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Research Article

A Third-Order Differential Equation and Starlikeness of a Double Integral Operator

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Functions $f(z) = z + \sum_2^\infty a_n z^n$ that are analytic in the unit disk and satisfy the differential equation $f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z)$ are considered, where g is subordinated to a normalized convex univalent function h. These functions f are given by a double integral operator of the form $f(z) = \int_0^1 \int_0^1 G(zt^\mu s^\nu) t^{-\mu} s^{-\nu} ds\, dt$ with G' subordinated to h. The best dominant to all solutions of the differential equation is obtained. Starlikeness properties and various sharp estimates of these solutions are investigated for particular cases of the convex function h.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f defined in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. Further, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions, and let $\mathcal{S}^*$ be its subclass of starlike functions. A starlike function f is characterized analytically by the condition $\operatorname{Re}(zf'(z)/f(z)) > 0$ in U, that is, the domain $f(U)$ is starlike with respect to origin. For two functions $f(z) = z + a_2 z^2 + \cdots$ and $g(z) = z + b_2 z^2 + \cdots$ in $\mathcal{A}$, the Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.1}$$

For f and g in $\mathcal{A}$, a function f is subordinate to g, written as $f(z) \prec g(z)$, if there is an analytic function w satisfying $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in U$.

When g is univalent in U, then f is subordinated to g which is equivalent to $f(U) \subset g(U)$ and $f(0) = g(0)$.

In a recent paper, Miller and Mocanu [1] investigated starlikeness properties of functions f defined by double integral operators of the form

$$f(z) = \int_0^1 \int_0^1 W(s,t,z)\,ds\,dt. \tag{1.2}$$

In this paper, conditions on a different kernel W are investigated from the perspective of starlikeness. Specifically, we consider functions $f \in \mathcal{A}$ given by the double integral operator of the form

$$f(z) = \int_0^1 \int_0^1 G(zt^{\mu}s^{\nu})t^{-\mu}s^{-\nu}\,ds\,dt. \tag{1.3}$$

In this case, it follows that

$$f'(z) = \int_0^1 \int_0^1 g(zt^{\mu}s^{\nu})\,ds\,dt, \tag{1.4}$$

where $G'(z) = g(z)$. Further, the function f satisfies a third-order differential equation of the form

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z) \tag{1.5}$$

for appropriate parameters α and γ. The investigation of such functions f can be seen as an extension to the study of the class

$$R(\alpha, h) = \{ f \in \mathcal{A} : f'(z) + \alpha z f''(z) \prec h(z),\ z \in U \}. \tag{1.6}$$

The class $R(\alpha, h)$ or its variations for an appropriate function h have been investigated in several works; see, for example, [2–10] and more recently [11, 12].

2. Results on Differential Subordination

We first recall the definition of best dominant solution of a differential subordination.

Definition 2.1 ((dominant and best dominant) [13]). Let $\Psi : \mathbb{C}^3 \times U \to \mathbb{C}$, and let h be univalent in U. If p is analytic in U and satisfies the differential subordination

$$\Psi\Big(p(z), zp'(z), z^2p''(z)\Big) \prec h(z), \tag{2.1}$$

then p is called a solution of the differential subordination. A univalent function q is called a dominant if $p \prec q$ for all p satisfying (2.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (2.1) is said to be the best dominant of (2.1).

In the following sequel, we will assume that h is an analytic convex function in U with $h(0)=1$. For $\alpha \geq \gamma \geq 0$, consider the third-order differential equation

$$f'(z)+\alpha z f''(z)+\gamma z^2 f'''(z)=g(z), \qquad g(z) \prec h(z). \tag{2.2}$$

We will denote the class consisting of all solutions $f \in \mathcal{A}$ as $R(\alpha,\gamma,h)$, that is,

$$R(\alpha,\gamma,h)=\left\{f \in \mathcal{A} : f'(z)+\alpha z f''(z)+\gamma z^2 f'''(z) \prec h(z),\ z \in U\right\}. \tag{2.3}$$

Let

$$\mu=\frac{(\alpha-\gamma)-\sqrt{(\alpha-\gamma)^2-4\gamma}}{2}, \qquad \nu+\mu=\alpha-\gamma,\ \mu\nu=\gamma. \tag{2.4}$$

The discriminant is denoted by $\Delta := (\alpha-\gamma)^2-4\gamma$. Note that $\operatorname{Re}\mu \geq 0$ and $\operatorname{Re}\nu \geq 0$.

We will rewrite the solution of

$$f'(z)+\alpha z f''(z)+\gamma z^2 f'''(z)=g(z) \tag{2.5}$$

in its equivalent integral form

$$f'(z)=\int_0^1\int_0^1 g(zt^{\mu}s^{\nu})\,ds\,dt. \tag{2.6}$$

It follows from relations (2.4) that

$$\begin{aligned} g(z) &= f'(z)+\big(\mu(1+\nu)+\nu\big)zf''(z)+\mu\nu z^2 f'''(z) \\ &= \nu z^{1-1/\nu}\left(\mu z^{1+1/\nu}f''(z)+z^{1/\nu}f'(z)\right)' \\ &= \nu z^{1-1/\nu}\left(\mu z^{1+1/\nu-1/\mu}\left(z^{1/\mu}f'(z)\right)'\right)'. \end{aligned} \tag{2.7}$$

Thus,

$$\mu z^{1+1/\nu-1/\mu}\left(z^{1/\mu}f'(z)\right)'=\frac{1}{\nu}\int_0^z w^{1/\nu-1}g(w)\,dw. \tag{2.8}$$

Making the substitution $w=zs^{\nu}$ in the above integral and integrating again, a change of variables yields

$$f'(z)=\int_0^1\int_0^1 g(zt^{\mu}s^{\nu})\,ds\,dt. \tag{2.9}$$

We will use the notation ϕ_λ for

$$\phi_\lambda(z) = \int_0^1 \frac{dt}{1 - zt^\lambda} = \sum_{n=0}^{\infty} \frac{z^n}{1 + \lambda n}. \tag{2.10}$$

From [14] it is known that ϕ_λ is convex in U provided $\operatorname{Re}\lambda \geq 0$.

Theorem 2.2. *Let μ and ν be given by* (2.4)*, and*

$$q(z) = \int_0^1 \int_0^1 h(zt^\mu s^\nu) dt\, ds. \tag{2.11}$$

Then the function $q(z) = (\phi_\nu * \phi_\mu) * h(z)$ is convex. If $f \in R(\alpha, \gamma, h)$, then

$$f'(z) \prec q(z) \prec h(z), \tag{2.12}$$

and q is the best dominant.

Proof. It follows from (2.10) that

$$h(z) * \phi_\mu(z) = \int_0^1 \frac{1}{1 - zt^\mu} dt * h(z) = \int_0^1 h(zt^\mu) dt := k(z). \tag{2.13}$$

Thus,

$$h(z) * \left(\phi_\mu(z) * \phi_\nu(z)\right) = k(z) * \phi_\nu(z) = \int_0^1 k(zs^\nu) ds = \int_0^1 \int_0^1 h(zt^\mu s^\nu) dt\, ds = q(z). \tag{2.14}$$

Since the convolution of two convex functions is convex [15], the function q is convex. Let

$$p(z) = f'(z) + \nu z f''(z). \tag{2.15}$$

Then,

$$p(z) + \mu z p'(z) = f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \prec h(z). \tag{2.16}$$

It is known from [16] that

$$p(z) \prec \frac{1}{\mu z^{1/\mu}} \int_0^z \zeta^{1/\mu - 1} h(\zeta) d\zeta = \left(\phi_\mu * h\right)(z) \prec h(z). \tag{2.17}$$

Similarly,

$$p(z) = f'(z) + \nu z f''(z) \prec \left(\phi_\mu * h\right)(z) \tag{2.18}$$

implies

$$\begin{aligned} f'(z) &\prec (\phi_\nu * \phi_\mu * h)(z) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{(1+\nu n)(1+\mu n)} * h(z) \\ &= \left(\int_0^1 \int_0^1 \frac{dt\,ds}{1 - zt^\mu s^\nu} \right) * h(z) \\ &= \int_0^1 \int_0^1 h(zt^\mu s^\nu) dt\,ds = q(z). \end{aligned} \tag{2.19}$$

The differential chain

$$f' \prec q \prec \phi_\mu * h \prec h \tag{2.20}$$

shows that $q \prec h$. Since $q(z) + \alpha z q'(z) + \gamma z^2 q''(z) = h(z)$, the function

$$Q(z) = \int_0^z q(w) dw \tag{2.21}$$

is a solution of the differential subordination $f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \prec h(z)$, and thus $q \prec \tilde{q}$ for all dominants $\tilde{q}$. Hence, q is the best dominant. □

Remark 2.3. (1) When $\gamma = 0$, then $\mu = 0$ and $\nu = \alpha$, and the above subordination reduces to the result of [16], that is,

$$f'(z) + \alpha z f''(z) \prec h(z) \Longrightarrow f'(z) \prec \int_0^1 h(zt^\alpha) dt. \tag{2.22}$$

(2) The above proof also reveals that

$$f \in R(\alpha, \gamma, h) \Longrightarrow f \in R(0, 0, h), \tag{2.23}$$

that is, $f'(z) \prec h(z)$.

Theorem 2.4. *Let μ, ν, and q be as given in Theorem 2.2. If $f \in R(\alpha, \gamma, h)$, then*

$$\begin{aligned} \frac{f(z)}{z} &\prec \int_0^1 q(tz) dt \\ &= \int_0^1 \int_0^1 \int_0^1 h(zrs^\mu t^\nu) dr\,ds\,dt. \end{aligned} \tag{2.24}$$

Proof. Let $p(z) = f(z)/z$. Then

$$p(z) + zp'(z) = f'(z) \prec q(z). \tag{2.25}$$

With ϕ_1 given by (2.10), this subordination implies

$$p(z) = \left(\phi_1 * (p + zp')\right)(z) \prec (\phi_1 * q)(z) = \int_0^1 q(tz)dt. \tag{2.26}$$

□

In this paper, starlikeness properties will be investigated for functions f given by a double integral operator of the form (1.3).

3. Applications

First, we consider a class of convex univalent functions h so that $h(U)$ is symmetric with respect to the real axis. Denote by $R(\alpha, \gamma, A, B)$ the class

$$R(\alpha, \gamma, A, B) = \left\{ f \in \mathcal{A} : f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \prec \frac{1 + Az}{1 + Bz},\ z \in U \right\}, \tag{3.1}$$

where $-1 \le B < A \le 1$, and let $h(z; A, B) = (1 + Az)/(1 + Bz)$. When $A = 1 - 2\beta$ and $B = -1$, let $h_\beta(z) := h(z; 1 - 2\beta, -1)$. The class of $R(\alpha, \gamma, h_\beta)$ is of particular significance, and we will simply denote it by

$$\begin{aligned} R(\alpha, \gamma, h_\beta) &:= R(\alpha, \gamma, \beta) \\ &= \left\{ f \in \mathcal{A} : f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \prec \frac{1 + (1 - 2\beta)z}{1 - z},\ z \in U \right\}. \end{aligned} \tag{3.2}$$

Equivalently,

$$R(\alpha, \gamma, \beta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \right) > \beta \right\}. \tag{3.3}$$

The following result is an immediate consequence of Theorems 2.2 and 2.4.

Theorem 3.1. *Under the assumptions of Theorem 2.2, if*

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \prec \frac{1 + Az}{1 + Bz}, \tag{3.4}$$

then

$$f'(z) \prec \begin{cases} q(z; A, B) \prec \dfrac{1 + Az}{1 + Bz}, & \text{if } B \neq 0, \\ q(z; A) \prec 1 + Az, & \text{if } B = 0, \end{cases} \tag{3.5}$$

where

$$q(z;A,B) := 1 + (A-B)\sum_{n=1}^{\infty}\frac{(-B)^{n-1}z^n}{(1+\mu n)(1+\nu n)},$$
$$q(z;A) := 1 + \frac{Az}{(1+\alpha)} \tag{3.6}$$

is the best dominant. Further,

$$\frac{f(z)}{z} \prec \frac{A}{B} - \frac{A-B}{B}\int_0^1\int_0^1\int_0^1 \frac{ds\,dt\,du}{1+Bzut^{\mu}s^{\nu}}$$
$$= 1 + (A-B)\sum_{n=1}^{\infty}\frac{(-B)^{n-1}z^n}{(1+n)(1+\mu n)(1+\nu n)} \tag{3.7}$$

if $B \neq 0$, *and*

$$\frac{f(z)}{z} \prec 1 + \frac{Az}{2(1+\alpha)} \tag{3.8}$$

if $B = 0$.

4. Starlikeness Property

Starlikeness properties of functions given by a double integral operator are investigated in this section. The following result will be required.

Lemma 4.1 (see [5]). *If* $f \in \mathcal{A}$ *satisfies*

$$\operatorname{Re}\left(f'(z) + \alpha z f''(z)\right) > \frac{(-1/\alpha)\int_0^1 t^{1/\alpha-1}((1-t)/(1+t))dt}{1-1/\alpha\int_0^1 t^{1/\alpha-1}((1-t)/(1+t))dt}, \quad z \in U, \tag{4.1}$$

for $\alpha \geq 1/3$, *then* $f \in S^*$. *This result is sharp.*

Theorem 4.2. *Let* μ *and* ν *be given by* (2.4) *with* $\Delta \geq 0$ *and* $\nu \geq 1/3$. *If*

$$f(z) = \int_0^1\int_0^1 G(zt^{\mu}s^{\nu})t^{-\mu}s^{-\nu}ds\,dt, \tag{4.2}$$

where $G'(z) \prec h_\beta(z) = h(z; 1-2\beta, -1)$, *and* β *satisfies*

$$\beta = 1 - \frac{1}{2\left(1 - (1/\nu)\int_0^1 t^{1/\nu-1}((1-t)/(1+t))dt\right)\left(1 - \int_0^1 (dt/(1+t^\mu))\right)}, \tag{4.3}$$

then $f \in S^*$.

Proof. The function f satisfies

$$f'(z) = \int_0^1 \int_0^1 g(zt^\mu s^\nu)ds\,dt, \qquad G'(z) = g(z) \prec h_\beta(z), \tag{4.4}$$

and thus

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) = g(z) \prec h_\beta(z). \tag{4.5}$$

Now, $\operatorname{Re} h_\beta(z) > \beta$ also implies that $\operatorname{Re} g(z) > \beta$, and so

$$\operatorname{Re}\left(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)\right) > \beta, \quad \beta < 1. \tag{4.6}$$

It follows from the proof of Theorem 2.2 that

$$f'(z) + \nu z f''(z) \prec (\phi_\mu * h_\beta)(z) := q_\mu(z), \tag{4.7}$$

where

$$q_\mu(z) = 2\beta - 1 + 2(1-\beta)\int_0^1 \frac{dt}{1 - zt^\mu}. \tag{4.8}$$

Since

$$\operatorname{Re}\, q_\mu(z) > 2\beta - 1 + 2(1-\beta)\int_0^1 \frac{dt}{1 + t^\mu}, \tag{4.9}$$

an application of Lemma 4.1 yields the result. □

Corollary 4.3. *Let* $\alpha \geq 3$ *and*

$$\operatorname{Re}\left(f'(z) + \alpha z f''(z) + \frac{\alpha - 1}{2} z^2 f'''(z)\right) > \beta, \quad \beta < 1. \tag{4.10}$$

If β satisfies

$$\beta = 1 - \frac{1}{2(1-\log 2)\left(1-(2/(\alpha-1))\int_0^1 t^{2/(\alpha-1)-1}((1-t)/(1+t))dt\right)}, \tag{4.11}$$

then $f \in S^*$.

Proof. In this case, $\mu = 1$, $\nu = (\alpha-1)/2$, and the result now follows from Theorem 4.2. □

Example 4.4. If

$$\mathrm{Re}\left(f'(z) + 3zf''(z) + z^2 f'''(z)\right) > \beta \tag{4.12}$$

and β satisfies

$$\beta = \frac{4(1-\log 2)^2 - 1}{4(1-\log 2)^2} \approx -1.65509, \tag{4.13}$$

then $f \in S^*$.

Theorem 4.5. *Let $f, g \in R(\alpha, \gamma, \beta)$ and let μ and ν be given by (2.4) with $\Delta \geq 0$. If β satisfies*

$$\beta = 1 - \frac{1}{4\left(1 - \int_0^1\int_0^1\int_0^1 (ds\,dt\,du/(1+ut^\mu s^\nu))\right)}, \tag{4.14}$$

then $f * g \in R(\alpha, \gamma, \beta)$.

Proof. Clearly,

$$(f*g)'(z) + \alpha z(f*g)''(z) + \gamma z^2 (f*g)'''(z) = \left(\left(f' + \alpha z f'' + \gamma z^2 f'''\right) * \frac{g}{z}\right)(z). \tag{4.15}$$

Since $f \in R(\alpha, \gamma, \beta)$, substituting $A = 1 - 2\beta$ and $B = -1$ in (3.7) gives

$$\mathrm{Re}\,\frac{g(z)}{z} > 2\beta - 1 + 2(1-\beta)\int_0^1\int_0^1\int_0^1 \frac{ds\,dt\,du}{1+ut^\mu s^\nu} = \frac{1}{2}. \tag{4.16}$$

Hence, it follows that

$$\mathrm{Re}\left((f*g)'(z) + \alpha z(f*g)''(z) + \gamma z^2 (f*g)'''(z)\right) > \beta. \tag{4.17}$$

□

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