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MIXED INTERFACE PROBLEMS OF THERMOELASTIC PSEUDO–OSCILLATIONS

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Abstract Three–dimensional basic and mixed interface problems of the mathematical theory of thermoelastic pseudo–oscillations are considered for piecewise homogeneous anisotropic bodies. Applying the method of boundary potentials and the theory of pseudodifferential equations existence and uniqueness theorems of solutions are proved in the the space of regular functions $C^{k+\alpha}$ and in the Bessel–potential (H_p^s) and Besov ($B_{p,q}^s$) spaces. In addition to the classical regularity results for solutions to the basic interface problems, it is shown that in the mixed interface problems the displacement vector and the temperature are Hölder continuous with exponent $0 < \alpha < 1/2$.

Introduction

The paper deals with the three–dimensional interface problems of the mathematical theory of thermoelastic pseudo–oscillations for piecewise homogeneous anisotropic bodies. The most general case of the structure of a piecewise homogeneous elastic body under consideration can be mathematicaly described as follows. In three–dimensional Euclidean space $I\!R^3$ we have some closed, smooth, connected non–self–intersecting surfaces $\widetilde{S}_1, \widetilde{S}_2, ..., \widetilde{S}_m$ $(\widetilde{S}_j \cap \widetilde{S}_k = \emptyset,\ \ j \neq k)$. By these surfaces the whole space $I\!R^3$ is devided into several connected domains $\Omega_1, ..., \Omega_l$. Each domain Ω_r is supposed to be filled up by an anisotropic material with corresponding, in general, different thermoelastic coefficients.

Common boundaries of the two distinct materials are called *interfaces* or *contact surfaces* of the piecewise homogeneous elastic body. If some domains represent empty inclusions, then corresponding to them surrounding surfaces are called *boundary surfaces* of the composed elastic body in question. Such type of piecewise homogeneous bodies encounter in many physical, mechanical and engineering applications, and the transmission problems for them have received considerable attention in the scientific literature.

We consider the following two groups of interface conditions:

I. On the whole contact surface there are given

a) jumps of the displacement vector, the temperature, the vector of thermal stresses, and the heat flux (*Problem C*) or

b) jumps of the temperature, the heat flux and the normal components of the displacement and the stress vectors; in additions to these conditions, the limits of either the tangent components of the stress vectors (*Problem G*) or the tangent components of the displacement vectors (*Problem H*) are given from both sides of the interface (cf. [13],[7],[8],[9]).

II. The contact (interface) surface S is devided into two disjoint parts S_1 and S_2 by a regular curve γ: $S = S_1 \cup S_2 \cup \gamma$. On S_1 the conditions of Problem C are prescribed, while on S_2 there are given:

a) the conditions of Problem G (*Problem C-G*) or

b) the conditions of Problem H (*Problem C-H*) or

c) the displacement vector and the temperature (*Problem C-DD*) or

d) the thermal stresses and the heat flux (*Problem C-NN*) or

e) the displacement [stress] vector and jumps of the temperature and the heat flux (*Problem C-DC* [*Problem C-NC*]) (cf. [19], [9], [10]).

Moreover, on the boundary surfaces there are given the displacement or the stress vector, and the temperature or the heat flux (see [22]).

We have studied the above problems by the classical potential methods and the theory of pseudodifferential equations (ΨDE) on manifolds. The investigation has been carried out in the regular $C^{k+\alpha}$ spaces and in the Bessel–potential (H^s_p) and Besov $B^s_{p,q}$ spaces. Besides the uniqueness and existence theorems we have established the regularity properties of solutions near singular points (in a vicinity of the curve γ). Using the embedding theorems it has been shown that solutions to the mixed interface problems (interface crack problems), in general, possess C^α–smoothness with $0 < \alpha < 1/2$.

Similar problems for elliptic equations and, in particular, for the system of classical elasticity theory (for isotropic and anisotrpic piecewise homogeneous bodies) are considered in [2], [4], [26], [1], [19], [12], [10], [20], [23], [21], [27], [18]. The basic and crack type problems of thermoelasticity for homogeneous anisotropic bodies are treated in [17], [16] and [5]. The present investigation generalizes results obtained in these works to the case of regular and mixed interface problems described above.

1 Mathematical Formulation of Problems

1.1. For illustration of the method suggested we consider the following model problems. We assume that the piecewise homogeneous composed anisotropic body consists of two elastic components occupying bounded domain $\Omega_1 = \Omega^+$ and its compliment $\Omega_2 = \Omega^- = I\!R^3 \setminus \overline{\Omega^+}$; $\partial\Omega^\pm = S$, $\overline{\Omega_j} = \Omega_j \cup S$, $j = 1, 2$. Thus the whole space $I\!R^3$ can be considered as a piecewise homogeneous anisotropic body with one contact (interface) surface S.

Let a smooth, connected, non–self–intersecting curve $\gamma \subset S$ devide the surface S into two parts S_1 and S_2: $S = S_1 \cup S_2 \cup \gamma$, $\overline{S}_j = S_j \cup \gamma$, $j = 1, 2$.

For simplicity in what follows we provide that S and γ are C^∞–regular (unless stated otherwise) though actually some finite regularity is sufficient.

1.2. Throughout the paper by $C^k(\Omega^\pm)$, $C^k(\overline{\Omega^\pm})$, $C^k(S)$ and $C^{k+\alpha}(\Omega^\pm)$, $C^{k+\alpha}(\overline{\Omega^\pm})$, $C^{k+\alpha}(S)$ are denoted usual k–smooth and Hölder (k,α)–smooth function spaces with integer $k \geq 0$ and $0 < \alpha < 1$; $W^1_p(\Omega^+)$, $W^1_{p,\mathrm{loc}}(\Omega^-)$, $W^1_{p,\mathrm{comp}}(\Omega^-)$ are well–known Sobolev spaces $(1 < p < \infty)$, while $B^s_{p,q}(\Omega^+)$, $B^s_{p,q,\mathrm{loc}}(\Omega^-)$, $B^s_{p,q}(S)$ and $H^s_p(\Omega^+)$, $H^s_{p,\mathrm{loc}}(\Omega^-)$, $H^s_p(S)$ denote the Besov and Bessel–potential spaces with $s \in I\!\!R$, $1 < p < \infty$, $1 \leq q \leq \infty$ (see [14], [29], [30]).

We need also the following function spaces defined on the submanifolds $S_j \subset S$ with boundary γ:

$$B^s_{p,q}(S_j) = \{f|_{S_j} \, : \, f \in B^s_{p,q}(S)\}, \quad H^s_p(S_j) = \{f|_{S_j} \, : \, f \in H^s_p(S)\},$$

$$\widetilde{B}^s_{p,q}(S_j) = \{f \in B^s_{p,q}(S) \, : \, \mathrm{supp}\, f \subset \overline{S}_j\}, \quad \widetilde{H}^s_p(S_j) = \{f \in H^s_p(S) \, : \, \mathrm{supp}\, f \subset \overline{S}_j\},$$

where $f|_{S_j}$ denotes the restriction of f to S_j.

Let us note here that in the sequel we will use the following notations (when no confusion can be caused by this):

a) if all elements of a vector $v = (v_1, v_2, ..., v_m)$ (matrix $a = ||a_{kj}||_{m\times n}$) belong to one and the same space X, we will write $v \in X$ $(a \in X)$ instead of $v \in [X]^m$ $(a \in [X]_{m\times n})$;

b) if $K : X_1 \times \cdots \times X_m \to Y_1 \times \cdots \times Y_n$ and $X_1 = \cdots = X_m$, $Y_1 = \cdots = Y_n$, we will write $K : X \to Y$ instead of $K : [X]^m \to [Y]^n$.

For the sake of simplicity sometimes we will use also the notation either $[a]_{m\times n}$ or $[a_{kj}]_{m\times n}$ for a matrix $||a_{kj}||_{m\times n}$.

1.3. The system of equations of the linear theory of thermoelastic pseudo–oscillations of homogeneous anisotropic medium reads [22]

$$\begin{aligned} c^r_{kjpq} D_j D_q u^r_p(x,\tau) - \tau^2 \rho_r u^r_k(x,\tau) - \beta^r_{kj} D_j u^r_4(x,\tau) &= 0, \quad k = 1,2,3, \\ \lambda^r_{pq} D_p D_q u^r_4(x,\tau) - \tau c_r u^r_4(x,\tau) - \tau T_r \beta^r_{pq} D_p u^r_q(x,\tau) &= 0, \quad x \in \Omega_r, \quad r = 1,2, \end{aligned} \tag{1.1}$$

where $c^r_{kjpq} = c^r_{pqkj} = c^r_{jkpq}$ are elastic constants, $\lambda^r_{pq} = \lambda^r_{qp}$ are heat conductivity coefficients, c_r is the thermal capacity, T_r is the temperature of the medium in the natural state, $\beta^r_{pq} = \beta^r_{qp}$ are expressed in terms of the thermal and elastic constants, ρ_r is the density of the medium; $u^r = (u^r_1, u^r_2, u^r_3)^\top$ is the displacement vector, u^r_4 is the temperature; $\tau = \sigma - i\omega$ is a complex parameter with $\omega \in I\!\!R$, and $\sigma \in I\!\!R\backslash\{0\}$; $D_p = \partial/\partial x_p$; here and in what follows the summantion over repeated indices is meant from 1 to 3, unless otherwise stated; the supperscript $\top$ denotes transposition.

We note that equations (1.1) are obtained from the corresponding dynamic equations by the formal Laplace transform [22].

In the thermoelasticity theory the stress tensor $\{\sigma^r_{kj}\}$, the strain tensor $\{\varepsilon^r_{kj} = 2^{-1}(D_k u^r_j + D_j u^r_k)\}$ and the temperature field u^r_4 are related by Duhamel-Neumann law

$$\sigma^r_{kj} = c^r_{kjpq} \varepsilon^r_{pq} - \beta^r_{kj} u^r_4;$$

the k-th component of the vector of thermostresses, acting on a surface element with the normal vector $n = (n_1, n_2, n_3)$, is calculated by the formula

$$\sigma^r_{kj} n_j = c^r_{kjpq} \varepsilon^r_{pq} n_j - \beta^r_{kj} n_j u^r_4 = c^r_{kjpq} n_j D_q u^r_p - \beta^r_{kj} n_j u^r_4. \tag{1.2}$$

In order to rewrite the above equations in the matrix form we set

$$
\begin{aligned}
U^r &= (u^r, u_4^r)^\top = (u_1^r, ..., u_4^r)^\top, \\
C^r(D) &= ||C_{kp}^r(D)||_{3\times3}, \quad C_{kp}^r(D) = c_{kjpq}^r D_j D_q, && (1.3)\\
\Lambda^r(D) &= \lambda_{pq}^r D_p D_q, \quad D = \nabla = (D_1, D_2, D_3), && (1.4)\\
T^r(D,n) &= ||T_{kj}^r(D,n)||_{3\times3}, \quad T_{kj}^r(D,n) = c_{kjpq}^r n_j D_q, \\
P^r(D,n) &= ||[T^r(D,n)]_{3\times3}\ ,\ [-\beta_{kj}^r n_j]_{3\times1}||_{3\times4}, \\
A^r(D,\tau) &= \left\| \begin{matrix} [C^r(D) - \tau^2\rho_r I_3]_{3\times3} & [-\beta_{kj}^r D_j]_{3\times1} \\ [-\tau T_r \beta_{kj}^r D_j]_{1\times3} & \Lambda^r(D) - \tau c_r \end{matrix} \right\|_{4\times4}, && (1.5)
\end{aligned}
$$

where $I_m = ||\delta_{kj}||_{m\times m}$ stands for the unit $m\times m$ matrix, δ_{kj} is Kronecker's symbol. Note that $T^r(D,n)$ and $P^r(D,n)$ are called the stress operators of the classical elasticity and the thermoelastisity, respectively.

From (1.2) it follows that

$$[P^r(D,n)U^r]_k = \sigma_{kj}^r n_j, \quad k = 1,2,3.$$

Further, equation (1.1) can be written as follows

$$A^r(D,\tau)U^r(x,\tau) = 0, \quad x \in \Omega_r, \quad r = 1,2. \tag{1.6}$$

From the physical considerations it follows that [22], [16]:
a) the matrix $[\lambda_{pq}^r]_{3\times3}$ is positive definite, i.e.,

$$\Lambda^r(\xi) = \lambda_{pq}^r \xi_p \xi_q \geq \delta_0|\xi|^2 \quad \text{for} \quad \xi \in I\!R^3 \quad \text{with} \quad \delta_0 = \text{const} > 0; \tag{1.7}$$

b) $c_{kjpq}^r e_{kj} e_{pq}$ is a positive definite quadratic form in the real symmetric variables $e_{kj} = e_{jk}$, which implies positive definiteness of the matrix $C^r(\xi)$, for $\xi \in I\!R^3\backslash\{0\}$ defined by (1.3), i.e.,

$$C_{kj}^r(\xi)\eta_j\eta_k \geq \delta_1|\xi|^2|\eta|^2 \quad \text{for} \quad \xi,\eta \in I\!R^3 \quad \text{with} \quad \delta_1 = \text{const} > 0. \tag{1.8}$$

Inequalities (1.7) and (1.8) along with the symmetry properties of the matrices $[\lambda_{pq}^r]_{3\times3}$ and $C^r(\xi)$ yield:

$$C^r(\xi)\eta\cdot\eta = C_{kj}^r(\xi)\eta_j\overline{\eta}_k \geq \delta_1|\xi|^2|\eta|^2 \quad \text{for} \quad \xi \in I\!R^3 \tag{1.9}$$

and

$$\lambda_{pq}^r \eta_p \overline{\eta}_q \geq \delta_0|\eta|^2 \tag{1.10}$$

for arbitrary complex vector $\eta \in C\!\!\!\!I^3$; throughout this paper $a\cdot b = \sum_{k=1}^m a_k\overline{b}_k$ denotes the scalar product of two vectors in $C\!\!\!\!I^m$ where upper bar means complex conjugate.

1.4. From particular problems of mathematical physics and mechanics it is well known that, in general, solutions (or their derivatives) to mixed boundary value problems have singularities near the curves on which different boundary conditions collide (solutions do not belong to $C^1(\overline{\Omega^\pm})$). Because of this fact, on one hand, and to involve a wide class of boundary data, on the other hand, we state the basic and mixed transmission problems in the Sobolev spaces W_p^1 ($W_{p,\text{loc}}^1$) with $p > 1$. If we note that the inclusion $U \in W_p^1(\Omega^+)$ $[U \in W_{p,\text{loc}}^1(\Omega^-)]$

implies $U|_S \in B^{1-1/p}_{p,p}(S)$, then the appearance of Besov spaces in the forthcoming formulation of transmission problems will become transparent. Clearly, here $U|_S$ exists in the trace sense [14], [29]. In what follows we denote by $n(x)$ the outward (to Ω^+) unit normal vector at $x \in S$, and by $l(x)$ and $m(x)$ orthogonal unit vectors in the tangent plane. The orthogonal local co–ordinate system n, l and m at $x \in S$ is oriented as follows: $l \times m = n$, where $\cdot \times \cdot$ denotes the vector product of two vectors. The symbols $[\cdot]^\pm$ denote limits on S from $\Omega^\pm$.

We will study the following problems.

Find a pair of vectors (U^1, U^2) with properties

$$U^1 \in W^1_p(\Omega_1), \quad U^2 \in W^1_{p,\mathrm{loc}}(\Omega_2), \quad |U^2_k(x)| < c|x|^N, \quad |D_p U^2_k(x)| < c|x|^N, \tag{1.11}$$
$$k = \overline{1,4}, \quad p = 1,2,3,$$

where $c = \mathrm{const} > 0$ and N is a certain real number; moreover, U^1 and U^2 satisfy equations (1.6) in the distributional sense in the corresponding domains Ω_1 and Ω_2, respectively, and one of the following transmission conditions on the interface S:

Problem C:

$$[U^1]^+ - [U^2]^- = f \quad \text{on} \quad S, \tag{1.12}$$

$$[\widetilde{B}^1 U^1]^+ - [\widetilde{B}^2 U^2]^- = F \quad \text{on} \quad S, \tag{1.13}$$

$$f = (\widetilde{f}, f_4)^\top, \quad \widetilde{f} = (f_1, f_2, f_3)^\top, \quad F = (\widetilde{F}, F_4)^\top, \quad \widetilde{F} = (F_1, F_2, F_3)^\top;$$

Problem G:

$$[P^1 U^1 \cdot l]^+ = \widetilde{F}^{(+)}_l, \quad [P^1 U^1 \cdot m]^- = \widetilde{F}^{(+)}_m \quad \text{on} \quad S, \tag{1.14}$$

$$[P^2 U^2 \cdot l]^- = \widetilde{F}^{(-)}_l, \quad [P^2 U^2 \cdot m]^- = \widetilde{F}^{(-)}_m \quad \text{on} \quad S, \tag{1.15}$$

$$[u^1 \cdot n]^+ - [u^2 \cdot n]^- = \widetilde{f}_n, \quad [P^1 U^1 \cdot n]^+ - [P^2 U^2 \cdot n]^- = \widetilde{F}_n \quad \text{on} \quad S, \tag{1.16}$$

$$[u^1_4]^+ - [u^2_4]^- = f_4, \quad [\widetilde{\lambda}^1 u^1_4]^+ - [\widetilde{\lambda}^2 u^2_4]^- = F_4 \quad \text{on} \quad S; \tag{1.17}$$

Problem H: conditions (1.16), (1.17) and

$$[u^1 \cdot l]^+ = \widetilde{f}^{(+)}_l, \quad [u^1 \cdot m]^+ = \widetilde{f}^{(+)}_m \quad \text{on} \quad S, \tag{1.18}$$

$$[u^2 \cdot l]^- = \widetilde{f}^{(-)}_l, \quad [u^2 \cdot m]^- = \widetilde{f}^{(-)}_m \quad \text{on} \quad S; \tag{1.19}$$

Problem C–DD:

$$[U^1]^+ - [U^2]^- = f^{(1)}, \quad [\widetilde{B}^1 U^1]^+ - [\widetilde{B}^2 U^2]^- = F^{(1)} \quad \text{on} \quad S_1, \tag{1.20}$$

$$[U^1]^+ = \varphi^{(+)}, \quad [U^2]^- = \varphi^{(-)} \quad \text{on} \quad S_2, \tag{1.21}$$

$$f^{(1)} = (\widetilde{f}^{(1)}, f^{(1)}_4)^\top, \quad \widetilde{f}^{(1)} = (f^{(1)}_1, f^{(1)}_2, f^{(1)}_3)^\top, \quad F^{(1)} = (\widetilde{F}^{(1)}, F^{(1)}_4)^\top,$$
$$\widetilde{F}^{(1)} = (F^{(1)}_1, F^{(1)}_2, F^{(1)}_3)^\top, \quad \varphi^{(\pm)} = (\widetilde{\varphi}^{(\pm)}, \varphi^{(\pm)}_4)^\top, \quad \widetilde{\varphi}^{(\pm)} = (\varphi^{(\pm)}_1, \varphi^{(\pm)}_2, \varphi^{(\pm)}_3)^\top;$$

Problem C–NN: conditions (1.20) on S_1 and

$$[\widetilde{B}^1U^1]^+ = \Phi^{(+)}, \quad [\widetilde{B}^2U^2]^- = \Phi^{(-)} \quad \text{on} \quad S_2, \tag{1.22}$$
$$\Phi^{(\pm)} = (\widetilde{\Phi}^{(\pm)}, \Phi_4^{(\pm)})^\top, \quad \widetilde{\Phi}^{(\pm)} = (\Phi_1^{(\pm)}, \Phi_2^{(\pm)}, \Phi_3^{(\pm)})^\top;$$

Problem C–DC: condition (1.17) on S and

$$[u^1]^+ - [u^2]^- = \widetilde{f}^{(1)}, \quad [P^1U^1]^+ - [P^2U^2]^- = \widetilde{F}^{(1)} \quad \text{on} \quad S_1, \tag{1.23}$$
$$[u^1]^+ = \widetilde{\varphi}^{(+)}, \quad [u^2]^- = \widetilde{\varphi}^{(-)} \quad \text{on} \quad S_2;$$

Problem C–NC: conditions (1.17) on S, (1.23) on S_1 and

$$[P^1U^1]^+ = \widetilde{\Phi}^{(+)}, \quad [P^2U^2]^- = \widetilde{\Phi}^{(-)} \quad \text{on} \quad S_2;$$

Problem C–G: conditions (1.17) on S, (1.23) on S_1 and

$$[u^1\cdot n]^+ - [u^2\cdot n]^- = \widetilde{f}_n^{(2)}, \quad [P^1U^1\cdot n]^+ - [P^2U^2\cdot n]^- = \widetilde{F}_n^{(2)} \quad \text{on} \quad S_2, \tag{1.24}$$
$$[P^1U^1\cdot l]^+ = \widetilde{\Phi}_l^{(+)}, \quad [P^1U^1\cdot m]^+ = \widetilde{\Phi}_m^{(+)} \quad \text{on} \quad S_2,$$
$$[P^2U^2\cdot l]^- = \widetilde{\Phi}_l^{(-)}, \quad [P^2U^2\cdot m]^- = \widetilde{\Phi}_m^{(-)} \quad \text{on} \quad S_2;$$

Problem C–H: conditions (1.17) on S, (1.23) on S_1, (1.24) on S_2, and

$$[u^1\cdot l]^+ = \widetilde{\varphi}_l^{(+)}, \quad [u^1\cdot m]^+ = \widetilde{\varphi}_m^{(+)} \quad \text{on} \quad S_2,$$
$$[u^2\cdot l]^- = \widetilde{\varphi}_l^{(-)}, \quad [u^2\cdot m]^- = \widetilde{\varphi}_m^{(-)} \quad \text{on} \quad S_2,$$

where

$$\widetilde{B}^r(D,n) = \left\| \begin{matrix} [T^r(D,n)]_{3\times 3} & [-\beta^r_{kj}n_j]_{3\times 1} \\ [0]_{1\times 3} & \widetilde{\lambda}^r(D,n) \end{matrix} \right\|_{4\times 4}, \tag{1.25}$$

$$\widetilde{\lambda}^r(D,n) = \frac{1}{T_r}\lambda^r_{pq}n_pD_q = \frac{1}{T_r}\lambda^r(D,n). \tag{1.26}$$

It is evident that first order derivatives of functions from $W^1_p(\Omega^+)$ and $W^1_{p,\mathrm{loc}}(\Omega^-)$ belong to $L_p(\Omega^+)$ and $L_{p,\mathrm{loc}}(\Omega^-)$, respectively, and they have no traces on S. However, for the vector functions (1.11), satisfying the condition

$$A^1(D,\tau)U^1 \in L_p(\Omega^+), \quad A^2(D,\tau)U^2 \in L_{p,\mathrm{loc}}(\Omega^-),$$

we can define correctly the functionals

$$[P^1U^1]^+_k, \quad [\lambda^1(D,n)u^1_4]^+, \quad [P^2U^2]^-_k, \quad [\lambda^2(D,n)u^2_4]^- \in B^{-1/p}_{p,p}(S)$$

by means of Green's formulae [17], [12]

$$\langle [B^1U^1]^+, [V^1]^+\rangle_S = \int\limits_{\Omega^+} A^1(D,\tau)U^1\cdot V^1dx + \int\limits_{\Omega^+} E^1(U^1,V^1)dx, \tag{1.27}$$

$$\langle [B^2U^2]^-, [V^2]^-\rangle_S = \int\limits_{\Omega^-} A^2(D,\tau)U^2\cdot V^2dx + \int\limits_{\Omega^-} E^2(U^2,V^2)dx, \tag{1.28}$$

where

$$V^r = (\tilde{v}^r, v_4^r)^\top, \quad \tilde{v}^r = (v_1^r, v_2^r, v_3^r)^\top, \quad V^1 \in W^1_{p'}(\Omega^+), \quad V^2 \in W^1_{p',\mathrm{comp}}(\Omega^-), \quad p' = \frac{p}{p-1},$$

$$\begin{aligned} B^r(D,n) &= \left\| \begin{matrix} [T^r(D,n)]_{3\times 3} & [-\beta^r_{kj} n_j]_{3\times 1} \\ [0]_{1\times 3} & \lambda^r(D,n) \end{matrix} \right\|_{4\times 4}, \qquad (1.29) \\ E^r(U^r, V^r) &= c^r_{kjpq} D_p u^r_q \, \overline{D_k v^r_j} + \tau^2 u^r_k \overline{v^r_k} - \beta^r_{kj} u^r_4 \, \overline{D_k v^r_j} \\ &+ \lambda^r_{pq} D_q u^r_4 \, \overline{D_p v^r_4} + c_r \tau u^r_4 \overline{v^r_4} + \tau T_r \beta^r_{kj} D_k u^r_j \, \overline{v^r_4}. \end{aligned}$$

Clearly, $[V^r]^\pm \in B^{1-1/p'}_{p',p'}(S) = B^{1/p}_{p',p'}(S)$. The symbol $\langle \cdot, \ \cdot \rangle_S$ in formulae (1.27) and (1.28) denotes the duality between $B^{-1/p}_{p,p}(S)$ and $B^{1/p}_{p',p'}(S)$, which for smooth vectors f and g reads as

$$\langle f, g \rangle_S = \int\limits_S f \cdot \overline{g} \, dS$$

(for details see [5]).

Note that

$$\widetilde{B}^r(D,n) = I_4^{(r)} B^r(D,n) \quad \text{with} \quad I_4^{(r)} = \mathrm{diag}\,\{1, 1, 1, T_r^{-1}\}, \qquad (1.30)$$

where $\mathrm{diag}\,\{a_1, \cdots, a_n\}$ denotes a diagonal $n \times n$ matrix with the entries $a_1, \cdots, a_n$ on the main diagonal. In the above formulations of interface problems the conditions for the displacement vector u^r and the temperature u^r_4 are understood in the trace sense, while the conditions for the stress vector $P^r U^r$ and the heat flux $\lambda^r(D,n) u^r_4$ are to be considered in the functional sense which has just been described. Therefore the functions given on the interface S are to meet the following natural restrictions stipulated by (1.11):

$$\begin{gathered} f_k, \widetilde{f}_l^{(\pm)}, \widetilde{f}_m^{(\pm)}, \widetilde{f}_n \in B^{1-1/p}_{p,p}(S), \quad F_k, \widetilde{F}_l^{(\pm)}, \widetilde{F}_m^{(\pm)}, \widetilde{F}_n \in B^{-1/p}_{p,p}(S), \\ f_k^{(1)} \in B^{1-1/p}_{p,p}(S_1), \quad F_k^{(1)} \in B^{-1/p}_{p,p}(S_1), \quad \varphi_k^{(\pm)}, \widetilde{f}_n^{(2)}, \widetilde{\varphi}_l^{(\pm)}, \widetilde{\varphi}_m^{(\pm)} \in B^{1-1/p}_{p,p}(S_2), \\ \Phi_k^{(\pm)}, \widetilde{F}_n^{(2)}, \widetilde{\Phi}_l^{(\pm)}, \widetilde{\Phi}_m^{(\pm)} \in B^{-1/p}_{p,p}(S_2), \quad k = \overline{1,4}. \end{gathered} \qquad (1.31)$$

The inclusions (1.11) imply also the following compatibility conditions for the given functions:

a) in Problem C–DD:

$$f^0 = \begin{cases} f^{(1)} & \text{on} \quad S_1, \\ \varphi^{(+)} - \varphi^{(-)} & \text{on} \quad S_2, \end{cases} \qquad f^0 \in B^{1-1/p}_{p,p}(S); \qquad (1.32)$$

b) in Problem C–NN:

$$F^0 = \begin{cases} F^{(1)} & \text{on} \quad S_1, \\ \Phi^{(+)} - \Phi^{(-)} & \text{on} \quad S_2, \end{cases} \qquad F^0 \in B^{-1/p}_{p,p}(S); \qquad (1.33)$$

c) in Problem C–DC:

$$\tilde{f}^{(0)} = \begin{cases} \tilde{f}^{(1)} & \text{on} \quad S_1, \\ \tilde{\varphi}^{(+)} - \tilde{\varphi}^{(-)} & \text{on} \quad S_2, \end{cases} \qquad \tilde{f}^0 \in B_{p,p}^{1-1/p}(S); \tag{1.34}$$

d) in Problem C–NC:

$$\tilde{F}^{(0)} = \begin{cases} \tilde{F}^{(1)} & \text{on} \quad S_1, \\ \tilde{\Phi}^{(+)} - \tilde{\Phi}^{(-)} & \text{on} \quad S_2, \end{cases} \qquad \tilde{F}^{(0)} \in B_{p,p}^{-1/p}(S); \tag{1.35}$$

e) in Problem C–G:

$$\tilde{f}_n^{(0)} = \begin{cases} \tilde{f}^{(1)} \cdot n & \text{on} \quad S_1, \\ \tilde{f}_n^{(2)} & \text{on} \quad S_2, \end{cases} \qquad \tilde{f}_n^{(0)} \in B_{p,p}^{1-1/p}(S), \tag{1.36}$$

$$\tilde{F}^{(0)} = \begin{cases} \tilde{F}^{(1)} & \text{on} \quad S_1, \\ [\tilde{\Phi}_l^{(+)} - \tilde{\Phi}_l^{(-)}]l + [\tilde{\Phi}_m^{(+)} - \tilde{\Phi}_m^{(-)}]m + \tilde{F}_n^{(2)}n & \text{on} \quad S_2, \end{cases} \qquad \tilde{F}^{(0)} \in B_{p,p}^{-1/p}(S); \tag{1.37}$$

f) in Problem C–H:

$$\tilde{f}^{(0)} = \begin{cases} \tilde{f}^{(1)} & \text{on} \quad S_1, \\ [\tilde{\varphi}_l^{(+)} - \tilde{\varphi}_l^{(-)}]l + [\tilde{\varphi}_m^{(+)} - \tilde{\varphi}_m^{(-)}]m + \tilde{f}_n^{(2)}n & \text{on} \quad S_2, \end{cases} \qquad \tilde{f}^{(0)} \in B_{p,p}^{1-1/p}(S), \tag{1.38}$$

$$\tilde{F}_n^{(0)} = \begin{cases} \tilde{F}^{(1)} \cdot n & \text{on} \quad S_1, \\ \tilde{F}_n^{(2)} & \text{on} \quad S_2, \end{cases} \qquad \tilde{F}_n^{(0)} \in B_{p,p}^{-1/p}(S). \tag{1.39}$$

In the sequel all these conditions are supposed to be fulfilled. Note that (1.32), (1.34), (1.36), and (1.38) [(1.33), (1.35), (1.37) and (1.39)] hold for arbitrary functions satisfying (1.31) with $1 < p < 2$ $\quad [2 < p < \infty]$, which follows from the multiplication properties of Besov spaces (see [30], Ch. 3, Section 3.3.2).

1.5. For the domains of general structure, described in Introduction, the basic and mixed transmission problems mathematically could be formulated quite similarly: on the contact surfaces the conditions one of the interface problems stated above are assigned, while on the boundary of the composed body the conditions of either basic or mixed boundary value problemes are given (for detailes concerning the formulation such type of problems see [13]). We observe that all principal difficulties arising in the study of problems for the composed bodies of general structure are presented in the above model problems as well.

We will investigate the above problems by making use of the boundary integral (pseudo-differential) equation methods. To this end we need some auxiliary material about properties of pseudo–oscillation potentials and operators generated by them. For the readers convenience all necessary results are collected together in the next section.

2 Properties of Pseudo–Oscillation Potentials

2.1. Fundamental matrix. Denote by $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ the generalized Fourier and the inverse Fourier transforms which for summable functions are defined as follows

$$\mathcal{F}_{x\to\xi}[f] = \int\limits_{R^n} f(x)e^{ix\xi}dx, \quad \mathcal{F}^{-1}_{\xi\to x}[g] = (2\pi)^{-n}\int\limits_{R^n} g(\xi)e^{-ix\xi}d\xi.$$

Further, let $\mathcal{A}^r(\xi,\tau)$ be the symbol matrix of the differential operator $A^r(D,\tau)$ (see (1.5)):

$$\mathcal{A}^r(\xi,\tau) = A^r(-i\xi,\tau), \quad \xi\in I\!R^3.$$

LEMMA 2.1 *Let* $\tau=\sigma-i\omega$, $\mathrm{Re}\,\tau=\sigma>0$, $\omega\in I\!R$, *and* $\xi\in I\!R^3$. *Then* $\det\mathcal{A}^r(\xi,\tau)\neq 0$ *and* $[\mathcal{A}^r(\cdot,\tau)]^{-1}\in L_2(I\!R^3)$.

Proof. The first part of the lemma follows from Lemma 1.1 of [12], while the second part is a consequence of the inequality

$$\{[\mathcal{A}^r(\xi,\tau)]^{-1}\}_{kj} \le \frac{c(\sigma)}{1+|\xi|^2} \quad \text{for} \quad \xi\in I\!R^3,$$

where the positive constant $c(\sigma)$ does not depend on ξ (it depends on σ and the thermoelastic constants of the medium in question). ∎

Applying Lemma 2.1 we can construct the fundamental matrix of the operator $A^r(D,\tau)$

$$\Psi^r(x,\tau) = \mathcal{F}^{-1}_{\xi\to x}[(\mathcal{A}^r(\xi,\tau))^{-1}] = \frac{1}{(2\pi)^3}\lim_{R\to\infty}\int\limits_{|\xi|\le R}[\mathcal{A}^r(\xi,\tau)]^{-1}e^{-ix\xi}d\xi, \quad x\in I\!R^3\setminus\{0\}.$$

Let $\Gamma^r(x)$ and $\gamma^r(x)$ be the homogeneous (of order -1) fundamental matrix and fundamental function of the differential operators $C^r(D)$ and $\Lambda^r(D)$, respectively, (see (1.3) and (1.4)):

$$\Gamma^r(x) = \mathcal{F}^{-1}_{\xi\to x}[(C^r(-i\xi))^{-1}] = -(8\pi^2|x|)^{-1}\int\limits_0^{2\pi}[C^r(a\eta)]^{-1}d\eta, \tag{2.1}$$

$$\gamma^r(x) = \mathcal{F}^{-1}_{\xi\to x}[(\Lambda^r(-i\xi))^{-1}] = -[4\pi|\Lambda_1^r|^{1/2}(\Lambda_1^r x\cdot x)^{1/2}]^{-1}, \quad x\in I\!R^3\setminus\{0\}, \tag{2.2}$$

where $a=||a_{kj}||_{3\times3}$ is an orthogonal matrix with property $a^\top x^\top=(0,0,|x|)^\top$, $\eta=(\cos\varphi,\sin\varphi,0)^\top$; $\Lambda_1^r=||\lambda_{pq}^r||_{3\times3}$, $|\Lambda_1^r|=\det\Lambda_1^r$ (see [16], [15]).

It is evident that

$$\Gamma^r(x)=\overline{\Gamma^r(x)}=\Gamma^r(-x)=[\Gamma^r(x)]^\top, \quad \Gamma^r(tx)=t^{-1}\Gamma^r(x),$$

$$\gamma^r(x)=\gamma^r(-x), \quad \gamma^r(tx)=t^{-1}\gamma^r(x),$$

for any $x\in I\!R^3\setminus\{0\}$ and $t>0$.

LEMMA 2.2 *Let* $\operatorname{Re}\tau=\sigma>0$. *Then entries of the matrix* $\Psi^r(\cdot,\tau)$ *belong to* $C^\infty(I\!R^3\setminus\{0\})$ *and together with all derivatives decrease more rapidly than any negative power of* $|x|$ *as* $|x|\to+\infty$.

In a neighbourhood of the origin ($|x|<1/2$) *the following inequalities*

$$|D^\beta\Psi^r_{kj}(x,\tau)-D^\beta\Psi^r_{kj}(x)|<c\,\varphi^{(kj)}_{|\beta|}(x)$$

hold, where $\beta=(\beta_1,\beta_2,\beta_3)$ *is an arbitrary multi–index,* $|\beta|=\beta_1+\beta_2+\beta_3$; *here*

$$\Psi^r(x)=\left\|\begin{array}{cc}[\Gamma^r(x)]_{3\times3} & [0]_{3\times1}\\ [0]_{1\times3} & \gamma^r(x)\end{array}\right\|_{4\times4}; \tag{2.3}$$

$$\varphi^{(kj)}_0(x)=1,\ \varphi^{(kj)}_1(x)=-\ln|x|,\ \varphi^{(kj)}_l(x)=|x|^{1-l},\ l\ge2,$$
$$\text{for}\ \ 1\le k,j\le3\ \ \text{and}\ \ k=j=4;$$
$$\varphi^{(k4)}_0(x)=\varphi^{(4k)}_0(x)=-\ln|x|,\ \varphi^{(k4)}_m(x)=\varphi^{(4k)}_m(x)=|x|^{-m},\ \ m\ge1,$$
$$\text{for}\ k=1,2,3.$$

Proof. Note that

$$D^\beta[\mathcal{A}^r(\xi,\tau)]^{-1}=O([1+|\xi|]^{-2-|\beta|}),$$

and

$$[\mathcal{A}^r(\xi,\tau)]^{-1}=\left\|\begin{array}{cc}[(C^r(-i\xi))^{-1}]_{3\times3} & [0]_{3\times1}\\ [0]_{1\times3} & [\Lambda^r(-i\xi)]^{-1}\end{array}\right\|+\left\|\begin{array}{cc}[O(|\xi|^{-4})]_{3\times3} & [O(|\xi|^{-3})]_{3\times1}\\ [O(|\xi|^{-3})]_{1\times3} & O(|\xi|^{-4})\end{array}\right\|,$$

hold for sufficiently large $|\xi|$.

Now the proof follows from Lemma 2.1 and equations (2.1), (2.2). ∎

Denote by $A^{*r}(D,\tau)$ the operator formally adjoint to $A^r(D,\tau)$:

$$A^{*r}(D,\tau)=\overline{[A^r(-D,\tau)]^\top}.$$

Clearly, the correspoding fundamental matrix is $\Psi^{*r}(x,\tau)=\overline{[\Psi^r(-x,\tau)]^\top}$.

By standard arguments we can derive the general integral representation formula for a regular solution of the equation (1.6)

$$\begin{aligned}U^r(x)\ &=\ (-1)^{r+1}\int_S\{Q^r(D_y,n(y),\tau)[\Psi^r(x-y,\tau)]^\top\}^\top[U^r(y)]^\pm dS_y\\ &-\ (-1)^{r+1}\int_S\Psi^r(x-y,\tau)[B^r(D_y,n)U^r(y)]^\pm dS_y,\quad x\in\Omega_r,\end{aligned} \tag{2.4}$$

where the operator $B^r(D,n)$ is defined by (1.29),

$$Q^r(D_y,n,\tau)=\left\|\begin{array}{cc}[T^r(D_y,n)]_{3\times3} & [\tau T_r\beta^r_{kj}n_j]_{3\times1}\\ [0]_{1\times3} & \lambda^r(D_y,n)\end{array}\right\|_{4\times4};$$

here U^2 is supposed to satisfy the inequality (1.11) (for details see [17], [12]).

From Lemma 2.2 and equations (2.4) and (1.11) it follows that any solution of equation (1.6) actually decrease, together with all its derivatives, more rapidly than any negative power of $|x|$ as $|x| \to \infty$. Therefore for any solution U^2 of (1.6) in Ω_2 the condition $U^2 \in W^1_{p,\mathrm{loc}}(\Omega_2)$ implies $U^2 \in W^1_p(\Omega_2)$ with $1 < p < \infty$. It is also evident that the vector $U^r \in W^1_p(\Omega_r)$ defined by (2.4) belongs to $C^\infty(\Omega_r)$.

2.2. Potential operators of pseudo–oscillations. Let

$$
\begin{aligned}
V^r(g)(x) &:= \int_S \Psi^r(x-y,\tau)g(y)dS_y, \ x \in I\!R^3 \backslash S, \\
W^r(g)(x) &:= \int_S \{Q^r(D_y, n(y), \tau)[\Psi^r(x-y,\tau)]^\top\}^\top g(y)dS_y, \ x \in I\!R^3 \backslash S,
\end{aligned}
$$

where $g = (\tilde{g}, g_4)^\top$ and $\tilde{g} = (g_1, g_2, g_3)^\top$, be the generalized single and double layer potentials.

Further, let us introduce the boundary integral (pseudodifferential) operators on S:

$$\mathcal{H}^r g(z) := \int_S \Psi^r(z-y,\tau)g(y)dS_y, \ z \in S, \tag{2.5}$$

$$\mathcal{K}^r g(z) := \int_S B^r(D_z, n(z))\Psi^r(z-y,\tau)g(y)dS_y, \ z \in S, \tag{2.6}$$

$$\tilde{\mathcal{K}}^r g(z) := \int_S \{Q^r(D_y, n(z), \tau)[\Psi^r(z-y,\tau)]^\top\}^\top g(y)dS_y, \ z \in S, \tag{2.7}$$

$$\mathcal{L}^r_\pm g(z) := \lim_{\Omega^\pm \ni x \to z \in S} B^r(D_x, n(z))W^r(g)(x). \tag{2.8}$$

Clearly, these operators are generated by the above potentials. Their mapping and Fredholm properties are described by the next two lemmata.

LEMMA 2.3 *[17], [5] Let $k \geq 0$ be an integer, $0 < \alpha < \alpha' < 1$, and $S \in C^{k+1+\alpha'}$. Then*

i) *the operators*

$$V^r : C^{k+\alpha}(S) \to C^{k+1+\alpha}(\overline{\Omega^\pm}), \tag{2.9}$$

$$W^r : C^{k+\alpha}(S) \to C^{k+\alpha}(\overline{\Omega^\pm}), \tag{2.10}$$

are bounded, and

$$[V^r(g)]^+ = [V^r(g)]^- \equiv \mathcal{H}^r g, \quad g \in C^\alpha(S), \tag{2.11}$$

$$[W^r(g)]^\pm = [\pm 2^{-1} I_4 + \tilde{\mathcal{K}}^r]g, \quad g \in C^\alpha(S), \tag{2.12}$$

$$[B^r(D,n)V^r(g)]^\pm = [\mp 2^{-1} I_4 + \mathcal{K}^r]g, \quad g \in C^\alpha(S), \tag{2.13}$$

$$\mathcal{L}^r_+ g = \mathcal{L}^r_- g \equiv \mathcal{L}^r g, \quad g \in C^{1+\alpha}(S); \tag{2.14}$$

ii) *the operators*

$$\mathcal{H}^r \ : \ C^{k+\alpha}(S) \to C^{k+1+\alpha}(S), \tag{2.15}$$

$$(\pm 2^{-1} I_4 + \tilde{\mathcal{K}}^r), \ (\pm 2^{-1} I_4 + \mathcal{K}^r) : C^{k+\alpha}(S) \to C^{k+\alpha}(S), \tag{2.16}$$

$$\mathcal{L}^r : C^{k+1+\alpha}(S) \to C^{k+\alpha}(S) \;\; [S \in C^{k+2+\alpha'}], \tag{2.17}$$

are bounded;

iii) *operators* (2.9), (2.10), *and* (2.15)–(2.17) *can be extended to the following bounded operators*

$$\begin{aligned} V^r \; &: \; B^s_{p,p}(S) \to H^{s+1+1/p}_p(\Omega^\pm) \;\; [B^s_{p,q}(S) \to B^{s+1+1/p}_{p,q}(\Omega^\pm)], \\ W^r \; &: \; B^s_{p,p}(S) \to H^{s+1/p}_p(\Omega^\pm) \;\; [B^s_{p,q}(S) \to B^{s+1/p}_{p,q}(\Omega^\pm)], \\ \mathcal{H}^r \; &: \; B^s_{p,q}(S) \to B^{s+1}_{p,q}(S) \;\; [H^s_p(S) \to H^{s+1}_p(S)], \end{aligned} \tag{2.18}$$

$$(\pm 2^{-1} I_4 + \mathcal{K}^r),\ (\pm 2^{-1} I_4 + \widetilde{\mathcal{K}}^r) \; : \; B^s_{p,q}(S) \to B^s_{p,q}(S) \;\; [H^s_p(S) \to H^s_p(S)], \tag{2.19}$$

$$\mathcal{L}^r \; : \; B^s_{p,q}(S) \to B^{s-1}_{p,q}(S) \;\; [H^s_p(S) \to H^{s-1}_p(S)], \tag{2.20}$$

for arbitrary $s \in I\!R$, $1 < p < \infty$, *and* $1 \le q \le \infty$, *provided* $S \in C^\infty$)*; for these extended operators formulae* (2.11) − (2.14) *remain valid in corresponding spaces.*

LEMMA 2.4 *[17], [12], [5] Operators* $\mathcal{H}^r$, $(\pm 2^{-1} I_4 + \mathcal{K}^r)$, $(\pm 2^{-1} I_4 + \widetilde{\mathcal{K}}^r)$ *and* $\mathcal{L}^r$ *are elliptic* ΨDOs *of order* -1, 0, 0 *and* 1, *respectively. The principal symbol matrices of* $-\mathcal{H}^r$ *and* $\mathcal{L}^r$ *are positive definite. Operators* (2.15), (2.16), (2.17) [(2.18), (2.19), (2.20)] *are isomorphisms. In particular,* $(\mathcal{H}^r)^{-1}$ *is a singular integro–differential operator.*

3 Problem C

3.1 The investigation of Problem C we begin with the following uniqueness

THEOREM 3.1 *The homogeneous Problem C* ($f = 0$, $F = 0$) *has only the trivial solution in the class of regular vectors.*

Proof. Let a pair (U^1, U^2), where $U^1 \in C^1(\overline{\Omega^+})$ and $U^2 \in C^1(\overline{\Omega^-})$, be an arbitrary solution of the homogeneous Problem C. Further, let us write the following Green formulae [17],[5]

$$\begin{aligned} &\int\limits_{\Omega_r} \{[A^r(D,\tau)U^r]_k \overline{u^r_k} + \tfrac{1}{\overline{\tau} T_r} \overline{[A^r(D,\tau)U^r]_4}\, u_4\} dx \\ &= (-1)^{r+1} \int\limits_S \{[B^r(D,n)U^r]^\pm_k \overline{[u^r_k]^\pm} + \tfrac{1}{\overline{\tau}} [u^r_4]^\pm \overline{[\widetilde{\lambda}^r(D,n)u^r_4]^\pm}\} dS \\ &- \int\limits_{\Omega_r} \{c^r_{kjpq} D_p u^r_q \overline{D_k u^r_j} + \rho_r \tau^2 |u^r|^2 + \tfrac{1}{\overline{\tau} T_r} \lambda^r_{pq} D_q u^r_4 \overline{D_p u^r_4} + \tfrac{c_r}{T_r} |u^r_4|^2\} dx, \quad r = 1, 2, \end{aligned} \tag{3.1}$$

where $A^r(D,\tau)$, $B^r(D,n)$, and $\widetilde{\lambda}^r(D,n)$ are defined by (1.5), (1.29), and (1.26), respectively; the superscript $+[-]$ corresponds to $r = 1$ $[r = 2]$.

Due to the homogeneity of the problem in question the equations (3.1) yield

$$\sum_{r=1}^{2} \int\limits_{\Omega_r} \{c^r_{kjpq} D_p u^r_q \overline{D_k u^r_j} + \rho_r \tau^2 |u^r|^2 + \frac{1}{\overline{\tau} T_r} \lambda^r_{pq} D_q u^r_4 \overline{D_p u^r_4} + \frac{c_r}{T_r} |u^r_4|^2\} dx = 0. \tag{3.2}$$

Recalling that $\tau = \sigma - i\omega$, and separating the real and imaginary parts, we obtain

$$\sum_{r=1}^{2} \int\limits_{\Omega_r} \{c^r_{kjpq} D_p u^r_q \overline{D_k u^r_j} + \rho_r(\sigma^2 - \omega^2)|u^r|^2$$
$$+\frac{\sigma}{|\tau^2|T_r} \lambda^r_{pq} D_q u^r_4 \, \overline{D_p u^r_4} + \frac{c_r}{T_r}|u^r|^2\} dx = 0,$$
$$\omega \sum_{r=1}^{2} \int\limits_{\Omega_r} \left\{2\sigma\rho_r|u^r|^2 + \frac{1}{|\tau|^2 T_r} \widetilde{\lambda}^r_{pq} D_q u^r_4 \overline{D_p u^r_4}\right\} dx = 0,$$

whence, by (1.9) and (1.10), $U^r = 0$ in Ω_r follows for arbitrary τ with $\operatorname{Re}\tau = \sigma > 0$. ∎

COROLLARY 3.2 . *Let $U^1 \in W^1_2(\Omega_1)$ and $U^2 \in W^1_2(\Omega_2)$ solve the homogeneous Problem C. Then $U^r = 0$ in Ω_r, $r = 1, 2$.*

3.2. We look for a solution to Problem C in the form of the single layer potentials

$$U^1(x) = V^1[(\mathcal{H}^1)^{-1} g^1](x), \quad x \in \Omega_1, \tag{3.3}$$
$$U^2(x) = V^2[(\mathcal{H}^2)^{-1} g^2](x), \quad x \in \Omega_2, \tag{3.4}$$

where $g^r = (\widetilde{g}^r, g^r_4)^\top$, $\widetilde{g}^r = (g^r_1, g^r_2, g^r_3)^\top$, $r = 1, 2$, are unknown densities and $(\mathcal{H}^r)^{-1}$ is the operator inverse to $\mathcal{H}^r$ (see Lemma 2.2).

Due to Lemma 2.3, the transmission conditions (1.12) and (1.13) lead to the following system of boundary equations on S:

$$g^1 - g^2 = f, \tag{3.5}$$
$$I^{(1)}_4(-2^{-1}I_4 + \mathcal{K}^1)(\mathcal{H}^1)^{-1}g^1 - I^{(2)}_4(2^{-1}I_4 + \mathcal{K}^2)(\mathcal{H}^2)^{-1}g^2 = F, \tag{3.6}$$

where $I^{(r)}_4$ and $\mathcal{K}^r$, $r = 1, 2$, are defined by (1.30) and (2.6), respectively.

Let

$$\mathcal{N}_1 = (-2^{-1}I_4 + \mathcal{K}^1)(\mathcal{H}^1)^{-1}, \quad \mathcal{N}_2 = -(2^{-1}I_4 + \mathcal{K}^2)(\mathcal{H}^2)^{-1},$$
$$\widetilde{\mathcal{N}}_1 = I^{(1)}_4 \mathcal{N}_1, \quad \widetilde{\mathcal{N}}_2 = I^{(2)}_4 \mathcal{N}_2, \quad \widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_1 + \widetilde{\mathcal{N}}_2. \tag{3.7}$$

Then equations (3.5) and (3.6) yield:

$$g^1 = f + g^2, \tag{3.8}$$
$$\widetilde{\mathcal{N}} g^2 = F - \widetilde{\mathcal{N}}_1 f. \tag{3.9}$$

Now we will study properties of the boundary operators $\widetilde{\mathcal{N}}_1$, $\widetilde{\mathcal{N}}_2$, and $\widetilde{\mathcal{N}}$.

LEMMA 3.3 *Let $S \in C^{k+2+\alpha'}$, $k \geq 0$ be an integer and $0 < \alpha' \leq 1$. Then*

$$\widetilde{\mathcal{N}}, \, \widetilde{\mathcal{N}}_j : C^{k+1+\alpha}(S) \to C^{k+\alpha}(S), \quad 0 < \alpha < \alpha', \quad j = 1, 2, \tag{3.10}$$

are bounded operators with the trivial null–spaces.

Operators $\widetilde{\mathcal{N}}$, $\widetilde{\mathcal{N}}_j$, $j = 1, 2$, defined by (3.10), are isomorphisms.

Proof. The mapping property (3.10) is an easy consequence of Lemma 2.3, item ii) since the operator $(\mathcal{H}^r)^{-1} : C^{k+1+\gamma}(S) \to C^{k+\gamma}(S)$ is an isomorphism due to Lemma 2.2.

From Lemma 2.4 it follows also that the equation $\widetilde{\mathcal{N}}_j h = 0$ has only the trivial solution. Therefore the operators $\widetilde{\mathcal{N}}_j$, $j = 1, 2$, defined by (3.10) are invertible and their inverses are bounded.

It remains to prove that the null-space of the operator $\widetilde{\mathcal{N}}$ is trivial as well. Let $h = (h_1, ...h_4)^\top \in C^{1+\gamma}(S)$ be an arbitrary solution of the equation $\widetilde{\mathcal{N}} h = 0$, i.e., $\widetilde{\mathcal{N}}_1 h + \widetilde{\mathcal{N}}_2 h = 0$. Then it is evident that the vectors $U^1(x) = V^1[(\mathcal{H}^1)^{-1}h](x)$, $x \in \Omega_1$ and $U^2(x) = V^2[(\mathcal{H}^2)^{-1}h](x)$, $x \in \Omega_2$, are regular and they solve the homogeneous Problem C, since $g^1 = h$ and $g^2 = h$ solve the homogeneous version of the system of equations (3.5), (3.6). Therefore by Theorem 3.1 we have $U^1 = 0$ in Ω_1 and $U^2 = 0$ in Ω_2, whence $h = 0$ follows immediately. ∎

LEMMA 3.4 *The principal simbol matrices of the operotors $\widetilde{\mathcal{N}}_1$, $\widetilde{\mathcal{N}}_2$ and $\widetilde{\mathcal{N}}$ are positive definite.*

Proof. Denote by $\sigma(\mathcal{P})(x, \xi)$ with $x \in S$ and $\xi \in I\!R^2 \setminus \{0\}$ the principal homogeneous symbol of the pseudodifferential operator $\mathcal{P}$.

Equations (3.7) imply

$$\sigma(\widetilde{\mathcal{N}}_1) = I_4^{(1)} \sigma(\mathcal{N}_1), \quad \sigma(\widetilde{\mathcal{N}}_2) = I_4^{(2)} \sigma(\mathcal{N}_2),$$
$$\sigma(\mathcal{N}_1) = \sigma(-2^{-1} I_4 + \mathcal{K}^1)[\sigma(\mathcal{H}^1)]^{-1}, \;\; \sigma(\mathcal{N}_2) = -\sigma(2^{-1} I_4 + \mathcal{K}^2)[\sigma(\mathcal{H}^2)]^{-1}, \tag{3.11}$$
$$\sigma(\widetilde{\mathcal{N}}) = \sigma(\widetilde{\mathcal{N}}_1) + \sigma(\widetilde{\mathcal{N}}_2).$$

Due to Lemmata 2.2, 2.4, and equation (1.29) we have

$$\sigma(\mathcal{H}^r) = \sigma(\mathcal{H}_0^r), \; \sigma(\mathcal{K}^r) = \sigma(\mathcal{K}_0^r),$$

where $\mathcal{H}_0^r$ and $\mathcal{K}_0^r$ are 4×4 matrix boundary operators on S:

$$\mathcal{H}_0^r g(x) := \int_S \Psi^r(x - y) g(y) dS_y, \;\; x \in S,$$
$$\mathcal{K}_0^r g(x) := \int_S B_0^r(D_x, n(x)) \Psi^r(x - y) g(y) dS_y, \;\; x \in S,$$

with $g = (\tilde{g}, g_4)^\top$ and $\tilde{g} = (g_1, g_2, g_3)^\top$; here Ψ^r is given by (2.3) and

$$B_0^r(D, n) = \begin{Vmatrix} [T^r(D, n)]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \lambda^r(D, n) \end{Vmatrix}_{4\times 4}.$$

Therefore

$$\mathcal{H}_0^r = \begin{Vmatrix} [\mathcal{H}_\Gamma^r]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{H}_\gamma^r \end{Vmatrix}_{4\times 4}, \tag{3.12}$$

$$\mathcal{K}_0^r = \begin{Vmatrix} [\mathcal{K}_\Gamma^r]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{K}_\gamma^r \end{Vmatrix}_{4\times 4}, \tag{3.13}$$

where $\mathcal{H}_\Gamma^r$ $[\mathcal{H}_\gamma^r]$ and $\mathcal{K}_\Gamma^r$ $[\mathcal{K}_\gamma^r]$ are 3×3 matrix [scalar] operators generated by the single layer potential constructed by the fundamental matrix $\Gamma^r(x)$ $[\gamma^r(x)]$ (see (2.3)):

$$\mathcal{H}_\Gamma^r\widetilde{g}(x)=\int_S \Gamma^r(x-y)\widetilde{g}(y)dS_y,\quad \mathcal{K}_\Gamma^r\widetilde{g}(x)=\int_S T^r(D_x,n(x))\Gamma^r(x-y)\widetilde{g}(y)dS_y, \tag{3.14}$$
$$\mathcal{H}_\gamma^r g_4(x)=\int_S \gamma^r(x-y)g_4(y)dS_y,\quad \mathcal{K}_\gamma^r g_4(x)=\int_S \lambda^r(D_x,n(x))\gamma^r(x-y)g_4(y)dS_y.$$

Taking into account the structure of the matrices (3.12) and (3.13), we get from (3.11)

$$\sigma(\mathcal{N}_1)=\sigma(-2^{-1}I_4+\mathcal{K}_0^1)[\sigma(\mathcal{H}_0^1)]^{-1}$$
$$=\begin{Vmatrix} [\sigma(-2^{-1}I_3+\mathcal{K}_\Gamma^1)[\sigma(\mathcal{H}_\Gamma^1)]^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(-2^{-1}I_1+\mathcal{K}_\gamma^1)[\sigma(\mathcal{H}_\gamma^1)]^{-1} \end{Vmatrix}_{4\times 4}, \tag{3.15}$$

$$\sigma(\mathcal{N}_2)=-\sigma(2^{-1}I_4+\mathcal{K}_0^2)[\sigma(\mathcal{H}_0^2)]^{-1}$$
$$=-\begin{Vmatrix} [\sigma(2^{-1}I_3+\mathcal{K}_\Gamma^2)[\sigma(\mathcal{H}_\Gamma^2)]^{-1}]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \sigma(2^{-1}I_1+\mathcal{K}_\gamma^2)[\sigma(\mathcal{H}_\gamma^2)]^{-1} \end{Vmatrix}_{4\times 4}. \tag{3.16}$$

It can be shown that $(-2^{-1}I_3+\mathcal{K}_\Gamma^1)(\mathcal{H}_\Gamma^1)^{-1}$ and $-(2^{-1}I_3+\mathcal{K}_\Gamma^2)(\mathcal{H}_\Gamma^2)^{-1}$ are non-negative 3×3 matrix pseudodifferential operators with positive definite principal symbol matrices, while $(-2^{-1}I_1+\mathcal{K}_\gamma^1)(\mathcal{H}_\gamma^1)^{-1}$ and $-(2^{-1}I_1+\mathcal{K}_\gamma^2)(\mathcal{H}_\gamma^2)^{-1}$ are non-negative scalar ΨDOs with positive principal symbol functions (for details see [19], Lemma 4.2 ii).

Therefore the equations (3.15) and (3.16) together with (3.11) yield that $\sigma(\widetilde{\mathcal{N}}_1)$, $\sigma(\widetilde{\mathcal{N}}_2)$, and $\sigma(\widetilde{\mathcal{N}})$ are positive definite matrices for arbitrary $x\in S$ and $\xi\in I\!R^2\backslash\{0\}$. ∎

COROLLARY 3.5 *Let S, k, α, and α' be as in Lemma* 3.3. *Then the operator $\widetilde{\mathcal{N}}$ (see* (3.10)) *is an isomorphism, and*

$$\widetilde{\mathcal{N}}^{-1}: C^{k+\alpha}(S)\to C^{k+1+\alpha}(S)$$

is a bounded operator.

Applying the above results we get from (3.8) and (3.9):

$$g^1=\widetilde{\mathcal{N}}^{-1}(F+\widetilde{\mathcal{N}}_2 f),\quad g^2=\widetilde{\mathcal{N}}^{-1}(F-\widetilde{\mathcal{N}}_1 f). \tag{3.17}$$

Clearly, $g^r\in C^{k+1+\alpha}(S)$, $r=1,2$, if

$$f\in C^{k+1+\alpha}(S),\ F\in C^{k+\alpha}(S). \tag{3.18}$$

Now we can formulate the following existence

THEOREM 3.6 *Let S, k, α and α', be as in Lemma* 3.3, *and let f and F meet the conditions* (3.18). *Then Problem C is uniquely solvable, and the solution is representable in the form of potentials*

$$U^1(x)=V^1[(\mathcal{H}^1)^{-1}\widetilde{\mathcal{N}}^{-1}(F+\widetilde{\mathcal{N}}_2 f)](x),\quad x\in\Omega_1, \tag{3.19}$$

$$U^2(x) = V^2[(\mathcal{H}^2)^{-1}\widetilde{\mathcal{N}}^{-1}(F - \widetilde{\mathcal{N}}_1 f)](x), \quad x \in \Omega_2. \tag{3.20}$$

Moreover,

$$U^r \in C^{k+1+\alpha}(\overline{\Omega}_r), \ r = 1, 2, \tag{3.21}$$

$$||U^r||_{(\Omega_r, k+1, \alpha)} \le C_0[||f||_{(S,k+1,\alpha)} + ||F||_{(S,k,\alpha)}], \ C_0 = \text{const} > 0, \tag{3.22}$$

where $||\cdot||_{(M,k,\alpha)}$ *denotes the norm in the space* $C^{k+\alpha}(M)$.

Proof. It follows from (3.3), (3.4), (3.17), Corollary 3.5 and Lemma 2.3, item i). ■

3.3. In this section we assume $S \in C^\infty$ and establish uniqueness and existence results in the Bessel–potential and Besov spaces.

LEMMA 3.7 *The operators* (3.10) *can be extended to the folloving bounded, elliptic* ΨDOs *(of order* -1*)*

$$\begin{aligned} \widetilde{\mathcal{N}}, \widetilde{\mathcal{N}}_j \ &: \ H_p^{s+1}(S) \to H_p^s(S) \\ &: \ B_{p,q}^{s+1}(S) \to B_{p,q}^s(S) \end{aligned} \tag{3.23}$$

for arbitrary $s \in I\!R$, $1 < p < \infty$, $1 \le q \le \infty$. *Moreover, the operator* $\widetilde{\mathcal{N}}$*, defined by* (3.23)*, is invertible.*

Proof. The boundedness, ellipticity and mapping properties (3.23) of the operators $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}}_j$ easily follow from Lemmata 2.1, 2.2 and 3.4.

The invertibility of the operator $\widetilde{\mathcal{N}}$ is a consequense of the embedding theorems for solutions of elliptic pseudodifferential equations (ΨDE) on closed manifold. In fact, any solution $h \in H_p^{s+1}(S)$ $[B_{p,q}^{s+1}(S)]$ of the homogeneous pseudodifferential equation $\widetilde{\mathcal{N}}h = 0$, belongs also to the space $C^{k+1+\gamma}(S)$, where $k \ge 0$ is an arbitrary integer and $0 < \gamma < 1$. Therefore $h = 0$ due to Corollary 3.5, whence the unique solvability in the spaces $H_p^{s+1}(S)$ $[B_{p,q}^{s+1}(S)]$ of the non–homogeneous equation $\widetilde{\mathcal{N}}h = f$ follows for the arbitrary right–hand side vector $f \in H_p^s(S)$ $[B_{p,q}^s(S)]$. ■

THEOREM 3.8 *Let*

$$f \in B_{p,p}^{s+1}(S) \ [B_{p,q}^{s+1}(S)], \quad F \in B_{p,p}^s(S) \ [B_{p,q}^s(S)]. \tag{3.24}$$

Then Problem C is uniquely solvable in the space $H_p^{s+1+1/p}(\Omega_r)$ $[B_{p,q}^{s+1+1/p}(\Omega_r)]$*, and the solution is representable by formulae* (3.19) *and* (3.20)*.*

Proof. Let conditions (3.24) be fulfilled. Then Lemmata 3.7 and 2.3, item iii) imply that the vectors U^r, $r = 1, 2$, defined by (3.19) and (3.20) represent a solution to Problem C of the class $H_p^{s+1+1/p}(\Omega_r)$ $[B_{p,q}^{s+1+1/p}(\Omega_r)]$.

As to the uniqueness of solution to Problem C in the above Bessel–potential and Besov spaces we can prove it as follows.

Let $U^r \in H_p^{s+1+1/p}(\Omega_r)$ $[B_{p,q}^{s+1+1/p}(\Omega_r)]$ $r = 1, 2$, be some solution to the homogeneous Problem C. We recall that $U^r \in C^\infty(\Omega_r)$. Then Lemma 2.3, item iii) and equations (1.30), (2.4) yield

$$\begin{aligned} U^1(x) &= W^1([U^1]^+)(x) - V^1([B^1U^1]^+)(x) \\ &= W^1([U^1]^+)(x) - V^1(J_4^{(1)}[\widetilde{B}^1U^1]^+)(x), \ x \in \Omega_1, \end{aligned} \tag{3.25}$$

$$\begin{aligned} U^2(x) &= -W^2([U^2]^-)(x) + V^2([B^2U^2]^-)(x) \\ &= -W^2([U^2]^-)(x) + V^2(J_4^{(2)}[\widetilde{B}^2U^2]^-)(x), \ x \in \Omega_2, \quad J_4^{(r)} = [I_4^{(r)}]^{-1}, \end{aligned} \tag{3.26}$$

where

$$[U^1]^+, \ [U^2]^- \in B_{p,p}^{s+1}(S) \ [B_{p,q}^{s+1}(S)],$$
$$[B^1U^1]^+, \ [B^2U^2]^- \in B_{p,p}^{s}(S) \ [B_{p,q}^{s}(S)].$$

The homogeneous transmission conditions read (see (1.12), (1.13))

$$[U^1]^+ = [U^2]^-, \ [\widetilde{B}^1U^1]^+ = [\widetilde{B}^2U^2]^-. \tag{3.27}$$

Denote

$$[U^1]^+ = g, \quad [\widetilde{B}^1U^1]^+ = h. \tag{3.28}$$

Then (3.27) along with (3.25), (3.26) and Lemma 2.3, item iii) implies (see (2.5)–(2.8), (2.14)) that h and g solve the homogeneous system of boundary ΨDEs:

$$-(\mathcal{H}^1 J_4^{(1)} + \mathcal{H}^2 J_4^{(2)})h + (\widetilde{\mathcal{K}}^1 + \widetilde{\mathcal{K}}^2)g = 0, \tag{3.29}$$

$$-(I_4^{(1)}\mathcal{K}^1 J_4^{(1)} + I_4^{(2)}\mathcal{K}^2 J_4^{(2)})h + (I_4^{(1)}\mathcal{L}^1 + I_4^{(2)}\mathcal{L}^2)g = 0. \tag{3.30}$$

From the positive definiteness of the principal symbol matrices $-\sigma(\mathcal{H}^r)$, $\sigma(\mathcal{L}^r)$ (see Lemma 2.4) and the equation $\sigma(\widetilde{\mathcal{K}}^r) = \overline{[\sigma(\mathcal{K}^r)]^\top}$, it follows that the system of ΨDEs (3.29) and (3.30) is strongly elliptic in the sense of Douglis–Nirenberg. Therefore by the embedding theorems we conclude that h and g are smooth vector–functions on S: $h \in C^{k+\alpha}(S)$, $g \in C^{k+1+\alpha}(S)$ for any $k \geq 0$ and $0 < \alpha < 1$. But then the vectors U^r, $r = 1, 2$, given by (3.25) and (3.26), are regular due to the representation formulae (3.27), (3.28), and Lemma 2.3, item i). Now the conditions (3.27) and Theorem 3.1 complete the proof. ∎

REMARK 3.9 *Using the representation formulas* (3.25) *and* (3.26) *we can solve Problem C by the so–called direct boundary integral equation method. This method reduces the transmission problem in question to the strongly elliptic* (*in the sense of Douglis–Nirenberg*) *system of ΨDEs on S*

$$G\psi = Q, \tag{3.31}$$

where $\psi = (\psi', \psi'')^\top$ *is the unknown vector with* $\psi' = [\widetilde{B}^1U^1]^+$ *and* $\psi'' = [U^1]^+$*; the matrix operator G is given by formula*

$$G = \left\| \begin{matrix} [-\mathcal{H}^1 J_4^{(1)} - \mathcal{H}^2 J_4^{(2)}]_{4\times 4} & [\widetilde{\mathcal{K}}^1 + \widetilde{\mathcal{K}}^2]_{4\times 4} \\ [-I_4^{(1)}\mathcal{K}^1 J_4^{(1)} - I_4^{(2)}\mathcal{K}^2 J_4^{(2)}]_{4\times 4} & [I_4^{(1)}\mathcal{L}^1 + I_4^{(2)}\mathcal{L}^2]_{4\times 4} \end{matrix} \right\|_{8\times 8},$$

while the given on S right hand–side 8–vector Q reads as

$$Q = \Big((2^{-1}I_4 + \widetilde{\mathcal{K}}^2) f - \mathcal{H}^2 J_4^{(2)} F \,,\; I_4^{(2)} \mathcal{L}^2 f + (2^{-1}I_4 - I_4^{(2)} \mathcal{K}^2 J_4^{(2)}) F \Big)^\top .$$

In fact, in the proof of Theorem 3.8 we have shown that the operators

$$\begin{aligned} G \;&:\; [C^{k+\gamma}(S)]^4 \times [C^{k+1+\gamma}(S)]^4 \to [C^{k+1+\gamma}(S)]^4 \times [C^{k+\gamma}(S)]^4 \\ &:\; [H_p^s(S)]^4 \times [H_p^{s+1}(S)]^4 \to [H_p^{s+1}(S)]^4 \times [H_p^s(S)]^4 \\ &:\; [B_{p,q}^s(S)]^4 \times [B_{p,q}^{s+1}(S)]^4 \to [B_{p,q}^{s+1}(S)]^4 \times [B_{p,q}^s(S)]^4 \end{aligned}$$

are invertible.

Therefore the unique solution of Problem C can be represented also in the form

$$\begin{aligned} U^1(x) &= W^1(\psi'')(x) - V^1(J_4^{(1)} \psi')(x), \\ U^2(x) &= -W^2(\psi'' - f)(x) + V^2[J_4^{(2)}(\psi' - F)](x), \end{aligned} \qquad (3.32)$$

where $\psi = (\psi', \psi'')^\top$ solves the system of ΨDEs (3.31).

Note that the conclusions of Theorems 3.6 and 3.8, concerning the smoothness properties of solutions, remain valid for the vectors defined by (3.32) if the conditions (3.18) and (3.24) hold.

4 Problem G

First let us rewrite the transmission conditions (1.14)-(1.17) in the following equivalent form

$$[P^1 U^1 \cdot l]^+ + [P^2 U^2 \cdot l]^- = \widetilde{F}_l^{(+)} + \widetilde{F}_l^{(-)}, \qquad (4.1)$$

$$[P^1 U^1 \cdot m]^+ + [P^2 U^2 \cdot m]^- = \widetilde{F}_m^{(+)} + \widetilde{F}_m^{(-)}, \qquad (4.2)$$

$$[P^1 U^1 \cdot l]^+ - [P^2 U^2 \cdot l]^- = \widetilde{F}_l^{(+)} - \widetilde{F}_l^{(-)}, \qquad (4.3)$$

$$[P^1 U^1 \cdot m]^+ - [P^2 U^2 \cdot m]^- = \widetilde{F}_m^{(+)} - \widetilde{F}_m^{(-)}, \qquad (4.4)$$

$$[u^1 \cdot n]^+ - [u^2 \cdot n]^- = \widetilde{f}_n, \qquad (4.5)$$

$$[P^1 U^1 \cdot n]^+ - [P^2 U^2 \cdot n]^- = \widetilde{F}_n, \qquad (4.6)$$

$$[u_4^1]^+ - [u_4^2]^- = f_4 \quad , \quad [\widetilde{\lambda}^1 u_4^1]^+ - [\widetilde{\lambda}^2 u_4^2]^- = F_4. \qquad (4.7)$$

Clearly, due to (4.3), (4.4), (4.6) and (4.7), the vector

$$[\widetilde{B}^1 U^1]^+ - [\widetilde{B}^2 U^2]^- = F$$

is a prescribed vector on S with

$$F = ((\widetilde{F}_l^{(+)} - \widetilde{F}_l^{(-)}) l + (\widetilde{F}_m^{(+)} - \widetilde{F}_m^{(-)}) m + \widetilde{F}_n n, F_4)^\top. \qquad (4.8)$$

Denote

$$[u^1 \cdot l]^+ - [u^2 \cdot l]^- = \psi_1, \quad [u^1 \cdot m]^+ - [u^2 \cdot m]^- = \psi_2, \qquad (4.9)$$

where ψ_1 and ψ_2 are the unknown scalar functions. Equations (4.5), (4.7), and (4.9) imply

$$[U^1]^+ - [U^2]^- = f,$$

where

$$f = (\psi_1 l + \psi_2 m + \widetilde{f}_n n, f_4)^\top. \tag{4.10}$$

Now let us look for a solution to Problem G in the form (3.19) and (3.20), where F and f are given by (4.8) and (4.10), respectively. Then from the results of Section 3 it follows that the transmission conditions (4.3)–(4.7) are automatically satisfied. It remains to satisfy the conditions (4.1) and (4.2). Taking into account Lemma 2.3, item i) and the equations (3.7), we get from (3.19) and (3.20):

$$\begin{aligned}
[\widetilde{B}^1(D,n)U^1]^+ &= [(P^1(D,n)U^1, \widetilde{\lambda}^1(D,n)u_4)^\top]^+ = \widetilde{\mathcal{N}}_1\widetilde{\mathcal{N}}^{-1}(F + \widetilde{\mathcal{N}}_2 f), \\
[\widetilde{B}^2(D,n)U^2]^- &= [(P^2(D,n)U^2, \widetilde{\lambda}^2(D,n)u_4)^\top]^- = -\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}(F - \widetilde{\mathcal{N}}_1 f).
\end{aligned}$$

Further, we set

$$l^* = [(l,0)^\top]_{4\times 1}, \quad m^* = [(m,0)^\top]_{4\times 1}, \quad n^* = [(n,0)^\top]_{4\times 1}, \tag{4.11}$$

where l, m and n are the tangent and the normal vectors introduced in Subsection 1.4.

Conditions (4.1) and (4.2) then imply

$$\begin{aligned}
[P^1U^1\cdot l]^+ &+ [P^2U^2\cdot l]^- \equiv [\widetilde{B}^1U^1\cdot l^*]^+ + [\widetilde{B}^2U^2\cdot l^*]^- \\
&\equiv (\widetilde{\mathcal{N}}_1 - \widetilde{\mathcal{N}}_2)\widetilde{\mathcal{N}}^{-1}F\cdot l^* + 2\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1 f\cdot l^* = \widetilde{F}_l^{(+)} + \widetilde{F}_l^{(-)}, \\
[P^1U^1\cdot m]^+ &+ [P^2U^2\cdot m]^- \equiv [\widetilde{B}^1U^1\cdot m^*]^+ + [\widetilde{B}^2U^2\cdot m^*]^- \\
&\equiv (\widetilde{\mathcal{N}}_1 - \widetilde{\mathcal{N}}_2)\widetilde{\mathcal{N}}^{-1}F\cdot m^* + 2\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1 f\cdot m^* = \widetilde{F}_m^{(+)} + \widetilde{F}_m^{(-)},
\end{aligned} \tag{4.12}$$

since $\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1 = \widetilde{\mathcal{N}}_1\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_2$. By virtue of (4.10) from (4.12) we have the following system of ΨDEs for the unknown functions ψ_1 and ψ_2:

$$\sum_{k,j=1}^{3}[(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1)_{kj}(\psi_1 l_j + \psi_2 m_j)]l_k = q_1, \tag{4.13}$$

$$\sum_{k,j=1}^{3}[(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1)_{kj}(\psi_1 l_j + \psi_2 m_j)]m_k = q_2, \tag{4.14}$$

where

$$\begin{aligned}
q_1 &= 2^{-1}\{\widetilde{F}_l^{(+)} + \widetilde{F}_l^{(-)} - (\widetilde{\mathcal{N}}_1 - \widetilde{\mathcal{N}}_2)\widetilde{\mathcal{N}}^{-1}F\cdot l^*\} \\
&- \sum_{k=1}^{3}[(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1)_{k4}f_4]l_k - \sum_{k,j=1}^{3}[(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1)_{kj}(\widetilde{f}_n n_j)]l_k, \\
q_2 &= 2^{-1}\{\widetilde{F}_m^{(+)} + \widetilde{F}_m^{(-)} - (\widetilde{\mathcal{N}}_1 - \widetilde{\mathcal{N}}_2)\widetilde{\mathcal{N}}^{-1}F\cdot m^*\} \\
&- \sum_{k=1}^{3}[(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1)_{k4}f_4]m_k - \sum_{k,j=1}^{3}[(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1)_{kj}(\widetilde{f}_n n_j)]m_k,
\end{aligned} \tag{4.15}$$

are given functions on S.

Now let

$$\mathcal{M}_G := \left\| \begin{matrix} l_k(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1)_{kj}l_j & l_k(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1)_{kj}m_j \\ m_k(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1)_{kj}l_j & m_k(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1)_{kj}m_j \end{matrix} \right\|_{2\times 2} .$$

We recall that the summation over repeated indices is meant from 1 to 3. Clearly, (4.13) and (4.14) can be written in the matrix form

$$\mathcal{M}_G\psi = q^* \tag{4.16}$$

with the unknown vector $\psi = (\psi_1, \psi_2)^\top$ and the right hand–side $q^* = (q_1, q_2)^\top$ given by formulae (4.15).

LEMMA 4.1 *The operator $\mathcal{M}_G$ is an elliptic ΨDO of order 1 with a positive definite principal symbol matrix and the index equal to zero.*

Proof. The equations (3.7), (3.15), and (3.16) imply that $\mathcal{M}_G$ is a ΨDO of order 1 with the principal symbol matrix

$$\sigma(\mathcal{M}_G) = \left\| \begin{matrix} l_kl_jE_{kj} & l_km_jE_{kj} \\ m_kl_jE_{kj} & m_km_jE_{kj} \end{matrix} \right\|_{2\times 2} = E_1EE_1^\top, \tag{4.17}$$

where

$$E_1 = \left\| \begin{matrix} l_1, & l_2, & l_3, & 0 \\ m_1, & m_2, & m_3, & 0 \end{matrix} \right\|_{2\times 4},$$

$$E = \sigma(\widetilde{\mathcal{N}}_2\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_1) = \sigma(\widetilde{\mathcal{N}}_2)\sigma(\widetilde{\mathcal{N}}^{-1})\sigma(\widetilde{\mathcal{N}}_1) = \sigma(\widetilde{\mathcal{N}}_2)[\sigma(\widetilde{\mathcal{N}}_1) + \sigma(\widetilde{\mathcal{N}}_2)]^{-1}\sigma(\widetilde{\mathcal{N}}_1).$$

By Lemma 3.4 we have that the matrices $\sigma(\widetilde{\mathcal{N}}_r)$, $r = 1, 2$ are positive definite for arbitrary $x \in S$ and $\xi \in \mathbb{R}^2\backslash 0$ (see (3.15), (3.16)). Therefore the matrix E is positive definite as well. Further, for arbitrary $\eta = (\eta_1, \eta_2)^\top \in \mathbb{C}^2$ we have

$$\sigma(\mathcal{M}_G)\eta\cdot\eta = (E_1EE_1^\top)\eta\cdot\eta = E(E_1^\top\eta)\cdot(E_1^\top\eta)$$
$$= E(l^*\eta_1 + m^*\eta_2)\cdot(l^*\eta_1 + m^*\eta_2) \geq c|\xi|\,|\eta_1 l^* + \eta_2 m^*|^2 = c|\xi|(|\eta_1|^2 + |\eta_2|^2),\ c > 0,$$

whence the positive definiteness of the matrix (4.17) follows. In turn, from this fact we conclude that the dominant singular part of the operator $\mathcal{M}_G$ is formally self-adjoint. This implies that the index of the operator $\mathcal{M}_G$ is equal to zero. ∎

By the arguments applied in Theorem 3.1 we can easily prove

LEMMA 4.2 *The homogeneous Problem G ($\widetilde{f}_n = \widetilde{F}_n = f_4 = F_4 = \widetilde{F}_l^{(\pm)} = \widetilde{F}_m^{(\pm)} = 0$) has only the trivial solution in the class of regular vectors.*

LEMMA 4.3 *Let S, k, α, and α' be as in Lemma 3.3. Then the operator*

$$\mathcal{M}_G : C^{k+1+\alpha}(S) \to C^{k+\alpha}(S) \tag{4.18}$$

is an isomorphism.

If $S \in C^\infty$, then (4.18) can be extended to the following bounded, invertible, elliptic ΨDO (of order 1)

$$\mathcal{M}_G : H_p^{s+1}(S) \to H_p^s(S)\ [B_{p,q}^{s+1}(S) \to B_{p,q}^s(S)],$$
$$s \in \mathbb{R},\ 1 < p < \infty,\ 1 \leq q \leq \infty.$$

Proof. It is quite similar to that of Lemmata 3.3 and 3.7. ∎

The above results lead to the following existence theorems.

THEOREM 4.4 *Let S, k, α, and α' be as in Lemma 3.3, and let*

$$\widetilde{F}_l^{(\pm)},\ \widetilde{F}_m^{(\pm)},\ \widetilde{F}_n,\ F_4 \in C^{k+\alpha}(S), \quad \widetilde{f}_n,\ f_4 \in C^{k+1+\alpha}(S).$$

Then Problem G is uniquelly solvable, and the solution is representable in the form of potentials (3.19), (3.20) *with F and f given by* (4.8) *and* (4.10)*, where $\psi_1,\ \psi_2 \in C^{k+1+\alpha}(S)$ are defined by the system of ΨDEs* (4.13) *and* (4.14) (*i.e.,* (4.16))*. Morover, the smoothness property* (3.21) *and the inequality* (3.22) *hold.*

THEOREM 4.5 *Let $S \in C^{\infty}$ and*

$$\widetilde{F}_l^{(\pm)},\ \widetilde{F}_m^{(\pm)},\ \widetilde{F}_n,\ F_4 \in B^s_{p,p}(S)\ [B^s_{p,q}(S)], \quad \widetilde{f}_n,\ f_4 \in B^{s+1}_{p,p}(S)\ [B^{s+1}_{p,q}(S)].$$

Then Problem G is uniquely solvable in the space $H_p^{s+1+1/p}(\Omega_r)$ $[B_{p,q}^{s+1+1/p}(\Omega_r)]$, and the solutions are representable by the formulae (3.19)*and*(3.20) *with F and f given by* (4.8) *and* (4.10)*, where $\psi_1,\ \psi_2 \in B^{s+1}_{p,p}(S)$ $[B^{s+1}_{p,q}(S)]$ are defined by the system of ΨDEs* (4.13) *and* (4.14) (*i.e.,* (4.16))*.*

Proof. It is verbatim the proof of Theorem 3.8. ∎

5 Problem H

As in the previous section let us rewrite the transmission conditions of Problem H (see (1.16)–(1.19)) in the equivalent form

$$[u^1\cdot l]^+ + [u^2\cdot l]^- = \widetilde{f}_l^{(+)} + \widetilde{f}_l^{(-)}, \tag{5.1}$$
$$[u^1\cdot m]^+ + [u^2\cdot m]^- = \widetilde{f}_m^{(+)} + \widetilde{f}_m^{(-)}, \tag{5.2}$$
$$[u^1\cdot l]^+ - [u^2\cdot l]^- = \widetilde{f}_l^{(+)} - \widetilde{f}_l^{(-)}, \tag{5.3}$$
$$[u^1\cdot m]^+ - [u^2\cdot m]^- = \widetilde{f}_m^{(+)} - \widetilde{f}_m^{(-)}, \tag{5.4}$$
$$[u^1\cdot n]^+ - [u^2\cdot n]^- = \widetilde{f}_n, \tag{5.5}$$
$$[P^1U^1\cdot n]^+ - [P^2U^2\cdot n]^- = \widetilde{F}_n, \tag{5.6}$$
$$[u_4^1]^+ - [u_4^2]^- = f_4 \quad , \quad [\widetilde{\lambda}^1 u_4^1]^+ - [\widetilde{\lambda}^2 u_4^2]^- = F_4. \tag{5.7}$$

Equations (5.3)–(5.5) imply

$$[U^1]^+ - [U^2]^- = f$$

where f is a given vector on S

$$f = ((\widetilde{f}_l^{(+)} - \widetilde{f}_l^{(-)})l + (\widetilde{f}_m^{(+)} - \widetilde{f}_m^{(-)})m + \widetilde{f}_n, f_4)^\top. \tag{5.8}$$

It is also evident that

$$[\widetilde{B}^1U^1]^+ - [\widetilde{B}^2U^2]^- = F$$

with

$$F = (\psi_1 l + \psi_2 m + \widetilde{F}_n n, F_4)^\top, \tag{5.9}$$

where $\widetilde{F}_n$ and F_4 are given functions on S, while

$$\psi_1 = [P^1U^1 \cdot l]^+ - [P^2U^2 \cdot l]^- \quad \text{and} \quad \psi_2 = [P^1U^1 \cdot m]^+ - [P^2U^2 \cdot m]^-,$$

are yet unknown scalar functions.

Now let us look for a solution to Problem H again in the form (3.19) and (3.20), with F and f defined by (5.8) and (5.9), respectively. It can be easily checked that the transmission conditions (5.3)–(5.7) are then automatically satisfied, while the equations (5.1) and (5.2) lead to the following system of ΨDEs for the unknown vector $\psi = (\psi_1, \psi_2)^\top$ on S:

$$\mathcal{M}_H \psi = q^*, \tag{5.10}$$

where

$$\mathcal{M}_H = \begin{Vmatrix} l_k(\widetilde{\mathcal{N}}^{-1})_{kj} l_j & l_k(\widetilde{\mathcal{N}}^{-1})_{kj} m_j \\ m_k(\widetilde{\mathcal{N}}^{-1})_{kj} l_j & m_k(\widetilde{\mathcal{N}}^{-1})_{kj} m_j \end{Vmatrix}_{2\times 2}, \tag{5.11}$$

and where the right hand–side vector $q^* = (q_1, q_2)^\top$ is defined by formulae:

$$\begin{aligned} q_1 &= 2^{-1}\{\widetilde{f}_l^{(+)} + \widetilde{f}_l^{(-)} - [\widetilde{\mathcal{N}}^{-1}(\widetilde{\mathcal{N}}_2 - \widetilde{\mathcal{N}}_1) f \cdot l^*]\} \\ &- [(\widetilde{\mathcal{N}}^{-1})_{kj}(\widetilde{F}_n n_j)] l_k - [(\widetilde{\mathcal{N}}^{-1})_{k4} F_4] l_k, \\ q_2 &= 2^{-1}\{\widetilde{f}_m^{(+)} + \widetilde{f}_m^{(-)} - [\widetilde{\mathcal{N}}^{-1}(\widetilde{\mathcal{N}}_2 - \widetilde{\mathcal{N}}_1) f \cdot m^*]\} \\ &- [(\widetilde{\mathcal{N}}^{-1})_{kj}(\widetilde{F}_n n_j)] m_k - [(\widetilde{\mathcal{N}}^{-1})_{k4} F_4] m_k; \end{aligned}$$

here l^* and m^* are given by (4.11).

By quite the same arguments as in Section 4 we can easily show that $\mathcal{M}_H$ is an elliptic, invertible ΨDO of order -1 with a positive definite principal symbol matrix.

Therefore the operators

$$\begin{aligned} \mathcal{M}_H \; &: \; C^{k+\alpha}(S) \to C^{k+1+\alpha}(S), \; S \in C^{k+2+\alpha}, \\ &: \; H_p^s(S) \to H_p^{s+1}(S), \; S \in C^\infty, \\ &: \; B_{p,q}^s(S) \to B_{p,q}^{s+1}(S), \; S \in C^\infty, \end{aligned}$$

are isomorphisms.

As a result we arrive to the following existence theorems.

THEOREM 5.1 *Let S, k, α, and α' be as in Lemma* 3.3, *and let*

$$\widetilde{f}_l^{(\pm)}, \; \widetilde{f}_m^{(\pm)}, \; \widetilde{f}_n, \; f_4 \in C^{k+1+\alpha}(S), \quad \widetilde{F}_n, \; F_4 \in C^{k+\alpha}(S).$$

Then Problem H has the unique solution representable in the form (3.19) *and* (3.20) *with f and F given by* (5.8) *and* (5.9), *where $\psi_1, \; \psi_2 \in C^{k+\alpha}(S)$ in* (5.9) *are defined by the system of ΨDEs* (5.10).

THEOREM 5.2 *Let $S \in C^\infty$ and*

$$\widetilde{f}_l^{(\pm)}, \; \widetilde{f}_m^{(\pm)}, \; \widetilde{f}_n, \; f_4 \in B_{p,p}^{s+1}(S) \; [B_{p,q}^{s+1}(S)], \quad \widetilde{F}_n, \; F_4 \in B_{p,p}^s(S) \; [B_{p,q}^s(S)].$$

Then Problem H is uniquely solvable in the space $H_p^{s+1+1/p}(\Omega_r)$ $[B_{p,q}^{s+1+1/p}(\Omega_r)]$, and the solution is representable by the formulae (3.19) *and* (3.20) *with f and F given by* (5.8) *and* (5.9), *where $\psi_1, \; \psi_2 \in B_{p,p}^s(S)$ $[B_{p,q}^s(S)]$ in* (5.9) *are defined by the system of ΨDEs* (5.10).

6 Problem C–DD

6.1. From now on, for simplicity, we assume that $S \in C^\infty$. First we prove the uniqueness of a solution in the particular case for $p = 2$. The general case $(1 < p < \infty)$ will be considered later on.

LEMMA 6.1 *Let $U^r \in W_2^1(\Omega_r)$ be a solution of the homogeneous Problem C–DD ($f^{(1)} = F^{(1)} = 0, \quad \varphi^{(\pm)} = 0$). Then $U^r = 0$ in Ω_r, $r = 1, 2$.*

Proof. Applying the formulae (1.27) and (1.28) with $V^r = U^r$, and taking into account the homogeneous transmission conditions, we arrive to the equation (3.2), whence, in the same way as in the proof of Theorem 3.1, it follows that $U^r = 0$ in Ω_r, $r = 1, 2$. ∎

The transmission conditions (1.21) imply

$$[U^1]^+ - [U^2]^- = \varphi^{(+)} - \varphi^{(-)} \quad \text{and} \quad [U^1]^+ + [U^2]^- = \varphi^{(+)} + \varphi^{(-)} \quad \text{on} \quad S_2. \tag{6.1}$$

On the other hand the vector $F^{(1)}$ (see (1.20) and (1.31)) can be extended from S_1 onto S_2 preserving the functional space $B_{p,p}^{-1/p}(S)$. Denote some fixed extension by $F^0 \in B_{p,p}^{-1/p}(S)$; clearly, $F^0|_{S_1} = F^{(1)}$. Evidently, an arbitrary extension F of the vector $F^{(1)}$ onto the whole surface S (preserving the functional space) can be represented as

$$F = F^0 + \varphi, \tag{6.2}$$

where $\varphi \in \widetilde{B}_{p,p}^{-1/p}(S_2)$.

Now we apply (6.1) and (6.2), and reformulate the interface conditions (1.20) and (1.21) in the following equivalent form:

$$\begin{aligned} [U^1]^+ - [U^2]^- &= f^0 \text{ on } S, &(6.3)\\ [\widetilde{B}^1 U^1]^+ - [\widetilde{B}^2 U^2]^- &= F \text{ on } S_1, &(6.4)\\ [U^1]^+ + [U^2]^- &= \varphi^{(+)} + \varphi^{(-)} \text{ on } S_2, &(6.5)\end{aligned}$$

where f^0 and F are given by formulae (1.32) and (6.2), respectively, with the only yet unknown vector φ.

We look for the solution of Problem C–DD in the form of potentials (see (3.19), (3.20))

$$\begin{aligned} U^1(x) &= V^1\{(\mathcal{H}^1)^{-1}\widetilde{\mathcal{N}}^{-1}[(F^0 + \varphi) + \widetilde{\mathcal{N}}_2 f^0]\}(x), \quad x \in \Omega_1, &(6.6)\\ U^2(x) &= V^2\{(\mathcal{H}^2)^{-1}\widetilde{\mathcal{N}}^{-1}[(F^0 + \varphi) - \widetilde{\mathcal{N}}_1 f^0]\}(x), \quad x \in \Omega. &(6.7)\end{aligned}$$

It can be easily checked that the conditions (6.3) and (6.4) are then automatically satisfied, while (6.5) leads to the ΨDE for $\varphi \in \widetilde{B}_{p,p}^{-1/p}(S_2)$

$$r_{S_2}\widetilde{\mathcal{N}}^{-1}\varphi = q \quad \text{on} \quad S_2, \tag{6.8}$$

where r_{S_2} is the restriction operator to S_2 and the right hand–side vector q is given by formula

$$q = 2^{-1}(\varphi^{(+)} + \varphi^{(-)}) - r_{S_2}[\widetilde{\mathcal{N}}^{-1}F^0 + 2^{-1}\widetilde{\mathcal{N}}^{-1}(\widetilde{\mathcal{N}}_2 - \widetilde{\mathcal{N}}_1)f^0] \in B_{p,p}^{1-1/p}(S_2). \tag{6.9}$$

ΨDEs of type (6.8) on manifolds with boundary have been investigated in [6], [3], [24], [5] (see also references cited therein). We need the following results about the Fredholm properties of ΨDEs defined on open surfaces.

THEOREM 6.2 *[5] Let $\mathcal{P}$ be an elliptic ΨDO of order $\alpha \in I\!R$ on S_0 with a positive definite principal symbol matrix for any $(x,\xi) \in \overline{S}_0 \times I\!R^2 \backslash \{0\}$, where S_0 is an open C^∞–smooth manifold with the C^∞–smooth boundary ∂S_0, and let*

$$\frac{1}{p} - 1 < s - \frac{\alpha}{2} < \frac{1}{p}, \quad s \in I\!R, \quad 1 < p < +\infty. \tag{6.10}$$

where S_0 is an open C^∞–smooth manifold with the C^∞–smooth boundary ∂S_0,

Then

i) *the operators*

$$\mathcal{P} \; : \; \widetilde{B}^s_{p,q}(S_0) \to B^{s-\alpha}_{p,q}(S_0), \;\; 1 \le q \le +\infty, \tag{6.11}$$

$$: \; \widetilde{H}^s_p(S_0) \to H^{s-\alpha}_p(S_0), \tag{6.12}$$

are Fredholm;

ii) *for any two pairs (p_1, s_1), $(p_2, s_2) \in (1, +\infty) \times I\!R$ satisfying the inequalities* (6.9), *there exists a common regularizer of the following operators:*

$$\begin{aligned} \mathcal{P} \; &: \; \widetilde{H}^{s_1}_{p_1}(S_0) \to H^{s_1-\alpha}_{p_1}(S_0), \\ &: \; \widetilde{H}^{s_2}_{p_2}(S_0) \to H^{s_2-\alpha}_{p_2}(S_0), \\ &: \; \widetilde{B}^{s_1}_{p_1,q_1}(S_0) \to B^{s_1-\alpha}_{p_1,q_1}(S_0), \quad 1 \le q_1 \le +\infty, \\ &: \; \widetilde{B}^{s_2}_{p_2,q_2}(S_0) \to B^{s_2-\alpha}_{p_2,q_2}(S_0), \quad 1 \le q_2 \le +\infty; \end{aligned}$$

iii) *null–spaces and indices of the operators* (6.11) *and* (6.12) *are the same for all values of the parameters $q \in [1, +\infty]$, $p \in (1, +\infty)$, and $s \in I\!R$ satisfying the conditions* (6.10).

Now we can investigate the equation (6.8).

LEMMA 6.3 *The operator*

$$r_{S_2} \widetilde{\mathcal{N}}^{-1} : \widetilde{B}^s_{p,q}(S_2) \to B^{s+1}_{p,q}(S_2), \; 1 < p < +\infty, \; 1 \le q \le +\infty, \; s \in I\!R, \tag{6.13}$$

is an isomorphism if the condition

$$\frac{1}{p} - \frac{3}{2} < s < \frac{1}{p} - \frac{1}{2} \tag{6.14}$$

holds.

Proof. Due to Lemma 3.4 and Theorem 6.2 it remains only to prove that the null–space of the operator (6.13) is trivial since the principal symbol matrix $\widetilde{\mathcal{N}}^{-1}$ is positive definite for arbitrary $(x,\xi) \in \overline{S}_2 \times I\!R^2 \backslash \{0\}$. This implies the self–adjointness of the main dominant part of the operator $\widetilde{\mathcal{N}}^{-1}$. So we have to show that the homogeneous version of equation (6.8)

$$r_{S_2} \widetilde{\mathcal{N}}^{-1} \varphi = 0 \tag{6.15}$$

has only the trivial solution. We begin with the case $p = 2$. Let $\varphi \in \widetilde{B}^{-\frac{1}{2}}_{2,2}(S_2)$ be some solution of equation (6.15), and construct the potentials

$$U^1(x) = V^1[(\mathcal{H}^1)^{-1} \widetilde{\mathcal{N}}^{-1} \varphi](x), \; x \in \Omega_1,$$

$$U^2(x) = V^2[(\mathcal{H}^2)^{-1}\widetilde{\mathcal{N}}^{-1}\varphi](x), \ x \in \Omega_2.$$

It is evident that $U^r \in W_2^1(\Omega_r)$, $r = 1,2$, due to Lemma 2.3, item iii). Further, from Lemma 6.1 it follows that $U^r(x) = 0$, $x \in \Omega_r$, $r = 1,2$. Therefore $[B^1(D,n)U^1]^+ = 0$ and $[B^2(D,n)U^2]^- = 0$ on S. Whence

$$\begin{aligned} 0 &= [\widetilde{B}^1(D,n)U^1]^+ - [\widetilde{B}^2(D,n)U^2]^- \\ &= I_4^{(1)}(-2^{-1}I_4 + \mathcal{K}^1)(\mathcal{H}^1)^{-1}\widetilde{\mathcal{N}}^{-1}\varphi - I_4^{(2)}(2^{-1}I_4 + \mathcal{K}^2)(\mathcal{H}^2)^{-1}\widetilde{\mathcal{N}}^{-1}\varphi = \varphi, \end{aligned}$$

due to the equations (3.7).

Thus equation (6.15) has only the trivial solution in the space $\widetilde{B}_{2,2}^{-1/2}(S_2)$. Now Theorem 6.2, item iii) implies the same uniqueness result in the space $\widetilde{B}_{p,q}^{s}(S_2)$ with arbitrary $s \in I\!R$, $p \in (1,+\infty)$, and $q \in [1,+\infty]$ satisfying the inequalities (6.14). ∎

Now we can formulate the following existence result for the problem in question.

THEOREM 6.4 *Let* $4/3 < p < 4$ *and* $f^{(1)}$, $F^{(1)}$, $\varphi^{(\pm)}$, f^0 *satisfy the conditions* (1.31) *and* (1.32). *Then Problem C–DD has the unique solution* $U^r \in W_p^1(\Omega_r)$, $r = 1,2$, *representable by the formulae* (6.6) *and* (6.7), *where* $\varphi \in \widetilde{B}_{p,p}^{-1/p}(S_2)$ *is defined by the uniquely solvable* ΨDE (6.8).

Proof. First we note that in accordance with Lemma 6.3 the ΨDE (6.8) is uniquely solvable in the space $\widetilde{B}_{p,p}^{-1/p}(S_2)$ for an arbitrary p satisfying inequality (6.14) with $s = -1/p$. Whence inequality $4/3 < p < 4$ follows. It is evident that the vectors U^r, $r = 1,2$, defined by (6.6) and (6.7), belong to the space $W_p^1(\Omega_r)$, $r = 1,2$.

It remains to prove the uniqueness of solution in the space $W_p^1(\Omega_r)$, $r = 1,2$, for $p \in (4/3,4)$. Let $U^r \in W_p^1(\Omega_r)$, $r = 1,2$, be some solution of the homogeneous Problem C–DD. Clearly, then $[U^1]^+, [U^2]^- \in B_{p,p}^{1-1/p}(S)$ and $[\widetilde{B}^1U^1]^+, [\widetilde{B}^2U^2]^- \in B_{p,p}^{-1/p}(S)$. In addition, $f = [U^1]^+ - [U^2]^- = 0$ on S and $F = [\widetilde{B}^1U^1]^+ - [\widetilde{B}^2U^2]^- = 0$ on S_1. Therefore $F \in \widetilde{B}_{p,p}^{-1/p}(S_2)$. But, due to Theorem 3.8, such solution is then representable by the formulae (3.19) and (3.20) which, in the case in question, take the form

$$U^r(x) = V^r[(\mathcal{H}^r)^{-1}\widetilde{\mathcal{N}}^{-1}F](x), \quad x \in \Omega_r, \quad r = 1,2, \tag{6.16}$$

with $F \in \widetilde{B}_{p,p}^{-1/p}(S_2)$.

Then the homogeneous version of conditions (1.21) on S_2 lead to the equation

$$r_{S_2}\widetilde{\mathcal{N}}^{-1}F = 0 \quad \text{on} \quad S_2,$$

from which $F = 0$ follows for arbitrary $p \in (4/3,4)$ due to Lemma 6.3. Now equations (6.16) show that U^r vanish in Ω_r ($r = 1,2$), which completes the proof. ∎

The next theorem establishes the almost best smoothness result for solutions of the mixed interface Problem C–DD (cf. [18], [5], [17], [12], [10]).

THEOREM 6.5 *Let* $f^{(1)}$, $F^{(1)}$, $\varphi^{(\pm)}$, *and* f^0 *be the same as in Theorem* 6.4. *Let*

$$\frac{4}{3} < p < 4, \quad 1 < t < \infty, \quad \frac{1}{t} - \frac{3}{2} < s < \frac{1}{t} - \frac{1}{2}, \tag{6.17}$$

and let $U^r \in W_p^1(\Omega_r)$, $r=1,2$, *be the solution to Problem C–DD.*

In addition to the above conditions,

i) *if*

$$f^{(1)} \in B_{t,t}^{s+1}(S_1),\ \varphi^{(\pm)} \in B_{t,t}^{s+1}(S_2),\ f^0 \in B_{t,t}^{s+1}(S),\ F^{(1)} \in B_{t,t}^{s}(S_1),$$

then

$$U^r \in H_t^{s+1+1/t}(\Omega_r),\ r=1,2;$$

ii) *if*

$$f^{(1)} \in B_{t,q}^{s+1}(S_1),\ \varphi^{(\pm)} \in B_{t,q}^{s+1}(S_2),\ f^0 \in B_{t,q}^{s+1}(S),\ F^{(1)} \in B_{t,q}^{s}(S_1),$$

then

$$U^r \in B_{t,q}^{s+1+1/t}(\Omega_r),\ r=1,2;$$

iii) *if, in particular,*

$$f^{(1)} \in C^\alpha(S_1),\ \varphi^{(\pm)} \in C^\alpha(S_2),\ f^0 \in C^\alpha(S),\ F^{(1)} \in B_{\infty,\infty}^{\alpha-1}(S_1), \tag{6.18}$$

for some $\alpha > 0$, *then*

$$U^r \in C^\beta(\overline{\Omega}_r),\ r=1,2,$$

with any $\beta \in (0,\alpha_0)$, $\alpha_0 = \min\{\alpha, 1/2\}$.

Proof. The assertions i) and ii) follow from Theorems 6.2, 6.4 and Lemmata 2.3, 6.3 along with the representation formulae (6.6) and (6.7).

The assertion of the theorem on the C^β–regularity of solution follows from the embeddings (see [29], [30])

$$C^\alpha(\Omega) = B_{\infty,\infty}^\alpha(\Omega) \subset B_{\infty,1}^{\alpha-\varepsilon}(\Omega) \subset B_{\infty,q}^{\alpha-\varepsilon}(\Omega) \subset B_{t,q}^{\alpha-\varepsilon}(\Omega) \subset C^{\alpha-\varepsilon-k/t}(\Omega), \tag{6.19}$$

where $\Omega \subset I\!R^3$ is a compact k–dimentional ($k=2,3$) C^∞–smooth manifold with a C^∞–smooth boundary, ε is an arbitrary small positive number, $1 \le q \le +\infty$, $1 < t < \infty$, $\alpha - \varepsilon - k/t > 0$ (here α and $\alpha-\varepsilon-k/t$ are supposed to be non–integer). Taking t sufficiently large and ε sufficiently small, we have:

$$f^{(1)} \in B_{t,q}^{\alpha-\varepsilon}(S_1) = B_{t,q}^{1/2+\sigma}(S_1), \quad \varphi^{(\pm)} \in B_{t,q}^{\alpha-\varepsilon}(S_2) = B_{t,q}^{1/2+\sigma}(S_2),$$
$$f^{(0)} \in B_{t,q}^{\alpha-\varepsilon}(S) = B_{t,q}^{1/2+\sigma}(S), \quad F^{(1)} \in B_{t,q}^{\alpha-1-\varepsilon}(S_1) = B_{t,q}^{-1/2+\sigma}(S_1),$$

with $\sigma = \alpha - \varepsilon - 1/2$, due to (6.18).

Lemmata 6.3, 2.3, item iii), and the conditions (6.17) then imply

$$U^r \in B_{t,q}^{1/2+\sigma+1/t}(\Omega_r) = B_{t,q}^{\alpha-\varepsilon+1/t}(\Omega_r),\ r=1,2, \tag{6.20}$$

if

$$\frac{1}{t} - \frac{3}{2} < -\frac{1}{2} + \sigma = \alpha - \varepsilon - 1 < \frac{1}{t} - \frac{1}{2}. \tag{6.21}$$

If the Hölder exponent α from (6.18) is such that $\alpha - \varepsilon - 1 = -1/2 + \sigma > 1/t - 1/2$, then, clearly, (6.18) holds also for arbitrary $\alpha' < \alpha$, and therefore

$$U^r \in B_{t,q}^{1/2+\sigma_0+1/t}(\Omega_r), \quad r=1,2, \tag{6.22}$$

for any σ_0 with $-1/2+\sigma_0 \in (1/t-3/2, 1/t-1/2)$ (i.e., $\sigma_0 \in (1/t-1, 1/t)$).

Now, the last embedding of (6.19) (with $k=3$) together with (6.20), (6.21), and (6.22) yields either

$$U^r \in C^{\alpha-\varepsilon-2/t}(\overline{\Omega}_r), \quad \text{if} \quad \alpha-\varepsilon-2/t < 1/2-1/t,$$

or

$$U^r \in C^{1/2-1/t}(\overline{\Omega}_r), \quad \text{if} \quad \alpha > 1/2,$$

which completes the proof. ∎

REMARK 6.6 *Problem C–DD includes, as a particular case, a crack type problem for homogeneous bodies with interior cut of the form of a two–dimentional nonclosed surface (similar to that of* S_2*), when on the cut–surface the Dirichlet type conditions are given. In fact, if both elastic anisotropic materials occupying domains* Ω_1 *and* Ω_2 *are the same (say, e.g., material with index* 1*), and the conditions* (1.20) *on* S_1 *are homogeneous* ($f^{(1)}=0$, $F^{(1)}=0$)*, then the surface* S_1 *becomes a formal interface since the displacement vector* u^1 *and the temperature* u_4^1 *satisfy equation* (1.6) *at point* $x \in S_1$ *which implies* $U^1 \in C^\infty(I\!R^3 \backslash \overline{S}_2)$*. This assertion can be easily proved by making use of the general integral representation formulae* (2.4)*. Thus the surface* S_1 *is erased and Problem C–DD is converted into the following crack type problem:*

$$A^1(D,\tau)U^1(x) = 0 \quad \text{in} \quad I\!R^3 \backslash \overline{S}_2, \tag{6.23}$$

$$[U^1]^\pm = \varphi^\pm \quad \text{on} \quad S_2, \tag{6.24}$$

where

$$U^1 \in W_p^1(I\!R^3 \backslash \overline{S}_2) \cap C^\infty(I\!R^3 \backslash \overline{S}_2), \; \varphi^{(\pm)} \in B_{p,p}^{1-1/p}(S_2), \; \varphi^{(+)} - \varphi^{(-)} \in \widetilde{B}_{p,p}^{1-1/p}(S_2). \tag{6.25}$$

It should be noted that all the results obtained in Section 6 *for Problem C–DD remain true* (*with evident slight modifications*) *for problem* (6.23) – (6.25) *as well* (*cf. [5]*)*.*

7 Problem C–NN

As in the previous sections we start the study of Problem C–NN with the reformulation of the interface conditions (1.22)–(1.23). They are equivalent to the following equations:

$$[\widetilde{B}^1 U^1]^+ - [\widetilde{B}^2 U^2]^- = F^0 \quad \text{on} \quad S, \tag{7.1}$$

$$[U^1]^+ - [U^2]^- = f^{(1)} \quad \text{on} \quad S_1, \tag{7.2}$$

$$[\widetilde{B}^1 U^1]^+ + [\widetilde{B}^2 U^2]^- = \Phi^{(+)} + \Phi^{(-)} \quad \text{on} \quad S_2, \tag{7.3}$$

where F^0 is defined by (1.33).

Further, denote by $f^0 \in B_{p,p}^{1-1/p}(S)$ some fixed extention of $f^{(1)}$ from S_1 onto the whole surface S. Then an arbitrary extention, preserving the functional space, reads

$$f = f^0 + \varphi \in B_{p,p}^{1-1/p}(S),$$

where $\varphi \in \widetilde{B}_{p,p}^{1-1/p}(S_1)$. Clearly, $f|_{S_1} = f^{(1)}$.

Now, let us look for a solution to Problem C–NN in the form of potentials (cf. (3.19), (3.20))

$$U^1(x) = V^1\{(\mathcal{H}^1)^{-1}\widetilde{\mathcal{N}}^{-1}[F^0 + \widetilde{\mathcal{N}}_2(f^0 + \varphi)]\}(x), \quad x \in \Omega_1, \tag{7.4}$$
$$U^2(x) = V^1\{(\mathcal{H}^2)^{-1}\widetilde{\mathcal{N}}^{-1}[F^0 - \widetilde{\mathcal{N}}_1(f^0 + \varphi)]\}(x), \quad x \in \Omega_2, \tag{7.5}$$

where φ is the only yet unknown vector.

It can be easily checked that conditions (7.1) and (7.2) are automatically satisfied, while (7.3) leads to the ΨDE for φ

$$r_{S_2}\widetilde{\mathcal{N}}_1\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_2\varphi = q \quad \text{on} \quad S_2; \tag{7.6}$$

here r_{S_2} is again the restrection operator to S_2, while the right hand–side in (7.6)

$$q = 2^{-1}(\Phi^{(+)} + \Phi^{(-)}) + 2^{-1}r_{S_2}(\widetilde{\mathcal{N}}_2 - \widetilde{\mathcal{N}}_1)\widetilde{\mathcal{N}}^{-1}F_0 - r_{S_2}\widetilde{\mathcal{N}}_1\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_2 f^0 \in B^{-1/p}_{p,p}(S_2)$$

is a given vector–function on S_2.

LEMMA 7.1 *The homogeneous Problem C–NN* ($\Phi^{(\pm)} = 0,\ f^{(1)} = 0,\ F^{(1)} = 0$) *has only the trivial solution in the space* $W^1_2(\Omega_r)$, $r = 1, 2$.

Proof. It is quite similar to that of Lemma 6.1. ∎

LEMMA 7.2 *The principal symbol matrix of the operator* $\widetilde{\mathcal{N}}_1\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_2$ *is positive definite for all* $(x, \xi) \in S \times \mathbb{R}^3 \setminus \{0\}$.

Proof. It is an easy consequence of Lemma 3.3, since from the positive definiteness of the matrices A_1 and A_2 the positive definitness of the matrix $A_1(A_1 + A_2)^{-1}A_2$ follows. ∎

LEMMA 7.3 *The operator*

$$r_{S_2}\widetilde{\mathcal{N}}_1\widetilde{\mathcal{N}}^{-1}\widetilde{\mathcal{N}}_2 : \widetilde{B}^{s+1}_{p,q}(S_2) \to B^{s}_{p,q}(S_2),\ 1 < p < \infty,\ 1 \le q \le \infty,\ s \in \mathbb{R},$$

is an isomorphism if the condition (6.14) *holds.*

Proof. It is verbatim the proof of Lemma 6.3. ∎

From the above lemata by quite the same arguments as in Section 6 we can prove the following existence and regularity results for the problem under consideration.

THEOREM 7.4 *Let* $4/3 < p < 4$ *and* $f^{(1)},\ F^{(1)},\ \varphi^{(\pm)}$, *and* F^0 *satisfy conditions* (1.31) *and* (1.33). *Then Problem C–NN has the unique solution* $U^r \in W^1_p(\Omega_r)$, $r = 1, 2$, *representable by formulae* (7.4) – (7.5), *where* $\varphi \in \widetilde{B}^{1-1/p}_{p,p}(S_2)$ *is defined by the uniquely solvable ΨDE* (7.6).

If, in particular,

$$f^{(1)} \in C^\alpha(S_1),\ \ \Phi^{(\pm)} \in B^{\alpha-1}_{\infty,\infty}(S_2),\ \ F^{(1)} \in B^{\alpha-1}_{\infty,\infty}(S_1),\ \ F^0 \in B^{\alpha-1}_{\infty,\infty}(S),$$

then

$$U^r \in C^\beta(\overline{\Omega_r}),\ r = 1, 2,$$

with any $\beta \in (0, \alpha_0)$, $\alpha_0 = \min\{\alpha, 1/2\}$.

REMARK 7.5 *If the both elastic materials in question are the same and the interface conditions* (1.20) *are homogeneous, then Problem C–NN is converted into the following crack type problem* (*cf. Remark* 6.6) :

$$A^1(D,\tau)U^1(x) = 0 \quad \text{in} \quad I\!R^3\setminus\overline{S}_2, \tag{7.7}$$

$$[\widetilde{B}^1 U^1]^{\pm} = \Phi^{(\pm)} \quad \text{on} \quad S_2, \tag{7.8}$$

where

$$U^1 \in W^1_p(I\!R^3\setminus\overline{S}_2)\cap C^\infty(I\!R^3\setminus\overline{S}_2),\ \Phi^{(\pm)}\in B^{-1/p}_{p,p}(S_2),\ \Phi^{(+)}-\Phi^{(-)}\in \widetilde{B}^{-1/p}_{p,p}(S_2). \tag{7.9}$$

It is obvious that the uniqueness, existence and regularity results obtained for Problem C–NN (*see Theorem* 7.4) *remain also valid for the problem* (7.7) – (7.9) (*cf. [5]*).

REMARK 7.6 *It is easy to see that the other problems formulated in Subsection* 1.4 (*Problem C–DC, C–NC, C–H and C–G*) *can be treated by the method developed in the previous sections. The representation formulae* (3.19) *and* (3.20) *reduce equivalently all these mixed interface problems to uniquely solvable, elliptic systems of* Ψ*DEs on* S_2 *for correspoding unknown vectors. Further, applying the same arguments as above we arrive at similar existence, uniqueness and regularity results for the original mixed interface problems in appropriate functional spaces* (*in particular, the Hölder–smoothness exponents are as in Theorems* 6.5 *and* 7.4).

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