RATIONAL CUBIC BALL INTERPOLANTS FOR SHAPE PRESERVING CURVES AND SURFACES

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RATIONAL CUBIC BALL INTERPOLANTS FOR SHAPE PRESERVING CURVES AND SURFACES

by

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ABSTRAK

Interpolan pengekalan bentuk adalah satu teknik rekabentuk lengkung/ permukaan yang sangat penting dalam CAD/-CAM dan rekabentuk geometrik. Ia mempunyai banyak kepentingannya di dalam pelbagai bidang kejuruteraan seperti rekabentuk dan pembuatan kapal, rekabentuk dan pembuatan rangka luar kereta, industri angkasa lepas, dan industri kejenteraan ketelitian, juga memainkan peranan yang sangat penting dalam aerografi , animasi dan permainan, dan juga di dalam bidang penyelidikan ba-ru seperti analisis data moden, matematik kewangan, pemprosesan imej, visualisasi dan teknik tera air digital. Interpolan pengekalan bentuk ditakrifkan untuk membentuk lengkung(permukaan) yang menginterpolasi titik data yang diberi dan mengekalkan bentuk yang tersirat dari titik data. Tujuan tesis ini dijalankan adalah untuk membi-na satu skema interpolasi alternatif yang mengekalkan ciri bentuk yang wujud untuk data lengkung dan permukaan. Untuk membina interpolasi pengekalan kepositifan, berekanada, kecembungan dan data kekangan (apabila data berada di atas garis lurus , lengkungan perlu berada di atas garis lurus) untuk data lengkung yang mana kebiasa-annya menggunakan ciri yang menjumlahkan bentuk sifat dalam titik data rational C^1, gambaran Ball kubik nisbah telah dihuraikan dengan empat bentuk parameter. Syarat-syarat pada dua parameter bentuk diperolehi untuk memelihara sifat-sifat bentuk yang wujud dari data, manakala dua parameter bentuk lagi kekal bebas bagi membolehkan

pengguna mengubah bentuk lengkung seperti yang mereka kehendaki dan membolehkan pengguna mengawal bentuk lengkung. Suatu fungsi dwi-kubik nisbah dengan dua belas parameter bentuk dan satu fungsi campuran separa bi-kubik nisbah dengan enam belas parameter bentuk telah dilanjutkan dari fungsi Ball kubik nisbah untuk mengekalkan sifat positif, berekanada , dan data dengan kekangan (apabila data berada di atas satu satah, permukaannya perlu berada di atas satu satah yang sama) dari permukaan data yang diberi yang telah disusun di atas grid segi empat tepat dengan mengambil kira syarat-syarat yang cukup pada beberapa parameter bentuk. Skema yang dibina telah diimplimentast dengan jayanya terhadap beberapa set data berjarak seragan dan tidak seragam dan keputusan menunjukkan yang ianya berfungsi dengan baik untuk set data yang diuji dan menghasilkan lengkung dan permukaan yang menyenangkan. Skema yang telah terhasil telah berjaya diimplementasikan ke atas beberapa bilangan data set sekata dan tidak sekata, dan keputusan adalah efisyen ke atas data dan lengkung dan permukaan yang terhasil memuaskan secara visual.

RATIONAL CUBIC BALL INTERPOLANTS FOR SHAPE PRESERVING CURVES AND SURFACES

ABSTRACT

Shape preserving interpolation is an essential curve/surface design technique in CAD/CAM and geometric design. It has a great significance in various areas of engineering such as ship design and manufacture, car body design and manufacture, aerospace industry, and precision mechanism industry. Furthermore, it plays a crucial role in aerography, animation and games, some emerging research fields, such as modern data analysis, mathematical finance, image processing, visualization, and digital watermarking technique. Shape preserving interpolation is defined as the method of constructing a curve (surface) to interpolate the given data points and preserve the shape implied by the data points. The focus of this thesis is to develop an alternative interpolating scheme that preserves the inherent shape features for curves and surfaces data. In order to develop the positivity, monotonicity, convexity and constrained data (when the data lies above a straight line the curve is required to lie above that straight line) preserving interpolant for curve data, which are the most often, used property to quantify the shape inherent in the data points a C^1 rational cubic Ball representation has been used with four shape parameters in its description. Conditions in two shape parameters are derived in such a way to preserve the shape properties inherent in the data, whereas the other two parameters remain free to enable the user to modify the shape of the curve as desired and to control the shape of the curves. A rational bi-cubic function with twelve shape parameters and a rational bi-cubic partially blended function with sixteen shape parameters have been extended from the rational cubic Ball functions to maintain positivity, monotonicity and constrained data (when the data lies above a plane, the surface is required to lie above the same plane) of a given surface data arranged on rectangular grid by deriving sufficient conditions on some of the shape parameters. The developed schemes have been implemented successfully on a number of regular equally and unequally spaced data sets and the results show that it is efficient for the tested data sets and gives visually pleasant curves and surfaces.

CHAPTER 1

INTRODUCTION

1.1 Introduction

Computer aided geometric design (CAGD) is concerned with the approximation and representation of curves and surfaces when they are subjected to computer processing. CAGD is a relatively new field. The work in this field was started in the mid-1960s. Barnhill and Riesenfeld have established the field of CAGD in 1974 when they organized a conference on the topic at the University of Utah in the United States of America (USA). The conference is considered as the founding event of the field. The first textbook of CAGD was "Computational Geometry for Design and Manufacture" by Faux and Pratt appeared in 1979. The first journal of "Computer Aided Geometric Design" was founded in 1984 by Barnhill and Boehm. Another early conference was held in Paris in 1971, which focused on automotive design. The conference was organized by Bézier. There was also a series of workshop started in 1982 at the Math-ematics Research Institute at Oberwolfach, which was organized by Barnhill (Farin, 2002).

A significant element is embodied by the design of curves and surfaces within the building of various items like vehicle bodies, wings and fuselages for aircraft, ship hulls, in addition to the definition of physical, geological and medical occurrences. Innovative aspects of CAGD use encompass computer vision and scrutiny of produced items, the film industry, and image evaluation for medical research. Spline functions in their simplest and most useful form are nothing more than pieces of polynomials joined together smoothly at certain knots, were first introduced into CAGD by Ferguson (1964) from Boeing in 1963. At about the same time de Boor and Gordon studied these curves at General Motors (Böhm et al., 1984).

In the efforts of engineers and scientists, spline functions comprise an essential and common elements. They comprise the key instrument in CAGD, which is essential to build curves and surfaces comprising particular shape features for most of CAGD applications. Splines remain very important tools in a multitude of applications involving curve fitting and design. The main reason for this is their excellent approximation properties. Also they are easy to manipulate, store, and evaluate on a computer.

Polynomial splines do not retain the shape properties of the data. This problem is known as the problem of shape preserving. During the last two decades, different authors have developed various algorithms of spline approximation with both local and global shape control. Based on spline functions, such methods are usually called methods of shape preserving spline approximation.

One of the main applications of shape preserving spline approximation is CAGD. The idea in CAGD is to find representations of curves and surfaces which are easy to treat on a computer, and to render on a graphical device such as a computer screen. To be of most use, these representations should have convenient handles consisting of a set of parameters which can be varied by the user to make well-defined changes in the curve or surface. Hence the main challenge is to develop algorithms that select these parameters automatically. Very strong requirements must be met in industrial design. Usually, a designer provides the envelopes of a car body, ship hull, airplane fuselage, engine details of complex shape as a discrete set of points. Hence to produce the body we need to describe these points as lying on some curves or some surfaces. Any discontinuities of the first and even second derivative may lead to flow separation that is to an increase in friction. By this reason, the designer is often interested in a very smooth approximation which preserves the shape of the data (Kvasov, 2000).

Preservation of shape features, inherent in the data, by an interpolant is one of important research subjects in CAGD. In particularly, when data from some scientific observation are considered, a user may be interested to visualize it graphically. There are splines which can produce smooth curves but unable to preserve the inherent shape of a given data.

In many interpolation problems, the solution that preserves some shape properties, such as positivity, monotonicity, and convexity is important. Numerous physical cases have entities that get meaning only when their values appear in a positive, a monotone, or a convex shape. Therefore, discussing shape preserving interpolation problems is important to provide a visually pleasing and computationally economical solution to various scientific events.

Positivity is an important property of the data that occurs in visualizing a physical quantity that cannot have a negative value, which may arise if the data is taken from some scientific, social or business environments, furthermore stability of radioactive substance and chemical reactions, solvability of solute in solvent, population statistics, observation of gas discharge when certain chemical experiment is in process (Hussain and Sarfraz, 2008), depreciation of the price of computers in the market (Abbas et al., 2014), monthly rainfall amounts, resistance offered by an electric circuit, probability distribution, volume and density (Hussain and Hussain, 2006b), dissemination rate of drugs in the blood, and half-life of a radioactive substance(Tahat et al., 2015a), are few examples of entities which are always positive. Therefor the negative graphical display of these physical quantities is meaningless.

Monotonicity is another important shape property that is applied in many scientific applications such as physical situations and engineering problems, where entities only have a meaning when their values are monotone. Data generated from stress of a material, uric acid level in patients suffering from gout (Tahat et al., 2015b), erythrocyte sedimentation rate (E.S.R.) in cancer patients, rate of dissemination of drug in blood (Hussain and Hussain, 2007), dose-response curves and surfaces in biochemistry and pharmacology, design of aggregation operators in multi criteria decision making and fuzzy logic, approximation of copulas and quasi-copulas in statistics, empirical option pricing models in finance, approximation of potential functions in physical and chemical systems (Beliakov, 2005) are few examples of entities which are always monotone.

Convexity is another important shape property and plays a major role in various applications including telecommunication systems designing, nonlinear programming, engineering, optimal control, optimization, parameter estimation, approximation theory and others (Sarfraz et al., 2012).

Many researchers have addressed the problem of data visualization. Schumaker (1983) used piecewise quadratic polynomial to preserve the shape of monotone data by introducing an additional knot in each subinterval where the shape of the data is not preserved. Brodlie and Butt (1991) developed a C^1 piecewise cubic interpolation to preserve the shape of convex data. They divided the interval where convexity was lost into two subintervals by inserting extra knots in that interval. Butt and Brodlie (1993) used the same technique to develop a C^1 positivity preserving scheme for 2D data. Goodman et al. (1991) developed two interpolating methods to maintain the shape of constrained data utilizing a rational cubic interpolant. Firstly they preserved the shape of the data by scaling the weights by some scale factors. Secondly they introduced a new data point to retain the shape of the data. Unlike Brodlie and Butt (1991), Butt and Brodlie (1993), and Goodman et al. (1991), the data visualization scheme for shape preserving curve developed in this thesis neither requires the specification of the interval in which the shape of data is lost nor scaling of weights. The schemes developed in this thesis, assure an automated selection of parameters in each subinterval.

Hussain et al. (2010) introduced a rational interpolant (cubic/linear) with one shape parameter to visualize the shape of positive, constrained and monotone data by imposing data dependent constraints on the shape parameter. Sarfraz et al. (2000, 2001), Sarfraz (2002) and Sarfraz and Hussain (2006) presented a C^1 rational cubic function with two shape parameters to maintain the shape properties of the shaped data. Data dependent conditions were derived on shape parameters to preserve the positivity of positive data (Sarfraz et al., 2000), positivity and monotonicity of shaped data (Sarfraz et al., 2001), positivity and convexity of 2D data (Sarfraz, 2002) and positivity, monotonicity and convexity (Sarfraz and Hussain, 2006).

The rational functions used in (Sarfraz, 2003) and (Hussain and Hussain, 2007) have also two parameters which are constrained to visualize the shape of data. However, no flexibility is provided for the user to refine the curves further if needed, whereas the schemes developed in this thesis have four parameters, where two of the parameters are constrained to visualize the data and the other two parameters provide the user with a degree of freedom to adjust the shape of the generated curve which is more suitable for interactive curve design.

Hussain and Sarfraz (2008, 2009) used a rational cubic function in its most generalized form (four shape parameters) to preserve the shape of positive and monotone data in (Hussain and Sarfraz, 2008) and (Hussain and Sarfraz, 2009) respectively. Sarfraz et al. (2012) proposed a piecewise rational function in a cubic/cubic form, which involves four shape parameters in each interval in its construction. Two of these shape parameters are constrained to preserve the shape of convex, monotone, and positive data while the other two parameters are used to modify positive, monotone and convex curves to obtain a visually pleasing curve.

A rational cubic Ball interpolant was developed by Piah and Unsworth (2013) with two shape parameters that can be used to generate the desired monotone curves from monotone data. However, no flexibility was provided for the user to refine the curves further if needed, so it is unsuitable for interactive curve design.

Shape preserving interpolation problem for visualization of 3D data is one of the basic problem in computer graphics, CAGD, data visualization and engineering. It also arises frequently in many fields including military, education, art, medicine, advertising, transport military, art and many other fields. Data are noticed from mathematical description, scientific phenomenon and real sciences, and one of the main interests for the designer in data visualization environment is to convert this data into any graphical representation that makes the content easier to understand and provides an insight into the noticed data. These data may have some special shape properties such as positivity, monotonicity and convexity.

In many shape preserving interpolation problems, it is required that the function exhibits the shape features ingrained in the data and the problem become critical when it fails to retain this shape property. Furthermore, smoothness is also required to demonstrate the data in a visual pleasant display. Ordinary spline methods usually ignore these characteristics thus exhibiting undesirable inflections or oscillations in resulting curves and surfaces. Due to this reason a good amount of work has been published that focuses on surfaces shape preserving.

Piah et al. (2005) have discussed the problem of positivity preserving for scattered data interpolation. Sufficient conditions are derived on the ordinates of the Bézier control points in each triangle to preserve the positivity of data. Hussain and Sarfraz (2008) utilized a C^1 rational cubic function to preserve the shape of positive data, then they extended it to an interpolating rational bi-cubic form, involving eight shape parameters. Constraints were derived on four shape parameters in the description of the rational bi-cubic function to visualize the shape of positive data in the view of positive surfaces and the remaining four shape parameters were left to the user to refine the shape of the surfaces.

The problem of visualization of constrained data which is a generalized case of problem of positive data visualization considered by few authors. This problem usually arises in the comparative study of data (Hussain et al., 2008). Brodlie et al. (2005) proposed the method of visualizing constrained data. They modified the quadratic Shepard method, which interpolates scattered data of any dimensionality to preserve positivity. Brodlie et al. (1995) discussed the problem of surface data interpolation subject to simple linear constraints. They developed a piecewise bi-cubic function from data on a rectangular grid. The problem of positivity was generalized to the case of linearly constrained interpolation, where it was required that the function lies between bounds which were linear functions.

Chan and Ong (2001) constructed a range restricted C^1 interpolant to scattered data, sufficient non-negativity conditions derived on the Bézier ordinates to ensure the non-negativity of a cubic Bézier triangular patch. Constraints are derived on derivatives and the gradients modified at the data points if needed to guarantee the achievement of non-negativity conditions. Carlson and Fritsch (1985) developed a bi-cubic polynomial interpolation scheme to preserve the shape of monotone data. Necessary and sufficient conditions were derived on derivatives, such that the resulting bi-cubic polynomial is monotone. Beatson and Ziegler (1985) presented a visualization of monotone data arranged over a rectangular grid by C^1 monotone quadratic spline.

In (Brodlie et al., 1995), (Chan and Ong, 2001), (Carlson and Fritsch, 1985), and (Beatson and Ziegler, 1985) the necessary and sufficient conditions were derived on derivatives values at grid points to preserve the shape of the 3D data. Thus the derivative values at the data sites were fixed and the proposed schemes were not applicable to data with derivatives at the data points. Hussain and Hussain (2006b) developed a rational bi-cubic interpolant to preserve the shape of positive surface data and the surface data that lies above a plane. Simple data dependent conditions were derived on shape parameters to conserve the shape of surface data. Hussain and Hussain (2006a) preserved the shape of monotonic surface data by utilizing a rational bi-cubic function with four shape parameters in it is description. Simple constraints are derived on shape parameters to preserve the shape of data. A smooth surface interpolation scheme for positive and convex data has been developed in (Hussain et al., 2011). The scheme has been extended from the rational quadratic spline function of Sarfraz to a rational bi-quadratic spline function. Simple data dependent constraints are derived on the shape parameters in the description of rational bi-quadratic spline function to preserve the shape of 3D positive and convex data.

Hussain and Hussain (2006b) extended the rational cubic function developed by Hussain and Ali (2006) to a rational bi-cubic partially blended function (Coons patches). Simple constraints are developed on the shape parameters in the description of rational bi-cubic function to visualize positive data and data that lies above the plane. Sarfraz et al. (2010) developed a C^1 piecewise rational cubic interpolant, with two shape parameters. Data dependent shape conditions are imposed on the shape parameters to preserve the shape of data. The rational cubic spline has been extended to a rational bi-cubic partially blended surface (Coons-patches) and derived constraints on parameters to visualize the shape of positive surface data. Shaikh et al. (2011) extended the rational cubic function developed by Hussain et al. (2011) to a rational bi-cubic partially blended function. Data dependent constraints are derived on shape parameters to visualize surface lies above the plane. Hussain and Hussain (2007) used piecewise rational cubic function to visualize monotone data in the view of monotone curves by making constraints on shape parameters in the description of rational cubic function. The rational cubic function is extended to rational bi-cubic partially blended function, simple constraints were derived on the parameters in the description of rational bi-cubic partially blended patches to visualize the monotone data in the view of monotone surfaces. Hussain et al. (2010) extended the piecewise rational cubic function for monotone curve design developed by Hussain and Sarfraz (2009) to rational bi-cubic partially blended function to preserve the shape of 3D monotone data. The rational cubic function presented in Sarfraz and Hussain (2006) has been extended to rational bi-cubic partially blended function to visualize the shape of 3D positive data by Hussain et al. (2011). Hussain et al. (2012) utilized the same rational bi-cubic function to preserve the shape of monotone and convex data. Simple data dependent constraints were developed on shape parameters in each rectangular patch to assure the preservation of the shape of data.

Hussain and Bashir (2011) presented surface data visualization scheme for the visualization of positive, constrained and monotone data using rational bi-cubic functions with linear denominator. To visualize surface data arranged over a rectangular mesh, a rational bi-cubic function has been developed which is an extension of the rational cubic function in Hussain et al. (2010). Data dependent conditions have been derived on shape parameters to preserve the shape of data. Hussain et al. (2015) extended a piecewise rational cubic function presented in (Sarfraz et al., 2012) to a bi-cubic partially blended rational function with eight shape parameters to preserve the inherent shape features of the shaped data. Data dependent sufficient constraints were developed on four of the parameters to preserve the shape of data while the remaining left free to refine the shape of data at user choice.

The C^1 rational cubic spline interpolant of Karim and Kong (2014) has been extended to a partially blended bi-cubic rational spline with 12 shape parameters in the descriptions by Karim et al. (2015). Sufficient conditions are derived on four shape parameters and the remaining 8 of them were free parameters which were used to change the shape of the final surfaces of the positive data.

From the previous discussion, positivity, monotonicity, and convexity are important shapes. They are independent shapes which are found inherited in data. Previous studies had discussed these shapes independently using different mathematical models and methodologies.

This thesis intends to discuss the three shapes within one mathematical model. It proposes a rational cubic Ball interpolant with four parameters in its description. The data dependent constraints have been developed on two parameters to introduce independent curve schemes to visualize positive, monotone and convex data. However, the other two parameters have been left as free parameter. It can assume any posi-tive value to further refine the curve schemes if needed, to obtain a visually pleasing curve. The problem of visualization constrained data is also addressed. When data is lying above a straight line the curve is required to lie on the same side of the line. A rational bi-cubic function and a rational bi-cubic partially blended function have been extended from the rational cubic Ball functions to maintain positivity, monotonicity and constrained data (when the data is lying above a plane the surface is required to lie above the plane) of a given surface data arranged on rectangular grid by deriving the sufficient conditions on some of the shape parameters. The developed schemes have been implemented successfully on a number of regular equally and unequally spaced data sets and the results shows that it works well for the tested data sets and obtain visually pleasant curves and surfaces.

1.2 Thesis Objective

The aim of this thesis is to develop an alternative interpolating method that preserves the inherent shape features for curves and surfaces data. In order to develop the positivity, monotonicity and convexity preserving interpolants for regular data, which are the most often used properties to quantify the shape rational cubic Ball interpolant, rational bi-cubic Ball interpolant and rational bi-cubic partially blended interpolant have been used. The parameters will be used to control the unwanted change in the shape of the curves and surfaces and to preserve the inherent shape properties of the data also to refine the shape of the curve and surfaces to obtain smooth and visually pleasing results according to the designer choice.

1.3 Thesis Outline

The layout of this thesis is organized in the following manner. Chapter 1 gives a brief introduction to the problem of shape preserving visualization and a comprehensive review of the literature. In Chapter 2, a review of the background of the study is given. In Chapter 3 algorithms based on rational cubic Ball basis function to visualize positive, constrained, monotone and convex curve data are developed. Numerical examples are presented to verify proposed algorithms. The rational cubic Ball interpolant developed in Chapter 3 is extended to rational bi-cubic Ball function and used for visualization of positive, monotone and constrained data in Chapter 4. Data dependent constraints are derived on shape parameters in the description of bi-cubic function in order to retain the shape of surface data. In Chapter 5 rational bi-cubic partially blended function, which is an extension of rational cubic Ball function in Chapter 3 is utilized to preserve the shape of positive, constrained and monotone surfaces of regular surface data by imposing sufficient data dependent conditions on shape parameters. Furthermore, the developed surface schemes are tested through different numerical examples, and finally Chapter 6 summarizes the major findings and concludes the work done in this thesis. Appendices *A* and *B* include some calculations introduced in Chapter 4.

CHAPTER 2

BACKGROUND

2.1 Introduction

One of the well-known mathematical representations for curves and surfaces used in computer graphics and computer-aided design are the Bézier curves and surfaces. Bézier methods were first developed by de Casteljau around 1959 and Bézier around 1962 independently. They developed their work as part of car manufacturers systems in French car companies, Renault and Citroen based on the Bernstein basis function. Many publications described The Renault system, UNISURF (by Bézier) this is why the entire field bears Bézier's name.

Bézier curves are deemed as the mathematical foundation of many computer-aided design (CAD) systems, they have also become the basis of the field of computer aided geometric design (CAGD) (Böhm et al., 1984). Bézier curves are used extensively because they have a particular mathematical representation. Their popularity is due to the fact that they possess a number of mathematical properties which facilitate their manipulation and analysis such as end-point interpolation, tangency to the control polygon at two end-points, and lying inside the convex hull (Tien, 1999).
In 1974 Alan Ball, a British mathematician used the cubic Ball basis to define his lofting surface program CONSURF at the British Aircraft Corporation using a method analogous to that of Bézier. Although the Ball basis functions are not the same as the Bernstein polynomials, they furnish the same shape-preserving features as Bernstein

polynomials. Later the basis was generalized for a polynomial of higher degree by Wang (1987) and Said (1989) respectively. These generalizations lead to the Wang-Ball curves (surfaces) and Said-Ball curves (surfaces). Goodman and Said pointed out the advantages of Said-Ball curves and surfaces in (Goodman and Said, 1991a). The shape preserving properties of the generalized Ball basis were discussed by Goodman and Said (1991b). Although Bézier curves have been used in shape preserving interpolation, Ball curves and surfaces have been found to be more suitable in some circumstances (Tien, 1999).

2.2 Ball Curves

Bézier's UNISURF utilized Bernstein polynomials as basis functions. These basis functions provide the shape-preserving features that are desired in free form curves and surfaces designing. Ball (1974, 1975, 1977) uses different basis functions to define his lofting surface program CONSURF at the British Aircraft Corporation.

The method is analogous to the Bézier method. The basis functions employed are cubic polynomials, and they are slightly different from the Bernstein polynomials used in the Bézier method. However, they inherit the same shape-preserving properties and the Bernstein polynomials (Goodman and Said, 1991b). One of the advantages of the Ball cubic method is that, if interior control points coalesce, then the method reduces to a quadratic.

A cubic curve is defined by Ball (1974) as

$$B_3(t) = \sum_{i=0}^{3} \beta_i^3(t) b_i, \qquad 0 \leq t \leq 1, \tag{2.1}$$

where $b_i,\ i=0,1,2,3,$ are called the control points and $\beta_i^3(t)$ the cubic basis functions which are defined as follows :

$$\begin{aligned}
\beta_0^3(t) &= (1-t)^2,\\
\beta_1^3(t) &= 2t(1-t)^2,\\
\beta_2^3(t) &= 2t^2(1-t),\\
\beta_3^3(t) &= t^2.
\end{aligned} \tag{2.2}$$

Figure 2.1 shows cubic Ball basis functions

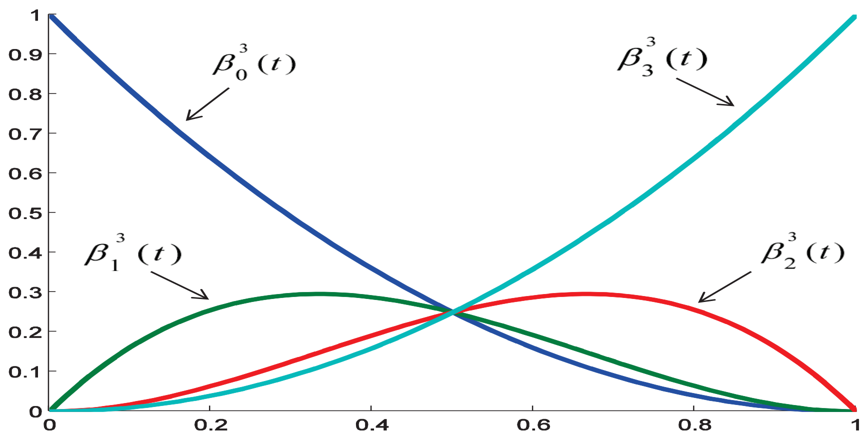

Figure 2.1: Cubic Ball basis functions

These basis functions have the following properties:

1. Non-negative: Since $0 \le t \le 1$, $\beta_i^3(t) \ge 0, \quad i=0,1,2,3.$

2. Partition of unity: An important property of the cubic Ball basis functions is the partition of unity. This means the sum of the functions is always one, for all values of $0 \le t \le 1$.

$$\sum_{i=0}^{3} \beta_i^3(t) = (1-t)^2 + 2t(1-t)^2 + 2t^2(1-t) + t^2$$

$$= 1 - 2t + t^2 + 2t - 4t^2 + 2t^3 + 2t^2 - 2t^3 + t^2$$

$$= 1.$$

Hence Ball curve possesses the following properties:

1. Convex hull property: Since Ball basis functions are non-negative and the sum of the functions are always one, for all values of $0 \leq t \leq 1$ so Ball curve lies completely in the convex hull of its control points.

2. End-point interpolation: The Ball curve interpolates the first and last points b_0 and b_3. $B(0) = b_0$ and $B(1) = b_3$. This property derived from the Ball basis functions, since at the endpoints it is equal to zero except at $b_0, \beta_0^3 = 1, \text{and at} b_3, \beta_3^3 = 1$.

3. Variation diminishing: Ball curve is variation diminishing. This means that no straight line intersects the curve more than its control polygon.

4. An interesting feature of the Ball curve is obtained by coalescing two interior control points, the cubic curve degenerates to quadratic if $b_1 = b_2$ as follows

$$B_3(t) = b_0(1-t)^2 + 2b_1t(1-t)^2 + 2b_2t^2(1-t) + b_3t^2$$

$$= b_0(1-t)^2 + 2b_1t(1-t)^2 + 2b_1t^2(1-t) + b_3t^2$$

$$= b_0(1-t)^2 + 2b_1t(1-t)[(1-t)+t] + b_3t^2$$

$$= b_0(1-t)^2 + 2b_1t(1-t) + b_3t^2$$

Figure 2.2 shows a cubic Ball curve, and Figure 2.3 shows a quadratic curve obtained by coalescing the interior control points of the cubic curve.

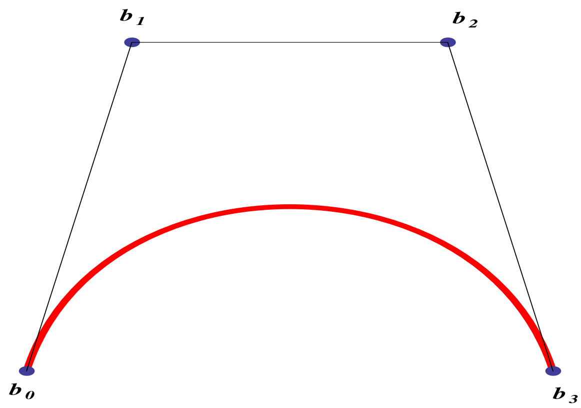

Figure 2.2: Cubic Ball curve

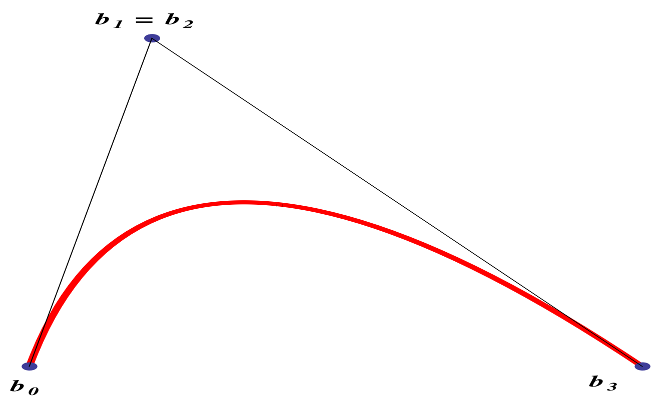

Figure 2.3: Quadratic curve obtained when $b_1 = b_2$

2.3 Generalized Ball Curves

2.3.1 Said-Ball Curve

The generalization by Said (1989) resulted in the Said-Ball curve, which can be expressed as

$$B_n(t) = \sum_{i=0}^{n} b_i S_i^n(t),\ 0 \leq t \leq 1 \tag{2.3}$$

where b_i are its control points, and $S_i^n(t)$ are the Said-Ball basis functions, which can be defined for both odd and even values of n as:

$$S_i^n(t) = \begin{cases} \binom{\lfloor \frac{n}{2} \rfloor + 1}{i} t^i (1-t)^{\lfloor \frac{n}{2} \rfloor + 1}, 0 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \\ \binom{n}{\frac{n}{2}} t^{\frac{n}{2}} (1-t)^{\frac{n}{2}}, i = \frac{n}{2} \\ S_{n-i}^n(1-t), \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n \end{cases} \tag{2.4}$$

where

$\lfloor x \rfloor =$ the greatest integer less than or equal to x

and

$\lceil x \rceil =$ the least integer greater than or equal to x

A rational Said-Ball curve of degree n can be defined by the following equation:

$$p(t) = \frac{\sum_{i=0}^{n} S_i^n(t) w_i b_i}{\sum_{i=0}^{n} S_i^n(t) w_i} \tag{2.5}$$

where w_i are called the weights.

2.3.2 Wang-Ball Curve

Another generalization of the cubic Ball curve was provided by Wang (1987), a Wang-Ball curve of degree n with control points can be expressed as

$$W(t) = \sum_{i=0}^{n} b_i A_i^n(t);\ 0 \le t \le 1 \tag{2.6}$$

where b_i are the control points and $A_i^n(t)$ are the Wang-Ball basis functions, which are defined as:

$$A_i^n(t) = \begin{cases} (2t)^i(1-t)^{i+2}, 0 \le i \le \lfloor n/2 \rfloor - 1 \\ 2t^{\lfloor n/2 \rfloor}(1-t)^{\lceil n/2 \rceil}, i = \lfloor n/2 \rfloor \\ (2(1-t))^{\lfloor n/2 \rfloor} t^{\lceil n/2 \rceil}, i = \lceil n/2 \rceil \\ A_{n-i}^n(1-t), \lceil n/2 \rceil + 1 \le i \le n. \end{cases} \tag{2.7}$$

Rational curves are becoming standard curves description in CAGD, CAD and computer graphics.

A rational Wang-Ball curve of degree n can be described by the following equation:

$$p(t) = \frac{\sum_{i=0}^{n} A_i^n(t) w_i b_i}{\sum_{i=0}^{n} A_i^n(t) w_i} \tag{2.8}$$

Remark: When all of the weights w_i are equal one, the denominator is identically equal to one and the standard non-rational Said-Ball curve and Wang-Ball curve will be obtained, and when $n = 3$, the Said-Ball curve and Wang-Ball curve reduce to the cubic Ball curve. (Dejdumrong et al., 2001).