposition we present explicit expressions for the VaR agent's optimal wealth and portfolio strategies before the horizon.

Proposition 4.2

Let $u(x)$ be the CRRA utility function given in (3.2). Then

(i) The VaR-optimal wealth at time t is given by

$$\begin{aligned} X_t^{VaR} = & \frac{e^{\Gamma(t)}}{(yH_t)^{\frac{1}{\gamma}}} - \left[\frac{e^{\Gamma(t)}}{(yH_t)^{\frac{1}{\gamma}}} \Phi(-d_1(\underline{h}, H_t)) - qe^{-r(T-t)} \Phi(-d_2(\underline{h}, H_t)) \right] \\ & + \left[\frac{e^{\Gamma(t)}}{(yH_t)^{\frac{1}{\gamma}}} \Phi(-d_1(\bar{h}, H_t)) - qe^{-r(T-t)} \Phi(-d_2(\bar{h}, H_t)) \right], \end{aligned} \quad (4.2)$$

where $\Phi(\cdot)$ is the standard-normal distribution function, y and \bar{h} are as in Proposition 4.1 and

$$\begin{aligned} \underline{h} &= \frac{1}{y_1 q^\gamma}, \\ \Gamma(t) &:= \frac{1-\gamma}{\gamma} \left(r + \frac{\kappa^2}{2\gamma} \right) (T-t), \\ d_1(z, H_t) &:= \frac{\ln \frac{z}{H_t} + (r - \frac{\kappa^2}{2})(T-t)}{\kappa \sqrt{T-t}}, \\ d_2(z, H_t) &:= d_1(z, H_t) + \frac{1}{\gamma} \kappa \sqrt{T-t}. \end{aligned}$$

(ii) The VaR-optimal fraction of wealth invested in stock is

$$\theta_t^{VaR} = \theta_t^B p_t^{VaR},$$

where the benchmark value θ_t^B and the exposure to risky assets relative to the benchmark p_t^{VaR} are

$$\begin{aligned} \theta_t^B &\equiv \theta^B = \frac{\kappa}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2} \\ p_t^{VaR} &= 1 - \frac{qe^{-r(T-t)}}{X_t^{VaR}} \left[\Phi(-d_2(\underline{h}, H_t)) - \Phi(-d_2(\bar{h}, H_t)) \right] \\ &\quad + \frac{\gamma}{\kappa\sqrt{T-t}X_t^{VaR}} \frac{e^{\Gamma(t)}}{(yH_t)^{\frac{1}{\gamma}}} \left[\varphi(d_1(\underline{h}, H_t)) - \varphi(d_1(\bar{h}, H_t)) \right] \\ &\quad - \frac{\gamma qe^{-r(T-t)}}{\kappa\sqrt{T-t}X_t^{VaR}} \left[\varphi(d_2(\bar{h}, H_t)) - \varphi(d_2(\underline{h}, H_t)) \right] \end{aligned}$$

respectively, and $\varphi(\cdot)$ is the standard-normal probability density function.

Proof.

(i) From Equations (2.2) and (2.3), Itô's lemma implies that the process $H_t X_t^{VaR}$ is \mathcal{F}_t -martingale:

$$\begin{aligned} X_t^{VaR} &= \mathbf{E} \left[\frac{H_T}{H_t} \xi^{VaR} \middle| \mathcal{F}_t \right] \\ &= \mathbf{E} \left[\frac{H_T}{H_t} I(yH_t) (1_{\{H_T < \underline{h}\}} + 1_{\{\bar{h} \leq H_T\}}) \middle| \mathcal{F}_t \right] + \mathbf{E} \left[\frac{H_T}{H_t} q 1_{\{\underline{h} \leq H_T < \bar{h}\}} \middle| \mathcal{F}_t \right]. \end{aligned}$$

These conditional expectations are computed by applying Markov's property of solution of stochastic differential equation and using the fact that $\ln H_T$ is normally distributed with mean $\ln H_t - (r + \frac{\kappa^2}{2})(T-t)$ and variance $\kappa^2(T-t)$. Hence, we get the form of the agent's optimal wealth before the horizon.

(ii) From Equality (4.2) it follows $X_t^{VaR} = f(H_t)$ where

$$\begin{aligned} f(x) &= \frac{e^{\Gamma(t)}}{(y_1 x)^{\frac{1}{\gamma}}} \left[1 - \Phi(-d_1(\underline{h}, x)) + \Phi(-d_1(\bar{h}, x)) \right] \\ &\quad + qe^{-r(T-t)} \left[\Phi(-d_2(\underline{h}, x)) - \Phi(-d_2(\bar{h}, x)) \right]. \end{aligned}$$

Applying Itô's lemma to the function $f(\cdot)$, we get

$$dX_t^{VaR} = [-rf'(H_t)H_t + \frac{\kappa^2}{2}f''(H_t)H_t^2]dt - \kappa f'(H_t)H_t dW_t,$$

and equating coefficients with Equation (2.3) leads to the following equality:

$$\theta_t^{VaR} = -\frac{\kappa f'(H_t)H_t}{\sigma f(H_t)} = -\theta^B \gamma \frac{f'(H_t)H_t}{f(H_t)}. \quad (4.3)$$

Computing the derivative of the $f(\cdot)$ we get

$$f'(x) = \frac{1}{\gamma x} \left[-f(x) + qe^{-r(T-t)} \left(\Phi(-d_2(\underline{h}, x)) - \Phi(-d_2(\bar{h}, x)) \right) \right] \\ - \frac{e^{\Gamma(t)}}{(yx)^{\frac{1}{\gamma}} \kappa \sqrt{T-tx}} \left[\varphi(d_1(\underline{h}, x)) - \varphi(d_1(\bar{h}, x)) \right] \\ + \frac{qe^{-r(T-t)}}{\kappa \sqrt{T-tx}} \left[\varphi(d_2(\bar{h}, x)) - \varphi(d_2(\underline{h}, x)) \right].$$

Substituting into (4.3), we get the final form of the optimal strategies before the horizon.

□

4.2 Expected Loss constraint

We consider in this section an agent who wishes to limit his Expected Loss. In this case the agent defines his strategy as one which fulfills the constraint

$$\mathbf{E}[(X_T - q)^-] \leq \varepsilon, \quad (4.4)$$

where $\varepsilon \geq 0$ is given. We will denote this strategy as EL-strategy. Our objective in this section is to solve the agent's optimization problem constrained by (4.4). The dynamic optimization problem of the EL-risk manager can be restated as the following static variational problem:

$$\begin{aligned} & \max_{\xi \in \mathcal{B}} \mathbf{E}[u(\xi)] \\ \text{subject to} & \quad \mathbf{E}[H_T \xi] \leq x \\ & \quad \mathbf{E}[(\xi - q)^-] \leq \varepsilon. \end{aligned}$$

The following proposition characterizes the optimal solution in the presence of the EL-constraint (4.4). We prove that if a terminal wealth satisfies (4.5), then it is the optimal solution.

Proposition 4.3

The EL-optimal terminal wealth is

$$\xi^{EL} = \begin{cases} I(y_1 H_T) & \text{if } H_T < \underline{h}, \\ q & \text{if } \underline{h} \leq H_T < \bar{h}, \\ I(y_1 H_T - y_2) & \text{if } \bar{h} \leq H_T, \end{cases} \quad (4.5)$$

where $\underline{h} = \frac{u'(q)}{y_1}$, $\bar{h} = \frac{u'(q)+y_2}{y_1}$ and $y_1, y_2 > 0$ solve the following system of equations

$$\begin{aligned} \mathbf{E}[H_T \xi^{EL}(T; y_1, y_2)] &= x, \\ \mathbf{E}[H_T (\xi^{EL}(T; y_1, y_2) - q)^-] &= \varepsilon \quad \text{or} \quad y_2 = 0. \end{aligned}$$

The EL-constraint(4.4) is binding if, and only if, $\underline{h} < \bar{h}$.

Proof.

Suppose that $\mathbf{E}[(X_T^{EL} - q)^-] < \varepsilon$, then it follows $y_2 = 0$, and $X^{EL}(T) = I(y_1 H_T)$ which is optimal since we have in this case a standard problem without a risk constraint. Otherwise, $\mathbf{E}[(X_T^{EL} - q)^-] = \varepsilon$, and $\underline{h} < \bar{h}$. In order to solve the optimization problem under EL-constraint, the common convex-duality approach is adapted by introducing the convex-conjugate of the utility function $u(\cdot)$ with an additional term capturing the EL-constraint as it is shown in the following lemma.

Lemma 4.4

The expression of ξ_T^{EL} solves the following pointwise problem $\forall H_T$:

$$u(\xi_T^{EL}) - y_1 H_T \xi_T^{EL} - y_2 (\xi_T^{EL} - q)^- = \max_{x \geq 0} \{u(x) - y_1 H_T x - y_2 (x - q)^-\}.$$

Proof.

Let $z > 0$ and consider the function $h(x) := u(x) - y_1 z x - y_2 (x - q)^-$. Defining the two functions

$$\begin{aligned} h_1(x) &:= u(x) - y_1 z x \\ h_2(x) &:= u(x) - y_1 z x + y_2 (x - q) = u(x) - (y_1 z - y_2)x - y_2 q, \end{aligned}$$

the function h can be written as

$$h(x) = \begin{cases} h_1(x) & \text{for } x \geq q, \\ h_2(x) & \text{for } x < q. \end{cases} \quad (4.6)$$

Since h_1 and h_2 are strictly concave and continuously differentiable, the function h is a continuous and strictly concave function which is differentiable in $[0, q)$ and (q, ∞) and possesses different one-sided derivatives in the point $x = q$ which are $h'(q - 0) = h_2'(q)$ and $h'(q + 0) = h_1'(q)$.

The functions h_1 and h_2 attain its maximum values at $x_1 := I(y_1 z)$ and $x_2 := I(y_1 z - y_2)$, respectively. Since the function $I(\cdot)$ is strictly decreasing and $y_2 > 0$ it follows $x_1 < x_2$. To find the maximum of h one has to consider the following three cases.

(i) $q < x_1$:

Since u' is strictly decreasing we have $u'(q) > u'(x_1) = u'(I(y_1 z)) = y_1 z$, hence $z < \frac{u'(q)}{y_1} = \underline{h}$. Considering the one-sided derivatives at $x = q$ one obtains

$$\begin{aligned} h'(q - 0) = h_2'(q) = u'(q) - (y_1 z - y_2) &> u'(q) - y_1 \frac{u'(q)}{y_1} + y_2 > 0 \\ \text{and } h'(q + 0) = h_1'(q) = u'(q) - y_1 z &> u'(q) - y_1 \frac{u'(q)}{y_1} = 0, \end{aligned}$$

i.e. the function h is increasing at $x = q$. It follows that the function h attains its maximum on (q, ∞) where $h(x) = h_1(x)$, i.e. the maximum is at $x = x_1 = I(y_1 z)$.

(ii) $\mathbf{x}_1 \leq \mathbf{q} < \mathbf{x}_2$:

Now the relation $q \geq x_1$ implies $z \geq \underline{h}$ while $q < x_2$ leads to

$$u'(q) > u'(x_2) = u'(I(y_1 z - y_2)) = y_1 z - y_2,$$

i.e. $z < \frac{u'(q) + y_2}{y_1} = \bar{h}$, which gives $\underline{h} \leq z < \bar{h}$. It follows that

$$h'(q - 0) = h'_2(q) = u'(q) - (y_1 z - y_2) > u'(q) - y_1 \frac{u'(q) + y_2}{y_1} + y_2 = 0$$

$$\text{and } h'(q + 0) = h'_1(q) = u'(q) - y_1 z \leq u'(q) - y_1 \frac{u'(q)}{y_1} = 0.$$

From the strict concavity of h we deduce that $h'(x) = h'_1(x) < h'_1(q) < 0$ for $x > q$. Thus the function h is strictly increasing for $x < q$ and strictly decreasing for $x > q$, hence h attains its maximum at $x = q$.

(iii) $\mathbf{q} \geq \mathbf{x}_2$:

This case is equivalent to $z \geq \bar{h} = \frac{u'(q) + y_2}{y_1}$. For the one-sided derivatives at $x = q$ one obtains

$$h'(q - 0) = h'_2(q) = u'(q) - (y_1 z - y_2) \leq u'(q) - y_1 \frac{u'(q) + y_2}{y_1} + y_2 = 0$$

$$\text{and } h'(q + 0) = h'_1(q) = u'(q) - y_1 z \leq u'(q) - y_1 \frac{u'(q) + y_2}{y_1} = -y_2 < 0.$$

It follows that the function h is decreasing at $x = q$ attains its maximum on $(0, q)$ where $h(x) = h_2(x)$ and hence the maximum is at $x = x_2 = I(y_1 z - y_2)$.

Since the above considerations hold for arbitrary $z > 0$ the assertion of the lemma holds pointwise for all $z = H_T$. \square

To complete the proof, let η be any admissible solution satisfying the static budget constraint and the EL-constraint (4.4). We have

$$\begin{aligned} \mathbf{E}[u(\xi_T^{EL})] - \mathbf{E}[u(\eta)] &= \mathbf{E}[u(\xi_T^{EL})] - \mathbf{E}[u(\eta)] - y_1 x + y_1 x - y_2 \varepsilon + y_2 \varepsilon \\ &\geq \mathbf{E}[u(\xi_T^{EL})] - \mathbf{E}[y_1 H_T \xi_T^{EL}] - y_2 \mathbf{E}[(\xi_T^{EL} - q)^-] \\ &\quad - \mathbf{E}[u(\eta)] + \mathbf{E}[y_1 H_T \eta] + y_2 \mathbf{E}[(\eta - q)^-] \\ &\geq 0, \end{aligned}$$

where the first inequality follows from the static budget constraint and the constraint for the risk holding with equality for ξ_T^{EL} , while holding with inequality for η . The last inequality is a consequence of the above lemma. Hence we obtain that ξ_T^{EL} is optimal. \square

We present in the following proposition the explicit expressions for the EL-optimal wealth and portfolio strategy before the horizon.

Proposition 4.5

Let $u(x)$ be the CRRA utility function given in (3.2). Then

(i) The EL-optimal wealth at time t is given by

$$X_t^{EL} = \frac{e^{\Gamma(t)}}{(y_1 H_t)^{\frac{1}{\gamma}}} \left[1 - \Phi(-d_1(\underline{h}, H_t)) \right] + qe^{-r(T-t)} \left[\Phi(-d_2(\underline{h}, H_t)) - \Phi(-d_2(\bar{h}, H_t)) \right] + G(H_t, \bar{h}), \quad (4.7)$$

where $\Phi(\cdot)$ is the standard-normal distribution function, y_1, y_2 are as in Proposition 4.3, $\Gamma(t), d_1, d_2$ are as in Proposition 4.2 and

$$\underline{h} = \frac{1}{y_1 q^\gamma} \quad \text{and} \quad \bar{h} = \frac{q^{-\gamma} + y_2}{y_1},$$

$$G(z, \bar{h}) := \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{c_2(\bar{h}, z)} \frac{e^{-\frac{1}{2}(u-b)^2}}{(y_1 t e^{a+bu} - y_2)^{\frac{1}{\gamma}}} du,$$

$$c_2(\bar{h}, z) = \frac{1}{b} (\ln(\frac{\bar{h}}{z}) - a), \quad a := -(r + \frac{\kappa^2}{2})(T-t) \quad \text{and} \quad b := -\kappa\sqrt{T-t}.$$

(ii) The fraction of wealth invested in stock is

$$\theta_t^{EL} = \theta^B p_t^{EL},$$

where θ^B is as in Proposition (4.2) and the exposure to risky assets relative to the benchmark p_t^{EL} is

$$p_t^{EL} = \frac{1}{X_t^{EL}} \frac{e^{\Gamma(t)}}{(y_1 H_t)^{\frac{1}{\gamma}}} \left[1 - \Phi(-d_1(\underline{h}, H_t)) + \frac{\gamma}{\kappa\sqrt{T-t}} \varphi(d_1(\underline{h}, H_t)) \right] - \frac{q\gamma e^{-r(T-t)}}{X_t^{EL} \kappa\sqrt{T-t}} \varphi(d_2(\underline{h}, H_t)) + \frac{y_1 H_t e^{(\kappa^2 - 2r)(T-t)}}{X_t^{EL}} \psi_0(c_2(\bar{h}, H_t), b, y_1 H_t e^a, y_2, 2b, 1, 1 + \frac{1}{\gamma}),$$

where $\varphi(\cdot)$ is the standard-normal probability density function and

$$\psi_0(\alpha, \beta, c_1, c_2, m, s, \delta) := \frac{1}{\sqrt{2\pi}s} \int_{-\infty}^{\alpha} \frac{\exp(-\frac{(u-m)^2}{2s^2})}{(c_1 e^{\beta u} - c_2)^\delta} du.$$

Proof.

(i) Taking into account Equations (2.2) and (2.3) and applying Itô's lemma one obtains that the process $H_t X_t^{EL}$ is an \mathcal{F}_t -martingale hence and

$$X_t^{EL} = \mathbf{E} \left[\frac{H_T}{H_t} \xi_T^{EL} \middle| \mathcal{F}_t \right] = J_1 + J_2,$$

where $J_1 := \mathbf{E} \left[\frac{H_T}{H_t} [I(y_1 H_t) 1_{\{H_T < \underline{h}\}} + q 1_{\{\underline{h} \leq H_T < \bar{h}\}}] \middle| \mathcal{F}_t \right],$

and $J_2 = G(H_t, \bar{h}) := \mathbf{E} \left[\frac{H_T}{H_t} I(y_1 H_t - y_2) 1_{\{\bar{h} \leq H_T\}} \middle| \mathcal{F}_t \right].$

The conditional expectation J_1 is computed by applying Markov's property of solution of stochastic differential equation and using the fact that $\ln H_T$ is normally distributed with mean $\ln H_t - (r + \frac{\kappa^2}{2})(T - t)$ and variance $\kappa^2(T - t)$, while the conditional expectation J_2 is computed numerically. Hence, we get the form of the optimal wealth before the horizon.

(ii) From Equation (4.7) we have $X_t^{EL} = f(H_t)$, where

$$f(x) = \frac{e^{\Gamma(t)}}{(y_1 x)^{\frac{1}{\gamma}}} \left[1 - \Phi(-d_1(\underline{h}, x)) \right] + q e^{-r(T-t)} \left[\Phi(-d_2(\underline{h}, x)) - \Phi(-d_2(\bar{h}, x)) \right] + G(x, \bar{h}).$$

Itô's lemma applied to the function $f(\cdot)$ leads to

$$dX_t^{EL} = [-r f'(H_t) H_t + \frac{\kappa^2}{2} f''(H_t) H_t^2] dt - \kappa f'(H_t) H_t dW_t.$$

Comparing with Equation (2.3), one gets the following equality:

$$\begin{aligned} \theta_t^{EL} &= -\frac{\kappa f'(H_t) H_t}{\sigma f(H_t)} \\ &= -\theta^B \gamma \frac{f'(H_t) H_t}{f(H_t)}. \end{aligned} \tag{4.8}$$

Computing the derivative of the function f , we obtain

$$\begin{aligned} f'(x) &= -\frac{e^{\Gamma(t)}}{x \gamma (y_1 x)^{\frac{1}{\gamma}}} \left[1 - \Phi(-d_1(\underline{h}, x)) + \frac{\gamma}{\kappa \sqrt{T-t}} \varphi(d_1(\underline{h}, x)) \right] \\ &\quad + \frac{q e^{-r(T-t)}}{\kappa \sqrt{T-t} x} \left[\varphi(d_2(\underline{h}, x)) - \varphi(d_2(\bar{h}, x)) \right] + \frac{\partial}{\partial x} G(x, \underline{h}). \end{aligned} \tag{4.9}$$

For the last term we have

$$\begin{aligned} \frac{\partial}{\partial x} G(x, \bar{h}) &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \frac{\partial}{\partial x} \left[\int_{\infty}^{c_2(\bar{h}, x)} l(x, u) du \right] \\ &= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \left[\int_{\infty}^{c_2(\bar{h}, x)} \frac{\partial}{\partial x} l(x, u) du + \frac{\partial}{\partial x} c_2(\bar{h}, x) l(x, c_2(\bar{h}, x)) \right], \end{aligned}$$

where

$$l(x, u) = \frac{e^{-\frac{1}{2}(u-b)^2}}{(y_1 x e^{a+bu} - y_2)^{\frac{1}{\gamma}}}.$$

Finally, we get

$$\begin{aligned} \frac{\partial}{\partial x} G(x, \bar{h}) &= \frac{-y_1}{\gamma} e^{(\kappa^2 - 2r)(T-t)} \psi_0(c_2(\bar{h}, x), b, y_1 x e^a, y_2, 2b, 1, 1 + \frac{1}{\gamma}) \\ &\quad + \frac{q e^{r(T-t)}}{\kappa \sqrt{T-t} x} \varphi(-d_2(\bar{h}, x)). \end{aligned}$$

Substituting in (4.8), we get the final form of the optimal strategies before the horizon.

□

5 Numerical results

This section illustrates the findings of the preceding sections with an example. Table 5.1 shows the parameters for the portfolio optimization problem and the underlying Black-Scholes model of the financial market. In this example the aim is to maximize the expected logarithmic utility ($\gamma = 1$) of the terminal wealth X_T of the portfolio with the horizon $T = 20$ years. The shortfall level q is set to be 80% of the terminal wealth of a pure bond portfolio, i.e. $q = 0.8 x e^{rT}$. We bound the shortfall probability $\mathbf{P}(X_T < q)$ by $\varepsilon = 10\%$, i.e. we consider the optimization with the VaR constraint described in Section 4.1.

stock	$\mu = 7\%, \quad \sigma = 20\%$
bond	$r = 4\%$
horizon	$T = 20$
initial wealth	$x = 1$
utility function	$u(x) = \ln x \quad \text{i.e. } \gamma = 1$
shortfall level	$q = 0.8 x e^{rT} = 1.78 \dots$
constraint	$\text{VaR}_\varepsilon(X_T - q) \leq 0 \quad \Leftrightarrow \quad \mathbf{P}(X_T < q) \leq \varepsilon$
shortfall probability	$\varepsilon = 10\%$

Table 5.1: Parameters of the optimization problem

First the solution of the static problem is considered, it leads to the optimal terminal wealth ξ^{VaR} . Figure 5.1 shows the probability density function and its cumulated counterpart - the distribution function - of this random variable. For the sake of comparison

we also give the corresponding functions for the terminal wealth resulting from portfolios managed by the

- pure bond strategy $\theta_t \equiv \theta^0 = 0$,
- pure stock strategy $\theta_t \equiv \theta^1 = 1$,
- optimal strategy of the unconstrained problem $\theta_t \equiv \theta^B = \frac{\mu-r}{\gamma\sigma^2} = 0.75$
(see Example 3.4).

Additionally, on the horizontal axes the expected terminal wealths $\mathbf{E}X_T$ for the considered portfolios are marked.

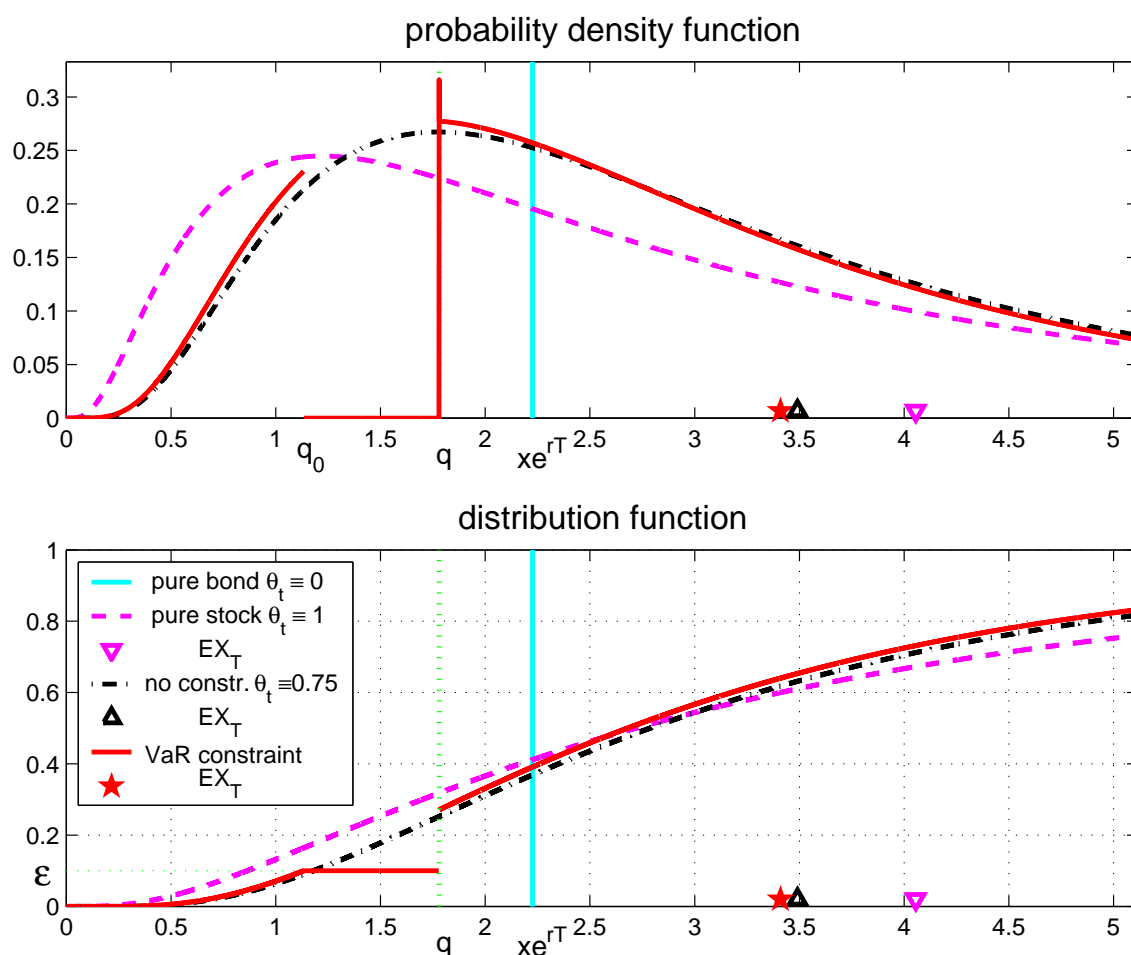


Figure 5.1: Distribution of the VaR-optimal terminal wealth

While in case of the pure bond portfolio the distribution of the terminal wealth $X_T^{\theta^0}$ is concentrated in the single point xe^{rT} , the terminal wealths $X_T^{\theta^1}$ and $X_T^{\theta^B}$ are absolutely continuous. It holds

$$e^{rT} = X_T^{\theta^0} < \mathbf{E}X_T^{\theta^B} < \mathbf{E}X_T^{\theta^1} = e^{\mu T}.$$

We mention that $\xi^B = X_T^{\theta^B}$ maximizes the expected utility $\mathbf{E}u(X_T^{\theta^B})$ and not the expected terminal wealth $\mathbf{E}X_T^{\theta^B}$ itself, therefore the latter inequality is not a contradiction. For parameter sets fulfilling $\theta^B = \frac{\mu-r}{\gamma\sigma^2} > 1$ the reverse inequality can be observed.

The distribution of the optimal terminal wealth ξ^{VaR} for the constrained problem contains a discrete as well as an absolutely continuous part. This follows from the representation of ξ^{VaR} in Proposition 4.1, which indicates that the probability $\mathbf{P}(\underline{h} \leq H_T < \bar{h}) = 0.1711 \dots$ is concentrated in the single point q . In the density plot this probability mass built up at the shortfall level q is marked by a vertical line at q . It is noted that there is a gap in the support of the absolutely continuous distribution, since an interval $(q_0, q) = (1.1343, 1.7804)$ of values below the shortfall level q (small losses) carries no probability while the interval $(0, q_0]$ (large losses) carries the maximum allowed probability of $\varepsilon = 10\%$. This effect demonstrates a serious drawback of the VaR constraint which bounds only the probability of the losses but does not care about the magnitude of losses.

The comparison of the expected terminal wealths yields that the VaR-optimal portfolio reaches an expected terminal wealth $\mathbf{E}\xi^{\text{VaR}}$ which is very close below of $\mathbf{E}\xi^B$ from the optimal portfolio of the unconstrained problem.

The solution of the representation problem, i.e. the optimal strategy $\theta_t^* = \theta_t^{\text{VaR}}$, is shown in Figure 5.2. Again we give for the sake of comparison the strategies of the other portfolios considered in Figure 5.1, i.e. the strategies $\theta^0 \equiv 0$, $\theta^1 \equiv 1$ and $\theta^B \equiv \frac{\mu-r}{\gamma\sigma^2} = 0.75$, which are constants. Contrary to this, the optimal strategy θ_t^{VaR} is a feedback strategy, i.e. it is a function of time t and the state X_t which is the wealth at time t . Proposition 4.2 (ii) gives an equivalent representation of θ_t^{VaR} in terms of t and the state price density H_t . Since H_t can be expressed in terms of t and S_t the optimal strategy can be written also as a function of t and the stock price S_t at time t . Figure 5.2 shows the dependence of θ_t^{VaR} on the stock price S_t for three instants $t = 0.25T = 5$, $t = 0.75T = 15$ and the time just before the horizon $T = 20$. Moreover the dependence of the EL-optimal strategy on time t and stock price S is visualized.

It can be seen that at the horizon T the optimal strategies θ_T^{VaR} and θ^B of the constrained and unconstrained problem, respectively, coincide for small and large stock prices, i.e. for $S_T \in (0, 0.8639) \cup (1.5759, \infty)$. In case of medium stock prices ($S_T \in (0.8639, 1.5759)$) it holds $\theta_t^{\text{VaR}} \rightarrow 0$ for $t \rightarrow T$, which indicates that in this case the complete capital is invested in the riskless bond, in order to ensure that the terminal wealth exceeds q with the required probability $1 - \varepsilon$. For prior instants t in case of very small stock prices the relation $\theta_t^{\text{VaR}} > \theta_t^B$ can be observed. This seems to be very risky and not rational but corresponds to the above described form of the distribution of the terminal wealth which concentrates the maximum of the allowed probability ε in the region of very small values of X_T , i.e. in a region of large losses.

Measuring the shortfall risk using the shortfall probability leads in case of the VaR-optimal portfolio to

$$\mathbf{P}(X_T^{\text{VaR}} < q) = \varepsilon = 0.1 \quad \text{or} \quad \text{VaR}_{0.1}(X_T - q) = 0.$$

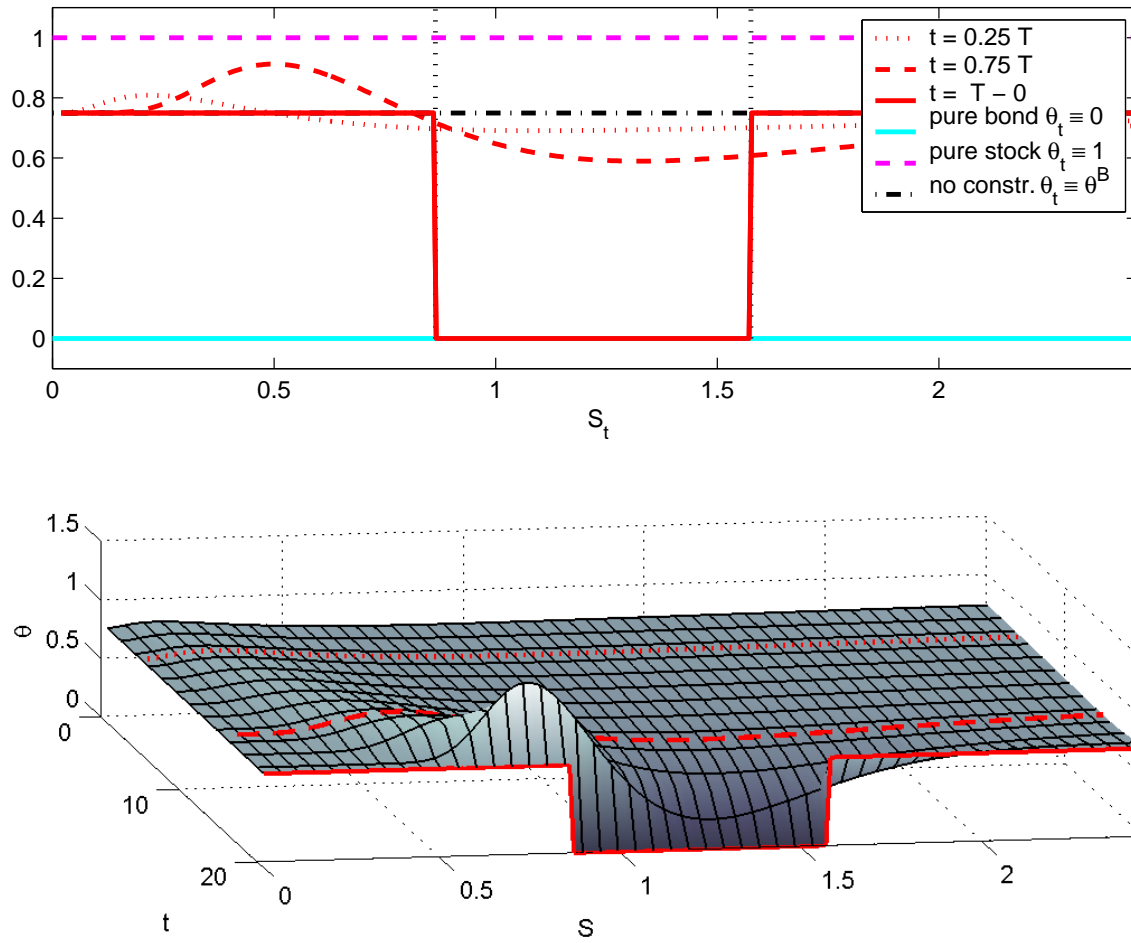


Figure 5.2: VaR-optimal strategy θ^{VaR} as a function of time t and the stock price S

Using the Expected Shortfall as a risk measure one obtains

$$\text{EL}(X_T^{\text{VaR}} - q) = \mathbf{E}(X_T^{\text{VaR}} - q)^- = 0.0926 \dots =: \bar{\varepsilon}.$$

For the sake of comparison we present results for the EL-optimal portfolio which maximizes the expected utility of the terminal wealth $\mathbf{E}u(X_T)$ but satisfies the constraint $\text{EL}(X_T - q) \leq \bar{\varepsilon}$ instead of $\mathbf{P}(X_T < q) = \varepsilon = 0.1$.

Figure 5.3 shows the probability density and distribution function of the EL-optimal terminal wealth $\xi^{\text{EL}} = X_T^{\text{EL}}$. As in the previous example there is a discrete as well as an absolutely continuous part of the distribution. The single point q carries the probability $\mathbf{P}(\underline{h} \leq H_T < \bar{h}) = 0.1073 \dots$. Contrary to the VaR-optimal terminal wealth now there is no gap in the support of the distribution.

While both (VaR- and EL-) optimal portfolios possess the same Expected Loss $\bar{\varepsilon}$ the shortfall probability of the EL-optimal terminal wealth is significantly higher, it holds

$$\mathbf{P}(X_T^{\text{EL}} < q) = 0.1664 \dots > 0.1 = \mathbf{P}(X_T^{\text{VaR}} < q).$$

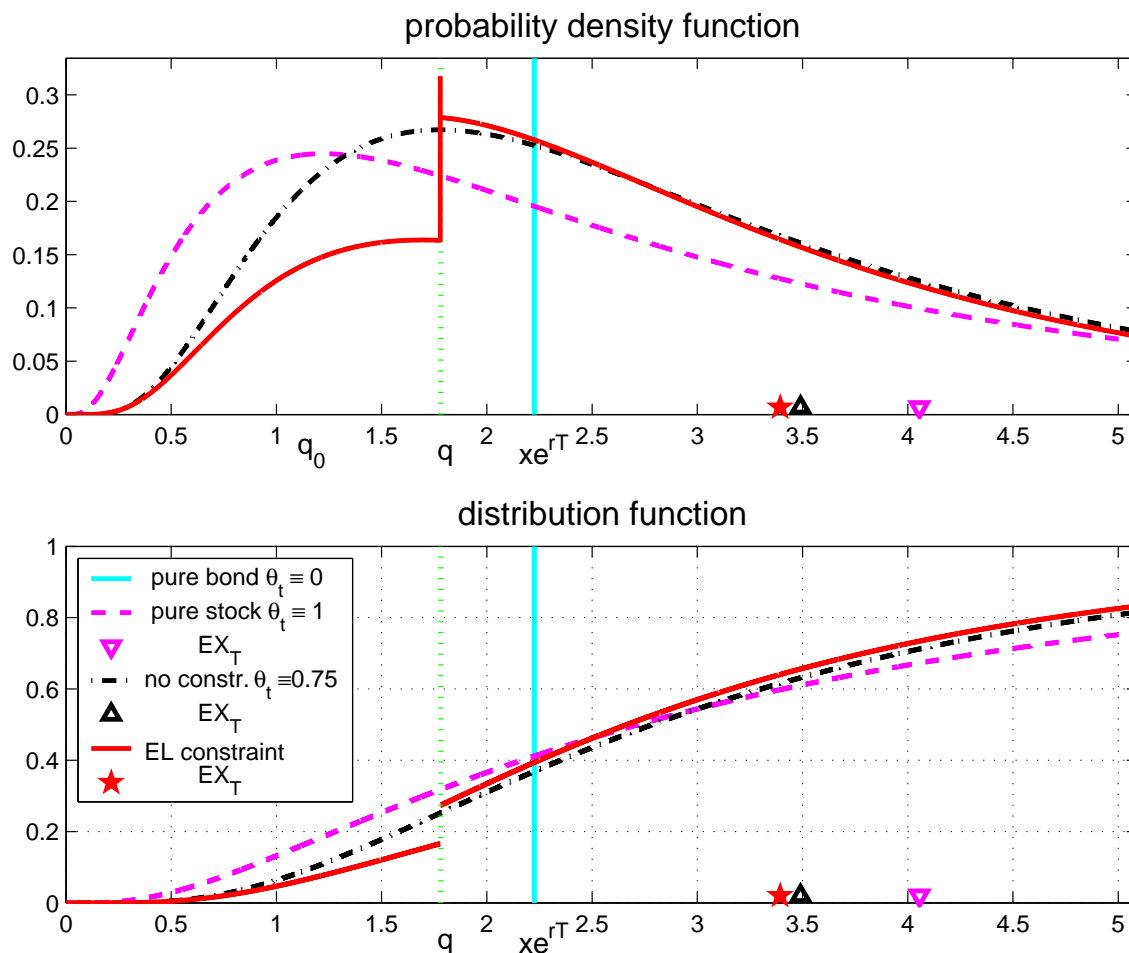


Figure 5.3: Distribution of the EL-optimal terminal wealth

On the other hand there is nearly no difference in the expected terminal wealths since

$$\mathbf{E}X_T^{\text{VaR}} = 3.4097\dots \approx 3.3938\dots = \mathbf{E}X_T^{\text{EL}}.$$

Both values are close to the expected optimal terminal $\mathbf{E}X_T^B = 3.4903$ wealth of the unconstrained problem.

Figure 5.4 is the analogue to Figure 5.2 and shows the EL-optimal strategy θ^{EL} as a function of time t and stock price S . There is a similar behavior for medium and large values of S . Differences can be observed for small values of S and if time t approaches the horizon T . For $t \rightarrow T$ the strategy θ_t^{EL} does not tend to the value θ^B of the optimal strategy of the unconstrained problem but remains larger. Moreover, the region of medium stock prices (1.1432, 1.5870) characterized by $\theta_t^{\text{EL}} \rightarrow 0$ for $t \rightarrow T$ is smaller than in case of the VaR-optimal strategy θ_t^{VaR} , where this region is the interval (0.8639, 1.5759).

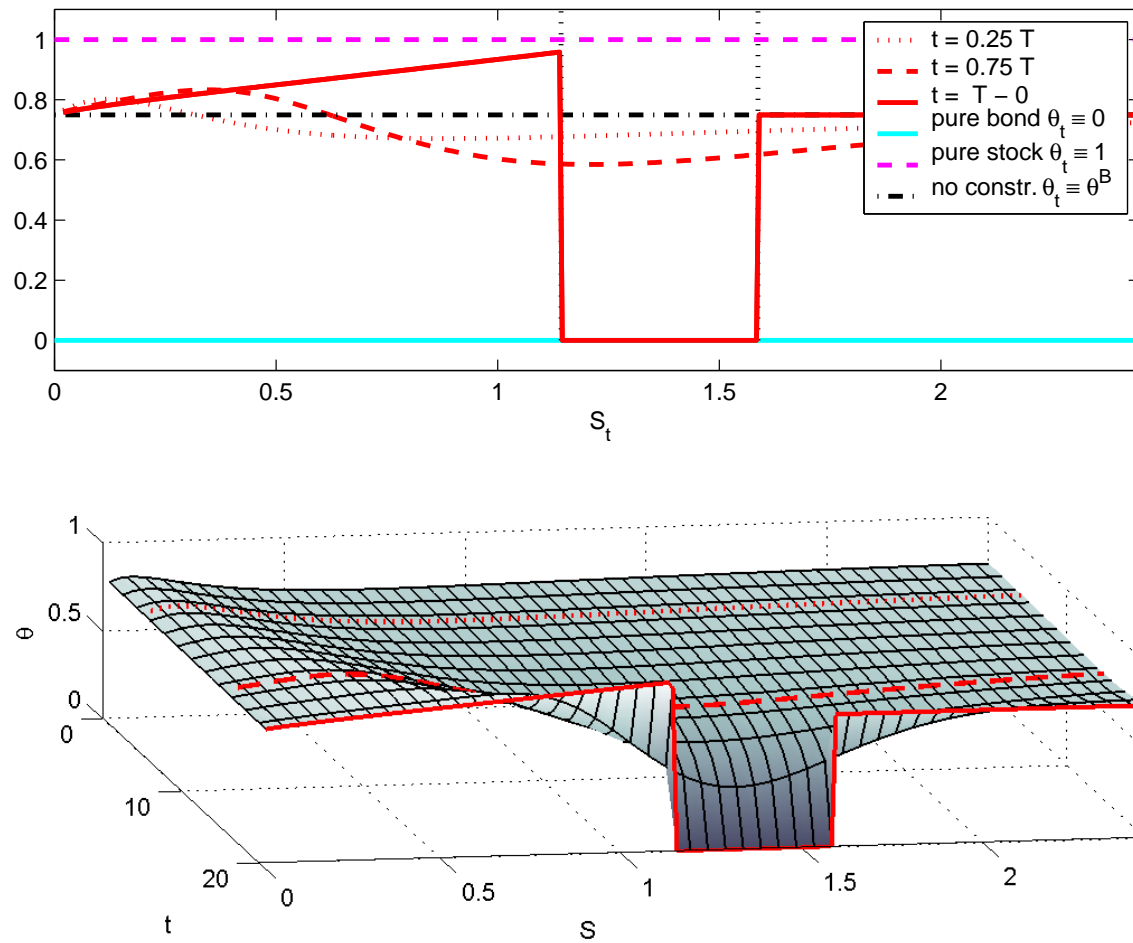


Figure 5.4: EL-optimal strategy θ^{EL} as a function of time t and the stock price S

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