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Research Article

Crank-Nicolson finite difference method for two-dimensional fractional sub-diffusion equation

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Abstract

A Crank-Nicolson finite difference method is presented to solve the time fractional two-dimensional sub-diffusion equation in the case where the Grünwald-Letnikov definition is used for the time-fractional derivative. The stability and convergence of the proposed Crank-Nicolson scheme are also analyzed. Finally, numerical examples are presented to test that the numerical scheme is accurate and feasible.

Keywords: Grünwald-Letnikov fractional derivative, 2D fractional sub-diffusion equation, Crank-Nicolson difference approximation, Stability, Convergence.

1 Introduction

Fractional calculus is essentially arbitrary order differentiation and integration. Comprehensive studies on fractional calculus and its applications can be found in [1-4]. Certain phenomena and processes can best be described by the fractional diffusion equation having fractional order derivatives in time or space or space-time [5]. Most papers on the numerical solution of the time fractional sub-diffusion equation have utilized the Caputo definition for the time fractional derivative [6, 7, 8, 9]. There have not been many studies that utilize the Grünwald-Letnikov definition. The limited studies that have used the Grünwald-Letnikov (or related to Grünwald-Letnikov) definition include [10, 11, 12, 13, 14].

This paper discusses the use of a Crank-Nicolson scheme for solving the two-dimensional time fractional sub-diffusion equation is constructed by applying the Grünwald-Letnikov definition instead of the Caputo definition for the time-fractional derivative. If the initial condition is zero then the Grünwald-Letnikov definition and Caputo definition are equivalent [15, 16]. It should be noted however that the Grünwald-Letnikov definition has the advantage of being less complex and more easily applied.

This paper considers the following two dimensional time fractional sub-diffusion equation

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + f(x,y,t), \qquad 0 \le t \le T, \tag{1.1}$$

subject to

$$u(x,y,0) = f_1(x,y), \tag{1.2}$$

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$$u(0,y,t) = f_2(y,t),\;\; u(x,0,t) = f_3(x,t),$$
$$u(1,y,t) = f_4(y,t),\;\; u(x,1,t) = f_5(x,t),$$
$$0 \leq x,y \leq L, \quad 0 \leq t \leq T, \tag{1.3}$$

where f_1, f_2, f_3, f_4 and f_5 are known functions, and u is the unknown dependent variable.
The time fractional derivative of order $\alpha (0 < \alpha \leq 1)$ of u can be defined by

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{du(x,y,s)}{dt} \frac{1}{(t-s)^{\alpha}} ds, & 0 < \alpha < 1 \\ \frac{\partial u(x,y,t)}{\partial t}, & \alpha = 1 \end{cases} \tag{1.4}$$

and

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t u(x,y,s) \frac{1}{(t-s)^{\alpha}} ds, & 0 < \alpha < 1 \\ \frac{\partial u(x,y,t)}{\partial t}, & \alpha = 1 \end{cases} \tag{1.5}$$

According to [9], (4) is known as the Caputo formula and (5) is known as the Riemann-Liouville formula. The Grünwald-Letnikov time fractional derivative formula is defined by [15]

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} u(x,y,0) + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{du(x,y,s)}{dt} \frac{1}{(t-s)^{\alpha}} ds. \tag{1.6}$$

The Caputo fractional derivative and Grünwald-Letnikov fractional derivative are equivalent if $u(x,y,0) = 0$.
The Grünwald-Letnikov formula can also be written as [13]

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = \lim_{n \to 0} 1/\tau^{\alpha} \sum_{k=0}^{[t/\tau]} \omega_k^{(\alpha)} u(x,y,t-k\tau) + O(\tau), t \geqslant 0, \tag{1.7}$$

where t/τ is an integer, $\omega_k^{(\alpha)} = 1$, $\omega_k^{(\alpha)} = (1 - \frac{\alpha+1}{k}) \omega_{k-1}^{(\alpha)}$ and $k = 0, 1, 2, ..t/\tau$.
The right shifted Grünwald-Letnikov formula can be defined as

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = 1/\tau^{\alpha} \sum_{k=0}^{[t/\tau]} \omega_k^{(\alpha)} u(x,y,t-(k-1)\tau) + O(\tau), t \geqslant 0. \tag{1.8}$$

Lemma 1.1. *The relation between Caputo and Reimann-Liouville fractional derivative is [16]:*

$$D_t^{1-\alpha} u(x,y,t) =_c D_t^{1-\alpha} u(x,y,t) + \frac{u(x,y,0)}{\Gamma(\alpha) t^{1-\alpha}}, \tag{1.9}$$

These two fractional derivatives are equivalent if and only if $u(x,y,0) = 0$. The proof is given by the Lemma 6.4.2 *in [18].*

Lemma 1.2. *In (1.8), the coefficients $\omega_k^{(\alpha)}, (k = 0, 1, 2, ...)$, satisfy (see[13]):*

$$(1)\;\; \omega_0^{(\alpha)} = 1, \omega_1^{(\alpha)} = -\alpha, \omega_k^{(\alpha)} < 0, k = 1, 2, ...,$$

$$(2)\;\; \sum_{k=0}^{\infty} \omega_k^{(\alpha)} = 0; \forall n \in N^+, -\sum_{k=1}^{n} \omega_k^{(\alpha)} < 1.$$

2 Crank-Nicolson method

For the discretization of the time fractional derivative we use specifically the right shifted Grünwald-Letnikov formula defined by (8) and replace the second order spatial derivatives in equation (1) by central difference approximation. The space steps are taken as $x_i = i(\Delta x)$, in the x-direction with$(i = 0,1,...,M_x - 1)$, $\Delta x = \frac{L}{M_x}$ and in the y-direction with $y_j = j(\Delta x)$,$(j = 0,1,...,M_y - 1)$, where $\Delta y = \frac{L}{M_y}$. The time stepping is $t_k = k\tau$,$(k = 0,1,...,N-1)$, where $\tau = \frac{T}{N}$. Let $u^k_{i,j}$ be the numerical approximation to $u(x_i, y_j, t_k)$. Neglecting the truncation error terms $O(\tau + \Delta x^2 + \Delta y^2)$, we obtain

$$u^{k+1}_{i,j} + \omega^{(\alpha)}_1 u^k_{i,j} + \sum_{s=2}^{k+1} \omega^{(\alpha)}_s u^{k-s+1}_{i,j} = S_1(u^{k+1}_{i+1,j} - 2u^{k+1}_{i,j} + u^{k+1}_{i-1,j} + u^k_{i+1,j} - 2u^k_{i,j} + u^k_{i-1,j}) + \\ S_2(u^{k+1}_{i,j+1} - 2u^{k+1}_{i,j} + u^{k+1}_{i,j-1} + u^k_{i,j+1} - 2u^k_{i,j} + u^k_{i,j-1}) + f^{k+\frac{1}{2}}_{i,j}, \quad (2.10)$$

where $S_1 = \frac{\tau^\alpha}{2(\Delta x)^2}, S_2 = \frac{\tau^\alpha}{2(\Delta y)^2}$.
The Crank-Nicolson finite difference scheme for the two-dimensional time fractional sub-diffusion equation (1.1)-(1.3) utilizing the right shifted, with associated initial and boundary conditions, is as follows

$$-S_1(u^{k+1}_{i+1,j} + u^{k+1}_{i-1,j}) + (1 + 2S_1 + 2S_2)u^{k+1}_{i,j} - S_2(u^{k+1}_{i,j+1} + u^{k+1}_{i,j-1}) = S_1(u^k_{i+1,j} + u^k_{i-1,j}) - \\ (\omega^{(\alpha)}_1 + 2S_1 + 2S_2)u^k_{i,j} + S_2(u^k_{i,j+1} + u^k_{i,j-1}) - \sum_{s=2}^{k+1} \omega^{(\alpha)}_k u^{k-s+1}_{i,j} + \tau^\alpha f^{k+\frac{1}{2}}_{i,j}, \quad (2.11)$$

where $i = 1,2,...,M_x - 1$, $j = 1,2,...,M_y - 1$ and $k = 0,1,2,...,N-1$
with

$$u^0_{i,j} = f_1(x_i, y_j), \quad (2.12)$$

$$u^k_{0,j} = f_2(y_j, t_k),\ u^k_{i,0} = f_3(x_i, t_k), \\ u^k_{M,j} = f_4(y_j, t_k), u^k_{i,M} = f_5(x_i, t_k), \\ 0 \le x, y \le L,\ \ 0 \le t \le T. \quad (2.13)$$

3 Stability analysis of Crank-Nicolson method

We follow the approach in [14] for the analysis of stability. Suppose that $U^k_{i,j}$, is the approximate solution of (11) and the error is defined as $\Psi^n_{i,j} = U^k_{i,j} - u^k_{i,j}$, $i = 0,1,2,...,M_x - 1, j = 0,1,2,...,M_y - 1, k = 0,1,2,...,N-1$. Due to linearity, the error satisfies equation (11) and we have

$$-S_1(\Psi^{k+1}_{i+1,j} + \Psi^{k+1}_{i-1,j}) + (1 + 2S_1 + 2S_2)\Psi^{k+1}_{i,j} - S_2(\Psi^{k+1}_{i,j+1} + \Psi^{k+1}_{i,j-1}) = S_1(\Psi^k_{i+1,j} + \Psi^k_{i-1,j}) - \\ (\omega^{(\alpha)}_1 + 2S_1 + 2S_2)\Psi^k_{i,j} + S_2(\Psi^k_{i,j+1} + \Psi^k_{i,j-1}) - \sum_{s=2}^{k+1} \omega^{(\alpha)}_k \Psi^{k-s+1}_{i,j}. \quad (3.14)$$

The error and initial conditions are given by

$$\Psi^k_0 = \Psi^k_M = \Psi^0_{i,j} = 0. \quad (3.15)$$

By defining the following grid functions for $k = 0,1,2,\ldots,N-1$

$$\Psi^k(x,y) = \begin{cases} \Psi^k_{i,j},\ when\ x_{i-\frac{\Delta x}{2}} < x \le x_{i+\frac{\Delta x}{2}}, y_{j-\frac{\Delta y}{2}} < y \le y_{i+\frac{\Delta y}{2}}, \\ 0\ \ when\ 0 \le x \le \frac{\Delta x}{2} or\ L - \frac{\Delta x}{2} \le x \le L, \\ 0\ \ when\ 0 \le y \le \frac{\Delta y}{2} or\ L - \frac{\Delta y}{2} \le y \le L, \end{cases} \quad (3.16)$$

then $\Psi^k(x,y)$ can be expanded as a Fourier series:

$$\Psi^k(x,y) = \sum_{l_1,l_2=-\infty}^{\infty} \lambda^k(l_1,l_2)e^{2\sqrt{-1}\pi(l_1x/L+l_2y/L)}, \tag{3.17}$$

where

$$\lambda^k(l_1,l_2) = \frac{1}{L}\int_0^L\int_0^L \Psi^k(x,y)e^{-2\sqrt{-1}\pi(l_1x/L+l_1y/L)}dxdy. \tag{3.18}$$

From the definition of l^2 norm and Parseval equality:

$$\| e^k \|_\infty^2 = \sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1} \Delta x\Delta y|e_{i,j}^k|^2 = \sum_{l_1,l_2=-\infty}^{\infty} |\lambda^k(l_1,l_2)|^2. \tag{3.19}$$

Supposing that

$$\Psi_{i,j}^k = \lambda^k e^{\sqrt{-1}(\sigma_1 i\Delta x+\sigma_2 i\Delta y)}, \tag{3.20}$$

where $\sigma_1 = 2\pi l_1/L, \sigma_2 = 2\pi l_2/L$ and substituting (3.20) in (3.14),

$$\lambda^{k+1} = \frac{\lambda^k(\alpha-\mu) - \sum_{s=2}^{k+1}\omega_s^{(\alpha)}\lambda^{k-s+1}}{(1+\mu)}, \tag{3.21}$$

where, $\mu = 4\Big(S_1 sin^2(\frac{\sigma_1\Delta x}{2}) + S_2 sin^2(\frac{\sigma_2\Delta y}{2})\Big) \geq 0,$

Proposition 3.1. *If λ^{k+1} $(k = 0,1,2,\ldots,N)$ satisfy (3.21), then $|\lambda^{k+1}| \leq |\lambda^0|$.*

Proof. The proof utilizes mathematical induction; take $k = 0$, in (3.21)

$$\lambda^1 = \frac{(\alpha-\mu)\lambda^0}{(1+\mu)},$$

and as $0 < \alpha < 1$, $\mu \geq 0$, then

$$|\lambda^1| \leq |\lambda^0|.$$

Now, assuming that

$$|\lambda^m| \leq |\lambda^0|; \quad m = 1,2,\ldots,k-1$$

and as $0 < \alpha < 1$ and $\mu \geq 0$, from (3.21) and **Lemma 1.2**, we obtain

$$\begin{aligned} |\lambda^{k+1}| \leq & \frac{|\lambda^k|(\alpha-\mu) + \sum_{s=2}^{k+1}|\omega_s^{(\alpha)}||\lambda^{k-s+1}|}{(1+\mu)}, \\ & \leq \Big(\frac{(\alpha-\mu)+\sum_{s=2}^{k+1}|\omega_s^{(\alpha)}|}{1+\mu}\Big)|\lambda^0|, \\ & = \Big(\frac{(\alpha-\mu)+(-\sum_{s=1}^{k+1}\omega_s^{(\alpha)}-\alpha)}{1+\mu}\Big)|\lambda^0|, \\ & = \Big(\frac{\alpha-\mu+(1-\alpha)}{1+\mu}\Big)|\lambda^0|, \\ & |\lambda^{k+1}| \leq |\lambda^0|. \end{aligned} \tag{3.22}$$

This complete the proof of **Proposition 3.1** by induction method. □

By using **Proposition 3.1** and (3.19), it can be seen that the solution of (2.11) satisfies

$$\| \lambda^{k+1} \|_2 \leq \| \lambda^0 \|_2,$$

which means that the Crank-Nicolson difference scheme in (2.11) is unconditionally stable.

4 Convergence analysis of Crank-Nicolson method

We follow the approach in [14] for analyzing the convergence. Let $u(x_i,y_j,t_k)$ be the exact solution represented by Taylor series. Then the truncation error of Crank-Nicolson method is

$$T_{i,j}^{k+1/2} = \frac{1}{\tau^\alpha}\sum_{s=0}^{k+1}\omega_s^{(\alpha)}u(x_i,y_j,t_{k-s+1}) - \frac{u(x_{i+1},y_j,t_{k+1})-2u(x_i,y_j,t_{k+1})+u(x_{i-1},y_j,t_{k+1})+u(x_i,y_j,t_k)-2u(x_i,y_j,t_k)+u(x_{i-1},y_j,t_k)}{2(\Delta x)^2} - \frac{u(x_i,y_{j+1},t_{k+1})-2u(x_i,y_j,t_{k+1})+u(x_i,y_{j-1},t_{k+1})+u(x_i,y_{j+1},t_k)-2u(x_i,y_j,t_k)+u(x_i,y_{j-1},t_k)}{2(\Delta y)^2} - f(x_i,y_j,t_{k+1/2}), \quad (4.23)$$

with $i=1,2,\ldots,M_x-1,\ \ j=1,2,\ldots,M_y-1,\ \ k=0,1,2,\ldots,N-1$.
From (1.1), we have

$$T_{i,j}^{k+1/2} = \frac{1}{\tau^\alpha}\sum_{s=0}^{n}\omega_s^{(\alpha)}u(x_i,y_j,t_{k-s+1}) - \frac{\partial^\alpha u(x_{i+1},y_j,t_{k+1})}{\partial t^\alpha} + \frac{\partial^2 u(x_i,y_j,t+k+1)}{\partial x^2} - \frac{u(x_{i+1},y_j,t_{k+1})-2u(x_i,y_j,t_{k+1})+u(x_{i-1},y_j,t_{k+1})+u(x_i,y_j,t_k)-2u(x_i,y_j,t_k)+u(x_{i-1},y_j,t_k)}{2(\Delta x)^2} - + \frac{\partial^2 u(x_i,y_j,t_{k+1})}{\partial y^2} - \frac{u(x_i,y_{j+1},t_{k+1})-2u(x_i,y_j,t_{k+1})+u(x_i,y_{j-1},t_{k+1})+u(x_i,y_{j+1},t_k)-2u(x_i,y_j,t_k)+u(x_i,y_{j-1},t_k)}{2(\Delta y)^2} = O(\tau+(\Delta x)^2+(\Delta y)^2). \quad (4.24)$$

Since $i,\ j$ and k are finite, a positive constant C_1 exists, for all $i,\ j$ and k, such that

$$|T_{i,j}^{k+1/2}| \le C_1(\tau+(\Delta x)^2+(\Delta y)^2), \quad (4.25)$$

with $i=1,2,...,M_x-1, i=1,2,...,M_y-1,\ k=0,1,2,...,N-1$.
The error is defined as

$$\phi_{i,j}^k = u(x_i,y_j,t_k) - u_{i,j}^k. \quad (4.26)$$

From (4.23), we have

$$-S_1(u(x_{i+1},y_j,t_{k+1})+u(x_{i-1},y_j,t_{k+1})) + (1+2S_1+2S_2)u(x_i,y_j,t_{k+1}) - S_2(u(x_i,y_{j+1},t_{k+1})+u(x_i,y_{j-1},t_{k+1})) = S_1(u(x_{i+1},y_j,t_k)+u(x_{i-1},y_j,t_k)) - (\omega_1^{(\alpha)}+2S_1+2S_2)u(x_i,y_j,t_k) + S_2(u(x_i,y_{j+1},t_k)+u(x_i,y_{j-1},t_k)) - \sum_{s=2}^{k+1}\omega_s^{(\alpha)}u(x_i,y_j,t_{k-s+1}) + \tau f(x_i,y_j,t_{k+1/2}). \quad (4.27)$$

To obtain the error equation, subtract (4.27) from (2.11) to obtain

$$-S_1(\phi_{i+1,j}^{k+1}+\phi_{i-1,j}^{k+1}) + (1+2S_1+2S_2)\phi_{i,j}^{k+1} - S_2(\phi_{i,j+1}^{k+1}+\phi_{i,j-1}^{k+1}) = S_1(\phi_{i+1,j}^{k}+\phi_{i-1,j}^{k}) - (\omega_1^{(\alpha)}+2S_1+2S_2)\phi_{i,j}^k + S_2(\phi_{i,j+1}^{k}+\phi_{i,j-1}^{k}) - \sum_{s=2}^{k+1}\omega_k^{(\alpha)}\phi_{i,j}^{k-s+1} + \tau^\alpha T_{i,j}^{k+\frac{1}{2}}, \quad (4.28)$$

with error boundary conditions

$$\phi_0^k = \phi_M^k = 0, \;\; k = 0,1,2,\ldots,N-1,$$

and the initial condition

$$\phi_{i,j}^0 = 0, \;\; i = 1,2,\ldots,M_x, \;\; j = 1,2,\ldots,M_y.$$

Next, the following grid functions are defined for $k = 0,1,2,\ldots,N-1$

$$\phi^k(x,y) = \begin{cases} \phi_{i,j}^k, \; when \; x_{i-\frac{\Delta x}{2}} < x \le x_{i+\frac{\Delta x}{2}}, \; y_{j-\frac{\Delta y}{2}} < y \le y_{j+\frac{\Delta y}{2}}, \\ 0 \;\; when \; 0 \le x \le \frac{\Delta x}{2} or \; L - \frac{\Delta x}{2} \le x \le L, \\ 0 \;\; when \; 0 \le y \le \frac{\Delta y}{2} or \; L - \frac{\Delta y}{2} \le y \le L, \end{cases}$$

and

$$T^k(x,y) = \begin{cases} T_{i,j}^k, \; when \; x_{i-\frac{\Delta x}{2}} < x \le x_{i+\frac{\Delta x}{2}}, \; y_{j-\frac{\Delta y}{2}} < y \le y_{j+\frac{\Delta y}{2}}, \\ 0 \;\; when \; 0 \le x \le \frac{\Delta x}{2} \; or \; L - \frac{\Delta x}{2} \le x \le L, \\ 0 \;\; when \; 0 \le y \le \frac{\Delta y}{2} \; or \; L - \frac{\Delta y}{2} \le y \le L, \end{cases}$$

$$i = 1,2...M_x - 1, \;\; j = 1,2,\ldots,M_y - 1, \;\; k = 0,1,2,\ldots,N-1.$$

Here, the $\phi^k(x,y)$ and $T^k(x,y)$ can be expanded in Fourier series such as

$$\phi^k(x,y) = \sum_{l_1,l_2=-\infty}^{\infty} \xi^k(l_1,l_2) e^{2\sqrt{-1}\pi(l_1 x/L + l_2 y/L)}, k = 0,1,2,\ldots,N,$$

$$T^k(x,y) = \sum_{l_1,l_2=-\infty}^{\infty} \Psi^k(l_1,l_2) e^{2\sqrt{-1}\pi(l_1 x/L + l_2 y/L)}, k = 0,1,2,\ldots,N,$$

where

$$\xi^k(l_1,l_2) = \frac{1}{L}\int_0^L \int_0^L \phi^k(x,y) e^{-2\sqrt{-1}\pi(l_1 x/L + l_2 y/L)} dxdy, \tag{4.29}$$

$$\Psi^k(l_1,l_2) = \frac{1}{L}\int_0^L \int_0^L T^k(x,y) e^{-2\sqrt{-1}\pi(l_1 x/L + l_2 y/L)} dxdy. \tag{4.30}$$

From the definition of l^2 norm and the Parseval equality:

$$\|\phi^k\|_{l^2}^2 = \sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1} \Delta x \Delta y |e_{i,j}^k|^2 = \sum_{l_1,l_2=-\infty}^{\infty} |\rho^k(l_1,l_2)|^2, \tag{4.31}$$

and

$$\|T^k\|_{l^2}^2 = \sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1} \Delta x \Delta y |e_{i,j}^k|^2 = \sum_{l_1,l_2=-\infty}^{\infty} |\Psi^k(l_1,l_2)|. \tag{4.32}$$

Based on the above, suppose that

$$\phi_i^k = \xi^k e^{\sqrt{-1}(\sigma_1 i \Delta x + \sigma_2 j \Delta y)}, \tag{4.33}$$

$$T_i^k = \Psi^k e^{\sqrt{-1}(\sigma_1 i \Delta x + \sigma_2 j \Delta y)}, \tag{4.34}$$

respectively, where $\sigma_1 = \frac{2\pi l_1}{L}$, $\sigma_2 = \frac{2\pi l_2}{L}$. Substituting (4.33) and (4.34) into (4.28), gives

$$\xi^{k+1} \;\; = \;\; \frac{\xi^k(\alpha - \mu) - \sum_{s=2}^{k+1} \omega_s^{(\alpha)} \xi^{k-s+1} + \tau^\alpha \Psi^{k+1/2}}{(1+\mu)}, \tag{4.35}$$

where μ is as mentioned in section 3.

Proposition 4.1. *Let $\xi^{k+1}(k=0,1,2,\ldots,N)$ be the solution of (4.35), then there is a positive constant C_2 such that*

$$|\xi^{k+1}| \leq C_2(k+1)\tau^{\alpha}|\Psi^{1/2}|,$$

Proof. From $\phi^0 = 0$ and (4.29), we have

$$\xi^0 = \xi^0(l_1,l_2) = 0. \tag{4.36}$$

From (4.30) and (4.32), then there is a positive constant C_2, such that

$$|\Psi^k| \leq C_2|\Psi^{1/2}(l_1,l_2)|. \tag{4.37}$$

Using mathematical induction, for $k=0$, then from (4.35) and (4.36) , we obtain

$$\xi^1 = \frac{1}{1+\mu}(\tau^{\alpha}\Psi^{1/2}).$$

Since $\mu \geq 0$, from (4.37), we get

$$|\xi^1| \leq \tau|\Psi^{1/2}| \leq C_2\tau^{\alpha}|\Psi^{1/2}|.$$

Now suppose that

$$|\xi^m| \leq C_2 m\tau^{\alpha}|\Psi^{1/2}|. \;\; m = 1,2,\ldots,k-1.$$

As $0<\alpha<1$, $\mu \geq 0$, from (4.34), (4.36) and **Lemma 1.2**, we have

$$\begin{aligned}
|\xi^{k+1}| \leq \frac{|\xi^k|(\alpha-\mu)+\sum_{s=2}^{k+1}|\omega_s^{(\alpha)}||\xi^{k-s+1}|+\tau^{\alpha}|\Psi^{k+1/2}|}{(1+\mu)}&,\\
\leq \Big[\frac{k(\alpha-\mu)+\sum_{s=2}^{k+1}|\omega_s^{(\alpha)}|(k-s+1)+1}{(1+\mu)}\Big]&C_2\tau^{\alpha}|\Psi^{1/2}|,\\
\leq \Big[\frac{k(\alpha-\mu)+k(-\sum_{s=1}^{k+1}\omega_s^{(\alpha)}-\alpha)+1}{(1+\mu)}\Big]&C_2\tau^{\alpha}|\Psi^{1/2}|,\\
= \Big[\frac{k(\alpha-\mu)+k(1-\alpha)+1}{(1+\mu)}\Big]&C_2\tau^{\alpha}|\Psi^{1/2}|,\\
\leq (k+1)&C_2\tau^{\alpha}|\Psi^{1/2}|.
\end{aligned} \tag{4.38}$$

This completes the proof of the **proposition**. □

Theorem 4.1. *The proposed Crank-Nicolson difference scheme is l^2 convergent and the order of convergence is $O(\tau+(\Delta x)^2+(\Delta y)^2)$.*

Proof. From (4.24) and (4.32), we obtain

$$\|T^{k+1}\| \leq \sqrt{M_x\Delta x}\sqrt{M_y\Delta y}C_1(\tau+(\Delta x)^2+(\Delta y)^2) = LC_1(\tau+(\Delta x)^2+(\Delta y)^2). \tag{4.39}$$

In view of **Proposition 4.1**, (4.31), (4.32) and (4.39)

$$\|\phi^{k+1}\|_{l^2} \leq (k+1)C_2\tau\|T^{1/2}\| \leq C_1C_2k\tau L(\tau+(\Delta x)^2+(\Delta y)^2),$$

as $k\tau \leq T$, thus

$$\|\phi^{k+1}\|_{l^2} \leq C_1C_2TL(\tau+(\Delta x)^2+(\Delta y)^2),$$

where $C = C_1C_2TL$. □

5 Numerical Experiments

Example 5.1. *Consider the following equation [7]*

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + \Big(\frac{2t^{2-t}}{\Gamma(3-\alpha)} + 2t^2\Big) sin(x)sin(y),$$

$$0<\alpha\leq 1,\ \ 0\leq t\leq T, \quad (5.40)$$

subject to the conditions

$$u(x,y,0) = u(0,y,t) = u(x,0,t) = 0, \quad (5.41)$$

$$u(1,y,t) \ = \ t^2 sin(1)sin(y), \qquad u(x,1,t) \ = \ t^2 sin(x)sin(1), \quad 0 \ \leq \ t \ \leq \ T. \quad (5.42)$$

The exact solution of (5.40) is given by

$$u(x,y,t) = t^2 sin(x)sin(y),\ \ 0\leq\ x,y\leq 1. \quad (5.43)$$

The error is defined as follows

$$E_\infty = \max_{0\leq i,j\leq M, 0\leq k\leq N} |u(x_i,y_j,t_k) - u^k_{i,j}|. \quad (5.44)$$

The proposed Crank-Nicolson scheme is applied to problem (5.40)-(5.42). Table 1 shows the errors E_∞ at $T = 1.0$ for different values of space step size($\Delta x, \Delta y$) and τ. Note that the time step, τ is defined by $\tau = \frac{T}{N}$.

Table 1: The errors E_∞ between the exact solution and the numerical solution of (TFSDE) at $T = 1.0$

τ	$\Delta x = \Delta y$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
1/4	1/2	1.6373 e-3	1.3083 e-3	9.7032 e-4	6.2414 e-4	2.7168 e-4
1/16	1/4	41098 e-4	3.0745 e-4	2.0061 e-4	9.2696 e-5	1.6877 e-5
1/64	1/8	1.1152 e-4	8.1593 e-5	5.0626 e-5	1.9344 e-5	1.1412 e-5
1/128	1/10	4.8991 e-5	3.4093 e-5	1.8680 e-5	3.0995 e-6	1.2233 e-6

Table 1 shows that, for various values of α, the errors decrease as we reduce the time and space step size τ and $(\Delta x, \Delta y)$. This indicates the method is convergent.
Figures 1 and 2 shows the numerical solution of the equation (5.40) and compares it with exact solution at $T = 1.0$.

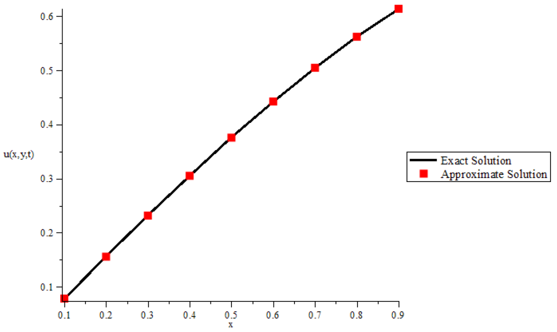

Figure 1: at $\alpha = 0.5$, $T = 1.0, y = 0.1$ and $N = 128$.

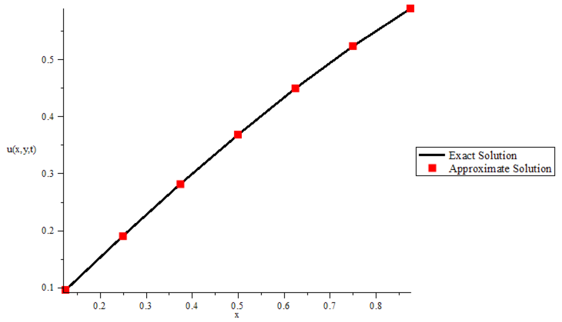

Figure 2: at $\alpha = 0.5$, $T = 1.0, y = 0.125$ and $N = 64$.

Clearly the numerical solution is in good agreement with the exact solution. These results seem to confirm the theoretical analysis.

Example 5.2. *Consider the following equation [21]*

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + \left(\Gamma(2+\alpha) - 2t^{1+\alpha}\right)e^{x+y},$$
$$0 < \alpha \le 1, \quad 0 \le t \le T, \quad (5.45)$$

with conditions

$$u(x,y,0) = 0, \qquad u(1,y,t) = e^{1+y}t^{1+\alpha},$$
$$u(0,y,t) = e^{y}t^{1+\alpha}, \qquad u(x,1,t) = e^{1+x}t^{1+\alpha},$$
$$u(x,0,t) = e^{x}t^{1+\alpha}, \quad 0 \le x,y \le 1,\ 0 \le t \le T. \quad (5.46)$$

The exact solution is

$$u(x,y,t) = e^{x+y}t^{1+\alpha}. \quad (5.47)$$

Table 2: The errors E_∞ between the exact solution and the numerical solution of (TFSDE) at $T = 1.0$

τ	$\Delta x = \Delta y$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
1/4	1/2	4.0157 e-3	2.3261 e-3	1.8037 e-5	6.2414 e-4	2.7168 e-4
1/16	1/4	2.0009 e-3	1.4263 e-3	7.5927 e-4	5.5386 e-5	9.8580 e-4
1/64	1/8	6.1583 e-4	4.7313 e-4	3.0411 e-4	1.0299 e-4	1.4257 e-4
1/128	1/10	2.8506 e-4	2.1330 e-4	1.2824 e-5	2.6784 e-5	9.7067 e-5

Example 5.3. *Consider the following equation on* $(0,1)^2 \times (0,1]$ *[22]*

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + \frac{2}{\Gamma(2-\alpha)}t^{2-\alpha}(x-x^2)^2(y-y^2)^2 -$$
$$2t^2(1-6x+6x^2)(y-y^2)^2 - 2t^2(1-6y+6y^2)(x-x^2)^2,$$
$$0 < \alpha \le 1, \quad 0 \le t \le T, \quad (5.48)$$

with conditions

$$u(x,y,0) = 0,$$
$$u(0,y,t) = u(1,y,t) = 0$$
$$u(x,0,t) = u(x,1,t) = 0, \quad 0 \le x,y \le 1,\ 0 \le t \le T. \quad (5.49)$$

The exact solution is

$$u(x,y,t) = t^2(x-x^2)^2(y-y^2)^2. \quad (5.50)$$

Table 3: The errors E_∞ between the exact solution and the numerical solution of (TFSDE) at $T = 1.0$

τ	$\Delta x = \Delta y$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
1/4	1/2	3.5866 e-3	3.5419 e-3	3.4934 e-3	3.4413 e-3	3.3860 e-3
1/16	1/4	9.3188 e-4	9.1046 e-4	8.8557 e-4	8.5714 e-4	8.2513 e-4
1/64	1/8	3.1952 e-4	3.0149 e-4	2.8018 e-4	2.5551 e-4	2.2748 e-4
1/128	1/10	2.4690 e-4	2.2923 e-4	2.0829 e-4	1.8402 e-4	1.5642 e-4

Example 5.4. *Consider the following equation on* $(0,1)^2 \times (0,1]$ *[22]*

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + \frac{2a\, tanh(\sqrt{t}/\sqrt{1+t})}{\Gamma 0.5\sqrt{1+t}} t^{2-\alpha}(x-x^2)^2(y-y^2)^2 - 2\ln(1+t)(1-6x+6x^2)(y-y^2)^2 - 2\ln(1+t)(1-6y+6y^2)(x-x^2)^2, \quad 0 < \alpha \le 1, \ 0 \le t \le T, \tag{5.51}$$

with conditions

$$u(x,y,0) = 0, \quad u(0,y,t) = u(1,y,t) = 0 \quad u(x,0,t) = u(x,1,t) = 0, \quad 0 \le x,y \le 1, \ 0 \le t \le T. \tag{5.52}$$

The exact solution is

$$u(x,y,t) = \ln(1+t)(x-x^2)^2(y-y^2)^2. \tag{5.53}$$

Table 4: The errors E_∞ between the exact solution and the numerical solution of (TFSDE) at $T = 1.0, N = 60$

$\Delta x = \Delta y$	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$
1/4	5.4065 e-4	5.4381 e-4	5.5738 e-4
1/6	2.1728 e-4	2.2009 e-4	2.3178 e-4
1/10	5.4219 e-5	5.6854 e-5	6.7629 e-5

The above numerical results show that the exact solution and the numerical solution are in good agreement. The result displayed and discussed in this section seems to confirm the results of our theoretical analysis.

6 Conclusions

The Crank-Nicolson difference method for two-dimensional sub-diffusion equation of fractional order has been described the Grünwald-Letnikov formula was used for time fractional derivative. The scheme was found to be convergent with order $(\tau + (\Delta x)^2 + (\Delta y)^2)$. Further it is unconditionally stable. The results of an application to certain examples indicated that the scheme is feasible and accurate.

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