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Research Article

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An algorithm for positive solution of boundary value problems of nonlinear fractional differential equations by Adomian decomposition method

Hytham. A. Alkresheh[1*], A. I. Md. Ismail[1]

(1) *School of Mathematical Sciences, Universiti Sains Malaysia (USM), 11800, Penang, Malaysia*

Abstract

In this paper, an algorithm based on a new modification, developed by Duan and Rach, for the Adomian decomposition method (ADM) is generalized to find positive solutions for boundary value problems involving nonlinear fractional ordinary differential equations. In the proposed algorithm the boundary conditions are used to convert the nonlinear fractional differential equations to an equivalent integral equation and then a recursion scheme is used to obtain the analytical solution components without the use of undetermined coefficients. Hence, there is no requirement to solve a nonlinear equation or a system of nonlinear equations of undetermined coefficients at each stage of approximation solution as per in the standard ADM. The fractional derivative is described in the Caputo sense. Numerical examples are provided to demonstrate the feasibility of the proposed algorithm.

Keywords: Adomian decomposition method, Fractional boundary value problems, Duan-Rach approach, Caputo derivative.

1 Introduction

Fractional order differential equations (FDEs) have been the subject of considerable interest during the last two decades. It is found to be effective in describing certain applications in the area of engineering, physics [10, 21], fluid-dynamics based traffic models [9], electromagnetism [8], continuum and statistical mechanics [17] and dynamics of viscoelastic materials [14]. Several methods have been presented to solve fractional nonlinear BVPs-Adomian decomposition method (ADM) [12], a shifted Legendre spectral method [20], homotopy analysis method [7], generalized differential transform method [18], a Chebyshev spectral method [4], Haar wavelet method [23], sinc-Galerkin method [25] and so on .

Some studies have been conducted on the positive solution of fractional nonlinear BVPs. The existence and multiplicity results of a positive solution with different types of boundary conditions for fractional differential equations can be found in the works of Xiaojie Xu [29], Sihualiang [15], De-Xiangma [16], Yige Zhao [31], Chengbo Zhai [30], Weihua Jiang [13], Muhammed Syam [26]. Jafary and Daftardar [12] used the standard ADM to find the positive solutions for a fractional nonlinear (Bratu-type) problem involving ordinary differential equations. Jafari and Bagherian [11] made a comparison between homotopy perturbation method (HPM) and the standard ADM method.They have shown that the standard ADM method is essentially the HPM method for the fractional nonlinear two point

*Corresponding author. Email address: hytham.k@hotmail.com, Tel:00962777442093

BVP with $1 < \alpha \leq 2$. The standard ADM approach can be improved by utilizing the approach of Duan and Rach [6] for solving BVPs. This approach allows the derivation of a modified recursion scheme for the approximate solution without any undetermined coefficients and avoids the need to solve a nonlinear sequence of algebraic equations for the undetermined coefficients. Dib and Haiahem [3] used the Duan and Rach approach to solve the governing partial differential equation of MHD Jeffery-Hamel flow problem and they showed this approach gives good agreement with the 4th-order Runge-Kutta algorithm and homotopy analysis method.

The purpose of this paper is to generalize the modification proposed by Duan and Rach [6] for the ADM to find a positive solution for the boundary value problems involving nonlinear fractional ordinary differential equations. As will be shown in the next sections, the generalized method is simpler to implement and easy to automate by computer programs. In particular the final solution does not contain any undetermined coefficient at each stage of approximation solution and hence we do not need to use numerical methods to evaluate the values of the undetermined coefficient as in the standard ADM.

There are various definitions of a fractional derivative of order α ($\alpha > 0$), but the two definitions that are most extensively used in applications of fractional calculus are the Riemann-Liouville and Caputo definition [22]. The Caputo fractional derivative which will be used in this study first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative while Riemann-Liouville fractional derivative is computed in the reverse order. Hence, the Caputo fractional derivative allows traditional initial and boundary conditions to be included in the formulation of the problem [19].
To provide the setting for this work, we list below some definitions and basic results of others. More details can be found in [22, 24].

Definition 1.1. *A real function $f(x)$, $x > 0$, is said to be in the space C_α, $\alpha \in \mathbb{R}$ if there exists a real number $p > \alpha$, such that $f(x) = x^p f_1(x)$ is continuous in $[0, \infty)$ and it is said to be in the space C^n_α iff $f^{(n)} \in C_\alpha$, $n \in \mathbb{N}_0$*

Definition 1.2. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of the function $f \in C_\mu$, $\mu > -1$ is defined as:*
$J^\alpha_a f(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-x)^{\alpha-1} f(x)\, dx, \;\; \alpha > 0,\; t > a,\; a \geq 0.$

1. *if $\alpha = 0$, $J^\alpha_0 f(t) = f(t)$ is the identity operator.*
2. *for $f \in C_\mu$, $\alpha, \beta > 0$, we have $(J^\alpha_a J^\beta_a)(t) = (J^{\alpha+\beta}_a)(t)$*
3. *if $f(t) = (t-a)^\beta$ for some $\beta > -1$ and $\alpha > 0$, then*
   $J^\alpha_a f(t) = \dfrac{\Gamma(\beta+1)}{\Gamma(n+\beta+1)} (t-a)^{n+\beta}.$

Definition 1.3. *The Caputo fractional derivative of $f(t)$ of order α with $a > 0$ is defined as:*
$(D^\alpha_{*a} f)(t) = (J^{n-\alpha}_a f^{(n)})(t) = \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t (t-x)^{n-\alpha-1} f_{(n)}(x)\, dx, n \in \mathbb{N}, n-1 < \alpha \leq n,$
$t > a$, $f(x) \in C^n_{-1}$. For this definition we have the following properties:

1. *$(J^\alpha_a D^\alpha_{*a} f)(t) = f(t) - \displaystyle\sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} f^k(a), n-1 < \alpha \leq n,\; n \in \mathbb{N}.$*
2. *If f is continuous and $\alpha \geq 0$ then, $D^\alpha_a J^\alpha_a f = f$.*
3. *If $f(t) = (t-a)^\beta$ for some $\beta \geq 0, m = \lceil \alpha \rceil$ then,*

$$D_{*a}^{\alpha} f(t) = \begin{cases} 0 & if\, \beta \in \{0,1,2,...,m-1\} \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}(t-a)^{\beta-\alpha} & if\, \beta \in \mathbb{N}_0\, and\, \beta \geq m \\ & or\, \beta \notin \mathbb{N}\, and\, \beta > m-1 \end{cases}$$

2 The proposed method

We first describe modification method of Duan and Rach [6] for the second-order nonlinear ordinary differential equations by using ADM. After that we will generalize this modification for the fractional differential equations case.

Consider the following two-point boundary value problem of the second-order ordinary differential equation

$$Lu + Nu = f(t), \quad a \leq t \leq b, \tag{2.1}$$

$$u(a) = c_1, \quad u(b) = c_2,$$

where $L(.) = \frac{d^2}{dt^2}$ is the linear differential operator, Nu is analytic nonlinear term and $f(t)$ known given function. Take the inverse linear operator $L^{-1}(.)$ to both sides of Eq.(2.1), where $L^{-1}(.)$ is defined as

$L^{-1}(.) = \int_a^t \int_\zeta^t (.)dtdt, \quad \zeta \in [a,b]$ is a prescribed value, yields

$$u(t) - u(a) - u'(\zeta)(t-a) = -L^{-1}Nu + L^{-1}f(t). \tag{2.2}$$

Let $t = b$ in Eq.(2.2). Then solve for $u'(\zeta)$, yields

$$u'(\zeta) = \frac{u(b)-u(a)}{b-a} + \frac{1}{b-a}\left(\left[L^{-1}Nu\right]_{t=b} - \left[L^{-1}f(t)\right]_{t=b}\right), \tag{2.3}$$

where $\left[L^{-1}(.)\right]_{t=b} = \int_a^b \int_\zeta^t (.)dtdt.$

Substituting Eq.(2.3) in to Eq.(2.2) yields

$$u(t) = u(a) + \frac{u(a)-u(b)}{b-a}(t-a) + L^{-1}f(t) - \frac{t-a}{b-a}\left[L^{-1}f(t)\right]_{t=b} - L^{-1}Nu + \frac{t-a}{b-a}\left[L^{-1}Nu\right]_{t=b}. \tag{2.4}$$

Thus the approximation solution components by using Duan and Rach modification for the Adomian decomposition method is given by

$$u_0(t) = u(a) + \frac{u(a)-u(b)}{b-a}(t-a) + L^{-1}f(t) - \frac{t-a}{b-a}\left[L^{-1}f(t)\right]_{t=b} \tag{2.5}$$

$$u_{n+1}(t) = -\int_a^t \int_\zeta^t (A_n)dtdt + \frac{t-a}{b-a}\left[\int_a^t \int_\zeta^t (A_n)dtdt\right]_{t=b}, \quad n \geq 0 \tag{2.6}$$

where A_n are the Adomian polynomials as will be discussed later in this section.

Now to generalize this modification, we consider the following nonlinear differential equation of fractional order:

$$D_*^{\alpha} u(t) + Nu = f(t), \quad m-1 < \alpha \leq m,\ m \in \mathbb{N},\ a \leq t \leq b \tag{2.7}$$

subject to the boundary conditions $u^{(q_k)}(t_k) = c_k, \quad t_k \in [a,b], \quad k = 0,1,2,...,m-1$.
The t_k are not all equal and $0 \le q_0 \le q_1 \le ... \le q_{m-1}$ such that $q_i \ne q_j$ if $t_i = t_j$.

Also we assume that the positive solution of Eq.(2.7) exists and is unique in the specified interval $[a,b]$.

Let $D_*^{\alpha}(.)$ represent the Caputo fractional derivative of order α, Nu The analytic nonlinear term, and f(t) is a known given function.

Applying $L^{-1}(.) = J^{\alpha}(.)$ to both sides of Eq.(2.7), where J^{α} is the Riemann-Liouville fractional integral yields

$J^{\alpha}D_*^{\alpha}u(t) = J^{\alpha}f(t) - J^{\alpha}(Nu)$.

Using property(1)of definition (1.3)we have

$$\begin{aligned} u(t) &= \sum_{k=0}^{m-1} \frac{D^k f(0)}{k!} t^k + J^{\alpha}(f(t)) - J^{\alpha}(Nu) \qquad (2.8) \\ &= u(0) + u'(0) + u''(0)\frac{t^2}{2} + ... + \frac{u^{(m-1)}}{(m-1)!} t^{m-1} + J^{\alpha}(f(t)) - J^{\alpha}(Nu). \end{aligned}$$

The Adomian decomposition method introduces the solution by decomposing $u(t)$ to an infinite series $u(t) = \sum_{n=0}^{\infty} u_n$ and the nonlinear term Nu by the infinite series $Nu = \sum_{n=0}^{\infty} A_n$ where A_n are the Adomian polynomials defined by

$A_n = A_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\left[\frac{d_n}{d\lambda^n} N\left(\sum_{n=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}$

The polynomials A_n are generated for each nonlinearity so that A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends on u_0, u_1, u_2 and so on. Many different algorithms to compute the Adomian polynomials have been proposed. See, for example, Adomian and Rach [2], Sheng Duan [5], Wazwaz [28].

The first five Adomian polynomials for the one variable $Nu = f(u(t))$ are given by

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= u_1 f'(u_0), \\ A_2 &= u_2 f'(u_0) + \frac{1}{2!}u_1^2 f''(u_0), \qquad (2.9) \\ A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!}u_1^3 f^{(3)}(u_0), \\ A_4 &= u_4 f'(u_0) + (u_1 u_3 + \frac{1}{2!}u_2^2) f''(u_0) + \frac{1}{2!}u_1^2 u_2 f^{(3)}(u_0) + \frac{1}{4!}u_1^4 f^{(4)}(u_0). \end{aligned}$$

There can also can be found from the formula

$$A_n = \sum_{\nu=1}^{n} C(\nu, n) f^{(\nu)}(u_0)$$

where the $c(\nu, n)$ are products (or sums of products) of ν components of u whose subscripts sum to n divided by the factorial of the number of repeated subscripts [1].

To clarify the method, we take a special case of Eq.(2.7)

$D_*^{\alpha}u(t)+Nu=f(t), \qquad 1<\alpha\leq 2, \quad a\leq t\leq b,$

subject to the boundary conditions, $u(a)=c_1$, $u(b)=c_2$.

According to Eq.(2.8) we have

$$\begin{aligned} u(t) &= u(0)+u'(0)t+L^{-1}(f(t))-L^{-1}(Nu)\\ &= u(0)+u'(0)t+J^{\alpha}(f(t))-J^{\alpha}(Nu). \end{aligned} \tag{2.10}$$

Let $t=a$ and then $t=b$ in Eq.(2.10) yields, respectively the two equations

$$u(a)=u(0)+u'(0)a+[J^{\alpha}(f(t))]_{t=a}-[J^{\alpha}(Nu)]_{t=a} \tag{2.11}$$
$$u(b)=u(0)+u'(0)b+[J^{\alpha}(f(t))]_{t=b}-[J^{\alpha}(Nu)]_{t=b}. \tag{2.12}$$

Equations (2.11)and(2.12) are two equations with two unknown coefficients $u(0)$ and $u'(0)$. Solving these two equations with respect to these unknown coefficients yields

$$u'(0)=\frac{u(b)-u(a)}{b-a}+\frac{1}{b-a}[J^{\alpha}f(t)]_{t=a}-\frac{1}{b-a}[J^{\alpha}f(t)]_{t=b}-\frac{1}{b-a}[J^{\alpha}Nu]_{t=a}+\frac{1}{b-a}[J^{\alpha}Nu]_{t=b} \tag{2.13}$$

and

$$\begin{aligned} u(0)=&u(a)-\frac{a}{b-a}[u(b)-u(a)]+\frac{a}{b-a}[J^{\alpha}f(t)]_{t=b}-\frac{a}{b-a}[J^{\alpha}f(t)]_{t=a}-\frac{a}{b-a}[J^{\alpha}Nu]_{t=b}+\frac{a}{b-a}[J^{\alpha}Nu]_{t=a}-\\ &[J^{\alpha}f(t)]_{t=a}+[J^{\alpha}Nu]_{t=a} \end{aligned}. \tag{2.14}$$

Substituting Eq.(2.13) and Eq.(2.14) in to Eq.(2.10) and simplifying we obtain

$$\begin{aligned} u(t)=&u(a)+\frac{t-a}{b-a}[u(b)-u(a)]+\frac{t-b}{b-a}[J^{\alpha}f(t)]_{t=a}-\frac{t-a}{b-a}[J^{\alpha}f(t)]_{t=b}+\frac{b-t}{b-a}[J^{\alpha}Nu]_{t=a}+\frac{t-a}{b-a}[J^{\alpha}Nu]_{t=b}+\\ &J^{\alpha}(f(t))-J^{\alpha}(Nu). \end{aligned} \tag{2.15}$$

Next the nonlinear term Nu will be equated to $\sum\limits_{n=0}^{\infty}A_n$ where A_n are the Adomian polynomials and decomposing the solution $u(t)$ into $\sum\limits_{n=0}^{\infty}u_n(t)$. Then applying ADM to Eg.(2.15) yields

$$u_0(t)=u(a)+\frac{t-a}{b-a}[u(b)-u(a)]+\frac{t-b}{b-a}[J^{\alpha}f(t)]_{t=a}-\frac{t-a}{b-a}[J^{\alpha}f(t)]_{t=b}+J^{\alpha}(f(t)), \tag{2.16}$$

$$u_{n+1}(t)=\frac{b-t}{b-a}[J^{\alpha}A_n]_{t=a}+\frac{t-a}{b-a}[J^{\alpha}A_n]_{t=b}-J^{\alpha}(A_n), \qquad n\geq 0. \tag{2.17}$$

We note that the recursion scheme (2.16) and (2.17) is without any undetermined coefficients and thus we avoid the complicated calculations to find the roots of matching nonlinear algebraic equation in each step of the approximation solutions as occurs in the undetermined coefficient method.

Finally the nth-term approximation solution for the Adomian decomposition method is given by

$$\phi_n = \sum_{k=0}^{n-1} u_k, \qquad n \geq 1 \tag{2.18}$$

and the solution $u(t) = \lim_{n\to\infty} \phi_n$

3 Illustrative examples

To demonstrate the effectiveness and the simplicity of the proposed method we give two examples with two and three boundary values of nonlinear fractional ordinary differential equation and make a comparison between the results obtained by the proposed method and the exact solution for some values of α.

3.1 Example [12]

Consider the following Bratu's type boundary value problem in the form

$$D_*^{\alpha} u(t) + e^{u(t)} = 0, \quad 1 < \alpha \leq 2, \ \ 0 \leq t \leq 1, \tag{3.19}$$
$$u(0) = 0, \ \ u(1) = 0.$$

The exact solution of this problem when $\alpha = 2$ is given in [27] by
$u(t) = -2\ln\left[\dfrac{\cosh\left(\frac{\theta t}{2} - \frac{\theta}{4}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right]$, where θ satisfies the equation $\theta = \sqrt{2}\cosh\left(\frac{\theta}{4}\right)$.

To solve this equation by the proposed method, according to the equations (3.16) and (3.17), we obtain

$$u_0(t) = 0, \tag{3.20}$$

$$u_{n+1}(t) = t\left[J^{\alpha}(A_n)\right]_{t=1} - \left[J^{\alpha}(A_n)\right], \qquad n \geq 0 \tag{3.21}$$

where A_n are the Adomian polynomials for the nonlinear term $Nu = e^u$ which are given by

$$\begin{aligned} A_0 &= e^{u_0}, \\ A_1 &= u_1 e^{u_0}, \\ A_2 &= \left(\frac{u_1^2}{2} + u_2\right) e^{u_0}, \\ A_3 &= \left(\frac{u_1^3}{6} + u_1 u_2 + u_3\right) e^{u_0}, \end{aligned} \tag{3.22}$$

.....

In view of (3.20), (3.21) and (3.22) we can write the components of the solution for Eq.(3.19) as follows

$$\begin{aligned} u_0(t) &= 0, \\ u_1(t) &= \frac{t - t^{\alpha}}{\alpha\Gamma(\alpha)}, \\ u_2(t) &= \frac{4^{-\alpha}\left(-4^{\alpha} t(-1+t^{\alpha})\Gamma(0.5+\alpha) + \sqrt{\pi}(-t+t^{2\alpha})\Gamma(2+\alpha)\right)}{\alpha\Gamma(\alpha)\Gamma(0.5+\alpha)\Gamma(2+\alpha)}, \end{aligned} \tag{3.23}$$

.....

In view of (3.23) the approximate solutions of Eq.(3.19) when $n = 5$ for various value of α are as follows

For $\alpha = 1.2$,

$$\begin{aligned}\phi(t) =&0.948641t - 0.907604t^{1.2} - 0.391384t^{2.2} + 0.335435t^{2.4} - 0.116044t^{3.2} + 0.299552t^{3.4} - \\ &0.166492t^{3.6} - 0.0259069t^{4.2} + 0.154842t^{4.4} - 0.224063t^{4.6} + 0.0930731t^{4.8} - \\ &0.00400549t^{5.2} + 0.0523022t^{5.4} - 0.153367t^{5.6} + 0.160541t^{5.8} - 0.0555197t^{6}.\end{aligned} \tag{3.24}$$

For $\alpha = 1.6$,

$$\begin{aligned}\phi(t) =&0.769751t - 0.699484t^{1.6} - 0.207002t^{2.6} + 0.128921t^{3.2} - 0.0439692t^{3.6} + 0.0848449t^{4.2} - \\ &0.00697325t^{4.6} - 0.0338418t^{4.8} + 0.0313441t^{5.2} - 0.000694494t^{5.6} - 0.0334611t^{5.8} + \\ &0.00686103t^{6.2} + 0.010058t^{6.4} - 0.0154317t^{6.8} + 0.0122721t^{7.4} - 0.00319406t^{8}.\end{aligned} \tag{3.25}$$

For $\alpha = 2$,

$$\begin{aligned}\phi(t) =&0.549288t - 0.5t^2 - 0.0915096t^3 + 0.0291811t^4 + 0.0169618t^5 - 0.00118634t^6 - \\ &0.00300926t^7 - 0.000272817t^8 + 0.000683422t^9 - 0.000136684t^{10}.\end{aligned} \tag{3.26}$$

The main advantage of using the Duan-Rach modification for solving fractional nonlinear BVP is that evaluating the inverse operator directly at the boundary conditions allows us to find the components of the solution without using numerical methods to calculate the values of the undetermined coefficients as in the standard ADM. For example in the given problem the matching algebraic equations for the approximate solutions by using the standard ADM when $\alpha = 2$ and $n = 2, 3$ are respectively given in [12] by

$$\phi_2(t) = \beta t - \frac{e^{\beta t} - \beta t - 1}{\beta^2}$$
$$\phi_3(t) = \beta t - \frac{e^{\beta t} - \beta t - 1}{\beta^2} - \frac{2\beta t - e^{\beta t}(e^{\beta t} - 4\beta t + 4) + 5}{4\beta^4}.$$

some of these equations need using numerical methods or some of commands in mathematical programs as the command (Find Root) in mathematica programme to find the values of the undetermined coefficient β. This cost more time and requires more complicated calculations. Furthermore the accuracy of the approximation solution depend in the accuracy of the values of the undetermined coefficient β.

Figure.1 shows the curves of the exact solution and the approximation solutions by the proposed method when $\alpha = 2$ and $n = 2, 3, 4$. We note that the curve of the exact solution is in a high agreement with the curve of approximation solution when $n = 4$ and $\phi_n(t)$ converge to the exact solution $u(t)$ when n increases in the interval $[0, 1]$. Figure.2 shows the curves of the approximation solution when $n = 6$ and various values of α. Table.1 shows the values of the maximum absolute error $ME_n(t)$ where
$ME_n(t) = ||\phi_n(t) - u_{exact}(t)||_\infty$ when $n = 3, 4, 5, 6, 7$ in the interval [0,1].

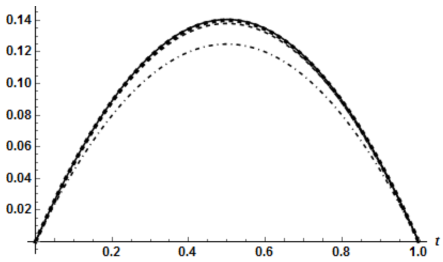

Figure 1: $u_{exact}(t)$ (solid line)and the approximation solution $\phi_n(t)$, $\phi_2(t)$ (dot-side line), $\phi_3(t)$ (dashed line), $\phi_4(t)$ (dot line).

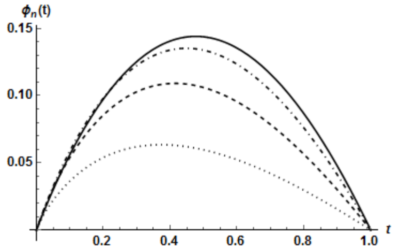

Figure 2: $\phi_6(t)$, $\alpha = 1.2$ (dot line), $\alpha = 1.4$ (dashed line), $\alpha = 1.6$ (dot-side line), $alpha = 1.8$ (solid line).

Table 1: The maximum absolute error function $ME_n(t)$ for $n = 3,4,5,6,7$ and $0 \le t \le 1$.

n	3	4	5	6	7
$\lvert \mathbf{ME_n(t)} \rvert$	2.51838×10^{-3}	4.78451×10^{-4}	9.92575×10^{-5}	2.17805×10^{-5}	4.9696×10^{-6}

3.2 Example

Consider the three point BVP for inhomogeneous fractional differential equation with $(u')^2$ nonlinearity and $3 < \alpha \le 4$.

$$D_*^\alpha u(t) - (u'(t))^2 + g(t) = 0, \quad 0 \le t \le 1, \quad 3 < \alpha \le 4. \tag{3.27}$$
$$u(0) = 0, \; u'(0) = 0, \; u''(0.5) = \gamma_1, \; u'''(1) = \gamma_2$$

where $g(t) = (\alpha+1)^2 t^{2\alpha} - t\Gamma(\alpha+2)$, $\gamma_1 = \left(\frac{1}{2}\right)^{\alpha-1} \alpha(\alpha+1)$, $\gamma_2 = \alpha(\alpha-1)(\alpha+1)$.

The exact solution for this problem $u(t) = t^{\alpha+1}$. Applying $L^{-1}(.) = J(.)$ to both sides of Eq (3.27) and then using boundary values $u(0) = 0$ and $u'(0) = 0$ yields

$$u(t) = u''(0)t + u'''(0)\frac{t^2}{2} - J^\alpha(g(t)) + J^\alpha((u'(t))^2). \tag{3.28}$$

Differentiating Eq.(3.28) two time and three time then using boundary value $u''(0.5) = \gamma_1$ and $u'''(1) = \gamma_2$, then solving the obtained equations with respect to $u''(0)$ and $u'''(0)$ yields

$$u''(0) = \gamma_1 - \frac{1}{2}\gamma_2 - \frac{1}{2}\left[J^{\alpha-3}(g(t))\right]_{t=1} + \left[J^{\alpha-2}(g(t))\right]_{t=\frac{1}{2}} + \frac{1}{2}\left[J^{\alpha-3}(u'(t))^2\right]_{t=1} - \left[J^{\alpha-2}(u'(t))^2\right]_{t=\frac{1}{2}}. \tag{3.29}$$

and

$$u'''(0) = \gamma_2 + \left[J^{\alpha-3}(g(t))\right]_{t=1} - \left[J_{\alpha-3}((u')^2)\right]_{t=1}. \tag{3.30}$$

Substituting Eq.(3.29) and Eq.(3.30) in to the Eq.(3.28) we obtain

$$\begin{aligned} u(t) =& \frac{t^2}{4}(2\gamma_1 - \gamma_2) + \frac{t^3}{6}\gamma_2 + \left(\frac{t^3}{6} - \frac{t^2}{4}\right)\left[J^{\alpha-3}(g(t))\right]_{t=1} + \frac{t^2}{2}\left[J^{\alpha-2}(g(t))\right]_{t=\frac{1}{2}} + \\ &\left(\frac{t^2}{4} - \frac{t^3}{6}\right)\left[J^{\alpha-3}(u'(t))^2\right]_{t=1} - \frac{t^2}{2}\left[J^{\alpha-2}((u'(t))^2)\right]_{t=\frac{1}{2}} - J^{\alpha}(g(t)) + J^{\alpha}((u'(t))^2). \end{aligned} \tag{3.31}$$

Applying the ADM in to the Eq.(3.31) we have

$$u_0(t) = \frac{t^2}{4}(2\gamma_1 - \gamma_2) + \frac{t^3}{6}\gamma_2 + \left(\frac{t^3}{6} - \frac{t^2}{4}\right)\left[J^{\alpha-3}(g(t))\right]_{t=1} + \frac{t^2}{2}\left[J^{\alpha-2}(g(t))\right]_{t=\frac{1}{2}} - J^{\alpha}(g(t)) \tag{3.32}$$

$$u_{n+1}(t) = \left(\frac{t^2}{4} - \frac{t^3}{6}\right)\left[J^{\alpha-3}(A_n)\right]_{t=1} - \frac{t^2}{2}\left[J^{\alpha-2}(A_n)\right]_{t=\frac{1}{2}} + J^{\alpha}(A_n), \quad n \geq 0 \tag{3.33}$$

where A_n are the Adomian polynomials for the nonlinearity $(u'(t))^2$ which given by

$$A_n = \sum_{k=0}^{n} u'_{n-k}u'_k, \quad n \geq 0 \tag{3.34}$$

Thus by using the equations (3.32), (3.33)and(3.34) we can calculate the solution components of equation (3.27) when $\alpha = 3.9$ as follows

$$u_0(t) = -0.851877t^2 + 0.568059t^3 + t^{4.9} - 0.00280675t^{11.7},$$

$$\begin{aligned} u_1(t) =& 0.754057t^2 - 0.504997t^3 + 0.00971651t^{5.9} - 0.00845123t^{6.9} + 0.00214008t^{7.9} - \\ & 0.00729485t^{8.8} + 0.00439289t^{9.8} + 0.00280675t^{11.7} + 3.68052 \times 10^{-6}t^{15.6} - \\ & 2.81652 \times 10^{-6}t^{16.6} - 5.09743 \times 10^{-6}t^{18.5} + 4.59557 \times 10^{-9}t^{25.3}, \end{aligned}$$

......

Clearly in this example if we use the standard ADM then we need to solve a system of nonlinear algebraic equations to find the undetermined coefficients $\beta_1 = u''(0)$, $\beta_2 = u'''(0)$ at each stage of the approximation solution. Some of these equations may possess non-physical roots. Thus the Duan-Rach modification is an efficient alternative method to the standard ADM for solving BVPs.

Figure.(3) shows the exact solution $u(t) = t^{4.9}$ of equation (3.27) when $\alpha = 3.9$ and the approximate solutions ϕ_n when $n = 4,6,8$. Figure.(4) shows the approximation solutions $\phi_n(t)$ at $n = 8$ and different values of α. In table (2) we computed the maximal absolute error for the error function $ME_n(t) = \|\phi_n(t) - u_{exact}(t)\|_\infty$ when $n = 3,4,5,6,7,8$ and $\alpha = 3.9$. We note the maximal absolute error decreasing monotonically with the increases of the integer n.
In figure.(5) we plot the curves of absolute error function $E_n(t) = |\phi_n(t) - u(t)|$ at $n = 8$ for various values of α.

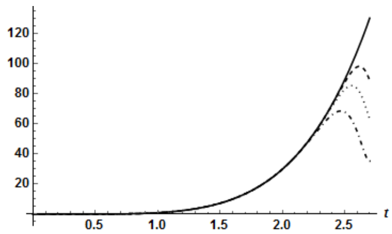

Figure 3: $u_{exact}(t)$ (solid line), and $\phi_n(t)$, $\phi_8(t)$ (dashed line), $\phi_6(t)$ (dot line), $\phi_4(t)$ (dot-side line).

Table 2: $|ME_n(t)|$

n	3	4	5	6	7	8
$\mathbf{|ME_n(t)|}$	0.0108606	0.00310907	0.0009581	0.000311657	0.000104246	0.0000356986

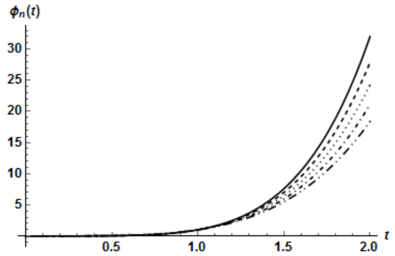

Figure 4: $\phi_8(t)$, $\alpha = 3.2$ (dot-dot-side line), $\alpha = 3.4$ (dot-side line), $\alpha = 3.6$ (dot line), $\alpha = 3.8$ (dashed line), $\alpha = 4$ (solid line)

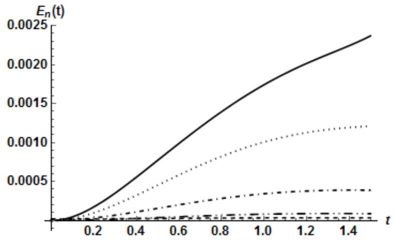

Figure 5: $E_8(t)$, $\alpha = 3.2$ (solid line), $\alpha = 3.4$ (dot line), $\alpha = 3.6$ (dot-side line), $\alpha = 3.8$ (dot-dot-side line), $\alpha = 4$ (dashed line)

4 Conclusions

In this paper, we utilize the modification proposed by Duan and Rach for the Adomian decomposition method to find positive solutions for ordinary nonlinear fractional differential equation with multi-point boundary value problems. This modification avoids the complicated calculations for the matching algebraic equations of undetermined coefficient that obtained at each stage of approximation solution and gives an approximate solution with a high agreement to the exact solution even for small values of n and with minimum number of calculation. The absolute error approaches zero when n increases as we showed in the given examples. Thus, the method is very convenient and efficient to solve boundary value problem of fractional order which gives it much wider applicability. Mathematica software was used for the two given examples.

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