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Advances on Belief Base Dynamics

DOCTORAL THESIS

Marco Paulo Ferreirinha Garapa

DOCTORATE IN MATHEMATICS
SPECIALTY OF LOGIC AND COMPUTER SCIENCE

UNIVERSIDADE da MADEIRA
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ORIENTATION
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UNIVERSIDADE da MADEIRA

ADVANCES ON BELIEF BASE DYNAMICS

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Thesis presented by

Marco Paulo Ferreirinha Garapa

and **unanimously approved**

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in Mathematics

Jury

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To my parents, wife and daughters
with love and gratitude.

Abstract

The main goal underlying the research area of *belief change* consists in finding appropriate ways of modelling the *belief state* of a rational agent and, additionally, the changes which occur in such a state when the agent receives new information. The most important model of belief change is the so-called *AGM model*, proposed in [AGM85]. In this model, the belief state of an agent is represented by a *belief set*—a deductively closed set of sentences. A change consists in adding or removing a specific sentence from a belief set to obtain a new belief set.
Two of the main shortcomings pointed out to the AGM model of belief change are the use of belief sets to represent belief states and the (unrealistic) acceptance of any new piece of information. In this thesis we address both those issues.
We present axiomatic characterizations for *ensconcement-based contractions* and for *brutal contractions*, two kinds of belief bases contraction operators introduced in [Wil94b] that are based on the concept of *ensconcement*, which is a generalization to the case of *belief bases* of the concept of *epistemic entrenchment* introduced in [Gär88, GM88]. We compare the axiomatic characterizations of these operators with those of other well-known base contraction operators and study the interrelations among the former and the contraction operators based on epistemic entrenchments.
We study non-prioritized base change operators, namely *shielded base contractions* and *credibility-limited base revisions*. We propose several different classes of shielded base contractions and obtain axiomatic characterizations for each one of them. Additionally we thoroughly investigate the interrelations (in the sense of inclusion) among all those classes. Afterwards we perform a similar study for credibility-limited base revisions. Finally, we study the interrelation between the different proposed classes of operators of credibility-limited base revision and of shielded contraction by means of the *consistency-preserving Levi identity* and the *Harper identity*.

Keywords:

Belief Change; Belief Bases; Ensconcement; Ensconcement-based Contractions; Shielded Contraction; Credibility-limited Revision.

Resumo

O objetivo principal da área de *revisão de crenças* é encontrar modelos que permitam modelar o *estado de crenças* de um agente racional, bem como as mudanças que ocorrem nesse estado de crenças quando o agente recebe novas informações. O modelo mais influente desta área é o chamado *modelo AGM* proposto em [AGM85]. Neste modelo, o estado de crenças de um agente é representado por um *conjunto de crenças*—conjunto de fórmulas dedutivamente fechado. Uma mudança consiste em adicionar ou remover uma fórmula específica de um conjunto de crenças para obter um novo conjunto de crenças. Dois dos principais problemas apontados ao modelo AGM são o uso de conjuntos de crenças para representar estados de crença e a aceitação (irrealista) de qualquer nova informação. Nesta tese abordamos ambas as questões.

Apresentamos caracterizações axiomáticas para *contrações baseadas em ensconcements* e para *contrações brutais*, dois tipos de operadores de contração em bases de crenças introduzidos em [Wil94b] e que se baseiam no conceito de *ensconcement*—generalização em *bases de crenças*, do conceito de *epistemic entrenchment* introduzido em [Gär88, GM88]. Comparamos as caracterizações axiomáticas destes operadores com as de outros operadores de contração em bases bem conhecidos e estudamos as inter-relações entre os primeiros e os operadores de contração baseados em *epistemic entrenchments*.

Estudamos operadores de mudanças de crenças não-priorizados em bases, nomeadamente *contrações protegidas* e *revisões com limite de credibilidade*. Propomos várias classes de operadores de contrações protegidas e obtemos teoremas de representação para cada uma dessas classes. Investigamos, igualmente, as inter-relações (no sentido de inclusão) entre todas essas classes. Posteriormente, realizamos um estudo semelhante para revisões com limite de credibilidade. Finalmente, estudamos a inter-relação entre as diferentes classes propostas de operadores (definidos em bases de crenças) de revisão com limite de credibilidade e de contrações protegidas através da identidade de *Levi conservadora-da-consistência* e da *identidade de Harper*.

Palavras-chave:

Mudanças de Crenças; Bases de Crenças; Ensconcement; Contrações Baseadas em Ensconcements; Contração Protegida; Revisão com Limite de Credibilidade.

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Chapter 1

Introduction

"The world around us changes all the time, and in order
to cope with it we must constantly change our beliefs:
acquire new ones and revise or give up old ones."
Hansson in [Han99b]

The research area that studies the dynamics of knowledge is known as *belief change* (also known as *belief revision* or *belief dynamics*). A rational agent is sometimes forced to adapt his/her beliefs when confronted with new information. This is the most general formulation of the main problem of belief change. This is a problem with applications to several areas. In the following we provide some examples of the situations where belief revision occurs:

Daily life: One day I woke up and heard in the weather forecast that it would rain on that day. Based on this information I decided that I should take my umbrella when I left home, since I was going out for a long time. As I was getting ready to leave home, I looked at the sky and I noticed that it was clear. Should I take my umbrella with me or should I give up my belief that it would rain on that day and leave my umbrella at home?

Robotics: [Was00] A mobile robot has a map of the environment that surrounds it. On that map, there is nothing in front of it. So it can move. Suppose now that its sensors detect a large object in front of it. How should the robot react? Should it doubt his sensors and continue moving straight ahead? Or, should it belief in its sensor and doubt the map?

Databases: [GR95] Suppose that a database contains the following information:
All European swans are white.
The bird caught in the trap is a swan.
The bird caught in the trap comes from Sweden.
Sweden is part of Europe.
From this set of propositions it can be deduced that the bird caught in the trap is white.
Now suppose that, the bird caught is black. If we simply add the new information to

the database it will become inconsistent. If it is intended that the database remains consistent how should it behave? Should it keep the old data, ignoring the new information? Should it add the new information to the database, removing some of the beliefs contained in the original database? In this case should all the beliefs be retracted, or just some of then? Which ones?

The main goal of *belief revision* is to model how a rational agent updates his/her set of beliefs/knowledge when confronted with new information. When facing new information an agent can change his/her set of beliefs: he/she can acquire some new ones and revise or give up some old ones. The main objective of most of the works in this area is to investigate and model how these changes occur. To do so operators of change are defined. Two approaches are usually followed: On the one hand these operators can be characterized by identifying properties—commonly designated by *postulates*—that they are naturally expected to satisfy. On the other hand, explicit constructive definitions can be presented for these operators. These two approaches can be considered to be complementary of each other. In fact, the studies of belief change tend to conciliate these two approaches by providing a constructive method for defining operators and, at the same time, a set of postulates that exactly characterizes the class of such operators. The results of the univocal identification of a given class of change operators with a set of postulates are called *representations theorems*. Representation theorems are also called *axiomatic characterizations*, since they characterize an operator in terms of axioms (or postulates).

One of the main contributions to the study of belief change is the so-called AGM model for belief change—named after the initial of its authors: Alchourrón, Gärdenfors and Makinson. This model was proposed in 1985 in [AGM85] and gained the status of standard model of belief change. Since then, this subject has been extensively studied and has developed rapidly and in many different directions influenced by different areas, in particular by computer science and philosophy.

As for potential applications for *belief change*, Hansson in [Han99b] stated that "There is no lack of potential applications for belief dynamics". In that reference Hansson mentioned the following examples:

- It may be useful in the development of models of learning and other mental processes;
- It may be useful in the development of economic models by providing a formal representation of the changing beliefs of economic agents;
- It may help legal theorists in the development of models of changes in legal systems;
- It may provide tools that allow to revise databases in a rational and efficient way.

1.1 Organization of the thesis

In the next chapter, we briefly present the main concepts of the Epistemological Theories and the rational criteria that govern the epistemic dynamics.

In **Chapter 3** we present the AGM model for belief change. We start this chapter by recalling the postulates that characterize the AGM operators of belief change, namely expansions, contractions and revisions. Afterwards, we present some explicit methods for the construction of contraction and revision operators on belief sets (by a single sentence) as well as axiomatic characterizations for each one of the classes of operators obtained through those methods. At the end of this chapter we present the logical relationships between the classes of contraction operators mentioned along this chapter.

In **Chapter 4** we discuss some criticisms to the AGM model that appear in the belief change literature as well as some of the proposals developed to overcome some of the identified problems of this model. In this chapter we present some examples where the use of the *recovery* postulate seems implausible, discuss the use of belief bases instead of belief sets to represent an agent's belief state, briefly mention some models for iterated revision and multiple contraction and mention several models of non-prioritized belief revision and contraction operators (*i.e.* operators that do not satisfy the *success* postulate). Regarding non-prioritized belief change operators we give special emphasis to the credibility-limited revision and shielded contraction. Credibility-limited revision was introduced in [HFCF01] and is based on the assumption that some inputs are accepted, others not. Those that are potentially accepted constitute the set C of credible sentences. If α is credible, then α is accepted in the revision process, otherwise no change is made to the belief set. Shielded contraction was defined in [FH01] and is based on the assumption that not every (non-tautological) belief is removed when contracting a given belief set by it. Those beliefs that can be removed when a contraction is perform constitute the set R of retractable sentences. If α is retractable, then α is always removed when contracting a belief set by it, otherwise no change is made to the belief set. We extend the work presented in [HFCF01, FH01], by axiomatically characterizing another classes of credibility-limited operators and by establishing the interrelation between different classes of credibility-limited revision operators and of shielded contraction operators by means of the consistency-preserving Levi identity (an adaptation of the Levi identity to the non-prioritized belief change context) and the Harper identity.

Chapter 5 is dedicated to belief base change. In this chapter we recall some construction methods for contraction and revision operators in the belief base context. For contraction we recall the following constructive methods: partial meet contraction, kernel and smooth kernel contraction, basic AGM-generated base contraction, as well as the axiomatic characterization for each one of these operators. We also present two operators proposed by Williams in [Wil94b], namely brutal and ensconcement-based contractions. These two operators are based on the notion of ensconcement that can be seen as an adaptation to the belief base context of the notion of epistemic entrenchment. In this chapter we also recall the definition of partial meet and kernel revision functions as well as the axiomatic characterization of each one of these revision functions. We also define and axiomatically characterize two new kinds of base revision functions, namely the smooth kernel base revisions and the basic AGM-generated base revisions. At the end of this chapter we briefly recall some operators of non-prioritized belief change on belief bases, namely: semi-revision (and consolidation), credibility-limited base revision and shielded base contraction.

The main contributions of the present thesis are exposed in Chapters 6 to 8.

In **Chapter 6** we present representation theorems for the brutal contractions and the ensconcement-based contractions. We compare the axiomatic characterization for these two contraction operators in the sense of identifying which postulates of each one of the mentioned axiomatic characterizations are (and which are not) satisfied by the other kind of operator. We also compare the axiomatic characterizations of brutal and ensconcement-based contractions with the axiomatic characterizations of other base contractions operators, namely with basic AGM-generated base contractions, kernel contractions and partial meet contractions. We present some results that clarify the interrelation among epistemic entrenchment-based contractions and ensconcement-based contractions, and the interrelation among sever withdrawals and brutal contractions.

In **Chapter 7** we start by thoroughly studying the interrelations among the postulates satisfied by a shielded contraction and the postulates and properties satisfied, respectively, by the (standard) contraction and by the set of retractable sentences that induce it. After that, we obtain representation theorems for several classes of shielded base contractions induced by some well-known kinds of base contractions and by sets of retractable sentences satisfying different sets of properties. Additionally, we thoroughly investigate the interrelations among all those classes. More precisely, we analyse whether each of those classes is or is not (strictly) contained in each of the remaining ones.

In **Chapter 8**, we conduct a study similar to the one made in Chapter 7, concerning credibility-limited base revision operators. Thus we start by studying the interrelations among the postulates satisfied by a credibility-limited revision and the postulates and properties satisfied, respectively, by the (standard) revision and by the set of credible sentences that induce it. Afterwards, we obtain axiomatic characterizations for classes of credibility-limited base revisions induced by different base revision functions and by sets of credible sentences satisfying different sets of properties. Additionally, we thoroughly investigate the interrelations among all those classes. We finish this chapter by establishing the interrelation between different classes of credibility-limited base revision operators and of shielded base contraction operators by means of the consistency-preserving Levi identity and the Harper identity.

In **Chapter 9** we present a brief overview of the main contributions of this thesis and point toward topics for future research.

In the **Appendix** we provide the proofs for the original results presented throughout this thesis.

Some of the original results presented in this thesis have already been published or submitted for publication. Parts of Chapter 6 were published as [GFR16], [GFR17b] and [FGR17]. Most of Chapter 7 is included in [GFR17a].

1.2 Formal preliminaries

We will assume a language $\mathcal{L}$ that is closed under truth-functional connectives: negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$) and equivalence ($\leftrightarrow$). By abuse of notation we shall use $\mathcal{L}$ to denote the language and the set of the well formed sentences that can be expressed in it. We say that a language $\mathcal{L}$ is finite if it is built from a finite set of propositional symbols (and the Boolean connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$). $\bot$ denotes an arbitrary contradiction and $\top$ an arbitrary tautology. The letters $\alpha, \alpha_i, \beta, \ldots$ will be used to denote sentences of $\mathcal{L}$. Lowercase Latin letters such as $p, p_i, q, q_i, \ldots$ will be used to denote atomic sentences of $\mathcal{L}$. $A, A', B, \ldots$ denote sets of sentences of $\mathcal{L}$. Cn denotes a consequence operation that satisfies the standard Tarskian properties [Tar56]:

- *inclusion:* $A \subseteq Cn(A)$.
- *monotony:* if $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$.
- *iteration:* $Cn(A) = Cn(Cn(A))$.

We will also assume that Cn satisfies:

- *Supraclassicality:* if α can be deduced from A by classical truth-functional logic, then $\alpha \in Cn(A)$.
- *Compactness:* if $\alpha \in Cn(A)$, then $\alpha \in Cn(A')$ for some finite subset A' of A.
- *Deduction:* if $\beta \in Cn(A \cup \{\alpha\})$, then $(\alpha \rightarrow \beta) \in Cn(A)$.

The following properties of Cn follow from the ones above:

- *Modus ponens:* if $\alpha \rightarrow \beta \in Cn(A)$ and $\alpha \in Cn(A)$, then $\beta \in Cn(A)$.
- *Contraposition:* if $\alpha \rightarrow \beta \in Cn(A)$, then $\neg\beta \rightarrow \neg\alpha \in Cn(A)$.

We will sometimes use $A \vdash \alpha$ as an alternative notation for $\alpha \in Cn(A)$, $\vdash \alpha$ for $\alpha \in Cn(\varnothing)$, $\alpha \vdash \beta$ for $\{\alpha\} \vdash \beta$ and $Cn(\alpha)$ for $Cn(\{\alpha\})$. Given a set of sentences A we say that A is logically closed or closed under logical consequence whenever $A = Cn(A)$. Such a set is called a *belief set* or *theory*. We will use bold uppercase letters such as $\mathbf{K}, \mathbf{H}, \ldots$ to denote belief sets. We will use $\mathbf{K}_\bot$ to denote the inconsistent belief set (containing all the sentences of $\mathcal{L}$). We shall denote the set of all theories of $\mathcal{L}$ by $\mathcal{T}_{\mathcal{L}}$.

We say that a set $X \subseteq \mathcal{L}$ is closed under double negation or that X satisfies closure under double negation if it holds that $\alpha \in X$ if and only if $\neg\neg\alpha \in X$.

Given $A \subseteq \mathcal{L}$, a total relation on A is a binary relation $\preceq$ such that for all $\alpha, \beta \in A$ it holds that either $\alpha \preceq \beta$ or $\beta \preceq \alpha$. A total pre-order $\leq$ is a binary relation that is reflexive, transitive and total. Given a binary relation $\leq$ on a set A and $S \subseteq A$, we denote by $\leq|_S$ the binary relation on S such that for all $\alpha, \beta \in S$, $\alpha \leq|_S \beta$ if and only if $\alpha \leq \beta$. Given a binary relation $\leq$ on a set A we shall write $\alpha < \beta$ to denote $\alpha \leq \beta$ and $\beta \nleq \alpha$, and $\alpha =_{\leq} \beta$ to denote $\alpha \leq \beta$ and $\beta \leq \alpha$. Given a set A we will denote the power set of A by $\mathcal{P}(A)$.

Chapter 2

Belief Change

"There is nothing permanent except change."
Heraclitus

In this chapter, which is essentially based on [Gär88], we present a brief introduction to the main epistemic factors[1] that form the core of the epistemological theories. One of the main goals of an epistemological theory is to provide a conceptual framework for investigating problems about changes of knowledge and beliefs. Such a theory must provide a representation of the epistemic elements (namely epistemic states, attitudes, inputs and changes) and the rational criteria that govern the epistemic dynamics.

2.1 Epistemic states, attitudes, inputs and changes

Epistemic states or belief states are representations of actual or possible cognitive states of an agent in a certain moment. There are several ways of modelling the epistemic state of an agent. Among them we mention the following ones:

(a) **Sentential Models:** In this kind of models an epistemic state is represented as a set of sentences of a given language. We can impose that this set of sentence is logically closed or not (necessarily logically closed). In the first case, the epistemic state is modelled by *belief sets* or *theories* and in the second by *belief bases*. In these models the beliefs of an agent are represented by the sentences of the language. In general, these sets are required to be consistent, since a rational agent is not expected to have contradictory beliefs.

(b) **Bayesian Models:** In these models an epistemic state is represented by a probabilistic measure defined over some object language or over some space of events.

(c) **Possible Worlds Models:** In these models an epistemic state is represented by a set of possible worlds. A possible world is a maximal consistent subset of the language under consideration. Sets of possible worlds are called

[1]Epistemic means related to knowledge.

propositions. As stated by Hansson: “A belief state can be represented by the propositions (set of possible worlds) that contains exactly those possible worlds that are compatible with the agents’ beliefs. In this way, propositions can represent belief states, just as belief sets” [Han99b].

The epistemic attitudes describe the state of several pieces of belief in an epistemic state. In formal systems, it is generally preferable to consider a small number of epistemic attitudes. This is due to the fact that, although the existence of more epistemic attitudes increases the expressiveness of the systems, it also increases its complexity making its theoretical study more difficult or even impossible. The epistemic attitudes depend on the model chosen to represent the epistemic state. For instance, in a probabilistic model, an agent may accept or reject a belief with a certain degree of probability. On the other hand, in a sentential model, where the epistemic states are represented as belief sets there are three epistemic attitudes to be considered regarding a given belief, represented by a sentence α, namely:

- *Acceptance:* When α belongs to the set representing the belief state;
- *Rejection:* When $\neg\alpha$ belongs to the set representing the belief state;
- *Indetermination:* When neither α nor $\neg\alpha$ belong to the set representing the belief state.

Epistemic inputs are pieces of information that may produce changes in the epistemic state. These inputs may trigger changes in the beliefs transforming the original epistemic state of an agent into a new epistemic state.

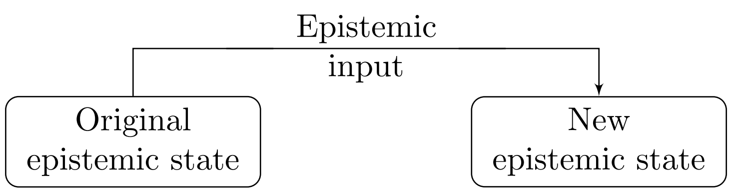

Figure 2.1: Epistemic change triggered by an epistemic input.

An epistemic input can lead to different kinds of epistemic change. If we consider the case where belief states are modelled by logically closed set of sentences, and the three epistemic attitudes mentioned above (acceptance, rejection and indetermination) that can be considered in such sentential model, there are six possible changes of epistemic attitudes towards a sentence, namely:

(a) From indetermined to accepted;

(b) From indetermined to reject;

(c) From accepted to rejected;

(d) From rejected to accepted;

(e) From accepted to indetermined;

(f) From reject to indetermined.

It is convenient to notice that, in general, a change of epistemic attitude towards a given sentence also implies a change in the epistemic attitude towards other sentences. To clarify this consider an epistemic input that causes a change in the epistemic attitude towards a sentence α, that is neither a tautology nor a contradiction, from accepted to rejected. Thus, assuming that the original belief set is consistent, we have that α belongs to that belief set but $\neg\alpha$ does not. On the other hand the new belief set contains $\neg\alpha$ but not α (assuming that we wish to preserve consistency). Hence while the epistemic attitude towards α changed from accepted to rejected, the opposite change occurred in the epistemic attitude towards $\neg\alpha$.

These six possible changes of epistemic attitudes towards a sentence can be identified with three types of belief change (operations), namely:

Expansion: An expansion occurs when new information is simply added to the set of the beliefs of an agent. Changes of the kinds (a) and (b) are *expansions*.

Revision: A revision occurs when new information is added to the set of the beliefs of an agent in a consistent matter. Changes of the kinds (c) and (d) are *revisions*.

Contraction: A contraction occurs when information is removed from the set of beliefs of an agent. Changes of the kinds (e) and (f) are *contractions*.

Figure 2.2 identifies the changes of epistemic attitudes towards a sentence by means of a belief change operator.

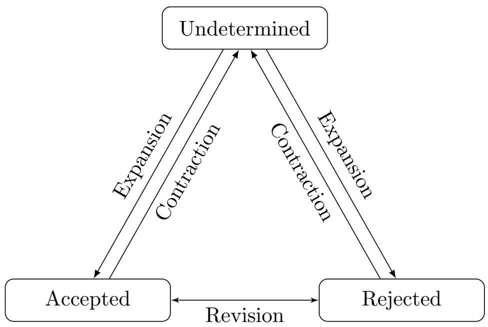

Figure 2.2: Schematic representation of belief change operations.

2.2 Criteria of rationality

The *rationality criteria* or *criteria of rationality* are basic principles which stand on the metalevel of an epistemological theory and that govern the other factors of the theory. Next we present a list of some of the most usual criteria of rationality (see [Dal88, Gär88, GR95]):

Principle of categorical matching or Adequacy of representation: After a change the new epistemic state should have the same type of representation as the previous one.

Irrelevance of syntax: The outcome of a change should not depend on the syntax/representation of either the previous beliefs or the new information that triggers that change.

Primacy (of the new information): The new information should always be accepted.

Consistency: If possible, the new epistemic state should be consistent.

Logical omniscience: The logical consequences of the beliefs that are accepted in a epistemic state should also be accepted by it.

Minimal change: The new epistemic state should retain as much as possible of the previous state.

Fairness: If there are several (valid) epistemic states that can be the outcome of a change, then none of them should be arbitrarily chosen.

Preference: Beliefs that are considered more important or entrenched should be kept in favour of less important ones.

It is important to notice that not all belief change process should obey the principles listed above. In fact, not all of the above criteria are desirable at all times and some of these principles contradict others in certain cases. However it is important that any given epistemological theory provides a clear indication of the order of priorities among the rational criteria that are assumed, in that context, to rule the dynamics of epistemic change.

2.3 Summary

In this chapter we presented the notions of *epistemic state, epistemic attitude, epistemic input* and *epistemic change*. We focused our attention in the *sentential model*, a model where epistemic states (a representation of actual or possible cognitive state of an agent in a certain moment) are represented as a set of sentences of a given language. We gave special emphasis to the analyses of the case where this set of sentences is a subset of $\mathcal{L}$ that is closed under logical consequences (a belief set). In this model there are three epistemic attitudes regarding a belief (represented by a sentence), namely *acceptance, rejection* and *indetermination* depending on whether the agent believes in a sentence, on its negation or in none of them. An epistemic input (new information) can provoke changes in the agents attitude toward a given belief and consequently on its set of beliefs. We identified three types of belief change operations: expansion (when new information is added to the set of beliefs of an agent), contraction (when beliefs are removed from the set of beliefs of an agent) and revision (where new information is added to the set of beliefs of an agent, but some beliefs may be removed in order to maintain consistence). We also presented some *criteria of rationality* namely: *principle of categorical matching, irrelevance of syntax, primacy (of the new information), consistency, logical omniscience, minimal change, fairness* and *preference.*

Chapter 3

The AGM Model of Theory Change

> "The models and results of the AGM paradigm are so neat, that one almost feels reluctant to change anything at all."
> Peppas in [Pep08]

One of the main contributions to the study of belief change is the so-called AGM model for belief change proposed by Alchourrón, Gärdenfors and Makinson. It was developed in a number of papers written in the 1970s and 1980s ([Gär78, Gär81, Gär82, Gär84, AM81, AM82, AM85, AGM85]). In this chapter we are going to present the AGM model for belief change. This model was proposed in [AGM85] and gained the status of standard model of belief change. As mentioned by Wassermann: "The original AGM framework is a theory about how highly idealized rational agents should revise their beliefs when receiving new information. The agents are idealized in that they have unlimited memory and ability of inference" [Was00]. In this framework, beliefs are represented by sentences of a propositional language, belief sets are used to model epistemic states, and epistemic inputs are represented by single sentences. There is only one inconsistent belief set, that we will represent by $\mathbf{K}_\perp$ and coincides with $\mathcal{L}$, the set of all formulae of the language. The AGM model considers three kinds of belief change operators, namely *expansion, contraction* and *revision.* In this chapter we will present a list of postulates that characterize each one of these operators. Of the three AGM operators of change, expansion is the only one that can be defined in a unique way. For the contraction and revision operators we will present some explicit methods for their construction, as well as the axiomatic characterizations for each one of the classes of operators obtained through those methods. At the end of this chapter we will present the interrelationships between the different classes of contraction operators obtained through the constructive methods presented throughout this chapter.

In the AGM model three kinds of belief change operators are considered:

Expansion: An expansion occurs when new information is simply added to the set of the beliefs of an agent. As a result of an expansion, the belief set of an agent can become inconsistent. The outcome of an expansion of a belief set $\mathbf{K}$ by a sentence α will be denoted by $\mathbf{K} + \alpha$.

Contraction: A contraction occurs when information is removed from the set of beliefs of an agent. The result of a contraction of a belief set $\mathbf{K}$ by a sentence α will be denoted by $\mathbf{K} \div \alpha$.

Revision: A revision occurs when new information is added to the set of the beliefs of an agent. When performing a revision some beliefs can be removed in order to ensure consistency. Contrary to expansion, revision preserves consistency (unless the new information is itself inconsistent). The result of a revision of a belief set $\mathbf{K}$ by a sentence α will be denoted by $\mathbf{K}\star\alpha$.

In the AGM framework a belief change operator is essentially a function taking a belief set $\mathbf{K}$ and a sentence α to a new belief set. Throughout this thesis, whenever considering the context of belief sets, we shall use the expression "contraction function" (or simply "contraction") to refer to a function

$$\begin{aligned} \div: \quad & \mathcal{T}_{\mathcal{L}} \times \mathcal{L} \to \mathcal{T}_{\mathcal{L}} \\ & (\mathbf{K}, \alpha) \mapsto \mathbf{K} \div \alpha \end{aligned}$$

Given a fixed belief set $\mathbf{K}$, we shall use the expression "contraction function on $\mathbf{K}$" (or simply "contraction on $\mathbf{K}$") to refer to a function

$$\begin{aligned} \div: \quad & \mathcal{L} \to \mathcal{T}_{\mathcal{L}} \\ & \alpha \mapsto \mathbf{K} \div \alpha \end{aligned}$$

Analogous notation and terminology will be used for expansions and revisions.

3.1 Postulates

In this section we will present each one of the AGM belief change functions through a set of postulates that determine the behaviour of each one of these functions, establishing conditions or constrains that they must satisfy. We will also present two identities, the Levi and Harper identities, that allows us to define revisions in terms of contractions and vice-versa.

3.1.1 Expansion

Expansion is the simplest of the AGM operators. It simply adds new information to the belief set. An AGM expansion operator satisfies the following set of postulates:

(+1) $\mathbf{K} + \alpha = Cn(\mathbf{K} + \alpha)$ (*i.e.*, $\mathbf{K} + \alpha$ is a belief set). (Closure)
(+2) $\alpha \in \mathbf{K} + \alpha$. (Success)
(+3) $\mathbf{K} \subseteq \mathbf{K} + \alpha$. (Inclusion)
(+4) If $\alpha \in \mathbf{K}$, then $\mathbf{K} + \alpha = \mathbf{K}$. (Vacuity)
(+5) If $\mathbf{K} \subseteq \mathbf{K}'$, then $\mathbf{K} + \alpha \subseteq \mathbf{K}' + \alpha$. (Monotony)

(+6) For all beliefs sets $\mathbf{K}$ and all sentences $\alpha \in \mathcal{L}$, $\mathbf{K} + \alpha$ is the smallest belief set that satisfies (+1) to (+5). (Minimality)

(+1) states that the outcome of an expansion is a belief set. This postulate is an expression of the principle of *categorical matching*. (+2) states that a sentence is always incorporated in the outcome of an expansion by it. (+2) is a formalization of the *primacy (of the new information)* criteria. (+3) says that the expanded belief set contains all the sentences of the original belief set, and thus that no information is removed when performing an expansion. (+4) says that if we expand a belief set by a sentence that already belongs to it, then the expansion should leave the original set unchanged.[1] (+5) states that if a belief set contains at least the same sentences as another, then its expansion by a given sentence should contain at least the same sentences of the expansion of the other belief set by the same sentence. (+6) insures that the outcome of an expansion does not contain more sentences than those that are needed to fulfill postulates (+1) to (+5). The postulates (+3), (+4) and (+6) can be considered as expressions of *minimal change*.

The following observation illustrates that, on a given a belief set, postulates (+1) to (+6) define an unique operator.

Observation 3.1.1 *[Gär88] An operator* + *on a belief set* $\mathbf{K}$ *satisfies postulates* (+1) *to* (+6) *if and only if (for all* $\alpha \in \mathcal{L}$*):*

$$\mathbf{K} + \alpha = Cn(\mathbf{K} \cup \{\alpha\}).$$

Based in this result we can define an AGM expansion as follows:

Definition 3.1.2 *[Lev77] Let* $\mathbf{K}$ *be a belief set and* α *be a sentence.* $\mathbf{K} + \alpha$*, the expansion of* $\mathbf{K}$ *by* α*, is defined as follows:*

$$\mathbf{K} + \alpha = Cn(\mathbf{K} \cup \{\alpha\}).$$

From the exposed we can conclude that the expansion of a belief set by a sentence is a two steps procedure: first that sentence is added to the belief set and, afterwards, the resulting set is closed by logical consequence.

3.1.2 Contraction

Contrary to expansions, the operators of contraction and revision are not defined in an unique way, but are constrained by a set of postulates. As stated by Meyer: "The idea is that these are the rational choices to be made" [Mey99].

A contraction of a belief set occurs when some beliefs are removed from it (and no new beliefs are added to that set). The following postulates, which were presented in [AGM85] (following [Gär78, Gär82]), are commonly known as *basic Gärdenfors*

[1] Note that the expansion of a belief set $\mathbf{K}$ by a sentence α that is not in $\mathbf{K}$ produces other changes to the belief set other than the simple addition of α. In order to satisfy (+1) all the consequences of $\mathbf{K} + \alpha$, that are not in $\mathbf{K}$ must be added too.

postulates for contraction or *basic AGM postulates for contraction*:

($\div$1) $\mathbf{K} \div \alpha = Cn(\mathbf{K} \div \alpha)$ (*i.e.* $\mathbf{K} \div \alpha$ is a belief set). (Closure)
($\div$2) $\mathbf{K} \div \alpha \subseteq \mathbf{K}$. (Inclusion)
($\div$3) If $\alpha \notin \mathbf{K}$, then $\mathbf{K} \subseteq \mathbf{K} \div \alpha$. (Vacuity)
($\div$4) If $\nvdash \alpha$, then $\alpha \notin \mathbf{K} \div \alpha$. (Success)
($\div$5) $\mathbf{K} \subseteq (\mathbf{K} \div \alpha) + \alpha$. (Recovery)
($\div$6) If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$. (Extensionality)

($\div$1) assures that the outcome of a contraction is always a belief set. ($\div$1) is a formalization of the *principle of categorical matching*. ($\div$2) states that no new sentences are added to the original set as a result of a contraction. ($\div$3) ensures that nothing is removed when contracting a belief set by a sentence that is not an element of that belief set. ($\div$4), also known as *success*, states that if a sentence is not a tautology, then it is not an element of the outcome of the contraction of the belief set by it. ($\div$6) states that the contraction of a belief set by logical equivalent sentences produces the same output. ($\div$6) is a formalization of the *irrelevance of syntax* criteria. ($\div$5), also known as *recovery*, states that if the result of contracting a belief set by a certain sentence is (subsequently) expanded by that same sentence then all the initial sentences are recovered. *Recovery* is an expression of the principle of *minimal change*. This is one of the most controversial postulates and its desirability is not consensual, since there are examples of contractions where *recovery* seems implausible (see Section 4.1).

The operators that satisfy postulates ($\div$1) to ($\div$6) are known as *basic AGM contractions*.

Definition 3.1.3 *An operator $\div$ for a belief set* $\mathbf{K}$ *is a basic AGM contraction if and only if it satisfies postulates* ($\div$1) *to* ($\div$6).

Next we present an example to illustrate the possible outcomes of a contraction by a basic AGM contraction function. This example and the diagram that follows it is adapted from [Rib10, Example 3.2, Figure 3.1].

Example 3.1.4 *Consider a language $\mathcal{L}$ that is built from the finite set of propositional symbols $\{p, q\}$ and the boolean connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$. Let $\mathbf{K} = Cn(\neg p \wedge \neg q)$. Assume that we wish to perform a basic AGM contraction in order to contract $\mathbf{K}$ by $\neg p$.*
In Figure 3.1 we represent all the possible belief sets of $\mathcal{L}$. Each node of the diagram represents a belief set and the arrows between nodes represent that the set at the beginning of the arrow contains the set that the arrow links it to.

Let $\div$ be a basic AGM contraction. Postulate ($\div$1) ensures that $\mathbf{K} \div \neg p$ is a belief set. Thus it should be one of the belief sets presented in the diagram. ($\div$4) assures that $\mathbf{K} \div \neg p$ does not contain $Cn(\neg p)$. Thus $\mathbf{K} \div \neg p$ cannot be $Cn(\neg p), Cn(\neg p \wedge q), Cn(\neg p \wedge \neg q)$ or $\mathcal{L}$. On the other hand ($\div$2) assures that $\mathbf{K} \div \neg p \subseteq \mathbf{K}$. This narrows the possible outcomes of $\mathbf{K} \div \neg p$ to $Cn(p \leftrightarrow q), Cn(\neg q), Cn(\neg p \vee q), Cn(\neg p \vee \neg q), Cn(p \vee \neg q)$ and $Cn(\varnothing)$. ($\div$5) ensures that $\mathbf{K} \div \neg p \cup \{\neg p\} \vdash \neg p \wedge \neg q$. Hence, by

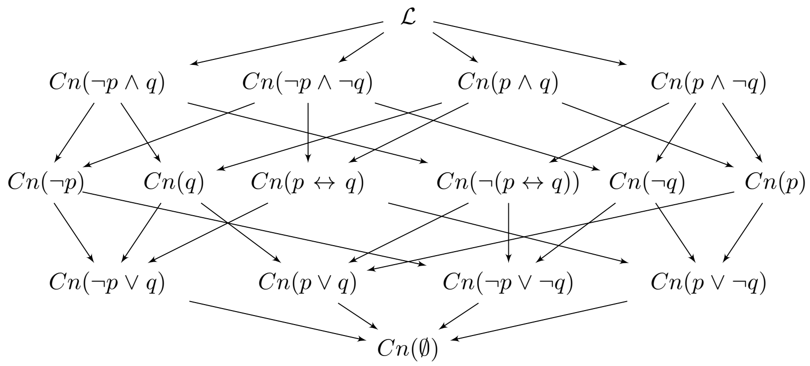

Figure 3.1: Diagram of Example 3.1.4.

deduction it must be the case that $\mathbf{K} \div \neg p \vdash \neg p \rightarrow (\neg p \wedge \neg q)$. *I.e.,* $\mathbf{K} \div \neg p \vdash \neg q \vee p$. *Therefore* $\mathbf{K} \div \neg p$ *must be one of the following:* $Cn(p \leftrightarrow q), Cn(\neg q)$ *or* $Cn(p \vee \neg q)$.

Due to the problematic character of the *recovery* postulate, Makinson proposed a wider class of contraction operators, that he designated by *withdrawals*. Makinson defined a *withdrawal* as an operator that satisfies the basic AGM postulates for contraction with the exception of *recovery*.

Definition 3.1.5 *[Mak87] An operator* $\div$ *for a belief set* $\mathbf{K}$ *is a withdrawal if and only if it satisfies postulates* $(\div 1)$ *to* $(\div 4)$ *and* $(\div 6)$.

In addition to the six basic AGM postulates for contraction, Alchourrón, Gärdenfors and Makinson presented in [AGM85] the following postulates for contraction by a conjunction:

$(\div 7)$ $(\mathbf{K} \div \alpha) \cap (\mathbf{K} \div \beta) \subseteq \mathbf{K} \div (\alpha \wedge \beta)$. (Conjunctive overlap)
$(\div 8)$ $\mathbf{K} \div (\alpha \wedge \beta) \subseteq \mathbf{K} \div \alpha$ whenever $\alpha \notin \mathbf{K} \div (\alpha \wedge \beta)$. (Conjunctive inclusion)

These are known as the supplementary AGM postulates. $(\div 7)$ states that all the sentences that are in $\mathbf{K} \div \alpha$ and in $\mathbf{K} \div \beta$ must be in $\mathbf{K} \div (\alpha \wedge \beta)$. $(\div 8)$ ensures that if α is removed in the process of contraction a belief set by $\alpha \wedge \beta$, then everything that is removed when contracting that belief set by α is also removed when contracting it by $\alpha \wedge \beta$.

Definition 3.1.6 *An operator* $\div$ *for a belief set* $\mathbf{K}$ *is an AGM contraction if and only if it satisfies postulates* $(\div 1)$ *to* $(\div 8)$.

The following postulate presented in [AGM85], known as *ventilation* or *conjunctive factoring*, is another postulate for contraction by a conjunction. It states that the outcome of a contraction by a conjunction is identical to either the contraction by one of the conjuncts or to the intersection of the outcomes of the contractions by each one of the conjuncts.

$(\div V)$ $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha$ or $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \beta$ or $\mathbf{K} \div (\alpha \wedge \beta) = \mathbf{K} \div \alpha \cap \mathbf{K} \div \beta$. (Conjunctive factoring)

The following postulate, known as *conjunctive trisection* [Han93a, Rot92b], is another postulate for contraction by a conjunction. It states that if a belief α is not removed when contracting a belief set by $\alpha \wedge \beta$, then α should be kept when contracting that belief set by $\alpha \wedge \beta \wedge \delta$. Intuitively, if α is not removed when a belief set is contracted by $\alpha \wedge \beta$, then α is (in some sense) better than β.[2] Hence, α should also be kept when contracting that belief set by $\alpha \wedge \beta \wedge \delta$.

(Conjunctive trisection) If $\alpha \in \mathbf{K} \div (\alpha \wedge \beta)$, then $\alpha \in \mathbf{K} \div (\alpha \wedge \beta \wedge \delta)$.

In the presence of the six basic AGM postulates for contraction, $(\div 7)$ and $(\div 8)$ are equivalent to $(\div V)$. In this case it also holds that $(\div 7)$ is equivalent to *conjunctive trisection.*

Observation 3.1.7 *Let* $\mathbf{K}$ *be a belief set and* $\div$ *an operator on* $\mathbf{K}$ *that satisfies* $(\div 1)$ *to* $(\div 6)$. *Then:*

(a) [AGM85] $\div$ *satisfies* $(\div 7)$ *and* $(\div 8)$ *if and only if* $\div$ *satisfies* $(\div V)$.

(b) [Rot92b] $\div$ *satisfies* $(\div 7)$ *if and only if* $\div$ *satisfies conjunctive trisection.*

3.1.3 Revision

The operation of revision of a belief set consists of the incorporation of new beliefs in that set. In a revision process, some previous beliefs may be retracted in order to preserve the consistency of the resulting belief set. The postulates for revision follow the same pattern as for contraction. The following six postulates, which were presented in [Gär88], are commonly known as *basic AGM postulates for revision*:[3]

$(\star 1)$	$\mathbf{K}\star\alpha = Cn(\mathbf{K}\star\alpha)$ (*i.e.* $\mathbf{K}\star\alpha$ is a belief set).	(Closure)
$(\star 2)$	$\alpha \in \mathbf{K}\star\alpha$.	(Success)
$(\star 3)$	$\mathbf{K}\star\alpha \subseteq \mathbf{K} + \alpha$.	(Inclusion)
$(\star 4)$	If $\neg\alpha \notin \mathbf{K}$, then $\mathbf{K} + \alpha \subseteq \mathbf{K}\star\alpha$.	(Vacuity)
$(\star 5)$	If α is consistent, then $\mathbf{K}\star\alpha$ is consistent.	(Consistency)
$(\star 6)$	If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K}\star\alpha = \mathbf{K}\star\beta$.	(Extensionality)

$(\star 1)$ assures that the outcome of a revision is always a belief set. This is a formalization of the *principle of categorical matching.* $(\star 2)$ states that the sentence by which the belief set is revised is an element of the revised belief set. This postulate formalizes the principle of *primacy (of the new information).* According to $(\star 3)$ what is added to the revision of a belief set $\mathbf{K}$ by a sentence α is at most the consequences of $\mathbf{K} \cup \{\alpha\}$ that are not in $\mathbf{K}$. $(\star 4)$ states that if a negation of a sentence is not in the belief set $\mathbf{K}$, then the revision of $\mathbf{K}$ by that sentence contains

[2]Except on the limiting case that $\vdash \alpha \wedge \beta$.

[3]The postulates were already presented in [AGM85] but with slightly different formulations.

the expansion **K** by that sentence. $(\star 5)$ assures that the revision of a belief set by a consistent sentence is itself consistent. $(\star 5)$ expresses the *consistency* criteria. $(\star 6)$ is a formalization of the *irrelevance of syntax* criteria. It states that the revision of a belief set by logical equivalent sentences produces the same output.

The operators that satisfy postulates $(\star 1)$ to $(\star 6)$ are known as *basic AGM revisions.*

Definition 3.1.8 *An operator $\star$ for a belief set $\mathbf{K}$ is a basic AGM revision if and only if it satisfies postulates $(\star 1)$ to $(\star 6)$.*

In the following example we revisit Example 3.1.4 to determine the possibles outcomes of performing a revision by a basic AGM revision function.

Example 3.1.9 *Consider a language $\mathcal{L}$ that is built from the finite set of propositional symbols $\{p, q\}$ and the boolean connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$. Let $\mathbf{K} = Cn(\neg p \wedge \neg q)$. Assume that we wish to perform a basic AGM revision in order to revise $\mathbf{K}$ by p.*
We recall that in Figure 3.1 we represent all the possible belief sets of $\mathcal{L}$. Each node of the diagram represents a belief set and the arrows between nodes represent that the set at the beginning of the arrow contains the set that the arrow links it to.
Let $\star$ be a basic AGM revision operator. Postulate $(\star 1)$ ensures that $\mathbf{K}\star p$ is a belief set. Thus it should be one of the belief sets presented in the diagram. $(\star 2)$ assures that $p \in \mathbf{K}\star p$. Thus $\mathbf{K}\star p$ must contain $Cn(p)$. $(\star 5)$ states that $\mathbf{K}\star p \neq \mathcal{L}$. This narrows the possible outcomes of $\mathbf{K}\star p$ to $Cn(p), Cn(p \wedge q)$ and $Cn(p \wedge \neg q)$. $(\star 3)$, $(\star 4)$ and $(\star 6)$ do not (in this case) narrow the possibles outcomes of $\mathbf{K}\star p$.[4]

As it was the case for contractions two other postulates were proposed to deal with the revision by conjunctions [Gär78, Gär82]:

$(\star 7)$ $\mathbf{K}\star(\alpha \wedge \beta) \subseteq (\mathbf{K}\star\alpha) + \beta$. (Superexpansion)
$(\star 8)$ If $\neg\beta \notin \mathbf{K}\star\alpha$, then $(\mathbf{K}\star\alpha) + \beta \subseteq \mathbf{K}\star(\alpha \wedge \beta)$. (Subexpansion)

$(\star 7)$ states that the outcome of the revision of a belief set by a conjunction is contained in the (belief) set that results of the expansion by one of the conjuncts of the outcome of revising the original belief set by the other conjunct. $(\star 8)$ states that if the outcome of the revision of a belief set by a conjunction does not contain the expansion by one of the conjuncts of the outcome of the revision by the other conjunct, then the revision by this (latter) conjunct contains the negation of the other one.[5]

Definition 3.1.10 *An operator $\star$ for a belief set $\mathbf{K}$ is an AGM revision if and only if it satisfies postulates $(\star 1)$ to $(\star 8)$.*

[4]For example, from $(\star 3)$ it follows that $\mathbf{K}\star p$ must be a subset of $Cn(Cn(\neg p \wedge \neg q) \cup \{p\}) = \mathcal{L}$.
[5]Note that if $\neg\beta \in \mathbf{K}\star\alpha$ and $(\mathbf{K}\star\alpha) + \beta \subseteq \mathbf{K}\star(\alpha \wedge \beta)$, then (assuming that $\star$ satisfies *success*) $\mathbf{K}\star(\alpha \wedge \beta)$ would be inconsistent.

The following postulates were proposed by Gärdenfors [Gär88] to deal with the revision by disjunctions:

(Disjunctive overlap) $(\mathbf{K}\star\alpha) \cap (\mathbf{K}\star\beta) \subseteq \mathbf{K}\star(\alpha \vee \beta)$.
(Disjunctive inclusion) If $\neg\beta \notin \mathbf{K}\star(\alpha \vee \beta)$, then $\mathbf{K}\star(\alpha \vee \beta) \subseteq \mathbf{K}\star\beta$.

Disjunctive overlap states that if an element is in the outcome of the revision of $\mathbf{K}$ by α and in the outcome of the revision of $\mathbf{K}$ by β, then it should also be in the outcome of the revision of $\mathbf{K}$ by $\alpha \vee \beta$.
Disjunctive inclusion ensures that if $\neg\beta$ does not belong to the revision of a belief set $\mathbf{K}$ by $\alpha \vee \beta$, then everything in the revision of $\mathbf{K}$ by $\alpha \vee \beta$ must be in the revision of $\mathbf{K}$ by β.

The following observation illustrates that in the presence of the basic AGM postulates for revision, *disjunctive overlap* and *disjunctive inclusion* are equivalent to $(\star 7)$ and $(\star 8)$, respectively.

Observation 3.1.11 *[Gär88] Let* $\mathbf{K}$ *be a logically closed set and* $\star$ *an operator on* $\mathbf{K}$ *that satisfies the basic AGM postulates for revision.*[6] *Then:*

(a) $\star$ *satisfies* $(\star 7)$ *if and only if* $\star$ *satisfies disjunctive overlap.*

(b) $\star$ *satisfies* $(\star 8)$ *if and only if* $\star$ *satisfies disjunctive inclusion.*

The following postulate, proposed by Alchourrón, Gärdenfors and Makinson in [AGM85], introduces a factoring condition on the revision by disjunctions.

$(\star V)$ $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}\star\alpha$ or $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}\star\beta$ or $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}\star\alpha \cap \mathbf{K}\star\beta$. (Disjunctive factoring)

$(\star V)$ states that the outcome of a revision by a disjunction is identical to either the revision by one of the disjuncts or to the intersection of the outcomes of the revisions by each one of the disjuncts.

The following observation illustrates that in the presence of the six basic AGM postulates for revision $(\star 7)$ and $(\star 8)$ are equivalent to *disjunctive factoring.*

Observation 3.1.12 *[Gär88] Let* $\mathbf{K}$ *be a logically closed set and* $\star$ *an operation for* $\mathbf{K}$ *that satisfies the basic AGM postulates for revision. Then* $\star$ *satisfies* $(\star V)$ *if and only if it satisfies both* $(\star 7)$ *and* $(\star 8)$.[7]

[6] In fact, according to [Han99b, Observations 3.49 and 3.50] (and their proofs), statements (a) and (b) of this result hold as long as $\star$ satisfies $(\star 1)$, $(\star 2)$ and $(\star 6)$.
[7] In fact, according to [Han99b, Observation 3.51] (and its proof), this statement holds as long as $\star$ satisfies $(\star 1)$, $(\star 2)$, $(\star 5)$ and $(\star 6)$.

3.1.4 Relations between contraction and revision

When revising a belief set by a sentence α one should incorporate α in that belief set, while removing some sentences during the process in order to ensure the consistency of the resulting belief set (whenever that is possible). This can be accomplished, if we first contract the belief set by $\neg\alpha$ and then expand its outcome by α.[8] Thus, it seems natural to define a revision operator through the following equality originally proposed in [Lev77]:

Levi identity: $\mathbf{K}\star\alpha = (\mathbf{K} \div \neg\alpha) + \alpha$.

According to Definition 3.1.2, this identity can be rewritten as follows:

Levi identity: $\mathbf{K}\star\alpha = Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\})$.

The Levi identity defines a revision in terms of a contraction. The following equality, which was originally presented in [Har76], defines a contraction operator in terms of a revision:[9]

Harper identity: $\mathbf{K} \div \alpha = (\mathbf{K}\star\neg\alpha) \cap \mathbf{K}$.

The Harper identity states that the outcome of contracting a belief set $\mathbf{K}$ by a sentence α consists of those beliefs that are kept from $\mathbf{K}$ when revising it by $\neg\alpha$. This follows from the fact that the revision of $\mathbf{K}$ by $\neg\alpha$ represents a minimal change of $\mathbf{K}$ required to incorporate $\neg\alpha$ in a consistent way. Thus it should contain as much as possible of the beliefs in $\mathbf{K}$ that fails to imply α.

The Levi and Harper identities make contraction and revision interchangeable. These identities allow us to define the revision and the contraction operators in terms of each other. The Levi (respectively Harper) identity enable the use of contraction (resp. revision) as primitive function and treat revision (resp. contraction) as defined in terms of contraction (resp. revision).

The following observation states that if an operator $\div$ satisfies postulates $(\div 2)$ to $(\div 4)$ and $(\div 6)$, then the operator obtained from $\div$ by means of the Levi identity is a basic AGM revision operator.

Observation 3.1.13 *[Gär78, Gär82]*[10] *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be an operator on* $\mathbf{K}$. *Let* $\star$ *be the operator defined from* $\div$ *by means of the Levi identity. If* $\div$ *satisfies* $(\div 2)$, $(\div 3)$, $(\div 4)$ *and* $(\div 6)$, *then* $\star$ *is a basic AGM revision operator on* $\mathbf{K}$.

[8] We note that if we exchange the order of the two operations, then after performing the expansion we may get an inconsistent belief set. Thus the outcome of such expansion can be $\mathcal{L}$. In this case we lose track of the original belief set.

[9] We note that the intersection of two belief sets is a belief set.

[10] The references here presented are the ones provided for this result in the proof of [AGM85, Observation 2.3] as well in [Mak87]. A proof for this observation is also provided in [Gär88] and in [GM88].

According to the last observation, it is not necessary that the operator $\div$ satisfies the postulates $(\div 1)$ and $(\div 5)$ in order to assure that the operator $\star$, obtained from $\div$ by means of the Levi identity, is a basic AGM revision operator. Hence if $\div$ is a withdrawal, then $\star$ is a basic AGM revision.

The following observation, when combined with Observation 3.1.13, illustrates that if $\div$ is an AGM contraction operator, then the operator obtained from it by means of the Levi identity is an AGM revision operator (in the sense of Definitions 3.1.6 and 3.1.10).

Observation 3.1.14 *[AGM85] Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a withdrawal operator on* $\mathbf{K}$*. Let* $\star$ *be the operator defined from* $\div$ *by means of the Levi identity. Then the following statements hold:*

(a) If $\div$ *also satisfies* $(\div 5)$ *and* $(\div 7)$*, then* $\star$ *satisfies* $(\star 7)$.[11]

(b) If $\div$ *also satisfies* $(\div 8)$*, then* $\star$ *satisfies* $(\star 8)$.

The following two observations provide similar results to the ones presented in the last two observations. From the following observation we can conclude that if $\star$ is a basic AGM revision, then the operator obtained from it by means of the Harper identity is a basic AGM contraction.

Observation 3.1.15 *[Gär78, Gär82]*[12] *Let* $\mathbf{K}$ *be a belief set and* $\star$ *be an operator on* $\mathbf{K}$*. Let* $\div$ *be the operator defined from* $\star$ *by means of the Harper identity. If* $\star$ *satisfies* $(\star 1)$, $(\star 2)$, $(\star 4)$, $(\star 5)$ *and* $(\star 6)$*, then* $\div$ *is a basic AGM contraction operator on* $\mathbf{K}$.

By combining the following observation with Observation 3.1.15 we can conclude that if $\star$ is an AGM revision operator, then the operator obtained from $\star$ by means of the Harper identity is an AGM contraction operator.

Observation 3.1.16 *[AGM85] Let* $\mathbf{K}$ *be a belief set and* $\star$ *be a basic AGM revision operator on* $\mathbf{K}$*. Let* $\div$ *be the operator defined from* $\star$ *by means of the Harper identity. Then the following statements hold:*

(a) If $\star$ *also satisfies* $(\star 7)$*, then* $\div$ *satisfies* $(\div 7)$.

(b) If $\star$ *also satisfies* $(\star 8)$*, then* $\div$ *satisfies* $(\div 8)$.

In the following two definitions we present functions that allow us to go from contractions to revisions and vice versa.

[11] In [Fer01] it was shown that if $\div$ satisfies postulates $(\div 1)$ to $(\div 4)$, $(\div 6)$ and $(\div 7)$, but does not satisfy $(\div 5)$, then in general, $\star$ does not satisfy $(\star 7)$.

[12] The references here presented are the ones provided for this result in the proof of [Mak87]. A proof for this observation is also provided in [Gär88] and in [GM88].

Definition 3.1.17 *[Mak87] Let* $\mathbf{K}$ *be a belief set. Then* $\mathcal{R}$ *is the function such that for every operator* $\div$ *for* $\mathbf{K}$*,* $\mathcal{R}(\div)$ *is the operator for* $\mathbf{K}$ *such that for all* α*:*

$$\mathbf{K}(\mathcal{R}(\div))\alpha = Cn((\mathbf{K} \div \neg\alpha) \cup \{\alpha\}).$$

Definition 3.1.18 *[Mak87] Let* $\mathbf{K}$ *be a belief set. Then* $\mathcal{C}$ *is the function such that for every operator* $\star$ *for* $\mathbf{K}$*,* $\mathcal{C}(\star)$ *is the operator for* $\mathbf{K}$ *such that for all* α*:*

$$\mathbf{K}(\mathcal{C}(\star))\alpha = \mathbf{K} \cap (\mathbf{K}\star\neg\alpha).$$

The following observation exposes that if we start with a basic AGM contraction operator (respectively basic AGM revision operator) on a belief set, and apply the Levi identity (resp. Harper identity), followed by the application of the Harper identity (resp. Levi identity), then we will obtain the contraction operator (resp. revision operator) that we started with.

Observation 3.1.19 *[Mak87] Let* $\mathbf{K}$ *be a belief set,* $\div$ *and* $\star$ *be operations on* $\mathbf{K}$ *and* $\mathcal{R}$ *and* $\mathcal{C}$ *the functions introduced in Definitions 3.1.17 and 3.1.18. Then the following statements hold:*

(a) If $\div$ *is a basic AGM contraction operator, then* $\mathcal{C}(\mathcal{R}(\div)) = \div$*.*[13]

(b) If $\star$ *is a basic AGM revision operator, then* $\mathcal{R}(\mathcal{C}(\star)) = \star$*.*[14]

3.2 Constructive models

The postulates that we presented in the last section, for contractions and revisions, do not determine in a unique way contraction and revision operators for a belief set. They only provide properties that these operators of change should satisfy. In this section we will present some explicit methods for the construction of such operators as well as the axiomatic characterization for each one of the classes of operators obtained through those methods.

3.2.1 Partial meet contractions and revisions

The first kind of contraction operations that we will present in this section are known as *partial meet contractions* and were originally presented in [AGM85]. This construction is based in the concept of *remainder set*, that is a set of maximal subsets (of a given set) that fails to imply a given sentence. Formally:

Definition 3.2.1 *[AM81] Let* A *be a set of sentences and* α *a sentence. The set* $A \perp \alpha$ *(*A *remainder* α*) is the set of sets such that* $B \in A \perp \alpha$ *if and only if:*

(a) $B \subseteq A$*.*

[13]In fact, according to [Han99b, Observation 3.56], this statement holds as long as $\div$ satisfies $(\div 1)$, $(\div 2)$, $(\div 3)$, $(\div 5)$ and $(\div 6)$.

[14]In fact, according to [Han99b, Observation 3.57], this statement holds as long as $\star$ satisfies $(\star 1)$, $(\star 2)$, $(\star 3)$ and $(\star 6)$.

(b) $B \nvdash \alpha$.

(c) *For all* B' *such that* $B \subset B' \subseteq A, B' \vdash \alpha$.

$A \perp \alpha$ is called *remainder set of* A *by* α and its elements are the *remainders of* A by α. From the definition of $A \perp \alpha$ if follows that:

- $A \perp \alpha = \{A\}$ if and only if $A \nvdash \alpha$.
- $A \perp \alpha = \varnothing$ if and only if $\vdash \alpha$.

The following observation exposes that if A is a logically closed set, then so are the elements of $A \perp \alpha$.

Observation 3.2.2 *[AGM85] If* $\mathbf{K}$ *is a belief set, then so are the elements of* $\mathbf{K} \perp \alpha$.

The partial meet contractions are obtained by intersecting some elements of the (associated) remainder set. The choice of those elements is performed by a *selection function*:

Definition 3.2.3 *[AGM85] Let* A *be a set of sentences. A selection function for* A *is a function* γ *such that, for all sentences* α:

(a) *If* $A \perp \alpha \neq \varnothing$, *then* $\gamma(A \perp \alpha)$ *is a non-empty subset of* $A \perp \alpha$.

(b) *If* $A \perp \alpha = \varnothing$, *then* $\gamma(A \perp \alpha) = \{A\}$.

A partial meet contraction is obtained by intersecting the elements chosen by the selection function:

Definition 3.2.4 *[AGM85] Let* A *be a set of sentences and* γ *a selection function for* A. *The partial meet contraction on* A *that is generated by* γ *is the operation* $\div_\gamma$ *such that for all sentences* α:

$$A \div_\gamma \alpha = \bigcap \gamma(A \perp \alpha).$$

An operator $\div$ *on* A *is a partial meet contraction if and only if there is a selection function* γ *for* A *such that for all sentences* α: $A \div \alpha = A \div_\gamma \alpha$.

There are two limiting cases of partial meet contractions:

Definition 3.2.5 *[AGM85] Let* $\div$ *be an operator on a set* A. *Then:*

(a) $\div$ *is an operator of maxichoice contraction if and only if it coincides with a partial meet contraction* $\div_\gamma$ *such that for all sentences* α, $\gamma(A \perp \alpha)$ *has exactly one element.*

(b) $\div$ *is an operator of full meet contraction if and only if it coincides with a partial meet contraction* $\div_\gamma$ *such that for all sentences* α, *if* $A \perp \alpha$ *is non-empty, then* $\gamma(A \perp \alpha) = A \perp \alpha$.

From the previous definition it follows immediately that, when performing a contraction of a set A by a sentence α, there is only one operation of full meet contraction, whereas several maxichoice contractions can be defined.

These limiting cases of partial meet contractions were considered unsatisfactory. The following example illustrates why *maxichoice contractions* were found to be unsatisfactory contraction functions.

Example 3.2.6 *[Han99b] I believed that John has a cat (α) and that he has a dog (β). Then I heard John saying that he would never have a dog and a cat at the same time. A cautious reasoner who does not know which of α and β should be withdrawn may choose to remove both of them. Maxichoice contractions does not allow such a cautious contraction. Note also that maxichoice contractions violate the fairness criteria.*

Full meet contraction, on the other hand, has the disadvantage of removing beliefs that are unrelated to the belief to be contracted. The following observation illustrates that the only sentences that remain after performing a full meet contraction of a belief set by a sentence α in that belief set, are the sentences of the belief set that are consequences of $\neg\alpha$.

Observation 3.2.7 *[AM82] Let $\mathbf{K}$ be a belief set and $\div$ the operator of full meet contraction on $\mathbf{K}$. Then for all $\alpha \in \mathbf{K}$*

$$\mathbf{K} \div \alpha = \mathbf{K} \cap Cn(\neg\alpha).$$

To see how implausible full meet contractions are, consider an agent that believes that there is milk in the fridge (α) and that Paris is the capital of France (β). After a full meet contraction of the agent's belief set by α, the sentence β is also removed (since it is not a logical consequence of $\neg\alpha$).

Partial meet contraction is an intermediate procedure "between the extreme caution of full meet contraction and the extreme incautiousness of maxichoice contraction" [Han99b].

Once again we revisit the Example 3.1.4, this time to illustrate the possible outcomes of a contraction by a partial meet contraction operator.

Example 3.2.8 *Consider a language $\mathcal{L}$ that is built from the finite set of propositional symbols $\{p, q\}$ and the boolean connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$. Let $\mathbf{K} = Cn(\neg p \wedge \neg q)$. Let $\div$ be a partial meet contraction operator on $\mathbf{K}$. We intend to determine the possible outcomes of $\mathbf{K} \div \neg p$. $\mathbf{K} \perp \neg p = \{Cn(p \leftrightarrow q), Cn(\neg q)\}$.[15] It follows that $\mathbf{K} \div \neg p = Cn(p \leftrightarrow q)$, $\mathbf{K} \div \neg p = Cn(\neg q)$ or $\mathbf{K} \div \neg p = Cn(p \leftrightarrow q) \cap Cn(\neg q)$ (in the first two cases $\div$ is maxichoice contraction and in the last case it is a full meet contraction). It follows that in the last case $\mathbf{K} \div \neg p = Cn(p \vee \neg q)$.*

[15] We note that by Observation 3.2.2 all the elements of $\mathbf{K} \perp \neg p$ must be belief sets.

Looking at Examples 3.1.4 and 3.2.8, it is possible to notice that the sets of possible outcomes of contracting $\mathbf{K}$ by $\neg p$ by a partial meet contraction and by a basic AGM contraction coincide. This is no coincidence. The following observation exposes that partial meet contractions on a belief set $\mathbf{K}$ satisfy all the basic AGM postulates for contraction. Furthermore it illustrates that if an operator $\div$ on a belief set satisfies all the basic AGM postulates for contraction then it is a partial meet contraction. Thus it elucidates that the class of partial meet contraction functions coincides exactly with the class of basic AGM contractions.

Observation 3.2.9 *[AGM85] Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a contraction function on* $\mathbf{K}$*. Then* $\div$ *is a partial meet contraction on* $\mathbf{K}$ *if and only if it satisfies the basic AGM postulates for contraction.*

A selection function is expected to pick up the best elements (in some sense) of the remainder set. If we consider a relation $\sqsubseteq$ on subsets of A such that $A_1 \sqsubseteq A_2$ holds if and only if A_2 is at least as much worth retaining as A_1, then we can define a selection function, based on $\sqsubseteq$, that chooses the "best" elements of $A \perp \alpha$, according to $\sqsubseteq$. A selection function that is based on a relation in this way is called relational:

Definition 3.2.10 *[AGM85] A selection function* γ *for a set* A *is relational if and only if there is a relation* $\sqsubseteq$ *over* $\bigcup\{A \perp \alpha : \alpha \in \mathcal{L}\}$ *such that for all sentences* α*, if* $A \perp \alpha$ *is non-empty, then:*

$$\gamma(A \perp \alpha) = \{B \in A \perp \alpha : C \sqsubseteq B \text{ for all } C \in A \perp \alpha\}.$$

γ *is transitively relational if and only if this holds for some transitive relation* $\sqsubseteq$*.*
A partial meet contraction function is relational (respectively transitively relational) if and only if it is determined by some relational (resp. transitively relational) selection function.

In the following example we revisit the Example 3.1.4. It illustrates that not every partial meet contraction is a transitively relational partial meet contraction.

Example 3.2.11 *Consider a language* $\mathcal{L}$ *that is built from the finite set of propositional symbols* $\{p, q\}$ *and the boolean connectives* $\neg, \wedge, \vee, \rightarrow$ *and* $\leftrightarrow$*. Let* $\mathbf{K} = Cn(\neg p \wedge \neg q)$*. It holds that:*
(1) $\mathbf{K} \perp \neg p = \{Cn(p \leftrightarrow q), Cn(\neg q)\}$*, (2)* $\mathbf{K} \perp \neg q = \{Cn(p \leftrightarrow q), Cn(\neg p)\}$ *and (3)* $\mathbf{K} \perp (\neg p \wedge \neg q) = \{Cn(p \leftrightarrow q), Cn(\neg p), Cn(\neg q)\}$ *. Let* $\div_1$ *be a partial meet contraction such that:*
(4) $\mathbf{K} \div_1 \neg p = \mathbf{K} \div_1 \neg q = Cn(p \leftrightarrow q)$ *and (5)* $\mathbf{K} \div_1 (\neg p \wedge \neg q) = Cn(\neg p)$*. Note that* $\div_1$ *cannot be a transitively relational partial meet contraction, since this would imply that there exists a transitive relation* $\sqsubseteq$ *over* $\bigcup\{\mathbf{K} \perp \alpha : \alpha \in \mathcal{L}\}$ *such that (from (3) and (5))* $Cn(p \leftrightarrow q) \sqsubseteq Cn(\neg p)$ *and (from (2) and (4)) that* $Cn(\neg p) \sqsubset Cn(p \leftrightarrow q)$*, i.e. that* $Cn(\neg p) \sqsubseteq Cn(p \leftrightarrow q)$ *and* $Cn(p \leftrightarrow q) \not\sqsubseteq Cn(\neg p)$*. Which leads to a contradiction. Hence* $\div_1$ *is a partial meet contraction but not a transitively relational partial meet contraction. Note also that* $\div_1$ *is a basic AGM contraction but is not an AGM contraction since it violates the supplementary postulates for contraction.*

From the above we can conclude that if $\div_2$ *is a transitively relational partial meet contraction such that (6)* $\mathbf{K} \div_2 \neg p = \mathbf{K} \div_2 \neg q = Cn(p \leftrightarrow q)$*, then* $\div_2$ *is determined by some transitively relational selection function* γ *based on a transitive relation* $\sqsubseteq_2$ *over* $\bigcup\{\mathbf{K} \perp \alpha : \alpha \in \mathcal{L}\}$ *such that* $Cn(\neg p) \sqsubset_2 Cn(p \leftrightarrow q)$ *and* $Cn(\neg q) \sqsubset_2 Cn(p \leftrightarrow q)$ *(from (1), (2) and (6)).*[16] *Therefore it must hold, having in mind (3), that* $\mathbf{K} \div_2 (\neg p \wedge \neg q) = Cn(p \leftrightarrow q)$*.*

The following observation provides an axiomatic characterization for the class of transitively relational partial meet contractions. It illustrates that such a class coincides exactly with the class of AGM contractions.

Observation 3.2.12 *[AGM85] Let* $\mathbf{K}$ *be a belief set and* $\div$ *be an operator on* $\mathbf{K}$*. Then* $\div$ *is a transitively relational partial meet contraction if and only if it satisfies both the basic and the supplementary AGM postulates for contraction.*

Partial meet revisions are obtained from partial meet contractions by means of the Levi identity:

Definition 3.2.13 *Let* $\mathbf{K}$ *be a belief set and* γ *a selection function for* $\mathbf{K}$*. For any sentence* α*, the partial meet revision operator over* $\mathbf{K}$ *determined by* γ *is defined as follows:*

$$\mathbf{K} \star_\gamma \alpha = Cn(\bigcap \gamma(\mathbf{K} \perp \neg\alpha) \cup \{\alpha\}).^{17}$$

An operator $\star$ *on* $\mathbf{K}$ *is a partial meet revision if and only if there is a selection function* γ *for* $\mathbf{K}$ *such that for all sentences* α*:* $\mathbf{K} \star \alpha = \mathbf{K} \star_\gamma \alpha$*.*
A partial meet revision function is relational (respectively transitively relational) over $\mathbf{K}$ *if and only if it is determined by some relational (resp. transitively relational) selection function.*

In the next observation we present the axiomatic characterizations of partial meet revision functions and of transitively relational partial meet revision functions. It illustrates that the class of partial meet revision functions coincides with the class of basic AGM revision functions. It also exposes that the class of transitively relational partial meet revision functions coincides with the class of AGM revision functions.

Observation 3.2.14 *[AGM85] Let* $\mathbf{K}$ *be a belief set. An operator on* $\mathbf{K}$ *is a partial meet revision function if and only if it satisfies postulates* $(\star 1)$ *to* $(\star 6)$*. It is a transitively relational partial meet revision function if and only if it also satisfies* $(\star 7)$ *and* $(\star 8)$*.*

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[16] Note also that $Cn(p \leftrightarrow q) \sqsubseteq_2 Cn(p \leftrightarrow q)$ must hold.

[17] The partial meet revision operator over a belief base A determined by a selection function γ for A is defined as follows:

$$A \star_\gamma \alpha = \bigcap \gamma(A \perp \neg\alpha) \cup \{\alpha\}.$$

3.2.2 Kernel/Safe contractions

The partial meet contraction operators on a set A are based on a selection among the maximal subsets of A that do not imply α. Another different proposal consists of constructing a contraction operator based on a selection of elements of A that imply α and then discarding them when contracting A by α. Following this approach, Hansson in [Han94] introduced a new contraction operator, the *kernel contraction*, which can be seen as a generalization of the *safe contraction* defined by Alchourrón and Makinson in [AM85].[18] We will start by presenting *kernel contractions* (the more general construction) and afterwards we will present *safe contractions* (which were chronologically presented first) as a special case of *kernel contractions*.

A kernel contraction is based on a selection among the sentences of a set A that contribute effectively to imply α; and on how to use this selection in contracting by α. Formally:

Definition 3.2.15 *[Han94] Let $A \subseteq \mathcal{L}$ and α a sentence. Then $A \perp\!\!\!\perp \alpha$ is the set such that $B \in A \perp\!\!\!\perp \alpha$ if and only if:*

(a) $B \subseteq A$.

(b) $B \vdash \alpha$.

(c) If $B' \subset B$, then $B' \nvdash \alpha$.

$A \perp\!\!\!\perp \alpha$ is called the kernel set of A with respect to α *and its elements are the* α-kernels *of A.*

To contract a belief α from a set A one must give up sentences in each α-kernel, otherwise α would continue being implied by A. The so-called incision functions selects the beliefs to be discarded.

Definition 3.2.16 *[Han94] Let A be a set of sentences. An* incision function *σ for A is a function such that for all sentences α:*

(a) $\sigma(A \perp\!\!\!\perp \alpha) \subseteq \bigcup(A \perp\!\!\!\perp \alpha)$.

(b) If $\emptyset \neq B \in A \perp\!\!\!\perp \alpha$, then $B \cap \sigma(A \perp\!\!\!\perp \alpha) \neq \emptyset$.

Definition 3.2.17 *[Han94] Let A be a set of sentences and σ an incision function for A. The* kernel contraction *$\div_\sigma$ for A is defined as:*

$$A \div_\sigma \alpha = A \setminus \sigma(A \perp\!\!\!\perp \alpha).$$

An operator $\div$ for A is a kernel contraction if and only if there is an incision function σ for A such that $A \div \alpha = A \div_\sigma \alpha$ for all sentences α.

[18] For a deep study of this kind of functions see [RH14].

Example 3.2.18 [19] *Let* $A = \{p, p \vee q, p \leftrightarrow q, r\}$. *Suppose that we intend to contract* A *by* $p \wedge q$. *The elements of the kernel set of* A *with respect to* $p \wedge q$ *are the minimal subsets of* A *that imply* $p \wedge q$. *Hence* $A \perp\!\!\!\perp (p \wedge q) = \{\{p, p \leftrightarrow q\}, \{p \vee q, p \leftrightarrow q\}\}$. *An incision function must choose at least one sentence from each element of* $A \perp\!\!\!\perp (p \wedge q)$. *An example of an incision function is* $\sigma(A \perp\!\!\!\perp (p \wedge q)) = \{p \vee q, p \leftrightarrow q\}$. *In this case* $A -_\sigma (p \wedge q) = A \setminus \{p \vee q, p \leftrightarrow q\} = \{p, r\}$.

Kernel contractions on belief sets satisfy all the basic AGM postulates with the exception of $(\div 1)$. In order to achieve the satisfaction of this postulate we need to impose a condition on the incision function.

Definition 3.2.19 *[Han94] An incision function* σ *for a set* A *is smooth if and only if it holds for all subsets* A' *of* A *that if* $A' \vdash \beta$ *and* $\beta \in \sigma(A \perp\!\!\!\perp \alpha)$ *then* $A' \cap \sigma(A \perp\!\!\!\perp \alpha) \neq \varnothing$. *A kernel contraction is smooth if and only if it is based on a smooth incision function.*

We note that the incision function presented in Example 3.2.18 is not smooth. To see this notice that in that example $\sigma(A \perp\!\!\!\perp (p \wedge q)) = \{p \vee q, p \leftrightarrow q\}$. Thus $p \vee q \in \sigma(A \perp\!\!\!\perp (p \wedge q))$. On the other hand it holds that $\{p\} \vdash p \vee q$, but $\{p\} \cap \sigma(A \perp\!\!\!\perp (p \wedge q)) = \varnothing$. This also allows us to conclude that not every kernel contraction is smooth.

Hansson in [Han94] proposed an alternative way of constructing smooth kernel contractions. More precisely, Hansson introduced a new contraction operator — the so called *saturated kernel contraction*, a particular kind of kernel contraction, and established that the class of smooth kernel contractions coincides with the class of saturated kernel contractions.

Definition 3.2.20 *[Han94] Let* σ *be an incision function for a set of sentences* A. *The saturated kernel contraction* $\div_\sigma^s$ *for* A *that is associated with* σ *is defined, for any sentence* α, *as follows:*

$$A \div_\sigma^s \alpha = A \cap Cn(A \div_\sigma \alpha),$$

where $\div_\sigma$ *is the kernel contraction for* A *based on* σ.[20] *An operator* $\div$ *for* A *is a saturated kernel contraction if and only if there is some incision function* σ *for* A *such that* $A \div \alpha = A \div_\sigma^s \alpha$ *for all sentences* α.

Observation 3.2.21 *[Han94] An operator* $\div$ *for a set of sentences* A *is a saturated kernel contraction if and only if it is a smooth kernel contraction.*

The following observation asserts that, in belief sets, there is no distinction between smooth kernel contractions and partial meet contractions.

Observation 3.2.22 *[Han94] Let* $\mathbf{K}$ *be a belief set. Then* $\div$ *is a smooth kernel contraction on* $\mathbf{K}$ *if and only if it is a partial meet contraction on* $\mathbf{K}$.

[19] Example adapted from [FKR08].

[20] We note that, according to Definition 3.2.17, $A \div_\sigma^s \alpha = A \cap Cn(A \setminus \sigma(A \perp\!\!\!\perp \alpha))$.

An operator of smooth kernel revision can be defined from an operator of smooth kernel contraction by means of the Levi identity. It follows from Observation 3.2.22 and Definition 3.2.13 that $\star$ is a smooth kernel revision if and only if it is a partial meet revision operator and consequently (by Observation 3.2.14) a basic AGM revision operator.

An incision function selects the sentences to remove, when a contraction is performed. This task can also be done by a selection function that selects at least one element of each kernel. In the following definition we formalize such a function:

Definition 3.2.23 *[Han99b] Let A be a set of sentences. A kernel selection function for A is a function s such that for all $X \in \{X : X \in A \perp\!\!\!\perp \alpha$ for some $\alpha\}$:*

(a) $s(X) \subseteq X$;

(b) If $X \neq \emptyset$, then $s(X) \neq \emptyset$.

According to Definition 3.2.16, given a kernel selection function s, the union of all $s(X)$ such that $X \in A \perp\!\!\!\perp \alpha$ is an incision function. This incision function is called *cumulative.*

Definition 3.2.24 *[Han99b] Let s be a kernel selection function for a set A. Then an incision function σ is the cumulation of s if and only if for all sentences α:*

$$\sigma(A \perp\!\!\!\perp \alpha) = \bigcup\{s(X) : X \in A \perp\!\!\!\perp \alpha\}.$$

A kernel selection function s can be seen as a function that selects the sentences of each kernel that should be removed when performing a contraction. It seems natural that these sentences should be the least valuable elements of each kernel. Thus it seems natural that s should be based on a binary relation that orders sentences according to its epistemic value.

Definition 3.2.25 *[Han99b] A kernel selection function s for a set A is based on a relation $\prec$ if and only if for all $X \in A \perp\!\!\!\perp \alpha$:*

$$\beta \in s(X) \text{ if and only if } \beta \in X \text{ and there is no } \delta \in X \text{ such that } \delta \prec \beta.$$

An incision function is based on a relation $\prec$ if and only if it is the cumulation of some kernel selection function that is based on $\prec$.

As shown by Hansson in [Han99b] not every binary relation can be used to construct a kernel selection function.[21] The following definition provides a sufficient condition for a relation $\prec$ to be adequate for defining a kernel selection function.

Definition 3.2.26 *[Han99b] Let A be a set of sentences and $\prec$ a relation on A. Then $\prec$ satisfies acyclicity if and only if for all positive integers n, if $\{\alpha_1, ..., \alpha_n\} \subseteq A$, then it is not the case that $\alpha_1 \prec \alpha_2 \prec ... \prec \alpha_n \prec \alpha_1$.*

[21] Hansson provided a counter-example that illustrates that a reflexive binary relation $\prec$ can not be used to construct a kernel selection function.

We will consider that $\prec$ treats logical equivalents sentences alike. An acyclic relation over a set A that satisfies this condition is called a *hierarchy over* A. Formally:

Definition 3.2.27 *[AM85] A relation $\prec$ over a set A is a hierarchy over A if and only if:*

(a) it is acyclic.

(b) if $\vdash \alpha \leftrightarrow \alpha'$ and $\vdash \beta \leftrightarrow \beta'$, then $\alpha \prec \beta$ holds if and only if $\alpha' \prec \beta'$ holds.

In the following definition we present the notion of *safe contraction*.

Definition 3.2.28 *[AM85] Let $\prec$ be a hierarchy over a set of sentences A. Let σ be the incision function that is based on $\prec$ and $\div_\sigma$ the kernel contraction based on σ. The operation of safe contraction $\div_s$ based on $\prec$ is defined as follows:*

$$A \div_s \alpha = A \cap Cn(A \div_\sigma \alpha).$$

We note that, according to Definitions 3.2.20 and 3.2.28, it holds that every safe contraction is a saturated kernel contraction.

We now recall from [AM85] the original construction of safe contraction proposed by Alchourrón and Makinson. This construction is based on the notion of *safe elements*, whose definition we present next.

Definition 3.2.29 *[AM85] Let A be a set of sentences and $\prec$ be a hierarchy over A. An element $\beta \in A$ is safe with respect to α (modulo $\prec$) if and only if β is not a minimal element (under $\prec$) of any minimal subset (under inclusion) B of A such that $B \vdash \alpha$. Equivalently, if and only if for all $B \in A \perp\!\!\!\perp \alpha$, either $\beta \notin B$ or there is some $\delta \in B$ such that $\delta \prec \beta$. We write A/α for the set of all elements of A that are safe with respect to α.*

We now present the original definition of safe contraction.

Definition 3.2.30 *[AM85] The operation of safe contraction $\div_s$ over a set A (modulo a hierarchy $\prec$) is defined as follows:*

$$A \div_s \alpha = A \cap Cn(A/\alpha).$$

We note that if the safe contraction operator is defined on a belief set $\mathbf{K}$, then the last equality can be simplified. In fact, it follows from $\mathbf{K}/\alpha \subseteq \mathbf{K}$, that $Cn(\mathbf{K}/\alpha) \subseteq Cn(\mathbf{K}) = \mathbf{K}$. Hence

$$\mathbf{K} \div_s \alpha = Cn(\mathbf{K}/\alpha).$$

The following observation exposes the equivalence between the approaches for the construction of safe contractions presented in Definitions 3.2.28 and 3.2.30.

Observation 3.2.31 *[Rei11] Let $A \subseteq \mathcal{L}$ and $\prec$ be a hierarchy over A. If σ is the incision function based on $\prec$, then:*

$$\sigma(A \perp\!\!\!\perp \alpha) = A \setminus A/\alpha.$$

Hence

$$A \div_{\sigma} \alpha = A/\alpha.$$

The following observation states that every safe contraction on a belief set $\mathbf{K}$ satisfies the basic AGM contraction postulates.

Observation 3.2.32 *[AM85] If $\div$ is a safe contraction function on a belief set $\mathbf{K}$, then $\div$ satisfies the basic AGM postulates for contraction.*

It follows from Observations and 3.2.9 and 3.2.32 that, when considering contraction functions on belief sets every safe contraction function is a partial meet contraction function. However, it was shown by Rott and Hansson in [RH14] that the converse does not hold.

If we impose further constrains to the hierarchy $\prec$, we can obtain new properties for the safe contraction operator constructed from $\prec$. Next we present some additional properties for $\prec$.

Transitivity: If $\alpha \prec \beta$ and $\beta \prec \delta$, then $\alpha \prec \delta$.
Strict Dominance: If $Cn(\beta) \subset Cn(\alpha)$, then $\alpha \prec \beta$.

It is natural to expect a hierarchy to satisfy transitivity: if we are more willing to give up α than β and more willing to give up β than δ, then we should be more willing to give up α than δ. On the other hand, if $\alpha \vdash \beta$ but $\beta \nvdash \alpha$, then in order to give up β we must give up α but to give up α we do not have to give up β. Thus is natural to expect that $\alpha \prec \beta$. The following two properties (presented in [AM85]) are satisfied by hierarchies that satisfy transitivity and strict dominance.

Continuing-up: If $\alpha \prec \beta$ and $\beta \vdash \delta$, then $\alpha \prec \delta$.
Continuing-down: If $\alpha \vdash \beta$ and $\beta \prec \delta$, then $\alpha \prec \delta$.

Definition 3.2.33 *[AM86] A relation $\prec$ over a set A is regular if and only if it satisfies continuing-up and continuing-down.*

The following observation asserts that if a hierarchy $\prec$ satisfies either *continuing-up* or *continuing-down*, then the safe contraction based on $\prec$ satisfies $(\div 7)$.

Observation 3.2.34 *[AM85] Let $\prec$ be a hierarchy over the belief set $\mathbf{K}$, and let $\div$ be the safe contraction on $\mathbf{K}$ that is based on $\prec$. Then:*

(a) If $\prec$ satisfies continuing-up, then $\div$ satisfies $(\div 7)$.

(b) If $\prec$ satisfies continuing-down, then $\div$ satisfies $(\div 7)$.

As stated in [AM85], for a safe contraction based on a hierarchy $\prec$ to satisfy ($\div$8) it is enough that $\prec$ satisfies either continuing-up or continuing-down as well as the following property:

Virtual connectivity: If $\alpha \prec \beta$, then either $\alpha \prec \delta$ or $\delta \prec \beta$.

Lemma 3.2.35 *[AM85] If a hierarchy satisfies virtual connectivity, then it satisfies continuing-up if and only if it satisfies continuing-down.*

In the following observation we present an axiomatic characterization for safe contractions operator functions on a belief set based on a regular and virtual connected hierarchy.

Observation 3.2.36 *[Rot92a]*[22] *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a contraction function on* $\mathbf{K}$. *Then* $\div$ *is a safe contraction, based on a regular and virtually connected hierarchy, if and only if it satisfies both the basic and the supplementary AGM postulates for contraction.*

3.2.3 Epistemic entrenchment-based operators of belief change

Epistemic entrenchment was introduced in [Gär88, GM88] and relies on the idea that contractions on a belief set $\mathbf{K}$ should be based on an ordering of its sentences according to their epistemic *importance*. When a belief set $\mathbf{K}$ is contracted it is preferred to give up beliefs with lower entrenchment over others with a higher entrenchment.

"Certain pieces of knowledge and belief about the world are more important than others when planning future actions, conducting scientific investigations or reasoning in general. We will say that some sentences in a knowledge system have a higher degree of epistemic entrenchment than other. The degree of entrenchment will, intuitively, have a bearing on what is abandoned from a knowledge set and what is retained, when a contraction or a revision is carried out" [GM88].

Gärdenfors proposed five postulates that an epistemic entrenchment order (with respect to a belief set $\mathbf{K}$) should satisfy.

Definition 3.2.37 *[Gär88, GM88] An ordering of epistemic entrenchment with respect to a belief set* $\mathbf{K}$ *is a binary relation* $\leq$ *on* $\mathcal{L}$ *which satisfies the following properties:*

(EE1) *For all* $\alpha, \beta, \delta \in \mathcal{L}$, *if* $\alpha \leq \beta$ *and* $\beta \leq \delta$ *then* $\alpha \leq \delta$. *(Transitivity)*
(EE2) *For all* $\alpha, \beta \in \mathcal{L}$, *if* $\alpha \vdash \beta$ *then* $\alpha \leq \beta$. *(Dominance)*
(EE3) *For all* $\alpha, \beta \in \mathcal{L}$, $\alpha \leq \alpha \wedge \beta$ *or* $\beta \leq \alpha \wedge \beta$. *(Conjunctiveness)*
(EE4) *When* $\mathbf{K} \nvdash \bot$, $\alpha \notin \mathbf{K}$ *iff* $\alpha \leq \beta$ *for all* $\beta \in \mathcal{L}$. *(Minimality)*
(EE5) *If* $\beta \leq \alpha$ *for all* $\beta \in \mathcal{L}$, *then* $\vdash \alpha$. *(Maximality)*

[22] For the case when the language is finite this result was proven in [AM86].

(EE1) states that an epistemic entrenchment order is transitive. (EE2) states that if α entails β, then α must be at most as entrenched as β. This must hold since it is not possible to give up β without given up α. (EE3) relies on the fact that to give up a conjunction we must give up (at least) one of the conjuncts. Thus the conjunction should be at least as entrenched as one of its conjuncts. (EE4) states that the less entrenched beliefs are the ones that are not in the belief set. (EE5) states that logical truths are (exactly) the beliefs with higher degree of entrenchment.

In the following lemmas we present some further properties that are satisfied by any epistemic entrenchment ordering. These properties will be useful further ahead.

Lemma 3.2.38 *[GM88] If the relation $\leq$ satisfies (EE1), (EE2) and (EE3), then it is a total relation i.e., for all $\alpha, \beta \in \mathcal{L}$ either $\alpha \leq \beta$ or $\beta \leq \alpha$.*

Lemma 3.2.39 *[Han99b] If $\mathbf{K}$ is a consistent belief set and $\leq$ is a total relation on $\mathbf{K}$ that satisfies (EE1) and (EE4), then: If $\alpha \notin \mathbf{K}$ and $\beta \in \mathbf{K}$, then $\alpha < \beta$.*

Lemma 3.2.40 *[Foo90] Let $\leq$ be a relation that satisfies (EE1), (EE2) and (EE5). If $\alpha \in Cn(\varnothing)$, then for all $\beta \in \mathcal{L} \setminus Cn(\varnothing)$, $\beta < \alpha$.*

Lemma 3.2.41 *[GR95] Let $\leq$ be a relation that satisfies (EE1) and (EE2). Then it also satisfies:*
(Intersubstitutivity) *If $\vdash \alpha \leftrightarrow \alpha'$ and $\vdash \beta \leftrightarrow \beta'$, then $\alpha \leq \beta$ if and only if $\alpha' \leq \beta'$.*

Lemma 3.2.42 *[Foo90] Let $\leq$ be a relation that satisfies (EE1), (EE2) and (EE3). Then it also satisfies:*
(Conjunction up) *If $\delta < \alpha$ and $\delta < \beta$, then $\delta < \alpha \wedge \beta$.*

Lemma 3.2.43 *(see e.g. [Han99b, Observation 2.75]) Let $\leq$ be a relation that satisfies transitivity. Then $\leq$ also satisfies:*

(a) If $\alpha \leq \beta$ and $\beta < \delta$, then $\alpha < \delta$.

(b) If $\alpha < \beta$ and $\beta \leq \delta$, then $\alpha < \delta$.

(c) If $\alpha < \beta$ and $\beta < \delta$, then $\alpha < \delta$.

In [Gär88, GM88] it was presented a way to define a contraction operator from an epistemic entrenchment with respect to a belief set $\mathbf{K}$:

Definition 3.2.44 *[Gär88, GM88] Let $\mathbf{K}$ be a belief set and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$.*

The $\leq$-based contraction *on $\mathbf{K}$ is the contraction operation $\div_{\leq}$ defined, for any $\alpha \in \mathcal{L}$, by:*

$$\mathbf{K} \div_{\leq} \alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha < \alpha \vee \beta\} & \text{,if } \nvdash \alpha \\ \mathbf{K} & \text{,if } \vdash \alpha. \end{cases} \qquad (\mathbf{C}_{\div_{\leq}})$$

An operation $\div$ on $\mathbf{K}$ is an epistemic entrenchment-based contraction *on $\mathbf{K}$ if and only if there is an epistemic entrenchment relation with respect to $\mathbf{K}$ such that, for all sentences $\alpha \in \mathcal{L}$, $\mathbf{K} \div \alpha = \mathbf{K} \div_{\leq} \alpha$.*

The following lemma establishes that the outcome of an epistemic entrenchment-based contraction on a belief set is itself a belief set.

Lemma 3.2.45 *[GM88] Let* $\mathbf{K}$ *be a belief set. Let* $\div$ *be an epistemic entrenchment-based contraction on* $\mathbf{K}$*. Then for all* $\alpha \in \mathcal{L}$ *it holds that* $Cn(\mathbf{K} \div \alpha) = \mathbf{K} \div \alpha$.

In the following example we revisit Example 3.1.4, this time to determine the possible outcomes of contracting by a given epistemic entrenchment-based contraction.

Example 3.2.46 *Consider a language* $\mathcal{L}$ *that is built from the finite set of propositional symbols* $\{p, q\}$ *and the boolean connectives* $\neg, \wedge, \vee, \rightarrow$ *and* $\leftrightarrow$*. Let* $\mathbf{K} = Cn(\neg p \wedge \neg q)$ *and* $\div$ *be an epistemic entrenchment-based contraction on* $\mathbf{K}$ *based on an epistemic entrenchment relation* $\leq$ *with respect to* $\mathbf{K}$*. We will determine the possibles outcome of* $\mathbf{K} \div \neg p$*. By Lemma 3.2.45 it follows that* $\mathbf{K} \div \neg p$ *must be one of the belief sets presented in the diagram of Figure 3.1. On the other hand, it follows from* $(\mathbf{C}_{\div_{\leq}})$*, that* $\mathbf{K} \div \neg p \subseteq \mathbf{K}$ *and* $\neg p \notin \mathbf{K} \div \neg p$ *(since, it follows by (EE2) that* $\neg p \nless \neg p$*). It also holds that* $p \vee \neg q \in \mathbf{K} \div \neg p$*, since* $\vdash \neg p \vee (p \vee \neg q)$*, thus by Lemma 3.2.40,* $\neg p < \neg p \vee (p \vee \neg q)$*. This reduces the possible outcomes of* $\mathbf{K} \div \neg p$ *to* $Cn(\neg q)$ *(if* $\neg p < \neg p \vee \neg q$*),* $Cn(p \leftrightarrow q)$ *(if* $\neg p < \neg p \vee (p \leftrightarrow q)$*) and* $Cn(p \vee \neg q)$ *(if* $\neg p \nless \neg p \vee \neg q$ *and* $\neg p \nless \neg p \vee (p \leftrightarrow q)$*).*[23] *Assume now that* $\div_2$ *is an epistemic entrenchment-based contraction on* $\mathbf{K}$ *such that* $\mathbf{K} \div_2 \neg p = \mathbf{K} \div_2 \neg q = Cn(p \leftrightarrow q)$*. Then* $\div_2$ *is based on an epistemic entrenchment relation* $\leq_2$ *with respect to* $\mathbf{K}$*. We will now determine, for this case, the possible outcomes of* $\mathbf{K} \div_2 (\neg p \wedge \neg q)$*. It follows from* $\mathbf{K} \div_2 \neg p = \mathbf{K} \div_2 \neg q = Cn(p \leftrightarrow q)$ *that* $\neg p <_2 \neg p \vee (p \leftrightarrow q)$ *and that* $\neg q <_2 \neg q \vee (p \leftrightarrow q)$*. By (EE2)* $\neg p \wedge \neg q \leq_2 \neg p$ *and* $\neg p \wedge \neg q \leq_2 \neg q$*. Thus by Lemma 3.2.43 it follows that* $\neg p \wedge \neg q <_2 \neg p \vee (p \leftrightarrow q)$ *and* $\neg p \wedge \neg q <_2 \neg q \vee (p \leftrightarrow q)$*. Therefore, by Lemma 3.2.42, it holds that* $\neg p \wedge \neg q <_2 (\neg p \vee (p \leftrightarrow q)) \wedge (\neg q \vee (p \leftrightarrow q))$*. From which it follows by (EE2) and Lemma 3.2.43 that* $\neg p \wedge \neg q <_2 (\neg p \wedge \neg q) \vee (p \leftrightarrow q)$*. Hence* $p \leftrightarrow q \in \mathbf{K} \div_2 (\neg p \wedge \neg q)$*. Since* $\mathbf{K} \div_2 (\neg p \wedge \neg q)$ *must be a belief set that does not contain* $\neg p \wedge \neg q$ *and is a subset of* $\mathbf{K}$*, it holds that* $\mathbf{K} \div_2 (\neg p \wedge \neg q) = Cn(p \leftrightarrow q)$*.*

In the next observation we recall the axiomatic characterization for the epistemic entrenchment-based contractions that was obtained by Gärdenfors and Makinsson [GM88]. It illustrates that the class of epistemic entrenchment-based contraction functions coincides with the class of AGM contraction functions.

Observation 3.2.47 *[GM88] Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a contraction function on* $\mathbf{K}$*. Then* $\div$ *is an epistemic entrenchment-based contraction if and only if it satisfies both the basic and the supplementary AGM postulates for contraction.*

We note here also that, apart from presenting a way of defining a contraction operation based on an epistemic entrenchment relation (by means of condition $(\mathbf{C}_{\div_{\leq}})$), Gärdenfors and Makinson [Gär88, GM88] have also exposed a way of proceeding in the converse direction. More precisely, in [GM88, Theorem 5], it is stated that if $\div$

[23] Note that $\neg p < \neg p \vee \neg q$ and $\neg p < \neg p \vee (p \leftrightarrow q)$ cannot hold, since this would imply, by Lemmas 3.2.42 and 3.2.43 and (EE2), that $\neg p < \neg p$.

is a contraction function on $\mathbf{K}$ that satisfies both the basic and the supplementary AGM postulates for belief set contraction, then the binary relation $\leq$ on $\mathcal{L}$ defined by the following condition:

$$\alpha \leq \beta \text{ iff } \alpha \notin \mathbf{K} \div \alpha \wedge \beta \text{ or } \vdash \alpha \wedge \beta, \tag{$\mathbf{C}_{\leq}$}$$

is an epistemic entrenchment relation with respect to $\mathbf{K}$ and, furthermore, it holds that $\mathbf{K} \div \alpha = \mathbf{K} -_{\leq} \alpha$.

Epistemic entrenchment-based revision is usually defined via epistemic entrenchment-based contraction through the Levi identity. However it is also possible to define an entrenchment-based revision directly from an epistemic ensconcement ordering, by means of the following conditions [LR91, Rot91, HFCF01]:

$(\mathbf{C}_{\leq_\star})$ $\alpha \leq \beta$ if and only if: If $\alpha \in \mathbf{K} \star \neg(\alpha \wedge \beta)$, then $\beta \in \mathbf{K} \star \neg(\alpha \wedge \beta)$.

$(\mathbf{C}_{\star_\leq})$ $\beta \in \mathbf{K} \star \alpha$ if and only if either $(\alpha \to \neg\beta) < (\alpha \to \beta)$ or $\alpha \vdash \bot$.

Next we recall the definition of the *severe withdrawals* (also known as mild contractions or Rott's contractions) which was introduced by Rott in [Rot91] and consists of an intuitively appealing simplification of the definition of *epistemic entrenchment-based contractions*.

Definition 3.2.48 *[Rot91] Let $\mathbf{K}$ be a belief set and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. The $\leq$-based severe withdrawal on $\mathbf{K}$ is the operation $\div^S_{\leq}$ defined, for any $\alpha \in \mathcal{L}$, by:*

$$\mathbf{K} \div^S_{\leq} \alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha < \beta\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{if } \vdash \alpha. \end{cases} \tag{$\mathbf{R}_{\div_\leq}$}$$

An operation $\div$ on $\mathbf{K}$ is a severe withdrawal if and only if there is an epistemic entrenchment relation $\leq$ with respect to $\mathbf{K}$ such that, for all sentences $\alpha \in \mathcal{L}$, $\mathbf{K} \div \alpha = \mathbf{K} \div^S_{\leq} \alpha$.

Severe withdrawals were axiomatically characterized independently by Rott and Pagnucco in [RP99] and by Fermé and Rodriguez in [FR98a].

Observation 3.2.49 *[RP99] Let $\mathbf{K}$ be a belief set and $\div$ be a contraction function on $\mathbf{K}$. Then $\div$ is a severe withdrawal if and only if it satisfies the following postulates:*[24] $(\div 1)$, $(\div 2)$, $(\div 3)$, $(\div 4)$,

$(\div 3')$ *If $\vdash \alpha$, then $\mathbf{K} \div \alpha = \mathbf{K}$.* *(Failure)*
$(\div 9)$ *If $\alpha \notin \mathbf{K} \div \beta$, then $\mathbf{K} \div \beta \subseteq \mathbf{K} \div \alpha$.* *(Strong Inclusion)*

[24] We note that the axiomatization of severe withdrawals presented in [RP99] consists of the postulates $(\div 1)$, $(\div 2)$, $(\div 4)$, $(\div 6)$, $(\div 9)$, and
$(\div 3)$ If $\alpha \notin \mathbf{K}$ or $\vdash \alpha$, then $\mathbf{K} \subseteq \mathbf{K} - \alpha$.
However, in the presence of $(\div 2)$, the postulate $(\div 3)$ is equivalent to the postulates $(\div 3)$ and $(\div 3')$ (taken together). On the other hand, as stated in Observation 3.2.50, $(\div 6)$ follows from $(\div 1)$, $(\div 3')$, $(\div 4)$ and $(\div 9)$.

The following observation exposes that severe withdrawals also satisfy (÷6).

Observation 3.2.50 *Let* **K** *be a belief set and* ÷ *an operator that satisfies* (÷1)*,* (÷3′)*,* (÷4) *and* (÷9)*. Then* ÷ *satisfies* (÷6)*.*

Proof: A proof for this observation can be found on page 173.

In the following observation we present some other properties that are satisfied by severe withdrawals.

Observation 3.2.51 *[RP99] Let* **K** *be a belief set and* ÷ *be a contraction function on* **K***. If* ÷ *is a severe withdrawal, then it satisfies:*

(a) If $\nvdash \alpha, \nvdash \beta$ *then* $\alpha \notin \mathbf{K} \div \beta$ *or* $\beta \notin \mathbf{K} \div \alpha$*. (Expulsiveness)*

(b) $\mathbf{K} \div \alpha \subseteq \mathbf{K} \div \beta$ *or* $\mathbf{K} \div \beta \subseteq \mathbf{K} \div \alpha$*. (Linearity)*

The properties listed in the previous observation are, respectively, known as *expulsiveness* and *linearity*. *Expulsiveness* was first presented in [Han99b] and, as it is mentioned there and also in [RP99, Rot01], it is a highly implausible property of belief contraction, since according to it two unrelated sentences influence the result of the contraction by each other. *Linearity*, which was originally presented in [FR98a, RP99], also suffers from this same excessive strength. Nevertheless, Rott and Pagnucco [RP99, Rot01] argue that the concept of severe withdrawal is still interesting and well-motivated. Lindström and Rabinowicz argued that severe withdrawals and Gärdenfors' entrenchment-based contractions should be taken as "lower" and "upper" bounds and that any realistic entrenchment-based contraction operator should be situated between them [LR91].

3.2.4 Possible worlds and spheres based operations of belief change

Along the previous sections we presented different methods to characterize the AGM operations of change, namely: through constructive methods, in which a series of steps are defined in order to obtain the change operation and through axiomatic characterizations, in which the change operators are defined by a set of properties that they should satisfy. In this section we will present an alternative method proposed by Adam Grove in [Gro88]. Grove developed a model for the change functions, based on a system of spheres, inspired by the semantics for counterfactuals proposed by Lewis in [Lew73]. As stated by Pagnucco: "Grove's idea can be viewed as a semantics insofar as it gives "a picture" for AGM belief change. Strictly speaking however, it deals with syntactic objects" [Pag96].

Definition 3.2.52 *A possible world is a maximal consistent subset of* $\mathcal{L}$*. The set of all possible worlds will be denoted by* $\mathcal{M}_{\mathcal{L}}$*.*[25] *Sets of possible worlds are called propositions.*

[25]Note that $\mathcal{M}_{\mathcal{L}} = \mathcal{L} \perp \perp$.

From the previous definition it follows that for all $\alpha \in \mathcal{L}$ and for all $M \in \mathcal{M}_{\mathcal{L}}$ it holds that $\alpha \in M$ if and only if $\neg\alpha \notin M$.

Definition 3.2.53 *[Gro88] Let R be a set of sentences of $\mathcal{L}$. The set of possible worlds that contain R is denoted by $\|R\|$, i.e.,*

$$\|R\| = \{M \in \mathcal{M}_{\mathcal{L}} : R \subseteq M\}.$$

If R is inconsistent, then $\|R\| = \varnothing$. The elements of R are designated R–worlds. For any sentence α, $\|\alpha\|$ is an abbreviation of $\|\{\alpha\}\|$ and its elements are designated by α-worlds.

Remark 3.2.54 *Let $\top_{\mathcal{L}}$ be the set of all tautologies of $\mathcal{L}$, then $\|\top_{\mathcal{L}}\| = \mathcal{M}_{\mathcal{L}}$. Thus if $\alpha \in Cn(\varnothing)$, then $\|\alpha\| = \mathcal{M}_{\mathcal{L}}$.*

Observation 3.2.55 *[Gro88, Han99b] Let* **K** *and* **H** *be belief sets and α and β sentences. Then the following properties hold:*

(a) If $w \in \mathcal{L} \perp_{\perp}$, then $w \in \|\alpha\|$ if and only if $w \notin \|\neg\alpha\|$.

(b) $\|\mathbf{K} \cup \mathbf{H}\| = \|\mathbf{K}\| \cap \|\mathbf{H}\|$.

(c) $\|\mathbf{K}\| \cup \|\mathbf{H}\| \subseteq \|\mathbf{K} \cap \mathbf{H}\|$.

(d) $\|\alpha\| \subseteq \|\beta\|$ if and only if $\vdash \alpha \rightarrow \beta$.

(e) $\|\alpha \wedge \beta\| = \|\alpha\| \cap \|\beta\|$.

(f) $\|\alpha \vee \beta\| = \|\alpha\| \cup \|\beta\|$.

Definition 3.2.56 *[Gro88] Let $V \subseteq \mathcal{M}_{\mathcal{L}}$. The theory associated to V is $Th(V) = \bigcap V$. If $V = \varnothing$, then $Th(V) = \mathcal{L}$.*

Observation 3.2.57 *[Gro88] Let* **K**, **H** *be belief sets and U, V be sets of possible worlds. Then:*

(a) $Th(\|\mathbf{K}\|) = \mathbf{K}$ (if the underlying logic is compact).

(b) $Th(V)$ is consistent if and only if V is non-empty.

(c) For any $\alpha \in \mathcal{L}$, $Th(V \cap \|\alpha\|) = Cn(Th(V) \cup \{\alpha\})$.

(d) If $U \subseteq V$, then $Th(V) \subseteq Th(U)$.

(e) If $\mathbf{K} \subseteq \mathbf{H}$, then $\|\mathbf{H}\| \subseteq \|\mathbf{K}\|$.

The following remark follows from the previous observation (more precisely from (a), (d) and (e)). It emphasizes an interesting relation between belief sets and possible worlds, namely that to greater number of beliefs corresponds a smaller set of possible worlds and vice-versa.

Remark 3.2.58 *Let* $\mathbf{K}$ *and* $\mathbf{H}$ *be belief sets. Then* $\mathbf{K} \subseteq \mathbf{H}$ *if and only if* $\|\mathbf{H}\| \subseteq \|\mathbf{K}\|$.

It follows from the previous remark that if a sentence α is an element of a belief set $\mathbf{K}$, then all $\mathbf{K}$-worlds are α-worlds. On the other hand, if $\{\alpha, \neg\alpha\} \cap \mathbf{K} = \varnothing$, then $\|\mathbf{K}\| \nsubseteq \|\alpha\|$ and $\|\mathbf{K}\| \nsubseteq \|\neg\alpha\|$, hence $\|\mathbf{K}\|$ contains some α-worlds and some $\neg\alpha$-worlds.

In what follows we will define the operations of expansion, contraction and revision on belief sets in terms of propositions. One of the main motivations for possible world models is that they can be represented graphically in a very intuitive way. Usually a rectangle is used to represent the set of all possible worlds ($\mathcal{M}_{\mathcal{L}}$). Every point on that rectangle represents a possible world. A region on the surface of the rectangle represents a set of possible worlds (*i.e.*, a proposition). As Hansson claimed: "Propositions provide us with a more intuitively clear picture of some aspects of belief change" [Han99b].

We will start by the operation of expansion. Expansion of a belief set $\mathbf{K}$ by a sentence α is defined in terms of possible worlds as the theory determined by the intersection of $\|\mathbf{K}\|$ with $\|\alpha\|$:

$$\mathbf{K} + \alpha = Th(\|\mathbf{K}\| \cap \|\alpha\|).$$

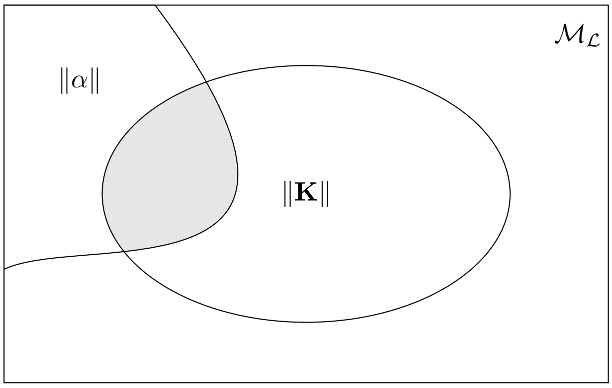

Figure 3.2: The outcome of $\mathbf{K} + \alpha$ is given by the intersection of all the worlds included in the shaded region.

Notice that if $\neg\alpha \in \mathbf{K}$, then $\|\mathbf{K}\| \cap \|\alpha\| = \varnothing$, from which it follows that $\mathbf{K} + \alpha = \mathcal{L}$ (according to Definition 3.2.56).

The outcome of contracting a belief set $\mathbf{K}$ by a sentence α such that $\alpha \in \mathbf{K} \setminus Cn(\varnothing)$ must be a subset of $\mathbf{K}$ that does not contain α. In terms of possible worlds, taking in consideration Remark 3.2.58, this means that the set of possible worlds for the contracted belief set should contain $\|\mathbf{K}\|$. In addition, it should also contain some $\neg\alpha$-worlds, in order to ensure the elimination of α. Hence, in possible world models, the contraction of the belief set $\mathbf{K}$ by α takes the form of an addition of some $\neg\alpha$-worlds to $\|\mathbf{K}\|$. The selection of these $\neg\alpha$-worlds can be performed by a propositional selection function:

Definition 3.2.59 *[Han99b] Let $M \subseteq \mathcal{M}_{\mathcal{L}}$. A propositional selection function f for M is a function such that for all sentences α:*

(a) $f(\|\alpha\|) \subseteq \|\alpha\|$.

(b) If $\|\alpha\| \neq \varnothing$, *then* $f(\|\alpha\|) \neq \varnothing$.

(c) If $M \cap \|\alpha\| \neq \varnothing$, *then* $f(\|\alpha\|) = M \cap \|\alpha\|$.

Definition 3.2.60 *[Gro88, Han99b] Let $\mathbf{K}$ be a belief set. An operation $\div$ on $\mathbf{K}$ is a possible worlds-based contraction if and only if there exists a propositional selection function f for $\|\mathbf{K}\|$ such that for all α:*

$$\mathbf{K} \div \alpha = Th(\|\mathbf{K}\| \cup f(\|\neg\alpha\|)).$$

Figure 3.3 contains a possible graphical representation of $\|\mathbf{K}\|$ and $f(\|\neg\alpha\|)$, from which $\mathbf{K} \div \alpha$ is defined (where f is a propositional selection function).

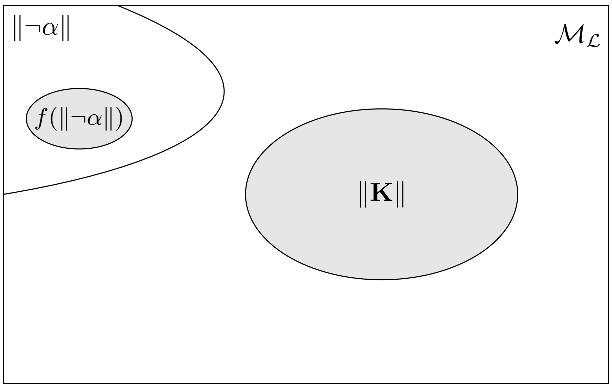

Figure 3.3: The outcome of $\mathbf{K} \div \alpha$ is given by the intersection of all worlds included in the shaded region.

Once more we revisit the Example 3.1.4, this time to illustrate the possible outcomes of a contraction by a possible worlds-based contraction operator.

Example 3.2.61 *Consider a language $\mathcal{L}$ that is built from the finite set of propositional symbols $\{p, q\}$ and the boolean connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$. Let $\mathbf{K} = Cn(\neg p \wedge \neg q)$. In this case it holds that $\mathcal{M}_{\mathcal{L}} = \mathcal{L} \perp \perp = \{Cn(\{p, q\}), Cn(\{\neg p, q\}), Cn(\{p, \neg q\}), Cn(\{\neg p, \neg q\})\}$. Hence: $\|\mathbf{K}\| = \{Cn(\{\neg p, \neg q\})\}$ and $\|p\| = \{Cn(\{p, q\}), Cn(\{p, \neg q\})\}$. If f is a propositional selection function for $\|\mathbf{K}\|$, then $f(\|p\|)$ must be either: $\{Cn(\{p, q\})\}, \{Cn(\{p, \neg q\})\}$ or $\{Cn(\{p, q\}), Cn(\{p, \neg q\})\}$.*
Let $\div$ be a possible worlds-based contraction operator on $\mathbf{K}$. We intend to determine the possible outcomes of $\mathbf{K} \div \neg p$. If follows, by Definition 3.2.60, that the outcome of $\mathbf{K} \div \neg p$ must be one of the following:

- $Th(\{Cn(\{\neg p, \neg q\})\} \cup \{Cn(\{p, q\})\}) = Th(\{Cn(\{\neg p, \neg q\}), Cn(\{p, q\})\} = Cn(\{\neg p, \neg q\}) \cap Cn(\{p, q\}) = Cn((\neg p \wedge \neg q) \vee (p \wedge q)) = Cn(p \leftrightarrow q)$.[26]

[26]Note that: $Cn(\{\alpha, \beta\}) = Cn(\alpha \wedge \beta)$ and $Cn(\alpha \vee \beta) = Cn(\alpha) \cap Cn(\beta)$.

- $Th(\{Cn(\{\neg p, \neg q\})\} \cup \{Cn(\{p, \neg q\})\}) = Cn(\{\neg p, \neg q\}) \cap Cn(\{p, \neg q\}) = Cn(\neg q)$.
- $Th(\{Cn(\{\neg p, \neg q\})\} \cup \{Cn(\{p, q\}), Cn(\{p, \neg q\})\}) = Cn(\{\neg p, \neg q\}) \cap Cn(\{p, q\}) \cap Cn(\{p, \neg q\}) = Cn(p \vee \neg q)$.

Note that, in this example, the sets of possible outcomes of contraction $\mathbf{K}$ by $\neg p$ by a possible worlds-based contraction and by a partial meet contraction coincide (as seen in Example 3.2.8). The relation between partial meet contractions and possible worlds-based contractions is clarified in the following observation.

Observation 3.2.62 *[Gro88, Han99b] Let* $\mathbf{K}$ *be a belief set and* $\div$ *be an operation on* $\mathbf{K}$*. Then the following statements are equivalent:*

(a) $\div$ *is a partial meet contraction.*

(b) $\div$ *is a possible worlds-based contraction.*

From the previous observation we can conclude that an operation $\div$ on $\mathbf{K}$ is a possible worlds-based contraction if and only if it satisfies the basic AGM-postulates for contraction.

In the process of revising a belief set $\mathbf{K}$ by a sentence α, α should be incorporated in the revised set. In terms of possible worlds this means that the set of possible worlds for the revised belief set should be a subset of $\|\alpha\|$. The selection of these α-worlds is performed by a propositional selection function:

Definition 3.2.63 *[Gro88, Han99b] Let* $\mathbf{K}$ *be a belief set. An operation* $\star$ *on* $\mathbf{K}$ *is a possible worlds-based revision if and only if there exists a propositional selection function* f *for* $\|\mathbf{K}\|$ *such that for all* α*:*

$$\mathbf{K} \star \alpha = Th(f(\|\alpha\|)).$$

Figure 3.4 contains a possible graphical representation of the set $f(\|\alpha\|)$, from which $\mathbf{K} \star \alpha$ is defined (where f is propositional selection function).

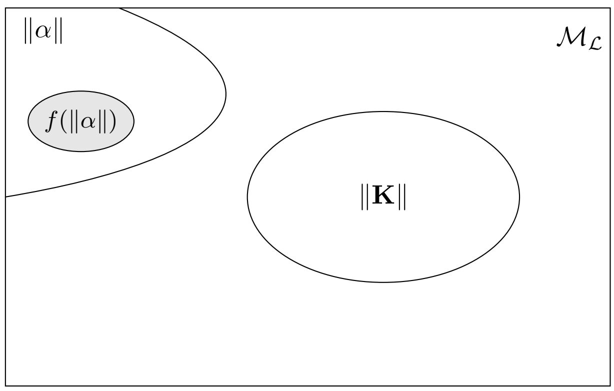

Figure 3.4: The outcome of $\mathbf{K} \star \alpha$ is given by the intersection of all worlds included in the shaded region.

The following observation illustrates that an operation on a belief set is a possible worlds-based revision if and only if it is a partial meet revision.

Observation 3.2.64 *[Gro88, Han99b] Let* $\mathbf{K}$ *be a belief set and* $\star$ *an operation on* $\mathbf{K}$*. Then the following statements are equivalent:*

(a) $\star$ *is a partial meet revision.*

(b) $\star$ *is a possible worlds-based revision.*

In order to capture also the supplementary postulates, for contractions and revisions, and thus transitively relational partial meet contractions and revisions we need to add more structure to the model. Grove in [Gro88] defined a sphere system centred on $\|\mathbf{K}\|$ as an ordering over sets of possible worlds where $\|\mathbf{K}\|$ is the innermost sphere. Figuratively, the distance between a possible world and the innermost sphere reflects its plausibility towards $\|\mathbf{K}\|$. The closer a possible world is to $\|\mathbf{K}\|$, the more plausible it is.

Definition 3.2.65 *Let* χ *be a subset of* $\mathcal{M}_{\mathcal{L}}$*. A system of spheres, or spheres' system, centred on* χ *is a collection* $\mathbb{S}$ *of subsets of* $\mathcal{M}_{\mathcal{L}}$*, i.e.,* $\mathbb{S} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$*, that satisfies the following conditions:*

($\mathbb{S}$1) $\mathbb{S}$ *is totally ordered with respect to set inclusion; that is, if* $U, V \in \mathbb{S}$*, then* $U \subseteq V$ *or* $V \subseteq U$*.*

($\mathbb{S}$2) $\chi \in \mathbb{S}$*, and if* $U \in \mathbb{S}$*, then* $\chi \subseteq U$ *(*χ *is the* $\subseteq$*-minimum of* $\mathbb{S}$*).*

($\mathbb{S}$3) $\mathcal{M}_{\mathcal{L}} \in \mathbb{S}$ *(*$\mathcal{M}_{\mathcal{L}}$ *is the largest element of* $\mathbb{S}$*).*

($\mathbb{S}$4) *For every* $\alpha \in \mathcal{L}$*, if there is any element in* $\mathbb{S}$ *intersecting* $\|\alpha\|$ *then there is also a smallest element in* $\mathbb{S}$ *intersecting* $\|\alpha\|$*.*

The elements of $\mathbb{S}$ *are called spheres. For any consistent sentence* $\alpha \in \mathcal{L}$*, the smallest sphere in* $\mathbb{S}$ *intersecting* $\|\alpha\|$ *is denoted by* $\mathbb{S}_\alpha$*.*

($\mathbb{S}$1) states that spheres are totally ordered by set inclusion, *i.e.* that they are concentric. ($\mathbb{S}$2) states that χ is the minimal sphere. In this thesis whenever we apply this definition we will assume that $\chi = \|\mathbf{K}\|$, for some belief set $\mathbf{K}$. ($\mathbb{S}$3) says that the set of all possible worlds, $\mathcal{M}_{\mathcal{L}}$, is the maximal sphere. ($\mathbb{S}$4), also known as the *limit assumption* states that if a sentence α is consistent,[27] then there is a smallest sphere that intersects $\|\alpha\|$.

In Figure 3.5 we present a graphical representation of a system of spheres centred on $\|\mathbf{K}\|$, as well as the sets $\|\alpha\|$ and $\mathbb{S}_\alpha$, for some α that is neither a tautology nor a contradiction.

[27] If α is inconsistent then, by Definition 3.2.53, $\|\alpha\| = \varnothing$.

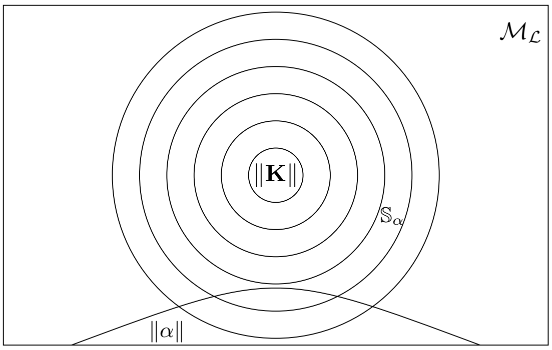

Figure 3.5: Schematic representation of a system of spheres centred on $\|\mathbf{K}\|$ displaying the sets $\|\alpha\|$ and $\mathbb{S}_\alpha$, for some α that is neither a tautology nor a contradiction.

Definition 3.2.66 *[Gro88] Let $M \subseteq \mathcal{M}_\mathcal{L}$. A propositional selection function f for M is sphere-based if and only there is a system of spheres $\mathbb{S}$ centred on M such that for all α:*
If $\|\alpha\| \neq \varnothing$, then $f(\|\alpha\|) = \mathbb{S}_\alpha \cap \|\alpha\|$.

Intuitively given a system of spheres $\mathbb{S}$ centred on M, a sphere-based propositional selection function f for M selects those α-worlds that are closer to M.

Definition 3.2.67 *[Gro88, Han99b] Let $\mathbf{K}$ be a belief set and $\div$ a possible worlds-based contraction on $\mathbf{K}$. Then $\div$ is an operator of sphere-based contraction if and only if it is based on a sphere-based propositional selection function.*

Example 3.2.68 *Consider a language $\mathcal{L}$ that is built from the finite set of propositional symbols $\{p,q\}$ and the boolean connectives $\neg, \wedge, \vee, \rightarrow$ and $\leftrightarrow$. Let $\mathbf{K} = Cn(\neg p \wedge \neg q)$. In this case it holds that $\mathcal{M}_\mathcal{L} = \mathcal{L} \perp\!\!\!\perp = \{Cn(\{p,q\}), Cn(\{\neg p,q\}), Cn(\{p,\neg q\}), Cn(\{\neg p,\neg q\})\}$. Hence: $\|\mathbf{K}\| = \{Cn(\{\neg p,\neg q\})\}$, $\|p\| = \{Cn(\{p,q\}), Cn(\{p,\neg q\})\}$, $\|q\| = \{Cn(\{p,q\}), Cn(\{\neg p,q\})\}$ and $\|p \vee q\| = \{Cn(\{p,q\}), Cn(\{p,\neg q\}), Cn(\{\neg p,q\})\}$. If f is a propositional selection function for $\|\mathbf{K}\|$, then $f(\|p\|)$ must be either: $\{Cn(\{p,q\})\}$, $\{Cn(\{p,\neg q\})\}$ or $\{Cn(\{p,q\}), Cn(\{p,\neg q\})\}$. $f(\|q\|)$ must be either: $\{Cn(\{p,q\})\}$, $\{Cn(\{\neg p,q\})\}$ or $\{Cn(\{p,q\}), Cn(\{\neg p,q\})\}$. Suppose now that f is a sphere-based propositional selection function for $\|\mathbf{K}\|$ (determined by a system of spheres $\mathbb{S}$), such that $f(\|p\|) = f(\|q\|) = \{Cn(\{p,q\})\}$. Note that, since $\|p \vee q\| = \|p\| \cup \|q\|$, $\mathbb{S}_{p\vee q}$ is the innermost one of the spheres $\mathbb{S}_p$ and $\mathbb{S}_q$, i.e. $\mathbb{S}_{p\vee q} = \mathbb{S}_p \cap \mathbb{S}_q$. Therefore $Cn(\{p,q\}) \in \mathbb{S}_{p\vee q}$ but $Cn(\{p,\neg q\}) \notin \mathbb{S}_{p\vee q}$ and $Cn(\{\neg p,q\}) \notin \mathbb{S}_{p\vee q}$. Thus $f(\|p \vee q\|) = \{Cn(\{p,q\})\}$. Let $\div$ be the sphere-based contraction operator based on f. Then, $\mathbf{K} \div \neg p = \mathbf{K} \div \neg q = \mathbf{K} \div (\neg p \wedge \neg q) = Th(\{Cn(\{\neg p,\neg q\})\} \cup \{Cn(\{p,q\})\}) = Cn(p \leftrightarrow q)$.*

The following observation asserts that the classes of transitively relational partial meet contractions and of sphere-based contractions coincide.

Observation 3.2.69 *[Gro88] Let $\mathbf{K}$ be a belief set and $\div$ be an operation on $\mathbf{K}$. Then the following statements are equivalent:*

(a) $\div$ *is a transitively relational partial meet contraction.*

(b) $\div$ *is a sphere-based contraction.*

From the previous observation we can conclude that an operation $\div$ on $\mathbf{K}$ is a sphere-based contraction if and only if it satisfies both the basic and supplementary AGM-postulates for contraction. It also asserts that the outcome of a transitively relational partial meet contraction of a belief set $\mathbf{K}$ by a sentence α consists of the theory determined by the set of possible worlds that results from the union of the most plausible $\neg\alpha$-worlds (with respect to some appropriate ordering) with the $\mathbf{K}$-worlds, i.e, $\mathbf{K} \div \alpha$ consists of the intersection of all the worlds included in the shaded regions of Figure 3.6.

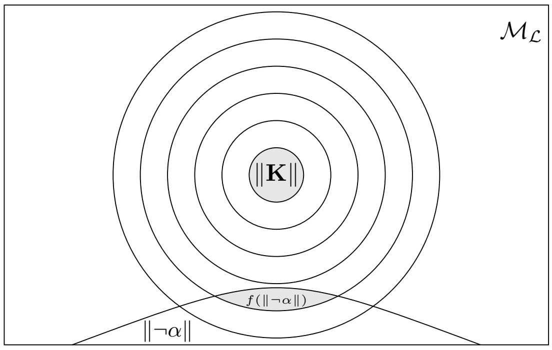

Figure 3.6: Transitively relational partial meet contraction. The outcome of $\mathbf{K} \div \alpha$ is the intersection of all the worlds contained in the shaded region.

The outcome of a transitively relational partial meet revision of a belief set $\mathbf{K}$ by a sentence α should consist of the theory determined by the set of the most plausible α-worlds. This is stated in the following observation and illustrated in Figure 3.7.

Observation 3.2.70 *[Gro88] Let* $\mathbf{K}$ *be a belief set. An operation* $\star$ *on* $\mathbf{K}$ *is a transitively relational partial meet revision if and only if there is a sphere-based proportional selection function* f *for* $\|\mathbf{K}\|$ *such that for all sentences* α*:*

$$\mathbf{K} \star \alpha = Th(f(\|\alpha\|)).$$

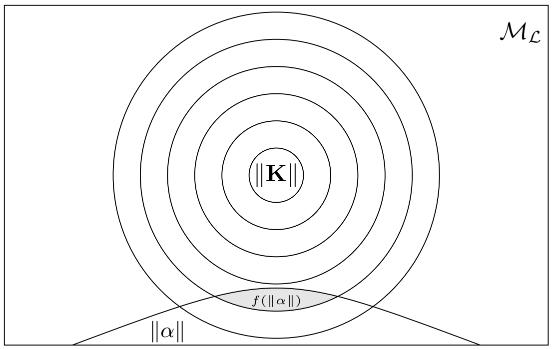

Figure 3.7: Transitively relational partial meet revision. The outcome of $\mathbf{K}\star\alpha$ is the intersection of all the worlds contained in the shaded region.

3.3 Maps between different belief set contraction functions

We finish this chapter by presenting in Figures 3.8 and 3.9 diagrams that summarize the logical relationships between operations of contraction on belief sets analysed in the previous sections. We note that the class of contraction functions represented in Figure 3.8 are contained in the class of contraction functions represented in Figure 3.9.

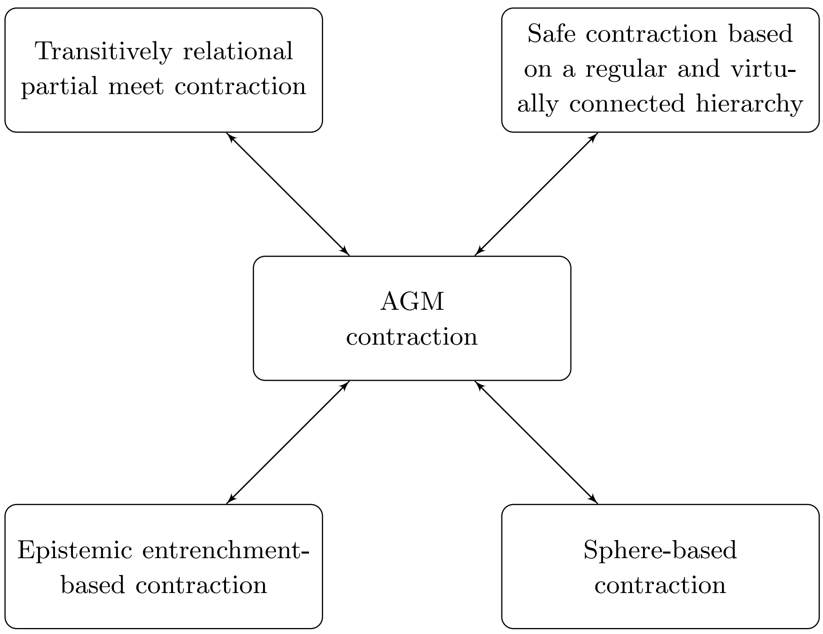

Figure 3.8: Equivalences between different operations of contraction on belief sets.

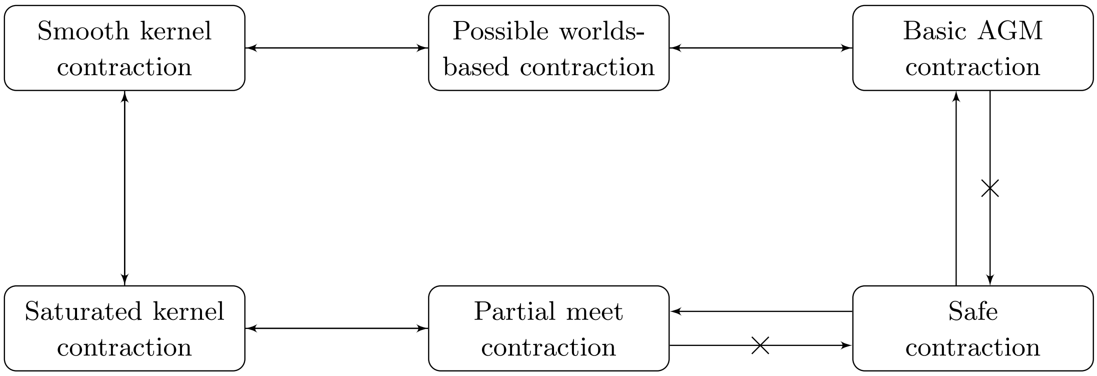

Figure 3.9: Logical relationships between different operations of contraction on belief sets.

3.4 Summary

In this chapter we presented postulates that characterize the AGM operations of belief change: expansions, contractions and revisions. We saw that postulates $(+1)$ to $(+6)$ uniquely characterize $\mathbf{K}+\alpha$ in terms of $\mathbf{K}$ and α ($\mathbf{K}+\alpha = Cn(\mathbf{K}\cup\{\alpha\})$). This does not happen regarding the set of postulates $(\div 1)$ to $(\div 8)$ for contractions nor for the set of postulates $(\star 1)$ to $(\star 8)$ for revisions. Afterwards we presented the Harper and the Levi identities that allow us to define contractions in terms of revisions and vice-versa. We saw that if an operator $\div$ satisfies the contraction postulates $(\div 2)$ to $(\div 4)$ and $(\div 6)$, then the operator $\star$ defined from $\div$ by means of the Levi identity is a basic AGM revision operator. And if $\div$ is an AGM contraction, then the operator $\star$ defined from $\div$ by means of the Levi identity is an AGM revision operator. Conversely, if $\star$ satisfies the revision postulates $(\star 1)$, $(\star 2)$, $(\star 4)$, $(\star 5)$ and $(\star 6)$, then the operator $\div$ defined from $\star$ by means of the Harper identity satisfies the basic AGM contraction postulates. Furthermore, if $\star$ is an AGM revision operator, then $\div$ is an AGM contraction operator. Afterwards we presented some explicit definitions of contraction and revision functions as well as the axiomatic characterization for each of them. Finally we presented the logical relationships between the classes of contraction functions mentioned along this chapter. In particular we saw that the (five) classes of contractions functions formed by transitively relational partial meet contractions, by safe contractions based on a regular and virtually connected hierarchy, by sphere-based contractions, by epistemic entrenchment-based contractions and by AGM contractions, coincide. We also saw that the classes of contraction functions formed by possible worlds-based contractions, by partial meet contractions and by basic AGM contractions coincide, and that every safe contraction function is a partial meet contraction function, but the converse does not hold.

Chapter 4

Some Extensions and Refinements of the AGM Framework

"The obstacles of your past can become the gateways that lead to new beginnings."
Ralph H. Blum

Although the AGM model has quickly acquired the status of standard model of theory change, several researchers (for an overview see [FH11]) have pointed out its inadequateness in several contexts and proposed several extensions and generalizations to that framework. In this chapter we present some of the criticisms made to the AGM model and some of the proposals developed to deal with its problems.

4.1 Contraction without recovery

One of the AGM postulates for contraction is *recovery* (or ($\div$5)). According to Alchourrón, Gärdenfors and Makinson *recovery* characterizes the notion of "minimal change" in the contraction process [Gär88, Mak85, Mak97a]. It is based in the intuition that "it is reasonable that we get all of the beliefs (...) back again after first contracting and then expanding with respect to the same belief" [Gär82]. This is one of the most controversial contraction postulates, since there are examples of contractions where *recovery* seems implausible:

Example 4.1.1 *[Han91b, Mak97a, Han99b] I believe that 'Cleopatra had a son' (p) and that 'Cleopatra had a daughter' (q), and thus also that 'Cleopatra had a child' ($p \vee q$, briefly r). Then I receive information that makes me give up my beliefs in r, and contract my belief set accordingly, forming $\mathbf{K} \div r$. Soon afterwards I learn from a reliable source that Cleopatra had a child. It seems reasonable to add r to my set of beliefs without also reintroducing either p or q.*

Example 4.1.2 *[Han96a, Han99b] I previously entertained the two beliefs 'George is a criminal' (p) and 'George is a mass murderer' (q). When I received information that induced me to give up the first of these beliefs (p), the second (q) had to go as well (since p would otherwise follow from q). Then I received new information that* *made me accept the belief 'George is a shoplifter' (r). The new resulting belief set is the expansion of* $\mathbf{K} \div p$ *by* r*. Since* p *follows from* r*,* $(\mathbf{K} \div p) + p$ *is a subset of* $(\mathbf{K} \div p) + r$*. By recovery* $q \in (\mathbf{K} \div p) + p$*, from which follows that* $q \in (\mathbf{K} \div p) + r$*. Thus, since I previously believed George to be a mass murderer, I cannot any longer believe him to be a shoplifter without believing him to be a mass murderer.*

A possible way to overcome the problematic nature of *recovery* is to try to replace it by other postulates capable of capturing the principle of "minimal change". Some postulates like *relevance* [Han89, Han92a], *core-retainment* [Han91b] and *disjunctive elimination* [FKR08] were proposed.[1] However, in the context of contractions of belief sets they are equivalent to *recovery* in the presence of the other basic AGM postulates for contraction [Han91b, FH94, FKR08]. Another postulate that captures the principle of "minimal change" is *failure*.[2] *Failure* states that contracting by a tautology leaves the set to be contracted unchanged. However, an operator $\div$ defined as follows:[3]

$$\mathbf{K} \div \alpha = \begin{cases} Cn(\varnothing) & \text{if } \alpha \in \mathbf{K} \setminus Cn(\varnothing) \\ \mathbf{K} & \text{otherwise} \end{cases}$$

satisfies *failure* and also the postulates $(\div 1)$ to $(\div 6)$ with the exception of $(\div 5)$.

According to this operation, whenever a belief is removed then all non-tautological beliefs are also discarded. This is far from being a desirable contraction.

Contraction operations that satisfy the basic AGM postulates for contraction with the exception of *recovery* are as known as *withdrawals* [Mak87]. Some alternative classes of contractions on belief sets which do not satisfy the *recovery* postulate are Levi contractions [Lev91], severe withdrawals (or mild contractions or Rott contractions) [Rot91, RP99], semi-contractions [Fer98, FR98b] and systematic withdrawal [MHLL02].

4.2 Non-prioritized belief change

Other ones of the AGM postulates that were criticized were the *success* postulates (both for revision and contraction). In this section we present some of the models that were proposed in the belief change literature in order to address this criticism.

4.2.1 Non-prioritized revision

The *success postulate* for revision (or $(\star 2)$) states that a belief is always incorporated when revising a belief set by it. This postulate characterizes the *principle of primacy of the new information.* Several authors have found this to be an implausible feature

[1] **Disjunctive Elimination:** If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K} \div \alpha$ then $\mathbf{K} \div \alpha \nvdash \alpha \vee \beta$.
Relevance: If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K} \div \alpha$, then there is a set H such that $\mathbf{K} \div \alpha \subseteq H \subseteq \mathbf{K}$ and $H \nvdash \alpha$ but $H \cup \{\beta\} \vdash \alpha$.
Core-retainment: If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K} \div \alpha$ then there is some set H such that $H \subseteq \mathbf{K}$ and $H \nvdash \alpha$ but $H \cup \{\beta\} \vdash \alpha$.
[2] **Failure:** If $\vdash \alpha$, then $\mathbf{K} \div \alpha = \mathbf{K}$.
[3] This operation was presented in [Han99b].

of belief revision.
"The AGM model always accepts the new information. This feature appears, in general, to be unrealistic, since rational agents, when confronted with information that contradicts previous beliefs, often reject it altogether or accept only parts of it" [FMT03].
This may happen for various reasons. For example, the new information may lack on credibility or it may contradict previous highly entrenched beliefs. Belief revision operators that do not satisfy the *success postulate* are called *non-prioritized belief revisions* [Han99a].

Screened revision

Makinson proposed in [Mak97b] an operator of non-prioritized belief revision designated by *screened revision*. Makinson defined a set A of sentences that are immune to revision. The belief set $\mathbf{K}$ should be revised by the input sentence if that sentence is consistent with $A \cap \mathbf{K}$, otherwise it should be left unchanged. Formally:

$$\mathbf{K}\#_A\alpha = \begin{cases} \mathbf{K}\star\alpha & \text{if } \alpha \text{ is consistent with } A \cap \mathbf{K} \\ \mathbf{K} & \text{otherwise} \end{cases}$$

where $\star$ is a basic AGM revision function and for all α, $A \cap \mathbf{K} \subseteq \mathbf{K}\star\alpha$.

A more general approach was proposed by Hansson in [Han97], by replacing A by a function f that is applied to the sentence to be revised by. This proposal was called *generalized screened revision*.

$$\mathbf{K}\#_{f(\alpha)}\alpha = \begin{cases} \mathbf{K}\star\alpha & \text{if } \alpha \text{ is consistent with } f(\alpha) \cap \mathbf{K} \\ \mathbf{K} & \text{otherwise} \end{cases}$$

where f is a function such for each sentence α, $f(\alpha)$ is a set of sentences and $\star$ is a (modified) basic AGM revision function such that for all α, $f(\alpha) \cap \mathbf{K} \subseteq \mathbf{K}\star\alpha$. Several properties can be added to f. For example, the following property was proposed: $f(\alpha) = \{\beta : \alpha < \beta\}$, where $<$ is a binary relation on $\mathcal{L}$.

Credibility-limited revision

Credibility-limited revision [HFCF01] is another operator of non-prioritized revision. When revising a belief set by a sentence, we need to analyse if this sentence is credible or not. When revising by a credible sentence, the operator works as a basic AGM revision operator, otherwise it leaves the original belief set unchanged. Formally:

Definition 4.2.1 *[HFCF01] Let* $\mathbf{K}$ *be a belief set,* $\star$ *a basic AGM revision operator on* $\mathbf{K}$ *and* C *a subset of* $\mathcal{L}$ *(the set of credible sentences). Then* $\odot$ *is a credibility-limited revision operator induced by* $\star$ *and* C *if and only if:*

$$\mathbf{K} \odot \alpha = \begin{cases} \mathbf{K}\star\alpha & \textit{if } \alpha \in C \\ \mathbf{K} & \textit{otherwise} \end{cases}$$

This construction can be further specified by adding constrains to the structure of C (the set of credible sentences). In [HFCF01], the following properties for C were proposed:

Closure Under Logical Equivalence: If $\vdash \alpha \leftrightarrow \beta$, and $\alpha \in C$, then $\beta \in C$.
Single Sentence Closure: If $\alpha \in C$, then $Cn(\alpha) \subseteq C$.
Disjunctive Completeness: If $\alpha \vee \beta \in C$, then either $\alpha \in C$ or $\beta \in C$.
Negation Completeness: $\alpha \in C$ or $\neg\alpha \in C$.
Element Consistency: If $\alpha \in C$, then $\alpha \nvdash \bot$.
Expansive Credibility: If $\mathbf{K} \nvdash \alpha$, then $\neg\alpha \in C$.
Revision Credibility: If $\alpha \in C$, then $\mathbf{K} \odot \alpha \subseteq C$.

Latter, in [FMT03], the following property was presented:

Strong Revision Credibility: If $\alpha \notin C$, then $\mathbf{K} \odot \beta \vdash \neg\alpha$.

Closure under logical equivalence states that logically equivalent sentences should be both elements of C or of $\mathcal{L} \setminus C$. *Single sentence closure* says that if a sentence is credible then all its logical consequences are also credible. *Single sentence closure* implies *closure under logical equivalence. Disjunctive completeness* states that if two sentences are not credible, then their disjunction is not credible. *Negation completeness* states that for any sentence it holds that either it is credible or its negation is credible. *Element consistency* states that contradictions are not credible. *Expansive credibility* informally states that sentences that are consistent with $\mathbf{K}$ are credible. *Revision credibility* states that sentences in the outcome of a revision by a credible sentence are credible. *Strong revision credibility* says that if a sentence is not credible, then its negation is implied by any revision outcome.

The following results highlight some interrelations among the above introduced properties of the set C.

Observation 4.2.2 *Let $C \subseteq \mathcal{L}$.*

(a) If C satisfies single sentence closure, then C also satisfies closure under logical equivalence.

(b) If C satisfies negation completeness and element consistency, then $Cn(\varnothing) \subseteq C$.

Proof: A proof for this observation can be found on page 175.

Observation 4.2.3 *Let C be a subset of $\mathcal{L}$ that satisfies disjunctive completeness and element consistency. Then C satisfies negation completeness if and only if $Cn(\varnothing) \subseteq C$.*

Proof: A proof for this observation can be found on page 175.

Observation 4.2.4 *Let* $\mathbf{K}$ *be a consistent belief set and* $C \subseteq \mathcal{L}$. *If* C *satisfies closure under logical equivalence and expansive credibility, then* $\mathbf{K} \subseteq C$.

Proof: A proof for this observation can be found on page 175.

Hansson *et al.* proposed also a construction of a credibility-limited revision operator adapting the construction suggested by Gärdenfors and Makinsson for an operator of epistemic-entrenchment revision. To do so, Hansson *et al.* considered a binary relation that satisfies (EE1) to (EE4) (but not necessarily (EE5)). Furthermore they use a variant of $(\mathbf{C}_{\star_{\leq}})$.

Definition 4.2.5 *[HFCF01] Let* $\mathbf{K}$ *be a belief set and* $\leq_{\mathbf{K}}$ *a relation satisfying (EE1) to (EE4) with respect to* $\mathbf{K}$. *Then* $\odot_{\leq_{\mathbf{K}}}$ *is an entrenchment-based credibility-limited revision operator based on* $\leq_{\mathbf{K}}$ *if and only if:*

$(\mathbf{C}_{\odot_{\leq}})$ $\beta \in \mathbf{K} \odot_{\leq_{\mathbf{K}}} \alpha$ *if and only if either* $(\alpha \rightarrow \neg\beta) <_{\mathbf{K}} (\alpha \rightarrow \beta)$ *or* $\beta \in \mathbf{K}$ *and* $\neg\alpha$ *is maximally entrenched.*

An operator $\odot$ *on* $\mathbf{K}$ *is an entrenchment-based credibility-limited revision if and only if there is a relation* $\leq_{\mathbf{K}}$ *that satisfies (EE1) to (EE4) with respect to* $\mathbf{K}$ *such that for all sentences* α:

$$\mathbf{K} \odot \alpha = \mathbf{K} \odot_{\leq_{\mathbf{K}}} \alpha.$$

Hansson *et al.* in [HFCF01] presented also a construction of a credibility-limited revision operator that consists on an adaptation of Grove's system of spheres [Gro88]. This adaptation consists on relaxing the standard requirements of the system of spheres, namely ($\mathbb{S}3$), allowing the existence of (non-credible) worlds outside the sphere system.

Definition 4.2.6 *[HFCF01] Let* χ *be a subset of* $\mathcal{M}_{\mathcal{L}}$. *An incomplete system of spheres, or incomplete spheres' system, centred on* χ *is a collection* $\mathcal{S}$ *of subsets of* $\mathcal{M}_{\mathcal{L}}$, *i.e.,* $\mathcal{S} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$, *that satisfies the following conditions:*

($\mathcal{S}1$) $\mathcal{S}$ *is totally ordered with respect to set inclusion; that is, if* $U, V \in \mathcal{S}$, *then* $U \subseteq V$ *or* $V \subseteq U$.

($\mathcal{S}2$) $\chi \in \mathcal{S}$, *and if* $U \in \mathcal{S}$, *then* $\chi \subseteq U$ *(*χ *is the* $\subseteq$*-minimum of* $\mathcal{S}$*).*

($\mathcal{S}3$) $\bigcup \mathcal{S} \in \mathcal{S}$.

($\mathcal{S}4$) *For every* $\alpha \in \mathcal{L}$, *if there is any element in* $\mathcal{S}$ *intersecting* $\|\alpha\|$, *then there is also a smallest element in* $\mathcal{S}$ *intersecting* $\|\alpha\|$.

The elements of $\mathcal{S}$ *are called spheres. For any sentence* $\alpha \in \mathcal{L}$ *such that* $\|\alpha\| \cap (\bigcup \mathcal{S}) \neq \emptyset$, *the smallest sphere in* $\mathcal{S}$ *intersecting* $\|\alpha\|$ *is denoted by* $\mathcal{S}_\alpha$. *If* $\|\alpha\| \cap (\bigcup \mathcal{S}) = \emptyset$, *then* $\mathcal{S}_\alpha$ *denotes the empty set.*

Definition 4.2.7 *[HFCF01] An operator* $\odot$ *on* $\mathbf{K}$ *is a sphere-based credibility-limited revision if and only if there is an incomplete system of spheres* $\mathcal{S}$ *centred on* $\|\mathbf{K}\|$ *such that for all sentences* α*:*

$$\mathbf{K} \odot \alpha = \begin{cases} Th(\|\alpha\| \cap \mathcal{S}_\alpha) & \text{if } \|\alpha\| \cap (\bigcup \mathcal{S}) \neq \varnothing \\ \mathbf{K} & \text{otherwise} \end{cases}$$

If α is credible, then the outcome of revising $\mathbf{K}$ by α consists of the theory determined by the intersection of $\|\alpha\|$ with the narrowest sphere around $\|\mathbf{K}\|$ (Figure 4.1). If α is not credible, then the outcome of revising $\mathbf{K}$ by α is $\mathbf{K}$ (Figure 4.2).

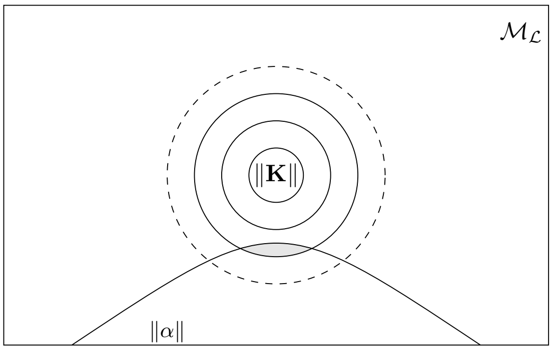

Figure 4.1: The outcome of the revision of $\mathbf{K}$ by a credible sentence α is the intersection of all worlds contained in the shaded region.

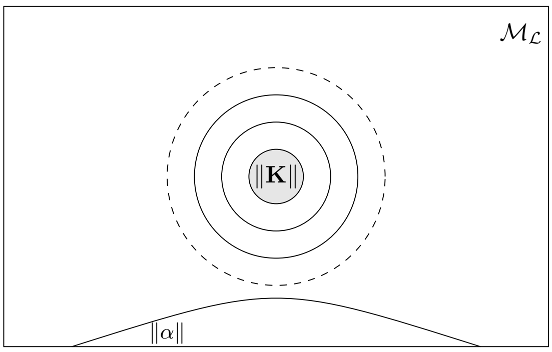

Figure 4.2: A belief set $\mathbf{K}$ is left unchanged when revising it by a non-credible sentence.

Representation theorems

In [HFCF01] the credibility-limited revision operators presented above were axiomatically characterized. Before presenting those axiomatic characterizations, we recall some postulates needed for that purpose.

When considering a credibility-limited revision the *success* postulate must be discarded. It must be replaced by weaker properties. The following postulates were formulated in [HFCF01] and in [FH01]:

(Relative Success) $\alpha \in \mathbf{K} \odot \alpha$ or $\mathbf{K} \odot \alpha = \mathbf{K}$.
(Disjunctive Success) Either $\alpha \in \mathbf{K} \odot \alpha$ or $\neg\alpha \in \mathbf{K} \odot \neg\alpha$.
(Strict Improvement) If $\alpha \in \mathbf{K} \odot \alpha$ and $\vdash \alpha \to \beta$, then $\beta \in \mathbf{K} \odot \beta$.
(Regularity) If $\beta \in \mathbf{K} \odot \alpha$, then $\beta \in \mathbf{K} \odot \beta$.
(Strong Regularity) If $\neg\beta \notin \mathbf{K} \odot \alpha$, then $\beta \in \mathbf{K} \odot \beta$.
(Disjunctive Distribution) If $\alpha \vee \beta \in \mathbf{K} \odot (\alpha \vee \beta)$, then $\alpha \in \mathbf{K} \odot \alpha$ or $\beta \in \mathbf{K} \odot \beta$.
(Disjunctive Constancy) If $\mathbf{K} \odot \alpha = \mathbf{K} \odot \beta = \mathbf{K}$, then $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K}$.

Relative success states that either a sentence is incorporated in the revision of a belief set by it, or the original belief set is left unchanged. *Disjunctive success* states that either a sentence belong to the revision of a belief set by it or the negation of that sentence belongs to the revision of that belief set by it. *Strict improvement* states that if a certain sentence is incorporated when revising a belief set by it, then the same thing happens regarding every logical consequence of that sentence. *Regularity* says that if a sentence does not belong to the revision of a belief set by it, then that sentence does not belong to the revision of that belief set by any other sentence. *Strong regularity* states that if a sentence does not belong to the revision of a belief set by it, then its negation belongs to the revision of that belief set by any given sentence. *Disjunctive distribution* states that if a disjunction belongs to the revision of a belief set by it, then the same thing happens regarding at least one of its disjuncts. *Disjunctive constancy* is, as *disjunctive distribution*, a postulate concerning revision by disjunctions. It states that a belief set is left unchanged when revising it by a disjunction, whenever the same thing occurs when revising that belief set by either one of the two disjuncts.

The following observation illustrates that in the presence of *relative success, vacuity* and *inclusion* the postulates *disjunctive distribution* and *disjunctive constancy* are equivalents.

Observation 4.2.8 *Let* $\mathbf{K}$ *be a consistent belief set and* $\odot$ *be an operator on* $\mathbf{K}$ *that satisfies relative success, vacuity and inclusion, then* $\odot$ *satisfies disjunctive distribution if and only if* $\odot$ *satisfies* disjunctive constancy*.*

Proof: A proof for this observation can be found on page 175.

The following postulates are related to consistency. They will also be used in the axiomatic characterizations that we will present.

(Weak Consistency Preservation) [KM92] If both $\mathbf{K}$ and α are consistent, then so is $\mathbf{K} \odot \alpha$.
(Strong Consistency) [Han96b] $\mathbf{K} \odot \alpha$ is consistent.
(Consistency Preservation) [Mak97b] If $\mathbf{K}$ is consistent, then $\mathbf{K} \odot \alpha$ is consistent.
(Consistent Expansion) [FH99] If $\mathbf{K} \nsubseteq \mathbf{K} \odot \alpha$, then $\mathbf{K} \cup (\mathbf{K} \odot \alpha) \vdash \bot$.

Additionally, in [HFCF01], the following postulate, that consists of an adaptation of *subexpansion* to the context of non-prioritized revision, was proposed:

(Guarded Subexpansion) If $\alpha \in \mathbf{K} \odot \alpha$ and $\mathbf{K} \odot \alpha \nvdash \neg\beta$, then $(\mathbf{K} \odot \alpha) + \beta \subseteq \mathbf{K} \odot (\alpha \wedge \beta)$.

Guarded subexpansion and *subexpansion* are equivalent in the presence of *success* and *closure*.

Now we present some relations between the postulates presented above.

Observation 4.2.9 *[HFCF01] Let* $\mathbf{K}$ *be a belief set and* $\odot$ *an operator on* $\mathbf{K}$*. If* $\odot$ *satisfies vacuity and relative success, then* $\odot$ *satisfies consistent expansion.*

Observation 4.2.10 *[Fal99] Let* $\mathbf{K}$ *a belief set and* $\odot$ *an operator on* $\mathbf{K}$*. Then,*

(a) If $\odot$ *satisfies strong consistency, then* $\odot$ *satisfies consistency.*

(b) If $\odot$ *satisfies consistency preservation, then* $\odot$ *satisfies weak consistency preservation.*

(c) If $\mathbf{K}$ *is consistent, then* $\odot$ *satisfies consistency preservation if and only if* $\odot$ *satisfies strong consistency.*

Observation 4.2.11 *[Fer99] Let* $\mathbf{K}$ *be a belief set and* $\odot$ *an operator on* $\mathbf{K}$ *that satisfies closure, vacuity, consistency, extensionality, strict improvement and relative success. Then* $\odot$ *satisfies disjunctive factoring if and only if* $\odot$ *satisfies both superexpansion and guarded subexpansion.*

Now we are in conditions to present the representations theorems for the credibility-limited revision operators mentioned above. We will start by presenting a minimal representation theorem. Then, we will add conditions on C, the set of credible sentences, obtaining more specific representation theorems.

Observation 4.2.12 *[HFCF01] Let* $\mathbf{K}$ *be a consistent belief set and* $\odot$ *an operator on* $\mathbf{K}$*. Then the following three conditions are equivalent:*

1. $\odot$ *satisfies closure, relative success, inclusion, weak consistency preservation, consistent expansion and extensionality.*

2. $\odot$ *is an operator of credibility-limited revision induced by a basic AGM revision operator for* $\mathbf{K}$ *and a set* $C \subseteq \mathcal{L}$ *that is closed under logical equivalence.*

3. $\odot$ *is an operator of credibility-limited revision induced by a basic AGM revision operator for* $\mathbf{K}$ *and a set* $C \subseteq \mathcal{L}$ *that satisfies* $\mathbf{K} \subseteq C$ *and is closed under logical equivalence.*

Observation 4.2.13 *[HFCF01] Let* $\mathbf{K}$ *be a consistent belief set and* $\odot$ *an operator on* $\mathbf{K}$*. Then:*[4]

$\odot$ is an operator of credibility-limited revision induced by a basic AGM operator on $\mathbf{K}$ and a set $C \subseteq \mathcal{L}$ that is closed under logical equivalence and satisfies	***if and only if $\odot$ satisfies closure, relative success, inclusion, weak consistency preservation, consistent expansion, extensionality and***
single sentence closure	*strict improvement*
disjunctive completeness	*disjunctive distribution*
negation completeness	*disjunctive success*
element consistency	*strong consistency*
expansive credibility	*vacuity*

By combining the results presented in the above observation (and in previous ones) we obtain the following representation theorem.

Theorem 4.2.14 *Let* $\mathbf{K}$ *be a consistent belief set and* $\odot$ *an operator on* $\mathbf{K}$*. Then the following three conditions are equivalent:*

1. $\odot$ *satisfies closure, relative success, inclusion, consistency preservation, extensionality, vacuity, strict improvement and disjunctive distribution.*

2. $\odot$ *is an operator of credibility-limited revision induced by a basic AGM revision operator for* $\mathbf{K}$ *and a set* $C \subseteq \mathcal{L}$ *that satisfies single sentence closure, disjunctive completeness, element consistency and expansive credibility.*

3. $\odot$ *is an operator of credibility-limited revision induced by a basic AGM revision operator for* $\mathbf{K}$ *and a set* $C \subseteq \mathcal{L}$ *that satisfies* $\mathbf{K} \subseteq C$*, single sentence closure, disjunctive completeness, element consistency and expansive credibility.*

Proof: A proof for this theorem can be found on page 176.

The above theorem presents three alternative ways of characterizing the same class of operators. This class is formally introduced in the following definition.

[4]The schema presented in this observation (and whenever a similar schema is used) should be interpreted as follows:

1. $\odot$ is an operator of credibility-limited revision induced by a basic AGM operator on $\mathbf{K}$ and a set $C \subseteq \mathcal{L}$ that is closed under logical equivalence and satisfies single sentence closure iff $\odot$ satisfies closure, relative success, inclusion, weak consistency preservation, consistent expansion, extensionality and strict improvement;

2. $\odot$ is an operator of credibility-limited revision induced by a basic AGM operator on $\mathbf{K}$ and a set $C \subseteq \mathcal{L}$ that is closed under logical equivalence and satisfies disjunctive completeness iff $\odot$ satisfies closure, relative success, inclusion, weak consistency preservation, consistent expansion, extensionality and disjunctive distribution;

3. ...

Definition 4.2.15 *Let* **K** *be a consistent belief set. An operator* $\odot$ *on* **K** *is a basic credibility-limited revision operator if and only if* $\odot$ *satisfies closure, relative success, inclusion, consistency preservation, extensionality, vacuity, strict improvement and disjunctive distribution.*

The following theorem presents an axiomatic characterization for operators of credibility-limited revision induced by an AGM revision operator (instead of by a basic AGM revision as in Theorem 4.2.14) and a set C, of credible sentences, that satisfies certain properties.

Theorem 4.2.16 *Let* **K** *be a consistent belief set and* $\odot$ *an operator on* **K**. *Then the following three conditions are equivalent:*

1. $\odot$ *satisfies closure, relative success, inclusion, consistency preservation, extensionality, vacuity, strict improvement, disjunctive distribution, strong regularity and disjunctive factoring.*

2. $\odot$ *is an operator of credibility-limited revision induced by an AGM revision operator for* **K** *and a set* $C \subseteq \mathcal{L}$ *that satisfies single sentence closure, disjunctive completeness, element consistency, expansive credibility and strong revision credibility.*

3. $\odot$ *is an operator of credibility-limited revision induced by an AGM revision operator for* **K** *and a set* $C \subseteq \mathcal{L}$ *that satisfies* $\mathbf{K} \subseteq C$*, single sentence closure, disjunctive completeness, element consistency, expansive credibility and strong revision credibility.*

Proof: A proof for this theorem can be found on page 176.

This theorem presents three alternative ways of characterizing the same class of operators. This class is formally introduced in the following definition and is a subclass of the class of operators introduced in Definition 4.2.15.

Definition 4.2.17 *Let* **K** *be a consistent belief set. An operator* $\odot$ *on* **K** *is a non-basic credibility-limited revision operator if and only if* $\odot$ *satisfies closure, relative success, inclusion, consistency preservation, extensionality, vacuity, strict improvement, disjunctive distribution, strong regularity and disjunctive factoring.*

We note that, having in mind Observation 4.2.10 (c), in the axiomatic characterization presented in Theorems 4.2.14 and 4.2.16, *consistency preservation* can be replaced by *strong consistency*.

The following observation illustrates that the classes of entrenchment-based credibility-limited revision operators, of sphere-based credibility-limited revision operators and of non-basic credibility-limited revision operators coincide.

Observation 4.2.18 *[HFCF01] Let* **K** *be a consistent belief set and* $\odot$ *an operator on* **K**. *Then the following three conditions are equivalent:*

1. *$\odot$ is a non-basic credibility-limited revision operator.*

2. *$\odot$ is an entrenchment-based credibility-limited revision operator.*

3. *$\odot$ is a sphere-based credibility-limited revision operator.*

Selective revision

In a standard (basic) AGM revision the new information is always accepted. In credibility-limited revisions, the new information is either fully accepted or completely rejected. In [FH99] Fermé and Hansson proposed a new operator that allows the acceptance of only part of the new information and the rejection of the rest of it. They called this operator *selective revision*. The following example, motivates the proposal of this operator:

Example 4.2.19 *[FH99] Suppose one day when I got home my youngest daughter tells me that a dinosaur broke her grandmother's vase in the living room. I will probably accept part of the information, namely that the vase is broken and I will certainly reject the part concerning the dinosaur.*

An operator of selective revision is constructed from a basic AGM revision and a transformation function f from $\mathcal{L}$ to $\mathcal{L}$:

Definition 4.2.20 *[FH99] Let* $\mathbf{K}$ *be a belief set,* $\star$ *a basic AGM revision operator for* $\mathbf{K}$ *and* f *a function from* $\mathcal{L}$ *to* $\mathcal{L}$*. The selective revision* $\circledcirc$*, based on* $\star$ *and* f*, is the operation such that for all sentences* α*:*

$$\mathbf{K} \circledcirc \alpha = \mathbf{K} \star f(\alpha).$$

f *is the transformation function on which* $\circledcirc$ *is based.*

Intuitively, the transformation function f selects the credible part of every sentence. A natural restriction is that $f(\alpha)$ should not contain more information that the one that is contained in α (*i.e.*, $\vdash \alpha \rightarrow f(\alpha)$). Some plausible properties that a transformation function may be expected to satisfy are:[5]

Implication: $\vdash \alpha \rightarrow f(\alpha)$.
Idempotence: $\vdash f(f(\alpha)) \leftrightarrow f(\alpha)$.
Monotony: If $\vdash \alpha \rightarrow \beta$, then $\vdash f(\alpha) \rightarrow f(\beta)$.
Extensionality: If $\vdash \alpha \leftrightarrow \beta$, then $\vdash f(\alpha) \leftrightarrow f(\beta)$.
Consistency Preservation: If $\nvdash \neg\alpha$, then $\nvdash \neg f(\alpha)$.
Consistency: $\nvdash \neg f(\alpha)$.
Weak Maximality: If $\mathbf{K} \nvdash \neg\alpha$, then $\vdash f(\alpha) \leftrightarrow \alpha$.

When adding properties to the transformation function f, several additional properties are obtained for the selective revision operator on which f is based. In [FH99] several representation theorems for operators of selective revision were presented depending on the properties that the associated transformation function f satisfies.

[5] Several other properties were proposed in [FH99].

4.2.2 Non-prioritized contraction

The *success* postulate for contractions states that a non-tautological sentence is always removed when contracting by it (*i.e.* all non-tautological sentences are retractable). As pointed out by Rott in [Rot92b] this is not a fully realistic requirement. An agent can have several non-tautological beliefs that he/she is not willing, for various reasons, to give up. In [FH01], Fermé and Hansson proposed an operator $\ominus$ on a belief set $\mathbf{K}$, based on a basic AGM contraction operator $\div$ and a set R of retractable sentences. The operator $\ominus$ has the same behaviour of the basic AGM contraction for the sentences in R, but is such that it leaves the set to be contracted unchanged whenever the belief, that is to be contracted, is not included in R. Formally:

Definition 4.2.21 *[FH01] Let* $\mathbf{K}$ *be a belief set,* $\div$ *be a basic AGM contraction on* $\mathbf{K}$ *and* R *be a subset of* $\mathcal{L}$ *(the set of retractable sentences). Then* $\ominus$ *is the shielded contraction operator induced by* $\div$ *and* R *if and only if:*

$$\mathbf{K}\ominus\alpha = \begin{cases} \mathbf{K}\div\alpha & \textit{if } \alpha\in R \\ \mathbf{K} & \textit{otherwise} \end{cases}$$

This construction can be further specified by adding constrains to the structure of R (the set of retractable sentences). In [FH01], the following rationality criteria for R were proposed:

Conjunctive Completeness: If $\alpha\wedge\beta\in R$, then $\alpha\in R$ or $\beta\in R$.
Non-retractability Propagation: If $\alpha\notin R$, then $Cn(\alpha)\cap R=\varnothing$.
Non-retractability Preservation: $\mathcal{L}\setminus R\subseteq\mathbf{K}\ominus\alpha$.

Conjunctive completeness states that if α and β are irretractable, then $\alpha\wedge\beta$ is also irretractable.[6] In fact, in order to remove a conjunction we must remove at least one of its conjuncts. Therefore, a conjunction of two irretractable sentences must itself be irretractable. *Non-retractability propagation* says that if a sentence α is irretractable, then all its logical consequences are also irretractable. *Non-retractability preservation* states that irretractable sentences cannot be removed, independently of the (shielded) contraction performed, i.e. irrectractable sentences should be (kept) in the outcome of the (shielded) contraction by any sentence.

Fermé and Hansson proposed also a construction of a shielded contraction operator adapting the construction suggested by Gärdenfors and Makinsson for an operator of epistemic-entrenchment contraction. To do so they considered a binary relation that satisfies (EE1) to (EE4) (but not necessarily (EE5)). Condition (EE5) was discarded since in shielded contractions (some) non-tautological sentences may be also considered maximally entrenched. Furthermore they imposed a slight modification on condition ($\mathbf{C}_{-_{\leq}}$), by replacing $\vdash\alpha$ by $\alpha\nless_{\mathbf{K}}\top$ (precisely because in shielded contractions not only tautologies are maximally entrenched).

[6]The sentences included in R are called *retractable* and the remaining ones are designated by *irretractable*.

Definition 4.2.22 *[FH01] Let* $\mathbf{K}$ *be a belief set and* $\leq_{\mathbf{K}}$ *a relation satisfying (EE1) to (EE4) with respect to* $\mathbf{K}$*. Then* $\ominus_{\leq_{\mathbf{K}}}$ *is the entrenchment-based shielded contraction based on* $\leq_{\mathbf{K}}$ *if and only if:*

$$\mathbf{K}\ominus_{\leq_{\mathbf{K}}}\alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha <_{\mathbf{K}} \alpha \vee \beta\} & \text{if } \alpha <_{\mathbf{K}} \top \\ \mathbf{K} & \text{otherwise .} \end{cases}$$

An operator $\ominus$ *on* $\mathbf{K}$ *is an entrenchment-based shielded contraction if and only if there exists a relation* $\leq_{\mathbf{K}}$ *that satisfies (EE1) to (EE4) with respect to* $\mathbf{K}$*, such that for all sentences* α*:*

$$\mathbf{K}\ominus\alpha = \mathbf{K}\ominus_{\leq_{\mathbf{K}}}\alpha$$

Fermé and Hansson presented also another construction based on Grove's systems of spheres [Gro88].

Definition 4.2.23 *[FH01] An operator* $\ominus$ *on* $\mathbf{K}$ *is a sphere-based shielded contraction if and only if there is an incomplete system of spheres* $\mathcal{S}$ *centred on* $\|\mathbf{K}\|$ *such that for all sentences* α*:*

$$\mathbf{K}\ominus\alpha = Th(\|\mathbf{K}\| \cup (\mathcal{S}_{\neg\alpha} \cap \|\neg\alpha\|)).$$

Representation theorems

In [FH01] the shielded contraction operators that we presented so far were axiomatically characterized. Before we present these axiomatic characterizations we will recall some postulates needed in those axiomatizations.

When considering shielded contractions (as opposed to standard contractions) the *success* postulate is the one that has to be removed. It must be replaced by weaker properties, that are capable of capturing the intuitions underlying shielded contractions, as it is the case of the following postulates.

(Relative Success) [Rot92b] $\mathbf{K}\ominus\alpha = \mathbf{K}$ or $\alpha \notin \mathbf{K}\ominus\alpha$.
(Persistence) [FH01] If $\mathbf{K}\ominus\beta \vdash \beta$, then $\mathbf{K}\ominus\alpha \vdash \beta$.
(Conjunctive Constancy) [FH01] If $\mathbf{K}\ominus\alpha = \mathbf{K}\ominus\beta = \mathbf{K}$, then $\mathbf{K}\ominus(\alpha \wedge \beta) = \mathbf{K}$.
(Success Propagation) [FH01] If $\mathbf{K}\ominus\beta \vdash \beta$ and $\vdash \beta \rightarrow \alpha$, then $\mathbf{K}\ominus\alpha \vdash \alpha$.

Relative success states that when contracting by any given sentence either that sentence is effectively removed, or the belief set is left unchanged. *Persistence* intuitively states that if a belief is kept when trying to contract a belief set $\mathbf{K}$ by it, then it should also be kept when contracting $\mathbf{K}$ by any other belief. *Conjunctive constancy* states that if the contraction by a given conjunction causes a change, then the same thing happens when contracting by at least one of its conjuncts. *Success propagation* states that if a certain sentence is not removed when trying to contract a belief set by it, then the same thing happens regarding every logical consequence of that sentence.

The following observation illustrates some relations between (shielded) contraction postulates.

Observation 4.2.24 *[FH01] Let* $\mathbf{K}$ *be a belief set and* $\ominus$ *an operator on* $\mathbf{K}$.

(a) If $\ominus$ *satisfies persistence, then* $\ominus$ *satisfies success propagation.*

(b) If $\ominus$ *satisfies inclusion and conjunctive overlap, then* $\ominus$ *satisfies conjunctive constancy.*

We are now in conditions to present the axiomatic characterizations for the shielded contraction operators that we mentioned above.

Observation 4.2.25 *[FH01] Let* $\mathbf{K}$ *be a consistent belief set and* $\ominus$ *an operator on* $\mathbf{K}$*. Then the following conditions are equivalent:*

1. $\ominus$ *satisfies closure, inclusion, vacuity, extensionality, recovery, relative success, success propagation and conjunctive constancy.*
2. $\ominus$ *is an operator of shielded contraction induced by a basic AGM operator for* $\mathbf{K}$ *and a set* $R \subseteq \mathcal{L}$ *that satisfies non-retractability propagation and conjunctive completeness.*
3. $\ominus$ *is an operator of shielded contraction induced by a basic AGM contraction operator for* $\mathbf{K}$ *and a set* $R \subseteq \mathcal{L}$ *that satisfies* $\mathcal{L} \setminus \mathbf{K} \subseteq R$*, non-retractability propagation and conjunctive completeness.*

The shielded contraction operators characterized in the above observation will be called *basic shielded contractions*. We formalize this concept in the following definition.

Definition 4.2.26 *Let* $\mathbf{K}$ *be a consistent belief set. An operator* $\ominus$ *on* $\mathbf{K}$ *is a basic shielded contraction operator if and only if* $\ominus$ *satisfies closure, inclusion, vacuity, extensionality, recovery, relative success, success propagation and conjunctive constancy.*

In the following observation we present an axiomatic characterization for a subclass of the class of operators introduced in the previous definition. We highlight that in this observation $\ominus$ is induced by an AGM contraction (instead of by a basic AGM contraction as the shielded contraction operators characterized in Observation 4.2.25).

Observation 4.2.27 *[FH01] Let* $\mathbf{K}$ *be a consistent belief set and* $\ominus$ *an operator on* $\mathbf{K}$*. Then the following conditions are equivalent:*

1. $\ominus$ *satisfies closure, inclusion, vacuity, extensionality, recovery, relative success, persistence, conjunctive inclusion and conjunctive overlap.*
2. $\ominus$ *is an operator of shielded contraction induced by an AGM contraction (or equivalently by a transitively relational partial meet contraction) operator for* $\mathbf{K}$ *and a set* $R \subseteq \mathcal{L}$ *that satisfies non-retractability propagation, conjunctive completeness and non-retractability preservation.*

3. *$\ominus$ is an operator of shielded contraction induced by an AGM contraction operator for* $\mathbf{K}$ *and a set* $R \subseteq \mathcal{L}$ *that satisfies* $\mathcal{L} \setminus \mathbf{K} \subseteq R$*, non-retractability propagation, conjunctive completeness and non-retractability preservation.*

The above observation presents three alternative ways of characterizing a certain class of operators. An operator in this class will be designated by *non-basic shielded contraction operator.*

Definition 4.2.28 *Let* $\mathbf{K}$ *be a consistent belief set. An operator* $\ominus$ *on* $\mathbf{K}$ *is a non-basic shielded contraction operator if and only if* $\ominus$ *satisfies closure, inclusion, vacuity, extensionality, recovery, relative success, persistence, conjunctive inclusion and conjunctive overlap.*

The following observation illustrates that the classes of entrenchment-based shielded contraction operators, of sphere-based shielded contraction operators and of non-basic shielded contraction operators coincide.

Observation 4.2.29 *[FH01] Let* $\mathbf{K}$ *be a consistent belief set and* $\ominus$ *an operator on* $\mathbf{K}$*. Then the following conditions are equivalent:*

1. *$\ominus$ is a non-basic shielded contraction operator.*
2. *$\ominus$ is an entrenchment-based shielded contraction operator.*
3. *$\ominus$ is a sphere-based shielded contraction operator.*

4.2.3 Generalized Levi and Harper identities

Revisions and contractions can be defined in terms of each other through the *Levi* identity and the *Harper* identity [Gär88]. In [FH01], Fermé and Hansson provided similar relationships between shielded contractions and credibility-limited revisions. To do so they needed to reformulate the Levi identity since an operator of revision defined in terms of the Levi identity always satisfies *success*. That is not the case in credibility-limited revisions. In [FH01] the following condition was proposed for defining an operator (of credibility-limited revision) $\odot$ by means of an operator (of shielded contraction) $\ominus$:

(Consistency-preserving Levi Identity) $\mathbf{K}\odot\alpha = \begin{cases} (\mathbf{K}\ominus\neg\alpha) + \alpha & \text{if } \mathbf{K}\ominus\neg\alpha \nvdash \neg\alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$

The following observation illustrates some interrelations between postulates of shielded contraction and of credibility-limited revision whenever the credibility-limited revision operator under consideration is obtained from the operator of shielded contraction through the consistency-preserving Levi identity.

Observation 4.2.30 *[FH01] Let* $\mathbf{K}$ *be a belief set and* $\ominus$ *be an operator on* $\mathbf{K}$*. Let* $\odot$ *be defined from* $\ominus$ *via the consistency-preserving Levi identity. Then:*

If* $\ominus$ *satisfies	***then* $\odot$ *satisfies***
—	*closure, consistency preservation and relative success*
inclusion	*inclusion*
inclusion and vacuity	*vacuity*
extensionality	*extensionality*
inclusion and persistence	*strong regularity*
relative success, vacuity, extensionality and conjunctive constancy	*disjunctive constancy*
inclusion and success propagation	*strict improvement*
extensionality, relative success conjunctive inclusion	*guarded subexpansion*
closure, inclusion, extensionality, recovery, persistence, relative success and conjunctive overlap	*superexpansion*

The following observation can be seen as the dual of the previous one. It illustrates some interrelations between postulates of shielded contraction and of credibility-limited revision whenever the shielded contraction operator under consideration is obtained from the operator of credibility-limited revision through the Harper identity:

$$\mathbf{K}\ominus\alpha = \mathbf{K} \cap \mathbf{K} \odot \neg\alpha.$$

Observation 4.2.31 *[FH01] Let* $\mathbf{K}$ *be a belief set and* $\odot$ *be an operator on* $\mathbf{K}$*. Let* $\ominus$ *be defined from* $\odot$ *via the Harper identity. Then:*

(a) It holds that:

If* $\odot$ *satisfies	***then* $\ominus$ *satisfies***
—	*inclusion*
closure	*closure*
vacuity	*vacuity*
extensionality	*extensionality*
closure and relative success	*recovery*
closure, consistency preservation and relative success	*relative success*
vacuity, consistency preservation, extensionality, relative success and disjunctive constancy	*conjunctive constancy*
closure, extensionality, relative success and superexpansion	*conjunctive overlap*
vacuity, strong regularity, strict improvement and guarded subexpansion	*conjunctive inclusion*

(b) If **K** *is a consistent belief set, then:*

If $\odot$ ***satisfies***	***then*** $\ominus$ ***satisfies***
closure, consistency preservation and strong regularity	*persistence*
strict improvement, relative success and consistency preservation	*success propagation*

The following result highlights the interrelations among the different classes of credibility-limited revision operators and of shielded contraction operators considered, provided that the former are obtained from the latter by means of the consistency-preserving Levi identity.

Theorem 4.2.32 *Let* **K** *be a consistent belief set,* $\ominus$ *be an operator on* **K** *and* $\odot$ *be the operator defined from* $\ominus$ *via the consistency-preserving Levi identity. Then:*

(a) If $\ominus$ *is a basic shielded contraction operator, then* $\odot$ *is a basic credibility-limited revision operator on* **K**.

(b) If $\ominus$ *is a non-basic shielded contraction operator, then* $\odot$ *is a non-basic credibility-limited revision operator on* **K**.

Proof: A proof for this theorem can be found on page 177.

The following result can be seen as the dual of the previous one. It highlights the interrelations among the classes of shielded contraction operators and the classes of credibility-limited revision operators considered, provided that the former are obtained from the latter by means of the Harper identity.

Theorem 4.2.33 *Let* **K** *be a consistent belief set,* $\odot$ *be an operator on* **K** *and* $\ominus$ *be an operator defined from* $\odot$ *via the Harper identity. Then:*

(a) If $\odot$ *is a basic credibility-limited revision operator, then* $\ominus$ *is a basic shielded contraction operator on* **K**.

(b) If $\odot$ *is a non-basic credibility-limited revision operator, then* $\ominus$ *is a non-basic shielded contraction operator on* **K**.

Proof: A proof for this theorem can be found on page 178.

Now we will illustrate that the operators of non-prioritized contraction and revision are interdefinable through the Harper and the consistency-preserving Levi identities. The following definition introduces functions that take us from contractions to revisions and vice-versa.

Definition 4.2.34 *[Mak87, FH01] For every operator* $\ominus$, $\mathbb{R}(\ominus)$ *is the operator generated from* $\ominus$ *through the consistency-preserving Levi identity. Furthermore, for every operator* $\odot$, $\mathbb{C}(\odot)$ *is the operator generated from* $\odot$ *by the Harper identity.*

Theorem 4.2.35 *[FH01] Let* $\mathbf{K}$ *be a belief set and* $\ominus$ *an operator for* $\mathbf{K}$ *that satisfies the contraction postulates closure, inclusion, recovery, extensionality and relative success. Then* $\mathbb{C}(\mathbb{R}(\ominus)) = \ominus$.[7]

Theorem 4.2.36 *[FH01] Let* $\mathbf{K}$ *be a belief set and* $\odot$ *an operator for* $\mathbf{K}$ *that satisfies the revision postulates closure, vacuity, relative success, extensionality and consistency preservation. Then* $\mathbb{R}(\mathbb{C}(\odot)) = \odot$.[8]

4.3 Belief bases

One criticism of the AGM framework is that it uses logically closed sets (or belief sets) to model the belief state of an agent. This can be considered undesirable for a number of reasons. Firstly, belief sets are very large entities, and its use is not adequate for computational implementations. Any attempt to computably implement the theory of belief change will have to be based on a finite representation. The logical closure of belief sets raises other issues not related to computational implementations. Rott pointed out in [Rot00b] that the AGM theory is unrealistic in its assumption that epistemic agents are "ideally competent regarding matters of logic. They should accept all the consequences of the beliefs they hold (that is, their set of beliefs should be logically closed), and they should rigorously see to it that their beliefs are consistent". In the AGM framework agents have unlimited memory and ability of inference. Furthermore, as Gärdenfors and Root pointed out "when we perform revisions or contractions, it seems that we never do it to the belief set itself (...) but rather on some typically finite base for the belief set" [GR95]. Formally, a *belief base* is a subset of $\mathcal{L}$ that is not (necessarily) logically closed. A set A is a *base* for a belief set $\mathbf{K}$ if and only if $Cn(A) = \mathbf{K}$. A sentence α is believed if and only if $\alpha \in Cn(A)$.

There are two distinct points of view on the use of belief bases to represent the belief state of an agent. In one of these approaches, supported by Dalal [Dal88], all the beliefs of the belief set have equal status and belief bases are a merely expressive resource. The fact that a sentence belongs to a belief base A does not distinguish

[7]We note that *extensionality* is not included in the list of postulates of the theorem presented in [FH01]. However, in the proof for this theorem there presented it is stated that

$$\mathbf{K} \odot \neg\alpha = \begin{cases} (\mathbf{K}\ominus\alpha) + \neg\alpha & \text{if } \mathbf{K}\ominus\alpha \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

and it seems that to justify this equality it is necessary to use $\ominus$ *extensionality*, since the equality that is obtained from the Levi-consistency identity is the following:

$$\mathbf{K} \odot \neg\alpha = \begin{cases} (\mathbf{K}\ominus\neg\neg\alpha) + \neg\alpha & \text{if } \mathbf{K}\ominus\neg\neg\alpha \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

[8]We note that *extensionality* is not included in the list of postulates of the theorem presented in [FH01]. However, in the proof for this theorem there presented it is stated that $\mathbf{K}\ominus\neg\alpha = \mathbf{K} \odot \alpha \cap \mathbf{K}$ and it seems that to justify this equality it is necessary to use $\odot$ *extensionality*, since the equality that is obtained from the Harper identity is the following: $\mathbf{K}\ominus\neg\alpha = \mathbf{K} \odot \neg\neg\alpha \cap \mathbf{K}$.

it from the other sentences of the correspondent belief set $Cn(A)$ that are not in the belief base. Dalal, in [Dal88], formulated the principle of *irrelevance of syntax*, according to which, the outcome of a belief change operation should not depend on the syntax (or representation) of either the old or the new information.

In other (more common) approach the elements of a belief base represent those beliefs of an agent that have independent standing. Beliefs that are not on the belief base, but are in its closure, have a different status, being merely derived beliefs. Merely derived beliefs are automatically removed when the beliefs of the underlying belief base that support them are withdrawn.

Example 4.3.1 *[Han91a, Han99b] I belief that Paris is the capital of France (α). I also belief that there is milk in my fridge (β). Thus I belief that Paris is the capital of France if and only if there is milk in my fridge ($\alpha \leftrightarrow \beta$). I open the fridge and noticed that there is no milk. Thus I have to replace my believe in β by $\neg\beta$. I cannot retain both my beliefs in α and $\alpha \leftrightarrow \beta$. In a belief set approach, α and $\alpha \leftrightarrow \beta$ are both elements of the belief set. $\alpha \leftrightarrow \beta$ is not removed immediately, it must be ensured by a mechanism (such as a selection function) that chooses between removing α or $\alpha \leftrightarrow \beta$. In a belief base approach, the merely derived belief $\alpha \leftrightarrow \beta$ is automatically removed when β is retracted.*

This use of belief bases provides more expressive power than the one that is possible by means of belief sets. As claimed by Hansson in [Han99b], for every belief base A there is only a belief set $Cn(A)$ that represents the set of beliefs held according to A. On the other hand, the same belief set can be represented by different belief bases. For example, the belief bases $A_1 = \{\alpha, \beta\}$ and $A_2 = \{\alpha, \alpha \leftrightarrow \beta\}$ generate the same belief set since $Cn(A_1) = Cn(A_2)$. The sets A_1 and A_2 are statically equivalent, in the sense that $Cn(A_1) = Cn(A_2)$, but are not dynamically equivalent, since they do not behave in the same way under operations of change [Han92a, Han99b]. The following example, presented in [Han99b], clarifies these concepts. Imagine that A_1 and A_2 represent the belief state of the agents x and y, respectively. Suppose now that these agents receive and accept the information that α is false. Then x ends up with the basic beliefs $\neg\alpha$ and β, while the resulting basic beliefs of the agent y are $\neg\alpha$ and $\alpha \leftrightarrow \beta$. Therefore, after receiving the new information agent x believes in β while agent y believes in $\neg\beta$.

The following example illustrates the "cost" of performing changes on belief sets rather than in belief bases.

Example 4.3.2 *[Han91a, Han99b] Suppose that an agent believes in α. Then for every sentence β in the language, it holds that $\alpha \vee \beta$ and $\alpha \vee \neg\beta$ are both in the belief set. In a belief set approach, in order to remove α we must retract either $\alpha \vee \beta$ or $\alpha \vee \neg\beta$ (or both). Thus we must give up at least as many beliefs as there are sentences in the language. In a belief base approach if α is in the belief base and is not implied by any subset of that belief base then α is the only sentence that has to be removed.*

Another important feature of belief bases is that they allow to distinguish between different inconsistent belief states. This does not happen in the belief set approach since there is only one inconsistent belief set, namely $\mathcal{L}$. Let A_1 and A_2 be the following inconsistent belief bases: $A_1 = \{p, \neg p, q, r, s\}$ and $A_2 = \{p, \neg p, \neg q, \neg r, \neg s\}$. The outcome of contracting these belief bases by $\neg p$ gives rise to two different consistent belief bases.

In general, contractions on belief bases do not satisfy *recovery*. This has been pointed out as another one of their main appealing features.

From the aspects pointed out in this section it seems better to represent belief states by belief bases instead of by belief sets. However, this may not be the case. Gärdenfors in [Gär90] claimed that many of the conceptual breakthroughs of bases for belief sets can be modelled by beliefs sets together with the notion of epistemic entrenchment of beliefs. Gärdenfors and Rott in [GR95], pointed out that the choice between belief bases and belief sets may depend on the particular application area. "Within computer science applications, bases seems easier to handle since they are mostly finite structures. (...) Changes of belief sets rather than bases represents what an ideal reasoner would or should do when forced to reorganize his beliefs. Belief set dynamics offers a competence model which helps us to understand what people — and AI systems, for that matter — should do if they were not bounded by limited logical reasoning capabilities" [GR95].

Chapter 5 is devoted to recalling some of the main models of belief base change so far presented in the literature.

4.4 Iterated change

The problem of constructing models for iterated change is probably one of the most studied problems in the context of belief change. An iterated (or repeated) change consists of the repeated application of change operations. For example, $((\mathbf{K} \star \alpha) \star \beta) \div \delta$. The AGM model has been criticized for not addressing the problem of iterated (or repeated) change. This may be surprising at first sight. If the AGM framework correctly formalizes one-step belief revision,[9] when attempting to execute a sequence of revisions, why not simply treat such a sequence as series of one-step revisions? This is based on the fact that in doing so, we are assuming that each one-step revision of the series is independent of the others. This fails to capture the fact that all such one-step revisions are carried out by the same rational agent and therefore must be related. Standard AGM operations of change take us from a complete belief state to an incomplete belief state (belief set only). This is not enough when performing iterated change. As stated by Darwiche and Pearl "The AGM theory is expressed as a set of one-step postulates which tell us what properties the next state of belief ought to have, given the current beliefs and the current evidence. However, the

[9]Since most of the literature on iterated change deals with revision (rather than contraction), here we will focus our attention on iterated revision.

language of one-step postulates is not rich enough to regulate sequential revisions because such a language deals only with transformation of beliefs and not with transformation of epistemic states. An agent's epistemic state contains not merely what the agent believes currently but also an encoding of how the agent would modify his/her beliefs given any hypothetical evidence" [DP94]. These facts have been discussed by the belief change community and several proposals were presented that have in common the following characteristic: the AGM model is expanded in such a way that the operation of change has to yield a complete representation of a belief state as its outcome, not only the modified belief set. There are several proposed ways to represent such an epistemic state. One of these approaches is an adaptation of Grove's system of spheres [Gro88].

Note that the AGM framework does not provide insight of how to perform iterated revision using systems of spheres. To see this, consider a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$. Suppose that we wish to revise $\mathbf{K}$ by α, where $\alpha \notin \mathbf{K}$. We will obtain a new belief set, $\mathbf{K}\star\alpha$, which is fully determined by $\mathbf{K}$, α and $\mathbb{S}$. Assume now that we wish to revise $\mathbf{K}\star\alpha$ by another sentence β such that $\mathbf{K}\star\alpha \vdash \neg\beta$ and $\nvdash \neg\beta$. Unfortunately, the AGM framework does not provide a system of spheres to perform this revision. $\mathbb{S}$ cannot be used because of condition ($\mathbb{S}$2). This condition states that the innermost sphere must be formed by the ($\mathbf{K}\star\alpha$)-worlds and not by the $\mathbf{K}$-worlds as it is the case in $\mathbb{S}$. Thus a new system of spheres $\mathbb{S}'$, centred on $\|\mathbf{K}\star\alpha\|$, is needed to perform the revision of $\mathbf{K}\star\alpha$ by β. The AGM postulates do not give insight of how to produce $\mathbb{S}'$. The AGM paradigm is focused only on one-step belief change, it does not address the problem of iterated change.

What we need is a change operation that gives rise to a new system of spheres, from which the new belief set can be inferred and which may itself be the subject of further changes. The most influential formulation of this approach for iterated revision is due to Darwiche and Pearl [DP97]. They formulated a revision operator on belief states instead of an operation on belief sets.

Definition 4.4.1 *[Han92b, DP97]*
Let ξ be a set of objects called belief states. A function $s : \xi \to \mathbb{P}(\mathcal{L})$ is a support function (from ξ to $\mathbb{P}(\mathcal{L})$) if and only if for all $\Psi \in \xi$, $s(\Psi)$ is a belief set.

Intuitively, given a belief state Ψ, $s(\Psi)$ represents its associated belief set. If the elements of ξ are finite-based, then s can be replaced by a function B from ξ to $\mathcal{L}$, such that $B(\Psi)$ is a single sentence that is equivalent with the conjunction of $s(\Psi)$. In what follows we will consider a finitary propositional language $\mathcal{L}$.

Darwiche and Pearl adapted, from [KM91], the Katsuno and Meldelzon postulates (R1) to (R6) for revision to the context of epistemic states whose associated belief sets are represented by a single sentence.[10] The following are the postulates proposed by Darwiche and Pearl in [DP97] for revision operators on belief states.

[10] In Katsuno and Meldelzon's framework ([KM91]) the beliefs of an agent are represented by a sentence ψ in a finitary propositional language $\mathcal{L}$. The outcome of revising ψ by a sentence μ is also a sentence, denoted by $\psi\star\mu$.

In these postulates Ψ is a belief state.

(CR1) $B(\Psi\star\mu) \vdash \mu$.
(CR2) If $B(\Psi) \wedge \mu \nvdash \bot$, then $\vdash B(\Psi\star\mu) \leftrightarrow B(\Psi) \wedge \mu$.
(CR3) If $\mu \nvdash \bot$, then $B(\Psi\star\mu) \nvdash \bot$.
(CR4) If $\vdash \mu_1 \leftrightarrow \mu_2$, then $\vdash B(\Psi\star\mu_1) \leftrightarrow B(\Psi\star\mu_2)$.[11]
(CR5) If $B(\Psi\star\mu) \wedge \phi \vdash B(\Psi\star(\mu \wedge \phi))$.
(CR6) If $B(\Psi\star\mu) \wedge \phi \nvdash \bot$, then $B(\Psi\star(\mu \wedge \phi)) \vdash B(\Psi\star\mu) \wedge \phi$.

Darwiche and Pearl [DP97] presented a representation theorem for postulates (CR1) to (CR6) with respect to a revision mechanism based on total pre-orders over possible worlds:

Definition 4.4.2 *[KM91, DP97, JT07] A function that maps each belief state Ψ to a total pre-order $\leq_\Psi$ on $\mathcal{M}_\mathcal{L}$ is called a faithful assignment over belief states if and only if:*

(a) If $w_1 \vdash B(\Psi)$ and $w_2 \vdash B(\Psi)$, then $w_1 =_\Psi w_2$.

(b) If $w_1 \vdash B(\Psi)$ and $w_2 \nvdash B(\Psi)$, then $w_1 <_\Psi w_2$.

where $w_1 =_\Psi w_2$ is defined as $w_1 \leq_\Psi w_2$ and $w_2 \leq_\Psi w_1$; and $w_1 <_\Psi w_2$ means that $w_1 \leq_\Psi w_2$ and $w_2 \nleq_\Psi w_1$.

The intuition meaning of $w_1 \leq_\Psi w_2$ is that w_1 is at least as plausible (or at least as preferred) as w_2 in Ψ.

Observation 4.4.3 *[DP97] A revision operator $\star$ satisfies postulates (CR1) to (CR6) if and only if there is a faithful assignment that maps a belief state Ψ to a total pre-order $\leq_\Psi$ such that:*

$$\|B(\Psi\star\alpha)\| = min(\|\alpha\|, \leq_\Psi).$$

Darwiche and Pearl showed that the Katsuno and Meldelzon postulates alone are too weak to adequately characterize iterated belief revision. To overcome this, Darwiche and Pearl proposed in [DP97] four additional postulates that are known as the DP-postulates for iterated revision:

(DP1) If $\phi \vdash \mu$, then $\vdash B((\Psi\star\mu)\star\phi) \leftrightarrow B(\Psi\star\phi)$.
(DP2) If $\phi \vdash \neg\mu$, then $\vdash B((\Psi\star\mu)\star\phi) \leftrightarrow B(\Psi\star\phi)$.
(DP3) If $B(\Psi\star\phi) \vdash \mu$, then $B((\Psi\star\mu)\star\phi) \vdash \mu$.
(DP4) If $B(\Psi\star\phi) \nvdash \neg\mu$, then $B((\Psi\star\mu)\star\phi) \nvdash \neg\mu$.

(DP1) states that if an agent obtains two pieces of information being the second more specific, then the first one should be ignored. (DP2) states that if an agent

[11] This postulate was originally formulated in [DP97] as: If $\Psi_1 = \Psi_2$ and $\vdash \mu_1 \leftrightarrow \mu_2$, then $\vdash B(\Psi_1\star\mu_1) \leftrightarrow B(\Psi_2\star\mu_2)$. Here we are considering the modified version of this postulate presented in [JT07].

receives two contradictory pieces of information, then the last one prevails. (DP3) states that if an agent acquires a piece of information μ but loses it after receiving ϕ, then μ should never have been part of the belief of that agent if he had acquired ϕ in the first place. (DP4) essentially states that a piece of information cannot contribute to its own demise. If the information μ is not contradicted after receiving ϕ, then it should remain that way when the information ϕ is preceded by μ.

Darwiche and Pearl showed that each one of the postulates (DP1) to (DP4) can be represented semantically as the following observation illustrates.

Observation 4.4.4 *[DP97] Suppose that a revision operator satisfies postulates (CR1) to (CR6). The operator satisfies postulates (DP1) to (DP4) if and only if the operator and its corresponding faithful assignment satisfy:*
(DPR1) *If* $w_1 \vdash \mu$ *and* $w_2 \vdash \mu$*, then* $w_1 \leq_\Psi w_2$ *if and only if* $w_1 \leq_{\Psi\star\mu} w_2$*.*
(DPR2) *If* $w_1 \vdash \neg\mu$ *and* $w_2 \vdash \neg\mu$*, then* $w_1 \leq_\Psi w_2$ *if and only if* $w_1 \leq_{\Psi\star\mu} w_2$*.*
(DPR3) *If* $w_1 \vdash \mu$*,* $w_2 \vdash \neg\mu$ *and* $w_1 <_\Psi w_2$*, then* $w_1 <_{\Psi\star\mu} w_2$*.*
(DPR4) *If* $w_1 \vdash \mu$*,* $w_2 \vdash \neg\mu$ *and* $w_1 \leq_\Psi w_2$*, then* $w_1 \leq_{\Psi\star\mu} w_2$*.*

(DPR1) (respectively (DPR2)) states that the order among the μ–worlds (respect. $\neg\mu$–worlds) remains unchanged after a revision by μ. (DPR3) (respectively (DPR4)) states that if a μ–world is strictly (respect. weakly) preferred to a $\neg\mu$-world then that strict (respect. weak) preference is kept after revising by μ.

Postulates (DP1) to (DP4) have become a reference for iterated revision, and all other related proposals are almost invariably compared to this one. However Jin and Thielscher have pointed out that these postulates are too permissive: "They support operators by which all newly acquired information is cancelled as soon as an agent learns a fact that contradicts some of its current beliefs" [JT07]. Jin and Thielscher [JT07] and Booth and Meyer [BM06] have proposed independently the following condition, known as the *independence postulate*, instead of (DP3) and (DP4):

(Ind) If $B(\Psi\star\phi) \nvdash \neg\mu$, then $B((\Psi\star\mu)\star\phi) \vdash \mu$.

(Ind) essentially states that if a piece of information μ is not contradicted after revising by ϕ, then it should be believed after revising first by μ and then by ϕ.

The following observation illustrates that in the presence of postulates (CR1) to (CR6), postulate (Ind) implies (DP3) and (DP4).

Observation 4.4.5 *[JT07] Suppose that a revision operator satisfies postulates (CR1) to (CR6). If the operator satisfies postulate (Ind), then it also satisfies postulates (DP3) and (DP4).*

Semantically, postulate (Ind) corresponds to the following condition:
(R-Ind) If $w_1 \vdash \mu$, $w_2 \vdash \neg\mu$ and $w_1 \leq_\Psi w_2$, then $w_1 <_{\Psi\star\mu} w_2$.
This stated in the next observation.

Observation 4.4.6 *Suppose that a revision operator satisfies postulates (CR1) to (CR6). The operator satisfies postulate (Ind) if and only if the operator and its corresponding faithful assignment satisfy (R-Ind).*

(R-Ind) essentially states that if a μ–world w_1 is weakly preferred to a $\neg\mu$–world w_2, then after revising by μ, w_1 should be strictly preferred to w_2.

In another approach to the problem of iterated revision, Nayak in [Nay94] proposed a model based on (a modified version of) the concept of epistemic entrenchment. In Nayak's model not only the belief states but also the inputs are (modified) epistemic entrenchments. In this model the initial epistemic entrenchment $\leq$ is revised by another epistemic entrenchment $\leq'$ producing a new epistemic entrenchment $(\leq * \leq')$. The input $\leq'$ encodes the belief set $\mathbf{K}'$ that it relates to. The ordering on $\mathbf{K}'$ is related to the relative strength of acceptance of the sentences in $\mathbf{K}'$. The resulting epistemic entrenchment encodes a new belief set as well as the preference structure associated with it.
Other works on iterated change can be found in [BM06, HD05, DDL06, KP00, Leh95, Wil95, Spo88, Bou93, Bou96, Bre91, FGKIS13, FR04, Rot03, Rot09, Seg97, Bon09, KI08]. For an overview on this subject see [Pep14].

4.5 Multiple change

In the original AGM framework the input is a single sentence. This is a clear limitation of the model since agents often receive at the same time more than one piece of information. Multiple contraction is an operation that performs a simultaneous contraction of a (not necessarily singleton) set of sentences. Similarly, multiple revision is a revision by a set of sentences instead of by a single sentence.

There are at least three possible ways to interpret multiple contraction. One is to remove all elements of the input set, this operation is called *package contraction* [FH94]. Another is to remove at least one of the sentences of the input set. This operation is called *choice contraction* [FH94]. The third alternative is called *set contraction* [ZF01, Zha96]. Instead of removing the input, *set contraction* consists of obtaining an outcome that is consistent with it. In this brief description of multiple change we will consider only the first of the mentioned generalizations of contraction.

It is important to distinguish package contraction from other operations involving standard contractions by single sentences. In [FH94] Furhmann and Hansson distinguish the package contraction by $\{\alpha, \beta\}$ from the following operations:

1. contracting by $\alpha \vee \beta$.

2. intersecting the results of contracting by α and of contracting by β.

3. first contracting by α and then by β, or vice versa.

4. contracting by $\alpha \wedge \beta$.

The first of the above mentioned operation is unsuitable. It is true that when contracting $\alpha \vee \beta$ both α and β have to be removed, but the converse does not hold. Contracting both α and β, does not require the removal $\alpha \vee \beta$. This becomes clear if we think about the case when $\beta = \neg\alpha$. In this case it holds that $\alpha \vee \beta$ is a tautology. Thus it cannot be removed. But it is possible to contract by $\{\alpha, \beta\}$. The second operation may lead to an unnecessary loss of information. The third operation would introduce asymmetry where there should be none. Contracting by α and then by β may produce a different outcome of contracting first by β and then by α. Hence as stated by Furhmann and Hansson, "sequential contraction cannot be the same operation as multiple contraction. Multiple contraction is simultaneous: it does not discriminate between items to be removed by some assignment of priority" [FH94]. The fourth operation is also unsuitable since in order to remove $\alpha \wedge \beta$ it is sufficient to remove one of its conjuncts.[12]

Most of the major classes of AGM-related contraction operators have been generalized to multiple package contraction. There is a wide variety of such operators including partial meet multiple contraction [Han89, FH94, Li98] and kernel multiple contraction [FSS03]. The possible worlds semantics for partial meet multiple contraction was presented in [Rei11, RF12]. Regarding the multiple contraction counterpart of Grove's system of spheres-based contraction several methods for defining partial meet multiple contractions by means of systems of spheres were presented in [Rei11, FR12, RFP16].[13] One of these methods was translated to epistemic entrenchment and axiomatically characterized in [Rei11, FR13, RPF16]. In [Fuh97] Fuhrmann presents a survey which compiles some constructions of multiple contraction functions.

Hansson in [Han91a, Han92a] proposed a generalization of the Levi identity to the case of multiple change, *i.e.* Hansson proposed a way of defining a multiple revision function in terms of any given (package) multiple contraction:

$$\mathbf{K} \star B = Cn((\mathbf{K} \div \neg B) \cup B),$$

where $\alpha \in \neg B$ if and only if α is either a contradiction, a negation of some sentence of B or a (finite) disjunction of sentences that are negations of elements of B.

For finite sets the latter equality can be simplified. Instead of contracting by $\neg B$ it is enough to contract by the *sentential negation of* B, whose concept is formalized in the following definition:

Definition 4.5.1 *[Han91a, Han92a, Han99b] Let B be any finite set. The sentential negation of B, denoted $n(B)$, is the set (of sentences) such that:*

(a) If $B = \emptyset$, then $n(B) = \{\bot\}$.

(b) If B is a singleton set, $B = \{\beta\}$, then $n(B) = \{\neg\beta\}$.

(c) If $B = \{\beta_1, ..., \beta_n\}$ for some $n > 1$, then $n(B) = \{\neg\beta_1 \vee ... \vee \neg\beta_n\}$.

Note that if B is finite then the multiple revision by B corresponds to perform a standard single-sentence revision by the conjunctions of the elements of B.

[12] We note that if B is finite, then choice contraction by $\{p_1, p_2, ..., p_n\}$ can be treated as a standard single-sentence contraction by $p_1 \wedge p_2 \wedge ... \wedge p_n$.

[13] We notice, however, that in [RFP16] it was proven that any method for constructing multiple contractions which is based on systems of spheres fails to generate the (whole) class of transitively relational partial meet multiple contractions.

4.6 Summary

In this chapter we pointed out and discussed some of the shortcomings of the AGM model that have been identified by the belief change community, namely:

- the recovery postulate for contraction (some examples were presented where this postulate seems implausible);
- the representation of an agent's belief state by logically closed set of sentences (belief sets);
- the inadequacy of the AGM framework for performing iterated changes such as $(((\mathbf{K}\star\alpha) \div \beta)\star\delta)$;
- the fact that the input in the original AGM model is a single sentence and not a set of sentences;
- the success postulates for both revision and contraction (it seems unrealistic that the new information is always incorporated in the agent's set of beliefs and that a belief is always removed from the agent's beliefs when contracting by it).

We presented some of the proposals developed to address these problems:

- we referred some classes of contraction functions that satisfy the basic AGM postulates for contraction with the exception of *recovery* – the so called *withdrawals*;
- we discussed the use of belief bases instead of belief sets to represent an agent's belief state;
- we briefly mentioned some models for iterated revision;
- we briefly referred some models for multiple contraction and presented a generalization of the Levi identity to the case of multiple change, proposed by Hansson in [Han91a, Han92a], that allows us to define multiple revision functions in terms of (package) multiple contraction functions;
- we mentioned several models of non-prioritized belief revision and contraction operators (revision and contraction operators that do not satisfy the *success* postulate for revision and contraction respectively), namely *screened revision, selective revision, credibility-limited revision* and *shielded contraction.*

Regarding non-prioritized belief change operations we gave special emphasis to *credibility-limited revisions* and *shielded contractions*. In particular, we extended the work presented in [HFCF01] and in [FH01] by axiomatically characterizing other classes of credibility-limited operators induced, respectively, by a basic AGM revision and by an AGM revision operator, and sets of credible sentences satisfying a given set of properties. We established that the class of credibility-limited revision operators induced by an AGM revision and a set C (of credible sentences) that satisfies *single sentence closure, disjunctive completeness, element consistency, expansive credibility* and *strong revision credibility* coincides with the class of entrenchment-based credibility-limited revision operators and the class of sphere-based credibility-limited revision operators presented in [HFCF01]. We also established the interrelation between different kinds of credibility-limited revision operators and shielded contraction operators by means of the consistency-preserving Levi identity and the Harper identity.

Chapter 5

Belief Base Change

"But it is through such simple, idealized representation of belief sets that we have begun to obtain the insights needed to tackle more complex ones without getting lost in intricacies and overheads. Having acquired a fairly good understanding of the former (...) we can now profitably give more attention to the latter."
Makinson in [Mak97a]

Several of the existing models of contraction for beliefs sets, mentioned in Chapter 3, have been adapted to the case when belief states are represented by sets of sentences not (necessarily) closed under logical consequence—the so-called belief bases: the partial meet contractions for belief bases were presented in [Han92a, Han93b, Han91a]; the kernel contractions—which can be seen as a generalization of safe contractions—were introduced in [Han94]; and, in [Wil94a], Mary-Anne Williams introduced the *ensconcement-based contractions* and the *brutal contractions* (of belief bases), which can be seen as adaptations to the case of belief bases of the *epistemic entrenchment-based contractions* and of the *severe withdrawals*, respectively. In fact, the definitions of the two classes of contraction functions proposed in [Wil94a] are based on the concept of ensconcement, which is a generalization to the case of belief bases of the concept of epistemic entrenchment introduced by Gärdenfors and Makinson in [Gär88, GM88]. In this chapter we recall, from the literature, several types of base contraction functions and their axiomatic characterizations. We also present explicit definitions for some base revision functions obtained by means of the Levi identity from the namesake contractions functions. We recall the representation theorems for partial meet and kernel revisions functions and present axiomatic characterizations for the smooth kernel base revision and for the basic AGM-generated base revision functions. We conclude this chapter by briefly presenting some works on non-prioritized belief change in belief bases.

In the belief base context, the (non-closing) expansion of a set A by a sentence α is defined as follows:

Definition 5.0.1 *[Han99b] Let A be a belief base and α a sentence. $A + \alpha$, the (non-closing) expansion of A by α, is defined as follows:*

$$A + \alpha = A \cup \{\alpha\}.$$

Throughout this thesis, for any not logically closed set of sentences A and any sentence α the notation "$A+\alpha$" will be used as a representation of the set "$A\cup\{\alpha\}$".[1]

We shall use the symbols $-$ and $*$ to denote the operators of base contraction and of base revision respectively. In the context of belief base change, we will use the expression "contraction function" (or simply "contraction") to refer to a function:

$$\begin{aligned} -: \quad & \mathcal{P}(\mathcal{L}) \times \mathcal{L} \to \mathcal{P}(\mathcal{L}) \\ & (A, \alpha) \mapsto A - \alpha \end{aligned}$$

Given a fixed belief base A, we shall use the expression "contraction function on A" (or simply "contraction on A") to refer to a function

$$\begin{aligned} -: \quad & \mathcal{L} \to \mathcal{P}(\mathcal{L}) \\ & \alpha \mapsto A - \alpha \end{aligned}$$

Analogous terminology and notations will be used for base revision.

5.1 Postulates

In this section we present several postulates for belief base change (both for revision and for contraction) as well as some results that highlight the interrelations among some of those postulates.

5.1.1 Base contraction postulates

We start by recalling the definition of a contraction operator in terms of postulates presented in [Han99b].

Definition 5.1.1 *[Han99b] An operator $-$ for a set A is an operator of contraction if and only if $-$ satisfies the following postulates:*
(Success) *If $\nvdash \alpha$, then $A - \alpha \nvdash \alpha$.*
(Inclusion) $A - \alpha \subseteq A$.

We note that *inclusion* has the same formulation as $(\div 2)$, while the *success* postulate consists of an adaptation to belief bases of postulate $(\div 4)$ (these postulates, $(\div 2)$ and $(\div 4)$, were already introduced, in the belief set context, in Chapter 3).

We now recall four postulates, that have the same formulation, respectively, as postulates $(\div 3')$, $(\div 6)$, $(\div 7)$, $(\div V)$ (introduced in Chapter 3).

(Failure) If $\vdash \alpha$, then $A - \alpha = A$.
(Extensionality) If $\vdash \alpha \leftrightarrow \beta$, then $A - \alpha = A - \beta$.
(Conjunctive overlap) $A - \alpha \cap A - \beta \subseteq A - (\alpha \wedge \beta)$.

[1] We shall use the symbol + both for expansions on belief sets and for expansions on belief bases. Its meaning will always be clear from the context.

(Conjunctive factoring) $A-(\alpha\wedge\beta)=A-\alpha$ or $A-(\alpha\wedge\beta)=A-\beta$ or $A-(\alpha\wedge\beta)=A-\alpha\cap A-\beta$.

The following postulates, result of adapting, respectively, the statements of postulates (÷1), (÷3), (÷8) and (÷9) in order to obtain similar properties suitable for belief base contractions (rather than only for belief set contractions).

(Relative Closure) $A\cap Cn(A-\alpha)\subseteq A-\alpha$.
(Vacuity) If $A\nvdash\alpha$, then $A\subseteq A-\alpha$.
(Conjunctive inclusion) If $A-(\alpha\wedge\beta)\nvdash\alpha$, then $A-(\alpha\wedge\beta)\subseteq A-\alpha$.
(Strong Inclusion) If $A-\beta\nvdash\alpha$, then $A-\beta\subseteq A-\alpha$.

Other classical base contraction postulates are:

(Disjunctive Elimination) If $\beta\in A$ and $\beta\notin A-\alpha$ then $A-\alpha\nvdash\alpha\vee\beta$.

Disjunctive elimination was proposed in [FKR08] and states that if a sentence β is removed in the process of contracting A by another sentence α then the disjunction of α and β is not deducible from the outcome of that contraction.

(Relevance) If $\beta\in A$ and $\beta\notin A-\alpha$, then there is a set A' such that $A-\alpha\subseteq A'\subseteq A$ and $A'\nvdash\alpha$ but $A'\cup\{\beta\}\vdash\alpha$.

The *relevance* postulate [Han89, Han92a] ensures that nothing is removed for no reason. It is an expression of the principle of *minimal change*. It states that if a sentence β is removed from A when contracting it by α, then β must contribute to deduce α from A.

(Logical Relevance) If $\beta\in A$ and $\beta\notin A-\alpha$, then there is a set A' such that $A-\alpha\subseteq A'\subseteq Cn(A)$ and $A'\nvdash\alpha$ but $A'\cup\{\beta\}\vdash\alpha$.

This postulate, that was presented in [RW08], is a weaker version of the *relevance* postulate. Instead of requiring the inclusion of A' on A, it only requires logical inclusion.

(Core-retainment) If $\beta\in A$ and $\beta\notin A-\alpha$ then there is some set A' such that $A'\subseteq A$ and $A'\nvdash\alpha$ but $A'\cup\{\beta\}\vdash\alpha$.

Core-retainment [Han91b] is also a weaker version of *relevance* since it does not require that $A-\alpha\subseteq A'$.

(Uniformity) If it holds for all subsets A' of A that $\alpha\in Cn(A')$ if and only if $\beta\in Cn(A')$ then $A-\alpha=A-\beta$.

This postulate, which was presented in [Han92a], states that if α and β are two sentences implied by exactly the same subsets of A, then the result of contracting

A by α is identical to the outcome of contracting A by β.

(Linearity) $A - \alpha \subseteq A - \beta$ or $A - \beta \subseteq A - \alpha$.

Linearity [FR98a, RP99] is an arguable property since it follows from it that the outputs of the contraction of a set A by two unrelated beliefs are connected by means of the set inclusion relation. For example, it requires that either $\{p,q\} - p \subseteq \{p,q\} - q$ or $\{p,q\} - q \subseteq \{p,q\} - p$. This is incompatible with Parikh's splitting principle [Par99], according to which $\{p,q\} - p = \{q\}$ and $\{p,q\} - q = \{p\}$.

(Expulsiveness) If $\nvdash \alpha, \nvdash \beta$ then $A - \beta \nvdash \alpha$ or $A - \alpha \nvdash \beta$.

This postulate states that for every non-tautological sentences α and β, either α is not implied by the contraction of A by β or β is not implied by the contraction of A by α. *Expulsiveness* was first presented in [Han99b, page 102] and, as it is mentioned there and also in [RP99, Rot01], it is a highly implausible property of belief contraction, since according to it two unrelated sentences influence the result of the contraction by each other.

(Decomposition) $A - (\alpha \wedge \beta) = A - \alpha$ or $A - (\alpha \wedge \beta) = A - \beta$.

According to *decomposition* [AGM85] (which is also known as *linear hierarchical ordering*) the output of contracting a set A by a conjunction of two beliefs is equal to the contraction of A by one of those two beliefs.

In the following observations we present some relations among the postulates presented above.

Observation 5.1.2 *Let A be a belief base and $-$ be an operator on A. Then:*

(a) *[Han99b] If $-$ satisfies relevance, then it satisfies relative closure and core-retainment.*

(b) *[Han99b] If $-$ satisfies inclusion and core-retainment, then it satisfies failure and vacuity.*

(c) *[Han99b] If $-$ satisfies uniformity, then it satisfies extensionality.*

(d) *[FKR08] If $-$ satisfies disjunctive elimination, then it satisfies relative closure. If $-$ also satisfies inclusion then it satisfies failure.*

(e) *[FKR08] If $-$ satisfies relevance, then it satisfies disjunctive elimination.*

In Figure 5.1 we present a diagram that summarizes all the interrelations among postulates that were stated in the above observation.

Observation 5.1.3 *Let A be a belief base and $-$ be an operator on A. Then:*

(a) If $-$ satisfies logical relevance, then it satisfies disjunctive elimination.

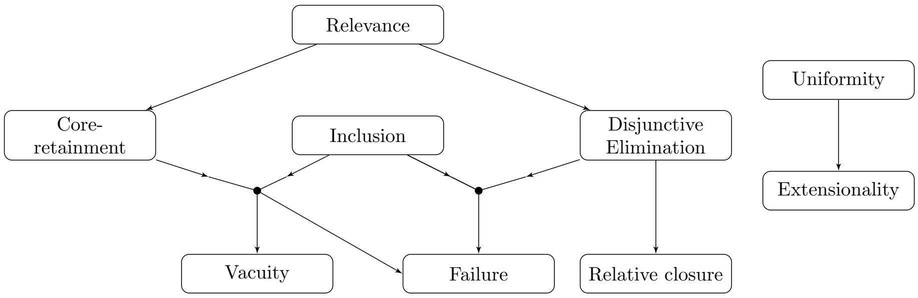

Figure 5.1: Diagram of the interrelations among postulates listed in Observation 5.1.2.

(b) If − satisfies inclusion, vacuity and disjunctive elimination, then it satisfies logical relevance.

(c) [RP99] If − satisfies strong inclusion, then it satisfies conjunctive inclusion.

(d) If − satisfies inclusion, failure, success and strong inclusion, then it satisfies extensionality, linearity, expulsiveness, conjunctive factoring, decomposition and uniformity.

Proof: A proof for this observation can be found on page 180.

In Figure 5.2 we present a diagram that summarizes all the interrelations among postulates that were stated in the above observation.

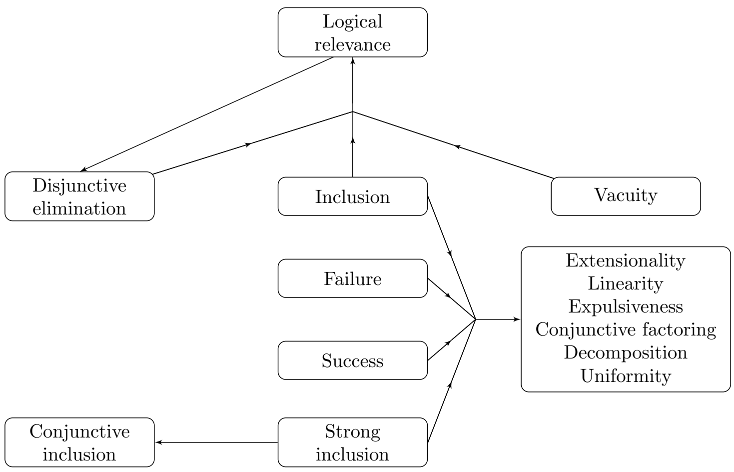

Figure 5.2: Diagram of the interrelations among postulates stated in Observation 5.1.3.

5.1.2 Internal and external revision

We saw in Subsection 3.1.4 that, in the belief set context, the Levi identity can be used to define revisions in terms of contractions (and expansions). This can also be done in the context of belief bases. Revising a set by α can be done by first contracting that set by $\neg\alpha$ and subsequently expanding its outcome by α. This is expressed by the *Levi identity*:

$$A * \alpha = (A - \neg\alpha) + \alpha.$$

An operator defined by the above equality is called an operator of *internal revision.*

It is also possible to define an operation of revision on belief bases by performing the expansions and contractions in a reverse order. That is, to revise a set by α we can first expand it by α and contract its outcome by $\neg\alpha$. This can be expressed by the *reverse Levi identity* [Han93b]:

$$A * \alpha = (A + \alpha) - \neg\alpha.$$

An operator defined by this last equality is called an operator of *external revision.* Note that, according to Definition 5.0.1, given an operator of contraction $-$ we can define an internal revision operator $*$ and a external revision operator $*'$ as follows:

$$A * \alpha = (A - \neg\alpha) \cup \{\alpha\} \qquad \text{and} \qquad A *' \alpha = (A \cup \{\alpha\}) - \neg\alpha.$$

5.1.3 Base revision postulates

We start this subsection by introducing a definition of a revision operator in terms of postulates. The following definition establishes the minimal set of postulates that a revision operator must satisfy.

Definition 5.1.4 *An operator $*$ for a set A is an operator of revision if and only if $*$ satisfies the following postulates:*
(Success) $\alpha \in A * \alpha$.
(Inclusion) $A * \alpha \subseteq A \cup \{\alpha\}$.
(Consistency) *If $\alpha \nvdash \bot$, then $A * \alpha \nvdash \bot$.*

We note that *success* and *consistency* have the same formulation as $(\star 2)$ and $(\star 5)$, respectively, while the formulation of *inclusion* consists in an adaptation to belief bases of postulate $(\star 3)$ (postulates $(\star 2)$, $(\star 3)$ and $(\star 5)$ were introduced, in the belief set context, in Chapter 3).

The following postulates are well known postulates for belief base revision:

(Vacuity) If $A \nvdash \neg\alpha$, then $A \cup \{\alpha\} \subseteq A * \alpha$.

The formulation of *vacuity* consists in an adaptation to belief bases of postulate $(\star 4)$.

(Uniformity) If for all subsets $A' \subseteq A$, $A' \cup \{\alpha\} \vdash \perp$ if and only if $A' \cup \{\beta\} \vdash \perp$, then $A \cap (A * \alpha) = A \cap (A * \beta)$.

Uniformity [Han93b] states that if two sentences are inconsistent with the same subsets of A, then the outcomes of the revisions of A by each of them should keep the same elements of A.

(Relevance) If $\beta \in A$ and $\beta \notin A * \alpha$, then there is some A' such that $A * \alpha \subseteq A' \subseteq A \cup \{\alpha\}$, $A' \nvdash \perp$ but $A' \cup \{\beta\} \vdash \perp$.

Relevance [Han93b] ensures that when revising a set A by a sentence, nothing is removed unless that removal contributes for keeping the outcome of the revision consistent.

(Core-retainment) If $\beta \in A$ and $\beta \notin A * \alpha$, then there is some $A' \subseteq A$ such that $A' \nvdash \neg\alpha$ and $A' \cup \{\beta\} \vdash \neg\alpha$.

Core-retainment [Was00] is, as *relevance*, an expression of the principle of *minimal change*. *Core-retainment* follows from *relevance* and *success*.

Additionally, we propose the following three postulates for belief base revision:[2]

(Disjunctive Elimination) If $\beta \in A$ and $\beta \notin A * \alpha$, then $A * \alpha \nvdash \neg\alpha \vee \beta$.

Disjunctive elimination states that if β is removed when revising a set A by α, then from the revision of A by α we can not deduce that α implies β.

(Relative Closure) $A \cap Cn(A \cap A * \alpha) \subseteq A * \alpha$.

Relative closure states that the set formed by the elements of A that are included in the outcome of revising A by α is logically closed relative to A.[3] We note that the intersection with the set A that appears in the argument of consequence operator Cn is not irrelevant as one might think. To see this consider the following example: Let $A = \{\alpha \to \beta, \beta, \beta \to \neg\alpha\}$ and $A * \alpha = \{\alpha \to \beta, \alpha\}$. Hence $\beta \in A \cap Cn(A * \alpha)$ but $\beta \notin A * \alpha$. On the other hand, $\alpha \to \beta$ is the only element of A that can be deduced from $A \cap A * \alpha$. It holds that $\alpha \to \beta \in A * \alpha$. Thus, $*$ satisfies *relative closure* but not the property $A \cap Cn(A * \alpha) \subseteq A * \alpha$.

(Weak Extensionality) If $\vdash \alpha \leftrightarrow \beta$, then $A \cap A * \alpha = A \cap A * \beta$.

Weak extensionality states that if α and β are two logically equivalent beliefs then every element of A that is kept when revising by α is also kept when revising by β. We note that *weak extensionality* is a weaker version of *extensionality*: If

[2] These three postulates are adaptations, for revision, of the contraction postulates: *Disjunction Elimination* [FKR08], *Relative Closure* [Han94] and *Extensionality* [Gär82].

[3] A set A is logically closed relative to B if and only if $Cn(A) \cap B \subseteq A$ ([Han91b]).

$\vdash \alpha \leftrightarrow \beta$, then $A * \alpha = A * \beta$. We also note that, in general, *extensionality* is not satisfied by belief base revisions. The following example illustrates this fact: Let α and β be two distinct sentences such that $\vdash \alpha \leftrightarrow \beta$. Let A be a belief base such that $A \cap \{\alpha, \beta\} = \emptyset$. Let $*$ be a revision operator on A, thus $*$ satisfies *success* and *inclusion*. Then $\alpha \in A * \alpha$ but $\alpha \notin A * \beta$, therefore $A * \alpha \neq A * \beta$.

5.2 Constructive models of base change operators

In this section we present some explicit definitions of base change functions and their axiomatic characterizations.

5.2.1 Partial meet contractions and revisions

The definition of an operator of partial meet contraction for belief bases was already presented in Definition 3.2.4.[4] Hansson characterized partial meet base contractions in terms of postulates:

Observation 5.2.1 *[Han91a] Let A be a belief base. An operator $-$ on A is a partial meet contraction if and only if $-$ satisfies success, inclusion, uniformity and relevance.*

Partial meet base revision is obtained from the partial meet base contraction by means of the Levi identity. Thus it can be defined as follows:

Definition 5.2.2 *[AGM85] Let A be a belief base. The partial meet revision operator on A based on a selection function γ is the operator $*_\gamma$ such that for all sentences α:*

$$A *_\gamma \alpha = \left(\bigcap \gamma(A \perp \neg\alpha)\right) \cup \{\alpha\}.$$

An operator $*$ on A is a partial meet revision if and only if there is a selection function γ for A such that for all sentences α: $A * \alpha = A *_\gamma \alpha$.

In the following observation we present an axiomatic characterization for partial meet base revision funtions.

Observation 5.2.3 *[Han91a] Let A be a belief base. An operator $*$ on A is a partial meet revision if and only if $*$ satisfies success, consistency, inclusion, uniformity and relevance.*

[4] Sometimes, when considering belief bases, we will refer to a partial meet contraction (revision) as partial meet *base* contraction (revision, respectively). The same will also occur regarding other kinds of contraction and revision operators.

5.2.2 Kernel contractions and revisions

The definition of an operator of kernel contraction for belief bases was already presented in Definition 3.2.17. Hansson also provided an axiomatic characterization for kernel base contractions.

Observation 5.2.4 *[Han94] Let A be a belief base. An operator $-$ on A is a kernel contraction if and only if $-$ satisfies success, inclusion, uniformity and core-retainment.*

In the following observation we recall an axiomatic characterization for a more conservative type of kernel base contractions, namely *smooth kernel base contractions* (presented in Definition 3.2.19). Sometimes, when contracting a set by means of a kernel contraction, some beliefs are removed without reason. For example if $\beta \in A$ and $\beta \in Cn(A - \alpha)$, then β should also be in $A - \alpha$. This can be solved if we ensure that an operator of kernel contraction satisfies *relative closure*. This is precisely the postulate that we need to add to the list of postulates that characterize kernel base contractions in order to obtain an axiomatic characterization for smooth kernel base contractions.

Observation 5.2.5 *[Han94] Let A be a belief base. An operator $-$ on A is a smooth kernel contraction if and only if it satisfies success, inclusion, uniformity, core-retainment and relative closure.*

Kernel base revision functions can be obtained from kernel base contraction functions by means of the Levi identity:

Definition 5.2.6 *Let A be a belief base. The kernel revision operator on A based on an incision function σ is the operator $*_\sigma$ such that for all sentences α:*

$$A *_\sigma \alpha = (A \setminus \sigma(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}.$$

An operator $*$ on A is a kernel revision if and only if there is an incision function σ for A such that for all sentences α: $A * \alpha = A *_\sigma \alpha$.

In the following observation we recall, from [Was00], an axiomatic characterization for kernel base revision functions.

Observation 5.2.7 *[Was00] Let A be a belief base. An operator $*$ on A is a kernel revision if and only if it satisfies success, consistency, inclusion, uniformity and core-retainment.*[5]

The following definition introduces the concept of smooth kernel base revision which is a kernel base revision based on a smooth incision function.

[5] To be precise, the axiomatic characterization for kernel base revisions presented in [Was00] is similar to this one but with *consistency* replaced by *non-contradiction*: If $\nvdash \neg\alpha$, then $A * \alpha \nvdash \neg\alpha$. However, it can be proven that, in the presence of *success*, the postulates of *non-contradiction* and *consistency* are equivalent.

Definition 5.2.8 *Let A be a belief base. An operator $*$ on A is a smooth kernel base revision if and only if it is a kernel base revision based on a smooth incision function.*

In the following observation we provide an axiomatic characterization for smooth kernel base revisions.

Observation 5.2.9 *Let A be a belief base. An operator $*$ on A is a smooth kernel revision if and only if it satisfies success, consistency, inclusion, uniformity, core-retainment and relative closure.*

Proof: A proof for this observation can be found on page 181.

5.2.3 Basic AGM-generated base contractions and revisions

In the following definition we recall the concept of *basic AGM-generated base contraction*, an operator of base contraction defined from an operator of basic AGM contraction (for belief sets).

Definition 5.2.10 *[FKR08] Let A be a belief base. An operator $-$ on A is a basic AGM-generated base contraction*[6] *if and only if there exists some basic AGM contraction $\div$ for $Cn(A)$,*[7] *such that for all $\alpha \in \mathcal{L}$:*

$$A - \alpha = (Cn(A) \div \alpha) \cap A.$$

In [FKR08], Fermé *et. al.* axiomatically characterized the basic AGM-generated base contraction functions.

Observation 5.2.11 *[FKR08] Let A be a belief base. An operator $-$ on A is a basic AGM-generated base contraction if and only if it satisfies success, inclusion, vacuity, extensionality and disjunctive elimination.*

We will now define and present an axiomatic characterization for *basic AGM-generated base revisions*, which are operators of base revision defined from operators of basic AGM revision (for belief sets).

Definition 5.2.12 *Let A be a belief base. An operator $*$ for A is a basic AGM-generated base revision if and only if there exists some basic AGM revision $\star$ for $Cn(A)$, such that for all $\alpha \in \mathcal{L}$:*

$$A * \alpha = (Cn(A) \star \alpha) \cap (A \cup \{\alpha\}).$$

Observation 5.2.13 *Let A be a belief base. An operator $*$ on A is a basic AGM-generated base revision if and only if it satisfies success, consistency, inclusion, vacuity, weak extensionality and disjunctive elimination.*

Proof: A proof for this observation can be found on page 182.

[6] In [FKR08] these operators were designated by basic related-AGM base contractions.

[7] *I.e.* an operator on $Cn(A)$ that satisfies the basic AGM postulates for belief set contractions.

5.2.4 Ensconcements and contractions based on ensconcements

We start this subsection by recalling the definition of *ensconcement*, which was originally proposed by Mary-Anne Williams.

Definition 5.2.14 *[Wil94a] An ensconcement is a pair $(A, \preceq)$ where A is a belief base and $\preceq$ is a transitive and total relation on A that satisfies the following three conditions:*
($\preceq$1) *If $\beta \in A \setminus Cn(\varnothing)$, then $\{\alpha \in A : \beta \prec \alpha\} \nvdash \beta$*
($\preceq$2) *If $\nvdash \alpha$ and $\vdash \beta$, then $\alpha \prec \beta$, for all $\alpha, \beta \in A$*
($\preceq$3) *If $\vdash \alpha$ and $\vdash \beta$, then $\alpha \preceq \beta$, for all $\alpha, \beta \in A$*

Informally an ensconcement relation establishes an order over the beliefs of an agent.[8] ($\preceq$1) says that, for any non-tautological β, the formulae that are strictly more ensconced than β do not (even conjointly) imply β. Conditions ($\preceq$2) and ($\preceq$3) say that tautologies are the most ensconced formulae.

The concept of ensconcement relation can be seen as a generalization of the notion of epistemic entrenchment to the context of belief bases.

In what follows we expose more formally the interrelation between these two kinds of binary relations.

The following result implies that the restriction of an epistemic entrenchment relation to a belief set is an ensconcement relation.

Observation 5.2.15 *Let $\mathbf{K}$ be a belief set and $\leq$ be a relation on $\mathcal{L}$ that satisfies (EE1), (EE2), (EE3) and (EE5). Then $(\mathbf{K}, \leq|_{\mathbf{K}})$ is an ensconcement.*

Proof: A proof for this observation can be found on page 183.

Since on the one hand an epistemic entrenchment is defined over the set of all $\mathcal{L}$-sentences while, on the other hand, an ensconcement relation is defined only over a set of sentences (which does not even need to be closed under logical consequence), we can immediately conclude that not every ensconcement relation is an epistemic entrenchment. Nevertheless, in [Wil94a], Mary-Anne Williams proposed a method for extending an ensconcement relation to the set of all $\mathcal{L}$-sentences which is such that the resulting relation is an epistemic entrenchment.

Let $(A, \preceq)$ be an ensconcement. In what follows we recall Williams' definition of an epistemic entrenchment $\leq_{\preceq}$ related to $Cn(A)$ such that for all $\alpha, \beta \in A$, it holds that $\alpha \leq_{\preceq} \beta$ if and only if $\alpha \preceq \beta$.

First we recall the notion of *cut* which was defined, in [Wil94a]. For any sentence $\alpha \in Cn(A)$ the cut of α, denoted by $cut_{\preceq}(\alpha)$ is the following subset of A:

$$cut_{\preceq}(\alpha) = \{\beta \in A : \{\gamma \in A : \beta \prec \gamma\} \nvdash \alpha\}.$$

[8] When forced to give up some of his/her beliefs an agent is more willing to remove the less ensconced ones.

In the following observation we present the above mentioned definition of an epistemic entrenchment $\leq_{\preceq}$ from the ensconcement relation $\preceq$.

Observation 5.2.16 *[Wil94a] Let $(A, \preceq)$ be an ensconcement and let $\leq_{\preceq}$ be the binary relation on $\mathcal{L}$ defined by: $\alpha \leq_{\preceq} \beta$ if and only if either*

(i) $\alpha \notin Cn(A)$, or

(ii) $\alpha, \beta \in Cn(A)$ and $cut_{\preceq}(\beta) \subseteq cut_{\preceq}(\alpha)$.

Then $\leq_{\preceq}$ is an epistemic entrenchment related to $Cn(A)$.

In what follows we recall the two kinds of base contraction functions defined by means of ensconcement relations that were proposed by Mary-Anne Williams in [Wil94a].

Both of the mentioned definitions are based on the *proper cut* operator, which is defined as follows:

Definition 5.2.17 *Given an* ensconcement $(A, \preceq)$*, for any sentence $\alpha \in \mathcal{L}$ the proper cut of α, denoted $cut_{\prec}(\alpha)$ is the subset of A defined by:*

$$cut_{\prec}(\alpha) = \{\beta \in A : \{\gamma \in A : \beta \preceq \gamma\} \nvdash \alpha\}.$$

The following observation states that when α is an explicit belief, its *proper cut* is the subset formed by the sentences of A which are strictly more ensconced than α.

Observation 5.2.18 *[Wil94a] If $\alpha \in A$, $cut_{\prec}(\alpha) = \{\beta \in A : \alpha \prec \beta\}$.*

In the following lemma we present some interesting and useful properties of the proper cut.

Lemma 5.2.19 *[FKR08]*

(a) *If $\vdash \alpha$, then $cut_{\prec}(\alpha) = \emptyset$.*

(b) *If $\nvdash \alpha, cut_{\prec}(\alpha) \nvdash \alpha$.*

(c) *If $A \nvdash \alpha, cut_{\prec}(\alpha) = A$.*

(d) *If $\beta \vdash \alpha$, then $cut_{\prec}(\alpha) \subseteq cut_{\prec}(\beta)$.*

(e) *If $\vdash \alpha \leftrightarrow \beta$, then $cut_{\prec}(\alpha) = cut_{\prec}(\beta)$.*

(f) *If $\alpha \preceq \beta$, then $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha)$.*

(g) *If $\alpha \prec \beta$, then $cut_{\prec}(\alpha) \vdash \beta$ and $cut_{\prec}(\beta) \nvdash \alpha$.*

(h) *If $\alpha \prec \beta$, then $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$.*

(i) *If $\beta =_{\preceq} \alpha$, then $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha) = cut_{\prec}(\beta)$.*

(j) *If* $cut_{\prec}(\alpha) \vdash \beta$, *then* $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$.

(k) *If* $cut_{\prec}(\alpha) \nvdash \beta$, *then* $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\beta)$.

Next we recall the definition of the so-called *ensconcement-based contractions*.

Definition 5.2.20 *[Wil94a] Let* $(A, \preceq)$ *be an ensconcement. The* $\preceq$*-based contraction on* A *is the operator* $-_{\preceq}$ *such that:*

$$A -_{\preceq} \alpha = \{\beta \in A : cut_{\prec}(\alpha) \vdash \alpha \vee \beta\}. \qquad \textbf{(EBC)}$$

An operator $-$ *on* A *is an ensconcement-based contraction if and only if there is an ensconcement* $(A, \preceq)$ *such that for all sentences* α: $A - \alpha = A -_{\preceq} \alpha$.

Note that if $\alpha \in Cn(\varnothing)$ and $-$ is an ensconcement-based contraction, then $A - \alpha = A$.

The other kind of base contraction functions introduced in [Wil94a], results from an intuitively appealing change in condition (**EBC**), used to define the ensconcement-based contractions. In fact, according to this condition in order for β to be preserved when contracting A by α it is necessary that $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. However, it seems more intuitive to require only that $\beta \in cut_{\prec}(\alpha)$ instead. Below we recall the definition of the class of contraction functions based on this simpler condition.

Definition 5.2.21 *[Wil94a] Let* $(A, \preceq)$ *be an ensconcement. The* $\preceq$*-based brutal contraction on* A *is the operator* $-_{\preceq}^{B}$ *such that:*

$$A -_{\preceq}^{B} \alpha = \begin{cases} cut_{\prec}(\alpha) & \text{if } \nvdash \alpha \\ A & \text{otherwise} \end{cases} \qquad \textbf{(BC)}$$

An operator $-$ *on* A *is a brutal contraction if and only if there is an ensconcement* $(A, \preceq)$ *such that for all sentences* α: $A - \alpha = A -_{\preceq}^{B} \alpha$.

The contraction functions presented in Definitions 5.2.20 and 5.2.21 will be axiomatically characterized in Chapter 6.

5.3 Non-prioritized change in belief bases

In this section we recall some non-prioritized change operators for belief bases. We give special emphasis to the operators of shielded contractions and of credibility-limited revisions that were presented in [FMT03]. These operators consist of the adaptation to the belief base context of the namesake operators for belief sets presented in [HFCF01] and in [FH01] (that were recalled in Section 4.2).

5.3.1 Shielded base contraction

In this subsection we summarize the main concepts and results, so far presented in the literature, concerning the adaptation to the case of belief bases of the operators of shielded contraction (on belief sets) that were recalled in Subsection 4.2.2.

Postulates for shielded contraction on belief bases

In [FMT03] the postulates proposed in [FH01] for shielded contractions on belief sets (that we recalled in Subsection 4.2.2) were adapted to the belief base context:

(Relative Success) [Rot92b] $A \sim \alpha = A$ or $\alpha \notin Cn(A \sim \alpha)$.
(Persistence) [FH01] If $\beta \in Cn(A \sim \beta)$, then $\beta \in Cn(A \sim \alpha)$.
(Success Propagation) [FH01] If $A \sim \beta \vdash \beta$ and $\vdash \beta \rightarrow \alpha$, then $A \sim \alpha \vdash \alpha$.
(Conjunctive Constancy) [FH01] If $A \sim \alpha = A \sim \beta = A$, then $A \sim (\alpha \wedge \beta) = A$.

The following two observations illustrate some relations between postulates (of contraction and) of shielded base contraction.

Observation 5.3.1 *[FH01] Let $\sim$ be an operator on A. If $\sim$ satisfies persistence, then it satisfies success propagation.*

Observation 5.3.2 *Let $\sim$ be an operator on A.*

(a) If $\sim$ satisfies relative success, then it satisfies failure.

(b) If $\sim$ satisfies inclusion, vacuity, persistence and relative success, then it satisfies conjunctive constancy.

Proof: A proof for this observation can be found on page 183.

Properties of the set of retractable sentences

In [FMT03] the properties that were considered as properties that may be desirable from a set R of retractable sentences were the following:

Non-retractability Propagation: If $\alpha \notin R$, then $Cn(\alpha) \cap R = \varnothing$.
Conjunctive Completeness: If $\alpha \wedge \beta \in R$, then $\alpha \in R$ or $\beta \in R$.
Non-retractability Preservation: $\mathcal{L} \backslash R \subseteq Cn(A \sim \alpha)$.
Non-retractability of Tautology: $R \cap Cn(\varnothing) = \varnothing$.

The first three ones of the above properties were proposed in [FH01]. The first two ones of the above properties were recalled in Subsection 4.2.2. *Non-retractability preservation* is a direct adaptation to the case of belief bases of the property with the same designation that was proposed in [FH01] for belief sets (and was recalled in Subsection 4.2.2). *Non-retractability of tautology* states that tautologies are irretractable sentences.

A constructive definition of shielded contractions on belief bases

In [FMT03], Fermé, Mikalef and Taboada adapted the definition of shielded contraction that we recalled in Definition 4.2.21 to the case of belief bases.

Definition 5.3.3 *[FMT03] Let A be a belief base. An operator $\sim$ on A is a shielded partial meet base contraction*[9] *if there exists a partial meet contraction $-$ on A and a set of (retractable) sentences R such that, for all $\alpha \in \mathcal{L}$:*

$$A \sim \alpha = \begin{cases} A - \alpha & \text{if } \alpha \in R \\ A & \text{otherwise} \end{cases}$$

The set R is called the retractable set associated to $\sim$.

Finally, in that same paper, the following axiomatic characterization was obtained for the class of shielded partial meet base contractions whose associated retractable set satisfies non-retractability propagation and non-retractability preservation.

Observation 5.3.4 *[FMT03] Let A be a belief base and $\sim$ an operator on A.Then the following conditions are equivalent:*

(a) *$\sim$ satisfies relative success, persistence, inclusion, relevance and uniformity.*

(b) *$\sim$ is an operator of shielded partial meet base contraction whose associated retractable set $R \subseteq \mathcal{L}$ satisfies non-retractability propagation and non-retractability preservation.*

Actually, we must note that in the representation theorem presented in [FMT03] the list of postulates consists of the postulates mentioned in (a) together with *vacuity* and *conjunctive constancy.* However, according to Observations 5.1.2 and 5.3.2, the latter two mentioned postulates follow from the remaining ones. Thus the axiomatic characterization presented in the above observation can, in fact, be seen as a refinement of the one presented in [FMT03].

In Subsection 7.3.2 we present axiomatic characterizations for classes of shielded base contractions induced by partial meet base contractions and by sets of retractable sentences that satisfy different sets of properties.

5.3.2 Credibility-limited base revision

In this subsection we recall, from [FMT03], some adaptations to the context of belief bases of the postulates and operators of credibility-limited revision (on belief sets) that were mentioned in Subsection 4.2.1.

Postulates for credibility-limited revision on belief bases

In [FMT03], the postulates pointed out in [HFCF01] as desirable properties that an operator of credibility-limited revision on belief sets should satisfy were adapted to

[9] In [FMT03] these operators were designated simply by *shielded base contraction* but here it is convenient to use this alternative designation, because in this thesis we shall use the expression *shielded base contraction* to designate a wider class of functions.

the belief base context:

(Relative Success) $\alpha \in A \circledast \alpha$ or $A \circledast \alpha = A$.
(Strict Improvement) If $\alpha \in A \circledast \alpha$ and $\vdash \alpha \rightarrow \beta$, then $\beta \in A \circledast \beta$.
(Regularity) If $A \circledast \alpha \vdash \beta$, then $\beta \in A \circledast \beta$.
(Strong Regularity) If $A \circledast \alpha \nvdash \neg\beta$, then $\beta \in A \circledast \beta$.
(Disjunctive Distribution) If $\alpha \vee \beta \in A \circledast (\alpha \vee \beta)$, then $\alpha \in A \circledast \alpha$ or $\beta \in A \circledast \beta$.
(Consistency Preservation) If $A \nvdash \bot$, then $A \circledast \alpha \nvdash \bot$.

Properties of the set of credible sentences

The following properties were pointed out as desirable properties for C, the set of credible sentences, in [FMT03]:

Single Sentence Closure: If $\alpha \in C$, then $Cn(\alpha) \subseteq C$.
Disjunctive Completeness: If $\alpha \vee \beta \in C$, then either $\alpha \in C$ or $\beta \in C$.
Element Consistency: If $\alpha \in C$, then $\alpha \nvdash \bot$.
Expansive Credibility: If $A \nvdash \alpha$, then $\neg\alpha \in C$.
Revision Credibility: If $\alpha \in C$, then $Cn(A \circledast \alpha) \subseteq C$.
Strong revision credibility: If $\alpha \notin C$, then $A \circledast \beta \vdash \neg\alpha$.

These properties were already recalled in the belief set context in Subsection 4.2.1.

A constructive definition of credibility-limited revision on belief bases

In Subsection 4.2.1 we recalled from [HFCF01] the construction of an operator of credibility-limited revision operator induced by a basic AGM revision operator $\star$ on a belief set $\mathbf{K}$ and a set $C \subseteq \mathcal{L}$ (the associated set of credible sentences). In [FMT03], Fermé, Mikalef and Taboada adapted this construction, proposing a model of credibility-limited revision for belief bases. They defined an operator $\circledast$ of credibility-limited base revision on a belief base A as follows:

Definition 5.3.5 *[FMT03] Let A be a belief base. An operator $\circledast$ on A is a credibility-limited partial meet base revision*[10] *if there exist a partial meet revision $*$ on A and a set of (credible) sentences C such that, for all $\alpha \in \mathcal{L}$:*

$$A \circledast \alpha = \begin{cases} A * \alpha & \text{if } \alpha \in C \\ A & \text{otherwise} \end{cases}$$

The set C is called the credible set associated to $\circledast$.

In Subsection 8.5.2 we present axiomatic characterizations for classes of credibility-limited base revisions induced by partial meet base revisions and by sets of credible sentences that satisfy different sets of properties.

[10]In [FMT03] these operators were designated simply by *credibility-limited base revision* but here it is convenient to use this alternative designation, because in this thesis we shall use the expression *credibility-limited base revision* to designate a wider class of functions.

5.3.3 Semi-revision and consolidation

Consolidation was introduced in [Han91a] and consists of making an inconsistent belief base consistent. It can be seen as a contraction by $\perp$ (falsum). Consolidation is an operator for belief bases and does not have a plausible counterpart for belief sets. This is due to the fact that there is only an inconsistent belief set and once it is obtained all distinctions are lost and consolidation can not restore them. The consolidation of a belief base A is denoted by $A!$.

Hansson in [Han97] proposed a modification of external revision that he designated by *semi-revision*. Instead of contracting by the negation of a sentence, the expanded set is contracted by $\perp$ (falsum).

$$A?\alpha = (A + \alpha) - \perp .$$

Thus, the semi-revision can be seen as the result of the consolidation of an expansion.

$$A?\alpha = (A + \alpha)!.$$

Note that in a semi-revision process the input sentence may be removed during the consolidation. Thus semi-revision is a non-prioritized change operator.

In [Han91a, Han97], Hansson defined and axiomatically characterized operators of kernel and partial-meet consolidation and of kernel and partial meet semi-revision, based in the namesake operations for contractions and revisions.

5.4 Summary

In this chapter we presented postulates for contraction and for revision defined on belief bases. We recalled some definitions of contraction and revision operators in the belief base context. For contraction we recalled the following constructive methods: partial meet contraction, kernel and smooth kernel contraction, basic AGM-generated base contraction, as well as the axiomatic characterization for each one of these operators. We also presented two operators proposed by Williams in [Wil94a] based on the notion of ensconcement, namely brutal and ensconcement-based contractions. An ensconcement can be seen as an adaptation to the belief base context of the notion of epistemic entrenchment. We will revisit these contraction functions in Chapter 6, where we will present axiomatic characterizations for each one of these operators.
We also recalled the definition of partial meet and kernel revision functions as well as the axiomatic characterization of each one of these revision functions. We also defined and axiomatically characterized two new kinds of base revision functions, namely the smooth kernel base revisions and the basic AGM-generated base revisions. These revision operators are based on their namesake contraction functions.
We ended this chapter by briefly recalling some operators of non-prioritized belief change on belief bases, namely: semi-revision (and consolidation), credibility-limited (partial meet) base revision and shielded (partial meet) base contraction. We also

refined the axiomatic characterization of shielded (partial meet) base contraction operators presented in [FMT03] by identifying a couple of postulates that follow from the remaining ones in the mentioned axiomatic characterization. In Chapter 7 and 8 we will revisit the operators of shielded base contraction and of credibility-limited base revision.

Chapter 6

On Ensconcement and Contraction

"Good order is the foundation of all things."
Edmund Burke

In this chapter we study the interrelation among the two kinds of belief base contraction operators introduced by Mary-Anne Williams in [Wil94a], namely *brutal contractions* and *ensconcement-based contractions*. We start by presenting axiomatic characterizations for the *brutal* and the *ensconcement-based contractions*. After that we compare these two axiomatic characterizations (for *brutal contractions* and for *ensconcement-based contractions*) in the sense of identifying which postulates of each one of the mentioned axiomatic characterizations are (and which are not) satisfied by the other kind of operators. This comparison allows us to determine which ones of the postulates used in those representation theorems can be considered characteristic properties of each one of those two kinds of contraction functions. We will also compare the axiomatic characterizations of *brutal* and *ensconcement-based contractions* with the axiomatic characterizations of other base contractions operators, namely with basic AGM-generated base contractions, kernel contractions and partial meet contractions. We also study the construction of ensconcement relations by means of each one of the two kinds of operations, based on ensconcements, considered. Furthermore, we will present some results which clarify the interrelation among epistemic entrenchment-based contractions and ensconcement-based contractions, as well as the interrelation among severe withdrawals and brutal contractions.

6.1 Some new postulates for belief base contractions

In this section we introduce some new postulates which will be useful afterwards in the process of obtaining axiomatic characterizations for brutal and for ensconcement-based contractions.

(Uniform Behaviour) If $\beta \in A$, $A \vdash \alpha$ and $A - \alpha = A - \beta$, then $\alpha \in Cn(A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\})$.

Uniform Behaviour asserts that if a sentence α that is deducible from A is such that the result of its contraction from A coincides with the result of contracting A by a sentence which is (explicitly) present in A then α should be deducible from the union of the set of all the sentences fulfilling that property with the set that results of contracting A by α. We note that this postulate is trivial when $\alpha \in A$.[1]

(Transitivity) If $\beta \in A$, $\alpha \notin A - (\alpha \wedge \beta)$ and $\beta \notin A - (\beta \wedge \delta)$, then $\alpha \notin A - (\alpha \wedge \delta)$.

The intuition behind this postulate (for the principal case) is the following: Provided that $\alpha, \beta \in A$ and $\nvdash \alpha \wedge \beta$, at least one of the sentences α or β is removed when contracting A by $\alpha \wedge \beta$. Thus, $\alpha \notin A - (\alpha \wedge \beta)$ can be interpreted as meaning that β is "at least as good as α". Having this interpretation in mind, the postulate of *transitivity* essentially states that if β is at least as good as α and δ is at least as good as β, then δ is at least as good as α.

Still considering the above reasoning, we notice that $\alpha \in A - (\alpha \wedge \beta)$ can be considered as meaning that "α is (strictly) better than β". Taking this into account two other postulates (similar to *transitivity*) which are natural to expect to be fulfilled by a contraction function are the following:

(ST) If $\delta \in A$, $\beta \in A - (\alpha \wedge \beta)$ and $\beta \notin A - (\beta \wedge \delta)$, then $\delta \in A - (\alpha \wedge \delta)$.
(SST) If $\alpha \in A$, $\beta \in A - (\alpha \wedge \beta)$ and $\delta \in A - (\beta \wedge \delta)$, then $\delta \in A - (\alpha \wedge \delta)$.

The *ST* postulate can be interpreted as follows: if β is (strictly) better than α and δ is at least as good as β, then δ is (strictly) better than α. While the *SST* postulate can be interpreted as: if β is (strictly) better than α and δ is (strictly) better than β, then δ is (strictly) better than α.

(EB1) If $\beta \in A$ and $\{\gamma \in A : \beta \notin A - (\beta \wedge \gamma)\} \nvdash \alpha$, then $\beta \in A - \alpha$.

The condition $\beta \notin A - (\beta \wedge \gamma)$ when $\beta, \gamma \in A$ can be seen as "it is at least as easy to give up the belief β as it is to give up γ". Therefore the set $\{\gamma \in A : \beta \notin A - (\beta \wedge \gamma)\}$, for a non tautological β can be seen as the set of formulae that are at least as "good" as β. Having this interpretation in mind, postulate *EB1* essentially states that if the subset of A formed by the sentences which are at least as "good" as β does not imply α, then β is kept when contracting A by α.

(EB2) If $\beta \in A - \alpha$ then $\{\gamma \in A : \gamma \in A - (\gamma \wedge \alpha)\} \vdash \alpha \vee \beta$.

We note that $\{\gamma \in A : \gamma \in A - (\gamma \wedge \alpha)\}$ is the set of formulae of A that are retained when a contraction by its conjunction with α occurs. In order to give up $\gamma \wedge \alpha$, either γ or α (or both) must be removed. If γ is kept, during the removal of $\gamma \wedge \alpha$ from A this means that γ is in some sense "better" than α. Hence, *EB2* can be read as follows: if β is kept when contracting A by α, then the set formed by the formulae

[1] This is explained in the proof of Observation 6.5.1 (i).

that are "better" than α implies $\alpha \vee \beta$.

The following observation clarifies that in the presence of extensionality, *transitivity* is equivalent to ST and, furthermore, that SST follows from *transitivity* provided that some other postulates also hold.

Observation 6.1.1 *Let A be a belief base and $-$ be an operator on A. Then:*

(a) If $-$ satisfies extensionality, then $-$ satisfies transitivity if and only if $-$ satisfies ST.

(b) If $-$ satisfies success, inclusion, extensionality, relative closure and transitivity, then it satisfies SST.

Proof: A proof for this observation can be found on page 187.

6.2 Representation theorems

In this section we present axiomatic characterizations for the operators of brutal contraction (*cf.* Definition 5.2.21) and for the operators of ensconcement-based contraction (*cf.* Definition 5.2.20).

6.2.1 Axiomatic characterization of brutal contraction functions

In this subsection we present an axiomatic characterization for the class of *brutal contractions*. We start by introducing the following condition which defines a binary relation on A by means of a contraction function $-$ on A:

$$\alpha \leq \beta \text{ if and only if } \alpha \notin A - \beta \text{ or } \vdash \beta. \qquad (\mathbf{C_{BR}} \leq)$$

The following theorem exposes that, provided that the contraction $-$ satisfies some of the postulates presented above, it holds that the binary relation $\leq$ defined by condition ($\mathbf{C_{BR}} \leq$) is an ensconcement relation on A and, furthermore, that operation $-$ satisfies condition (**BC**) presented in Definition 5.2.21.

Theorem 6.2.1 *Let A be a belief base and $-$ an operator on A. If $-$ satisfies success, inclusion, vacuity, failure, relative closure, strong inclusion and uniform behaviour, then the binary relation $\leq$ on A defined by ($\mathbf{C_{BR}} \leq$) is an ensconcement relation and $-$ satisfies (**BC**).*

Proof: A proof for this theorem can be found on page 187.

The next result attests that a brutal contraction satisfies all the postulates mentioned in the previous theorem and that condition ($\mathbf{C_{BR}} \leq$) holds whenever $(A, \leq)$ is an ensconcement and $-$ is the $\leq$-based brutal contraction.

Theorem 6.2.2 *Let* $(A, \leq)$ *be an ensconcement and* $-$ *be the* $\leq$*-based brutal contraction on* A*. Then* $-$ *satisfies success, inclusion, vacuity, failure, relative closure, strong inclusion and uniform behaviour as well as the condition (*$\mathbf{C_{BR}} \leq$*).*

Proof: A proof for this theorem can be found on page 189.

It follows immediately from the two previous theorems that brutal contractions are axiomatically characterized by the postulates of *success, inclusion, vacuity, failure, relative closure, strong inclusion* and *uniform behaviour*.

From Theorem 6.2.2 it also follows that all the properties listed in Observation 5.1.3 (d), namely *extensionality, linearity, expulsiveness, conjunctive factoring, decomposition* and *uniformity*, are satisfied by brutal contractions.

6.2.2 Axiomatic characterization of ensconcement-based contraction functions

Our main goal in the present section is to obtain an axiomatic characterization for the ensconcement-based contractions and our first step in that direction is to introduce the following condition which defines a binary relation on A by means of a contraction function $-$ on A:

$$\alpha \leq \beta \text{ if and only if } \alpha \notin A - (\alpha \wedge \beta) \text{ or } \vdash \alpha \wedge \beta. \qquad (\mathbf{C_{EB}} \leq)$$

We note that this construction is similar to condition ($\mathbf{C}_{\leq}$) proposed by Gärdenfors and Makinson [GM88] to define an epistemic entrenchment relation by means of a given operator of belief set contraction.

The following theorem exposes that, provided that the contraction $-$ satisfies some of the postulates presented above, it holds that the binary relation $\leq$ defined by condition ($\mathbf{C_{EB}} \leq$) is an ensconcement relation on A and, furthermore, that the operation $-$ is the $\leq$-based contraction on A.

Theorem 6.2.3 *Let* A *be a belief base and* $-$ *an operator on* A*. If* $-$ *satisfies inclusion, vacuity, success, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1 and EB2, then the binary relation* $\leq$ *on* A *defined by (*$\mathbf{C_{EB}} \leq$*) is an ensconcement relation and* $-$ *satisfies the condition (***EBC***) presented in Definition 5.2.20.*

Proof: A proof for this theorem can be found on page 190.

The next result attests that an ensconcement-based contraction satisfies all the postulates mentioned in the previous theorem and that condition ($\mathbf{C_{EB}} \leq$) holds whenever $(A, \leq)$ is an ensconcement and $-$ is the $\leq$-based contraction.

Theorem 6.2.4 *Let* $(A, \leq)$ *be an ensconcement and* $-$ *be the* $\leq$*-based contraction on* A*. Then* $-$ *satisfies inclusion, vacuity, success, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1 and EB2 as well as the condition (*$\mathbf{C_{EB}} \leq$*).*

Proof: A proof for this theorem can be found on page 191.

We note that it follows immediately from the two previous theorems that ensconcement-based contractions are axiomatically characterized by the postulates of *inclusion, vacuity, success, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1* and *EB2*.

We highlight that, since it follows from Observation 5.1.3 (a) and (b) that, in the presence of *inclusion* and *vacuity*, *disjunctive elimination* is equivalent to *logical relevance*, we can conclude that if *disjunctive elimination* is replaced by *logical relevance* in the axiomatization highlighted in the above paragraph, the resulting list of postulates consists also of an (alternative) axiomatic characterization of the ensconcement-based contraction. At this point we must remark that in [FKR08, Theorem 14] an axiomatic characterization for the ensconcement-based contractions was presented which consisted only of the postulates of *success*, *inclusion*, *vacuity*, *extensionality*, *conjunctive factoring* and *disjunctive elimination*. However, in personal communications, Zhiqiang Zhuang pointed out two gaps in the *From postulates to Ensconcement-based contraction* part of the proof of the mentioned theorem.

6.3 Ensconcement-based versus brutal contraction

In this section we will compare the two kinds of base contraction functions presented in [Wil94a], namely brutal contractions and ensconcement-based contractions. More precisely we will check which postulates of the axiomatic characterization of the ensconcement-based contraction functions are satisfied by the brutal contraction functions and which are not and vice-versa. We end this section by comparing the two methods for constructing an ensconcement from a contraction operator presented in the previous section. On the one hand we will establish that the method used in the previous section to define an ensconcement from an ensconcement-based contraction (condition ($\mathbf{C_{EB}} \preceq$)) works also when the operation under consideration is a brutal contraction. On the other hand we will show that condition ($\mathbf{C_{BR}} \preceq$) works if the operation under consideration is a brutal contraction but does not work if it is an ensconcement-based contraction.

We start by presenting an example that clarifies the difference between the definitions of brutal contraction and of ensconcement-based contraction.

Example 6.3.1 *Let $(A, \preceq)$ be an ensconcement where $A = \{p \vee q, q, q \rightarrow p\}$ and $\preceq$ is the three-level ensconcement relation on A defined by: $q \prec q \rightarrow p \prec p \vee q$. Let $-$ be the $\preceq$-based contraction and $-^B$ be the $\preceq$-based brutal contraction on A. Hence $cut_\prec(p) = \{p \vee q\}$. Therefore, $A -^B p = \{p \vee q\}$ and $A - p = \{p \vee q, q\}$.*

It is clear that given an ensconcement $(A, \preceq)$, if $-$ is the $\preceq$-based contraction and $-^B$ is the $\preceq$-based brutal contraction, then for all sentences α: $A -^B \alpha \subseteq A - \alpha$.

The following example illustrates that in general brutal contractions do not satisfy *disjunctive elimination*.

Example 6.3.2 *Let* $(A, \preceq)$ *be an ensconcement where* $A = \{p, q, p \vee q\}$ *and* $\preceq$ *is the two-level ensconcement relation on* A *defined by:* $p =_{\preceq} q \prec p \vee q$*. Let* $-$ *be the* $\preceq$*-based brutal contraction on* A*. By Definition 5.2.21 it holds that* $A - p = \{p \vee q\}$*. Hence* $q \in A$*,* $q \notin A - p$ *and* $A - p \vdash p \vee q$*. Hence* $-$ *does not satisfy disjunctive elimination.*

The two following examples illustrate that ensconcement-based contractions, in general, do not satisfy two of the postulates that axiomatically characterize brutal contractions, namely *strong inclusion* and *uniform behaviour*.

Example 6.3.3 *Let* $-$ *be an ensconcement-based contraction on* $A = \{p, q, r, q \vee r\}$*. Let* $\preceq$ *be an ensconcement relation on* A *defined by:* $r \prec p \prec q \prec q \vee r$*. By Definition 5.2.20 it follows that* $A - p = \{q, q \vee r\}$ *and* $A - q = \{r, q \vee r\}$*. Hence* $A - q \nvdash p$ *but* $A - q \nsubseteq A - p$*. Hence* $-$ *does not satisfy strong inclusion.*

Example 6.3.4 *Let* $-$ *be an ensconcement-based contraction on* $A = \{q, r, r \to p\}$ *and let* $\preceq$ *be an ensconcement relation on* A *such that all the formulae of* A *are at the same level. By Definition 5.2.20 it follows that* $A - q = A - p = \emptyset$*,* $A - r = \{r \to p\}$ *and* $A - (r \to p) = \{r\}$ *(since* $\vdash r \vee (r \to p)$*). Hence* $q \in A$*,* $A \vdash p$*,* $A - p = A - q$*, but* $p \notin Cn(A - q \cup \{\gamma \in A : A - q = A - \gamma\})$*. Hence* $-$ *does not satisfy uniform behaviour.*

In the two following observations we expose which postulates of the axiomatic characterization of the ensconcement-based contraction functions are satisfied by brutal contractions functions and which are not and vice- versa.

Observation 6.3.5 *Let* A *be a belief base. If* $-$ *is a brutal contraction on* A*, then:*

(a) $-$ *satisfies success, inclusion, vacuity, extensionality, conjunctive factoring, transitivity, EB1 and EB2;*

(b) $-$ *in general does not satisfy disjunctive elimination.*

Proof: A proof for this observation can be found on page 193.

Observation 6.3.6 *Let* A *be a belief base. If* $-$ *is an ensconcement-based contraction on* A*, then:*

(a) $-$ *satisfies success, inclusion, vacuity, failure and relative closure;*

(b) $-$ *in general does not satisfy strong inclusion nor uniform behaviour.*

Proof: A proof for this observation can be found on page 194.

The following observation states that if $-$ is a brutal contraction, then condition $(\mathbf{C_{EB}} \preceq)$ is equivalent to condition $(\mathbf{C_{BR}} \preceq)$ and defines an ensconcement relation on A whereas, if $-$ is an ensconcement-based contraction on A, then in general condition $(\mathbf{C_{BR}} \preceq)$ does not define an ensconcement relation on A.

Observation 6.3.7 *Let A be a belief base. Then:*

(a) *If an operator $-$ on A is a brutal contraction, then condition ($\mathbf{C_{EB}} \leq$) is equivalent to condition ($\mathbf{C_{BR}} \leq$) and defines an ensconcement relation on A.*

(b) *If an operator $-$ on A is an ensconcement-based contraction, then condition ($\mathbf{C_{BR}} \leq$) in general does not define an ensconcement relation on A.*

Proof: A proof for this observation can be found on page 194.

To finish this section we briefly summarize the main results here presented. Hence, according to Observations 6.3.5 and 6.3.6, the postulates of success, inclusion, vacuity, failure, relative closure, extensionality, conjunctive factoring, transitivity, EB1 and EB2 are satisfied by both ensconcement-based contractions and brutal contractions. Furthermore, those two results allow us to conclude that the properties that distinguish those two kinds of base contractions are disjunctive elimination, strong inclusion and uniform behaviour. More precisely disjunctive elimination (or, equivalently, logical relevance) can be considered the main characteristic property of ensconcement-based contractions since it is the only postulate included in the axiomatic characterization that is not satisfied by the related operation of brutal contraction. Analogously, the postulates that can be considered characteristic properties of brutal contractions (in the sense that they are not satisfied by ensconcement-based contractions) are strong inclusion and uniform behaviour. On the other hand, the remaining results of the present section allow us to conclude that, while the binary relation defined by condition ($\mathbf{C_{EB}} \leq$) is an ensconcement relation on A whether the belief contraction $-$ there considered is an ensconcement-based contraction or a brutal contraction, condition ($\mathbf{C_{BR}} \leq$) gives rise to an ensconcement relation on A when $-$ is a brutal contraction, but in general do not define an ensconcement relation if $-$ is an ensconcement-based contraction.

6.4 Maps between different base contraction functions

The following observation exposes interrelations among the different classes of contractions recalled in Section 5.2. These interrelations follow trivially from the axiomatic characterizations presented in Observations 5.2.1, 5.2.4, 5.2.5, 5.2.11 and in Theorem 6.2.4 as well as the interrelations among postulates that we presented in Observation 5.1.2 (and that are represented in the diagram of Figure 5.1).

Observation 6.4.1 *Let A be a belief base and $-$ be an operator on A. Then:*

(a) *If $-$ is an operator of partial meet contraction, then it is an operator of smooth kernel contraction.*

(b) *If $-$ is an operator of smooth kernel contraction, then it is an operator of kernel contraction.*

(c) *If $-$ is an operator of partial meet contraction, then it is a basic AGM-generated base contraction.*

(d) *If $-$ is an operator of ensconcement-based contraction, then it is a basic AGM-generated base contraction.*

We have already seen that not every operator of kernel base contraction is an operator of smooth kernel base contraction (*cf.* Example 3.2.18). On the other hand, in Observation 6.3.5 we saw that brutal contractions do not in general satisfy *disjunctive elimination*, from which (together with Observation 5.2.11) it follows that not all brutal contractions are basic AGM-generated base contractions. Furthermore, in [FKR08], Fermé *et. al.* provided an example that shows that not every operator of smooth kernel base contraction is an operator of partial meet base contraction. These facts are stated in the following observation.

Observation 6.4.2

(a) There are kernel base contraction operators that are not smooth kernel base contractions.

(b) There are smooth kernel base contraction operators that are not partial meet base contractions.

(c) There are brutal contraction operators that are not basic AGM-generated base contractions (nor partial meet contractions by Observation 6.4.1 (c)).

The following examples will be useful to clarify further interrelations between the base contraction functions mentioned in Section 5.2.

Example 6.4.3 *Consider a language that consists of p, q and r, and their truth-functional combinations. Let $A = \{p, q, r\}$ and $\epsilon = p \wedge q$.
It holds that $Cn(p \wedge (q \leftrightarrow r)) \in Cn(A)\bot\epsilon$. Let $\div$ be a partial meet contraction on $Cn(A)$ such that $Cn(A) \div \epsilon = Cn(p \wedge (q \leftrightarrow r))$ and let $-$ be an operator on A defined for all $\theta \in \mathcal{L}$ by $A - \theta = (Cn(A) \div \theta) \cap A$. Hence $-$ is a basic AGM-generated base contraction (since every partial meet contraction on belief sets is a basic AGM contraction, by Observation 3.2.9). On the other hand, it holds that $A - \epsilon = Cn(p \wedge (q \leftrightarrow r)) \cap A = \{p\}$. Therefore $-$ does not satisfy* core-retainment*, since $r \in A \setminus A - \epsilon$. Hence, according to Observation 5.2.4, $-$ is not a kernel contraction.*

Example 6.4.4 *Let $A = \{p, p \vee q, p \to q\}$. It holds that $A \Perp q = \{\{p, p \to q\}, \{p \vee q, p \to q\}\}$. Let $-_\sigma$ be the smooth kernel base contraction based on a smooth incision function σ such that: $\sigma(A \Perp q) = \{p, p \to q\}$. Hence $A -_\sigma q = \{p \vee q\}$. Thus $-_\sigma$ does not satisfy* disjunctive elimination *(since $p \in A \setminus A -_\sigma q$ and $A -_\sigma q \vdash p \vee q$). Therefore, by Observation 5.2.11, $-_\sigma$ is not a basic AGM-generated base contraction.*

Example 6.4.5 *Let $A = \{p, q, r\}$. It holds that $A \bot (p \wedge q) = \{\{p, r\}, \{q, r\}\}, A \bot (q \wedge r) = \{\{p, q\}, \{p, r\}\}$ and $A \bot (p \wedge r) = \{\{p, q\}, \{q, r\}\}$. Let $-_\gamma$ be the partial meet contraction based on a selection function γ for A such that: $\gamma(A \bot (p \wedge q)) = \{\{q, r\}\}$*

and $\gamma(A \perp (q \wedge r)) = \{\{p, r\}\}$ *and* $\gamma(A \perp (p \wedge r)) = \{\{p, q\}\}$*. Hence* $A -_\gamma (p \wedge q) = \{q, r\}, A -_\gamma (q \wedge r) = \{p, r\}$ *and* $A -_\gamma (p \wedge r) = \{p, q\}$*. Therefore* $-_\gamma$ *does not satisfy* transitivity *since* $p \notin A -_\gamma (p \wedge q), q \notin A -_\gamma (q \wedge r)$ *but* $p \in A -_\gamma (p \wedge r)$*. Thus, according to Theorem 6.2.4,* $-_\gamma$ *is not an ensconcement-based contraction. Furthermore, according to Observation 6.3.5 (a),* $-_\gamma$ *is not a brutal contraction either.*

Example 6.4.6 *Let* $A = \{p, q, q \to r\}$ *and* $\leq$ *be an ensconcement relation on* A *defined by:* $q \prec p \prec q \to r$*. If* $-$ *is either the* $\leq$*-based contraction or the* $\leq$*-based brutal contraction for* A*, then it holds that* $A - p = \{q \to r\}$*. It follows that* $-$ *does not satisfy* core-retainment *since* $q \in A \setminus A - p$*.*

The statements of the following observation follow from the examples presented above.

Observation 6.4.7

(a) There are operators of basic AGM-generated base contraction that are not kernel base contractions (nor smooth kernel contractions nor partial meet contractions)—Example 6.4.3.

(b) There are operators of smooth kernel base contraction that are not basic AGM-generated base contractions—Example 6.4.4.

(c) There are operators of partial meet base contractions that are not ensconcement-based contractions nor brutal contractions—Example 6.4.5.

(d) There are operators of ensconcement-based contractions and also of brutal contractions that are not kernel base contractions (nor smooth kernel contractions nor partial meet contractions)—Example 6.4.6.

It is worth to notice that Example 6.4.6 also illustrates, as pointed out by Rott in [Rot00a], that ensconcement-based contractions may provoke a loss of independent beliefs with a low priority in the belief base.

In Figure 6.1 we present a diagram that summarizes the logical relationships between the operators of base contraction analysed in this and in the previous section. This diagram is inspired in the diagram presented in [FKR08, Figure 1]. In this diagram an arrow between two kinds of contractions means that the class formed by all the operators of the kind mentioned at the origin of the arrow is strictly contained in the class formed by the operators of the kind mentioned at the end of the arrow. The inexistence of an arrow between two kinds of contractions means that the corresponding classes are not related by means of inclusion.

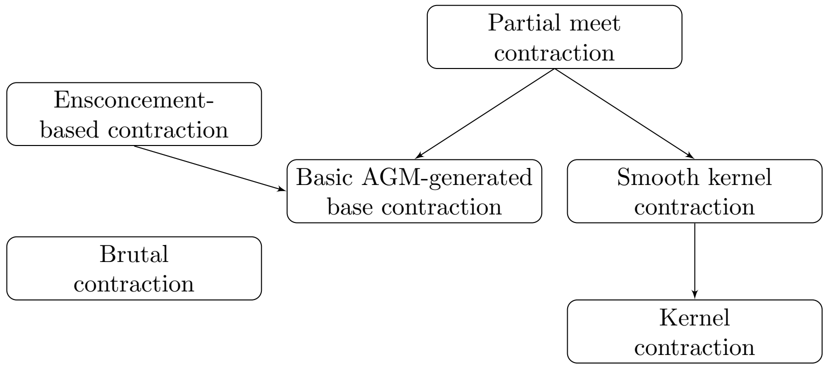

Figure 6.1: Logical relationships between different operations of base contraction.

6.5 Connections between base contractions based on ensconcements and belief set contractions based on epistemic entrenchments

In this section we study the interrelations among ensconcement-based contractions and epistemic entrenchment-based contractions and among brutal contractions and severe withdrawals.

We start by highlighting, in the two following observations, some interrelations among the postulates included in the axiomatizations of those four classes of contraction functions on logically closed sets. The first of these observations exposes which of the belief set contraction postulates are enough to assure the fulfilment of some of the belief base contraction postulates, while the second one highlights which belief base contraction postulates are enough to imply certain belief set contraction postulates.

Observation 6.5.1 *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be an operator on* $\mathbf{K}$.

(a) If $\div$ *satisfies* $(\div 1)$, *then it satisfies relative closure.*

(b) If $\div$ *satisfies* $(\div 3)$, *then it satisfies vacuity.*

(c) If $\div$ *satisfies* $(\div 1)$ *and* $(\div 4)$, *then it satisfies success.*

(d) If $\div$ *satisfies* $(\div 1)$ *and* $(\div 9)$, *then it satisfies strong inclusion.*

(e) If $\div$ *satisfies* $(\div 1)$, $(\div 2)$, $(\div 3)$ *and* $(\div 5)$, *then it satisfies disjunctive elimination.*

(f) If $\div$ *satisfies* $(\div 1)$, $(\div 2)$, $(\div 3)$, $(\div 4)$, $(\div 5)$, $(\div 6)$, *and* $(\div V)$, *then it satisfies transitivity.*

(g) If $\div$ *satisfies* $(\div 1)$, $(\div 3)$, $(\div 4)$ *and* $(\div V)$, *then it satisfies EB1.*

(h) If $\div$ *satisfies* $(\div 1)$, $(\div 2)$ *and* $(\div 6)$, *then it satisfies EB2.*

(i) $\div$ *satisfies uniform behaviour.*

Proof: A proof for this observation can be found on page 194.

Observation 6.5.2 *Let A be a logically closed set and $-$ be an operator on A.*

(a) If $-$ satisfies success, then it satisfies $(\div 4)$.[2]

(b) If $-$ satisfies inclusion and vacuity, then it satisfies $(\div 3)$.

(c) If $-$ satisfies inclusion and relative closure, then it satisfies $(\div 1)$.

(d) If $-$ satisfies inclusion and disjunctive elimination, then it satisfies $(\div 1)$.

(e) If $-$ satisfies inclusion, vacuity and disjunctive elimination, then it satisfies $(\div 5)$.

(f) If $-$ satisfies inclusion, relative closure and strong inclusion, then it satisfies $(\div 9)$.

Proof: A proof for this observation can be found on page 195.

Using the results presented above, we can easily identify which postulates of the axiomatic characterization of the ensconcement-based contraction functions are satisfied by epistemic entrenchment-based contractions and which are not and vice-versa, as we expose below.

The most significant conclusion that follows from the two previous observations is that, in the context of belief set contraction, the class of epistemic entrenchment contractions coincides with the class of ensconcement-based contractions. This fact is formally stated in the following theorem, which highlights also some new axiomatic characterizations for the epistemic entrenchment-based contractions and for the ensconcement-based contractions on belief sets.

Theorem 6.5.3 *Let $\mathbf{K}$ be a belief set and $\div$ be a contraction on $\mathbf{K}$. The following statements are equivalent:*

1. *$\div$ is an epistemic entrenchment-based contraction on $\mathbf{K}$.*
2. *$\div$ is an ensconcement-based contraction on $\mathbf{K}$.*
3. *$\div$ satisfies the postulates $(\div 1)$–$(\div 6)$ and $(\div V)$.*
4. *$\div$ satisfies the postulates of* inclusion, vacuity, success, extensionality, conjunctive factoring *and* disjunctive elimination.

Proof: A proof for this theorem can be found on page 195.

We notice that it follows from the above theorem that every epistemic entrenchment-based contraction satisfies all the postulates included in the axiomatic characterization of ensconcement-based contractions obtained in Subsection 6.2 and,

[2] This statement holds even if A is a not logically closed.

moreover, that the postulates of *transitivity, EB1* and *EB2* are redundant in that axiomatic characterization if the set on which the contraction is defined is logically closed. Conversely, according to the above result it holds also that every ensconcement-based contraction on a belief set satisfies the postulates $(\div 1)$–$(\div 6)$ and $(\div V)$. Nevertheless, this is not the case regarding ensconcement-based contraction on a belief base that is not logically closed, as we clarify in the following observation.

Observation 6.5.4 *Let A be a belief base and $-$ be an ensconcement-based contraction on A. Then*

(a) $-$ satisfies postulates $(\div 2)$, $(\div 4)$, $(\div 6)$ and $(\div V)$.

(b) If A is not logically closed, then $-$ in general does not satisfy $(\div 1)$ nor $(\div 3)$ nor $(\div 5)$.

Proof: A proof for this observation can be found on page 196.

Above we have concluded that the class of epistemic entrenchment-based contractions and the class of ensconcement-based contractions on belief sets are identical, by means of a comparison among axiomatic characterizations. However, this fact can be proven directly as we expose in the two following observations.

From the following observation we can conclude in a direct way that every epistemic entrenchment-based contraction is an ensconcement-based contraction. We notice that this result is essentially based on the fact that, according to Observation 5.2.15, the restriction of an epistemic entrenchment relation to a belief set is an ensconcement relation. Therefore, any given epistemic entrenchment relation can be used to define both an epistemic entrenchment-based contraction and an ensconcement-based contraction.

Observation 6.5.5 *Let $\mathbf{K}$ be a belief set. Let $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$ and $\div$ be the $\leq$-based contraction on $\mathbf{K}$ defined by condition $(\mathbf{C}_{\div_\leq})$. Let $-$ be the ensconcement-based contraction on $\mathbf{K}$ defined from $\leq|_\mathbf{K}$ by means of condition $(\mathbf{EBC})$. Then, for all $\alpha \in \mathcal{L}$, $\mathbf{K} \div \alpha = \mathbf{K} - \alpha$.*

Proof: A proof for this observation can be found on page 196.

Conversely, the following observation allows us to conclude in a direct way that every ensconcement-based contraction on a belief set is an epistemic entrenchment-based contraction.[3]

Observation 6.5.6 *Let $\mathbf{K}$ be a belief set. Let $(\mathbf{K}, \preceq)$ be an ensconcement and $-$ be the $\preceq$-based contraction on $\mathbf{K}$ defined by condition $(\mathbf{EBC})$. Let $\leq_\preceq$ be the epistemic entrenchment related to $\mathbf{K}$ defined from $\preceq$ as exposed in Observation 5.2.16 and $\div$ be the epistemic entrenchment-based contraction on $\mathbf{K}$ defined from $\leq_\preceq$ by means of condition $(\mathbf{C}_{\div_\leq})$. Then, for all $\alpha \in \mathcal{L}$, $\mathbf{K} - \alpha = \mathbf{K} \div \alpha$.*

[3]We notice that, however this result follows immediately from [Wil94a, Theorem 14], we provide a direct proof for it in the Appendix.

Proof: A proof for this observation can be found on page 197.

A comparison between brutal contractions and severe withdrawals analogous to the one that was presented above regarding ensconcement-based contractions and epistemic entrenchment-based contractions is in order.

The main conclusion, concerning the interrelation among brutal contractions and severe withdrawals, that follows from Observations 6.5.1 and 6.5.2 is that the class of severe withdrawals coincides with the class of brutal contractions on belief sets. This fact as well as some axiomatic characterizations for the above mentioned class of contractions are formally presented in the following theorem.

Theorem 6.5.7 *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a contraction on* $\mathbf{K}$*. The following statements are equivalent:*

1. $\div$ *is a severe withdrawal on* $\mathbf{K}$*.*
2. $\div$ *is a brutal contraction on* $\mathbf{K}$*.*
3. $\div$ *satisfies the postulates* $(\div 1)$*,* $(\div 2)$*,* $(\div 3)$*,* $(\div 3')$*,* $(\div 4)$ *and* $(\div 9)$*.*
4. $\div$ *satisfies the postulates of* relative closure, inclusion, vacuity, failure, success *and* strong inclusion*.*

Proof: A proof for this theorem can be found on page 198.

It follows immediately from the above theorem (and also from Observation 6.5.1) that severe withdrawals satisfy all the postulates included in the axiomatic characterization of brutal contractions presented in Theorems 6.2.1 and 6.2.2 (including the postulate of *uniform behaviour*, which is not necessary for the axiomatic characterization of brutal contractions on belief sets). Conversely, every brutal contraction on a belief set satisfies all the postulates included in the axiomatic characterization of severe withdrawals presented in Observation 3.2.49. However, brutal contractions on sets that are not logically closed, in general do not satisfy all those postulates, as we clarify in the following observation.

Observation 6.5.8 *Let* A *be a belief base and* $-$ *be a brutal contraction on* A*.*

(a) $-$ *satisfies postulates* $(\div 2)$*,* $(\div 3')$ *and* $(\div 4)$*.*

(b) If A *is not logically closed, then* $-$ *in general does not satisfy* $(\div 1)$ *nor* $(\div 3)$ *nor* $(\div 9)$*.*

Proof: A proof for this observation can be found on page 198.

The two following observations show in a direct way (rather than by means of a comparison among axiomatizations) that every brutal contraction on a belief set is a severe withdrawal and vice-versa, by means of a procedure analogous to the one that was used above to prove explicitly that a contraction function on a belief set is an epistemic entrenchment-based contraction if and only if it is an ensconcement-based contraction.

Observation 6.5.9 *Let* $\mathbf{K}$ *be a belief set. Let* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$ *and* $\div$ *be the* $\leq$*-based severe withdrawal on* $\mathbf{K}$ *defined by condition* ($\mathbf{R}_{\div_{\leq}}$)*. Let* $-$ *be the brutal contraction on* $\mathbf{K}$ *defined from* $\leq|_{\mathbf{K}}$ *by means of condition* ($\mathbf{BC}$)*. Then, for all* $\alpha \in \mathcal{L}$, $\mathbf{K} \div \alpha = \mathbf{K} - \alpha$.

Proof: A proof for this observation can be found on page 198.

Observation 6.5.10 *Let* $\mathbf{K}$ *be a belief set. Let* $(\mathbf{K}, \preceq)$ *be an ensconcement and* $-$ *be the* $\preceq$*-based brutal contraction on* $\mathbf{K}$ *defined by condition* ($\mathbf{BC}$)*. Let* $\leq_{\preceq}$ *be the epistemic entrenchment related to* $\mathbf{K}$ *defined from* $\preceq$ *as exposed in Observation* 5.2.16 *and* $\div$ *be the severe withdrawal on* $\mathbf{K}$ *defined from* $\leq_{\preceq}$ *by means of condition* ($\mathbf{R}_{\div_{\leq}}$)*. Then, for all* $\alpha \in \mathcal{L}$, $\mathbf{K} - \alpha = \mathbf{K} \div \alpha$.

Proof: A proof for this observation can be found on page 199.

To close this section we remark that the main conclusion that raises from the results presented here is the fact that in the belief sets context the class of ensconcement-based contractions coincides with the class of epistemic entrenchment-based contractions as well as the class of brutal contractions is identical to the class of severe withdrawals. These facts, together with the results of Subsection 5.2.4, contribute to assert that Williams' concept of ensconcement relation is an adequate generalization to the context of belief bases of Gärdenfors and Makinson's notion of epistemic entrenchment relation (on a belief set).

6.6 Summary

We have presented, in Section 6.2 axiomatic characterizations for the brutal and for the ensconcement-based contractions. The representation theorems obtained opened up the possibility of comparing the ensconcement-based contractions and the brutal contractions with each other as well as with some other well-know contraction operations through an axiomatic perspective. In particular, in Section 6.3, we have made such a comparison among the ensconcement-based contractions and the brutal contractions. That comparison allows us to conclude, in particular, that neither one of those two classes of functions contains the other. We have also presented a method for defining an ensconcement relation by means of either an ensconcement-based or a brutal contraction operation and we have show that, given a contraction of either one of those two types, the ensconcement thus obtained coincides exactly with the ensconcement on which the operation under consideration is based. Furthermore, we have also proven that, when considering only brutal contractions, the mentioned construction of an ensconcement relation from a contraction operation can be significantly simplified. In Section 6.4 we compare through an axiomatic perspective the contraction base function presented in Section 5.2, obtaining a map that relates the contraction functions there presented. Finally, in Section 6.5, we have presented some results relating base contraction postulates and belief set contraction postulates and we have investigated the connections between the ensconcement-based contractions and the epistemic entrenchment-based contractions as well as between the brutal contractions and the severe withdrawals. The main conclusion that we have achieved in this regard was that a contraction function on a belief set is an ensconcement-based contraction (respectively, a brutal contraction) if and only if it is an epistemic entrenchment-based contraction (respectively, a severe withdrawal).

Chapter 7

Shielded Contraction on Belief Bases

"Old beliefs die hard even when demonstrably false."
Edward O. Wilson

This chapter is dedicated to the study of shielded contraction on belief bases. As mentioned before, two of the main shortcomings pointed out to the AGM model of belief change are the (impractical) use of belief sets to represent belief states and the (unrealistic) acceptance of any new piece of information. In this chapter we study a kind of operators — known as shielded base contractions — which address both those issues. Indeed, on the one hand, these operators are defined on belief bases (rather than belief sets) and, on the other hand, they are constructed with the underlying idea that not all new information are accepted.

The motivation for the proposal of this kind of operators was the fact that, as pointed out by Rott [Rot92b], the *success* postulate is not a fully realistic requirement since an agent can have several (non-tautological) beliefs that he/she is not willing, for various reasons, to give up. Shielded contractions are operators that for some inputs behave just as (*standard*) contractions and for other inputs just do not have any effect at all — in the sense that simple return (as output) the belief state received as input.

In [FH01], a shielded contraction is defined by means of an AGM contraction and a set of sentences R satisfying certain properties, named *set of retractable sentences*, which models the set of sentences that the agent is willing to give up (if needed). Informally speaking, the shielded contraction is a function that receives (just as a *standard* contraction does) a belief set and a sentence and returns:

- The received belief set (unchanged), if the received sentence is not included in R;

- The output produced by the associated AGM contraction (when it receives those two inputs), if the received sentence is in R.

In the present chapter we shall study shielded contractions defined for belief bases (rather than for belief sets). Until now, only one class of such operators has been considered in the belief change literature, namely a class of shielded contractions on belief bases defined by means of a partial meet contraction and a set of retractable sentences (satisfying certain properties), which was presented and axiomatically characterized in [FMT03] (and which was recalled in Subsection 5.3.1). In this chapter we consider classes of shielded base contraction induced by several well-known kinds of *standard* contractions (not only partial meet contractions) and several kinds of sets of retractable sentences (*i.e.* we consider several different, and non-equivalent, sets of properties for characterizing a set of retractable sentences). We axiomatically characterize all the classes of shielded base contractions considered and study the interrelations among them, namely by investigating if each of those classes is or is not (strictly) contained in each one of the remaining classes considered.

7.1 Shielded base contractions

The basic idea of shielded contractions is to define a function in two steps. In the first step, one needs to define which sentences are retractable, *i.e.*, the sentences that an agent is willing to give up when performing a contraction. Afterwards the function should:
- leave the set of beliefs unchanged when contracting it by an irretractable sentence;
- work as a "standard" contraction when contracting by a retractable sentence.
The following definition presents a class of shielded base contractions which generalizes the class mentioned in Definition 5.3.3 in the sense that the underlying contraction does not need to be a partial meet contraction.

Definition 7.1.1 *Let $-$ be a contraction operator on a belief base A (i.e. an operator that satisfies success and inclusion). Let R be a set of sentences (the associated set of retractable sentences). Then $\sim$ is the* shielded base contraction *induced by $-$ and R if and only if:*

$$A \sim \alpha = \begin{cases} A - \alpha & \text{if } \alpha \in R \\ A & \text{otherwise} \end{cases}$$

7.1.1 Postulates for shielded base contractions

For convenience, we recall the postulates for (shielded) base contraction that we have already mentioned in Subsections 5.1.1 and 5.3.1.

(Success) If $\nvdash \alpha$ then $A \sim \alpha \nvdash \alpha$.
(Inclusion) $A \sim \alpha \subseteq A$.
(Failure) If $\vdash \alpha$ then $A \sim \alpha = A$.
(Vacuity) If $A \nvdash \alpha$, then $A \subseteq A \sim \alpha$.
(Uniformity) If it holds for all subsets A' of A that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$, then $A \sim \alpha = A \sim \beta$.
(Extensionality) If $\vdash \alpha \leftrightarrow \beta$, then $A \sim \alpha = A \sim \beta$.
(Relevance) If $\beta \in A$ and $\beta \notin A \sim \alpha$ then there is some set A' such that $A \sim \alpha \subseteq$

$A' \subseteq A$ and $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\beta\})$.
(Core-retainment) If $\beta \in A$ and $\beta \notin A \sim \alpha$ then there is some set A' such that $A' \subseteq A$ and $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\beta\})$.
(Disjunctive Elimination) If $\beta \in A$ and $\beta \notin A \sim \alpha$, then $A \sim \alpha \nvdash \alpha \vee \beta$.
(Relative Closure) $A \cap Cn(A \sim \alpha) \subseteq A \sim \alpha$.
(Relative Success) $A \sim \alpha = A$ or $\alpha \notin Cn(A \sim \alpha)$.
(Success Propagation) If $A \sim \beta \vdash \beta$ and $\vdash \beta \to \alpha$, then $A \sim \alpha \vdash \alpha$.
(Conjunctive Constancy) If $A \sim \alpha = A \sim \beta = A$, then $A \sim \alpha \wedge \beta = A$.
(Persistence) If $\beta \in Cn(A \sim \beta)$, then $\beta \in Cn(A \sim \alpha)$.

7.1.2 The set of retractable sentences

We start this subsection by recalling the properties proposed in [FH01, FMT03] for a set of retractable sentences R (these properties were already mentioned in Subsection 5.3.1).

Non-retractability Propagation: If $\alpha \notin R$, then $Cn(\alpha) \cap R = \emptyset$.
Conjunctive Completeness: If $\alpha \wedge \beta \in R$, then $\alpha \in R$ or $\beta \in R$.
Non-retractability Preservation: $\mathcal{L} \backslash R \subseteq Cn(A \sim \alpha)$.
Non-retractability of Tautology: $R \cap Cn(\emptyset) = \emptyset$.

We now propose some new properties that may naturally be required from a set of retractable sentences and, after that, we present some results exposing interrelations among different properties associated to sets of retractable sentences.

The following property states that two logically equivalent sentences should be both retractable or both irretractable.

Retractability of Logical Equivalents: If $\vdash \alpha \leftrightarrow \beta$, then $\alpha \in R$ if and only if $\beta \in R$.

Another property that is natural to expect is the following one, which attests that a conjunction is retractable as long as one of its conjuncts is so.

Converse Conjunctive Completeness: If $\alpha \in R$, then $\alpha \wedge \beta \in R$.

Additionally, we propose some other properties interconnecting R and A that are natural to expect.

The following property states that if two sentences α and β are implied by exactly the same subsets of A, then they are both retractable or both irretractable.

Uniform Retractability: If it holds for all subsets A' of A that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$, then $\alpha \in R$ if and only if $\beta \in R$.

The following property states that all irretractable sentences are deducible from the set to be contracted.

Non-retractability Upper Bounding: $\mathcal{L}\backslash R \subseteq Cn(A)$.

We notice that *non-retractability upper bounding* is equivalent to $\mathcal{L} \smallsetminus Cn(A) \subseteq R$ which is mentioned in the representation results of [FH01] that we have mentioned in Subsection 4.2.2.[1]

At this point we propose the following condition that relates a set of retractable sentences R and a contraction function $-$:

$$\text{If } \alpha \notin R \text{ and } \beta \in R, \text{ then } A - \beta \vdash \alpha. \qquad (\mathbf{R} - -)$$

The intuition underlying this condition is the following:
Ideally the irretractable sentences should not be removed from the belief base when a shielded contraction (by any sentence) is performed. Then, if a set (of retractable sentences) R and a contraction $-$ are intended to be used to define a shielded contraction it should hold that if a sentence is not in R (*i.e.*, is considered irretractable), then it should not be removed when using $-$ to contract by a sentence included in R (*i.e.*, by a sentence that is considered retractable).[2] The following example clarifies this reasoning.

Example 7.1.2 *Let* $A = \{p, q\}$ *and* $R = \mathcal{L}\backslash Cn(p)$.[3] *Consider a contraction operator* $-$ *such that* $A - (p \wedge q) = \{q\}$. *If* $\sim$ *is the shielded contraction induced by* $-$ *and* R, *it holds that:*

$$A \sim (p \wedge q) = A - (p \wedge q) = \{q\} \nvdash p.$$

This contradicts one of the main goals underlying the concept of "irretractability": irretractable sentences should be kept (more precisely, should be implied) after performing the related shielded contraction (by any sentence). We note that this happens because R *and* $-$ *do not satisfy condition* ($\mathbf{R} - -$). *Indeed,* $p \notin R$ *and* $p \wedge q \in R$ *but* $A - (p \wedge q) \nvdash p$.

From a different perspective, we can say that condition ($\mathbf{R} - -$) imposes that R and $-$ are such that the complement of R consists precisely of the sentences which are more difficult to remove by means of $-$.

Throughout this thesis we shall often say, by abuse of language, that R satisfies condition ($\mathbf{R} - -$), instead of saying that R and $-$ satisfy condition ($\mathbf{R} - -$).

---

[1] We note that more rigorously the expression "with respect to A" should be added to the designation of the last two properties presented (namely, *uniform retractability* and *non-retractability upper bounding*), since these relate R and A. However we will use the shorter designation of these properties since there is no risk of ambiguity whenever these properties are mentioned along this thesis.

[2] Note that if $\beta \notin R$ and $\sim$ is a shielded base contraction operator on a set A induced by a contraction operator and R, then $A \sim \beta = A$. Therefore if R satisfies *non-retractability upper bounding* then $A \sim \beta \vdash \alpha$ for every irretractable sentence α.

[3] Note that the set R satisfies *non-retractability of tautology, non-retractability propagation, conjunctive completeness, uniform retractability* and *non-retractability upper bounding.*

Interrelations among properties for the set of retractable sentences

The following observation highlights that if a set R, of retractable sentences, satisfies *non-retractability of tautology*, *non-retractability propagation* and *conjunctive completeness*, then its complement $\mathcal{L} \setminus R$, the set of irretractable sentences, is logically closed.

Observation 7.1.3 *Let R be a set that satisfies non-retractability of tautology, non-retractability propagation and conjunctive completeness. Then:*
$\alpha \in \mathcal{L} \setminus R$ *if and only if* $\mathcal{L} \setminus R \vdash \alpha$.

Proof: A proof for this observation can be found on page 202.

We note that, if $R \neq \mathcal{L}$ satisfies *non-retractability propagation*, then it is not logically closed *i.e.*, the fact that $R \vdash \alpha$ does not imply that $\alpha \in R$. To see this it is enough to note that if $p \in R$ and $q \notin R$, then $R \vdash p \vee q$. But, by *non-retractability propagation*, $p \vee q \notin R$.

The following observation illustrates that *retractability of logical equivalents* follows from *uniform retractability* and also follows from *non-retractability propagation*. It also highlights that *converse conjunctive completeness* is implied by *non-retractability propagation*.

Observation 7.1.4 *Let R be a set of sentences. Then:*

(a) *If R satisfies uniform retractability, then R satisfies retractability of logical equivalents.*

(b) *If R satisfies non-retractability propagation, then R satisfies retractability of logical equivalents and converse conjunctive completeness.*

Proof: A proof for this observation can be found on page 202.

The following theorem presents an explicit definition for the set of retractable sentences R in terms of the shielded contraction operator $\sim$, which is induced by it, provided that R satisfies *non-retractability of tautology* and *non-retractability upper bounding*.

Theorem 7.1.5 *Let A be a belief base and $\sim$ be a shielded base contraction induced by a contraction operator on A and a set $R \subseteq \mathcal{L}$. Then R satisfies non-retractability of tautology and non-retractability upper bounding if and only if* $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$.

Proof: A proof for this theorem can be found on page 203.

The following observation exposes that a shielded contraction $\sim$ is induced by a contraction operator $-$ and a set R that satisfies *non-retractability upper bounding* and which are such that condition (**R - $-$**) holds, if and only if R satisfies non-retractability preservation.

Observation 7.1.6 *Let A be a belief base and $\sim$ be a shielded base contraction induced by a contraction operator $-$ on A and a set $R \subseteq \mathcal{L}$. Then, R satisfies non-retractability preservation if and only if R satisfies non-retractability upper bounding and $-$ and R satisfy condition (**R - −**).*

Proof: A proof for this observation can be found on page 203.

The following observation exposes some other relations among the properties of a set of retractable sentences.

Observation 7.1.7 *Let A be a belief base and $\sim$ be a shielded base contraction induced by a contraction operator on A and a set $R \subseteq \mathcal{L}$. If R satisfies non-retractability preservation and non-retractability of tautology, then R satisfies conjunctive completeness, non-retractability propagation, uniform retractability and retractability of logical equivalents.*

Proof: A proof for this observation can be found on page 204.

7.2 Relations between base contractions and shielded base contractions

The following theorem illustrates some properties that an operator of shielded base contraction, induced by a contraction operator $-$ and a set R, satisfies whenever $-$ and R satisfy some given properties.

Theorem 7.2.1 *Let A be a belief base, $-$ be a contraction on A, $R \subseteq \mathcal{L}$, and $\sim$ be the shielded base contraction induced by $-$ and R. Then:*[4]

(a) *It holds that:*

[4] The schema presented in this theorem (and whenever a similar schema is used) should be read as follows: Let A be a belief base, $\div$ be a contraction on A, $R \subseteq \mathcal{L}$, and $\sim$ be the shielded base contraction induced by $\div$ and R. Then:

(a) It holds that:

1. $\sim$ satisfies inclusion.
2. If $\div$ satisfies vacuity, then $\sim$ satisfies vacuity.
3. If $\div$ satisfies failure, then $\sim$ satisfies relative success.
   ...

...

If − satisfies	***then ~ satisfies***
—	*inclusion*
vacuity	*vacuity*
failure	*relative success*
relative closure	*relative closure*
relevance	*relevance*
core-retainment	*core-retainment*
disjunctive elimination	*disjunctive elimination*

(b) *If R and − satisfy condition (***R - −***), then:*

If − satisfies	***then ~ satisfies***
—	*persistence*
failure and extensionality	*extensionality*
failure and uniformity	*uniformity*

(c) *If R satisfies non-retractability preservation, then:*

If − satisfies	***then ~ satisfies***
—	*persistence*
failure and extensionality	*extensionality*
failure and uniformity	*uniformity*

(d) *If R satisfies non-retractability of tautology, then:*

If − satisfies	***then ~ satisfies***
—	*relative success*

(e) *If R satisfies retractability of logical equivalents, then:*

If − satisfies	***then ~ satisfies***
extensionality	*extensionality*

(f) *If R satisfies uniform retractability, then:*

If − satisfies	***then ~ satisfies***
uniformity	*uniformity*

(g) *If R satisfies non-retractability propagation, then:*

If − satisfies	***then ~ satisfies***
—	*success propagation*
extensionality	*extensionality*

(h) *If R satisfies conjunctive completeness and retractability of logical equivalents (or uniform retractability), then:*

If − satisfies	***then ~ satisfies***
vacuity and failure	*conjunctive constancy*

(i) *If R satisfies non-retractability propagation and conjunctive completeness, then:*

If* − *satisfies	***then* ~ *satisfies***
vacuity and extensionality (or uniformity)	*conjunctive constancy*

Proof: A proof for this theorem can be found on page 204.

At this point we note that, given a shielded contraction ~, we can use it to define a set R of sentences that may be considered retractable in that context. As Theorem 7.1.5 suggests, a natural way to define such a set is by $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$. The next theorem illustrates some properties that this set R satisfies whenever ~ satisfies some of the postulates presented in Subsection 7.1.1.

Theorem 7.2.2 *Let A be a belief base, ~ be an operator on A and $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$. Then:*

(a) *It holds that:*

If* ~ *satisfies	***then* R *satisfies***
—	*non-retractability of tautology*
inclusion	*non-retractability upper bounding*
extensionality	*retractability of logical equivalents*
inclusion and uniformity	*uniform retractability*
success propagation	*non-retractability propagation*
relative success and conjunctive constancy	*conjunctive completeness*
persistence	*non-retractability preservation, conjunctive completeness, non-retractability propagation and retractability of logical equivalents*
persistence and inclusion	*uniform retractability*

(b) *If ~ is a shielded base contraction induced by a contraction operator − and R, then:*

If* ~ *satisfies	***then* R *satisfies***
persistence	*non-retractability upper bounding and condition (**R - −**)*

Proof: A proof for this theorem can be found on page 207.

We have already explored the properties that an operator of shielded base contraction induced by an operator − and a set R satisfies whenever − and R satisfy some given properties. In the next theorem we see that it is possible to construct an operator − in terms of ~ and $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$ and investigate the properties that such an operator satisfies taking into account the properties satisfied by ~.

Theorem 7.2.3 *Let A be a belief base, $\sim$ be an operator on A and $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$. Then there exists an operator $-$ on A such that:*

(a) *If $\sim$ satisfies relative success and inclusion, then $-$ is a contraction operator and $\sim$ is the shielded base contraction induced by $-$ and R.*

(b) *It holds that:*

If $\sim$ satisfies	***then $-$ satisfies***
vacuity	*vacuity*
extensionality	*extensionality*
failure	*failure*
relative closure	*relative closure*
inclusion and uniformity	*uniformity*
relevance	*relevance*
core-retainment	*core-retainment*
disjunctive elimination	*disjunctive elimination*

(c) *If $\sim$ satisfies persistence, then $-$ and R satisfy condition (**R - $-$**).*

Proof: A proof for this theorem can be found on page 208.

7.3 Axiomatic characterizations of different kinds of shielded base contractions

In this section we propose and axiomatically characterize several classes of shielded contraction functions. We start by presenting, in Subsection 7.3.1 a representation theorem for the operators that satisfy *relative success* and *inclusion*, which constitute the most general class of shielded contractions that we will consider. Afterwards in Subsections 7.3.2 – 7.3.4 we consider some more specific classes of shielded contractions. More precisely we consider the classes of shielded contractions induced by different well-known kinds of contraction functions (namely, partial meet, (smooth) kernel and basic AGM-generated base contractions) and by several alternative types of sets of retractable sentences (*i.e.* considering different sets of properties associated to the related set of retractable sentences). We note that the proofs provided in the Appendix for the representation theorems included here rely very strongly on the results presented in the previous section.

All the Subsections 7.3.2–7.3.4 have a similar structure. In each one of these subsections firstly a theorem is presented which provides axiomatic characterizations for five classes of shielded contractions all induced by a same kind of contraction function but each of them with a different type of associated set of retractable sentences. Afterwards, a definition is presented where a designation is proposed for each one of the operators that were axiomatically characterized in the previously presented theorem. The reason for providing a designation for each of the considered classes of operators only after presenting the corresponding axiomatic characterization is the fact that the designation proposed for each class is based on the names of some of the postulates that are included in that axiomatic characterization.

7.3.1 Basic shielded base contractions

In Definition 7.1.1 we have introduced, from a constructive perspective, the concept of shielded base contraction induced by a contraction operator $-$ and a set of retractable sentences R. At this point we notice that if, alternatively, we wished to define shielded base contractions in terms of postulates then, having in mind (i) the definition for contraction recalled in Definition 5.1.1, and (ii) the postulates for shielded contraction presented in Subsection 7.1.1, the most natural proposal would be to define a shielded base contraction as an operator that satisfies the postulates of *relative success* and *inclusion.*

It follows trivially from Definition 7.1.1 that any shielded contraction satisfies inclusion. However, somehow surprisingly, a shielded contraction operator induced by a (general) contraction operator $-$ (*i.e.* an operator $-$ that satisfies success and inclusion) and a set of retractable sentences $R \subseteq \mathcal{L}$ (which is not required to satisfy any properties at all) does not satisfy the postulate of relative success. Indeed, in order to assure that a shielded base contraction $\sim$ induced by a contraction $-$ and a set R satisfies relative success, it is necessary to impose that $-$ satisfies the postulate of *failure* and/or that the set R satisfies *non-retractability of tautology.* This fact is formally stated in the following theorem.

Theorem 7.3.1 *Let A be a belief base and $\sim$ an operator on A. Then the following conditions are equivalent:*

1. *$\sim$ satisfies relative success and inclusion.*
2. *$\sim$ is an operator of shielded base contraction induced by a contraction operator and a set $R \subseteq \mathcal{L}$ that satisfies non-retractability of tautology.*
3. *$\sim$ is an operator of shielded base contraction induced by a contraction operator that satisfies failure and a set $R \subseteq \mathcal{L}$.*

Proof: A proof for this theorem can be found on page 209.

Having in mind the above theorem and the discussion that proceeded it, we are led to consider that the most general kind of shielded contractions that are worth considering (in the sense that these are the most general operators that satisfy the minimal set of properties that are intuitively associated to the notion of shielded contraction) are the ones that we introduce in the following definition.

Definition 7.3.2 *A shielded base contraction $\sim$ on a belief base A induced by a contraction $-$ and a set $R \subseteq \mathcal{L}$ is a* basic shielded contraction *if and only if $-$ satisfies* failure *or R satisfies* non-retractability of tautology*.*

In the following result we present an axiomatic characterization for the basic shielded contractions, which follows trivially from Theorem 7.3.1.

Corollary 7.3.3 *Let ~ be an operator on A. Then ~ is a basic shielded contraction if and only if ~ satisfies relative success and inclusion.*

In the remainder of this section we will obtain axiomatic characterizations for other less general classes of shielded contractions (which are strict subclasses of the class of basic shielded contractions). More precisely, we will consider the shielded contractions on belief bases induced by partial meet contractions, by (smooth) kernel contractions and by basic AGM-generated base contractions and, additionally, we will take into account different sets of properties regarding the associated set of retractable sentences R.

7.3.2 Shielded partial meet base contractions

The following theorem presents axiomatic characterizations for five kinds of operators of shielded base contraction. All these operators are induced by partial meet contractions but each one of them has a different type of associated set of retractable sentences.

Theorem 7.3.4 *Let A be a belief base and ~ an operator on A. Then:*

~ is an operator of shielded base contraction induced by a partial meet contraction − and a set $R \subseteq \mathcal{L}$ that satisfies	***if and only if ~ satisfies relative success, inclusion, uniformity, relevance and***
uniform retractability	—
uniform retractability and non-retractability propagation	*success propagation*
uniform retractability and conjunctive completeness	*conjunctive constancy*
uniform retractability, non-retractability propagation and conjunctive completeness	*success propagation and conjunctive constancy*
condition (**R - −**)	*persistence*

Proof: A proof for this theorem can be found on page 210.

In the following definition we attribute designations to the different kinds of shielded contractions that were axiomatically characterized in the above theorem.

Definition 7.3.5 *A shielded base contraction ~ on a belief base A induced by a partial meet contraction − and a set $R \subseteq \mathcal{L}$ is a:*

Designation	***if and only if R satisfies***
Shielded partial meet contraction (SPMC)	*uniform retractability*
Success propagant shielded partial meet contraction (SP-SPMC)	*unif. retractability and non-retractability propagation*

Continued on next page

Continued from previous page

Conjunctive constant shielded partial meet contraction (CC-SPMC)	*unif. retractability and conjunctive completeness*
Success propagant conjunctive constant shielded partial meet contraction (SP+CC-SPMC)	*unif. retractability, non-retr. propagation and conj. completeness*
Persistent shielded partial meet contraction (P-SPMC)	*condition* (**R** - −)

Throughout this chapter we will sometimes use the acronyms introduced in the first column of the table above, to designate the whole class of the corresponding kind of operators (instead of only one of the elements of that class). It will always be clear by the context whether the acronym is being used to denote a class of operators or a single operator.

It is worth to mention here that, since in the proof of the right-to-left part of Theorem 7.3.4 we used the set $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$, it follows from Observation 7.1.4 (a) and Theorem 7.2.2 that in each row of the tables presented in Theorem 7.3.4 and in Definition 7.3.5, to the list of properties of R (there presented) we can add the following ones: *non-retractability of tautology, non-retractability upper-bounding* and *retractability of logical equivalents.* Furthermore, in the case of the last row, besides the three properties mentioned above, *uniform retractability, non-retractability propagation, conjunctive completeness* and *non-retractability preservation* can also be added to the list of properties of R (there presented). In fact, according to Observation 7.1.6, in the last row of the tables presented in Theorem 7.3.4 and in Definition 7.3.5, condition (**R** - −) can even be replaced by *non-retractability preservation.*

We note that it follows from the above remark that there are several alternative (equivalent) definitions for the classes introduced in Definition 7.3.5, more precisely several equivalent definitions, but each one with a different set of properties associated to the set R.

This same situation occurs regarding the classes of shielded contractions introduced in Definitions 7.3.10, 7.3.14 and 7.3.18 (that will appear in the following subsections).

The following corollary follows trivially from the above definition and Theorem 7.3.4.

Corollary 7.3.6 *Let ~ be an operator on A. Then:*

~ is a	***if and only if ~ satisfies relative success, inclusion, uniformity, relevance and***
SPMC	—
SP-SPMC	*success propagation*
CC-SPMC	*conjunctive constancy*
SP+CC-SPMC	*success propagation and conjunctive constancy*
P-SPMC	*persistence*

At this point it is worth to notice that the axiomatic characterization given in the last row of the above table is identical to the axiomatic characterization presented in [FMT03] (that we recalled in Observation 5.3.4). Therefore, the class formed by the operators designated in Definition 7.3.5 by *persistent shielded partial meet contraction* coincides with the class of shielded contractions that was axiomatically characterized in [FMT03].

Next we present examples of shielded contraction operators that belong to some of the classes introduced in Definition 7.3.5 but do not belong to others. These examples allow us to conclude that the classes mentioned in Definition 7.3.5 are all different from each other.

Example 7.3.7 *Let $A = \{p, q\}$ and $-$ be a partial meet base contraction on A such that $A - (p \wedge q) = \{p\}$ and $A - (p \vee q) = \varnothing$. Let $\sim$ be the operator of shielded base contraction induced by $-$ and a set R.*

(a) *If $R = \mathcal{L} \setminus (Cn(p) \cup Cn(q))$, then R satisfies* uniform retractability *and* non-retractability propagation.[5] *Therefore, by Definition 7.3.5, $\sim$ is a SPMC and a SP-SPMC. On the other hand it holds that $p \notin R$, $q \notin R$ and $p \wedge q \in R$. Hence $A \sim p = A \sim q = A$ but $A \sim (p \wedge q) = A - (p \wedge q) = \{p\} \neq A$. Thus $\sim$ does not satisfy* conjunctive constancy. *Therefore, according to Corollary 7.3.6, $\sim$ is not a CC-SPMC nor a SP+CC-SPMC.*

(b) *If $R = \mathcal{L} \setminus (Cn(\varnothing) \cup (Cn(p) \setminus Cn(q)))$, then R satisfies* uniform retractability *and* conjunctive completeness.[6]
Therefore, according to Definition 7.3.5 $\sim$ is a SPMC and a CC-SPMC. On the other hand it holds that $p \vee q \in R$ and $p \notin R$. From the latter it follows that $A \sim p = A$, thus $A \sim p \vdash p$. It holds that $\vdash p \rightarrow (p \vee q)$ and that $A \sim (p \vee q) = A - (p \vee q) = \varnothing$. Hence $A \sim (p \vee q) \nvdash p \vee q$. Thus $\sim$ does not satisfy success propagation. *Therefore, according to Corollary 7.3.6, $\sim$ is not a SP-SPMC nor a SP+CC-SPMC.*

(c) *If $R = \mathcal{L} \setminus Cn(q)$, then R satisfies* conjunctive completeness, non-retractability propagation *and* uniform retractability.[7]
Therefore, according to Definition 7.3.5 $\sim$ is a SP+CC-SPMC (and also a SP-SPMC, a CC-SPMC and a SPMC). On the other hand it holds that $p \wedge q \in R$ and $q \notin R$. Hence $A \sim q = A \vdash q$ and $A \sim (p \wedge q) = A - (p \wedge q) \nvdash q$. Thus $\sim$ does not satisfy persistence. *Therefore, according to Corollary 7.3.6, $\sim$ is not a P-SPMC.*

Next we establish the relations between the shielded base contractions introduced in Definition 7.3.5. It is assumed that the classes of operators mentioned in each item of the following observation are formed by operators defined on the same belief base. The same also applies regarding Observations 7.3.12, 7.3.16 and 7.3.20.

[5] See Lemma E.3 in the Appendix.
[6] See Lemma E.4 in the Appendix.
[7] See Lemma E.5 in the Appendix.

Observation 7.3.8

(a) *P-SPMC* $\subset$ *SP+CC-SPMC.*

(b) *SP+CC-SPMC* $\subset$ *CC-SPMC.*

(c) *SP+CC-SPMC* $\subset$ *SP-SPMC.*

(d) *CC-SPMC* $\nsubseteq$ *SP-SPMC and SP-SPMC* $\nsubseteq$ *CC-SPMC.*

(e) *CC-SPMC* $\subset$ *SPMC.*

(f) *SP-SPMC* $\subset$ *SPMC.*

Proof: A proof for this observation can be found on page 211.

In Figure 7.1 we present a diagram that summarizes the results established in the above observation. In that diagram an arrow between two boxes symbolizes that the class of shielded contractions at the origin of the arrow is a strict subclass of the class of shielded contractions at the end of that arrow.

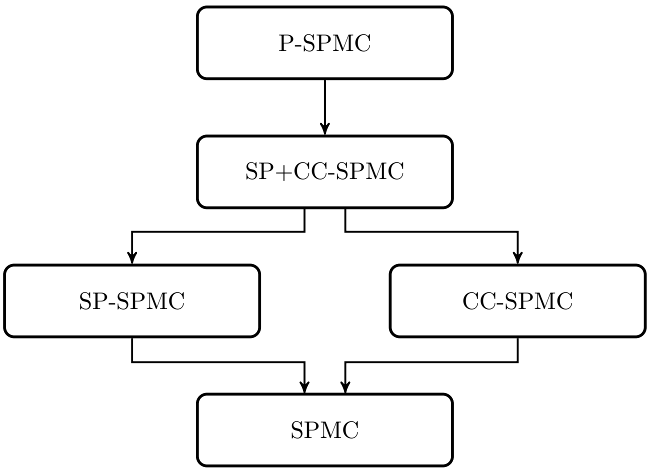

Figure 7.1: Map among different classes of shielded base contraction functions induced by partial meet contractions.

7.3.3 Shielded kernel base contractions

The following representation theorem axiomatically characterizes five kinds of operators of shielded base contraction. All these operators are induced by kernel contractions but each one of them has a different type of associated set of retractable sentences.

Theorem 7.3.9 *Let A be a belief base and $\sim$ an operator on A. Then:*

$\sim$ is an operator of shielded base contraction induced by a kernel contraction $-$ and a set $R \subseteq \mathcal{L}$ that satisfies	**if and only if $\sim$ satisfies relative success, inclusion, uniformity, core-retainment and**
uniform retractability	—
uniform retractability and non-retractability propagation	*success propagation*
uniform retractability and conjunctive completeness	*conjunctive constancy*
uniform retractability, non-retractability propagation and conjunctive completeness	*success propagation and conjunctive constancy*
condition (**R - $-$**)	*persistence*

Proof: A proof for this theorem can be found on page 210.

In the next definition we introduce designations for the different kinds of shielded contractions that were axiomatically characterized in the above theorem.

Definition 7.3.10 *A shielded base contraction $\sim$ on a belief base A induced by a kernel contraction $-$ and a set $R \subseteq \mathcal{L}$ is a:*

Designation	**if and only if R satisfies**
Shielded kernel contraction (SKC)	*uniform retractability*
Success propagant shielded kernel contraction (SP-SKC)	*unif. retractability and non-retractability propagation*
Conjunctive constant shielded kernel contraction (CC-SKC)	*unif. retractability and conjunctive completeness*
Success propagant conjunctive constant shielded kernel contraction (SP+CC-SKC)	*unif. retractability, non-retr. propagation and conj. completeness*
Persistent shielded kernel contraction (P-SKC)	*condition* (**R - $-$**)

As expected the difference between the axiomatic characterizations of the classes of shielded partial meet contractions, presented in the previous subsection, and the axiomatic characterizations of the classes of shielded kernel contractions, presented in Theorem 7.3.9, is the replacement of *relevance* by *core-retainment*, which also means, by Observation 5.1.2, that every class of shielded partial meet contractions is a subclass of the corresponding class of shielded kernel contractions, *i.e.* P-SPMC $\subseteq$ P-SKC, CC-SPMC $\subseteq$ CC-SKC, SP-SPMC $\subseteq$ SP-SKC, SP+CC-SPMC $\subseteq$ SP+CC-SKC and SPMC $\subseteq$ SKC.
The following corollary follows trivially from Definition 7.3.10 and Theorem 7.3.9.

Corollary 7.3.11 *Let $\sim$ be an operator on A. Then:*

$\sim$ is a	***if and only if $\sim$ satisfies relative success, inclusion, uniformity, core-retainment and***
SKC	—
SP-SKC	*success propagation*
CC-SKC	*conjunctive constancy*
SP+CC-SKC	*success propagation and conjunctive constancy*
P-SKC	*persistence*

In the following observation we establish the relations between the shielded base contractions introduced in Definition 7.3.10.

Observation 7.3.12

(a) *P-SKC* $\subset$ *SP+CC-SKC.*

(b) *SP+CC-SKC* $\subset$ *CC-SKC.*

(c) *SP+CC-SKC* $\subset$ *SP-SKC.*

(d) *CC-SKC* $\nsubseteq$ *SP-SKC and SP-SKC* $\nsubseteq$ *CC-SKC.*

(e) *CC-SKC* $\subset$ *SKC.*

(f) *SP-SKC* $\subset$ *SKC.*

Proof: A proof for this observation can be found on page 211.

In Figure 7.2 we present a diagram that summarizes the results established in the above observation. We note that a similar diagram was presented in Figure 7.1 regarding the classes of shielded contractions functions induced by partial meet base contractions.

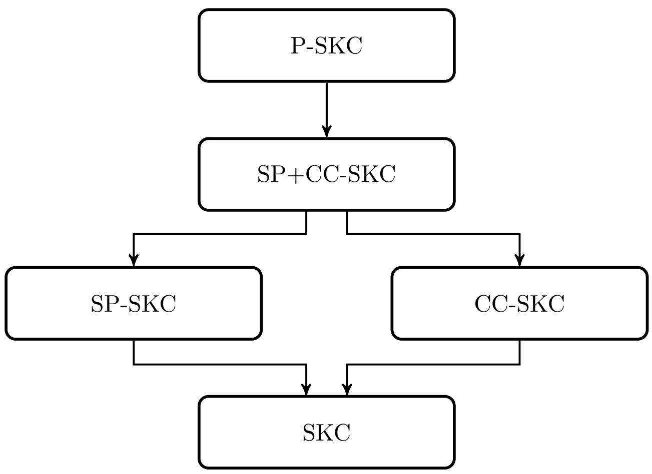

Figure 7.2: Map among different classes of shielded base contraction functions induced by kernel contractions.

Shielded smooth kernel base contractions

The following theorem presents axiomatic characterizations for five kinds of operators of shielded base contraction that are all induced by smooth kernel contractions but each one of them has a different type of associated set of retractable sentences.

Theorem 7.3.13 *Let A be a belief base and $\sim$ an operator on A. Then:*

$\sim$ is an operator of shielded base contraction induced by a smooth kernel contraction $-$ and a set $R \subseteq \mathcal{L}$ that satisfies	***if and only if $\sim$ satisfies relative success, inclusion, uniformity, core-retainment, relative closure and***
uniform retractability	—
uniform retractability and non-retractability propagation	*success propagation*
uniform retractability and conjunctive completeness	*conjunctive constancy*
uniform retractability, non-retractability propagation and conjunctive completeness	*success propagation and conjunctive constancy*
condition (**R - $-$**)	*persistence*

Proof: A proof for this theorem can be found on page 210.

In the following definition we introduce designations for the different kinds of shielded contractions that were axiomatically characterized in the above theorem.

Definition 7.3.14 *A shielded base contraction $\sim$ on a belief base A induced by a smooth kernel contraction $-$ and a set $R \subseteq \mathcal{L}$ is a:*

Designation	***if and only if R satisfies***
Shielded smooth kernel contraction (SSKC)	*uniform retractability*
Success propagant shielded smooth kernel contraction (SP-SSKC)	*unif. retractability and non-retractability propagation*
Conjunctive constant shielded smooth kernel contraction (CC-SSKC)	*unif. retractability and conjunctive completeness*
Success propagant conjunctive constant shielded smooth kernel contraction (SP+CC-SSKC)	*unif. retractability, non-retr. propagation and conj. completeness*
Persistent shielded smooth kernel contraction (P-SSKC)	*condition* (**R - $-$**)

The following corollary follows trivially from Definition 7.3.14 and Theorem 7.3.13.

Corollary 7.3.15 *Let $\sim$ be an operator on A. Then:*

$\sim$ ***is a***	***if and only if*** $\sim$ ***satisfies relative success, inclusion, uniformity, core-retainment, relative closure and***
SSKC	—
SP-SSKC	*success propagation*
CC-SSKC	*conjunctive constancy*
SP+CC-SSKC	*success propagation and conjunctive constancy*
P-SSKC	*persistence*

In the following observation we establish the relations between the classes of shielded base contractions introduced in Definition 7.3.14.

Observation 7.3.16

(a) *P-SSKC* $\subset$ *SP+CC-SSKC.*

(b) *SP+CC-SSKC* $\subset$ *CC-SSKC.*

(c) *SP+CC-SSKC* $\subset$ *SP-SSKC.*

(d) *CC-SSKC* $\nsubseteq$ *SP-SSKC and SP-SSKC* $\nsubseteq$ *CC-SSKC.*

(e) *CC-SSKC* $\subset$ *SSKC.*

(f) *SP-SSKC* $\subset$ *SSKC.*

Proof: A proof for this observation can be found on page 212.

In Figure 7.3 we present a diagram that summarizes the results established in the above observation. Similar diagrams were presented in Figures 7.1 and 7.2 regarding, respectively, the classes of shielded contractions functions induced by partial meet base contractions and by kernel contractions.

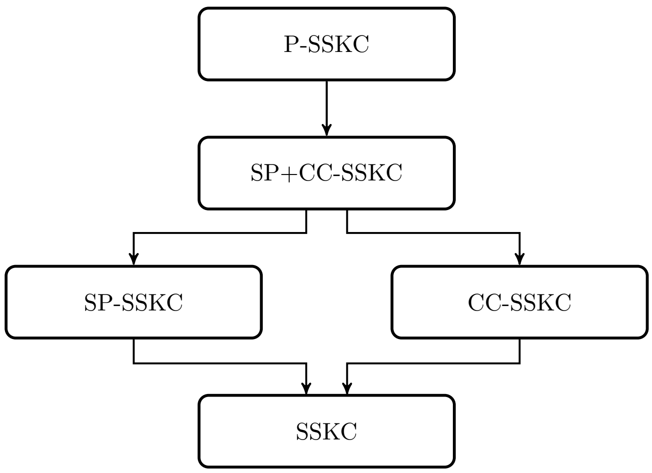

Figure 7.3: Map among different classes of shielded base contraction functions induced by smooth kernel contractions.

7.3.4 Shielded basic AGM-generated base contractions

The following representation theorem axiomatically characterizes five kinds of operators of shielded contraction. All these operators are induced by basic AGM-generated base contractions but each one of them has a different type of associated set of retractable sentences.

Theorem 7.3.17 *Let A be a belief base and $\sim$ an operator on A. Then:*

$\sim$ is an operator of shielded base contraction induced by a basic AGM-generated base contraction $-$ and a set $R \subseteq \mathcal{L}$ that satisfies	***if and only if $\sim$ satisfies relative success, inclusion, vacuity, extensionality, disjunctive elimination and***
retractability of logical equivalents	—
non-retractability propagation	*success propagation*
retr. of logical equivalents and conjunctive completeness	*conjunctive constancy*
non-retractability propagation and conjunctive completeness	*success propagation and conjunctive constancy*
condition (**R - $-$**)	*persistence*

Proof: A proof for this theorem can be found on page 212.

In the following definition we introduce designations for the different kinds of shielded contractions that were axiomatically characterized in the above theorem.

Definition 7.3.18 *A shielded base contraction $\sim$ on a belief base A induced by a basic AGM-generated base contraction $-$ and a set $R \subseteq \mathcal{L}$ is a:*

Designation	***if and only if R satisfies***
Shielded basic AGM-generated base contraction (SbAGMC)	*retractability of logical equivalents*
Success propagant shielded basic AGM-generated base contraction (SP-SbAGMC)	*non-retractability propagation*
Conjunctive constant shielded basic AGM-generated base contraction (CC-SbAGMC)	*retractability of logical equivalents and conjunctive completeness*
Success propagant conjunctive constant shielded basic AGM-generated base contraction (SP+CC-SbAGMC)	*non-retr. propagation and conj. completeness*
Persistent shielded basic AGM-generated base contraction (P-SbAGMC)	*condition* (**R - $-$**)

It is worth to notice that in the proof of the right-to-left part of Theorem 7.3.17 it was used the set $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$. Therefore, from Theorem 7.2.2, it follows that in each row of the tables presented in Theorem 7.3.17 and in Definition 7.3.18, to the list of properties of R (there presented) we can add: *non-retractability of tautology* and *non-retractability upper-bounding*. We can also add *retractability of logical equivalents* to the list of properties of R presented in rows 2, 4 and 5. Furthermore, *uniform retractability, non-retractability propagation* and *conjunctive completeness* can be also added to the list of properties of R presented in the last row. In fact, according to Observation 7.1.6, in the last row condition (**R - –**) can even be replaced by *non-retractability preservation*.

The following corollary follows trivially from Definition 7.3.18 and Theorem 7.3.17.

Corollary 7.3.19 *Let $\sim$ be an operator on A. Then:*

$\sim$ is a	***if and only if $\sim$ satisfies relative success, inclusion, vacuity, extensionality, disjunctive elimination and***
SbAGMC	—
SP-SbAGMC	*success propagation*
CC-SbAGMC	*conjunctive constancy*
SP+CC-SbAGMC	*success propagation and conjunctive constancy*
P-SbAGMC	*persistence*

In the following observation we establish the relations between the classes of shielded base contractions introduced in Definition 7.3.18.

Observation 7.3.20

(a) *P-SbAGMC $\subset$ SP+CC-SbAGMC.*

(b) *SP+CC-SbAGMC $\subset$ CC-SbAGMC.*

(c) *SP+CC-SbAGMC $\subset$ SP-SbAGMC.*

(d) *CC-SbAGMC $\nsubseteq$ SP-SbAGMC and SP-SbAGMC $\nsubseteq$ CC-SbAGMC.*

(e) *CC-SbAGMC $\subset$ SbAGMC.*

(f) *SP-SbAGMC $\subset$ SbAGMC.*

Proof: A proof for this observation can be found on page 213.

In Figure 7.4 we present a diagram that summarizes the results established in the above observation. This diagram is similar to the ones presented in Figures 7.1, 7.2 and 7.3 regarding, respectively, the classes of shielded base contractions functions induced by partial meet contractions, by kernel contractions and by smooth kernel contractions.

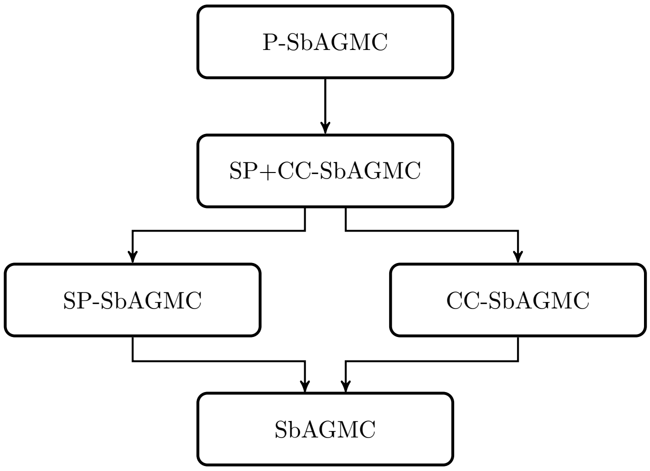

Figure 7.4: Map among different classes of shielded base contraction functions induced by basic AGM-generated base contractions.

Table 7.1 summarizes the results obtained in the representation theorems presented in this section. Given a shielded contraction ~ the white cells that are on the top of the same column represent the properties that R (the associated set of retractable sentences) satisfies. The white cells that are placed on the right of the same row indicate the properties that ~ satisfies. Considering, for example, the class SP+CC-SSKC, by observing this table we can see that these operators satisfy *success propagation, conjunctive constancy, relative closure, uniformity, core-retainment, relative success, inclusion, vacuity* and *extensionality*. Furthermore, we can also see that a SP+CC-SSKC is a shielded contraction induced by (a smooth kernel contraction and) a set of retractable sentences that satisfies *non-retractability propagation, conjunctive completeness, uniform-retractability* and *retractability of logical equivalents*.

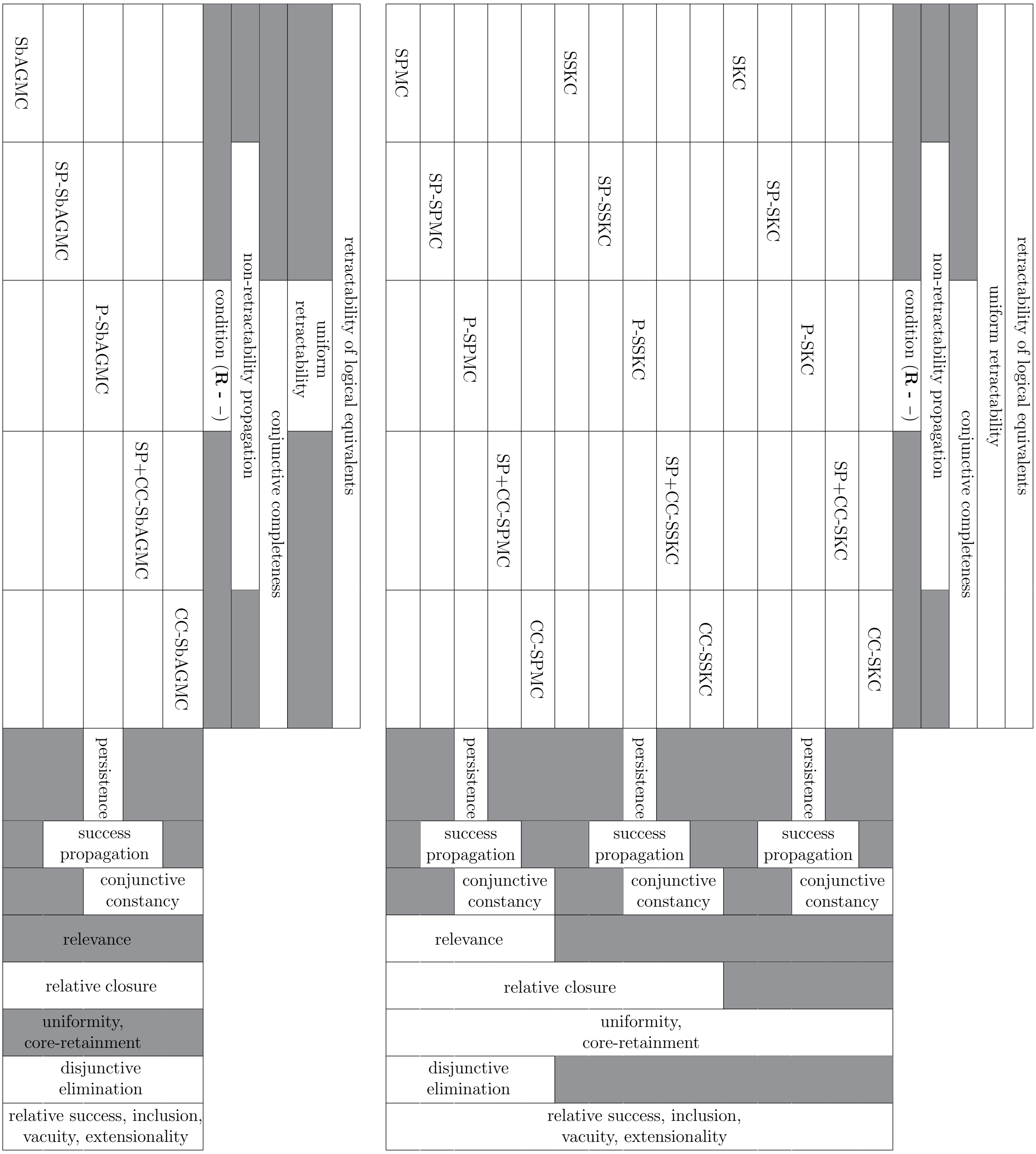

Table 7.1: Schematic representation of the main postulates satisfied by each one of the twenty classes of shielded contraction considered and also of the properties satisfied by the set of retractable sentences by which each of those operators is induced.

7.4 More maps between classes of shielded base contraction functions

In this section we study the interrelations among classes of shielded contractions induced by different kinds of contractions. Throughout this section we assume all classes of shielded contraction mentioned are formed by operators defined on the same belief base.

We start by presenting an example that shows that P-SSKC $\nsubseteq$ SPMC and that P-SKC $\nsubseteq$ SSKC.

Example 7.4.1 (Adapted from [FKR08, Example 22])
Let $A = \{p, p \vee q, p \leftrightarrow q, r\}$ *and* $R = \mathcal{L} \backslash Cn(\varnothing)$*. It holds that* $A \perp\!\!\!\perp (p \wedge q) = \{\{p, p \leftrightarrow q\}, \{p \vee q, p \leftrightarrow q\}\}$*.*

(a) Let $-$ *be a smooth kernel contraction based on a smooth incision function* σ_1 *such that* $\sigma_1(A \perp\!\!\!\perp (p \wedge q)) = \{p, p \leftrightarrow q\}$*. Let* $\sim$ *be the shielded base contraction induced by* $-$ *and* R*. It holds that* R *and* $-$ *satisfy condition* ($\mathbf{R}$ **-** $-$)*. Thus, by Definition 7.3.14,* $\sim$ *is a P-SSKC. On the other hand,* $p \wedge q \in R$*. Thus* $A \sim (p \wedge q) = A - (p \wedge q) = \{p \vee q, r\}$*. Therefore* $p \notin A \sim (p \wedge q)$*. Thus* $\sim$ *does not satisfy* relevance*. Hence, according to Corollary 7.3.6,* $\sim$ *is not a SPMC.*

(b) Let $-$ *be a kernel contraction based on an incision function* σ_2 *such that* $\sigma_2(A \perp\!\!\!\perp (p \wedge q)) = \{p \vee q, p \leftrightarrow q\}$*. Hence* $A - (p \wedge q) = \{p, r\}$*. Let* $\sim$ *be the shielded base contraction induced by* $-$ *and* R*. It holds that* R *and* $-$ *satisfy condition* ($\mathbf{R}$ **-** $-$)*. Thus, by Definition 7.3.10,* $\sim$ *is a P-SKC.*
On the other hand, $p \wedge q \in R$*. Thus* $A \sim (p \wedge q) = A - (p \wedge q) = \{p, r\}$*. Therefore* $p \vee q \in A \cap Cn(A \sim (p \wedge q))$ *but* $p \vee q \notin A \sim (p \wedge q)$*. Therefore* $\sim$ *does not satisfy* relative closure*. Thus, by Corollary 7.3.15,* $\sim$ *is not a SSKC.*

The following result exposes that each one of the classes of SSKCs that we have considered in the previous section, on the one hand, is subsumed by the corresponding class of SKCs and, on the other hand, contains the corresponding class of SPMCs.

Observation 7.4.2

(a) *SPMC* $\subset$ *SSKC* $\subset$ *SKC.*

(b) *SP-SPMC* $\subset$ *SP-SSKC* $\subset$ *SP-SKC.*

(c) *CC-SPMC* $\subset$ *CC-SSKC* $\subset$ *CC-SKC.*

(d) *SP+CC-SPMC* $\subset$ *SP+CC-SSKC* $\subset$ *SP+CC-SKC.*

(e) *P-SPMC* $\subset$ *P-SSKC* $\subset$ *P-SKC.*

Proof: A proof for this observation can be found on page 213.

The following example provides a shielded contraction that is a P-SbAGMC but not a SPMC nor a SKC. Therefore this example shows that P-SbAGMC $\notin$ SPMC and that P-SbAGMC $\notin$ SKC.

Example 7.4.3 *Let $A = \{p, q, r\}$. It holds that $Cn(p \wedge (q \leftrightarrow r)) \in Cn(A) \perp (p \wedge q)$. Let $\div$ be a partial meet contraction on $Cn(A)$ such that $Cn(A) \div (p \wedge q) = Cn(p \wedge (q \leftrightarrow r))$ and let $-$ be an operator on A defined for all $\alpha \in \mathcal{L}$ by $A - \alpha = (Cn(A) \div \alpha) \cap A$. Hence $-$ is a basic AGM-generated base contraction (since every partial meet contraction on a belief set is a basic AGM contraction [AGM85]). On the other hand, it holds that $A - (p \wedge q) = Cn(p \wedge (q \leftrightarrow r)) \cap A = \{p\}$. Consider the set $R = \mathcal{L} \backslash Cn(\varnothing)$ and let $\sim$ be the shielded base contraction induced by $-$ and R. It holds that R and $-$ satisfy condition (**R - $-$**). Thus, by Definition 7.3.18, $\sim$ is a P-SbAGMC. On the other hand $p \wedge q \in R$. Thus $r \notin A \sim (p \wedge q) = A - (p \wedge q)$. Therefore $\sim$ does not satisfy* core-retainment *nor* relevance*. Hence, according to Corollaries 7.3.6 and 7.3.11, $\sim$ is not a SPMC nor a SKC.*

The following observation highlights that each one of the classes of SPMCs that we have considered above is contained in the corresponding class of SbAGMCs.

Observation 7.4.4

(a) *SPMC $\subset$ SbAGMC.*

(b) *SP-SPMC $\subset$ SP-SbAGMC.*

(c) *CC-SPMC $\subset$ CC-SbAGMC.*

(d) *SP+CC-SPMC $\subset$ SP+CC-SbAGMC.*

(e) *P-SPMC $\subset$ P-SbAGMC.*

Proof: A proof for this observation can be found on page 213.

In the following example we present a shielded contraction that is a P-SSKC (and consequently a P-SKC) but not a SbAGMC.

Example 7.4.5 *Let $A = \{p, p \vee q, p \rightarrow q\}$. It holds that $A \perp\!\!\!\perp q = \{\{p, p \rightarrow q\}, \{p \vee q, p \rightarrow q\}\}$. Let $-$ be a smooth kernel contraction based on a smooth incision function σ such that: $\sigma(A \perp\!\!\!\perp q) = \{p, p \rightarrow q\}$. Hence $A - q = \{p \vee q\}$. Consider the set $R = \mathcal{L} \backslash Cn(\varnothing)$ and let $\sim$ be the shielded base contraction induced by $-$ and R. It holds that R and $-$ satisfy condition (**R - $-$**).*
Thus, by Definition 7.3.14, $\sim$ is a P-SSKC. On the other hand, $q \in R$. Thus $A \sim q = A - q = \{p \vee q\}$, from which it follows that $\sim$ does not satisfy disjunctive elimination *(since $p \in A \setminus A \sim q$ and $A \sim q \vdash p \vee q$). Therefore, by Corollary 7.3.19, $\sim$ is not a SbAGMC.*

The last result of this section exposes that each one of the classes of SbAGMCs is not related, in terms of inclusion, neither with the corresponding class of SSKCs nor with the corresponding class of SKCs.

Observation 7.4.6

(a) *SKC* $\nsubseteq$ *SbAGMC, SbAGMC* $\nsubseteq$ *SKC, SSKC* $\nsubseteq$ *SbAGMC and SbAGMC* $\nsubseteq$ *SSKC.*

(b) *SP-SKC* $\nsubseteq$ *SP-SbAGMC, SP-SbAGMC* $\nsubseteq$ *SP-SKC, SP-SSKC* $\nsubseteq$ *SP-SbAGMC and SP-SbAGMC* $\nsubseteq$ *SP-SSKC.*

(c) *CC-SKC* $\nsubseteq$ *CC-SbAGMC, CC-SbAGMC* $\nsubseteq$ *CC-SKC, CC-SSKC* $\nsubseteq$ *CC-SbAGMC and CC-SbAGMC* $\nsubseteq$ *CC-SSKC.*

(d) *SP+CC-SKC* $\nsubseteq$ *SP+CC-SbAGMC, SP+CC-SbAGMC* $\nsubseteq$ *SP+CC-SKC, SP+CC-SSKC* $\nsubseteq$ *SP+CC-SbAGMC and SP+CC-SbAGMC* $\nsubseteq$ *SP+CC-SSKC.*

(e) *P-SKC* $\nsubseteq$ *P-SbAGMC, P-SbAGMC* $\nsubseteq$ *P-SKC, P-SSKC* $\nsubseteq$ *P-SbAGMC and P-SbAGMC* $\nsubseteq$ *P-SSKC.*

Proof: A proof for this observation can be found on page 213.

In Figure 7.5 we present a diagram that summarizes the results presented in this section. The X in that diagram is either a blank space or an element of the following set of strings: {SP-, CC-, SP+CC-, P-}.

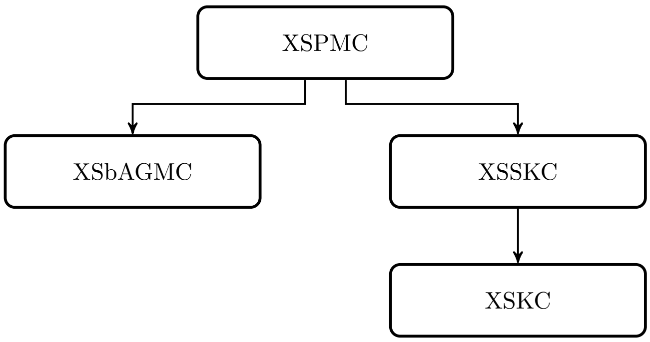

Figure 7.5: Map among different kinds of shielded base contraction functions. The X must be replaced either by a blank space or by one of the following strings: SP-, CC-, SP+CC- or P-.

7.5 Summary

Standard contraction operators are always successful, *i.e.* they are such that the result of contracting a (non-tautological) belief from a given belief base is a new belief base which (is contained in the original one and) does not imply that belief. However, as discussed before, this is not a realistic feature of belief contraction. An agent may have a set of beliefs (not necessarily tautologies) that he/she is not

willing to give up independently of the contraction to be performed. The basic idea of shielded contraction is to define a function in two steps. The first step consists of determining which beliefs are suitable to be contracted and which are not, *i.e.* given a belief, first it is analysed if it belongs to the set of retractable sentences (set of beliefs suitable to be contracted) or not. Afterwards the function should:
- leave the set of beliefs unchanged when the belief to be contracted is considered irretractable;
- behave as a standard contraction when contracting by a retractable belief.

This kind of operators is useful for modelling the behaviour of a rational agent when it receives some new information that forces the disbelief in one of its current beliefs. In a context where the belief states of an agent are represented by belief bases, a shielded base contraction can be used to obtain the new belief state of the agent, after such information is received. We notice that by means of this kind of operators it is possible to obtain more realistic models than those that can be obtained using (only) standard contractions, since it is naturally expectable that a rational agent will not always be willing to give up any of its present beliefs even if some external new information compels it to do so.

The present chapter constitutes a thorough study of shielded contractions on belief bases. In Section 7.2 we presented several results highlighting some direct relations among the postulates satisfied by a shielded contraction function $\sim$ induced by a standard contraction $-$ and a set of retractable sentences R and the postulates satisfied by $-$ and the properties of the set R. From the conclusions that can be drawn from those results we highlight the two following ones, concerning the postulates of *relative success* and *persistence*:

(i) The shielded base contraction $\sim$ satisfies relative success if and only if $-$ satisfies *failure* or the set R satisfies *non-retractability of tautology.*

(ii) If R and $-$ satisfy condition ($\mathbf{R}$ - $-$) then $\sim$ satisfies *persistence*. Furthermore if $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$ and $\sim$ satisfies *persistence* then R and $-$ satisfy condition ($\mathbf{R}$ - $-$).

We are giving special attention here to the postulates of *relative success* and *persistence* since these two postulates can be considered to be the most characteristic (and intuitive) properties of shielded contraction (together with inclusion). On the one hand *relative success* can be thought of as the most natural weakened version of the success postulate that complies with the idea underlying the notion of shielded contraction. On the other hand, among the postulates here considered to characterize shielded contractions, *persistence* can be seen as the one that more accurately describes the behaviour that is expected from a shielded contraction.

In this chapter we have considered that a reasonable shielded contraction operator should satisfy at least the postulates of *inclusion* and *relative success*, and we presented a representation theorem—which is essentially based on the result mentioned in (i) above—for the class of (shielded contraction) operators that satisfy exactly those two postulates. In what concerns *persistence*, according to (ii) above, in order to assure that the shielded contraction, that is built from a contraction $-$

and a set R, satisfies that postulate, it is enough to impose that R and $-$ satisfy condition (**R - $-$**).

Sections 7.3 and 7.4 contain the other contributions of this chapter, namely the proposal and axiomatic characterization of twenty kinds of shielded contractions and the study of the interrelations among all those classes in terms of inclusion. All the shielded contractions considered are defined by means of a base contraction function (of some well known class of such operators) and a certain set R formed by the so-called retractable sentences. Among these classes there are four which are formed by shielded contraction operators that satisfy the above highlighted postulate of persistence.

By means of the provided results it is possible to predict the behaviour of any of the functions constructed as indicated in each of the definitions presented. On the other hand, it is also possible to use these results in the converse direction, that is, for certain sets of properties that are desirable from a shielded contraction function, our results allow to identify an explicit construction of a function that will satisfy all the properties included in that set.

Chapter 8

Credibility-limited Revision on Belief Bases

"It is undesirable to believe a proposition when there
is no ground whatsoever for believing it true."
Bertrand Russel

This chapter is devoted to the study of credibility-limited revision on belief bases. Credibility-limited revision is based on the assumption that in a revision process some inputs are accepted, others not. Those that are potentially accepted constitute the set C of credible sentences. If α is credible, then α is potentially accepted in the revision process, otherwise no change is made to the set of beliefs. This model was proposed and characterized for a single revision step for belief sets in [HFCF01] and extended to cover iterated revision in [BFKP12]. As we mentioned in Subsection 5.3.2, Fermé *et al.*, in [FMT03], extended the work presented in [HFCF01] to the belief base context. In this chapter we present axiomatic characterizations for operators of credibility-limited base revision induced by several types of revision operators (namely, by partial meet, kernel, smooth kernel and basic AGM-generated revisions) and by sets of credible sentences satisfying different properties. We study the interrelations between the classes formed by these operators. We also establish the relation between different kinds of operators of shielded contraction and of credibility-limited revision by means of the consistency-preserving Levi identity and the Harper identity.

8.1 Credibility-limited base revisions

The basic idea of credibility-limited base revision is to define a function in two steps. In the first step, one needs to define which sentences are credible, *i.e.*, the sentences that an agent is willing to incorporate when performing a revision. Afterwards the function should:
- leave the set of beliefs unchanged when revising it by a non-credible sentence;
- work as a base revision when revising by a credible sentence.
The following definition formalizes this concept:

Definition 8.1.1 *Let $*$ be a revision operator (i.e., an operator that satisfies success, inclusion and consistency) on a belief base A. Let C be a set of sentences (the associated set of credible sentences). Then $\circledast$ is the credibility-limited base revision induced by $*$ and C if and only if:*

$$A \circledast \alpha = \begin{cases} A * \alpha & \text{if } \alpha \in C \\ A & \text{otherwise} \end{cases}$$

The above definition is an extension of the one presented for credibility-limited base revision in [FMT03] and reproduced in Definition 5.3.5, since it defines a credibility-limited base revision induced by a generic revision operator (and a set C) instead of by a partial meet revision as in Definition 5.3.5.

8.2 Postulates for credibility-limited base revisions

In this section we present postulates for (credibility-limited) base revision and some relations among these postulates. For convenience, we start by recalling the postulates for (credibility-limited) base revision that we have already mentioned in Subsections 5.1.3 and 5.3.2.

(Success) $\alpha \in A \circledast \alpha$.
(Inclusion) $A \circledast \alpha \subseteq A \cup \{\alpha\}$.
(Consistency) If $\alpha \nvdash \bot$, then $A \circledast \alpha \nvdash \bot$.
(Vacuity) If $A \nvdash \neg\alpha$, then $A \cup \{\alpha\} \subseteq A \circledast \alpha$.
(Consistency Preservation) If $A \nvdash \bot$, then $A \circledast \alpha \nvdash \bot$.
(Uniformity) If for all subsets $A' \subseteq A$, $A' \cup \{\alpha\} \vdash \bot$ if and only if $A' \cup \{\beta\} \vdash \bot$, then $A \cap (A \circledast \alpha) = A \cap (A \circledast \beta)$.
(Weak Extensionality) If $\vdash \alpha \leftrightarrow \beta$, then $A \cap A \circledast \alpha = A \cap A \circledast \beta$.
(Relevance) If $\beta \in A$ and $\beta \notin A \circledast \alpha$, then there is some A' such that $A \circledast \alpha \subseteq A' \subseteq A \cup \{\alpha\}$, $A' \nvdash \bot$ but $A' \cup \{\beta\} \vdash \bot$.
(Core-retainment) If $\beta \in A$ and $\beta \notin A \circledast \alpha$, then there is some $A' \subseteq A$ such that $A' \nvdash \neg\alpha$ and $A' \cup \{\beta\} \vdash \neg\alpha$.
(Disjunctive Elimination) If $\beta \in A$ and $\beta \notin A \circledast \alpha$, then $A \circledast \alpha \nvdash \neg\alpha \vee \beta$.
(Relative Closure) $A \cap Cn(A \cap A \circledast \alpha) \subseteq A \circledast \alpha$.
(Relative Success) $\alpha \in A \circledast \alpha$ or $A \circledast \alpha = A$.
(Strict Improvement) If $\alpha \in A \circledast \alpha$ and $\vdash \alpha \rightarrow \beta$, then $\beta \in A \circledast \beta$.
(Regularity) If $A \circledast \alpha \vdash \beta$, then $\beta \in A \circledast \beta$.
(Strong Regularity) If $A \circledast \alpha \nvdash \neg\beta$, then $\beta \in A \circledast \beta$.
(Disjunctive Distribution) If $\alpha \vee \beta \in A \circledast (\alpha \vee \beta)$, then $\alpha \in A \circledast \alpha$ or $\beta \in A \circledast \beta$.
(Consistency Preservation) If $A \nvdash \bot$, then $A \circledast \alpha \nvdash \bot$.

We note that some of the postulates mentioned above can be considered arguable. For example, *vacuity* is a postulate that is natural to expect to be satisfied in the context of standard revisions. From *vacuity* it follows that if α is consistent with a given set A, then α is incorporated when revising A by it (even if *success* does not hold). This argument seems to be more arguable in the context of credibility-limited revisions. We do not always incorporate information consistent with our own beliefs, since they may come from unreliable sources or lack of sufficient plausibility. However there are some contexts where *vacuity* seems more plausible. For example, in the context of databases, if C is considered to be the set of all propositions that are consistent with the set of integrity constraints, then *vacuity* should be naturally satisfied.

We propose also the following postulate:

(Persistence) If $\neg\beta \in Cn(A \cap A \circledast \beta)$, then $\neg\beta \in Cn(A \cap A \circledast \alpha)$.

Persistence is based on the namesake postulate for shielded contraction. Roughly speaking the pre-condition of this postulate, in terms of the Harper identity, means that $A \sim \neg\beta \vdash \neg\beta$. This intuitively means that $\neg\beta$ is irretractable. Thus, as the *persistence* postulate for contraction states, $\neg\beta$ should be kept independently of the contraction performed. Hence $\neg\beta$ should also be kept when revising by any sentence α.

The following observations relate some of the postulates mentioned above.

Observation 8.2.1 *Let A be a belief base and $\circledast$ be an operator on A.*

(a) If $\circledast$ satisfies relevance and relative success, then $\circledast$ satisfies disjunctive elimination.

(b) If $\circledast$ satisfies uniformity, then $\circledast$ satisfies weak extensionality.

(c) If $\circledast$ satisfies persistence, relative success and vacuity, then $\circledast$ satisfies strong regularity.

(d) If $\circledast$ satisfies relevance and relative success, then $\circledast$ satisfies core-retainment.

(e) If $\circledast$ satisfies success and core-retainment, then $\circledast$ satisfies vacuity.

(f) If $\circledast$ satisfies disjunctive elimination, then $\circledast$ satisfies relative closure.

(g) If $\circledast$ satisfies relevance and success, then $\circledast$ satisfies disjunctive elimination.

(h) If $\circledast$ satisfies core-retainment, relative success and strong regularity, then $\circledast$ satisfies vacuity.

Proof: A proof for this observation can be found on page 216.

Observation 8.2.2 *Let A be a consistent belief base and $\circledast$ be an operator on A.*

(a) If $\circledast$ satisfies consistency preservation, persistence, relative success and vacuity, then $\circledast$ satisfies disjunctive distribution.

(b) If $\circledast$ satisfies consistency preservation and strong regularity, then $\circledast$ satisfies strict improvement and regularity.

Proof: A proof for this observation can be found on page 217.

8.3 The set of credible sentences

In the credibility-limited model the set of sentences that an agent is willing to accept when a revision is performed is called *set of credible sentences*. This set will be denoted by C. In this section we propose some new properties that are natural to expect to be satisfied by a set of credible sentences and present some interrelations among these properties. We start by recalling, from [HFCF01] and [FMT03], some of the proposed properties for C (the set of credible sentences). These properties were already mentioned in Subsections 4.2.1 and 5.3.2.

Credibility of Logical Equivalents: If $\vdash \alpha \leftrightarrow \beta$, then $\alpha \in C$ if and only if $\beta \in C$.[1].
Single Sentence Closure: If $\alpha \in C$, then $Cn(\alpha) \subseteq C$.
Disjunctive Completeness: If $\alpha \vee \beta \in C$, then either $\alpha \in C$ or $\beta \in C$.
Negation Completeness: $\alpha \in C$ or $\neg\alpha \in C$.
Element Consistency: If $\alpha \in C$, then $\alpha \nvdash \bot$.
Expansive Credibility: If $A \nvdash \alpha$, then $\neg\alpha \in C$.
Revision Credibility: If $\alpha \in C$, then $Cn(A \circledast \alpha) \subseteq C$.
Strong Revision Credibility: If $\alpha \notin C$, then $A \circledast \beta \vdash \neg\alpha$.

We propose other additional properties for C:

Closure Under Double Negation: $\alpha \in C$ if and only if $\neg\neg\alpha \in C$.
Credibility Lower Bounding: If $A \nvdash \bot$, then $Cn(A) \subseteq C$.
Uniform Credibility: If it holds for all subsets A' of A that $A' \cup \{\alpha\} \vdash \bot$ if and only if $A' \cup \{\beta\} \vdash \bot$, then $\alpha \in C$ if and only if $\beta \in C$.

Closure under double negation states that a sentence α is credible if and only its double negation, $\neg\neg\alpha$, is credible. Note that if a set satisfies *credibility of logical equivalents*, then it satisfies closure under double negation. *Credibility lower bounding* states that the logical consequences of the set of beliefs under consideration are credible. *Uniform Credibility* states that if the negation of two sentences α and β are implied by exactly the same subsets of A, then α and β are both credible or both non-credible.[2]

[1] This property is equivalent to the one designated by *closure under logical equivalence* in [HFCF01] (If $\vdash \alpha \leftrightarrow \beta$, and $\alpha \in C$, then $\beta \in C$).

[2] We note that more rigorously the expression "with respect to A" should be added to the designation of the last two properties, since they relate C with A. This will be omitted since there is no risk of ambiguity whenever these properties are mentioned along this thesis. The same also applies to *expansive credibility*.

We also propose the following condition that relates a set of credible sentences C and a revision function $*$:

$$\text{If } \alpha \notin C \text{ and } \beta \in C, \text{ then } A \cap A * \beta \vdash \neg\alpha. \qquad \textbf{(C - *)}$$

Condition (**C - $*$**) states that if a sentence α is not credible, then any possible outcome of revising a set A by a credible sentence contains a subset of A that implies $\neg\alpha$.

The following observation illustrates some relations between the properties mentioned above.

Observation 8.3.1 *Let C be a set of sentences.*

(a) *If C satisfies single sentence closure, then C satisfies credibility of logical equivalents.*

(b) *If C satisfies uniform credibility, then C satisfies credibility of logical equivalents.*

(c) *If C satisfies expansive credibility and credibility lower bounding with respect to a consistent set A, then C satisfies negation completeness.*

(d) *If $\alpha \in Cn(\varnothing)$ and C satisfies negation completeness and element consistency, then $\alpha \in C$.*

(e) *If C satisfies credibility of logical equivalents, then C is closed under double negation.*

Proof: A proof for this observation can be found on page 218.

In the following theorem we present an explicit definition for the set of credible sentences C in terms of a credibility-limited revision operator $\circledast$, which is induced by it, provided that C satisfies *expansive credibility* and *closure under double negation.*

Theorem 8.3.2 *Let A be a consistent belief base and $\circledast$ be an operator of credibility-limited revision induced by a revision operator for A and a set $C \subseteq \mathcal{L}$. If C satisfies expansive credibility and closure under double negation, then $C = \{\alpha : \alpha \in A \circledast \alpha\}$.*

Proof: A proof for this observation can be found on page 218.

8.4 Relations between base revisions and credibility-limited base revisions

The following theorem illustrates some properties that an operator of credibility-limited base revision, induced by a revision operator $*$ and a set C, satisfies whenever $*$ and C satisfy some given properties.

Theorem 8.4.1 *Let A be a belief base, $C \subseteq \mathcal{L}$, and $\circledast$ be a credibility-base revision induced by a revision operator $*$ and C. Then:*

(a) *It holds that:*

If $*$ satisfies	***then $\circledast$ satisfies***
—	*inclusion and relative success*
relevance	*relevance*
core-retainment	*core-retainment*
disjunctive elimination	*disjunctive elimination*
relative closure	*relative closure*

(b) *If C satisfies element consistency, then:*

If $*$ satisfies	***then $\circledast$ satisfies***
—	*consistency preservation*

(c) *If C satisfies uniform credibility, then:*

If $*$ satisfies	***then $\circledast$ satisfies***
uniformity	*uniformity*

(d) *If C satisfies credibility of logical equivalents (or single sentence closure), then:*

If $*$ satisfies	***then $\circledast$ satisfies***
weak extensionality	*weak extensionality*

(e) *If C satisfies expansive credibility and is closed under double negation (or satisfies either credibility of logical equivalents or uniform credibility or single sentence closure), then:*

If $*$ satisfies	***then $\circledast$ satisfies***
vacuity	*vacuity*

(f) *Let $A \nvdash \bot$. If C satisfies expansive credibility and single sentence closure, then:*

If $*$ satisfies	***then $\circledast$ satisfies***
—	*strict improvement*

(g) *Let $A \nvdash \bot$. If C satisfies expansive credibility, closure under double negation (or credibility of logical equivalents or uniform credibility or single sentence closure) and disjunctive completeness, then:*

If $*$ satisfies	***then $\circledast$ satisfies***
—	*disjunctive distribution*

(h) *If C and $*$ satisfy condition (**C - $*$**) and C satisfies element consistency, then:*

If $*$ satisfies	***then $\circledast$ satisfies***
—	*persistence*
weak extensionality	*weak extensionality*
uniformity	*uniformity*

(i) *If C and $*$ satisfy condition (**C** - $*$) and C satisfies expansive credibility, then:*

If $*$ satisfies	***then $\circledast$ satisfies***
vacuity	*vacuity*

Proof: A proof for this theorem can be found on page 218.

If α is credible, then it should be an element of the outcome of the revision of a set A by it. Therefore, a natural way to define a set of credible sentences C is by $C = \{\alpha : \alpha \in A \circledast \alpha\}$, where $\circledast$ is a credibility-limited revision. The next theorem illustrates some properties that such a set satisfies whenever $\circledast$ satisfies some of the postulates mentioned in the beginning of Section 8.2.

Theorem 8.4.2 *Let A be a consistent belief base, $\circledast$ be an operator on A and $C = \{\alpha : \alpha \in A \circledast \alpha\}$. Then:*

If $\circledast$ satisfies	***then C satisfies***
consistency preservation	*element consistency*
strict improvement	*single sentence closure*
disjunctive distribution	*disjunctive completeness*
vacuity	*expansive credibility and credibility lower bounding*
relative success, consistency preservation, vacuity and uniformity	*uniform credibility*
relative success, consistency preservation, vacuity and weak extensionality	*credibility of logical equivalents*
consistency preservation, persistence, relative success and vacuity	*single sentence closure, disjunctive completeness, revision credibility and uniform credibility*
persistence, relative success and vacuity	*strong revision credibility*

Proof: A proof for this theorem can be found on page 221.

In the next theorem we will see that it is possible to construct an operator $*$ in terms of $\circledast$ and $C = \{\alpha : \alpha \in A \circledast \alpha\}$ and investigate the properties that such an operator satisfies taking into account the properties satisfied by $\circledast$.

Theorem 8.4.3 *Let A be a consistent belief base, $\circledast$ be an operator on A and $C = \{\alpha : \alpha \in A \circledast \alpha\}$. Then there exists an operator $*$ on A such that:*

(a) *If $\circledast$ satisfies relative success, consistency preservation and inclusion, then $*$ is a revision operator and $\circledast$ is the credibility-limited base revision induced by $*$ and C.*

(b) *It holds that:*

If $\circledast$ satisfies	***then $*$ satisfies***
vacuity	*vacuity*
relevance	*relevance*
core-retainment	*core-retainment*
disjunctive elimination	*disjunctive elimination*
uniformity, relative success, vacuity, consistency preservation	*uniformity*
weak extensionality, relative success, vacuity, consistency preservation	*weak extensionality*
relative closure	*relative closure*

(c) *If $\circledast$ satisfies persistence, relative success and vacuity, then $*$ and C satisfy condition (**C - $*$**).*

Proof: A proof for this theorem can be found on page 222.

8.5 Axiomatic characterizations of different kinds of credibility-limited base revision

In this section we present axiomatic characterizations for several classes of credibility-limited base revision functions. We start by presenting a representation theorem for the most general class of credibility-limited base revision operators that we will consider. Afterwards we will consider more specific classes of credibility-limited base revision operators induced by different kinds of revision functions (namely, by partial meet, (smooth) kernel, and basic AGM-generated base revisions) and by different types of sets of credible sentences. The structure used in this section is similar to the one adopted in Section 7.3. In each one of the Subsections 8.5.2 — 8.5.5 we start by presenting a representation theorem for five classes of credibility-limited base revisions all induced by the same kind of revision functions but each one of then with a different type of associated set of credible sentences. Afterwards, we present a definition where a designation is proposed for each one of the operators mentioned in the previously presented representation theorem.

8.5.1 Basic credibility-limited base revision

In this subsection we present a representation theorem for the most general class of credibility-limited base revision that we will consider. This class is characterized by the following postulates: *relative success, inclusion* and *consistency preservation.* The operators in this class will be designated by *basic credibility-limited revisions.*

Theorem 8.5.1 *Let A be a consistent belief base and $\circledast$ be an operator on A. Then the following pair of conditions are equivalent:*

(a) $\circledast$ satisfies relative success, consistency preservation and inclusion.

(b) $\circledast$ is an operator of credibility-limited base revision induced by a revision operator for A and a set $C \subseteq \mathcal{L}$ that satisfies element consistency.

Proof: A proof for this theorem can be found on page 224.

Definition 8.5.2 *A credibility-limited base revision $\circledast$ on a consistent belief base A induced by a revision operator $*$ and a set $C \subseteq \mathcal{L}$ is a* basic credibility-limited revision *if and only if C satisfies element consistency.*

In the remainder of this section we will obtain representation theorems for other less general classes of credibility-limited revisions. More precisely, we will consider the credibility-limited revision operators induced by partial meet base revisions, by (smooth) kernel base revisions and by basic AGM-generated base revisions and, additionally, we will take into account different sets of properties regarding the associated set of credible sentences.

8.5.2 Credibility-limited partial meet base revisions

The following theorem presents axiomatic characterizations for five kinds of operators of credibility-limited base revision. All these operators are induced by partial meet revisions but each one of them has a different type of associated set of credible sentences.

Theorem 8.5.3 *Let A be a consistent belief base and $\circledast$ an operator on A. Then:*

$\circledast$ is an operator of credibility-limited base revision induced by a partial meet revision operator $*$ and a set C that satisfies element consistency, expansive credibility and	***if and only if $\circledast$ satisfies relative success, consistency preservation, inclusion, vacuity, uniformity, relevance and***
uniform credibility	—
uniform credibility and single sentence closure	*strict improvement*
uniform credibility and disjunctive completeness	*disjunctive distribution*
uniform credibility, single sentence closure and disjunctive completeness	*strict improvement and disjunctive distribution*
condition (**C - $*$**)	*persistence*

Proof: A proof for this theorem can be found on page 224.

In the following definition we attribute designations to the different kinds of credibility-limited base revisions that were axiomatically characterized in the above theorem.

Definition 8.5.4 *A credibility-limited base revision $\circledast$ on a consistent belief base A induced by a partial meet revision $*$ and a set $C \subseteq \mathcal{L}$ is a:*

Designation	***if and only if C satisfies element consistency, expansive credibility and***
Credibility-limited partial meet revision (CLPMR)	*uniform credibility*
Strictly improving credibility-limited partial meet revision (SI-CLPMR)	*uniform credibility and single sentence closure*
Disjunctive distributive credibility-limited partial meet revision (DD-CLPMR)	*uniform credibility and disjunctive completeness*
Strictly improving disjunctive distributive credibility-limited partial meet revision (SI+DD-CLPMR)	*uniform credibility, single sentence closure and disjunctive completeness*
Persistent credibility-limited partial meet revision (P-CLPMR)	*condition* ($\mathbf{C}$ - $*$)

Throughout this chapter we will sometimes use the acronyms presented in the above definition (and whenever a similar definition is presented) to designate the whole class of the corresponding operators (rather than only one of its elements).

Note that in the proof of the right-to-left part of Theorem 8.5.3 it was used the set $C = \{\alpha : \alpha \in A \circledast \alpha\}$. Therefore, it follows immediately from Theorem 8.4.2 and Observation 8.3.1 that, to the list of properties of C mentioned in each row of the tables presented in Theorem 8.5.3 and in Definition 8.5.4, we can add the following ones: *credibility lower-bounding, credibility of logical equivalents, negation completeness* and *closure under double negation*. Furthermore, in the case of the last row, besides the properties mentioned above we can also add (to the list of properties of C there presented): *single sentence closure, disjunctive completeness, revision credibility, strong revision credibility* and *uniform credibility*.

As discussed above there are several alternative (equivalent) definitions for the classes introduced in Definition 8.5.4, more precisely several equivalent definitions, but each one with a different set of properties associated to the set C. This same situation occurs regarding the classes of credibility-limited revisions introduced in Definitions 8.5.7, 8.5.10 and 8.5.13.

Note that in [FMT03, Theorem 4.3] it was presented an axiomatic characterization of credibility-limited base revision operators induced by a partial meet revision and a set C satisfying *element consistency, single sentence closure, disjunctive completeness, expansive credibility, revision credibility, strong revision credibility* and *uniform credibility*.[3] This axiomatic characterization is formed by the following set

[3]In fact, in the mentioned result, *uniform credibility* is not included among the properties that the set C is assumed to satisfy. However there is a small gap in the proof of that theorem which can

of postulates: *relative success, consistency preservation, inclusion, vacuity, disjunctive distribution, relevance, strong regularity* and *uniformity.*

The following corollary follows trivially from the above definition and Theorem 8.5.3.

Corollary 8.5.5 *Let A be a consistent belief base and* $\circledast$ *an operator on A. Then:*

$\circledast$ ***is a***	***if and only if*** $\circledast$ ***satisfies relative success, consistency preservation, inclusion, vacuity, uniformity, relevance and***
CLPMR	—
SI-CLPMR	*strict improvement*
DD-CLPMR	*disjunctive distribution*
SI+DD-CLPMR	*strict improvement and disjunctive distribution*
P-CLPMR	*persistence*

8.5.3 Credibility-limited kernel base revisions

In the following theorem we present axiomatic characterizations for five kinds of operators of credibility-limited base revision that are all induced by kernel revisions but each one of them has a different type of associated set of credible sentences.

Theorem 8.5.6 *Let A be a consistent belief base and* $\circledast$ *an operator on A. Then:*

$\circledast$ ***is an operator of credibility-limited base revision induced by a kernel revision operator and a set*** C ***that satisfies element consistency, expansive credibility and***	***if and only if*** $\circledast$ ***satisfies relative success, consistency preservation, inclusion, vacuity, uniformity, core-retainment and***
uniform credibility	—
uniform credibility and single sentence closure	*strict improvement*
uniform credibility and disjunctive completeness	*disjunctive distribution*
uniform credibility, single sentence closure and disjunctive completeness	*strict improvement and disjunctive distribution*
condition (**C - ***)	*persistence*

Proof: A proof for this theorem can be found on page 224.

In the next definition we introduce designations for the different kinds of credibility-limited revisions that were axiomatically characterized in the above theorem.

be easily corrected if we add *uniform credibility* to the list of properties that the set C is required to satisfy.

Definition 8.5.7 *A credibility-limited base revision $\circledast$ on a consistent belief base A induced by a kernel revision $*$ and a set $C \subseteq \mathcal{L}$ is a:*

Designation	***if and only if C satisfies element consistency, expansive credibility and***
Credibility-limited kernel revision (CLKR)	*uniform credibility*
Strictly improving credibility-limited kernel revision (SI-CLKR)	*uniform credibility and single sentence closure*
Disjunctive distributive credibility-limited kernel revision (DD-CLKR)	*uniform credibility and disjunctive completeness*
Strictly improving disjunctive distributive credibility-limited kernel revision (SI+DD-CLKR)	*uniform credibility, single sentence closure and disjunctive completeness*
Persistent credibility-limited kernel revision (P-CLKR)	*condition (**C** - $*$)*

The difference between the axiomatic characterizations of the classes of credibility-limited revision induced by partial meet revision operators, presented in the previous subsection, and the ones for the classes of credibility-limited induced by kernel revision operators, presented in Theorem 8.5.6, is the replacement of *relevance* by *core-retainment*, therefore, according to Observation 8.2.1 (d), every class of credibility-limited revision induced by partial meet revision operators is a subclass of the corresponding class of credibility-limited revision induced by kernel revision operators, *i.e.*, P-CLPMR $\subseteq$ P-CLKR, DD-CLPMR $\subseteq$ DD-CLKR, SI-CLPMR $\subseteq$ SI-CLKR, SI+DD-CLPMR $\subseteq$ SI+DD-CLKR and CLPMR $\subseteq$ CLKR.

The following corollary follows trivially from the above definition and Theorem 8.5.6.

Corollary 8.5.8 *Let A be a consistent belief base and $\circledast$ an operator on A. Then:*

$\circledast$ is a	***if and only if $\circledast$ satisfies relative success, consistency preservation, inclusion, vacuity, uniformity, core-retainment and***
CLKR	—
SI-CLKR	*strict improvement*
DD-CLKR	*disjunctive distribution*
SI+DD-CLKR	*strict improvement and disjunctive distribution*
P-CLKR	*persistence*

8.5.4 Credibility-limited smooth kernel base revisions

The following representation theorem axiomatically characterizes five kinds of operators of credibility-limited revisions. All these operators are induced by smooth kernel revisions but each one of them has a different type of associated set of credible sentences.

Theorem 8.5.9 *Let A be a consistent belief base and $\circledast$ an operator on A. Then:*

$\circledast$ is an operator of credibility-limited base revision induced by a smooth kernel revision operator and a set C that satisfies element consistency, expansive credibility and	***if and only if $\circledast$ satisfies relative success, consistency preservation, inclusion, vacuity, uniformity, core-retainment, relative closure and***
uniform credibility	—
uniform credibility and single sentence closure	*strict improvement*
uniform credibility and disjunctive completeness	*disjunctive distribution*
uniform credibility, single sentence closure and disjunctive completeness	*strict improvement and disjunctive distribution*
condition (**C - $*$**)	*persistence*

Proof: A proof for this theorem can be found on page 224.

In the following definition we introduce designations for the different kinds of credibility-limited revisions that were axiomatically characterized in the above theorem.

Definition 8.5.10 *A credibility-limited base revision $\circledast$ on a consistent belief base A induced by a smooth kernel revision $*$ and a set $C \subseteq \mathcal{L}$ is a:*

Designation	***if and only if C satisfies element consistency, expansive credibility and***
Credibility-limited smooth kernel revision (CLSKR)	*uniform credibility*
Strictly improving credibility-limited smooth kernel revision (SI-CLSKR)	*uniform credibility and single sentence closure*
Disjunctive distributive credibility-limited smooth kernel revision (DD-CLSKR)	*uniform credibility and disjunctive completeness*

Continued on next page

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Strictly improving disjunctive distributive credibility-limited smooth kernel revision (SI+DD-CLSKR)	*uniform credibility, single sentence closure and disjunctive completeness*
Persistent credibility-limited smooth kernel revision (P-CLSKR)	*condition* $(\mathbf{C} - *)$

The following corollary follows trivially from the above definition and Theorem 8.5.9.

Corollary 8.5.11 *Let A be a consistent belief base and $\circledast$ an operator on A. Then:*

$\circledast$ ***is a***	$\circledast$ ***satisfies relative success, consistency preservation, inclusion, vacuity, uniformity, core-retainment, relative closure and***
CLSKR	—
SI-CLSKR	*strict improvement*
DD-CLSKR	*disjunctive distribution*
SI+DD-CLSKR	*strict improvement and disjunctive distribution*
P-CLSKR	*persistence*

It follows from Corollaries 8.5.5, 8.5.8 and 8.5.11 and Observation 8.2.1 that CLPMR ⊆ CLSKR ⊆ CLKR, SI-CLPMR ⊆ SI-CLSKR ⊆ SI-CLKR, DD-CLPMR ⊆ DD-CLSKR ⊆ DD-CLKR, SI+DD-CLPMR ⊆ SI+DD-CLSKR ⊆ SI+DD-CLKR and P-CLPMR ⊆ P-CLSKR ⊆ P-CLKR. These results will be refined in Observation 8.6.2 where it is stated that (all of) these set inclusions are in fact strict.

8.5.5 Credibility-limited basic AGM-generated base revisions

In the following representation theorem we axiomatically characterize five kinds of operators of credibility-limited revisions, all induced by basic AGM-generated base revisions but each of them with a different type of associated set of credible sentences.

Theorem 8.5.12 *Let A be a consistent belief base and $\circledast$ an operator on A. Then:*

$\circledast$ is an operator of credibility-limited base revision induced by a basic AGM-generated base revision operator and a set C that satisfies element consistency, expansive credibility and	**if and only if $\circledast$ satisfies relative success, consistency preservation, inclusion, vacuity, weak extensionality, disjunctive elimination and**
credibility of logical equivalents	—
single sentence closure	*strict improvement*
credibility of logical equivalents and disjunctive completeness	*disjunctive distribution*
single sentence closure and disjunctive completeness	*strict improvement and disjunctive distribution*
condition $(\mathbf{C} - *)$	*persistence*

Proof: A proof for this theorem can be found on page 225.

In the following definition we attribute designations to the different kinds of credibility-limited base revisions that were axiomatically characterized in the above theorem.

Definition 8.5.13 *A credibility-limited base revision* $\circledast$ *on a consistent belief base* A *induced by a basic AGM-generated revision* $*$ *and a set* $C \subseteq \mathcal{L}$ *is a:*

Designation	***if and only if C satisfies element consistency, expansive credibility and***
Credibility-limited basic AGM-generated revision (CLbAGMR)	*credibility of logical equivalents*
Strictly improving credibility-limited basic AGM-generated revision (SI-CLbAGMR)	*single sentence closure*
Disjunctive distributive credibility-limited basic AGM-generated revision (DD-CLbAGMR)	*credibility of logical equivalents and disjunctive completeness*
Strictly improving disjunctive distributive credibility-limited basic AGM-generated revision (SI+DD-CLbAGMR)	*single sentence closure and disjunctive completeness*
Persistent credibility-limited basic AGM-generated revision (P-CLbAGMR)	*condition* $(\mathbf{C} - *)$

It is worth to notice that in the proof of the right-to-left part of Theorem 8.5.12 it was used the set $C = \{\alpha : \alpha \in A \circledast \alpha\}$. Therefore, from Theorem 8.4.2 and Observation

8.3.1, it follows that in each row of the tables presented in Theorem 8.5.12 and in Definition 8.5.13, we can also add *credibility lower bounding, negation completeness* and *closure under double negation* to the list of properties of C (there presented). We can also add *credibility of logical equivalents* to the list of properties of C presented in rows 2, 4 and 5. Furthermore, according to Theorem 8.4.2, *single sentence closure, disjunctive completeness, revision credibility, strong revision credibility* and *uniform credibility* can be also added to the list of properties of C presented in the last row.

The following corollary follows trivially from the above definition and Theorem 8.5.12.

Corollary 8.5.14 *Let A be a consistent belief base and $\circledast$ an operator on A. Then:*

$\circledast$ is a	***if and only if $\circledast$ satisfies relative success, consistency preservation, inclusion, vacuity, weak extensionality, disjunctive elimination and***
CLbAGMR	—
SI-CLbAGMR	*strict improvement*
DD-CLbAGMR	*disjunctive distribution*
SI+DD-CLbAGMR	*strict improvement and disjunctive distribution*
P-CLbAGMR	*persistence*

It follows from Corollaries 8.5.5 and 8.5.14 and Observation 8.2.1 that CLPMR $\subseteq$ CLbAGMR, SI-CLPMR $\subseteq$ SI-CLbAGMR, DD-CLPMR $\subseteq$ DD-CLbAGMR, SI+DD-CLPMR $\subseteq$ SI+DD-CLbAGMR and P-CLPMR $\subseteq$ P-CLbAGMR. In fact, as it will be stated in Observation 8.6.3, the inclusions above are (all) strict.

Table 8.2 summarizes the results obtained in the representation theorems presented in this section. Given a credibility limited revision $\circledast$ the white cells that are on the top of the same column represent the properties that C (the associated set of credible sentences) satisfies. The white cells that are placed on the right of the same row indicates the properties that $\circledast$ satisfies.

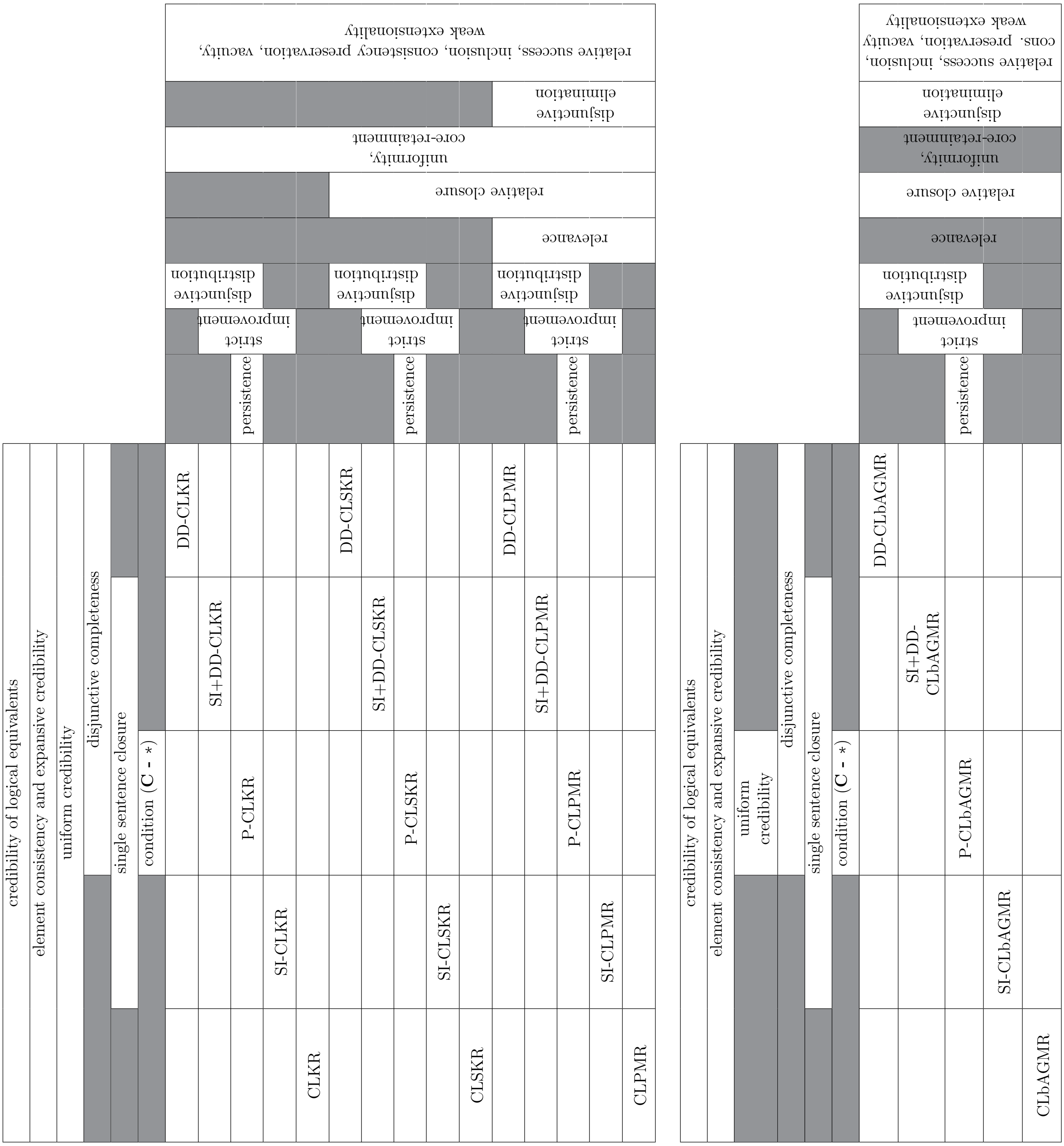

Table 8.2: Schematic representation of the main postulates satisfied by each one of the twenty classes of credibility limited revisions considered and also of the properties satisfied by the set of credible sentences by which each of those operators is induced.

8.6 Maps between credibility-limited base revision functions

We start this section by presenting an observation that illustrates the interrelations among classes of credibility-limited base revisions induced by the same type of revision function, but each one of them with a different type of associated set of credible sentences. Through this section we assume that the classes of credibility-limited revision mentioned are formed by operators defined on the same belief base.

Observation 8.6.1 *Let X be any one of the element that belong to following set of strings:* $\{\mathrm{CLPMR}, \mathrm{CLSKR}, \mathrm{CLKR}, \mathrm{CLbAGMR}\}$*. Then:*

(a) *P-X* $\subset$ *SI+DD-X.*

(b) *SI+DD-X* $\subset$ *DD-X.*

(c) *SI+DD-X* $\subset$ *SI-X.*

(d) *DD-X* $\nsubseteq$ *SI-X and SI-X* $\nsubseteq$ *DD-X.*

(e) *DD-X* $\subset$ *X.*

(f) *SI-X* $\subset$ *X.*

Proof: A proof for this observation can be found on page 226.

In Figure 8.1 we present a diagram that summarizes the results presented in Observation 8.6.1. The X in this diagram must be replaced by an element of the following set of strings $\{\mathrm{CLPMR}, \mathrm{CLSKR}, \mathrm{CLKR}, \mathrm{CLbAGMR}\}$.

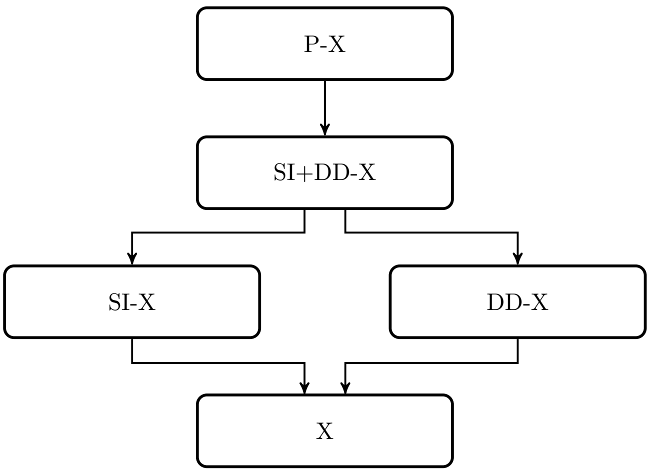

Figure 8.1: Map among different classes of credibility-limited base revision functions induced by the same kind of revisions. The X must be replaced by one of the following strings CLPMR, CLSKR, CLKR, CLbAGMR.

The following observation highlights that each one of the classes of CLSKRs that we have considered in Definition 8.5.10 is subsumed by the corresponding class of CLKRs and, on the other hand, contains the corresponding class of CLPMRs.

Observation 8.6.2

(a) *CLPMR* $\subset$ *CLSKR* $\subset$ *CLKR.*

(b) *SI-CLPMR* $\subset$ *SI-CLSKR* $\subset$ *SI-CLKR.*

(c) *DD-CLPMR* $\subset$ *DD-CLSKR* $\subset$ *DD-CLKR.*

(d) *SI+DD-CLPMR* $\subset$ *SI+DD-CLSKR* $\subset$ *SI+DD-CLKR.*

(e) *P-CLPMR* $\subset$ *P-CLSKR* $\subset$ *P-CLKR.*

Proof: A proof for this observation can be found on page 227.

The following observation highlights that each one of the classes of CLPMRs that we have considered in Subsection 8.5.2 is contained in the corresponding class of CLbAGMRs.

Observation 8.6.3

(a) *CLPMR* $\subset$ *CLbAGMR.*

(b) *SI-CLPMR* $\subset$ *SI-CLbAGMR.*

(c) *DD-CLPMR* $\subset$ *DD-CLbAGMR.*

(d) *SI+DD-CLPMR* $\subset$ *SI+DD-CLbAGMR.*

(e) *P-CLPMR* $\subset$ *P-CLbAGMR.*

Proof: A proof for this observation can be found on page 227.

The following observation exposes that each one of the classes of CLbAGMRs is not related, in terms of inclusion, neither with the corresponding class of CLSKRs nor with the corresponding class of CLKRs.

Observation 8.6.4

(a) *CLKR* $\nsubseteq$ *CLbAGMR, CLbAGMR* $\nsubseteq$ *CLKR, CLSKR* $\nsubseteq$ *CLbAGMR and CLbAGMR* $\nsubseteq$ *CLSKR.*

(b) *SI-CLKR* $\nsubseteq$ *SI-CLbAGMR, SI-CLbAGMR* $\nsubseteq$ *SI-CLKR, SI-CLSKR* $\nsubseteq$ *SI-CLbAGMR and SI-CLbAGMR* $\nsubseteq$ *SI-CLSKR.*

(c) *DD-CLKR* $\nsubseteq$ *DD-CLbAGMR, DD-CLbAGMR* $\nsubseteq$ *DD-CLKR, DD-CLSKR* $\nsubseteq$ *DD-CLbAGMR and DD-CLbAGMR* $\nsubseteq$ *DD-CLSKR.*

(d) *SI+DD-SKC* $\notin$ *SI+DD-CLbAGMR, SI+DD-CLbAGMR* $\notin$ *SI+DD-SKC, SI+DD-CLSKR* $\notin$ *SI+DD-CLbAGMR and SI+DD-CLbAGMR* $\notin$ *SI+DD-CLSKR.*

(e) *P-CLKR* $\notin$ *P-CLbAGMR, P-CLbAGMR* $\notin$ *P-SKC, P-CLSKR* $\notin$ *P-CLbAGMR and P-CLbAGMR* $\notin$ *P-CLSKR.*

Proof: A proof for this observation can be found on page 228.

In Figure 8.2 we present a diagram that summarizes the results presented in Observations 8.6.2, 8.6.3 and 8.6.4. The X in this diagram is either a blank space or one of the following strings: SI-, DD-, SI+DD- or P-.

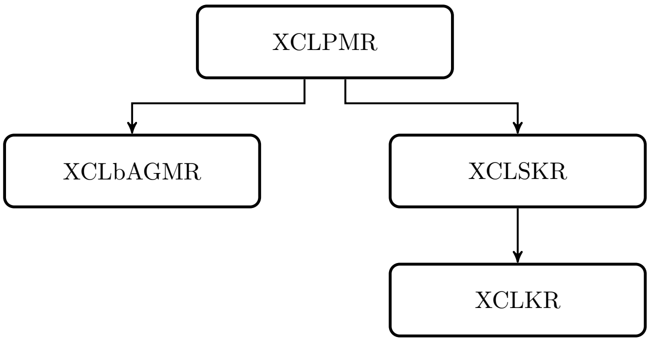

Figure 8.2: Map among different classes of credibility-limited base revision operators. The X in the diagram must be replaced by a blank space or by an element of the following set of strings: $\{SI-, DD-, SI+DD-, P-\}$.

8.7 Relations between sets of credible and of retractable sentences

In this section we study the relation between credible and retractable sentences of an agent. If we want to ensure that our credibility-limited revision operators satisfy *consistency preservation*, then we must assure that a sentence is credible only if its negation can be removed during the revision process, otherwise the outcome of this revision will be inconsistent. Hence we can relate the sets R and C by the following condition: if $\alpha \in C$, then $\neg\alpha \in R$. In [FMT03], a stronger condition relating the set of credible and of retractable sentences was presented:[4]

$$\alpha \in C \text{ if and only if } \neg\alpha \in R. \qquad \textbf{(C-R)}$$

The following condition can be seen as the dual of the previous one:

$$\alpha \in R \text{ if and only if } \neg\alpha \in C. \qquad \textbf{(R-C)}$$

The following observation illustrates that conditions (**C-R**) and (**R-C**) are equivalent provided that R and C are closed under double negation.

[4]The left-to-right implication of the condition (**C-R**) may be considered a little more arguable, namely if one considers that there are some cases in which an agent is willing to discard the negation of a sentence without being the case that he/she considers that sentence to be credible.

Observation 8.7.1 *Let R and C be subsets of $\mathcal{L}$. Then:*
*If R and C are closed under double negation, then condition (*__C-R__*) holds if and only if condition (*__R-C__*) also holds.*

Proof: A proof for this observation can be found on page 228.

Having in mind conditions (**C-R**) and (**R-C**), between credible and retractable sentences, the following observations establish the relation between properties of sets of retractable sentences and of credible sentences.

Observation 8.7.2 *Let A be a belief base, R and C be set of sentences that satisfy condition (*__C-R__*).*

(a) If R is closed under double negation, then:

R satisfies	***if and only if C satisfies***
retractability of logical equivalents	*credibility of logical equivalents*
non-retractability of tautology	*element consistency*
non-retractability propagation	*single sentence closure*
uniform retractability with respect to A	*uniform credibility with respect to A*
non-retractability upper bounding with respect to A	*expansive credibility with respect to A*

(b) If R satisfies retractability of logical equivalents, *then:*

R satisfies	***if and only if C satisfies***
conjunctive completeness	*disjunctive completeness*

Proof: A proof for this observation can be found on page 228.

Observation 8.7.3 *Let A be a belief base, R and C be set of sentences that satisfy condition (*__R-C__*).*

(a) If C is closed under double negation, then:

C satisfies	***if and only if R satisfies***
credibility of logical equivalents	*retractability of logical equivalents*
element consistency	*non-retractability of tautology*
single sentence closure	*non-retractability propagation*
uniform credibility with respect to A	*uniform retractability with respect to A*
expansive credibility with respect to A	*non-retractability upper bounding with respect to A*

(b) If C satisfies credibility of logical equivalents, *then:*

C satisfies	***if and only if R satisfies***
disjunctive completeness	*conjunctive completeness*

Proof: A proof for this observation can be found on page 230.

The following observation relates conditions (**R - −**) and (**C - ∗**), when C and R are related though condition (**C-R**) and the revision operator $*$ is defined by means of the Levi identity from $-$.

Observation 8.7.4 *Let A be a belief base, R and C be sets of sentences that satisfy condition (**C-R**). Let $*$ be a revision operator defined from the contraction operator $-$ on A by means of the Levi identity. If R and $-$ satisfy condition (**R - −**), then C and $*$ satisfy condition (**C - ∗**).*

Proof: A proof for this observation can be found on page 230.

The following observation relates conditions (**C - ∗**) and (**R - −**), whenever the contraction operator $-$ is defined by means of the Harper identity from $*$ and, C and R are related through condition (**R-C**):

Observation 8.7.5 *Let A be a belief base, R and C be sets of sentences that satisfy condition (**R-C**). Let $-$ be a contraction operator defined from the revision operator $*$ on A by means of the Harper identity. If C and $*$ satisfy condition (**C - ∗**), then R and $-$ satisfy condition (**R - −**).*

Proof: A proof for this observation can be found on page 230.

8.8 Generalized Levi and Harper identities

In this section we establish several results that relate credibility-limited revisions and shielded contractions through the consistency-preserving Levi and the Harper identities:

$$A \circledast \alpha = \begin{cases} (A \sim \neg\alpha) + \alpha & \text{if } A \sim \neg\alpha \nvdash \neg\alpha \\ A & \text{otherwise} \end{cases}$$

$$A \sim \alpha = (A \circledast \neg\alpha) \cap A.$$

The following theorem illustrates that if a contraction operator $-$ is defined from a revision operator $*$ by means of the Harper identity, then the shielded contraction induced by $-$ and R can be obtained by means of the Harper identity from the credibility-limited revision operator induced by $*$ and C, provided that the sets R and C are related by through condition (**R-C**).

Theorem 8.8.1 *Let A be a belief base and $*$ be a revision operator on A. Let $-$ be the contraction operator on A defined from $*$ by means of the Harper identity. Let $C \subseteq \mathcal{L}$ and $\circledast$ be the credibility-limited revision induced by $*$ and C. Let $R \subseteq \mathcal{L}$ be the set defined from C by means of condition (**R-C**). Let $\sim$ be the shielded base contraction on A induced by $-$ and R. Then $\sim$ can be defined from $\circledast$ by means of the Harper identity.*

Proof: A proof for this theorem can be found on page 230.

In the following theorem it is presented a result that can be seen as the dual of the previous one. The second item of this theorem states that if a revision operator $*$ is defined from a contraction operator $-$ by means of the Levi identity, then the credibility-limited revision induced by $*$ and C can be obtained by means of the consistency-preserving Levi identity from the shielded contraction operator induced by $-$ and R, provided that the sets R and C are related by the condition (**C-R**) and R satisfies *non-retractability of tautology* and *non-retractability upper bounding.*

Theorem 8.8.2 *Let A be a belief base and $-$ be a contraction operator on A. Let $*$ be the revision operator on A defined from $-$ by means of the Levi identity. Let $R \subseteq \mathcal{L}$ and $\sim$ be the shielded base contraction induced by $-$ and R. Let $C \subseteq \mathcal{L}$ be the set defined from R by means of condition (**C-R**). Let $\circledast$ be the credibility-limited revision induced by $*$ and C. Then:*

(a)

$$A \circledast \alpha = \begin{cases} (A \sim \neg\alpha) \cup \{\alpha\} & \text{if } \alpha \in C \\ A & \text{otherwise} \end{cases}$$

(b) If R satisfies non-retractability of tautology and non-retractability upper bounding, then $\circledast$ can be defined from $\sim$ by means of the consistency-preserving Levi identity.

Proof: A proof for this theorem can be found on page 230.

The following two observations illustrate some relations between postulates of shielded contraction and of credibility-limited revision whenever one of these operators is obtained from the other by means of the Harper or the consistency-preserving Levi identities. These two observations are adaptations to the belief base context of Observations 4.2.30 and 4.2.31 (which concern operations on belief sets and were originally presented in [FH01]).

Observation 8.8.3 *Let A be a consistent belief base and $\sim$ be a shielded base contraction on A. Let $\circledast$ be defined from $\sim$ via the consistency-preserving Levi identity. Then:*

If $\sim$ satisfies	***then $\circledast$ satisfies***
—	*relative success and consistency preservation*
inclusion	*inclusion*
inclusion and persistence	*disjunctive distribution, persistence and strong regularity*
inclusion, vacuity and uniformity	*uniformity*
relevance	*relevance*
core-retainment	*core-retainment*

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conjunctive constancy, relative success and extensionality	*disjunctive distribution*
inclusion and success propagation	*strict improvement*
inclusion and vacuity	*vacuity*
disjunctive elimination	*disjunctive elimination*
extensionality, inclusion and vacuity	*weak extensionality*
inclusion, vacuity and relative closure	*relative closure*

Proof: A proof for this observation can be found on page 230.

Observation 8.8.4 *Let A be a consistent belief base and $\circledast$ be a credibility-limited base revision on A. Let $\sim$ be defined from $\circledast$ via the Harper identity. Then:*

If $\circledast$ satisfies	***then $\sim$ satisfies***
—	*inclusion*
relative success and consistency preservation	*relative success*
persistence	*persistence*
relative success and relevance	*relevance*
core-retainment	*core-retainment*
uniformity	*uniformity*
vacuity	*vacuity*
vacuity, relative success, consistency preservation, disjunctive distribution and weak extensionality	*conjunctive constancy*
disjunctive elimination	*disjunctive elimination*
weak extensionality	*extensionality*
consistency preservation, strict improvement and relative success	*success propagation*
relative closure	*relative closure*

Proof: A proof for this observation can be found on page 233.

The following two corollaries clarify that each element of one of the classes of shielded contraction considered in Chapter 7 gives rise, by means of the consistency-preserving Levi identity, to an element of one of the classes of credibility-limited base revision operators considered in the present chapter.

Corollary 8.8.5 *Let A be a consistent belief base. Let $\sim$ be a basic shielded base contraction operator on A. Let $\circledast$ be defined from $\sim$ via the consistency-preserving Levi identity. Then $\circledast$ is a basic credibility-limited base revision operator.*

Proof: A proof for this corollary can be found on page 235.

Corollary 8.8.6 *Let A be a consistent belief base and ∼ be a shielded base contraction operator on A. Let ⊛ be defined from ∼ via the consistency-preserving Levi identity. Then:*

(a)

If ∼ is a	*then ⊛ is a*
SPMC	*CLPMR*
SP-SPMC	*SI-CLPMR*
CC-SPMC	*DD-CLPMR*
SP+CC-SPMC	*SI+DD-CLPMR*
P-SPMC	*P-CLPMR*

(b)

If ∼ is a	*then ⊛ is a*
SKC	*CLKR*
SP-SKC	*SI-CLKR*
CC-SKC	*DD-CLKR*
SP+CC-SKC	*SI+DD-CLKR*
P-SKC	*P-CLKR*

(c)

If ∼ is a	*then ⊛ is a*
SSKC	*CLSKR*
SP-SSKC	*SI-CLSKR*
CC-SSKC	*DD-CLSKR*
SP+CC-SSKC	*SI+DD-CLSKR*
P-SSKC	*P-CLSKR*

(d)

If ∼ is a	*then ⊛ is a*
SbAGMC	*CLbAGMR*
SP-SbAGMC	*SI-CLbAGMR*
CC-SbAGMC	*DD-CLbAGMR*
SP+CC-SbAGMC	*SI+DD-CLbAGMR*
P-SbAGMC	*P-CLbAGMR*

Proof: A proof for this corollary can be found on page 235.

The following two corollaries illustrate that each element of one of the classes of credibility-limited base revision operators considered in this chapter gives rise, by means of the Harper identity, to an element of the classes of shielded contractions considered in Chapter 7.

Corollary 8.8.7 *Let A be a consistent belief base. Let ⊛ be a basic credibility-limited base revision operator on A. Let ∼ be defined from ⊛ via the Harper identity. Then ∼ is a basic shielded base contraction operator.*

Proof: A proof for this corollary can be found on page 235.

Corollary 8.8.8 *Let A be a consistent belief base and ⊛ be a credibility-limited base revision operator on A. Let ~ be defined from ⊛ via the Harper identity. Then:*

(a)

If ⊛ is a	***then ~ is a***
CLPMR	*SPMC*
SI-CLPMR	*SP-SPMC*
DD-CLPMR	*CC-SPMC*
SI+DD-CLPMR	*SP+CC-SPMC*
P-CLPMR	*P-SPMC*

(b)

If ⊛ is a	***then ~ is a***
CLKR	*SKC*
SI-CLKR	*SP-SKC*
DD-CLKR	*CC-SKC*
SI+DD-CLKR	*SP+CC-SKC*
P-CLKR	*P-SKC*

(c)

If ⊛ is a	***then ~ is a***
CLSKR	*SSKC*
SI-CLSKR	*SP-SSKC*
DD-CLSKR	*CC-SSKC*
SI+DD-CLSKR	*SP+CC-SSKC*
P-CLSKR	*P-SSKC*

(d)

If ⊛ is a	***then ~ is a***
CLbAGMR	*SbAGMC*
SI-CLbAGMR	*SP-SbAGMC*
DD-CLbAGMR	*CC-SbAGMC*
SI+DD-CLbAGMR	*SP+CC-SbAGMC*
P-CLbAGMR	*P-SbAGMC*

Proof: A proof for this corollary can be found on page 235.

The following theorems illustrate that the operators of non-prioritized base contraction and revision are interdefinable through the Harper and the consistency-preserving Levi identities. These theorems are adaptations to the belief base context of Theorems 4.2.35 and 4.2.36 (which concern operations on belief sets and were originally presented in [FH01]). The functions $\mathbb{C}$ and $\mathbb{R}$ mentioned in these theorems, were presented in Definition 4.2.34.

Theorem 8.8.9 *Let A be a consistent belief base and $\sim$ be an operator for A that satisfies the (shielded contraction) postulates of inclusion, vacuity, extensionality and relative success. Then $\mathbb{C}(\mathbb{R}(\sim)) = \sim$.*

Proof: A proof for this theorem can be found on page 236.

Theorem 8.8.10 *Let A be a consistent belief base and $\circledast$ be an operator for A that satisfies the (credibility-limited revision) postulates of relative success, consistency preservation, inclusion, vacuity and weak extensionality. Then $\mathbb{R}(\mathbb{C}(\circledast)) = \circledast$.*

Proof: A proof for this theorem can be found on page 236.

8.9 Summary

Standard revision operators are always successful in the sense that the sentence by which the original belief base is revised is always incorporated in the resulting revised set. However this is not a realistic feature of a belief revision process. An intelligent agent, when facing new information, should be able to rejected it, for instance, if that information has insufficient plausibility or comes from an unreliable source.

The basic idea of credibility-limited revision is to define a function in two steps. The first step consists of determining if a given belief is credible or not. Then, the credibility-limited revision operator should:
- leave the set of beliefs unchanged when the belief by which it is revised is considered non-credible;
- behave as a standard revision when revising by a credible belief.

In this chapter we performed a deep study of credibility-limited revisions on belief bases. We presented several results highlighting some direct relations among the postulates satisfied by a credibility-limited revision function $\circledast$ induced by a base revision $*$ and a set of credible sentences C and the postulates satisfied by $*$ and the properties of the set C.

We axiomatically characterized several kinds of credibility-limited base revisions and studied the interrelations among all those classes in terms of inclusion.

We established the relation between properties of a set of retractable sentences R and of a set of credible sentences C.

We investigated the relation between the postulates satisfied by an operator of credibility-limited base revision and by an operator of shielded base contraction whenever one of these operators is obtained from the other by means of the consistency-preserving Levi identity or of the Harper identity. Based on these relations, we established the interrelation between classes of non-prioritized contractions and of non-prioritized revisions, whenever the operators of the latter (respectively, former) classes are obtained from the operators of the former (respectively, latter) via the consistency-preserving Levi (respectively, Harper) identity.

Chapter 9

Conclusion and Future Work

"Every new beginning comes from
some other beginning's end."
Seneca

This final chapter is devoted to the presentation of a brief overview of the work presented along this thesis, referring its main contributions and mentioning some potential topics for future research.

9.1 Summary

One of the main goals underlying the research area of *belief change* consists in finding appropriate ways to model the belief state of an agent and study the changes that occur in that belief state when the agent receives some new information. In Chapter 1 we presented a brief introduction to the research area of belief change, presenting some of its main motivations and goals.

In Chapter 2 we presented the main concepts of *epistemological theories*, namely *epistemic state* (or *belief state*), *epistemic attitude*, *epistemic input* and *epistemic change*. We saw that there are several ways of modelling a belief state of an agent and, from among those, we gave special emphasis to models where epistemic states are represented by sets of sentences of a propositional language (sentential models) and gave a first glimpse of the three operations of change that we addressed in the following chapters:

- **Expansion:** when new information is simply added to the set of the beliefs of an agent.
- **Revision:** when new information is added to the set of the beliefs of an agent in a consistent matter.
- **Contraction:** when information is removed from the set of beliefs of an agent.

We ended Chapter 2 by recalling some of the rationality criteria proposed in the literature: *principle of categorical matching, irrelevance of syntax, primacy (of the* *new information), consistency, logical omniscience, minimal change, fairness* and *preference.*

In the third chapter we presented the one that is currently considered the standard model in the belief change literature, known as the AGM model — after the initials of its three creators: Carlos Alchourrón, Peter Gärdenfors and David Makinson — that was originally presented in [AGM85]. In this framework, each *belief* of an agent is represented by a sentence (of a propositional language $\mathcal{L}$) and the *belief state* of an agent is represented by a logically closed set of (belief-representing) sentences — called *belief sets.* We presented the three change operators of this model: *expansions* (+), *contractions* (÷) and *revisions* ($\star$) and the postulates (properties) that characterize each one of these operators. Expansion is the simplest of the three operations. The expansion of a belief set $\mathbf{K}$ by a sentence α consists simply in adding α to $\mathbf{K}$ and then closing the resulting set by logical consequence, *i.e.*, $\mathbf{K} + \alpha = Cn(\mathbf{K} \cup \{\alpha\})$. The rest of this chapter was dedicated to the study of operators of contraction and revision. We presented the Levi identity and Harper identity that allow the inter-definition between operators of contraction and revision and afterwards we presented some explicit methods for constructing contraction and revision operators as well as the axiomatic characterization for each of the classes of operators obtained through these methods. We presented a model based on a system of spheres that allows us to study the change operations from a semantic point of view allowing a more intuitive understanding of the operations of change. We dedicated the last part of Chapter 3 to study the logical relationships between the classes of contraction operators mentioned along this chapter. We saw that *transitively relational partial meet contractions, safe contractions based on a regular and virtually connected hierarchy, sphere-based contractions, epistemic entrenchment-based contractions* and *AGM contractions* can be seen as (five) alternative ways of defining the same class of contraction functions and that the classes of contraction functions formed by *possible worlds-based contractions, partial meet contractions* and *basic AGM contractions* coincide, and that every *safe contraction* is a *partial meet contraction*, but that the converse does not hold.

In Chapter 4 we discussed some of the problems of the AGM model identified by the belief change community as well as some proposals to deal with those problems:

- The inadequacy of representing an agent's belief state by belief sets. We discussed the disadvantages of such a representation as well as the use of belief bases instead of belief sets for that purpose.

- The inadequacy of the AGM model to deal with iterated change and with multiple change (the input in the original AGM model is a single sentence and not a set of sentences). We briefly mentioned some of the models proposed in the literature for iterated revision and for multiple contraction.

- Some of the proposed postulates can be considered inadequate to characterize change operators, namely *recovery* and *success* (for both contraction and revision). In what concerns *recovery* we pointed out the existence of some classes of contraction functions that satisfy the basic AGM postulates for contraction with the exception of *recovery* — the so-called *withdrawals*. For *success* we mentioned several models of non-prioritized change operators.

Regarding non-prioritized belief change, we dedicated special attention to operators of *credibility-limited revision* and of *shielded contraction*. We extended the work presented in [HFCF01] and in [FH01] by axiomatically characterizing classes of credibility-limited operators that were not characterized in [HFCF01] and by establishing the interrelation between different credibility-limited revision and shielded contraction operators by means of the consistency-preserving Levi identity and the Harper identity.

In Chapter 5 we presented postulates for contraction and for revision in the belief base context. We recalled some constructive methods for contraction operators on belief bases, namely, partial meet contractions, (smooth) kernel contractions, basic AGM-generated base contractions as well as the axiomatic characterization for each one of these operators. We also presented two operators proposed by Williams in [Wil94a] based on the notion of ensconcement, namely brutal and ensconcement-based contractions. We recalled some construction methods for revision operators on belief bases, namely, partial meet revisions and kernel revisions as well as the axiomatic characterization of each one of these revision functions. We also defined and presented representation theorems for other two kinds of base revision functions, namely smooth kernel revisions and basic AGM-generated base revisions. These revision operators are based on their namesake contraction functions. At the end of Chapter 5 we briefly recalled some operators of non-prioritized belief change on belief bases, namely: semi-revision (and consolidation), credibility-limited base revision and shielded base contraction. We also refined the representation theorem for shielded base contraction operators presented in [FMT03] by identifying a couple of redundant postulates in that representation theorem.

The main contributions of this thesis were presented in Chapters 6 to 8.

In Chapter 6 we presented representation theorems for ensconcement-based contractions and for brutal contractions. For that purpose we proposed some new postulates. Ensconcement-based contractions and brutal contractions were introduced by Mary-Anne Williams in [Wil94a] and are based on the concept ensconcement. An ensconcement is an ordering of sentences and can be seen as an adaptation to the belief base context of epistemic entrenchments (introduced in [Gär88, GM88]). We compared the axiomatic characterizations of *brutal contractions* and of *ensconcement-based contractions* in order to identify the postulates that can be considered characteristic properties of each one of those two kinds of contraction functions, in the sense that they are satisfied by only one of those two kinds of operators. We also compared the representation theorems of the other base contraction operators presented in Section 5.2. This allowed us to build the map presented in Figure 6.1. In particular we concluded that every ensconcement-based contraction is a basic AGM-generated base contraction but in general is not a (smooth) kernel nor a partial meet contraction. Afterwards we studied the construction of an ensconcement relation by means of *brutal* and *ensconcement-based contractions*. We finished Chapter 6 by presenting some results that relate base contraction postulates and belief set contraction postulates and by investigating the connections between the ensconcement-based contractions (respectively, brutal contractions) and the epistemic entrenchment-based contractions (respectively, severe withdrawals). Regarding this study, the main conclusion that we achieved was that a contraction function on a belief set is an ensconcement-based contraction (respectively, a brutal contraction) if and only if it is an epistemic entrenchment-based contraction (respectively, a severe withdrawal).

Chapter 7 was devoted to the study of shielded base contraction operators. The definition of shielded contractions was motivated by the fact that an agent can have several (non-tautological) beliefs that he/she is not willing to give up (even when trying to contract by it). The *set of retractable sentences* R models the set of sentences that the agent is willing to give up (if needed). Informally speaking, the shielded contraction is a function that receives (just as a *standard* contraction does) a set and a sentence and returns:

- The received set (unchanged), if the sentence is not included in R;
- The output produced by the associated contraction, if the sentence is in R.

The concept of shielded contraction for belief bases was introduced in [FMT03] by adapting the original definition for shielded contractions on belief sets proposed in [FH01]. In [FMT03] a shielded contraction was defined in terms of a partial meet contraction and a set R of retractable sentences. Therefore, the outcome of performing a shielded contraction of a belief base A by a sentence, returns:

- The belief base A (unchanged), if the sentence is not included in R;
- The output produced by the partial meet contraction that induces it, if the sentence is in R.

In this chapter we extended the definition presented in [FMT03] by defining shielded base contractions induced by any contraction function instead of only by partial meet base contractions as in [FMT03]. We presented several results highlighting some direct relations among the postulates satisfied by a shielded contraction function $\sim$ induced by a base contraction $-$ and a set of retractable sentences R and the postulates satisfied by $-$ and the properties of the set R. From the conclusions obtained we highlight that:

(i) The shielded base contraction $\sim$ satisfies relative success if and only if $-$ satisfies *failure* or the set R satisfies *non-retractability of tautology*.

(ii) If R and $-$ satisfy condition (**R** - $-$) then $\sim$ satisfies *persistence*. Furthermore if $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$ and $\sim$ satisfies *persistence* then R and $-$ satisfy condition (**R** - $-$).

Afterwards we studied classes of shielded base contraction induced by several well-known kinds of base contractions (not only by partial meet base contractions) and several kinds of sets of retractable sentences (*i.e.* we consider several different, and non-equivalent, sets of properties for characterizing a set of retractable sentences). We axiomatically characterized all the classes of shielded base contractions considered and investigated the interrelations among those classes in terms of strict inclusion. *I.e.*, we investigated whether each of those classes is or is not strictly contained in each of the remaining ones. Four of the classes studied satisfy *persistence*, one of the postulates that we believe best captures the underlying insights of shielded contractions (and of irretractable sentences).

Chapter 8 was dedicated to the study of credibility-limited base revision operators. The definition of credibility-limited revision operators was motivated by the existence of beliefs that a given agent is unwilling to incorporate when performing a revision. The *set of credible sentences* C models the set of sentences that the agent is willing to incorporate during a revision process. Informally speaking, the credibility-limited revision is a function that receives (just as a *standard* revision does) a set and a sentence and returns:

- The received set (unchanged), if the sentence is not included in C;
- The output produced by the associated revision, if the sentence is in C.

The concept of credibility-limited revision for belief bases was introduced in [FMT03] by adapting the original definition for credibility-limited revisions on belief sets proposed in [HFCF01]. Following the same structure as in Chapter 7 we start this chapter by extending the definition of credibility-limited revisions operators presented in [FMT03] by defining credibility-limited revisions operators induced by any revision function instead of only by partial meet base revision as in [FMT03]. We also proposed some new properties for the set of credible sentences and the *persistence* postulate for credibility-limited revision based on the namesake postulate for shielded contractions. We presented several results highlighting some direct relations among the postulates satisfied by a credibility-limited base revision function $\circledast$ induced by a base revision $*$ and a set of credible sentences C and the postulates satisfied by $*$ and the properties of the set C. Based on the results obtained, we thoroughly studied several classes of credibility-limited revision operators induced by different kinds of base revisions (partial meet, (smooth) kernel, basic AGM-generated base revisions) and several kinds of sets of credible sentences (*i.e.* we consider several different, and non-equivalent, sets of properties for characterizing such a set) and presented for each one of those classes of operators a representation theorem. We then investigated the interrelations among those classes in terms of strict inclusion. We finished this chapter by studying the interrelation between the different classes of shielded base contraction and of credibility-limited base revision operators, presented respectively in Chapters 7 and 8, by means of the consistency-preserving Levi identity and the Harper identity. We ended Chapter 8 showing, that shielded contractions and credibility-limited revisions can be interdefined by means of the mentioned identities as long as the operators under consideration satisfy some of the postulates of shielded contractions and of credibility-limited revisions.

9.2 Future work

In this section, we present a list of potential future research topics that arise naturally in the sequence of the investigation reported in this thesis.

1. To find more natural postulates than (EB1) and (EB2) to be used (instead of these two postulates) in the axiomatic characterization of ensconcement-based contractions.

2. To define and present a representation theorem for operators of revisions (on belief bases) based on the concept of ensconcement.

3. To combine the work presented in Chapters 6 and 7 in order to define and axiomatically characterize operators of non-prioritized contraction on belief bases based in the notion of ensconcement. To do so we could adapt the concept of ensconcement relation in order to allow the existence of non-tautological sentences as ensconced as tautologies. An obvious way to do so is by weakening the definition of ensconcement relation, namely property ($\preceq$2) (a similar process was followed in the definition of shielded entrenchment-based contractions in [FH01]).

4. To extend the concepts of ensconcement-based contractions and shielded contractions to the case of contractions by sets of sentences rather than only by a single sentence, *i.e.*, multiple contraction.

5. To define operators on belief bases that allow partial acceptance of the new information. Operators mentioned along this thesis either accept a revision or completely reject it, if the new information is insufficiently credible. However an alternative natural behaviour would be to accept only (the credible) part of the new information. This perspective is inspired in the operators of selective revision for belief sets presented in [FH99].

6. To implement computationally the belief change operators proposed in this thesis and analyse its computational complexity.

Appendix

Appendix A

Proofs of Chapter 3

Proof of Observation 3.2.50.
Let $\mathbf{K}$ be a belief set and $\div$ an operator that satisfies $(\div 1)$, $(\div 3')$, $(\div 4)$ and $(\div 9)$. Assume that $Cn(\alpha) = Cn(\beta)$. We intent to prove that $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$. Assume first that $\vdash \alpha \wedge \beta$. Hence $\vdash \alpha$ and $\vdash \beta$. Therefore, by $(\div 3')$, it follows that $\mathbf{K} \div \alpha = \mathbf{K} \div \beta = \mathbf{K}$. Assume now that $\nvdash \alpha \wedge \beta$. From $Cn(\alpha) = Cn(\beta)$ it follows that $\nvdash \alpha$ and $\nvdash \beta$ furthermore it follows that $\beta \in Cn(\alpha)$ and $\alpha \in Cn(\beta)$. From $(\div 4)$ and $(\div 1)$ it follows that $\mathbf{K} \div \alpha \nvdash \alpha$ and $\mathbf{K} \div \beta \nvdash \beta$. Therefore $\beta \notin \mathbf{K} \div \alpha$ and $\alpha \notin \mathbf{K} \div \beta$. Hence, by $(\div 9)$, it follows that $\mathbf{K} \div \alpha = \mathbf{K} \div \beta$. ■

Appendix B

Proofs of Chapter 4

Proof of Observation 4.2.2.

(a) Trivial.

(b) Let $\alpha \in \mathcal{L}$ be such that $\vdash \alpha$. Hence $\neg\alpha \vdash \bot$. Therefore, by *element consistency*, $\neg\alpha \notin C$, from which it follows by *negation completeness* that $\alpha \in C$. ■

Proof of Observation 4.2.3.
Assume first that it holds that $Cn(\varnothing) \subseteq C$. Hence $(\neg\alpha \vee \alpha) \in C$. Therefore, by *disjunctive completeness* it follows that $\neg\alpha \in C$ or $\alpha \in C$.
Assume now that C satisfies *negation completeness*. It follows by Observation 4.2.2 (b) that $Cn(\varnothing) \subseteq C$. ■

Proof of Observation 4.2.4.
Let $\mathbf{K}$ be a consistent belief set and $\beta \in \mathbf{K}$. Assume by *reductio ad absurdum*, that $\beta \notin C$. By *closure under logical equivalence* it follows that $\neg\neg\beta \notin C$. Hence by *expansive credibility* $\mathbf{K} \vdash \neg\beta$. Contradiction, since $\mathbf{K} \nvdash \bot$. ■

Proof of Observation 4.2.8.
Let $\mathbf{K}$ be a consistent belief set and $\odot$ be an operator on $\mathbf{K}$ that satisfies *relative success, vacuity, inclusion* and *disjunctive distribution.* Suppose that $\mathbf{K}\odot\alpha = \mathbf{K}\odot\beta = \mathbf{K}$. If $\alpha\vee\beta \notin \mathbf{K}\odot(\alpha\vee\beta)$, then by *relative success* $\mathbf{K}\odot(\alpha\vee\beta) = \mathbf{K}$. If $\alpha\vee\beta \in \mathbf{K}\odot(\alpha\vee\beta)$, then by *disjunctive distribution* it holds that either $\alpha \in \mathbf{K}\odot\alpha$ or $\beta \in \mathbf{K}\odot\beta$. In both cases it yields that $\alpha\vee\beta \in \mathbf{K}$ (since $\mathbf{K}\odot\alpha = \mathbf{K}\odot\beta = \mathbf{K}$ and $\mathbf{K}$ is logically closed). $\mathbf{K}$ is consistent, hence $\neg(\alpha\vee\beta) \notin \mathbf{K}$. Thus, by *vacuity* and *inclusion*, it follows that $\mathbf{K} + (\alpha\vee\beta) = \mathbf{K}\odot(\alpha\vee\beta)$. Hence, since $\alpha\vee\beta \in \mathbf{K}$ it follows that $\mathbf{K} = \mathbf{K}\odot(\alpha\vee\beta)$.
Assume now that $\odot$ is an operator on $\mathbf{K}$ that satisfies *relative success, vacuity, inclusion* and *disjunctive constancy.* Suppose that $\alpha \notin \mathbf{K}\odot\alpha$ and $\beta \notin \mathbf{K}\odot\beta$. By *relative success* it follows that $\mathbf{K}\odot\alpha = \mathbf{K}\odot\beta = \mathbf{K}$. Thus, by *disjunctive constancy* it follows that $\mathbf{K}\odot(\alpha\vee\beta) = \mathbf{K}$. On the other hand, from $\alpha \notin \mathbf{K}\odot\alpha$ and $\beta \notin \mathbf{K}\odot\beta$ it follows by *vacuity* that $\mathbf{K} \vdash \neg\alpha$ and $\mathbf{K} \vdash \neg\beta$. Thus $\mathbf{K} \vdash \neg(\alpha\vee\beta)$. Since $\mathbf{K}$ is consistent it holds that $\alpha\vee\beta \notin \mathbf{K}$. Therefore $\alpha\vee\beta \notin \mathbf{K}\odot(\alpha\vee\beta)$. ■

Proof of Theorem 4.2.14.
Follows trivially by Observations 4.2.2, 4.2.4, 4.2.9, 4.2.10, 4.2.12 and 4.2.13. ■

Proof of Theorem 4.2.16.
(1 → 3) Let $C = \{\alpha : \alpha \in \mathbf{K} \odot \alpha\}$ and let $\star$ be such that:

(i) if $\alpha \in \mathbf{K} \odot \alpha$, then $\mathbf{K}\star\alpha = \mathbf{K} \odot \alpha$;

(ii) if $\alpha \notin \mathbf{K} \odot \alpha$, then $\mathbf{K}\star\alpha = \mathbf{K}\star'\alpha$, for some AGM revision operator $\star'$.

C and $\star$ are defined in the same way as in the corresponding parts of the proofs of Observations 4.2.12 and 4.2.13, and (consequently) also of the proof of Theorem 4.2.14. Thus $\star$ is a basic AGM revision operator. By Theorem 4.2.14, $\odot$ is the operator of credibility-limited revision induced by $\star$ and a set C that satisfies $\mathbf{K} \subseteq C$, *single sentence closure, disjunctive completeness, element consistency* and *expansive credibility*. We will now show that C satisfies *strong revision credibility*. Let $\alpha \notin C$ and $\beta \in \mathcal{L}$. Hence $\alpha \notin \mathbf{K}\odot\alpha$. Thus, by *strong regularity*, $\neg\alpha \in \mathbf{K}\odot\beta$. Thus C satisfies *strong revision credibility*.
It remains to prove that $\star$ satisfies *disjunctive factoring* (Observation 3.1.12).
If $\alpha \vee \beta \notin \mathbf{K} \odot (\alpha \vee \beta)$, then by $\odot$ *strict improvement* $\alpha \notin \mathbf{K} \odot \alpha$ and $\beta \notin \mathbf{K} \odot \beta$. Therefore $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}\star'(\alpha \vee \beta)$, $\mathbf{K}\star\alpha = \mathbf{K}\star'\alpha$ and $\mathbf{K}\star\beta = \mathbf{K}\star'\beta$. The rest of the proof for this case follows by $\star'$ *disjunctive factoring*.
Suppose now that $\alpha \vee \beta \in \mathbf{K} \odot (\alpha \vee \beta)$. Hence, by definition of $\star$ it follows that $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K} \odot (\alpha \vee \beta)$. By $\odot$ *disjunctive factoring* it follows that $\mathbf{K}\star(\alpha \vee \beta)$ is either $\mathbf{K} \odot \alpha$, $\mathbf{K} \odot \beta$ or $\mathbf{K} \odot \alpha \cap \mathbf{K} \odot \beta$. On the other hand from $\alpha \vee \beta \in \mathbf{K} \odot (\alpha \vee \beta)$ it follows, by *disjunctive distribution*, that $\alpha \in \mathbf{K} \odot \alpha$ or $\beta \in \mathbf{K} \odot \beta$. There are three cases to consider:
Case 1) $\alpha \in \mathbf{K} \odot \alpha$ and $\beta \in \mathbf{K} \odot \beta$. Therefore, by definition of $\star$, it follows that $\mathbf{K}\star\alpha = \mathbf{K}\odot\alpha$ and $\mathbf{K}\star\beta = \mathbf{K}\odot\beta$. Hence, $\mathbf{K}\star(\alpha\vee\beta)$ is either $\mathbf{K}\star\alpha$, $\mathbf{K}\star\beta$ or $\mathbf{K}\star\alpha\cap\mathbf{K}\star\beta$.
Case 2) $\alpha \in \mathbf{K} \odot \alpha$ and $\beta \notin \mathbf{K} \odot \beta$. By $\odot$ *relative success* it follows that $\mathbf{K} \odot \beta = \mathbf{K}$. By definition of $\star$ it follows that $\mathbf{K}\star\alpha = \mathbf{K} \odot \alpha$. On the other hand, from $\beta \notin \mathbf{K} \odot \beta$ it follows, by $\odot$ *vacuity*, that $\neg\beta \in \mathbf{K}$. By $\odot$ *disjunctive factoring* it follows that $\mathbf{K}\star(\alpha\vee\beta)$ is either $\mathbf{K}\star\alpha$, $\mathbf{K}$ or $\mathbf{K}\star\alpha\cap\mathbf{K}$. Hence, there are two sub-cases to consider:
Case 2.1) $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}$. Therefore, by $\star$ *success* and since $\mathbf{K}$ is logically closed, it follows that $\neg\beta \wedge (\alpha \vee \beta) \in \mathbf{K}$. Hence $\alpha \in \mathbf{K}$. Thus $\neg\alpha \notin \mathbf{K}$, since $\mathbf{K}$ is consistent. Hence, by $\odot$ *vacuity* and *inclusion*, it follows that $\mathbf{K} \odot \alpha = \mathbf{K} + \alpha = \mathbf{K}$. From which it follows that $\mathbf{K}\star\alpha = \mathbf{K} \odot \alpha = \mathbf{K}$. Therefore, $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}\star\alpha$.
Case 2.2) $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}\star\alpha \cap \mathbf{K}$. By $\star$ *success* it follows that $\alpha \vee \beta \in \mathbf{K}\star(\alpha \vee \beta)$. Therefore $\alpha \vee \beta \in \mathbf{K}$. It follows as in the previous case that $\mathbf{K}\star\alpha = \mathbf{K}$. Thus $\mathbf{K}\star(\alpha \vee \beta) = \mathbf{K}\star\alpha$.
Case 3) $\alpha \notin \mathbf{K} \odot \alpha$ and $\beta \in \mathbf{K} \odot \beta$. Follows as in case 2) by symmetry.
(3 → 2) Trivial.
(2 → 1) Let $\mathbf{K}$ be a consistent belief set and $\star$ an AGM revision operator on $\mathbf{K}$. Let C be a set of sentences that satisfies single sentence closure, disjunctive completeness, element consistency, expansive credibility and strong revision credibility. Let $\odot$ be an operator such that:

$$\mathbf{K} \odot \alpha = \begin{cases} \mathbf{K}\star\alpha & \text{if } \alpha \in C \\ \mathbf{K} & \text{otherwise} \end{cases}$$

By Theorem 4.2.14, $\odot$ satisfies closure, relative success, inclusion, consistency preservation, extensionality, vacuity, strict improvement and disjunctive distribution. It remains to show that $\odot$ satisfies *strong regularity* and *disjunctive factoring.*

Strong regularity: Let $\alpha \in \mathcal{L}$. Suppose that $\beta \notin \mathbf{K} \odot \beta$. Thus by $\star$ *success* it follows that $\mathbf{K} \odot \beta \neq \mathbf{K} \star \beta$. Therefore, by definition of $\odot$, $\beta \notin C$. Hence, by *strong revision credibility*, it follows that $\mathbf{K} \odot \alpha \vdash \neg\beta$. Hence, by $\odot$ closure, $\neg\beta \in \mathbf{K} \odot \alpha$.

Disjunctive factoring: We will consider four cases:

Case 1) $\alpha \notin C$ and $\beta \notin C$. By *disjunctive completeness* it follows that $\alpha \vee \beta \notin C$. Thus, by definition of $\odot$, $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K} \odot \alpha = \mathbf{K} \odot \beta = \mathbf{K}$.

Case 2) $\alpha \in C$ and $\beta \in C$. Then, by *single sentence closure*, $\alpha \vee \beta \in C$. Therefore, by definition of $\odot$, $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K} \star (\alpha \vee \beta), \mathbf{K} \odot \alpha = \mathbf{K} \star \alpha$ and $\mathbf{K} \odot \beta = \mathbf{K} \star \beta$. The rest of the proof for this case follows by $\star$ *disjunctive factoring* (Observation 3.1.12).

Case 3) $\alpha \notin C$ and $\beta \in C$. By *single sentence closure*, it follows that $\alpha \vee \beta \in C$. Hence, from the definition of $\odot$, it follows that $\mathbf{K} \odot \alpha = \mathbf{K}$, $\mathbf{K} \odot \beta = \mathbf{K} \star \beta$ and $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K} \star (\alpha \vee \beta)$. By $\star$ *disjunctive factoring* it follows that $\mathbf{K} \odot (\alpha \vee \beta)$ is either $\mathbf{K} \star \alpha$, $\mathbf{K} \odot \beta$ or $\mathbf{K} \star \alpha \cap \mathbf{K} \odot \beta$. If $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K} \odot \beta$, then we are done.

On the other hand, from $\alpha \notin C$ it follows by *strong revision credibility* that $\mathbf{K} \odot (\alpha \vee \beta) \vdash \neg\alpha$. Thus, by $\odot$ *consistency preservation*, it follows that $\alpha \notin \mathbf{K} \odot (\alpha \vee \beta)$. Thus by $\star$ *success* $\mathbf{K} \odot (\alpha \vee \beta) \neq \mathbf{K} \star \alpha$.

If $\mathbf{K} \odot (\alpha \vee \beta) = \mathbf{K} \star \alpha \cap \mathbf{K} \odot \beta$, then $\neg\alpha \in \mathbf{K} \star \alpha$. Thus, by $\star$ *success*, $\mathbf{K} \star \alpha \vdash \bot$. By $\star$ *closure*, it holds that $\mathbf{K} \star \alpha$ is a belief set, hence $\mathbf{K} \star \alpha = \mathcal{L}$, from which it follows that $\mathbf{K} \odot (\alpha \vee \beta) = \mathcal{L} \cap \mathbf{K} \odot \beta = \mathbf{K} \odot \beta$.

Case 4) $\alpha \in C$ and $\beta \notin C$. Follows as in case 3) by symmetry. ∎

Proof of Theorem 4.2.32.

(a) Let $\ominus$ be a basic shielded contraction operator on $\mathbf{K}$. Then, by Definition 4.2.26, $\ominus$ satisfies *closure, inclusion, vacuity, extensionality, recovery, relative success, success propagation* and *conjunctive constancy.* Hence, by Observation 4.2.30, the operator $\odot$ defined from $\ominus$ via the consistency-preserving Levi identity satisfies *closure, relative success, inclusion, consistency preservation, extensionality, vacuity, strict improvement* and *disjunctive constancy.* It follows by Observation 4.2.8 that $\odot$ also satisfies *disjunctive distribution.* Thus, by Definition 4.2.15, $\odot$ is a basic credibility-limited revision operator on $\mathbf{K}$.

(b) Let $\ominus$ be a non-basic shielded contraction operator on $\mathbf{K}$. Then, by Definition 4.2.28, $\ominus$ satisfies *closure, inclusion, vacuity, extensionality, recovery, relative success, persistence, conjunctive inclusion* and *conjunctive overlap.* By Observation 4.2.24 $\ominus$ also satisfies *success propagation* and *conjunctive constancy.* Hence, by Observation 4.2.30, the operator $\odot$ defined from $\ominus$ via the consistency-preserving Levi identity satisfies *closure, relative success, inclusion, consistency preservation, extensionality, vacuity, strict improvement, disjunctive constancy, strong regularity, guarded subexpansion* and *superexpansion.* By Observation 4.2.10, $\odot$ also satisfies *consistency.* It follows by Observations 4.2.11 and 4.2.8 that $\odot$ also satisfies *disjunctive distribution* and *disjunctive factoring.* Thus, by Definition 4.2.17, $\odot$ is a non-basic credibility-limited revision operator on $\mathbf{K}$. ∎

Proof of Theorem 4.2.33.

(a) Let $\odot$ be a basic credibility-limited revision operator. Then, by Definition 4.2.15, $\odot$ satisfies *closure, relative success, inclusion, consistency preservation, extensionality, vacuity, strict improvement* and *disjunctive distribution*. It follows by Observation 4.2.8 that $\odot$ also satisfies *disjunctive constancy*. Hence, by Observation 4.2.31, the operator $\ominus$ defined from $\odot$ via the Harper identity satisfies *closure, inclusion, vacuity, extensionality, recovery, relative success, success propagation* and *conjunctive constancy*. Hence, by Definition 4.2.26, $\ominus$ is a basic shielded contraction operator on $\mathbf{K}$.

(b) Let $\odot$ be a non-basic credibility-limited revision operator. Then, by Definition 4.2.17, $\odot$ satisfies *closure, relative success, inclusion, consistency preservation, extensionality, vacuity, strict improvement, disjunctive distribution, strong regularity* and *disjunctive factoring*. By Observation 4.2.10, $\odot$ satisfies *consistency*. It follows by Observations 4.2.11 and 4.2.8 that $\odot$ also satisfies *disjunctive constancy, superexpansion* and *guarded subexpansion*. Hence, by Observation 4.2.31, the operator $\ominus$ defined from $\odot$ via the Harper identity satisfies *closure, inclusion, vacuity, extensionality, recovery, relative success, persistence, conjunctive inclusion* and *conjunctive overlap*. Thus, by Definition 4.2.28, $\ominus$ is a non-basic shielded contraction operator on $\mathbf{K}$. ■

Appendix C

Proofs of Chapter 5

Lemma C.1 *Let A be a belief base and $-$ an operator on A that satisfies success and strong inclusion. Then $-$ satisfies: If $\nvdash \alpha$, then $A - \alpha \subseteq A - (\alpha \wedge \beta)$.*

Proof.

Let $\nvdash \alpha$. Hence, by *success* $A - \alpha \nvdash \alpha$, and so $A - \alpha \nvdash \alpha \wedge \beta$. Therefore, by *strong inclusion*, $A - \alpha \subseteq A - (\alpha \wedge \beta)$. ■

Lemma C.2 *Let A be a belief base and $-$ be an operator on A that satisfies success, inclusion, failure and strong inclusion. Then $-$ satisfies:*

(a) *If $-$ also satisfies relative closure, then: if $\alpha \in A \setminus A - \beta$, then $A - \beta \subseteq A - \alpha$.*

(b) *If $\nvdash \alpha, \nvdash \beta$ and $A - \alpha \vdash \beta$, then $A - \beta \subseteq A - \alpha$.*

(c) *If $\nvdash \alpha$ and $\alpha \in A - \beta$, then $A - \alpha \subset A - \beta$.*

(d) *If $A - \alpha \subset A - \beta$, then $A - \beta \vdash \alpha$.*

Proof.

(a) Let $\alpha \in A \setminus A - \beta$, then it follows by *relative closure* that $A - \beta \nvdash \alpha$ and so, by *strong inclusion*, $A - \beta \subseteq A - \alpha$.

(b) It follows from $\nvdash \alpha$ and Lemma C.1 that $A-\alpha \subseteq A-(\alpha\wedge\beta)$. Hence, $A-(\alpha\wedge\beta) \vdash \beta$. Therefore, since $\nvdash \alpha \wedge \beta$, due to *success*, it follows that $A - (\alpha \wedge \beta) \nvdash \alpha$. From *strong inclusion* it follows that $A-(\alpha\wedge\beta) \subseteq A-\alpha$, and so $A-(\alpha\wedge\beta) = A-\alpha$. On the other hand, since $\nvdash \beta$ it follows from Lemma C.1 that $A-\beta \subseteq A-(\alpha\wedge\beta) = A - \alpha$.

(c) Let $\nvdash \alpha$ and $\alpha \in A - \beta$, then $A - \alpha \neq A - \beta$, since from *success* $\alpha \notin A - \alpha$. We will prove by cases:
Case 1) $A - \alpha \nvdash \beta$. By *strong inclusion*, $A - \alpha \subseteq A - \beta$. Hence $A - \alpha \subset A - \beta$.
Case 2) $\vdash \beta$. It follows from *failure* that $A - \beta = A$ and from *inclusion* that $A - \alpha \subseteq A - \beta$. Hence $A - \alpha \subset A - \beta$.
Case 3) $A - \alpha \vdash \beta$ and $\nvdash \beta$. It follows from $\nvdash \alpha$ and (b) that $A - \beta \subseteq A - \alpha$. Contradiction, since $\alpha \in A - \beta$ and $\nvdash \alpha$.

(d) Follows by *strong inclusion.* ■

Lemma C.3 *Let A be a belief base and $-$ an operator on A that satisfies success, inclusion, vacuity, failure, relative closure and strong inclusion. Then $-$ satisfies: If $\alpha \in A - \beta$ and $\nvdash \beta$, then $\beta \notin A - \alpha \wedge \beta$.*

Proof.
Let $-$ be an operator on A that satisfies *success, inclusion, vacuity, failure, relative closure* and *strong inclusion.* Then by Observation 5.1.3 (d) $-$ satisfies *extensionality, expulsiveness* and *decomposition.* Let $\alpha \in A - \beta$ and $\nvdash \beta$. If $\vdash \alpha$, then $A - \alpha \wedge \beta = A - \beta$, by *extensionality.* Hence, by *success* $\beta \notin A - \alpha \wedge \beta$. Consider now that $\nvdash \alpha$ and assume by *reductio ad absurdum* that $\beta \in A - \alpha \wedge \beta$. By *decomposition* and *success*, it follows that $A - \alpha \wedge \beta = A - \alpha$. Thus $\beta \in A - \alpha$. Contradiction, by *expulsiveness.* ■

Proof of Observation 5.1.3.

(a) Let $\beta \in A$ and $\beta \notin A - \alpha$. Then, by *logical relevance*, there is some set A' such that $A - \alpha \subseteq A' \subseteq Cn(A)$ and $A' \nvdash \alpha$ but $A' \cup \{\beta\} \vdash \alpha$. From $A' \cup \{\beta\} \vdash \alpha$ it follows, by deduction, that $A' \vdash \beta \rightarrow \alpha$. Hence $A' \vdash \neg\beta \vee \alpha$. If it was the case that $A' \vdash \alpha \vee \beta$ it would follow that $A' \vdash \alpha$. Hence $A' \nvdash \alpha \vee \beta$. Therefore, by monotony, $A - \alpha \nvdash \alpha \vee \beta$.

(b) Let $\beta \in A$, $\beta \notin A - \alpha$ and consider $A' = A - \alpha \cup \{\neg\beta \vee \alpha\}$. From $\beta \in A$ and $\beta \notin A - \alpha$ it follows, by *vacuity*, that $A \vdash \alpha$. Hence $\neg\beta \vee \alpha \in Cn(A)$. By *inclusion*, $A - \alpha \subseteq A \subseteq Cn(A)$. Therefore $A' \subseteq Cn(A)$. On the other hand, since $\neg\beta \vee \alpha \in A'$, it follows that $A' \cup \{\beta\} \vdash \alpha$. It remains to prove that $A' \nvdash \alpha$. Assume by *reductio ad absurdum* that $A' \vdash \alpha$. Hence, by deduction, it follows that $A - \alpha \vdash (\neg\beta \vee \alpha) \rightarrow \alpha$. Since $(\neg\beta \vee \alpha) \rightarrow \alpha$ is logically equivalent to $\alpha \vee \beta$, it follows that $A - \alpha \vdash \alpha \vee \beta$ which contradicts *disjunctive elimination.* Hence $A' \nvdash \alpha$.

(d) **Linearity:** We will prove by cases:
Case 1) $\vdash \alpha$, it follows from *failure* that $A - \alpha = A$ and so (by *inclusion*) $A - \beta \subseteq A - \alpha$.
Case 2) $\vdash \beta$, due to the symmetry of the case, it follows that $A - \alpha \subseteq A - \beta$.
Case 3) $A - \alpha \nvdash \beta$, then by *strong inclusion* $A - \alpha \subseteq A - \beta$.
Case 4) $\nvdash \alpha, \nvdash \beta$ and $A - \alpha \vdash \beta$. It follows from Lemma C.2 (b) that $A - \beta \subseteq A - \alpha$.
Expulsiveness: Follows by *success* and *linearity.*
Extensionality: If $\vdash \alpha \wedge \beta$ it follows trivially from *failure.* Assume now that $\nvdash \alpha \wedge \beta$. It follows from $\vdash \alpha \leftrightarrow \beta$ that $\nvdash \alpha$, $\nvdash \beta$, $\vdash \alpha \rightarrow \beta$ and $\vdash \beta \rightarrow \alpha$. Then, due to *success*, $A - \beta \nvdash \alpha$ and $A - \alpha \nvdash \beta$. Hence, by *strong inclusion*, $A - \alpha = A - \beta$.
Decomposition: We will prove by cases:
Case 1) $\vdash \alpha \wedge \beta$. Follows trivially from *failure.*
Case 2) $\vdash \alpha$ and $\nvdash \beta$. It follows from *success* that $A - (\alpha \wedge \beta) \nvdash \beta$. Then, by Lemma C.1 and *conjunctive inclusion* (Observation 5.1.3 (c)) it follows that $A - \beta = A - (\alpha \wedge \beta)$.
Case 3) $\nvdash \alpha$ and $\vdash \beta$. Due to the symmetry of the case, it follows that

$A - \alpha = A - (\alpha \wedge \beta)$.
Case 4) $\nvdash \alpha$ and $\nvdash \beta$. It follows from Lemma C.1 that $A - \alpha \subseteq A - (\alpha \wedge \beta)$ and $A - \beta \subseteq A - (\alpha \wedge \beta)$. On the other hand, by *success*, it follows that $A - (\alpha \wedge \beta) \nvdash \alpha$ or $A - (\alpha \wedge \beta) \nvdash \beta$. Then by *conjunctive inclusion*, $A - (\alpha \wedge \beta) \subseteq A - \alpha$ or $A - (\alpha \wedge \beta) \subseteq A - \beta$. Hence, $A - (\alpha \wedge \beta) = A - \alpha$ or $A - (\alpha \wedge \beta) = A - \beta$.
Conjunctive factoring: Follows trivially by *decomposition*.
Uniformity: Let α and β be two sentences such that it holds for all subsets A' of A that $A' \vdash \alpha$ if and only if $A' \vdash \beta$.
Case 1) $\vdash \alpha$ and $\vdash \beta$. It follows trivially from *failure* that $A - \alpha = A - \beta$.
Case 2) $\nvdash \alpha$ and $\nvdash \beta$. It follows from *inclusion* that $A - \alpha \subseteq A$ and $A - \beta \subseteq A$. By *success* it follows that $A - \alpha \nvdash \alpha$. Then, by hypothesis, $A - \alpha \nvdash \beta$. By symmetry of the case it follows that $A - \beta \nvdash \alpha$. Therefore from *strong inclusion* it follows that $A - \alpha = A - \beta$. ■

Proof of Observation 5.2.9.
(Construction-to-postulates)
Let $*$ be an operator of smooth kernel revision on A. It follows from Observation 5.2.7 that $*$ satisfies *consistency, success, inclusion, uniformity* and *core-retainment.* It remains to prove that $*$ satisfies *relative closure.* Since $*$ is a kernel revision operator it follows that $*$ is based on a incision function σ such that for all sentences α:

$$A * \alpha = (A \setminus \sigma(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$$

On the other hand, σ is smooth. Hence it holds for all subsets A' of A that if $A' \vdash \beta$ and $\beta \in \sigma(A \perp\!\!\!\perp \alpha)$ then $A' \cap \sigma(A \perp\!\!\!\perp \alpha) \neq \emptyset$.

Assume that $\beta \notin A * \alpha$. By $*$ *success* it follows that $\beta \neq \alpha$. We intend to prove that $\beta \notin A \cap Cn(A \cap A * \alpha)$. It follows trivially if $\beta \notin A$. Consider now that $\beta \in A$. Hence, by definition of $*$, $\beta \in \sigma(A \perp\!\!\!\perp \neg\alpha)$. Assume by *reductio ad absurdum* that $\beta \in A \cap Cn(A \cap A * \alpha)$. Then $((A \setminus \sigma(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}) \cap A \vdash \beta$. Thus $((A \setminus \sigma(A \perp\!\!\!\perp \neg\alpha)) \cup (A \cap \{\alpha\})) \vdash \beta$. We will consider two cases:
Case 1) $\alpha \notin A$. Hence $A \setminus \sigma(A \perp\!\!\!\perp \neg\alpha) \vdash \beta$. Let $X = A \setminus \sigma(A \perp\!\!\!\perp \neg\alpha)$. Hence $X \subseteq A$, $X \vdash \beta$, $\beta \in \sigma(A \perp\!\!\!\perp \neg\alpha)$ and $X \cap \sigma(A \perp\!\!\!\perp \neg\alpha) = \emptyset$. Which contradicts the fact that σ is smooth.
Case 2) $\alpha \in A$. Hence, by deduction, $A \setminus \sigma(A \perp\!\!\!\perp \neg\alpha) \vdash \alpha \to \beta$. Suppose that $\alpha \in \sigma(A \perp\!\!\!\perp \neg\alpha)$. Hence there exists $Y \subseteq A \perp\!\!\!\perp \neg\alpha$ such that $\alpha \in Y$. Let $Y' = Y \setminus \{\alpha\}$. Thus $Y' \nvdash \neg\alpha$ but $Y' \cup \{\alpha\} \vdash \neg\alpha$. Hence by deduction, $Y' \vdash \alpha \to \neg\alpha$. Contradiction, since $\vdash \neg\alpha \leftrightarrow (\alpha \to \neg\alpha)$. Thus $\alpha \notin \sigma(A \perp\!\!\!\perp \neg\alpha)$, from which it follows that $\alpha \in A \setminus \sigma(A \perp\!\!\!\perp \neg\alpha)$ and consequently that $A \setminus \sigma(A \perp\!\!\!\perp \neg\alpha) \vdash \beta$. The rest of the proof for this case follows as in the previous one.
(Postulates-to-construction)
Let $*$ be an operator that satisfies all the postulates listed in the observation. Let $\sigma(A \perp\!\!\!\perp \neg\alpha) = A \setminus (A \cap (A * \alpha))$. This is the same construction that is used in the proof of Observation 5.2.7 (presented in [Was00, Proof of Theorem 5.2.14]). Hence σ is an incision function for A and $A * \alpha = (A \setminus \sigma(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$. It remains to show that σ is smooth. Let A' be a subset of A such that $A' \vdash \beta$ and $\beta \in \sigma(A \perp\!\!\!\perp \neg\alpha)$. We intend to prove that $A' \cap \sigma(A \perp\!\!\!\perp \neg\alpha) \neq \emptyset$.

Assume by *reductio ad absurdum* that $A \cap A * \alpha \vdash \beta$. From $\beta \in \sigma(A {\perp\!\!\!\perp} \neg\alpha)$ and $\sigma(A {\perp\!\!\!\perp} \neg\alpha) = A \setminus (A \cap (A * \alpha))$ it follows that $\beta \in A$. Thus by $*$ *relative closure* it follows that $\beta \in A * \alpha$. Contradiction, since $\beta \in \sigma(A {\perp\!\!\!\perp} \neg\alpha) = A \setminus (A \cap (A * \alpha))$. Therefore $A \cap A * \alpha \nvdash \beta$. From $A' \vdash \beta$ it follows that $A' \notin A \cap A * \alpha$. Hence there is some sentence $\delta \in A' \setminus (A \cap A * \alpha)$. Thus $\delta \in \sigma(A {\perp\!\!\!\perp} \neg\alpha)$. Therefore $\delta \in A' \cap \sigma(A {\perp\!\!\!\perp} \neg\alpha)$, from which it follows that $A' \cap \sigma(A {\perp\!\!\!\perp} \neg\alpha) \neq \varnothing$. ■

Proof of Observation 5.2.13.
(Construction-to-postulates)
Let $A * \alpha = (Cn(A) \star \alpha) \cap (A \cup \{\alpha\})$ and $\star$ be a basic AGM revision for $Cn(A)$. Hence $\star$ satisfies *success, inclusion, vacuity, consistency, extensionality* and *closure.* We will now prove that $*$ satisfies *consistency, success, inclusion, vacuity, weak extensionality* and *disjunctive elimination.*
Success: Follows trivially by $*$ definition and $\star$ *success.*
Inclusion: Follows trivially by $*$ definition.
Vacuity: Let $A \nvdash \neg\alpha$ and $\beta \in A \cup \{\alpha\}$. If $\beta = \alpha$, then $\beta \in A * \alpha$ by definition of $*$ and $\star$ *success.* Assume now that $\beta \neq \alpha$. Then $\beta \in A$. Hence, by $\star$ *vacuity* $Cn(Cn(A) \cup \{\alpha\}) \subseteq Cn(A) \star \alpha$. On the other hand,[1] $Cn(Cn(A) \cup \{\alpha\}) = Cn(A \cup \{\alpha\})$ therefore $A \cup \{\alpha\} = Cn(A \cup \{\alpha\}) \cap (A \cup \{\alpha\}) \subseteq (Cn(A) \star \alpha) \cap (A \cup \{\alpha\}) = A * \alpha$.
Weak extensionality: Let $\vdash \alpha \leftrightarrow \beta$. Then $A \cap A * \alpha = A \cap ((Cn(A) \star \alpha) \cap (A \cup \{\alpha\})) = A \cap (Cn(A) \star \alpha)$. Thus, by $\star$ *extensionality*, $A \cap A * \alpha = A \cap (Cn(A) \star \beta)$. Hence, by definition of $*$, it follows that $A \cap A * \alpha = A \cap A * \beta$.
Disjunctive elimination: Let $\beta \in A$ and $\beta \notin A * \alpha$. Then, by definition of $*$, $\beta \notin (Cn(A) \star \alpha) \cap (A \cup \{\alpha\})$. Hence $\beta \notin Cn(A) \star \alpha$. On the other hand, by $\star$ *success*, $\alpha \in Cn(A) \star \alpha$. Thus, by $\star$ *closure*, $Cn(A) \star \alpha \nvdash \neg\alpha \vee \beta$. Therefore, by definition of $*$ it follows that $A * \alpha \nvdash \neg\alpha \vee \beta$.
Consistency: Let $\alpha \nvdash \bot$. By $\star$ consistency it follows that $Cn(A) \star \alpha \nvdash \bot$. Hence $A * \alpha \nvdash \bot$.
(Postulates-to-construction)
Let $*$ be an operator on A that satisfies *consistency, success, inclusion, vacuity, weak extensionality* and *disjunctive elimination.* Let $\star$ be an operator on $Cn(A)$ defined, for all $\alpha \in \mathcal{L}$, as follows:

$$Cn(A) \star \alpha = Cn(A * \alpha)$$

We must prove that:
a) $\star$ satisfies *success, consistency, extensionality, inclusion, vacuity* and *closure*;
b) $A * \alpha = (Cn(A) \star \alpha) \cap (A \cup \{\alpha\})$.
Proof of a) *Closure* follows trivially from $\star$ definition. *Success*, *consistency* and *inclusion* follow trivially from $\star$ definition and $*$ *success, consistency* and *inclusion* respectively.
Vacuity: Let $\neg\alpha \notin Cn(A)$. By $*$ *vacuity* it follows that $A \cup \{\alpha\} \subseteq A * \alpha$. Thus $Cn(Cn(A) \cup \{\alpha\}) = Cn(A \cup \{\alpha\}) \subseteq Cn(A * \alpha) = Cn(A) \star \alpha$.
Extensionality: Let $\vdash \alpha \leftrightarrow \beta$. It follows, by $*$ *weak extensionality*, that $A \cap A * \alpha = A \cap A * \beta$. We will prove by double inclusion that $Cn(A) \star \alpha = Cn(A) \star \beta$.

[1] $Cn(A \cup B) = Cn(A \cup Cn(B))$ [Han99b].

We will start by proving that $A * \alpha = (A \cap A * \alpha) \cup \{\alpha\}$.
$(A \cap A * \alpha) \cup \{\alpha\} = (A \cup \{\alpha\}) \cap (A * \alpha \cup \{\alpha\}) = A * \alpha$ (the last equality follows from $*$ *success* and *inclusion*).
Let $\delta \in Cn(A) \star \alpha$. Then $A * \alpha \vdash \delta$. Thus $(A \cap A * \alpha) \cup \{\alpha\} \vdash \delta$. Therefore, by deduction $(A \cap A * \alpha) \vdash \alpha \rightarrow \delta$. It follows, from $\vdash \alpha \leftrightarrow \beta$ and $*$ *weak extensionality*, that $(A \cap A * \beta) \vdash \beta \rightarrow \delta$. Hence $(A \cap A * \beta) \cup \{\beta\} \vdash \delta$. Therefore $A * \beta \vdash \delta$. Hence $\delta \in Cn(A) \star \beta$. Thus $Cn(A) \star \alpha \subseteq Cn(A) \star \beta$. By symmetry of the case it holds that $Cn(A) \star \beta \subseteq Cn(A) \star \alpha$. Therefore $Cn(A) \star \alpha = Cn(A) \star \beta$.
Proof of b) That $A * \alpha \subseteq (Cn(A) \star \alpha) \cap (A \cup \{\alpha\})$, follows from $\star$ definition and $*$ *inclusion*. Let $\delta \in (Cn(A) \star \alpha) \cap (A \cup \{\alpha\})$. If $\delta = \alpha$, then it follows from $*$ *success* that $\delta \in A * \alpha$. Assume now that $\delta \neq \alpha$. Hence $\delta \in A$ and $A * \alpha \vdash \delta$. From the latter it follows that $A * \alpha \vdash \neg\alpha \vee \delta$. Hence, by *disjunctive elimination*, it follows that $\delta \in A * \alpha$. Thus $(Cn(A) \star \alpha) \cap (A \cup \{\alpha\}) \subseteq A * \alpha$. Therefore $(Cn(A) \star \alpha) \cap (A \cup \{\alpha\}) = A * \alpha$. ■

Proof of Observation 5.2.15.
In order to prove that $(\mathbf{K}, \leq |_{\mathbf{K}})$ is an ensconcement we must show that $\leq |_{\mathbf{K}}$ is a transitive and total relation on $\mathbf{K}$ that satisfies (≤ 1), (≤ 2) and (≤ 3).
According to (EE1) $\leq$ is a transitive relation and it follows from (EE1), (EE2) and (EE3) that $\leq$ is a total relation (Lemma 3.2.38). Thus $\leq |_{\mathbf{K}}$ is a transitive and total relation on $\mathbf{K}$.
(≤ 1) Let $\beta \in \mathbf{K} \setminus Cn(\varnothing)$. Assume by *reductio ad absurdum* that $\{\alpha \in \mathbf{K} : \beta < |_{\mathbf{K}}\alpha\} \vdash \beta$. Hence $\{\alpha \in \mathbf{K} : \beta < \alpha\} \vdash \beta$. By compactness, and since $\nvdash \beta$, it follows that there exists a non-empty finite subset $A' = \{\alpha_1, \ldots, \alpha_n\}$ of $\{\alpha \in \mathbf{K} : \beta < \alpha\}$ such that $A' \vdash \beta$. Hence $\alpha_1 \wedge \ldots \wedge \alpha_n \vdash \beta$, from which it follows by (EE2) that $\alpha_1 \wedge \ldots \wedge \alpha_n \leq \beta$. It follows from (EE3) and (EE1) that there exists $\alpha_i \in A'$ such that $\alpha_i \leq \alpha_1 \wedge \ldots \wedge \alpha_n$. Thus, from (EE1) $\alpha_i \leq \beta$. Contradiction.
(≤ 2) Let $\alpha, \beta \in \mathbf{K}$ be such that $\nvdash \alpha$ and $\vdash \beta$. It follows from (EE2) that $\alpha \leq \beta$. Thus $\alpha \leq |_{\mathbf{K}}\beta$. Assume, by *reductio ad absurdum*, that $\beta \leq |_{\mathbf{K}}\alpha$. Hence $\beta \leq \alpha$. Let $\theta \in \mathcal{L}$, then by (EE2) $\theta \leq \beta$. Hence by (EE1) $\theta \leq \alpha$. Hence for all $\delta \in \mathcal{L}$, $\delta \leq \alpha$. Therefore, by (EE5), it holds that $\vdash \alpha$. Contradiction. Thus $\alpha < |_{\mathbf{K}}\beta$.
(≤ 3) Let $\alpha, \beta \in \mathbf{K}$ be such that $\vdash \alpha$ and $\vdash \beta$. Hence, by (EE2), $\alpha \leq \beta$. Therefore $\alpha \leq |_{\mathbf{K}}\beta$. ■

Proof of Observation 5.3.2.

(a) Let A be a belief base and $\sim$ an operator on A that satisfies *relative success*. Let $\alpha \in \mathcal{L}$ be such that $\vdash \alpha$. By *relative success* it follows that $A \sim \alpha = A$ or $A \sim \alpha \nvdash \alpha$. The latter does not hold since α is a tautology. Hence $A \sim \alpha = A$.

(b) Assume that $A \sim \alpha = A \sim \beta = A$. If $A \vdash \alpha$ and $A \vdash \beta$, then $A \sim \alpha \vdash \alpha$ and $A \sim \beta \vdash \beta$. Hence, by *persistence*, $A \sim (\alpha \wedge \beta) \vdash \alpha$ and $A \sim (\alpha \wedge \beta) \vdash \beta$, from which it follows that $A \sim (\alpha \wedge \beta) \vdash \alpha \wedge \beta$. Therefore by *relative success* it follows that $A \sim (\alpha \wedge \beta) = A$.
If $A \nvdash \alpha$ or $A \nvdash \beta$ it follows that $A \nvdash \alpha \wedge \beta$. From which it follows, by *inclusion* and *vacuity*, that $A \sim (\alpha \wedge \beta) = A$. ■

Appendix D

Proofs of Chapter 6

Lemma D.1 *Let* $(A, \leq)$ *be an ensconcement. If* $\alpha, \beta \in A$*, then* $\alpha =_{\leq} \beta$ *if and only if* $cut_{\prec}(\alpha) = cut_{\prec}(\beta)$*.*

Proof.
If $\vdash \alpha$, then from ($\leq$ 2) it follows that $\vdash \beta$. Hence, the proof follows from ($\leq$ 3) and Lemma 5.2.19 (a). Assume now that $\nvdash \alpha$ and consequently that $\nvdash \beta$. From left to right it follows from Lemma 5.2.19 (i). For the other direction: Let $\alpha, \beta \in A$, if $\alpha \prec \beta$, then by Lemma 5.2.19 (g) $cut_{\prec}(\alpha) \vdash \beta$ and so $cut_{\prec}(\beta) \vdash \beta$ which contradicts Lemma 5.2.19 (b). Due to the symmetry of the case we may conclude that $\beta \nprec \alpha$. Since $\alpha \nprec \beta, \beta \nprec \alpha$, and $\leq$ is connected, we can conclude that $\alpha \leq \beta$ and $\beta \leq \alpha$. ■

Lemma D.2 *Let* $(A, \leq)$ *be an ensconcement. Let* $-$ *be the* $\leq$*-based contraction on* A*. Then:*

(a) $cut_{\prec}(\alpha) \subseteq A - \alpha$*.*

(b) *If* $A - \alpha \nvdash \beta$*, then* $cut_{\prec}(\alpha) \nvdash \beta$*.*

(c) *If* $A - \alpha \vdash \beta$*, then* $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$*.*

(d) *If* $\beta \in cut_{\prec}(\alpha)$*, then* $\beta \in A - \alpha \wedge \beta$*.*

Proof.

(a) Let $\beta \in cut_{\prec}(\alpha)$. It follows that $\beta \in A$ and $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. Thus, by definition of $\leq$-based contraction, $\beta \in A - \alpha$.

(b) It follows trivially from (a).

(c) It is trivial if $\vdash \beta$ or $\vdash \alpha$. Assume now that $\nvdash \beta$ and $\nvdash \alpha$. From $A - \alpha \vdash \beta$ by compactness there exists a finite subset of $A - \alpha$, $H = \{\gamma_1, \ldots, \gamma_k\}$, such that $H \vdash \beta$. It follows, by definition of $\leq$-based contraction, that for each $\gamma_i \in H$, $cut_{\prec}(\alpha) \vdash \alpha \vee \gamma_i$. Hence $cut_{\prec}(\alpha) \vdash (\alpha \vee \gamma_1) \wedge \ldots \wedge (\alpha \vee \gamma_k)$. Therefore, $cut_{\prec}(\alpha) \vdash \alpha \vee (\gamma_1 \wedge \ldots \wedge \gamma_k)$, from which it follows that $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$.

(d) Let $\beta \in cut_{\prec}(\alpha)$. If $\vdash \beta$, then $cut_{\prec}(\alpha \wedge \beta) \vdash (\alpha \wedge \beta) \vee \beta$. Hence, $\beta \in A - \alpha \wedge \beta$. Consider now that $\nvdash \beta$. Assume by *reductio ad absurdum* that $\beta \notin A - \alpha \wedge \beta$. Hence, by Definition 5.2.20 it follows that $cut_{\prec}(\alpha \wedge \beta) \nvdash \beta$ and by Lemma 5.2.19 (k) and (e) that $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\beta)$. We will consider two cases:
Case 1) $A - \alpha \wedge \beta \vdash \alpha$. Hence, by (c), $cut_{\prec}(\alpha \wedge \beta) \vdash \alpha$. Thus $cut_{\prec}(\beta) \vdash \alpha$. Therefore, by Observation 5.2.18, $\{\gamma \in A : \beta \prec \gamma\} \vdash \alpha$. Hence $\{\gamma \in A : \beta \preceq \gamma\} \vdash \alpha$, from which it follows that $\beta \notin cut_{\prec}(\alpha)$. Contradiction.
Case 2) $A - \alpha \wedge \beta \nvdash \alpha$. It follows from (b) that $cut_{\prec}(\alpha \wedge \beta) \nvdash \alpha$. Hence, by Lemma 5.2.19 (k) and (e), $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$. Therefore $cut_{\prec}(\alpha) = cut_{\prec}(\beta)$. Hence $\beta \in cut_{\prec}(\beta)$ which contradicts Lemma 5.2.19 (b). ■

Lemma D.3 *Let $(A, \preceq)$ be an ensconcement. If $\alpha \in A \setminus Cn(\varnothing)$, then $cut_{\preceq}(\alpha) = \{\gamma \in A : \alpha \preceq \gamma\}$.*

Proof.
Let $\beta \in cut_{\preceq}(\alpha)$. Hence $\{\gamma \in A : \beta \prec \gamma\} \nvdash \alpha$. Since $\alpha, \beta \in A$ and $\preceq$ is a total relation on A, it follows that, $\alpha \preceq \beta$ or $\beta \prec \alpha$. In the latter case, it follows that $\{\gamma \in A : \beta \prec \gamma\} \vdash \alpha$. Hence $\alpha \preceq \beta$.
Let $\beta \in \{\gamma \in A : \alpha \preceq \gamma\}$. Hence $\alpha \preceq \beta$. $(A, \preceq)$ is an ensconcement. Hence by $(\preceq 1)$ it follows that $\{\gamma \in A : \alpha \prec \gamma\} \nvdash \alpha$. Hence $\{\gamma \in A : \beta \prec \gamma\} \nvdash \alpha$. Therefore $\beta \in cut_{\preceq}(\alpha)$. ■

Lemma D.4 *[FH94, Han99b] Let* **K** *be a belief set and $\div$ be an operator on* **K**. *Then:*

(a) If $\div$ satisfies relevance, then it satisfies $(\div 5)$.

(b) If $\div$ satisfies $(\div 1)$, $(\div 2)$, $(\div 3)$, and $(\div 5)$, then it satisfies relevance.

Lemma D.5 *[FKR08] Let* **K** *be a belief set and $\div$ be an operator on* **K** *that satisfies inclusion, vacuity and disjunctive elimination. Then $\div$ satisfies relevance.*

Lemma D.6 *If $-$ is a brutal contraction on a belief base A, then for $\alpha, \beta \in A$:*

(a) *($\alpha \notin A - (\alpha \wedge \beta)$ or $\vdash \alpha \wedge \beta$) if and only if ($\alpha \notin A - \beta$ or $\vdash \beta$).*

(b) *Condition ($\mathbf{C_{EB}} \preceq$) is equivalent to condition ($\mathbf{C_{BR}} \preceq$).*

Proof.

(a) Let $(A, \preceq)$ be an ensconcement, $\alpha, \beta \in A$ and $-$ be the $\preceq$-based brutal contraction. We intend to prove that $\alpha \notin A - \alpha \wedge \beta$ or $\vdash \alpha \wedge \beta$ holds if and only if $\alpha \notin A - \beta$ or $\vdash \beta$ holds. Assume first that $\alpha \notin A - \alpha \wedge \beta$ or $\vdash \alpha \wedge \beta$ holds. We will prove by cases:
Case 1) $\vdash \alpha \wedge \beta$. Hence $\vdash \beta$.
Case 2) $\nvdash \alpha \wedge \beta$. Hence $\alpha \notin A - \alpha \wedge \beta$. We can consider two cases $\vdash \beta$ or $\nvdash \beta$. In the latter, from $\alpha \notin A - \alpha \wedge \beta$ it follows that $\alpha \notin cut_{\prec}(\alpha \wedge \beta)$. Thus, by Lemma 5.2.19 (d), it follows that $\alpha \notin cut_{\prec}(\beta)$. Hence, $\alpha \notin A - \beta$.
Assume now that $\alpha \notin A - \beta$ or $\vdash \beta$. We will show that $\alpha \notin A - \alpha \wedge \beta$ or $\vdash \alpha \wedge \beta$. We will prove by cases:

Case 1) $\vdash \beta$. If $\vdash \alpha$, then $\vdash \alpha \wedge \beta$. Consider now that $\nvdash \alpha$. Hence, by Lemma 5.2.19 (e), $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$. Therefore, by Lemma 5.2.19 (b), $\alpha \notin cut_{\prec}(\alpha \wedge \beta)$. Hence $\alpha \notin A - \alpha \wedge \beta$.
Case 2) $\nvdash \beta$. Then $\alpha \notin A - \beta$. Hence $\alpha \notin cut_{\prec}(\beta)$. It follows, by Observation 5.2.18, that $\beta \npreceq \alpha$. Thus $\nvdash \alpha$, by ($\preceq$ 2). Furthermore, since $\preceq$ is total, $\alpha \preceq \beta$. By Lemma 5.2.19 (h) and (i) it follows that $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$. Hence, by Lemma 5.2.19 (b), $\alpha \notin cut_{\prec}(\alpha \wedge \beta)$. Therefore $\alpha \notin A - \alpha \wedge \beta$.

(b) Follows trivially from (a). ■

Proof of Observation 6.1.1.

(a) Assume that $-$ is an operator on A that satisfies *transitivity* we will show that $-$ satisfies *ST*. Let $\delta \in A$, $\beta \in A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$. Assume by *reductio ad absurdum* that $\delta \notin A - \alpha \wedge \delta$, then by *transitivity* $\beta \notin A - \alpha \wedge \beta$. Contradiction. Hence $\delta \in A - \alpha \wedge \delta$.
Assume now that $-$ is an operator on A that satisfies *ST* we will show that $-$ satisfies *transitivity*. Let $\beta \in A$, $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$. Assume by *reductio ad absurdum* that $\alpha \in A - \alpha \wedge \delta$, then by *ST* $\beta \in A - \beta \wedge \delta$. Contradiction. Hence $\alpha \notin A - \alpha \wedge \delta$.

(b) Let $\alpha \in A$, $\beta \in A - \alpha \wedge \beta$ and $\delta \in A - \beta \wedge \delta$. From *inclusion* it follows that $\beta, \delta \in A$ and from (a) it follows that $-$ satisfies *ST*. We will prove by cases:
Case 1) $\vdash \alpha$. From $\beta \in A - \alpha \wedge \beta$ it follows, by *extensionality*, that $\beta \in A - \beta$. Thus, by *success*, $\vdash \beta$. Therefore, by *extensionality* and *success*, and proceeding as before, it follows from $\delta \in A - \beta \wedge \delta$ that $\vdash \delta$. Thus, by *relative closure*, $\delta \in A - \alpha \wedge \delta$.
Case 2) $\nvdash \alpha$. Then $\nvdash \alpha \wedge \beta$. From $\beta \in A - \alpha \wedge \beta$, it follows, by *success*, that $\alpha \notin A - \alpha \wedge \beta$. Assume, by *reductio ad absurdum*, that $\delta \notin A - \alpha \wedge \delta$. It follows, by *ST*, that $\alpha \in A - \alpha \wedge \beta$. Contradiction. Hence $\delta \in A - \alpha \wedge \delta$. ■

Proof of Theorem 6.2.1.

Let $-$ be an operator on A that satisfies *success, inclusion, vacuity, failure, relative closure, strong inclusion* and *uniform behaviour*. We will start by proving that $\preceq$, defined by ($\mathbf{C_{BR}}$ $\preceq$), is an ensconcement relation. Let $\preceq$ be defined for $\alpha, \beta \in A$ as follows:

$$\alpha \preceq \beta \text{ iff } \alpha \notin A - \beta \text{ or } \vdash \beta.$$

($\preceq$ 1) Let $\gamma \in A \setminus Cn(\varnothing)$, we must show that $H = \{\alpha \in A : \gamma \prec \alpha\} \nvdash \gamma$. Let $\alpha \in A$ and $\gamma \prec \alpha$, then, according to our construction ($\gamma \notin A - \alpha$ or $\vdash \alpha$) and ($\alpha \in A - \gamma$ and $\nvdash \gamma$). Hence $\alpha \in A - \gamma$. Hence $H \subseteq A - \gamma$. Therefore, by *success*, it follows that $H \nvdash \gamma$.
($\preceq$ 2) Let $\alpha, \beta \in A$ such that $\nvdash \alpha$ and $\vdash \beta$. We need to prove that $\alpha \preceq \beta$ and $\beta \npreceq \alpha$. That $\alpha \preceq \beta$ follows from the definition of $\preceq$. That $\beta \npreceq \alpha$ follows from the definition of $\preceq$ and $-$ *relative closure*.
($\preceq$ 3) Follows trivially from the definition of $\preceq$.
($\preceq$ is total) Let $\alpha, \beta \in A$. Assume that $\alpha \npreceq \beta$. By the definition of $\preceq$, it follows that

$\alpha \nleq \beta$ if and only if $\alpha \in A - \beta$ and $\nvdash \beta$. We will prove by cases:
Case 1) $\vdash \alpha$. Then by definition of $\leq$, $\beta \leq \alpha$.
Case 2) $\nvdash \alpha$. By *success* it follows that $\alpha \notin A - \alpha$. Therefore $A - \beta \nsubseteq A - \alpha$. From which it follows by *linearity* (Observation 5.1.3 (d)) that $A - \alpha \subseteq A - \beta$. Therefore by *success* $\beta \notin A - \alpha$. Hence, by definition of $\leq$, it follows that $\beta \leq \alpha$.
($\leq$ is transitive) Let $\alpha, \beta, \gamma \in A$ and assume that $\alpha \leq \beta$ and $\beta \leq \gamma$. By definition of $\leq$, it follows that $(\alpha \notin A - \beta$ or $\vdash \beta)$ and $(\beta \notin A - \gamma$ or $\vdash \gamma)$.
If $\vdash \gamma$, then by definition of $\leq$ it follows that $\alpha \leq \gamma$.
Assume now that $\nvdash \gamma$. Hence $\beta \notin A - \gamma$. By *relative closure* it follows that $\nvdash \beta$. Hence $\alpha \notin A - \beta$. Again, by *relative closure* it follows that $\nvdash \alpha$. By Lemma C.2 (a) it follows that $A - \beta \subseteq A - \alpha$ and $A - \gamma \subseteq A - \beta$. Therefore, since $\subseteq$ is transitive $A - \gamma \subseteq A - \alpha$. Hence by *success* $\alpha \notin A - \gamma$. Therefore, by definition of $\leq$ it follows that $\alpha \leq \gamma$.
It remains to prove that:

$$A - \alpha = \begin{cases} cut_{<}(\alpha) & \text{if } \nvdash \alpha \\ A & \text{otherwise} \end{cases}$$

We will prove by cases:
1. $\vdash \alpha$. Follows trivially by *failure*.
2. $\nvdash \alpha$
2.1. $A \nvdash \alpha$. It follows from *vacuity, inclusion* and Lemma 5.2.19 (c) that $cut_{<}(\alpha) = A = A - \alpha$.
2.2. $A \vdash \alpha$.
2.2.1. $\alpha \in A$. We will prove that $A - \alpha = cut_{<}(\alpha)$ by double inclusion.
($\supseteq$) Let $\beta \in cut_{<}(\alpha)$ and assume by *reductio ad absurdum* that $\beta \notin A - \alpha$. From $\beta \notin A - \alpha$ it follows, by definition of $\leq$ that $\beta \leq \alpha$. On the other hand, since $\beta \in cut_{<}(\alpha)$, it follows from Observation 5.2.18 that $\alpha < \beta$. Contradiction.
It follows that $cut_{<}(\alpha) \subseteq A - \alpha$.
($\subseteq$) Let $\beta \in A - \alpha$ and assume by *reductio ad absurdum* that $\beta \notin cut_{<}(\alpha)$. By *inclusion* it follows that $\beta \in A$. We will prove by cases:
Case 1) $\vdash \beta$. Then, by (≤ 2), $\alpha < \beta$. Therefore, from Observation 5.2.18, it follows that $\beta \in cut_{<}(\alpha)$. Contradiction.
Case 2) $\nvdash \beta$. Since $\beta \notin cut_{<}(\alpha)$, by Observation 5.2.18, it follows that $\alpha \nless \beta$. Therefore, since $\leq$ is a total relation, it follows that $\beta \leq \alpha$. According to the definition of $\leq$ this means that $\beta \notin A - \alpha$ or $\vdash \alpha$. In both cases we obtain a contradiction.
Therefore $cut_{<}(\alpha) = A - \alpha$.
2.2.2. $\alpha \notin A$. We will prove that $A - \alpha = cut_{<}(\alpha)$ by double inclusion.
($\supseteq$) Let $\beta \in cut_{<}(\alpha)$ and assume by *reductio ad absurdum* that $\beta \notin A - \alpha$. From $\beta \in cut_{<}(\alpha)$ it follows that $\beta \in A$. Hence $\beta \in A \setminus A - \alpha$, from which it follows, by *relative closure*, that $\nvdash \beta$. From Lemma C.2 (a), it follows that $A - \alpha \subseteq A - \beta$.
We will consider two cases:
Case 1) $A - \alpha \subset A - \beta$. Hence, by Lemma C.2 (d), $A - \beta \vdash \alpha$. Therefore, since $\beta \in A$, from the case 2.2.1. it follows that $cut_{<}(\beta) \vdash \alpha$. Hence, from Observation 5.2.18, $\{\gamma \in A : \beta < \gamma\} \vdash \alpha$. Contradiction, since $\beta \in cut_{<}(\alpha)$.
Case 2) $A - \alpha = A - \beta$. From $\beta \in cut_{<}(\alpha)$ it follows that $\{\gamma \in A : \beta \leq \gamma\} \nvdash \alpha$. Hence, by Observation 5.2.18, it follows that $cut_{<}(\beta) \cup \{\gamma \in A : \beta =_{\leq} \gamma\} \nvdash \alpha$. Hence, by

Lemma D.1, it follows that $cut_{\prec}(\beta) \cup \{\gamma \in A : cut_{\prec}(\beta) = cut_{\prec}(\gamma)\} \nvdash \alpha$. Therefore, from case 2.2.1., it follows that $(A - \beta) \cup \{\gamma \in A : A - \beta = A - \gamma\} \nvdash \alpha$. Contradiction, from *uniform behaviour*.
It follows that $cut_{\prec}(\alpha) \subseteq A - \alpha$.
($\subseteq$) Let $\beta \in A - \alpha$ and assume by *reductio ad absurdum* that $\beta \notin cut_{\prec}(\alpha)$ (note that it follows from *inclusion* that $\beta \in A$). We will consider two cases:
Case 1) $\vdash \beta$. Then, by ($\preceq 2$), it follows that, if $\gamma \in A$ and $\beta \preceq \gamma$, then $\vdash \gamma$. Therefore $\{\gamma \in A : \beta \preceq \gamma\} \subseteq Cn(\varnothing)$. Hence, since $\nvdash \alpha$, it follows that $\{\gamma \in A : \beta \preceq \gamma\} \nvdash \alpha$. Therefore $\beta \in cut_{\prec}(\alpha)$. Contradiction.
Case 2) $\nvdash \beta$. From $\beta \notin cut_{\prec}(\alpha)$ it follows that $\{\gamma \in A : \beta \preceq \gamma\} \vdash \alpha$. Then according to the definition of $\preceq$, $\{\gamma \in A : \beta \notin A - \gamma \text{ or } \vdash \gamma\} \vdash \alpha$. Hence $H = \{\gamma \in A : \beta \notin A - \gamma\} \vdash \alpha$. Let $\delta \in H$. Hence $\delta \in A$ and $\beta \notin A - \delta$. Thus $A - \alpha \nsubseteq A - \delta$ (since $\beta \in A - \alpha$). Therefore, by Lemma C.2 (a), it follows that $\delta \in A - \alpha$. Hence $H \subseteq A - \alpha$, from which it follows by *success* that $H \nvdash \alpha$. Contradiction. ∎

Proof of Theorem 6.2.2.

Success Let $\nvdash \alpha$ and assume by *reductio ad absurdum* that $A - \alpha \vdash \alpha$. Then it follows from the definition of $-$ that $cut_{\prec}(\alpha) \vdash \alpha$. Contradiction by Lemma 5.2.19 (b).

Inclusion and **Failure** follow trivially.

Vacuity follows trivially from Lemma 5.2.19 (c).

Relative Closure If $\vdash \alpha$, trivial from *failure*. Let $\nvdash \alpha$ and assume by *reductio ad absurdum* that $\beta \in A, A - \alpha \vdash \beta$ and $\beta \notin A - \alpha$. It follows from the definition of $-$ that $cut_{\prec}(\alpha) \vdash \beta$ and $\beta \notin cut_{\prec}(\alpha)$. From $\beta \notin cut_{\prec}(\alpha)$ it follows that $\{\gamma \in A : \beta \preceq \gamma\} \vdash \alpha$. If $\vdash \beta$, then it follows from ($\preceq$2) that $\vdash \alpha$. Contradiction.
Assume that $\nvdash \beta$. Since $cut_{\prec}(\alpha) \vdash \beta$ it follows that $cut_{\prec}(\alpha) \neq \varnothing$. Let $\delta \in cut_{\prec}(\alpha)$.
$\delta, \beta \in A$ and $\preceq$ is a total relation, thus $\delta \preceq \beta$ or $\beta \prec \delta$.
If $\delta \preceq \beta$, then $\{\gamma \in A : \beta \preceq \gamma\} \subseteq \{\gamma \in A : \delta \preceq \gamma\}$. Contradiction, since $\{\gamma \in A : \beta \preceq \gamma\} \vdash \alpha$ and $\delta \in cut_{\prec}(\alpha)$.
If $\beta \prec \delta$, and since δ is an arbitrary element of $cut_{\prec}(\alpha)$, then for all $\gamma \in cut_{\prec}(\alpha)$, we have that $\beta \prec \gamma$. Contradiction, since $cut_{\prec}(\alpha) \vdash \beta$ contradicts ($\preceq$1).
Strong Inclusion Let $\vdash \beta$, then by *failure* it follows that $A - \beta = A$. Hence $A \nvdash \alpha$ and, by *vacuity* and *inclusion*, it follows that $A - \alpha = A$. Therefore $A - \beta \subseteq A - \alpha$.
Assume now that $\nvdash \beta$. Let $A - \beta \nvdash \alpha$, it follows by the definition of $-$ that $cut_{\prec}(\beta) \nvdash \alpha$. By Lemma 5.2.19 (k) it follows that $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$. By Lemma 5.2.19 (d) $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha \wedge \beta)$, then $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha)$. Thus $A - \beta \subseteq A - \alpha$ (by $-$ definition).
Uniform Behaviour Let $\beta \in A, A \vdash \alpha$ and $A - \alpha = A - \beta$. It follows trivially from *failure* if $\vdash \beta$. Let $\nvdash \beta$ and assume by *reductio ad absurdum* that $\alpha \notin Cn(A - \beta \cup \{\gamma \in A : A - \beta = A - \gamma\})$.
Since $\nvdash \beta$ and $\beta \in A$, by *success* it follows that $A - \beta \neq A$. Hence, for all δ, if $A - \delta = A - \beta$ it follows that $A - \delta \neq A$ and, by *failure*, $\nvdash \delta$. Then it follows from the hypothesis and the definition of $-$ that $cut_{\prec}(\alpha) = cut_{\prec}(\beta)$ and $(cut_{\prec}(\beta) \cup \{\gamma \in A : cut_{\prec}(\gamma) = cut_{\prec}(\beta)\}) \nvdash \alpha$. From Observation 5.2.18 and Lemma D.1 it follows that $(\{\gamma \in A : \beta \prec \gamma\} \cup \{\gamma \in A : \gamma =_{\preceq} \beta\}) \nvdash \alpha$. Hence, $\{\gamma \in A : \beta \preceq \gamma\} \nvdash \alpha$, and so $\beta \in cut_{\prec}(\alpha) = cut_{\prec}(\beta)$. Contradiction (Lemma 5.2.19 (b)).

($\mathbf{C_{BR}} \leq$) ($\Rightarrow$) Let $\alpha, \beta \in A$ be such that $\alpha \leq \beta$. Suppose that $\nvdash \beta$. We intend to prove that $\alpha \notin A - \beta$. It follows from (≤ 2) that $\nvdash \alpha$. By $-$ definition it follows that $A - \alpha = cut_{\prec}(\alpha)$ and $A - \beta = cut_{\prec}(\beta)$. From $\alpha \leq \beta$ it follows, by Lemma 5.2.19 (f), that $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha)$. Hence, by Lemma 5.2.19 (b), $\alpha \notin cut_{\prec}(\beta)$. Therefore, $\alpha \notin A - \beta$.
($\Leftarrow$) Let $\alpha, \beta \in A$ be such that $\alpha \notin A - \beta$ or $\vdash \beta$.
If $\vdash \beta$, then, by (≤ 2) and (≤ 3), $\alpha \leq \beta$.
Consider now that $\nvdash \beta$. Hence $\alpha \notin A - \beta$. It also holds, by definition of $-$, that $A - \beta = cut_{\prec}(\beta)$. Assume by *reductio ad absurdum* that $\alpha \nleq \beta$. Hence $\beta \prec \alpha$, since $\leq$ is a total relation. From which it follows, by Observation 5.2.18, that $\alpha \in cut_{\prec}(\beta)$. Thus $\alpha \in A - \beta$. Contradiction. ∎

Proof of Theorem 6.2.3.
Let $-$ be an operator to A that satisfies *success, inclusion, vacuity, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1* and *EB2*. We will start by proving that $\leq$, defined by ($\mathbf{C_{EB}} \leq$), is an ensconcement relation.
($\leq$ is total) Let $\alpha, \beta \in A$ be such that $\alpha \nleq \beta$. By ($C_{EB} \leq$), it follows that $\alpha \in A - \alpha \wedge \beta$ and $\nvdash \alpha \wedge \beta$. It follows, by *success*, that $\beta \notin A - \alpha \wedge \beta$. Therefore $\beta \leq \alpha$, by ($\mathbf{C_{EB}} \leq$).
($\leq$ is transitive) Let $\alpha, \beta, \delta \in A$ be such that $\alpha \leq \beta$ and $\beta \leq \delta$. We wish to prove that $\alpha \leq \delta$. It follows trivially by ($\mathbf{C_{EB}} \leq$) if $\vdash \alpha \wedge \delta$. Assume now that $\nvdash \alpha \wedge \delta$. From $\alpha \leq \beta$ and $\beta \leq \delta$ it follows, by ($\mathbf{C_{EB}} \leq$), that ($\alpha \notin A - \alpha \wedge \beta$ or $\vdash \alpha \wedge \beta$) and ($\beta \notin A - \beta \wedge \delta$ or $\vdash \beta \wedge \delta$). By *relative closure* (Observation 5.1.2) there are only two possible cases to consider:
Case 1) $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$. Hence, by *transitivity*, $\alpha \notin A - \alpha \wedge \delta$. Therefore, by ($C_{EB} \leq$), $\alpha \leq \delta$.
Case 2) $\alpha \notin A - \alpha \wedge \beta$ and $\vdash \beta \wedge \delta$. It follows, by *failure* (Observation 5.1.2), that $\nvdash \alpha$ and, by *extensionality*, that $A - \alpha \wedge \delta = A - \alpha$. Thus, by *success*, $\alpha \notin A - \alpha \wedge \delta$. Therefore $\alpha \leq \delta$, by ($\mathbf{C_{EB}} \leq$).
($\leq$1) Let $\gamma \in A \setminus Cn(\varnothing)$ and let $H = \{\alpha \in A : \gamma \prec \alpha\}$. We will show that $H \subseteq A - \gamma$. Let $\alpha \in H$. If $\vdash \alpha$, then by *relative closure* $\alpha \in A - \gamma$. Assume now that $\nvdash \alpha$. From $\alpha \in H$ it follows that $\gamma \prec \alpha$. By ($\mathbf{C_{EB}} \leq$) this means that:
($\gamma \notin A - \alpha \wedge \gamma$ or $\vdash \alpha \wedge \gamma$) and ($\alpha \in A - \alpha \wedge \gamma$ and $\nvdash \alpha \wedge \gamma$). This condition holds if and only if ($\gamma \notin A - \alpha \wedge \gamma$, $\alpha \in A - \alpha \wedge \gamma$ and $\nvdash \alpha \wedge \gamma$) or ($\vdash \alpha \wedge \gamma$, $\alpha \in A - \alpha \wedge \gamma$ and $\nvdash \alpha \wedge \gamma$). In the latter case we have a contradiction. From the former, and since $\nvdash \alpha$, it follows, by *conjunctive factoring* and *success* that $A - \alpha \wedge \gamma = A - \gamma$. Hence $\alpha \in A - \gamma$. Therefore $H \subseteq A - \gamma$ and thus $H \nvdash \gamma$, by *success*.
($\leq$2) Let $\alpha, \beta \in A$, $\nvdash \alpha$ and $\vdash \beta$. We wish to prove that $\alpha \prec \beta$. From $\vdash \beta$, it follows by *extensionality* that $A - \alpha \wedge \beta = A - \alpha$. Hence, by *success*, $\alpha \notin A - \alpha \wedge \beta$. Therefore $\alpha \leq \beta$, by ($\mathbf{C_{EB}} \leq$). It remains to prove that $\beta \nleq \alpha$. Assume by *reductio ad absurdum* that $\beta \leq \alpha$. By ($\mathbf{C_{EB}} \leq$), this means that $\beta \notin A - \alpha \wedge \beta$ or $\vdash \alpha \wedge \beta$. The latter contradicts $\nvdash \alpha$ and the former contradicts *relative closure*. Hence $\beta \nleq \alpha$, from which it follows that $\alpha \prec \beta$.
($\leq$3) Follows trivially by ($\mathbf{C_{EB}} \leq$).
(EBC) It remains to show that $-$ satisfies (**EBC**), *i.e.*, that $A - \alpha = \{\beta \in A : cut_{\prec}(\alpha) \vdash \alpha \vee \beta\}$.

We will prove by cases:

1. $\vdash \alpha$. Follows trivially by *failure*.

2. $\nvdash \alpha$

2.1 $A \nvdash \alpha$. By *vacuity* and *inclusion* it follows that $A - \alpha = A$. On the other hand, by Lemma 5.2.19 (c), $cut_{<}(\alpha) = A$, from which it follows that $\{\beta \in A : cut_{<}(\alpha) \vdash \alpha \vee \beta\} = A$.

2.2 $A \vdash \alpha$.

We will prove (**EBC**), *i.e.* $A - \alpha = \{\beta \in A : cut_{<}(\alpha) \vdash \alpha \vee \beta\}$, by double inclusion.
($\subseteq$) Let $\beta \in A - \alpha$. It follows, from *inclusion* that $\beta \in A$. We intend to prove that $cut_{<}(\alpha) \vdash \alpha \vee \beta$.
It is trivial if $\vdash \alpha \vee \beta$. Consider now that $\nvdash \alpha \vee \beta$ and assume by *reductio ad absurdum* that $cut_{<}(\alpha) \nvdash \alpha \vee \beta$. It follows that $\{\delta \in A : \{\gamma \in A : \delta \leq \gamma\} \nvdash \alpha\} \nvdash \alpha \vee \beta$. From ($\mathbf{C_{EB}} \leq$) it holds that $Z \nvdash \alpha \vee \beta$ where $Z = \{\delta \in A : \{\gamma \in A : \delta \notin A - \delta \wedge \gamma$ or $\vdash \delta \wedge \gamma\} \nvdash \alpha\}$. According to *EB2*, $Y \vdash \alpha \vee \beta$, where $Y = \{\gamma \in A : \gamma \in A - \gamma \wedge \alpha\}$. Let $\theta \in Y$. Hence $\theta \in A - \theta \wedge \alpha$. We will prove that $\theta \in Z$ *i.e.*, that $W \nvdash \alpha$ where $W = \{\gamma \in A : \theta \notin A - \theta \wedge \gamma$ or $\vdash \theta \wedge \gamma\}$. Let $\lambda \in W$. Hence $\lambda \in A$ and $\theta \notin A - \theta \wedge \lambda$ or $\vdash \theta \wedge \lambda$.
Now we will show that, $\lambda \in A - \alpha$. If $\vdash \lambda$ then, by *relative closure*, $\lambda \in A - \alpha$. Assume now that $\nvdash \lambda$. Hence, since $\vdash \theta \wedge \lambda$ can not hold, it follows that $\theta \notin A - \theta \wedge \lambda$, and thus $\nvdash \theta$ by *failure*. To prove that $\lambda \in A - \alpha$ it is enough to show, by *EB1*, that $X = \{\gamma \in A : \lambda \notin A - \lambda \wedge \gamma\} \nvdash \alpha$. We will show that $X \subseteq A - \alpha$. If $\delta \in X$ is such that $\vdash \delta$, then, by *relative closure*, $\delta \in A - \alpha$.
For all $\delta \in X \setminus Cn(\varnothing)$ it follows that $\lambda \notin A - \lambda \wedge \delta$. By *transitivity* and *ST* (Observation 6.1.1) it follows, from $\theta \notin A - \lambda \wedge \theta$, $\lambda \notin A - \lambda \wedge \delta$ and $\theta \in A - \theta \wedge \alpha$, that $\delta \in A - \alpha \wedge \delta$. It follows, by *success* and *conjunctive factoring*, that $\delta \in A - \alpha$. Hence $X \subseteq A - \alpha$. Thus, by *success*, $X \nvdash \alpha$.
Therefore, $\lambda \in A - \alpha$. Hence $W \subseteq A - \alpha$. By *success* it follows that $W \nvdash \alpha$. Hence $Y \subseteq Z$. Contradiction, since $Y \vdash \alpha \vee \beta$ and $Z \nvdash \alpha \vee \beta$.
($\supseteq$) We will start by proving that $cut_{<}(\alpha) \subseteq A - \alpha$. Let $\delta \in cut_{<}(\alpha)$. If $\vdash \delta$, then $\delta \in A - \alpha$ by *relative closure*. Consider now that $\nvdash \delta$ and assume by *reductio ad absurdum* that $\delta \notin A - \alpha$. Hence, by *EB1*, it follows that $\{\gamma \in A : \delta \notin A - \delta \wedge \gamma\} \vdash \alpha$. Let $\psi \in A$ such that $\delta \notin A - \delta \wedge \psi$. Thus, by ($\mathbf{C_{EB}} \leq$), $\delta \leq \psi$. Therefore $\{\gamma \in A : \delta \notin A - \delta \wedge \gamma\} \subseteq \{\gamma \in A : \delta \leq \gamma\}$. It follows that $\{\gamma \in A : \delta \leq \gamma\} \vdash \alpha$. Hence $\delta \notin cut_{<}(\alpha)$. Contradiction.
Hence $cut_{<}(\alpha) \subseteq A - \alpha$. Therefore, if $\beta \in A$ and $cut_{<}(\alpha) \vdash \alpha \vee \beta$, then $A - \alpha \vdash \alpha \vee \beta$ from which, by *disjunctive elimination*, it follows that $\beta \in A - \alpha$. ∎

Proof of Theorem 6.2.4.

Success Let $\nvdash \alpha$ and assume by *reductio ad absurdum* that $A - \alpha \vdash \alpha$. Then it follows by compactness that there exists a finite subset of $A - \alpha$, $A' = \{\beta_1, \ldots, \beta_k\}$, such that $A' \vdash \alpha$. Then it follows from the definition of $-$ that $cut_{<}(\alpha) \vdash \alpha \vee \beta_i$, $i = 1, \ldots, k$. Then $cut_{<}(\alpha) \vdash \alpha \vee (\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k)$. Hence $cut_{<}(\alpha) \vdash \alpha$. Contradiction by Lemma 5.2.19 (b).
Inclusion Trivial.
Vacuity Let $A \nvdash \alpha$ and let $\beta \in A$. By Lemma 5.2.19 (c) it follows that $cut_{<}(\alpha) = A$, from which it follows that $cut_{<}(\alpha) \vdash \alpha \vee \beta$, hence, according to the definition of $-$, $\beta \in A - \alpha$. Therefore $A \subseteq A - \alpha$.

Extensionality Let $\vdash \alpha \leftrightarrow \beta$. Then $cut_{\prec}(\alpha) = cut_{\prec}(\beta)$ by Lemma 5.2.19 (e), and the rest follows trivially.

Disjunctive Elimination Let $\beta \in A$ and $\beta \notin A - \alpha$. Then it follows from the definition of $-$ that $cut_{\prec}(\alpha) \nvdash \alpha \vee \beta$. Assume by *reductio ad absurdum* that $A - \alpha \vdash \alpha \vee \beta$. Then compactness yields that there exists a finite subset of $A - \alpha$, $A' = \{\beta_1, \ldots, \beta_k\}$, such that $A' \vdash \alpha \vee \beta$. It follows from the definition of $-$ that $cut_{\prec}(\alpha) \vdash \alpha \vee \beta_i$, $i = 1, \ldots, k$. Then $cut_{\prec}(\alpha) \vdash \alpha \vee (\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k)$. Hence $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. Contradiction.

Conjunctive Factoring First we will consider the case when $\vdash \alpha \wedge \beta$. Hence $\vdash \alpha$ and $\vdash \beta$. It follows, by definition of $-$, that $A - \alpha \wedge \beta = A - \alpha = A - \beta = A$.

Let $\nvdash \alpha \wedge \beta$. We will prove by cases:

Case 1) $cut_{\prec}(\alpha) \vdash \beta$. Then by Lemma 5.2.19 (j) $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$. We will prove by double inclusion that $A - \alpha \wedge \beta = A - \alpha$. Let $\gamma \in A - \alpha \wedge \beta$. It follows from the definition of $-$ that $\gamma \in A$ and $cut_{\prec}(\alpha \wedge \beta) \vdash (\alpha \wedge \beta) \vee \gamma$, then $cut_{\prec}(\alpha \wedge \beta) \vdash \alpha \vee \gamma$. Hence $cut_{\prec}(\alpha) \vdash \alpha \vee \gamma$, from which we can conclude that $\gamma \in A - \alpha$.

For the other inclusion, let $\gamma \in A - \alpha$. Then it follows from the definition of $-$ that $\gamma \in A$ and $cut_{\prec}(\alpha) \vdash \alpha \vee \gamma$. Then, by Lemma 5.2.19 (d), $cut_{\prec}(\alpha \wedge \beta) \vdash \alpha \vee \gamma$. On the other hand $cut_{\prec}(\alpha) \vdash \beta$ yields $cut_{\prec}(\alpha) \vdash \beta \vee \gamma$, then $cut_{\prec}(\alpha \wedge \beta) \vdash \beta \vee \gamma$. Hence $cut_{\prec}(\alpha \wedge \beta) \vdash (\alpha \wedge \beta) \vee \gamma$. Therefore $\gamma \in A - \alpha \wedge \beta$.

Case 2) $cut_{\prec}(\beta) \vdash \alpha$. Due to the symmetry of the case, it follows that $A - \alpha \wedge \beta = A - \beta$.

Case 3) $cut_{\prec}(\alpha) \nvdash \beta$ and $cut_{\prec}(\beta) \nvdash \alpha$. It follows by Lemma 5.2.19 (k) that $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha) = cut_{\prec}(\beta)$. Let $\gamma \in A - \alpha \wedge \beta$. According to the definition of $-$, $\gamma \in A$ and $cut_{\prec}(\alpha \wedge \beta) \vdash (\alpha \wedge \beta) \vee \gamma$ iff $cut_{\prec}(\alpha \wedge \beta) \vdash \alpha \vee \gamma$ and $cut_{\prec}(\alpha \wedge \beta) \vdash \beta \vee \gamma$ iff $cut_{\prec}(\alpha) \vdash \alpha \vee \gamma$ and $cut_{\prec}(\beta) \vdash \beta \vee \gamma$ iff $\gamma \in A - \alpha$ and $\gamma \in A - \beta$. Hence $A - \alpha \wedge \beta = A - \alpha \cap A - \beta$.

Transitivity Let $\beta \in A$, $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$. If $\alpha \notin A$, then by *inclusion* it follows that $\alpha \notin A - \alpha \wedge \delta$. Assume now that $\alpha \in A$. From $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$ it follows, by definition of $-$, that $cut_{\prec}(\alpha \wedge \beta) \nvdash \alpha$ and $cut_{\prec}(\beta \wedge \delta) \nvdash \beta$ and thus $\nvdash \alpha$ and $\nvdash \beta$. We will prove by cases:

Case 1) $\vdash \delta$. It follows trivially by *extensionality* and *success* that $\alpha \notin A - \alpha \wedge \delta$.

Case 2) $\nvdash \delta$. From $cut_{\prec}(\alpha \wedge \beta) \nvdash \alpha$ and $cut_{\prec}(\beta \wedge \delta) \nvdash \beta$ it follows by Lemma 5.2.19 (k) and (e) that $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$ and $cut_{\prec}(\beta \wedge \delta) = cut_{\prec}(\beta)$. Hence, from Lemma 5.2.19 (d), $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha)$ and $cut_{\prec}(\delta) \subseteq cut_{\prec}(\beta)$ from which it follows that $cut_{\prec}(\delta) \subseteq cut_{\prec}(\alpha)$. Assume by *reductio ad absurdum* that $\alpha \in A - \alpha \wedge \delta$. Hence, by *success*, it follows that $A - \alpha \wedge \delta \nvdash \delta$. Therefore, by Lemma D.2 (b) and (c), it follows that $cut_{\prec}(\alpha \wedge \delta) \vdash \alpha$ and $cut_{\prec}(\alpha \wedge \delta) \nvdash \delta$. From which it follows, by Lemma 5.2.19 (k) and (e), that $cut_{\prec}(\alpha \wedge \delta) = cut_{\prec}(\delta)$. Hence, $cut_{\prec}(\delta) \vdash \alpha$. Contradiction by Lemma 5.2.19 (b), since $cut_{\prec}(\delta) \subseteq cut_{\prec}(\alpha)$.

EB1 Let $\beta \in A$. If $\vdash \beta$, then $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. From which it follows, by definition of $-$ that $\beta \in A - \alpha$. Assume now that $\nvdash \beta$. Let $X = \{\gamma \in A : \beta \notin A - \beta \wedge \gamma\} \nvdash \alpha$. We wish to prove that $\beta \in A - \alpha$, *i.e.*, by definition of $-$, that $\beta \in A$ and $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. To do so, it is enough to prove that $\beta \in cut_{\prec}(\alpha)$, *i.e.*, that $\{\gamma \in A : \beta \leq \gamma\} \nvdash \alpha$. Let $\theta \in A$ be such that $\beta \leq \theta$. If $\vdash \theta$ then $\theta \in X$ by *extensionality* and *success*. Consider now that $\nvdash \theta$ and assume by *reductio ad absurdum*, that $\theta \notin X$. Hence $\beta \in A - \beta \wedge \theta$. Thus, by *success*, $\theta \notin A - \beta \wedge \theta$. By definition of $-$, it follows that $cut_{\prec}(\beta \wedge \theta) \nvdash \theta \vee (\beta \wedge \theta)$. Hence $cut_{\prec}(\beta \wedge \theta) \nvdash \theta$. By Lemma 5.2.19 (d) it follows that $cut_{\prec}(\beta) \nvdash \theta$. Therefore $\theta \notin cut_{\prec}(\beta)$ from which it follows, by Observation 5.2.18, that $\beta \nleq \theta$. Since $\leq$ is

a total relation it follows that $\theta \leq \beta$. Hence $\theta =_{\leq} \beta$. From Lemma 5.2.19 (i), it follows that $cut_{<}(\theta) = cut_{<}(\beta) = cut_{<}(\beta \wedge \theta)$. On the other hand, from $\beta \in A - \beta \wedge \theta$ and Definition 5.2.20 it follows that $cut_{<}(\beta \wedge \theta) \vdash \beta$. And so, $cut_{<}(\beta) \vdash \beta$ which contradicts Lemma 5.2.19 (b). Hence $\theta \in X$. Therefore $\{\gamma \in A : \beta \leq \gamma\} \subseteq X$, from which it follows that $\{\gamma \in A : \beta \leq \gamma\} \nvdash \alpha$ and consequently that $\beta \in A - \alpha$.
EB2 Let $\beta \in A - \alpha$ we will prove that $\{\gamma \in A : \gamma \in A - \gamma \wedge \alpha\} \vdash \alpha \vee \beta$.
It is trivial if $\vdash \alpha \vee \beta$. Assume now that $\nvdash \alpha \vee \beta$. From $\beta \in A - \alpha$ it follows that $cut_{<}(\alpha) \vdash \alpha \vee \beta$. Let $\delta \in cut_{<}(\alpha)$, then by Lemma D.2 (d), $\delta \in X = \{\gamma \in A : \gamma \in A - \alpha \wedge \gamma\}$. It follows that $cut_{<}(\alpha) \subseteq X$. Therefore $X \vdash \alpha \vee \beta$.
($\mathbf{C_{EB}} \leq$) ($\Rightarrow$) Let $\alpha, \beta \in A$ be such that $\alpha \leq \beta$. If $\vdash \alpha$, then by (≤ 2), $\vdash \beta$. Hence $\vdash \alpha \wedge \beta$. Consider now that $\nvdash \alpha$. Assume by *reductio ad absurdum* that $\alpha \in A - \alpha \wedge \beta$. Hence, $cut_{<}(\alpha \wedge \beta) \vdash \alpha \vee (\alpha \wedge \beta)$. Therefore, $cut_{<}(\alpha \wedge \beta) \vdash \alpha$. Thus, by Lemma 5.2.19 (b), $cut_{<}(\alpha \wedge \beta) \nvdash \beta$. Hence, by Lemma 5.2.19 (k) and (e), it follows that $cut_{<}(\alpha \wedge \beta) = cut_{<}(\beta)$. Hence $cut_{<}(\beta) \vdash \alpha$. Therefore, by Observation 5.2.18, it follows that $\{\gamma \in A : \beta < \gamma\} \vdash \alpha$. Thus, since $\alpha \leq \beta$ and $\leq$ is a transitive relation on A, it follows that $\{\gamma \in A : \alpha < \gamma\} \vdash \alpha$, which contradicts ($\leq 1$).
($\Leftarrow$) We will consider two cases:
Case 1) $\vdash \alpha \wedge \beta$. Then, by ($\leq 3$), $\alpha \leq \beta$.
Case 2) $\alpha \notin A - \alpha \wedge \beta$. Assume by *reductio ad absurdum* that $\alpha \nleq \beta$. Hence $\beta < \alpha$, since $\leq$ is a total relation. From which it follows, by Observation 5.2.18, that $\alpha \in cut_{<}(\beta)$. Thus, by Lemma 5.2.19 (d), $\alpha \in cut_{<}(\alpha \wedge \beta)$. Therefore, by Lemma D.2 (a), $\alpha \in A - \alpha \wedge \beta$. Contradiction. ∎

Proof of Observation 6.3.5.

(a) *Success, inclusion* and *vacuity* follow trivially by Observation 6.2.2. From Observation 6.2.2 we also know that $-$ satisfies: *strong inclusion, failure, relative closure* and *uniform behaviour*. *Extensionality* and *conjunctive factoring* follow trivially by Observations 5.1.3 (d).
Transitivity: Let $\beta \in A$, $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$. Hence, by *relative closure* $\nvdash \beta$. We intend to prove that $\alpha \notin A - \alpha \wedge \delta$. It follows trivially by *inclusion* if $\alpha \notin A$. Assume now that $\alpha \in A$. In this case, by *relative closure* $\nvdash \alpha$. We will prove by cases:
Case 1) $\vdash \delta$. It follows by *extensionality* and *success*.
Case 2) $\nvdash \delta$. From $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$ it follows by *relative closure*, Lemma C.1 and Observation 5.1.3 (c) that $A - \beta \subseteq A - \alpha \wedge \beta \subseteq A - \alpha$ and $A - \delta \subseteq A - \beta \wedge \delta \subseteq A - \beta$. Hence $A - \delta \subseteq A - \alpha$. Thus, by *decomposition* (Observation 5.1.3 (d)), it follows that $A - \alpha \wedge \delta \subseteq A - \alpha$. If $\alpha \in A - \alpha \wedge \delta$, then $\alpha \in A - \alpha$ which contradicts *success*. Hence $\alpha \notin A - \alpha \wedge \delta$.
EB1: Let $\beta \in A$ and $\{\gamma \in A : \beta \notin A - \beta \wedge \gamma\} \nvdash \alpha$. We will show that $\beta \in A - \alpha$. We will prove by cases:
Case 1) $A \nvdash \alpha$. It follows by *vacuity.*
Case 2) $\vdash \beta$. It follows by *relative closure.*
Case 3) $A \vdash \alpha$ and $\nvdash \beta$. Let $X = \{\gamma \in A : \beta \notin A - \beta \wedge \gamma\}$. Assume by *reductio ad absurdum* that $\beta \notin A - \alpha$. By *relative closure* and *strong inclusion*, it follows that $A - \alpha \subseteq A - \beta$. We will consider two cases:
Case 3.1) $A - \alpha \subset A - \beta$. It follows that $A - \beta \nsubseteq A - \alpha$. From which it follows, by

strong inclusion, that $A-\beta \vdash \alpha$. On the other hand, by Lemma C.3 $A-\beta \subseteq X$. Therefore $X \vdash \alpha$. Contradiction.
Case 3.2) $A-\alpha = A-\beta$. By *uniform behaviour* it follows that $\alpha \in Cn(A-\beta \cup \{\gamma \in A : A-\beta = A-\gamma\})$. Let $Y = A-\beta \cup \{\gamma \in A : A-\beta = A-\gamma\}$. We will now show that $Y \subseteq X$. Let $\psi \in Y$. If $\vdash \psi$, then $\psi \in X$, by *extensionality* and *success*. Assume now that $\nvdash \psi$. We will consider two cases:
Case 3.2.1) $A-\beta = A-\psi$. Hence, by *decomposition* (Observation 5.1.3 (d)), $A-\beta \wedge \psi \subseteq A-\beta$.
Case 3.2.2) $\psi \in A-\beta$. By *linearity* (Observation 5.1.3 (d)) and *success*, $A-\psi \subseteq A-\beta$. Hence, by *decomposition*, $A-\beta \wedge \psi \subseteq A-\beta$.
Hence in both cases $\psi \in X$, since from $A-\beta \wedge \psi \subseteq A-\beta$ it follows, by *success*, that $\beta \notin A-\beta \wedge \psi$. Therefore $Y \subseteq X$. But this leads to a contradiction since $Y \vdash \alpha$ and $X \nvdash \alpha$.
EB2: Let $\beta \in A-\alpha$. We intend to prove that $X = \{\gamma \in A : \gamma \in A-\gamma \wedge \alpha\} \vdash \alpha \vee \beta$. It is trivial if $\vdash \alpha \vee \beta$. Consider now that $\nvdash \alpha \vee \beta$. From $\beta \in A-\alpha$ it follows by *inclusion* that $\beta \in A$ and by Lemma C.1 that $\beta \in A-\alpha \wedge \beta$. Hence $\beta \in X$. Therefore $X \vdash \alpha \vee \beta$.

(b) Follows trivially from Example 6.3.2. ■

Proof of Observation 6.3.6.

(a) *Success, inclusion* and *vacuity* follow trivially by Theorem 6.2.4. On the other hand, by Theorem 6.2.4, $-$ satisfies *disjunctive elimination*. Hence, by Observation 5.1.2, it also satisfies *failure* and *relative closure*.

(b) Follows trivially from Examples 6.3.3 and 6.3.4. ■

Proof of Observation 6.3.7.

(a) It follows from Lemma D.6 (b) and Theorems 6.2.1 and 6.2.2.

(b) Let $A = \{p, q, p \vee q\}$ and $\preceq$ be an ensconcement relation on A defined by: $p \prec q \prec p \vee q$. Let $-$ be the $\preceq$-based contraction on A. Hence $A-p = \{q, p \vee q\}$ and $A-q = \{p, p \vee q\}$. The relation $\preceq'$ defined by condition ($\mathbf{C_{BR}} \preceq$) based on the operation $-$ is not an ensconcement. Indeed, since $q \in A-p$ and $p \in A-q$, it follows from ($\mathbf{C_{BR}} \preceq$), that $p \npreceq' q$ and $q \npreceq' p$. Therefore $\preceq'$ is not total and thus, $\preceq'$ is not an ensconcement relation. ■

Proof of Observation 6.5.1.

(a)–(d) Straightforward.

(e) Let $\mathbf{K}$ be a belief set and $\div$ be an operator on $\mathbf{K}$ that satisfies $(\div 1)$, $(\div 2)$, $(\div 3)$ and $(\div 5)$. It follows from Lemma $D.4$ (b) that $\div$ satisfies relevance. Therefore, since logical relevance follows trivially from relevance, it holds that $\div$ satisfies logical relevance. Finally, it follows from Observation 5.1.3 (a) that $\div$ satisfies disjunctive elimination.

(f) Let $\mathbf{K}$ be a belief set, $\beta \in \mathbf{K}$, $\alpha \notin \mathbf{K} \div \alpha \wedge \beta$, $\beta \notin \mathbf{K} \div \beta \wedge \delta$ and assume by *reductio ad absurdum* that $\alpha \in \mathbf{K} \div \alpha \wedge \delta$. From the latter it follows by *conjunctive trisection* (which is satisfied due to Observation 3.1.7 (a) and (b)) that $\alpha \in \mathbf{K} \div \alpha \wedge \delta \wedge \beta$. On the other hand, by $(\div V)$ and $(\div 6)$, it follows that $\mathbf{K} \div \alpha \wedge \beta \wedge \delta$ is identical to: (i) $\mathbf{K} \div \alpha \wedge \beta$, (ii) $\mathbf{K} \div \alpha \wedge \beta \cap \mathbf{K} \div \beta \wedge \delta$ or (iii) $\mathbf{K} \div \beta \wedge \delta$. The first two cases can not hold, since $\alpha \in \mathbf{K} \div \alpha \wedge \beta \wedge \delta$ and $\alpha \notin \mathbf{K} \div \alpha \wedge \beta$. Thus $\mathbf{K} \div \alpha \wedge \beta \wedge \delta = \mathbf{K} \div \beta \wedge \delta$. Hence $\beta \notin \mathbf{K} \div \alpha \wedge \beta \wedge \delta$, from which it follows, by $(\div 1)$, that $\alpha \wedge \beta \notin \mathbf{K} \div \alpha \wedge \beta \wedge \delta$. Hence, by $(\div 8)$ (which is satisfied due to Observation 3.1.7 (a)), $\mathbf{K} \div \alpha \wedge \beta \wedge \delta \subseteq \mathbf{K} \div \alpha \wedge \beta$. Contradiction, since $\alpha \in \mathbf{K} \div \alpha \wedge \beta \wedge \delta$ and $\alpha \notin \mathbf{K} \div \alpha \wedge \beta$.

(g) Let $\mathbf{K}$ be a belief set. Let $\beta \in \mathbf{K}$ and $X = \{\gamma \in \mathbf{K} : \beta \notin \mathbf{K} \div \beta \wedge \gamma\} \nvdash \alpha$. We wish to prove that $\beta \in \mathbf{K} \div \alpha$. This follows trivially by $(\div 1)$ if $\vdash \beta$ and by $(\div 3)$ if $\alpha \notin \mathbf{K}$. Assume now that $\nvdash \beta$ and $\alpha \in \mathbf{K}$. From $X \nvdash \alpha$ it follows that $\alpha \notin X$. Therefore, $\beta \in \mathbf{K} \div \beta \wedge \alpha$, from which it follows by $(\div 4)$ and $(\div V)$ that $\beta \in \mathbf{K} \div \alpha$.

(h) Let $\mathbf{K}$ be a belief set and $\beta \in \mathbf{K} \div \alpha$. We intend to show that $X = \{\gamma \in \mathbf{K} : \gamma \in \mathbf{K} \div \gamma \wedge \alpha\} \vdash \alpha \vee \beta$. It is trivial if $\vdash \alpha \vee \beta$. Assume now that $\nvdash \alpha \vee \beta$. From $\vdash \alpha \leftrightarrow (\alpha \vee \beta) \wedge \alpha$ it follows, by $(\div 6)$, that $\mathbf{K} \div \alpha = \mathbf{K} \div (\alpha \vee \beta) \wedge \alpha$. Hence $\beta \in \mathbf{K} \div (\alpha \vee \beta) \wedge \alpha$. Therefore, by $(\div 1)$, $\alpha \vee \beta \in \mathbf{K} \div (\alpha \vee \beta) \wedge \alpha$. Thus, by $(\div 2)$, $\alpha \vee \beta \in \mathbf{K}$. Hence $\alpha \vee \beta \in X$.

(i) Let $\mathbf{K}$ be a belief set and assume that $\mathbf{K} \vdash \alpha$ and $\mathbf{K} - \alpha = \mathbf{K} - \beta$. Then $\alpha \in \mathbf{K}$ and, furthermore, $\alpha \in \{\gamma \in \mathbf{K} : \mathbf{K} - \beta = \mathbf{K} - \gamma\}$. Therefore $\alpha \in Cn(\mathbf{K} - \beta \cup \{\gamma \in \mathbf{K} : \mathbf{K} - \beta = \mathbf{K} - \gamma\})$. ∎

Proof of Observation 6.5.2.

(a)–(c) Straightforward.

(d) Follows trivially from (c) and Observation 5.1.2.

(e) Let A be a belief set and $-$ be an operator on A that satisfies inclusion, vacuity and disjunctive elimination. Then, it follows from Lemma $D.5$ that $-$ satisfies relevance. Therefore, by Lemma $D.4$ (a) we can conclude that $-$ satisfies $(\div 5)$.

(f) Let A be a belief set and $-$ be an operator on A that satisfies inclusion, relative closure and strong inclusion. Then, it follows from (c) that $-$ satisfies $(\div 1)$. Therefore, we can conclude that $-$ satisfies $(\div 9)$, since this property follows trivially from strong inclusion and $(\div 1)$. ∎

Proof of Theorem 6.5.3.

We will prove that statements 1.–4. are equivalent by showing that 1. ⇔ 3., 3. ⇔ 4., 2. ⇒ 4. and 3. ⇒ 2..

(1. ⇔ 3.) Follows trivially from Observations 3.2.47 and 3.1.7(a).

(3. $\Leftrightarrow$ 4.) Follows from Observations 6.5.1 and 6.5.2 and the fact that *inclusion* and ($\div$2) are two alternative designations of the same property, and this is also the case regarding *extensionality* and ($\div$6) as well as *conjunctive factoring* and ($\div V$).

(2. $\Rightarrow$ 4.) Follows trivially from Theorem 6.2.4.

(3. $\Rightarrow$ 2.) Let $\mathbf{K}$ be a belief set and $\div$ be a contraction on $\mathbf{K}$ that satisfies ($\div$1)–($\div$6) and ($\div V$). Then it follows from Observation 6.5.1 (and from the fact that *inclusion* and ($\div$2), *extensionality* and ($\div$6), and *conjunctive factoring* and ($\div V$) are pairs of alternative designations for one same property) that $\div$ satisfies the postulates of *inclusion, vacuity, success, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1* and *EB2*. Therefore, according to Theorem 6.2.3, $\div$ is an ensconcement-based contraction on $\mathbf{K}$. ∎

Proof of Observation 6.5.4.

(a) Follows trivially from Theorem 6.2.4, Observation 6.5.2 and the fact that *inclusion* and ($\div$2), *extensionality* and ($\div$6), and *conjunctive factoring* and ($\div V$) are pairs of alternative designations for one same property.

(b) We will show that in general $-$ does not satisfy ($\div$1) nor ($\div$3) nor ($\div$5). Consider the following counter-example: Let $(A, \preceq)$ be an ensconcement where $A = \{q, q \to p\}$ and $\preceq$ is the two-level ensconcement relation on A defined by: $q \prec q \to p$. Hence $cut_{\prec}(p) = \{q \to p\}$. Let $-$ be the $\preceq$-based contraction for A. According to Definition 5.2.20, it holds that $A - p = \{q \to p\}$. Therefore $A - p \neq Cn(A - p)$ (hence $-$ does not satisfy ($\div$1)) and $A \nsubseteq Cn(A - p \cup \{p\})$ (hence $-$ does not satisfy ($\div$5)). Furthermore $p \notin A$ and $A - p \neq A$ (hence $-$ does not satisfy ($\div$3)). ∎

Proof of Observation 6.5.5.

Let $\mathbf{K}$ be a belief set and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. Let $\div$ be the $\leq$-based contraction on $\mathbf{K}$ and $-$ be the ensconcement-based contraction on $\mathbf{K}$ defined from $\leq|_{\mathbf{K}}$. We start by recalling that (according to conditions ($\mathbf{C}_{\div_\leq}$) and (**EBC**)) it holds that:

$$\mathbf{K} \div \alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha < \alpha \vee \beta\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

and

$$\mathbf{K} - \alpha = \{\beta \in \mathbf{K} : cut_{<|_{\mathbf{K}}}(\alpha) \vdash \alpha \vee \beta\}.$$

We recall also that, since $\leq$ is an epistemic entrenchment relation with respect to $\mathbf{K}$, it follows from Observation 5.2.15 that $(\mathbf{K}, \leq|_{\mathbf{K}})$ is an ensconcement.

We will prove by cases that, for all $\alpha \in \mathcal{L}$, $\mathbf{K} - \alpha = \mathbf{K} \div \alpha$.

Case 1) $\alpha \notin \mathbf{K}$. Hence $\mathbf{K} \nvdash \alpha$ and $\mathbf{K} \nvdash \bot$. Thus by Lemma 5.2.19 (c) $cut_{<|_{\mathbf{K}}}(\alpha) = \mathbf{K}$. Therefore $\mathbf{K} - \alpha = \mathbf{K}$.

On the other hand, by definition of the operator $\div$, it holds that $\mathbf{K} \div \alpha \subseteq \mathbf{K}$. Let $\beta \in \mathbf{K}$. Since $\mathbf{K}$ is a logically closed set it holds that $\alpha \vee \beta \in \mathbf{K}$. Hence by Lemmas 3.2.38 and 3.2.39 it follows that $\alpha < \alpha \vee \beta$. Thus $\mathbf{K} \subseteq \mathbf{K} \div \alpha$. Hence $\mathbf{K} - \alpha = \mathbf{K} = \mathbf{K} \div \alpha$.

Case 2) $\vdash \alpha$. Hence $\mathbf{K} - \alpha = \mathbf{K} \div \alpha = \mathbf{K}$.
Case 3) $\alpha \in \mathbf{K}$ and $\nvdash \alpha$.
Let $\beta \in \mathbf{K} - \alpha$. Then, since $\mathbf{K}$ is a logically closed set, $\alpha \vee \beta \in \mathbf{K}$. If $\vdash \alpha \vee \beta$, then $\alpha < |_{\mathbf{K}} \alpha \vee \beta$ (by (≤ 2)). Then $\alpha < \alpha \vee \beta$. Hence $\beta \in \mathbf{K} \div \alpha$. Therefore $\mathbf{K} - \alpha \subseteq \mathbf{K} \div \alpha$. Assume now that $\nvdash \alpha \vee \beta$. It follows, by definition of $-$ that $cut_{<|_{\mathbf{K}}}(\alpha) \vdash \alpha \vee \beta$. Hence, by Observation 5.2.18, $\{\gamma \in \mathbf{K} : \alpha < |_{\mathbf{K}} \gamma\} \vdash \alpha \vee \beta$. Since $\leq |_{\mathbf{K}}$ is a total relation on $\mathbf{K}$ it follows that $\alpha < |_{\mathbf{K}} \alpha \vee \beta$ or $\alpha \vee \beta \leq |_{\mathbf{K}} \alpha$. In the latter case it would follow that $\{\gamma \in \mathbf{K} : \alpha \vee \beta < |_{\mathbf{K}} \gamma\} \vdash \alpha \vee \beta$, which contradicts ($\leq 1$). Therefore $\alpha < |_{\mathbf{K}} \alpha \vee \beta$. Then $\alpha < \alpha \vee \beta$, from which it follows that $\beta \in \mathbf{K} \div \alpha$. Hence $\mathbf{K} - \alpha \subseteq \mathbf{K} \div \alpha$.
Let $\beta \in \mathbf{K} \div \alpha$. Hence $\beta \in \mathbf{K}$ and $\alpha < \alpha \vee \beta$. Since $\mathbf{K}$ is a logically closed set it follows that $\alpha \vee \beta \in \mathbf{K}$. Thus $\alpha < |_{\mathbf{K}} \alpha \vee \beta$. Therefore, by Observation 5.2.18, $\alpha \vee \beta \in cut_{<|_{\mathbf{K}}}(\alpha)$. Hence $\beta \in \mathbf{K} - \alpha$.
Therefore $\mathbf{K} - \alpha = \mathbf{K} \div \alpha$. ∎

Proof of Observation 6.5.6.
Let $\mathbf{K}$ be a belief set and $(\mathbf{K}, \preceq)$ be an ensconcement. Let $-$ be the $\preceq$-based contraction on $\mathbf{K}$ and $\div$ be the epistemic entrenchment-based contraction on $\mathbf{K}$ defined from the epistemic entrenchment relation $\leq_{\preceq}$. We start by recalling that (according to conditions ($\mathbf{C}_{\div_{\leq}}$) and (**EBC**)) it holds that:

$$\mathbf{K} \div \alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha <_{\preceq} \alpha \vee \beta\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

and

$$\mathbf{K} - \alpha = \{\beta \in \mathbf{K} : cut_{\prec}(\alpha) \vdash \alpha \vee \beta\}.$$

We will prove that, for all $\alpha \in \mathcal{L}$, $\mathbf{K} \div \alpha = \mathbf{K} - \alpha$.
If $\vdash \alpha$, then $\mathbf{K} \div \alpha = \mathbf{K}$ and $\vdash \alpha \vee \beta$. From the latter it follows that $\mathbf{K} - \alpha = \mathbf{K}$. Thus $\mathbf{K} \div \alpha = \mathbf{K} - \alpha$.
Assume now that $\nvdash \alpha$. Let $\alpha \notin \mathbf{K}$ (it follows that $\mathbf{K} \nvdash \alpha$ since $\mathbf{K}$ is a logically closed set). By definition of $\div$, it follows that $\mathbf{K} \div \alpha \subseteq \mathbf{K}$. Let $\beta \in \mathbf{K}$. Thus $\alpha \vee \beta \in \mathbf{K}$ (since $\mathbf{K}$ is a logically closed set). Therefore, by Lemmas 3.2.38 and 3.2.39, $\alpha <_{\preceq} \alpha \vee \beta$. Hence $\mathbf{K} \subseteq \mathbf{K} \div \alpha$. Thus $\mathbf{K} \div \alpha = \mathbf{K}$. On the other hand, by Lemma 5.2.19 (c), $cut_{\prec}(\alpha) = \mathbf{K}$. Thus $\mathbf{K} - \alpha = \mathbf{K}$.
Assume now that $\alpha \in \mathbf{K}$ and $\nvdash \alpha$.
Let $\beta \in \mathbf{K} \div \alpha$. Hence $\beta \in \mathbf{K}$ and $\alpha <_{\preceq} \alpha \vee \beta$. We will prove by cases that $\beta \in \mathbf{K} - \alpha$.
Case 1) $\vdash \alpha \vee \beta$. Then $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. Hence $\beta \in \mathbf{K} - \alpha$.
Case 2) $\nvdash \alpha \vee \beta$. Since $\mathbf{K}$ is a logically closed set and $\beta \in \mathbf{K}$ it follows that $\alpha \vee \beta \in \mathbf{K}$. From $\alpha <_{\preceq} \alpha \vee \beta$ it follows, by definition of $\leq_{\preceq}$, that $cut_{\preceq}(\alpha \vee \beta) \subset cut_{\preceq}(\alpha)$. Therefore there exists $\delta \in cut_{\preceq}(\alpha)$, such that $\delta \notin cut_{\preceq}(\alpha \vee \beta)$. Thus $\{\gamma \in \mathbf{K} : \delta \prec \gamma\} \vdash \alpha \vee \beta$. By ($\preceq 1$) and, since $\preceq$ is a total relation on $\mathbf{K}$, it follows that $\delta \prec \alpha \vee \beta$. From $\delta \in cut_{\preceq}(\alpha)$ it follows that $\{\gamma \in \mathbf{K} : \delta \prec \gamma\} \nvdash \alpha$. Hence $\{\gamma \in \mathbf{K} : \alpha \vee \beta \preceq \gamma\} \nvdash \alpha$. Therefore $\alpha \vee \beta \in cut_{\prec}(\alpha)$. Thus $\beta \in \mathbf{K} - \alpha$.
Let $\beta \in \mathbf{K} - \alpha$. We will prove that $\beta \in \mathbf{K} \div \alpha$. From $\beta \in \mathbf{K} - \alpha$ it follows that $\beta \in \mathbf{K}$ and $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$. Furthermore, since $\mathbf{K}$ is a logically closed set it follows that $\alpha \vee \beta \in \mathbf{K}$. If $\vdash \alpha \vee \beta$ it follows, by Lemma 3.2.40, that $\alpha <_{\preceq} \alpha \vee \beta$. Thus $\beta \in \mathbf{K} \div \alpha$.

Assume now that $\nvdash \alpha \vee \beta$.
From $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$, and since $\alpha \in \mathbf{K}$, it follows by Observation 5.2.18, that $\{\gamma \in \mathbf{K} : \alpha \prec \gamma\} \vdash \alpha \vee \beta$. Hence, by $(\preceq 1)$ and since $\preceq$ is a total relation on $\mathbf{K}$, it follows that $\alpha \prec \alpha \vee \beta$.
Assume by *reductio ad absurdum* that $\alpha \vee \beta \leq_{\preceq} \alpha$. Thus, by the definition of $\leq_{\preceq}$, it follows that $cut_{\preceq}(\alpha) \subseteq cut_{\preceq}(\alpha \vee \beta)$. Since $\alpha \in cut_{\preceq}(\alpha)$, it follows that $\alpha \in cut_{\preceq}(\alpha \vee \beta)$. Hence, by Lemma D.3, $\alpha \vee \beta \preceq \alpha$. Contradiction. Hence $\alpha \vee \beta \nleq_{\preceq} \alpha$. Thus, by Lemma 3.2.38, $\alpha <_{\preceq} \alpha \vee \beta$, from which it follows that $\beta \in \mathbf{K} \div \alpha$. ∎

Proof of Theorem 6.5.7.
We start by noticing that Observation 3.2.49 states exactly that 1. ⇔ 3.. Hence, in order to prove that statements 1. – 4. are equivalent, we will show that 3. ⇔ 4., 2. ⇒ 4. and 3. ⇒ 2..

(3. ⇔ 4.) Follows from Observations 6.5.1 and 6.5.2 and the fact that *inclusion* and $(\div 2)$ are two alternative designations of the same property, and this is also the case regarding *failure* and $(\div 3')$.

(2. ⇒ 4.) Follows trivially from Observation 6.2.2.

(3. ⇒ 2.) Let $\mathbf{K}$ be a belief set and $\div$ be a contraction on $\mathbf{K}$ that satisfies $(\div 1)$, $(\div 2)$, $(\div 3)$, $(\div 3')$, $(\div 4)$ and $(\div 9)$. Then it follows from Observation 6.5.1 (and from the fact that *inclusion* and $(\div 2)$, and *failure* and $(\div 3')$ are pairs of alternative designations for one same property) that $\div$ satisfies the postulates of *relative closure, inclusion, vacuity, failure, success, strong inclusion* and *uniform behaviour*. Therefore, according to Observation 6.2.1, $\div$ is a brutal contraction on $\mathbf{K}$. ∎

Proof of Observation 6.5.8.

(a) Follows trivially from Observation 6.2.2, Observation 6.5.2 and the fact that *inclusion* and $(\div 2)$, and *failure* and $(\div 3')$ are pairs of alternative designations for one same property.

(b) We will show that in general $-$ does not satisfy $(\div 1)$ nor $(\div 3)$ nor $(\div 9)$. Consider the following counter-example: Let $(A, \preceq)$ be an ensconcement where $A = \{q, p \vee q, q \rightarrow p\}$ and $\preceq$ is the three-level ensconcement relation on A defined by: $q \prec p \vee q \prec q \rightarrow p$. Let $-$ be the $\preceq$-based brutal contraction on A. According to Definition 5.2.21, it holds that $A - p = cut_{\prec}(p) = \{q \rightarrow p\}$ and $A - q = cut_{\prec}(q) = \{p \vee q, q \rightarrow p\}$. Therefore $A - q \neq Cn(A - q)$ (hence $-$ does not satisfy $(\div 1)$) and, however $p \notin A - q$, it does not hold that $A - q \subseteq A - p$ (hence $-$ does not satisfy $(\div 9)$). Furthermore $p \notin A$ and $A - p \neq A$ (hence $-$ does not satisfy $(\div 3)$). ∎

Proof of Observation 6.5.9.
Let $\mathbf{K}$ be a belief set and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. Let $\div$ be the $\leq$-based severe withdrawal on $\mathbf{K}$ and $-$ be the brutal contraction on $\mathbf{K}$ defined from $\leq|_{\mathbf{K}}$ (notice that, according to Observation 5.2.15, $(\mathbf{K}, \leq|_{\mathbf{K}})$ is an ensconcement). We start by recalling that (according to conditions $(\mathbf{R}_{\div_{\leq}})$ and

($\mathbf{BC}$)) it holds that:

$$\mathbf{K} \div \alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha < \beta\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

and

$$\mathbf{K} - \alpha = \begin{cases} cut_{<|_{\mathbf{K}}}(\alpha) & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

We will prove by cases that, for all $\alpha \in \mathcal{L}$, $\mathbf{K} - \alpha = \mathbf{K} \div \alpha$.
Case 1) $\alpha \notin \mathbf{K}$. Hence $\mathbf{K} \nvdash \alpha$ and $\mathbf{K} \nvdash \bot$. Thus by Lemma 5.2.19 (c) $cut_{<|_{\mathbf{K}}}(\alpha) = \mathbf{K}$. Therefore $\mathbf{K} - \alpha = \mathbf{K}$.
On the other hand, by definition of the operator $\div$, it holds that $\mathbf{K} \div \alpha \subseteq \mathbf{K}$. Let $\beta \in \mathbf{K}$. Hence by Lemmas 3.2.38 and 3.2.39 it follows that $\alpha < \beta$. Thus $\mathbf{K} \subseteq \mathbf{K} \div \alpha$. Hence $\mathbf{K} - \alpha = \mathbf{K} = \mathbf{K} \div \alpha$.
Case 2) $\vdash \alpha$. Hence $\mathbf{K} - \alpha = \mathbf{K} \div \alpha = \mathbf{K}$.
Case 3) $\alpha \in \mathbf{K}$ and $\nvdash \alpha$.
Let $\beta \in \mathbf{K} - \alpha$. Hence $\beta \in cut_{<|_{\mathbf{K}}}(\alpha)$. Therefore, by Observation 5.2.18, $\alpha < |_{\mathbf{K}}\beta$. Hence $\alpha < \beta$. Thus $\beta \in \mathbf{K} \div \alpha$.
Let $\beta \in \mathbf{K} \div \alpha$. Then $\alpha < \beta$. Thus, $\alpha < |_{\mathbf{K}}\beta$. By Observation 5.2.18, it follows that $\beta \in cut_{<|_{\mathbf{K}}}(\alpha)$. Therefore $\beta \in \mathbf{K} - \alpha$. Hence $\mathbf{K} - \alpha = \mathbf{K} \div \alpha$. ∎

Proof of Observation 6.5.10.
Let $\mathbf{K}$ be a belief set and $(\mathbf{K}, \preceq)$ be an ensconcement. Let $-$ be the $\preceq$-based brutal contraction on $\mathbf{K}$ and $\div$ be the severe withdrawal on $\mathbf{K}$ defined from the epistemic entrenchment relation $\leq_{\preceq}$. We start by recalling that (according to conditions ($\mathbf{R}_{\div_{\leq}}$) and ($\mathbf{BC}$)) it holds that:

$$\mathbf{K} \div \alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha <_{\preceq} \beta\} & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

and

$$\mathbf{K} - \alpha = \begin{cases} cut_{\prec}(\alpha) & \text{if } \nvdash \alpha \\ \mathbf{K} & \text{otherwise} \end{cases}$$

We will prove that, for all $\alpha \in \mathcal{L}$, $\mathbf{K} \div \alpha = \mathbf{K} - \alpha$.
If $\vdash \alpha$, then $\mathbf{K} - \alpha = \mathbf{K} \div \alpha = \mathbf{K}$. Assume now that $\nvdash \alpha$.
Let $\beta \in \mathbf{K} - \alpha$. Hence $\beta \in \mathbf{K}$. We will prove by cases that $\beta \in \mathbf{K} \div \alpha$.
Case 1) $\vdash \beta$. Then, by Lemma 3.2.40, $\alpha <_{\preceq} \beta$. Thus $\beta \in \mathbf{K} \div \alpha$.
Case 2) $\mathbf{K} \nvdash \alpha$. Then $\alpha \notin \mathbf{K}$ and, by Lemmas 3.2.38 and 3.2.39, $\alpha <_{\preceq} \beta$. Thus $\beta \in \mathbf{K} \div \alpha$.
Case 3) $\nvdash \beta$ and $\mathbf{K} \vdash \alpha$. Hence $\alpha \in \mathbf{K}$. From $\beta \in \mathbf{K} - \alpha$ it follows that $\beta \in cut_{\prec}(\alpha)$. Therefore, by Observation 5.2.18, $\alpha \prec \beta$. Assume by *reductio ad absurdum* that $\beta \leq_{\preceq} \alpha$. Thus, by the definition of $\leq_{\preceq}$, $cut_{\preceq}(\alpha) \subseteq cut_{\preceq}(\beta)$. Since $\alpha \in cut_{\preceq}(\alpha)$ it follows that $\alpha \in cut_{\preceq}(\beta)$. Hence, by Lemma D.3, $\beta \preceq \alpha$. Contradiction. Hence by Lemma 3.2.38, $\alpha <_{\preceq} \beta$. Therefore $\beta \in \mathbf{K} \div \alpha$.

Let $\beta \in \mathbf{K} \div \alpha$. Then $\beta \in \mathbf{K}$ and $\alpha <_{\preceq} \beta$. We will prove by cases that $\beta \in \mathbf{K} - \alpha$.
Case 1) $\mathbf{K} \nvdash \alpha$. Then $cut_{\prec}(\alpha) = \mathbf{K}$ (by Lemma 5.2.19 (c)). Then $\beta \in \mathbf{K} - \alpha$.
Case 2) $\vdash \beta$. Hence, by ($\preceq 2$) and ($\preceq 3$), $H = \{\gamma \in \mathbf{K} : \beta \preceq \gamma\} \subseteq Cn(\varnothing)$. Thus $H \nvdash \alpha$. Hence $\beta \in cut_{\prec}(\alpha)$. Therefore $\beta \in \mathbf{K} - \alpha$.
Case 3) $\mathbf{K} \vdash \alpha$ and $\nvdash \beta$. From $\alpha <_{\preceq} \beta$ it follows, by the definition of $\leq_{\preceq}$, that $cut_{\preceq}(\beta) \subset cut_{\preceq}(\alpha)$. Therefore, there exists $\delta \in cut_{\preceq}(\alpha)$ such that $\delta \notin cut_{\preceq}(\beta)$. By Lemma D.3 it follows that $\beta \npreceq \delta$. Thus, since $\preceq$ is a total relation on $\mathbf{K}$, it follows that $\delta \prec \beta$. Hence $\beta \in cut_{\prec}(\alpha)$. Therefore $\beta \in \mathbf{K} - \alpha$. ∎

Appendix E

Proofs of Chapter 7

Lemma E.1 ([Han94]) *The following two conditions are equivalent:*

1. $A \perp\!\!\!\perp \alpha = A \perp\!\!\!\perp \beta$
2. *For all subsets B of A: $B \vdash \alpha$ if and only if $B \vdash \beta$.*

Lemma E.2 *Let A be a belief base. Let $-$ be an operator on A such that (for all α) $A - \alpha = A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$, then $-$ satisfies relevance, success, inclusion, vacuity, extensionality, uniformity, core-retainment, disjunctive elimination, failure and relative closure.*

Proof.
Let $-$ be an operator on a belief base A defined (for all α) by $A - \alpha = A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$. We will start by showing that $-$ satisfies *relevance*. Let $\beta \in A$ and $\beta \notin A - \alpha$. Thus $\beta \in \bigcup(A \perp\!\!\!\perp \alpha)$. Hence there exists $Y \in A \perp\!\!\!\perp \alpha$ such that $\beta \in Y$. Let $X = Y \setminus \{\beta\} \cup (A \setminus \bigcup(A \perp\!\!\!\perp \alpha))$. Hence $X \subseteq A$, $A - \alpha = A \setminus \bigcup(A \perp\!\!\!\perp \alpha) \subseteq X$ and $X \cup \{\beta\} \vdash \alpha$. It remains to prove that $X \nvdash \alpha$. Assume by *reductio ad absurdum* that $X \vdash \alpha$. Hence it follows by compactness that there exists a finite subset $H = \{\gamma_1, ..., \gamma_n\}$ of X such that $H \vdash \alpha$. Where $\gamma_1, ..., \gamma_k \in Y \setminus \{\beta\}$ and $\gamma_{k+1}, ..., \gamma_n \in A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$, for some $1 \leq k < n$. Hence $\{\gamma_1, ..., \gamma_k\} \cup \{\gamma_{k+1}, ..., \gamma_n\} \vdash \alpha$ but $\{\gamma_1, ..., \gamma_k\} \nvdash \alpha$. Thus there is some inclusion-minimal subset W of H such that $W \vdash \alpha$ but no proper set of W implies α. Hence $W \in A \perp\!\!\!\perp \alpha$. On the other hand, since $\{\gamma_1, ..., \gamma_k\} \nvdash \alpha$, W contains at least one of the $\gamma_i \in \{\gamma_{k+1}, ..., \gamma_n\}$. Contradiction since $\{\gamma_{k+1}, ..., \gamma_n\} \subseteq A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$. Hence $-$ satisfies *relevance*.
On the other hand, according to Definition 3.2.17, $-$ is a kernel contraction. Hence, it follows from Observations 5.2.4 and 5.1.2, that $-$ satisfies *success, inclusion, vacuity, extensionality, uniformity, core-retainment, disjunctive elimination, failure* and *relative closure.* ■

Lemma E.3 *Let $A = \{p, q\}$ and $R = \mathcal{L} \setminus (Cn(p) \cup Cn(q))$. Then R satisfies* uniform retractability *and* non-retractability propagation*.*

Proof.
Uniform retractability: Assume that it holds for all subsets A' of A that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$. We will prove that $\alpha \in R$ if and only if $\beta \in R$.

Let $\alpha \notin R$. Then $\alpha \in Cn(p)$ or $\alpha \in Cn(q)$. Therefore $\beta \in Cn(p)$ or $\beta \in Cn(q)$. In both cases it follows that $\beta \notin R$. By symmetry of the case it follows that if $\beta \notin R$, then $\alpha \notin R$. Therefore $\alpha \in R$ if and only if $\beta \in R$.
Non-retractability propagation: Let $\alpha \notin R$ and $\beta \in Cn(\alpha)$. From $\alpha \notin R$ it follows that $\alpha \in Cn(p)$ or $\alpha \in Cn(q)$. Hence $\beta \in Cn(p)$ or $\beta \in Cn(q)$. In both cases it follows that $\beta \notin R$. Thus R satisfies *non-retractability propagation*. ∎

Lemma E.4 *Let* $A = \{p, q\}$ *and* $R = \mathcal{L} \setminus (Cn(\varnothing) \cup (Cn(p) \setminus Cn(q)))$. *Then* R *satisfies* uniform retractability *and* conjunctive completeness.

Proof.
Uniform retractability: Assume that it holds for all subsets A' of A that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$. We will prove that $\alpha \in R$ if and only if $\beta \in R$.
Let $\alpha \notin R$. Then $\alpha \in Cn(\varnothing)$ or $\alpha \in Cn(p) \setminus Cn(q)$. In the first case it follows that $\beta \in Cn(\varnothing)$. Therefore $\beta \notin R$. Assume now that $\alpha \in Cn(p) \setminus Cn(q)$. Therefore $\alpha \in Cn(p)$ and $\alpha \notin Cn(q)$. Hence $\beta \in Cn(p)$ and $\beta \notin Cn(q)$. Hence $\beta \notin R$. By symmetry of the case it follows that if $\beta \notin R$, then $\alpha \notin R$. Therefore $\alpha \in R$ if and only if $\beta \in R$.
Conjunctive completeness Let $\alpha \wedge \beta \in R$. Thus $\alpha \wedge \beta \notin Cn(\varnothing)$ and $\alpha \wedge \beta \notin Cn(p) \setminus Cn(q)$. Hence $\alpha \wedge \beta \notin Cn(p)$ or $\alpha \wedge \beta \in Cn(q)$. In the former case $\alpha \notin Cn(p)$ or $\beta \notin Cn(p)$. Hence $\alpha \in R$ or $\beta \in R$. In the latter case, $\alpha \in Cn(q)$ and $\beta \in Cn(q)$. Since $\alpha \notin Cn(\varnothing)$ or $\beta \notin Cn(\varnothing)$ (because $\alpha \wedge \beta \notin Cn(\varnothing)$), it follows that $\alpha \in R$ or $\beta \in R$. Therefore R satisfies *conjunctive completeness*. ∎

Lemma E.5 *Let* $A = \{p, q\}$ *and* $R = \mathcal{L} \setminus Cn(q)$. *Then* R *satisfies* conjunctive completeness, non-retractability propagation *and* uniform retractability.

Proof.
Conjunctive completeness: Let $\alpha \wedge \beta \in R$. Hence $\{q\} \nvdash \alpha \wedge \beta$. Therefore $\{q\} \nvdash \alpha$ or $\{q\} \nvdash \beta$, from which it follows that $\alpha \in R$ or $\beta \in R$.
Non-retractability propagation: Let $\alpha \notin R$. Then $\{q\} \vdash \alpha$. Let $\beta \in Cn(\alpha)$. Hence $\{q\} \vdash \beta$, from which it follows that $\beta \notin R$.
Uniform retractability: Assume that it holds for all subsets A' of A that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$. Let $\alpha \notin R$. Hence $\{q\} \vdash \alpha$, from which it follows that $\{q\} \vdash \beta$. Thus $\beta \notin R$. It also holds, by symmetry of the case, that if $\beta \notin R$, then $\alpha \notin R$. Hence $\alpha \in R$ if and only if $\beta \in R$. ∎

Proof of Observation 7.1.3.
Let R be a set that satisfies non-retractability of tautology, non-retractability propagation and conjunctive completeness. We intend to prove that: $\alpha \in \mathcal{L} \backslash R$ if and only if $\mathcal{L} \setminus R \vdash \alpha$.
Left to right is trivial. For the other direction consider that $\mathcal{L} \setminus R \vdash \alpha$. If $\vdash \alpha$, then $\alpha \notin R$ by *non-retractability of tautology*. Assume now that $\nvdash \alpha$. Hence by compactness there is a non-empty finite set $H = \{\gamma_1, ..., \gamma_n\}$ such that $H \subseteq \mathcal{L} \setminus R$ and $H \vdash \alpha$. Hence $\{\gamma_1 \wedge ... \wedge \gamma_n\} \vdash \alpha$. By repeated use of *conjunctive completeness* it follows that $\gamma_1 \wedge ... \wedge \gamma_n \notin R$. Hence, by *non-retractability propagation*, $\alpha \notin R$. ∎

Proof of Observation 7.1.4.

(a) Let α, β be such that $\vdash \alpha \leftrightarrow \beta$. Hence α and β are implied by exactly the same subsets of A. Thus, by *uniform retractability*, $\alpha \in R$ if and only if $\beta \in R$.

(b) **Retractability of logical equivalents:** Let α, β be such that $\vdash \alpha \leftrightarrow \beta$. Hence $\alpha \in Cn(\beta)$ and $\beta \in Cn(\alpha)$. If $\alpha \notin R$, then by *non-retractability propagation* $\beta \notin R$. It also holds that if $\beta \notin R$, then $\alpha \notin R$. Therefore $\alpha \in R$ if and only if $\beta \in R$.

Converse conjunctive completeness: Let $\alpha \in R$ and assume by *reductio ad absurdum* that $\alpha \wedge \beta \notin R$. From the latter, by *non-retractability propagation*, it follows that $Cn(\alpha \wedge \beta) \cap R = \varnothing$. Hence $\alpha \notin R$, which is a contradiction. Therefore $\alpha \wedge \beta \in R$. ■

Proof of Theorem 7.1.5.

Let $\sim$ be an operator of shielded base contraction induced by a contraction operator $-$ for A and a set $R \subseteq \mathcal{L}$. We intend to prove that R satisfies non-retractability of tautology and non-retractability upper bounding if and only if $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$.

(Left to right)

Let $\beta \in R$. Hence, $A \sim \beta = A - \beta$. Furthermore, by *non-retractability of tautology*, $\nvdash \beta$. Therefore, by $-$ *success*, $A \sim \beta \nvdash \beta$.

Let $\beta \notin R$. Hence, $A \sim \beta = A$ and by *non-retractability upper bounding* $A \vdash \beta$. Therefore $A \sim \beta \vdash \beta$.

Hence $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$.

(Right to left)

Let $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$. From the definition of R it follows trivially that R satisfies *non-retractability of tautology*. Let $\beta \in \mathcal{L} \backslash R$. Hence $A \sim \beta \vdash \beta$ and $A \sim \beta = A$. Therefore $A \vdash \beta$. Hence R satisfies *non-retractability upper bounding.* ■

Proof of Observation 7.1.6.

Let A be a belief base, $R \subseteq \mathcal{L}$ and $\sim$ be a shielded base contraction induced by a contraction operator $-$ and R. We intend to prove that R satisfies non-retractability preservation if and only if R satisfies non-retractability upper bounding, and $-$ and R satisfy condition (**R -** $-$).

Assume first that R satisfies *non-retractability preservation.*

Condition (R - $-$): Assume that $\alpha \notin R$ and $\beta \in R$. By *non-retractability preservation* it follows that $A \sim \beta \vdash \alpha$. On the other hand $A \sim \beta = A - \beta$, by definition of $\sim$. Therefore $A - \beta \vdash \alpha$.

Non-retractability upper bounding: By *non-retractability preservation* $\mathcal{L} \backslash R \subseteq Cn(A \sim \alpha)$. On the other hand, by definition of shielded base contraction, $A \sim \alpha \subseteq A$. Hence, by monotony, $\mathcal{L} \backslash R \subseteq Cn(A)$.

Assume now that R satisfies *non-retractability upper bounding* and that $-$ and R are related through condition (**R -** $-$). Let $\alpha \in \mathcal{L} \backslash R$ and $\beta \in \mathcal{L}$. We intend to prove that $A \sim \beta \vdash \alpha$.

If $\beta \in R$, then it follows by definition of $\sim$ that $A \sim \beta = A - \beta$. On the other hand, by condition (**R -** $-$) it follows that $A - \beta \vdash \alpha$. Hence $A \sim \beta \vdash \alpha$.

Assume now that $\beta \notin R$. Hence, by definition of $\sim$, it follows that $A \sim \beta = A$. On the other hand by *non-retractability upper bounding* it follows that $A \vdash \alpha$. Thus

$A \sim \beta \vdash \alpha$. ■

Proof of Observation 7.1.7.
Let A be a belief base, $-$ be a contraction on A, $R \subseteq \mathcal{L}$, and $\sim$ be the shielded base contraction induced by $-$ and R. Assume that R satisfies non-retractability preservation and non-retractability of tautology.
Conjunctive completeness: Let $\alpha \wedge \beta \in R$ and assume by *reductio ad absurdum* that $\alpha \notin R$ and $\beta \notin R$. By *non-retractability of tautology* $\nvdash \alpha \wedge \beta$. On the other hand, by *non-retractability preservation* $\mathcal{L} \backslash R \subseteq Cn(A \sim (\alpha \wedge \beta))$. Hence $A \sim (\alpha \wedge \beta) \vdash \alpha$ and $A \sim (\alpha \wedge \beta) \vdash \beta$. Therefore $A \sim (\alpha \wedge \beta) \vdash \alpha \wedge \beta$. By definition of $\sim$, $A \sim (\alpha \wedge \beta) = A - (\alpha \wedge \beta)$. Thus $A - (\alpha \wedge \beta) \vdash \alpha \wedge \beta$, which contradicts $-$ *success.*
Non-retractability propagation: Let $\alpha \notin R$ and $\beta \in Cn(\alpha)$. If $\vdash \beta$, then by *non-retractability of tautology* $\beta \notin R$. Consider now that $\nvdash \beta$. By *non-retractability preservation* $\mathcal{L} \backslash R \subseteq Cn(A \sim \beta)$. Hence $A \sim \beta \vdash \alpha$. Therefore $A \sim \beta \vdash \beta$. Hence $\beta \notin R$ (otherwise it would follow that $A - \beta \vdash \beta$, which contradicts $-$ *success*).
Uniform retractability: Assume that for all subsets A' of A, $A' \vdash \alpha$ if and only if $A' \vdash \beta$. Assume that $\alpha \notin R$. By *non-retractability preservation* $\mathcal{L} \backslash R \subseteq Cn(A \sim \beta)$. Hence $A \sim \beta \vdash \alpha$. Since $\sim$ is a shielded base contraction it follows that $A \sim \beta \subseteq A$. Hence, by hypothesis, $A \sim \beta \vdash \beta$. Assume by *reductio ad absurdum* that $\beta \in R$. Thus by *non-retractability of tautology* $\nvdash \beta$. On the other hand, by definition of $\sim$, $A \sim \beta = A - \beta$. Therefore $A - \beta \vdash \beta$, which contradicts $-$ *success.* Hence if $\alpha \notin R$, then $\beta \notin R$. By symmetry of the case it holds that if $\beta \notin R$, then $\alpha \notin R$. Hence $\alpha \in R$ if and only if $\beta \in R$.
Retractability of logical equivalents: Follows by Observation 7.1.4 since, as shown above, R satisfies *uniform retractability.* ■

Proof of Theorem 7.2.1.
Let A be a belief base, $-$ be a contraction on A, $R \subseteq \mathcal{L}$, and $\sim$ be the shielded base contraction induced by $-$ and R.
Thus,

$$A \sim \alpha = \begin{cases} A - \alpha & \text{if } \alpha \in R \\ A & \text{otherwise} \end{cases}$$

where $-$ is an operator on A that satisfies *success* and *inclusion.*

(a) It follows trivially from its definition that $\sim$ satisfies *inclusion.*

Assume that $-$ satisfies *vacuity.* It follows trivially from its definition that $\sim$ satisfies *vacuity.*

Assume that $-$ satisfies *failure.* If $\alpha \notin R$, then $A \sim \alpha = A$. Assume now that $\alpha \in R$. Hence $A \sim \alpha = A - \alpha$. If $\nvdash \alpha$, then it follows, by $-$ *success*, that $A \sim \alpha \nvdash \alpha$. If $\vdash \alpha$, then by $-$ *failure* $A \sim \alpha = A - \alpha = A$. Therefore $\sim$ satisfies *relative success.*

Assume that $-$ satisfies *relative closure.* If $\alpha \notin R$ it follows trivially that $A \cap Cn(A \sim \alpha) \subseteq A \sim \alpha$. Assume now that $\alpha \in R$. Then $A \sim \alpha = A - \alpha$, and it follows trivially by $-$ *relative closure* that $A \cap Cn(A \sim \alpha) \subseteq A \sim \alpha$. Therefore $\sim$ satisfies *relative closure*.

Assume that $-$ satisfies *relevance*. Let $\beta \in A$ and $\beta \notin A \sim \alpha$. Hence $A \sim \alpha = A - \alpha$, and it follows trivially from $-$ *relevance* that $\sim$ satisfies *relevance*.

Assume that $-$ satisfies *core-retainment*. Let $\beta \in A$ and $\beta \notin A \sim \alpha$. Hence $A \sim \alpha = A - \alpha$, and it follows trivially from $-$ *core-retainment* that $\sim$ satisfies *core-retainment*.

Assume that $-$ satisfies *disjunctive elimination*. Let $\beta \in A$ and $\beta \notin A \sim \alpha$. Hence $A \sim \alpha = A - \alpha$, and it follows trivially from $-$ *disjunctive elimination* that $\sim$ satisfies *disjunctive elimination*.

(b) Let R and $-$ be such that condition ($\mathbf{R}$ **-** $-$) is satisfied.

Now we will show that $\sim$ satisfies *persistence*. Let $\alpha, \beta \in \mathcal{L}$. Assume that $A \sim \beta \vdash \beta$. We intend to prove that $A \sim \alpha \vdash \beta$. It follows trivially if $\vdash \beta$. Assume now that $\nvdash \beta$. By $-$ *success* it follows that $A - \beta \nvdash \beta$. Hence $A \sim \beta \neq A - \beta$. Thus, by definition of $\sim$, it follows that $\beta \notin R$. Therefore $A \sim \beta = A$, from which it follows that $A \vdash \beta$. If $\alpha \in R$, then it follows by condition ($\mathbf{R}$ **-** $-$) that $A \sim \alpha \vdash \beta$. If $\alpha \notin R$, then $A \sim \alpha = A$. Hence $A \sim \alpha \vdash \beta$.

Assume that $-$ satisfies *failure* and *extensionality*. Let α, β be such that $\vdash \alpha \leftrightarrow \beta$. We must prove that $A \sim \alpha = A \sim \beta$. It holds that $\vdash \alpha$ if and only if $\vdash \beta$. Therefore there are two cases to consider:
Case 1) $\vdash \alpha$ and $\vdash \beta$. Hence, by $-$ *failure*, $A - \alpha = A - \beta = A$. Then $A \sim \alpha = A \sim \beta = A$.
Case 2) $\nvdash \alpha$ and $\nvdash \beta$. Assume by *reductio ad absurdum*, without loss of generality, that $\alpha \notin R$ and $\beta \in R$. By condition ($\mathbf{R}$ **-** $-$) it follows that $A - \beta \vdash \alpha$. Thus $A - \beta \vdash \beta$, which contradicts $-$ *success*. Hence $\alpha \in R$ if and only if $\beta \in R$. We will prove by cases:
Case 2.1) $\alpha \in R$ and $\beta \in R$. Hence $A \sim \alpha = A - \alpha$ and $A \sim \beta = A - \beta$. Thus, by $-$ *extensionality*, $A \sim \alpha = A \sim \beta$.
Case 2.2) $\alpha \notin R$ and $\beta \notin R$. Thus $A \sim \alpha = A \sim \beta = A$.

Assume that $-$ satisfies *failure* and *uniformity*. Assume that for all subsets A' of A, $A' \vdash \alpha$ if and only if $A' \vdash \beta$. We must prove that $A \sim \alpha = A \sim \beta$. If $\vdash \alpha$, then $\vdash \beta$ (since $\emptyset \subseteq A$). By $-$ *failure*, $A - \alpha = A - \beta = A$. Hence, by definition of $\sim$, it follows that $A \sim \alpha = A \sim \beta = A$.
Consider now that $\nvdash \alpha$. Hence $\nvdash \beta$. Assume without loss of generality, that $\alpha \notin R$ and $\beta \in R$. By condition ($\mathbf{R}$ **-** $-$) it follows that $A - \beta \vdash \alpha$. On the other hand, it follows from $-$ *inclusion* that $A - \beta \subseteq A$. Hence by hypothesis, $A - \beta \vdash \beta$, which contradicts $-$ *success*. Hence $\alpha \in R$ if and only if $\beta \in R$.
So there are two cases to consider:
Case 1) $\alpha \in R$ and $\beta \in R$. Then $A \sim \alpha = A - \alpha$ and $A \sim \beta = A - \beta$. By $-$ *uniformity* $A - \alpha = A - \beta$. Hence $A \sim \alpha = A \sim \beta$.
Case 2) $\alpha \notin R$ and $\beta \notin R$. Then $A \sim \alpha = A \sim \beta = A$.

(c) Let R be a set that satisfies *non-retractability preservation*.
It follows from Observation 7.1.6 that $-$ and R satisfy condition ($\mathbf{R}$ - $-$). The rest of the proof follows trivially from (b).

(d) It follows trivially if $\alpha \notin R$. Assume now that $\alpha \in R$. Hence $A \sim \alpha = A - \alpha$. By *non-retractability of tautology*, it follows that $\nvdash \alpha$. Thus, from $-$ *success* it follows that $A \sim \alpha \nvdash \alpha$.

(e) Let $\vdash \alpha \leftrightarrow \beta$. By *retractability of logical equivalents* it follows that $\alpha \in R$ if and only if $\beta \in R$. Thus, there are two cases to consider:
Case 1) $\alpha \notin R$ and $\beta \notin R$. Then $A \sim \alpha = A \sim \beta = A$.
Case 2) $\alpha \in R$ and $\beta \in R$. Then $A \sim \alpha = A - \alpha$ and $A \sim \beta = A - \beta$. Thus by $-$ *extensionality* it follows that $A \sim \alpha = A \sim \beta$.

(f) Assume that for all subsets A' of A, $A' \vdash \alpha$ if and only if $A' \vdash \beta$. By *uniform retractability* $\alpha \in R$ if and only if $\beta \in R$. There are two cases to consider:
Case 1) $\alpha \in R$ and $\beta \in R$. Then $A \sim \alpha = A - \alpha$ and $A \sim \beta = A - \beta$. By $-$ *uniformity* $A - \alpha = A - \beta$. Hence $A \sim \alpha = A \sim \beta$.
Case 2) $\alpha \notin R$ and $\beta \notin R$. Then $A \sim \alpha = A \sim \beta = A$.

(g) Let $A \sim \beta \vdash \beta$ and $\vdash \beta \rightarrow \alpha$.
Case 1) $\beta \notin R$. Hence $A \sim \beta = A$ and $A \vdash \alpha$. It follows by *non-retractability propagation* that $\alpha \notin R$. Therefore $A \sim \alpha = A$ and $A \sim \alpha \vdash \alpha$.
Case 2) $\beta \in R$. Hence $A \sim \beta = A - \beta$. Thus, by $-$ *success*, it follows that $\vdash \beta$. Thus $\vdash \alpha$ and (consequently) $A \sim \alpha \vdash \alpha$.

Assume that $-$ satisfies *extensionality*. Let α, β be such that $\vdash \alpha \leftrightarrow \beta$. Hence $\beta \in Cn(\alpha)$ and $\alpha \in Cn(\beta)$, from which it follows, by *non-retractability propagation*, that $\alpha \in R$ if and only if $\beta \in R$. Thus if $\alpha \notin R$, then $A \sim \alpha = A = A \sim \beta$. If $\alpha \in R$, then by $-$ *extensionality*, $A \sim \alpha = A - \alpha = A - \beta = A \sim \beta$.

(h) We start by noticing that, according to Observation 7.1.4, if R satisfies *uniform retractability*, then it also satisfies *retractability of logical equivalents*. Now assume that $-$ satisfies *vacuity* and *failure* and let R be a set that satisfies *conjunctive completeness* and *retractability of logical equivalents*. We will prove that $\sim$ satisfies *conjunctive constancy*. Let $A \sim \alpha = A \sim \beta = A$. If $A \nvdash \alpha \wedge \beta$, then by $-$ *inclusion* and *vacuity* $A \sim (\alpha \wedge \beta) = A$. Consider that $A \vdash \alpha \wedge \beta$. Hence $A \vdash \alpha$ and $A \vdash \beta$. If $\alpha \wedge \beta \notin R$, then $A \sim (\alpha \wedge \beta) = A$. Assume now that $\alpha \wedge \beta \in R$. Hence, by *conjunctive completeness* $\alpha \in R$ or $\beta \in R$. Assume without loss of generality that $\alpha \in R$. Hence $A - \alpha \vdash \alpha$. Thus, by $-$ *success*, $\vdash \alpha$, from which it follows that $\vdash (\alpha \wedge \beta) \leftrightarrow \beta$. Therefore, by *retractability of logical equivalents*, $\beta \in R$. Hence $A - \beta \vdash \beta$. Thus, by $-$ *success*, $\vdash \beta$. Therefore $\vdash \alpha \wedge \beta$, from which it follows by $-$ *failure* that $A \sim (\alpha \wedge \beta) = A$.

(i) We start by noticing that, according to Observation 5.1.2, if $-$ satisfies *uniformity*, then it also satisfies *extensionality*. Now assume that $-$ satisfies *vacuity* and *extensionality* and let R be a set that satisfies *non-retractability propagation* and *conjunctive completeness*. We will show that $\sim$ satisfies *conjunctive constancy*. Let $A \sim \alpha = A \sim \beta = A$. We will prove by cases:

Case 1) $\vdash \alpha$. Then $\vdash \beta \leftrightarrow (\alpha \wedge \beta)$. By *non-retractability propagation* $\alpha \wedge \beta \in R$ if and only if $\beta \in R$, from which it follows by the definition of $\sim$ and $-$ *extensionality* that $A \sim (\alpha \wedge \beta) = A \sim \beta = A$.
Case 2) $\vdash \beta$. The proof is symmetrical to the one presented in the previous case.
Case 3) $\nvdash \alpha$ and $\nvdash \beta$.
Case 3.1) $\alpha \in R$. Then $A \sim \alpha = A - \alpha$. Hence by $-$ *success* $A \sim \alpha \nvdash \alpha$. Thus $A \nvdash \alpha$ (since $A \sim \alpha = A$). Therefore, $A \nvdash \alpha \wedge \beta$, from which it follows by $-$ *vacuity* and *inclusion* that $A - (\alpha \wedge \beta) = A$. Thus, by definition of $\sim$, $A \sim (\alpha \wedge \beta) = A$.
Case 3.2) $\beta \in R$. The proof is symmetrical to the one presented in the previous case.
Case 3.3) $\alpha \notin R$ and $\beta \notin R$. Then by *conjunctive completeness* $\alpha \wedge \beta \notin R$. Hence $A \sim (\alpha \wedge \beta) = A$. ■

Proof of Theorem 7.2.2.

(a) That R satisfies *non-retractability of tautology* follows trivially by definition of R.

Assume that $\sim$ satisfies *inclusion*. We will show that R satisfies *non-retractability upper bounding*. Let $\alpha \in \mathcal{L} \setminus R$. Hence $A \sim \alpha \vdash \alpha$. Thus, by $\sim$ *inclusion*, $A \vdash \alpha$.

Assume that $\sim$ satisfies *extensionality*. Consider $\alpha, \beta \in \mathcal{L}$ such that $\vdash \alpha \leftrightarrow \beta$. Then by $\sim$ *extensionality* $A \sim \alpha = A \sim \beta$. Let $\alpha \notin R$. Hence $A \sim \alpha \vdash \alpha$. Thus $A \sim \beta \vdash \alpha$, from which it follows that $A \sim \beta \vdash \beta$. Hence $\beta \notin R$. By symmetry of the case, if $\beta \notin R$, then $\alpha \notin R$. Hence $\alpha \in R$ if and only if $\beta \in R$. Thus R satisfies *retractability of logical equivalents*.

Assume that $\sim$ satisfies *inclusion* and *uniformity*. We will prove that R satisfies *uniform retractability*. Assume that for all subsets A' of A, $A' \vdash \alpha$ if and only if $A' \vdash \beta$. Let $\alpha \notin R$. Hence $A \sim \alpha \vdash \alpha$, from which it follows (by hypothesis and $\sim$ *inclusion*) that $A \sim \alpha \vdash \beta$. Thus by $\sim$ *uniformity* $A \sim \beta \vdash \beta$. Therefore $\beta \notin R$. It follows by symmetry of the case that if $\beta \notin R$, then $\alpha \notin R$. Therefore $\alpha \in R$ if and only if $\beta \in R$. Hence R satisfies *uniform retractability*.

Assume that $\sim$ satisfies *success propagation*. We will show that R satisfies *non-retractability propagation*. Consider α such that $\alpha \notin R$. It follows that $A \sim \alpha \vdash \alpha$. Consider β such that $\beta \in Cn(\alpha)$. Hence by deduction $\vdash \alpha \to \beta$. Thus, by $\sim$ *success propagation*, $A \sim \beta \vdash \beta$. Therefore $\beta \notin R$. Hence $Cn(\alpha) \cap R = \varnothing$.

Assume that $\sim$ satisfies *relative success* and *conjunctive constancy*. We will show that R satisfies *conjunctive completeness*. Let $\alpha \notin R$ and $\beta \notin R$. It follows by definition of R that $A \sim \alpha \vdash \alpha$ and $A \sim \beta \vdash \beta$. Hence, by $\sim$ *relative success*, $A \sim \alpha = A \sim \beta = A$. Thus, by $\sim$ *conjunctive constancy* $A \sim (\alpha \wedge \beta) = A$, from which it follows that $A \sim (\alpha \wedge \beta) \vdash \alpha \wedge \beta$. Therefore $\alpha \wedge \beta \notin R$.

Assume that $\sim$ that satisfies *persistence*. We will show that R satisfies *non-retractability preservation, conjunctive completeness, non-retractability propagation* and *retractability of logical equivalents*.
Non-retractability preservation: Let α be an arbitrary sentence and $\beta \in$

$\mathcal{L} \setminus R$. Hence $A \sim \beta \vdash \beta$. By *persistence* $A \sim \alpha \vdash \beta$. Thus $\mathcal{L} \setminus R \subseteq Cn(A \sim \alpha)$.
Conjunctive completeness: Let $\alpha \notin R$ and $\beta \notin R$. Hence, by definition of R, $A \sim \alpha \vdash \alpha$ and $A \sim \beta \vdash \beta$. Thus, by $\sim$ *persistence*, $A \sim (\alpha \wedge \beta) \vdash \alpha$ and $A \sim (\alpha \wedge \beta) \vdash \beta$. Therefore $A \sim (\alpha \wedge \beta) \vdash \alpha \wedge \beta$. Thus $\alpha \wedge \beta \notin R$.
Non-retractability propagation: Let $\alpha \notin R$ and $\beta \in Cn(\alpha)$. Hence, by definition of R, $A \sim \alpha \vdash \alpha$. By *persistence* it follows that $A \sim \beta \vdash \alpha$. Therefore $A \sim \beta \vdash \beta$. Hence, by definition of R, $\beta \notin R$.
Retractability of logical equivalents: Follows trivially by Observation 7.1.4, since, as shown above, R satisfies *non-retractability propagation*.

Assume that $\sim$ satisfies *inclusion* and *persistence*. We will show that R satisfies *uniform retractability*. Assume that for all subsets A' of A, $A' \vdash \alpha$ if and only if $A' \vdash \beta$. Let $\alpha \notin R$. Hence, by definition of R, $A \sim \alpha \vdash \alpha$. By $\sim$ *persistence* it follows that $A \sim \beta \vdash \alpha$ and by $\sim$ *inclusion* $A \sim \beta \subseteq A$. Therefore by hypothesis $A \sim \beta \vdash \beta$. Hence $\beta \notin R$, by definition of R. It follows by symmetry of the case that if $\beta \notin R$, then $\alpha \notin R$. Hence $\alpha \in R$ if and only if $\beta \in R$.

(b) Follows trivially from (a) and Observation 7.1.6. ∎

Proof of Theorem 7.2.3.
Let A be a belief base, $\sim$ be an operator on A and $R = \{\alpha : A \sim \alpha \nvdash \alpha\}$. Let $-$ be the operator on A defined by:

$$A - \alpha = \begin{cases} A \sim \alpha & \text{if } \alpha \in R \\ A \setminus \bigcup(A \perp\!\!\!\perp \alpha) & \text{otherwise} \end{cases}.$$

In what follows we show that this operator satisfies statements (a), (b) and (c).

(a) Assume that $\sim$ satisfies *relative success* and *inclusion*. We start by showing that $-$ satisfies *success* and *inclusion*. It follows from $\sim$ *inclusion* and definition of $-$ that $-$ satisfies *inclusion*. We will now show that $-$ satisfies *success*. Let α be such that $\nvdash \alpha$. If $\alpha \in R$, then $A - \alpha = A \sim \alpha$ and by definition of R, $A \sim \alpha \nvdash \alpha$. Thus $A - \alpha \nvdash \alpha$. If $\alpha \notin R$, then $A - \alpha = A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$. Therefore, by Lemma E.2, $A - \alpha \nvdash \alpha$. Hence, according to Definition 5.1.1, $-$ is a contraction operator.

Finally we show that the following equality holds:

$$A \sim \alpha = \begin{cases} A - \alpha & \text{if } \alpha \in R \\ A & \text{otherwise} \end{cases}$$

If $\alpha \in R$, then by definition of $-$, $A \sim \alpha = A - \alpha$. Assume now that $\alpha \notin R$. Hence $A \sim \alpha \vdash \alpha$, from which it follows, by $\sim$ *relative success*, that $A \sim \alpha = A$.

(b) Assume that $\sim$ satisfies *vacuity*. We will prove that $-$ satisfies *vacuity*. Consider that $A \nvdash \alpha$. If $\alpha \in R$, then $A - \alpha = A \sim \alpha$. By $\sim$ *vacuity* it follows that $A \subseteq A - \alpha$. If $\alpha \notin R$, then $A - \alpha = A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$. Thus, by Lemma E.2, $A \subseteq A - \alpha$.

Assume that $\sim$ satisfies *extensionality*. We will prove that $-$ satisfies *extensionality*. Let α and β be such that $\vdash \alpha \leftrightarrow \beta$. Let $\alpha \in R$. Hence $A \sim \alpha \nvdash \alpha$.

Thus, by $\sim$ *extensionality*, $A \sim \beta \nvdash \alpha$. From which it follows that $A \sim \beta \nvdash \beta$. Therefore $\beta \in R$. Hence $A - \alpha = A \sim \alpha = A \sim \beta = A - \beta$. Let $\alpha \notin R$. Hence $\beta \notin R$. From $\vdash \alpha \leftrightarrow \beta$ it follows, by Lemma E.1, that $A \perp\!\!\!\perp \alpha = A \perp\!\!\!\perp \beta$. Hence $A - \alpha = A - \beta$.

Assume that $\sim$ satisfies *failure*. We will prove that $-$ satisfies *failure*. Let $\vdash \alpha$. If $\alpha \in R$, then $A - \alpha = A \sim \alpha$. By $\sim$ *failure* it follows that $A - \alpha = A$. If $\alpha \notin R$, then $A - \alpha = A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$. Thus, by Lemma E.2, $A - \alpha = A$.

Assume that $\sim$ satisfies *relative closure*. We will prove that $-$ satisfies *relative closure*. If $\alpha \in R$, then $A - \alpha = A \sim \alpha$. The rest of the proof for this case follows by $\sim$ *relative closure*. Consider now that $\alpha \notin R$. Hence $A - \alpha = A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$. The rest of the proof follows trivially by Lemma E.2.

Assume that $\sim$ satisfies *inclusion* and *uniformity*. We will prove that $-$ satisfies *uniformity*. Assume that it holds for all subsets A' of A that $A' \vdash \alpha$ if and only if $A' \vdash \beta$. By $\sim$ *uniformity* it follows that $A \sim \alpha = A \sim \beta$. Consider the case that $\alpha \in R$. Hence $A \sim \alpha \nvdash \alpha$. Thus $A \sim \beta \nvdash \alpha$. By $\sim$ *inclusion* $A \sim \beta \subseteq A$. Therefore, by hypothesis, $A \sim \beta \nvdash \beta$. Hence, $\beta \in R$. Thus $A - \alpha = A - \beta$. Consider now that $\alpha \notin R$. Hence $\beta \notin R$. By Lemma E.1, it follows that $A \perp\!\!\!\perp \alpha = A \perp\!\!\!\perp \beta$. Hence $A - \alpha = A - \beta$.

Assume that $\sim$ satisfies *relevance*. We will prove that $-$ satisfies *relevance*. If $\alpha \in R$, then $A - \alpha = A \sim \alpha$ and the rest of the proof for this case follows by $\sim$ *relevance*. Let $\alpha \notin R$. Hence $A - \alpha = A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$. The rest of the proof follows trivially by Lemma E.2.

Assume that $\sim$ satisfies *core-retainment*. We will prove that $-$ satisfies *core-retainment*. If $\alpha \in R$, then $A - \alpha = A \sim \alpha$. The rest of the proof for this case follows by $\sim$ *core-retainment*. Let $\alpha \notin R$. Hence $A - \alpha = A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$. The rest of the proof follows trivially by Lemma E.2.

Assume that $\sim$ satisfies *disjunctive elimination*. We will prove that $-$ satisfies *disjunctive elimination*. If $\alpha \in R$, then $A - \alpha = A \sim \alpha$. Let $\beta \in A$ and $\beta \notin A - \alpha$. Hence $\beta \notin A \sim \alpha$. Thus, by $\sim$ *disjunctive elimination*, $A \sim \alpha \nvdash \alpha \vee \beta$. Therefore $A - \alpha \nvdash \alpha \vee \beta$. Let $\alpha \notin R$. Hence $A - \alpha = A \setminus \bigcup(A \perp\!\!\!\perp \alpha)$. The rest of the proof follows trivially by Lemma E.2.

(c) Assume that $\sim$ satisfies *persistence*. Let $\alpha \notin R$ and $\beta \in R$. From $\alpha \notin R$ it follows that $A \sim \alpha \vdash \alpha$. Hence, by $\sim$ *persistence* it follows that $A \sim \beta \vdash \alpha$. From $\beta \in R$ it follows that $A - \beta = A \sim \beta$. Hence $A - \beta \vdash \alpha$. ■

Proof of Theorem 7.3.1.

Let A be a belief base. We will prove this theorem by showing that condition (1) is equivalent to condition (2) and to condition (3).

$(1 \to 2)$ Let $\sim$ be an operator on A that satisfies *relative success* and *inclusion*. Let R be the set defined by:

$$R = \{\alpha : A \sim \alpha \nvdash \alpha\}$$

According to Theorem 7.2.2 (a) R satisfies *non-retractability of tautology*. On the other hand, it follows from Theorem 7.2.3 (a) that there exists an operator $-$ such

that $\sim$ is the shielded base contraction induced by $-$ and R.
($2 \rightarrow 1$) Let $\sim$ be the operator of shielded base contraction induced by a contraction operator $-$ and a set $R \subseteq \mathcal{L}$ that satisfies *non-retractability of tautology.* Hence by Theorem 7.2.1 (a) and (d), $\sim$ satisfies *relative success* and *inclusion.*
($1 \rightarrow 3$) Let $\sim$ be an operator on A that satisfies *relative success* and *inclusion.* By Observation 5.3.2 (a) it follows that $\sim$ also satisfies *failure.* Hence, according to Theorem 7.2.3 ((a) and (b)), $\sim$ is a shielded base contraction induced by an operator $-$ that satisfies *failure* and R.
($3 \rightarrow 1$) Let $\sim$ be an operator of shielded base contraction induced by a contraction operator $-$ that satisfies failure and a set $R \subseteq \mathcal{L}$. By Theorem 7.2.1 (a), $\sim$ satisfies *relative success* and *inclusion.* ■

Proof of Theorems 7.3.4, 7.3.9 and 7.3.13.
(Right-to-left)
Let A be a belief base and $\sim$ an operator that satisfies *relative success, inclusion* and *uniformity.* Let R be the set defined by:

$$R = \{\alpha : A \sim \alpha \nvdash \alpha\}$$

It follows from Theorem 7.2.2 that R satisfies *uniform retractability.*
Furthermore, from Theorem 7.2.2, it follows that:
- If $\sim$ satisfies *success propagation*, then R satisfies *non-retractability propagation*;
- If $\sim$ satisfies *conjunctive constancy*, then R satisfies *conjunctive completeness*;
From Theorem 7.2.3 (a) it follows that there exists a contraction operator $-$ such that $\sim$ is the shielded base contraction induced by $-$ and R. Furthermore, from Theorem 7.2.3 (c), if $\sim$ satisfies *persistence*, then $-$ and R satisfy condition (**R - $-$**).
For Theorem 7.3.4
From Theorem 7.2.3 and Observation 5.2.1 it follows that $-$ is a partial meet contraction.
For Theorem 7.3.9
From Theorem 7.2.3 and Observation 5.2.4 it follows that $-$ is a kernel contraction.
For Theorem 7.3.13
From Theorem 7.2.3 and Observation 5.2.5 it follows that $-$ is a smooth kernel contraction.
(Left-to-right)
Let A be a belief base, $-$ be an operator on A and $R \subseteq \mathcal{L}$. Let $\sim$ be such that:

$$A \sim \alpha = \begin{cases} A - \alpha & \text{if } \alpha \in R \\ A & \text{otherwise} \end{cases}$$

From Theorem 7.2.1 it follows that if $-$ satisfies *success, inclusion, failure, vacuity* and *uniformity*, then:
- If R satisfies uniform retractability, then $\sim$ satisfies uniformity (Theorem 7.2.1 (f)).
- If R satisfies non-retractability propagation, then $\sim$ satisfies success propagation (Theorem 7.2.1 (g)).
- If R satisfies *conjunctive completeness* and *uniform retractability*, then $\sim$ satisfies *conjunctive constancy* (Theorem 7.2.1 (h)).

- If R and $-$ satisfy condition (**R - $-$**), then $\sim$ satisfies *persistence* and *uniformity* (Theorem 7.2.1 (b)).

For Theorem 7.3.4

If $-$ is a partial meet contraction operator on A, then from Observation 5.2.1 $-$ satisfies *success, inclusion, uniformity* and *relevance.* From Observation 5.1.2 it follows that $-$ satisfies *failure* and *vacuity.* Hence from Theorem 7.2.1 (a) it holds also that $\sim$ satisfies *relative success, inclusion* and *relevance.*

For Theorem 7.3.9

If $-$ is a kernel contraction operator on A, then from Observation 5.2.4 $-$ satisfies *success, inclusion, uniformity* and *core-retainment.* From Observation 5.1.2 it follows that $-$ satisfies *failure* and *vacuity.* Hence from Theorem 7.2.1 (a) it holds also that $\sim$ satisfies *relative success, inclusion* and *core-retainment.*

For Theorem 7.3.13

If $-$ is a smooth kernel contraction operator on A, then from Observation 5.2.5 $-$ satisfies *success, inclusion, uniformity, core-retainment* and *relative closure.* From Observation 5.1.2 it follows that $-$ satisfies *failure* and *vacuity.* Hence from Theorem 7.2.1 (a) it holds also that $\sim$ satisfies *relative success, inclusion, core-retainment* and *relative closure.* ■

Proof of Observation 7.3.8.

It follows from Example 7.3.7 that:

(i) SPMC $\nsubseteq$ CC-SPMC, SP-SPMC $\nsubseteq$ CC-SPMC and SP-SPMC $\nsubseteq$ SP+CC-SPMC;

(ii) SPMC $\nsubseteq$ SP-SPMC, CC-SPMC $\nsubseteq$ SP-SPMC and CC-SPMC $\nsubseteq$ SP+CC-SPMC;

(iii) SP+CC-SPMC $\nsubseteq$ P-SPMC.

On the other hand it follows from Corollary 7.3.6 that SP+CC-SPMC $\subseteq$ CC-SPMC, SP+CC-SPMC $\subseteq$ SP-SPMC, CC-SPMC $\subseteq$ SPMC and SP-SPMC $\subseteq$ SPMC. Furthermore, combining Corollary 7.3.6 and Observations 5.1.2, 5.3.1 and 5.3.2 (b) we can conclude that P-SPMC $\subseteq$ SP+CC-SPMC. Therefore SP+CC-SPMC $\subset$ CC-SPMC, SP+CC-SPMC $\subset$ SP-SPMC, CC-SPMC $\subset$ SPMC, SP-SPMC $\subset$ SPMC and P-SPMC $\subset$ SP+CC-SPMC. ■

Proof of Observation 7.3.12.

That P-SKC $\subseteq$ SP+CC-SKC follows trivially from Corollary 7.3.11 and Observations 5.1.2, 5.3.1 and 5.3.2 (b).

That SP+CC-SKC $\subseteq$ CC-SKC, SP+CC-SKC $\subseteq$ SP-SKC, CC-SKC $\subseteq$ SKC and SP-SKC $\subseteq$ SKC follow trivially from Corollary 7.3.11.

To prove that SP+CC-SKC $\nsubseteq$ P-SKC, CC-SKC $\nsubseteq$ SP+CC-SKC, SP-SKC $\nsubseteq$ SP+CC-SKC, SP-SKC $\nsubseteq$ CC-SKC, CC-SKC $\nsubseteq$ SP-SKC, SKC $\nsubseteq$ CC-SKC and SKC $\nsubseteq$ SP-SKC it is enough to consider the shielded contractions presented in Example 7.3.7, attending to Definition 7.3.10, Corollary 7.3.11 and to the fact that every partial meet contraction is a kernel contraction (Observation 6.4.1). ■

Proof of Observation 7.3.16.
That P-SSKC ⊆ SP+CC-SSKC follows trivially from Corollary 7.3.15 and Observations 5.1.2, 5.3.1 and 5.3.2 (b).
That SP+CC-SSKC ⊆ CC-SSKC, SP+CC-SSKC ⊆ SP-SSKC, CC-SSKC ⊆ SSKC and SP-SSKC ⊆ SSKC follow trivially from Corollary 7.3.15.
To prove that SP+CC-SSKC ⊈ P-SSKC, CC-SSKC ⊈ SP+CC-SSKC, SP-SSKC ⊈ SP+CC-SSKC, SP-SSKC ⊈ CC-SSKC, CC-SSKC ⊈ SP-SSKC, SSKC ⊈ CC-SSKC and SSKC ⊈ SP-SSKC it is enough to consider the shielded contractions presented in Example 7.3.7, attending to Definition 7.3.14, Corollary 7.3.15 and to the fact that every partial meet contraction is a smooth kernel contraction (Observation 6.4.1). ■

Proof of Theorem 7.3.17.
(Right-to-left)
Let A be a belief base and $\sim$ an operator that satisfies *relative success, inclusion, vacuity, extensionality* and *disjunctive elimination*. Let R be the set defined by:

$$R = \{\alpha : A \sim \alpha \nvdash \alpha\}$$

It follows from Theorem 7.2.2 that R satisfies *retractability of logical equivalents*. Furthermore, from Theorem 7.2.2 it follows that:
- If $\sim$ satisfies *success propagation*, then R satisfies *non-retractability propagation*;
- If $\sim$ satisfies *conjunctive constancy*, then R satisfies *conjunctive completeness*;
From Theorem 7.2.3 and Observation 5.2.11 it follows that there exists a basic AGM-generated base contraction $-$ such that $\sim$ is the shielded base contraction induced by $-$ and R. Furthermore, from Theorem 7.2.3 (c), if $\sim$ satisfies *persistence*, then $-$ and R satisfy condition condition (**R - $-$**).
(Left-to-right)
Let A be a belief base, $-$ be a basic AGM-generated base contraction operator on A and $R \subseteq \mathcal{L}$. Let $\sim$ be such that:

$$A \sim \alpha = \begin{cases} A - \alpha & \text{if } \alpha \in R \\ A & \text{otherwise} \end{cases}$$

From Observation 5.2.11 $-$ satisfies *success, inclusion, vacuity, extensionality* and *disjunctive elimination*. From Observation 5.1.2 (d) it holds that $-$ also satisfies *failure*. Hence, from Theorem 7.2.1 (a) it follows that $\sim$ satisfies *relative success, inclusion, vacuity* and *disjunctive elimination*. Furthermore, it follows from Theorem 7.2.1 that:
- If R satisfies *retractability of logical equivalents*, then $\sim$ satisfies *extensionality* (Theorem 7.2.1 (d)).
- If R satisfies *non-retractability propagation*, then $\sim$ satisfies *success propagation* and *extensionality* (Theorem 7.2.1 (g)).
- If R satisfies *conjunctive completeness* and *retractability of logical equivalents*, then $\sim$ satisfies *conjunctive constancy* (Theorem 7.2.1 (h)).
- If R satisfies *non-retractability propagation* and *conjunctive completeness*, then $\sim$ satisfies *conjunctive constancy* (Theorem 7.2.1 (i)).
- If R and $-$ satisfy condition (**R - $-$**), then $\sim$ satisfies *persistence* and *extensionality*

(Theorem 7.2.1 (b)). ■

Proof of Observation 7.3.20.
That P-SbAGMC ⊆ SP+CC-SbAGMC follows trivially from Corollary 7.3.19 and Observations 5.3.1 and 5.3.2 (b).
That SP+CC-SbAGMC ⊆ CC-SbAGMC, SP+CC-SbAGMC ⊆ SP-SbAGMC, CC-SbAGMC ⊆ SbAGMC and SP-SbAGMC ⊆ SbAGMC follow trivially from Corollary 7.3.19.
To prove that SP+CC-SbAGMC ⊈ P-SbAGMC, CC-SbAGMC ⊈ SP+CC-SbAGMC, SP-SbAGMC ⊈ SP+CC-SbAGMC, SP-SbAGMC ⊈ CC-SbAGMC, CC-SbAGMC ⊈ SP-SbAGMC, SbAGMC ⊈ CC-SbAGMC and SbAGMC ⊈ SP-SbAGMC it is enough to consider the shielded contractions presented in Example 7.3.7, attending to Definition 7.3.18, Corollary 7.3.19, Observation 7.1.4 and to the fact that every partial meet contraction is a basic AGM-generated base contraction (Observation 6.4.1). ■

Proof of Observation 7.4.2.
According to Example 7.4.1 (a) it holds that P-SSKC ⊈ SPMC. Hence, from Observations 7.3.8 and 7.3.16, it follows that SSKC ⊈ SPMC, SP-SSKC ⊈ SP-SPMC, CC-SSKC ⊈ CC-SPMC, SP+CC-SSKC ⊈ SP+CC-SPMC and P-SSKC ⊈ P-SPMC. On the other hand, by Corollaries 7.3.6 and 7.3.15 and Observation 5.1.2 (a), it follows that SPMC ⊆ SSKC, SP-SPMC ⊆ SP-SSKC, CC-SPMC ⊆ CC-SSKC, SP+CC-SPMC ⊆ SP+CC-SSKC and P-SPMC ⊆ P-SSKC. Therefore SPMC ⊂ SSKC, SP-SPMC ⊂ SP-SSKC, CC-SPMC ⊂ CC-SSKC, SP+CC-SPMC ⊂ SP+CC-SSKC and P-SPMC ⊂ P-SSKC.

According to Example 7.4.1 (b) it holds that P-SKC ⊈ SSKC. Hence, from Observations 7.3.12 and 7.3.16 it follows that SKC ⊈ SSKC, SP-SKC ⊈ SP-SSKC, CC-SKC ⊈ CC-SSKC, SP+CC-SKC ⊈ SP+CC-SSKC and P-SKC ⊈ P-SSKC. On the other hand, by Corollaries 7.3.11 and 7.3.15, it follows that SSKC ⊆ SKC, SP-SSKC ⊆ SP-SKC, CC-SSKC ⊆ CC-SKC, SP+CC-SSKC ⊆ SP+CC-SKC and P-SSKC ⊆ P-SKC. Therefore SSKC ⊂ SKC, SP-SSKC ⊂ SP-SKC, CC-SSKC ⊂ CC-SKC, SP+CC-SSKC ⊂ SP+CC-SKC and P-SSKC ⊂ P-SKC. ■

Proof of Observation 7.4.4.
According to Example 7.4.3 it holds that P-SbAGMC ⊈ SPMC. Hence it follows, from Observations 7.3.8 and 7.3.20, that SbAGMC ⊈ SPMC, SP-SbAGMC ⊈ SP-SPMC, CC-SbAGMC ⊈ CC-SPMC, SP+CC-SbAGMC ⊈ SP+CC-SPMC and P-SbAGMC ⊈ P-SPMC. On the other hand, according to Corollaries 7.3.6 and 7.3.19 and Observation 5.1.2, it holds that SPMC ⊆ SbAGMC, SP-SPMC ⊆ SP-SbAGMC, CC-SPMC ⊆ CC-SbAGMC, SP+CC-SPMC ⊆ SP+CC-SbAGMC and P-SPMC ⊆ P-SbAGMC. Therefore SPMC ⊂ SbAGMC, SP-SPMC ⊂ SP-SbAGMC, CC-SPMC ⊂ CC-SbAGMC, SP+CC-SPMC ⊂ SP+CC-SbAGMC and P-SPMC ⊂ P-SbAGMC. ■

Proof of Observation 7.4.6.
According to Examples 7.4.3 and 7.4.5 it holds, respectively, that P-SbAGMC ⊈ SKC and that P-SSKC ⊈ SbAGMC. Therefore it follows from Observations 7.3.12,

7.3.16, 7.3.20 and 7.4.2 that all the statements of this result hold. ∎

Appendix F

Proofs of Chapter 8

Lemma F.1 *Let A be a belief base. Let $*$ be an operator on A such that (for all α) $A * \alpha = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$, then $*$ satisfies relevance, consistency, success, inclusion, vacuity, weak extensionality, uniformity, core-retainment, disjunctive elimination and relative closure.*

Proof.
Let $*$ be an operator on a belief base A defined (for all α) by $A*\alpha = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$. We will start by showing that $*$ satisfies *relevance*. Let $\beta \in A$ and $\beta \notin A * \alpha$. Thus $\beta \in \bigcup(A \perp\!\!\!\perp \neg\alpha)$. Hence there exists $Y \in A \perp\!\!\!\perp \neg\alpha$ such that $\beta \in Y$. Let $X = Y \setminus \{\beta\} \cup (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$. Hence $X \subseteq A \cup \{\alpha\}$, $A*\alpha = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\} \subseteq X$ and $X \cup \{\beta\} \vdash \bot$ (since $X \cup \{\beta\} \vdash \neg\alpha$ and $\alpha \in X$). It remains to prove that $X \nvdash \bot$. Assume by *reductio ad absurdum* that $X \vdash \bot$. Hence, by deduction it follows that $X' = Y \setminus \{\beta\} \cup (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \vdash \neg\alpha$. It follows by compactness that there exists a finite subset $H = \{\gamma_1, ..., \gamma_n\}$ of X' such that $H \vdash \neg\alpha$. Where $\gamma_1, ..., \gamma_k \in Y \setminus \{\beta\}$ and $\gamma_{k+1}, ..., \gamma_n \in A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)$, for some $1 \leq k < n$. Hence $\{\gamma_1, ..., \gamma_k\} \cup \{\gamma_{k+1}, ..., \gamma_n\} \vdash \neg\alpha$ but $\{\gamma_1, ..., \gamma_k\} \nvdash \neg\alpha$. Thus there is some inclusion-minimal subset W of H such that $W \vdash \neg\alpha$ but no proper set of W implies $\neg\alpha$. Hence $W \in A \perp\!\!\!\perp \neg\alpha$. On the other hand, since $\{\gamma_1, ..., \gamma_k\} \nvdash \neg\alpha$, W contains at least one of the $\gamma_i \in \{\gamma_{k+1}, ..., \gamma_n\}$. Contradiction since $\{\gamma_{k+1}, ..., \gamma_n\} \subseteq A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)$. Hence $*$ satisfies *relevance*.
On the other hand, by definition of kernel revision, $*$ is a kernel revision. Hence, by Observations 5.2.7, $*$ satisfies *consistency, success, inclusion, uniformity and core-retainment*. On the other hand, by Observation 8.2.1, $*$ satisfies *vacuity, weak extensionality, disjunctive elimination* and *relative closure*. ■

Lemma F.2 *Let R and C be subsets of $\mathcal{L}$.*

*(a) R is closed under double negation and condition (***C-R***) holds if and only if C is closed under double negation and condition (***R-C***) holds.*

*(b) R satisfies retractability of logical equivalents and condition (***C-R***) holds if and only if C satisfies credibility of logical equivalents and condition (***R-C***) holds.*

Proof. [1]
Let R and C be subsets of $\mathcal{L}$.

[1]In this proof we will use Observation 8.7.1, whose proof is presented after this one. However, this is not an issue because the result that is proven here is not used in the proof of that Observation.

(a) We intend to prove that R is closed under double negation and condition (**C-R**) holds if and only if C is closed under double negation and condition (**R-C**) holds.
(From left to right) $\alpha \in C$ iff $\neg\alpha \in R$ iff $\neg\neg\neg\alpha \in R$ iff $\neg\neg\alpha \in C$. Therefore C is closed under double negation. Thus, according to Observation 8.7.1, condition (**R-C**) holds.
(From right to left) $\alpha \in R$ iff $\neg\alpha \in C$ iff $\neg\neg\neg\alpha \in C$ iff $\neg\neg\alpha \in R$. Therefore R is closed under double negation. Thus, according to Observation 8.7.1, condition (**C-R**) holds.

(b) We intend to prove that R satisfies *retractability of logical equivalents* and condition (**C-R**) holds if and only if C satisfies *credibility of logical equivalents* and condition (**R-C**) holds.
(From left to right) Let $\vdash \alpha \leftrightarrow \beta$. Hence $\vdash \neg\alpha \leftrightarrow \neg\beta$. Let $\alpha \in C$. Then, by condition (**C-R**), $\neg\alpha \in R$. Therefore, by *retractability of logical equivalents*, $\neg\beta \in R$, from which it follows, by condition (**C-R**), that $\beta \in C$. By symmetry of the case it holds that if $\beta \in C$, then $\alpha \in C$. Hence C satisfies *credibility of logical equivalents*. On the other hand, since R satisfies *retractability of logical equivalents*, then R also satisfies *closure under double negation*. Thus from (a) condition (**R-C**) holds.
(From right to left) Let $\vdash \alpha \leftrightarrow \beta$. Hence $\vdash \neg\alpha \leftrightarrow \neg\beta$. Let $\alpha \in R$. Then, by condition (**R-C**), $\neg\alpha \in C$. Therefore, by *credibility of logical equivalents*, $\neg\beta \in C$, from which it follows, by condition (**R-C**), that $\beta \in R$. By symmetry of the case it holds that if $\beta \in R$, then $\alpha \in R$. Hence R satisfies *retractability of logical equivalents*. On the other hand, since C satisfies *retractability of logical equivalents*, then C also satisfies *closure under double negation*. Thus from (a) condition(**C-R**) holds. ■

Proof of Observation 8.2.1.

(a) Let A be a belief base and $\circledast$ be an operator on A that satisfies *relevance* and *relative success*. Let $\beta \in A$ and $\beta \notin A \circledast \alpha$. It follows by *relative success* that $\alpha \in A \circledast \alpha$. Assume by *reductio ad absurdum* that $A \circledast \alpha \vdash \neg\alpha \vee \beta$. Hence $A \circledast \alpha \vdash \beta$. On the other hand, by *relevance*, there exists some A' such that $A \circledast \alpha \subseteq A' \subseteq A \cup \{\alpha\}, A' \nvdash \bot$ but $A' \cup \{\beta\} \vdash \bot$. Contradiction, since every set that contains $A \circledast \alpha$ implies β.

(b) Trivial.

(c) Let A be a belief base and $\circledast$ be an operator on A that satisfies *persistence, relative success* and *vacuity*. Assume that $A \circledast \alpha \nvdash \neg\beta$. Hence $A \cap A \circledast \alpha \nvdash \neg\beta$. From which it follows by *persistence* that $A \cap A \circledast \beta \nvdash \neg\beta$. We will prove by cases that $\beta \in A \circledast \beta$.
Case 1) $A \nvdash \neg\beta$. Hence, by *vacuity*, $A \cup \{\beta\} \subseteq A \circledast \beta$. Thus $\beta \in A \circledast \beta$.
Case 2) $A \vdash \neg\beta$. Hence $A \neq A \circledast \beta$. Therefore it follows, by *relative success*, that $\beta \in A \circledast \beta$.

(d) Let A be a belief base and $\circledast$ be an operator on A that satisfies *relative success* and *relevance*. Let $\beta \in A$ and $\beta \notin A \circledast \alpha$. It follows by *relative success* that $\alpha \in A \circledast \alpha$. On the other hand, by *relevance*, there exists some A' such that $A \circledast \alpha \subseteq A' \subseteq A \cup \{\alpha\}, A' \nvdash \bot$ but $A' \cup \{\beta\} \vdash \bot$. Since $\alpha \in A \circledast \alpha$ it holds that $\alpha \in A'$. Let $B = A' \setminus \{\alpha\}$. Hence $B \subseteq A$. On the other hand, since $A' \nvdash \bot$ it follows that $B \cup \{\alpha\} \nvdash \bot$. Thus $B \nvdash \neg\alpha$. From $A' \cup \{\beta\} \vdash \bot$ it follows that $(B \cup \{\alpha\}) \cup \{\beta\} \vdash \bot$. Hence, by deduction $B \cup \{\beta\} \vdash \neg\alpha$.

(e) Let $A \nvdash \neg\alpha$. Assume by *reductio ad absurdum* that $A \cup \{\alpha\} \nsubseteq A \circledast \alpha$. By *success* it follows that $\alpha \in A \circledast \alpha$. Hence, there exists some β such that $\beta \in A$ and $\beta \notin A \circledast \alpha$. Thus, by *core-retainment*, there exists $A' \subseteq A$ such that $A' \nvdash \neg\alpha$ and $A' \cup \{\beta\} \vdash \neg\alpha$. Contradiction, since $A' \cup \{\beta\} \subseteq A$. Therefore, $A \cup \{\alpha\} \subseteq A \circledast \alpha$.

(f) Let $\beta \in A \cap Cn(A \circledast \alpha \cap A)$ and assume by *reductio ad absurdum* that $\beta \notin A \circledast \alpha$. Then, by *disjunctive elimination*, $A \circledast \alpha \nvdash \neg\alpha \vee \beta$. On the other hand, from $\beta \in Cn(A \circledast \alpha \cap A)$ it follows by monotony that $A \circledast \alpha \vdash \beta$ then $A \circledast \alpha \vdash \neg\alpha \vee \beta$. Contradiction. Therefore, $\beta \in A \circledast \alpha$. Hence $A \cap Cn(A \circledast \alpha \cap A) \subseteq A \circledast \alpha$.

(g) Let $\beta \in A$ and $\beta \notin A \circledast \alpha$. By *success* it follows that $\alpha \in A \circledast \alpha$. Assume by *reductio ad absurdum* that $A \circledast \alpha \vdash \neg\alpha \vee \beta$. Thus $A \circledast \alpha \vdash \beta$. On the other hand, by *relevance*, there is some set A' such that $A \circledast \alpha \subseteq A' \subseteq A \cup \{\alpha\}$ and $A' \nvdash \bot$ but $A' \cup \{\beta\} \vdash \bot$. Contradiction, since $A' \vdash \beta$. Therefore, $A \circledast \alpha \nvdash \neg\alpha \vee \beta$.

(h) Let $A \nvdash \neg\alpha$. Assume by *reductio ad absurdum* that $A \cup \{\alpha\} \nsubseteq A \circledast \alpha$. There are two cases to consider:
Case 1) $\alpha \notin A \circledast \alpha$. Hence, by *relative success*, it follows that $A \circledast \alpha = A$ and by *strong regularity* it holds that $A \circledast \alpha \vdash \neg\alpha$. Hence $A \vdash \neg\alpha$. Contradiction.
Case 2) There exists $\beta \in A$ such that $\beta \notin A \circledast \alpha$. Hence by *core-retainment*, there exists $A' \subseteq A$ such that $A' \nvdash \neg\alpha$ and $A' \cup \{\beta\} \vdash \neg\alpha$. Contradiction since $A' \cup \{\beta\} \subseteq A$ and $A \nvdash \neg\alpha$. ∎

Proof of Observation 8.2.2.

(a) Let $\alpha \notin A \circledast \alpha$ and $\beta \notin A \circledast \beta$. By *relative success* it follows that $A \circledast \alpha = A$ and $A \circledast \beta = A$. If $A \nvdash \neg\alpha$, then by *vacuity* $A \cup \{\alpha\} \subseteq A \circledast \alpha$, from which it follows that $\alpha \in A \circledast \alpha$. Contradiction. Hence $A \vdash \neg\alpha$. By symmetry of the case it follows that $A \vdash \neg\beta$. Thus $A \cap A \circledast \alpha \vdash \neg\alpha$ and $A \cap A \circledast \beta \vdash \neg\beta$. Hence, by *persistence*, $A \cap A \circledast (\alpha \vee \beta) \vdash \neg\alpha$ and $A \cap A \circledast (\alpha \vee \beta) \vdash \neg\beta$. Therefore $A \cap A \circledast (\alpha \vee \beta) \vdash \neg\alpha \wedge \neg\beta$. Thus $A \cap A \circledast (\alpha \vee \beta) \vdash \neg(\alpha \vee \beta)$. Hence by monotony $A \circledast (\alpha \vee \beta) \vdash \neg(\alpha \vee \beta)$. It follows, by *consistency preservation*, that $\alpha \vee \beta \notin A \circledast (\alpha \vee \beta)$.

(b) **Strict improvement** Let $\alpha \in A \circledast \alpha$ and $\vdash \alpha \rightarrow \beta$. Thus $A \circledast \alpha \vdash \beta$. By *consistency preservation* $A \circledast \alpha \nvdash \neg\beta$. Hence by *strong regularity* $\beta \in A \circledast \beta$.
Regularity Let $A \circledast \alpha \vdash \beta$. By *consistency preservation* $A \circledast \alpha \nvdash \neg\beta$. Hence by *strong regularity* $\beta \in A \circledast \beta$. ∎

Proof of Observation 8.3.1.

(a) Let $\vdash \alpha \leftrightarrow \beta$. If $\alpha \in C$, then by *single sentence closure* $\beta \in C$. By symmetry of the case it follows that is $\beta \in C$, then $\alpha \in C$. Thus $\alpha \in C$ if and only if $\beta \in C$.

(b) Let $\vdash \alpha \leftrightarrow \beta$. It follows trivially by *uniform credibility*, that $\alpha \in C$ if and only if $\beta \in C$.

(c) Let $A \nvdash \perp$ and $\neg\alpha \notin C$. Then by *expansive credibility* it follows that $A \vdash \alpha$. Therefore by *credibility lower bounding* $\alpha \in C$.

(d) Let $\alpha \in Cn(\varnothing)$. Then $\neg\alpha \vdash \perp$. From which it follows by *element consistency* that $\neg\alpha \notin C$. Therefore, by *negation completeness*, $\alpha \in C$.

(e) Trivial. ∎

Proof of Theorem 8.3.2.
Let A be a consistent belief base and $\circledast$ be an operator of credibility-limited revision induced by a revision operator $*$ for A and a set $C \subseteq \mathcal{L}$ that satisfies *expansive credibility* and *closure under double negation.* We will prove by double inclusion that $C = \{\alpha : \alpha \in A \circledast \alpha\}$.
Let $\alpha \in C$. Then $A \circledast \alpha = A * \alpha$, from which it follows by $*$ *success* that $\alpha \in A \circledast \alpha$. Therefore $C \subseteq \{\alpha : \alpha \in A \circledast \alpha\}$.
Let $\alpha \in A \circledast \alpha$. If $A \circledast \alpha \neq A$, then by definition of $\circledast$ it follows that $\alpha \in C$. Assume now that $A \circledast \alpha = A$. Thus $A \nvdash \neg\alpha$ (since $A \nvdash \perp$). By *expansive credibility* it follows that $\neg\neg\alpha \in C$. Therefore $\alpha \in C$ (since C satisfies *closure under double negation*). Hence $\{\alpha : \alpha \in A \circledast \alpha\} \subseteq C$. ∎

Proof of Theorem 8.4.1.
Let A be a belief base, $C \subseteq \mathcal{L}$, and $\circledast$ be a credibility-limited base revision induced by a revision operator $*$ and C.
Thus,

$$A \circledast \alpha = \begin{cases} A * \alpha & \text{if } \alpha \in C \\ A & \text{otherwise} \end{cases}$$

where, by Definition 5.1.4, $*$ is an operator on A that satisfies *success, consistency* and *inclusion.*

(a) That $\circledast$ satisfies *inclusion* follows trivially by its definition and $*$ *inclusion.*

That $\circledast$ satisfies *relative success* follows trivially by its definition and $*$ *success.*

Let $*$ be an operator on A that satisfies *relevance.* Let $\beta \in A$ and $\beta \notin A \circledast \alpha$. Hence $A \circledast \alpha = A * \alpha$, and it follows trivially from $*$ *relevance* that $\circledast$ satisfies *relevance.*

Let $*$ be an operator on A that satisfies *core-retainment.* Let $\beta \in A$ and $\beta \notin A \circledast \alpha$. Hence $A \circledast \alpha = A * \alpha$, and it follows trivially from $*$ *core-retainment* that $\circledast$ satisfies *core-retainment*.

Let $*$ be an operator on A that satisfies *disjunctive elimination*. Let $\beta \in A$ and $\beta \notin A \circledast \alpha$. Hence $A \circledast \alpha = A * \alpha$, and it follows trivially from $*$ *disjunctive elimination* that $\circledast$ satisfies *disjunctive elimination*.

Let $*$ be an operator on A that satisfies *relative closure*. If $\alpha \in C$, then $A \circledast \alpha = A * \alpha$. Thus, by $*$ *relative closure* it follows that $A \cap Cn(A \cap A \circledast \alpha) \subseteq A \circledast \alpha$. If $\alpha \notin C$, then $A \circledast \alpha = A$. Thus $A \cap Cn(A \cap A \circledast \alpha) = A = A \circledast \alpha$.

(b) Let C be a set that satisfies *element consistency* and $\alpha \in \mathcal{L}$. We will show that $\circledast$ satisfies *consistency preservation*. Let $A \nvdash \bot$. It follows trivially if $\alpha \notin C$. Assume now that $\alpha \in C$. Then $A \circledast \alpha = A * \alpha$. On the other hand, by *element consistency*, $\alpha \nvdash \bot$. Hence, by $*$ *consistency* $A \circledast \alpha \nvdash \bot$.

(c) Let C be a set that satisfies *uniform credibility* and $*$ a revision operator that satisfies uniformity. Let it be the case that for all subsets $A' \subseteq A$, $A' \cup \{\alpha\} \vdash \bot$ if and only if $A' \cup \{\beta\} \vdash \bot$. By *uniform credibility* $\alpha \in C$ if and only if $\beta \in C$. If $\alpha \in C$, then $\beta \in C$. The rest follows from $*$ *uniformity*.
If $\alpha \notin C$, then $\beta \notin C$. Thus $A \cap A \circledast \alpha = A = A \cap A \circledast \beta$.

(d) We start by noticing that, according to Observation 8.3.1, if C satisfies *single sentence closure*, then it also satisfies *credibility of logical equivalents*. Let C be a set that satisfies credibility of logical equivalents and $*$ a revision operator that satisfies *weak extensionality*. We intend to prove that $\circledast$ satisfies *weak extensionality*. Let $\vdash \alpha \leftrightarrow \beta$. By *credibility of logical equivalents* $\alpha \in C$ if and only if $\beta \in C$. If $\alpha \in C$, then $\beta \in C$. The rest follows from $*$ *weak extensionality*. If $\alpha \notin C$, then $\beta \notin C$, from which it follows that $A \circledast \alpha = A \circledast \beta = A$. Thus $A \cap A \circledast \alpha = A \cap A \circledast \beta$.

(e) We start by noticing that, according to Observation 8.3.1, if C satisfies either *credibility of logical equivalents* or *uniform credibility* or *single sentence closure*, then it also satisfies *closure under double negation*. Let C be a set that satisfies *expansive credibility* and *closure under double negation* and $*$ a revision operator that satisfies *vacuity*. We intend to prove that $\circledast$ satisfies *vacuity*. Consider that $A \nvdash \neg\alpha$. By *expansive credibility* $\neg\neg\alpha \in C$. Thus by *closure under double negation* $\alpha \in C$. Hence $A \circledast \alpha = A * \alpha$. Therefore, it follows by $*$ *vacuity* that $A \cup \{\alpha\} \subseteq A \circledast \alpha$.

(f) Let $A \nvdash \bot$ and let C be a set that satisfies *expansive credibility* and *single sentence closure*. We intend to prove that $\circledast$ satisfies *strict improvement*. Let $\alpha \in A \circledast \alpha$ and $\vdash \alpha \rightarrow \beta$. Suppose that $\alpha \notin C$. By *single sentence closure* $\neg\neg\alpha \notin C$. Thus by *expansive credibility* $A \vdash \neg\alpha$. On the other hand, by definition of $\circledast$, it follows that $A \circledast \alpha = A$. Hence $A \vdash \bot$. Contradiction. Hence $\alpha \in C$. Thus by *single sentence closure* $\beta \in C$. Therefore $A \circledast \beta = A * \beta$, from which it follows by $*$ *success* that $\beta \in A \circledast \beta$.

(g) Let $A \nvdash \bot$. We start by noticing that, according to Observation 8.3.1, if C satisfies either *credibility of logical equivalents* or *uniform credibility* or *single sentence closure*, then it also satisfies *closure under double negation*. Let C be a set that satisfies *expansive credibility*, *closure under double negation* and *disjunctive completeness*. We intend to prove that $\circledast$ satisfies *disjunctive distribution*. Let $\alpha \notin A \circledast \alpha$ and $\beta \notin A \circledast \beta$. Thus, by $*$ *success*, it follows that $\alpha \notin C$ and $\beta \notin C$. By *closure under double negation* it follows that $\neg\neg\alpha \notin C$ and $\neg\neg\beta \notin C$, from which it follows, by *expansive credibility* that $A \vdash \neg(\alpha \vee \beta)$. On the other hand, by *disjunctive completeness* $\alpha\vee\beta \notin C$. Therefore $A\circledast(\alpha\vee\beta) = A$, thus $\alpha \vee \beta \notin A \circledast (\alpha \vee \beta)$ (since $A \nvdash \bot$).

(h) Let C and $*$ satisfy condition (**C - $*$**) and assume C satisfies *element consistency*.

Let $A \cap A \circledast \beta \vdash \neg\beta$ and $\alpha \in \mathcal{L}$. It holds that $A \vdash \neg\beta$. If $\beta \in C$, then by *element consistency* $\beta \nvdash \bot$. On the other hand, $A \circledast \beta = A * \beta$, from which it follows, by $*$ *success*, that $A * \beta \vdash \bot$ (since it holds that $A \circledast \beta \vdash \neg\beta$). This contradicts $*$ *consistency*. Hence $\beta \notin C$. If $\alpha \in C$, then $A \circledast \alpha = A * \alpha$. On the other hand, by condition (**C - $*$**), it follows that $A \cap A * \alpha \vdash \neg\beta$. Therefore $A \cap A \circledast \alpha \vdash \neg\beta$. If $\alpha \notin C$, then $A \circledast \alpha = A$. Hence $A \cap A \circledast \alpha \vdash \neg\beta$.

Assume that $*$ is a revision operator that satisfies *weak extensionality*. We intend to prove that $\circledast$ satisfies *weak extensionality*. Let $\vdash \alpha \leftrightarrow \beta$. Assume that $\alpha \notin C$. If $\beta \in C$, then by *element consistency* $\beta \nvdash \bot$. On the other hand, $A \circledast \beta = A * \beta$, from which it follows, by $*$ *success* and *consistency*, that $A \circledast \beta \nvdash \neg\beta$. From $\alpha \notin C$ it follows, by condition (**C - $*$**), that $A \cap A \circledast \beta \vdash \neg\alpha$. Therefore $A \circledast \beta \vdash \neg\alpha$ from which it follows that $A \circledast \beta \vdash \neg\beta$. Contradiction. Hence $\beta \notin C$. By symmetry of the case it also follows that if $\beta \notin C$, then $\alpha \notin C$. Therefore $\alpha \in C$ if and only if $\alpha \in C$.
Let $\alpha \in C$. Then $\beta \in C$. Hence $A \circledast \alpha = A * \alpha$ and $A \circledast \beta = A * \beta$. From which it follows, by $*$ *weak extensionality*, that $A \cap A \circledast \alpha = A \cap A \circledast \beta$.
Let $\alpha \notin C$. Then $\beta \notin C$. Hence $A \circledast \alpha = A \circledast \beta = A$. Therefore $A \cap A \circledast \alpha = A \cap A \circledast \beta = A$.

Assume that $*$ is a revision operator that satisfies *uniformity*. We intend to prove that $\circledast$ satisfies *uniformity*. Let it be the case that for all subsets $A' \subseteq A$, $A' \cup \{\alpha\} \vdash \bot$ if and only if $A' \cup \{\beta\} \vdash \bot$. If $\alpha, \beta \in C$, then $A \circledast \alpha = A * \alpha$ and $A \circledast \beta = A * \beta$. By $*$ *uniformity* $A \cap A * \alpha = A \cap A * \beta$. Hence $A \cap A \circledast \alpha = A \cap A \circledast \beta$. If $\alpha \notin C$ and $\beta \notin C$, then $A \circledast \alpha = A \circledast \beta = A$. Hence $A \cap A \circledast \alpha = A \cap A \circledast \beta$.
Assume now by *reductio ad absurdum*, without loss of generality, that $\alpha \in C$ and $\beta \notin C$. Hence by condition (**C - $*$**) it follows that $A \cap A * \alpha \vdash \neg\beta$. On the other hand, from $\alpha \in C$, it follows that $A \circledast \alpha = A * \alpha$ and, by *element consistency*, that $\alpha \nvdash \bot$. Hence, by $*$ *consistency* $A \circledast \alpha \nvdash \bot$. By $*$ *success* it follows that $A \circledast \alpha \nvdash \neg\alpha$. Thus $A \cap A \circledast \alpha \nvdash \neg\alpha$. By hypothesis it follows that $A \cap A \circledast \alpha \nvdash \neg\beta$. Contradiction.

(i) Let C and $*$ satisfy condition (**C - $*$**) and assume C satisfies *expansive credibil-*

ity. Let $*$ be a revision operator that satisfies *vacuity*. Consider that $A \nvdash \neg\alpha$. It follows by *expansive credibility* that $\neg\neg\alpha \in C$. Assume by *reductio ad absurdum* that $\alpha \notin C$. Hence by condition (**C - $*$**) it follows that $A \cap A * \neg\neg\alpha \vdash \neg\alpha$. From which it follows that $A \vdash \neg\alpha$. Contradiction. Hence $\alpha \in C$. Therefore $A \circledast \alpha = A * \alpha$. Thus, by $*$ *vacuity* it follows that $A \cup \{\alpha\} \subseteq A \circledast \alpha$. ■

Proof of Theorem 8.4.2.

Let $\circledast$ be an operator on A that satisfies *consistency preservation*. We will show that C satisfies *element consistency*. Let $\alpha \in C$. Then $\alpha \in A \circledast \alpha$. On the other hand from $A \nvdash \bot$ it follows by $\circledast$ *consistency preservation* that $A \circledast \alpha \nvdash \bot$. Hence $\alpha \nvdash \bot$.

Let $\circledast$ be an operator on A that satisfies *strict improvement*. We will show that C satisfies *single sentence closure*. Let $\alpha \in C$ and $\beta \in Cn(\alpha)$. Hence $\alpha \in A \circledast \alpha$ and $\vdash \alpha \to \beta$. Thus, by *strict improvement*, $\beta \in A \circledast \beta$. Therefore $\beta \in C$.

Let $\circledast$ be an operator on A that satisfies *disjunctive distribution*. We will show that C satisfies *disjunctive completeness*. Let $\alpha \vee \beta \in C$. Hence $\alpha \vee \beta \in A \circledast (\alpha \vee \beta)$. Thus, by *disjunctive distribution*, $\alpha \in A \circledast \alpha$ or $\beta \in A \circledast \beta$. From which it follows that $\alpha \in C$ or $\beta \in C$.

Let $\circledast$ be an operator on A that satisfies *vacuity*. We will show that C satisfies *expansive credibility* and *credibility lower bounding*.
Expansive credibility: Let $A \nvdash \alpha$. Then $A \nvdash \neg\neg\alpha$. By $\circledast$ *vacuity* it follows that $\neg\alpha \in A \circledast \neg\alpha$. Therefore $\neg\alpha \in C$.
Credibility lower bounding: Let $A \vdash \alpha$. It follows that $A \nvdash \neg\alpha$, since $A \nvdash \bot$. Thus, by $\circledast$ *vacuity*, $\alpha \in A \circledast \alpha$. Hence $\alpha \in C$.

Let $\circledast$ be an operator on A that satisfies *relative success, uniformity, vacuity* and *consistency preservation*. We will show that C satisfies *uniform credibility*. Assume that it holds for all subsets A' of A that $A' \cup \{\alpha\} \vdash \bot$ if and only if $A' \cup \{\beta\} \vdash \bot$. Hence, by $\circledast$ *uniformity*, $A \cap A \circledast \alpha = A \cap A \circledast \beta$. Let $\alpha \notin C$. Thus $\alpha \notin A \circledast \alpha$. Therefore by $\circledast$ *vacuity* $A \vdash \neg\alpha$ and by $\circledast$ *relative success* $A \circledast \alpha = A$. Thus $A = A \cap A \circledast \beta$. Therefore $A \cap A \circledast \beta \vdash \neg\alpha$. From which it follows, by hypothesis, that $A \cap A \circledast \beta \vdash \neg\beta$. Thus $A \circledast \beta \vdash \neg\beta$. By $\circledast$ *consistency preservation* it follows that $\beta \notin A \circledast \beta$. Hence $\beta \notin C$. By symmetry of the case it follows from $\beta \notin C$ that $\alpha \notin C$. Therefore $\alpha \in C$ if and only if $\beta \in C$.

Let $\circledast$ be an operator on A that satisfies *relative success, vacuity, consistency preservation* and *weak extensionality*. We will show that C satisfies *credibility of logical equivalents*. Let $\vdash \alpha \leftrightarrow \beta$. Suppose that $\alpha \notin C$. Hence $\alpha \notin A \circledast \alpha$. Thus by *relative success* $A \circledast \alpha = A$. Assume by *reductio ad absurdum* that $\beta \in C$. Thus $\beta \in A \circledast \beta$. Thus, by $\circledast$ *consistency preservation*, $A \circledast \beta \nvdash \neg\beta$. By *weak extensionality* it follows that $A \cap A \circledast \alpha = A \cap A \circledast \beta$. Thus $A \cap A \circledast \alpha \nvdash \neg\beta$, from which it follows that $A \cap A \circledast \alpha \nvdash \neg\alpha$. Therefore, $A \nvdash \neg\alpha$. From which it follows by *vacuity* that $\alpha \in A \circledast \alpha$. Contradiction. Hence if $\alpha \notin C$, then $\beta \notin C$. By symmetry of the case it follows that

$\beta \notin C$, then $\alpha \notin C$. Therefore $\alpha \in C$ if and only if $\beta \in C$.

Let $\circledast$ be an operator on A that satisfies *consistency preservation, persistence, relative success* and *vacuity.* By Observation 8.2.1 (c) it follows that $\circledast$ satisfies *strong regularity.* By Observation 8.2.2 it follows that $\circledast$ also satisfies *disjunctive distribution, strict improvement* and *regularity.* Thus as shown above C satisfies *single sentence closure* and *disjunctive completeness.* We will now show that C also satisfies *revision credibility.* Let $\alpha \in C$ and $\beta \in Cn(A \circledast \alpha)$. Hence, by $\circledast$ *regularity*, it follows that $\beta \in A \circledast \beta$. Hence $\beta \in C$, by definition of C. It remains to show that C satisfies *uniform credibility.*
Assume that it holds for all subsets A' of A that $A' \cup \{\alpha\} \vdash \bot$ if and only if $A' \cup \{\beta\} \vdash \bot$. We intend to prove that $\alpha \in C$ holds if and only if $\beta \in C$ holds.
Consider that $\beta \notin C$. Hence $\beta \notin A \circledast \beta$. By $\circledast$ *relative success* and *vacuity* it follows, respectively, that $A \circledast \beta = A$ and $A \vdash \neg\beta$. Hence $A \cap A \circledast \beta \vdash \neg\beta$. By *persistence* it follows that $A \cap A \circledast \alpha \vdash \neg\beta$. From which it follows by hypothesis that $A \cap A \circledast \alpha \vdash \neg\alpha$. Hence $A \circledast \alpha \vdash \neg\alpha$. By *consistency preservation* it follows that $\alpha \notin A \circledast \alpha$. Thus $\alpha \notin C$. By symmetry of the case it follows that if $\alpha \notin C$, then $\beta \notin C$.

Let $\circledast$ be an operator on A that satisfies *persistence, relative success* and *vacuity.* By Observation 8.2.1 (c) it follows that $\circledast$ satisfies *strong regularity.* We will show that C satisfies *strong revision credibility.*
Let $A \circledast \alpha \nvdash \neg\beta$. Then by $\circledast$ *strong regularity*, $\beta \in A \circledast \beta$. Therefore, $\beta \in C$, by definition of C. ■

Proof of Theorem 8.4.3.
Let A be a consistent belief base, $\circledast$ be an operator on A and $C = \{\alpha : \alpha \in A \circledast \alpha\}$. Let $*$ be the operator on A defined by:

$$A * \alpha = \begin{cases} A \circledast \alpha & \text{if } \alpha \in C \\ (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\} & \text{otherwise} \end{cases}.$$

In what follows we show that this operator satisfies statements (a), (b) and (c).

(a) We start by showing that the following equality holds:

$$A \circledast \alpha = \begin{cases} A * \alpha & \text{if } \alpha \in C \\ A & \text{otherwise} \end{cases}$$

If $\alpha \in C$, then by definition of $*$, $A \circledast \alpha = A * \alpha$. Assume now that $\alpha \notin C$. Hence $\alpha \notin A \circledast \alpha$, from which it follows, by $\circledast$ *relative success*, that $A \circledast \alpha = A$.
It remains to show that $*$ is a revision operator. According to Definition 5.1.4 we must to show that $*$ satisfies *success, inclusion and consistency.*
That $*$ satisfies *success* follows trivially by $*$ definition.
That $*$ satisfies *inclusion* follows trivially by $*$ definition and $\circledast$ *inclusion.*
Let $\alpha \nvdash \bot$. If $\alpha \in C$, then $A * \alpha = A \circledast \alpha$. Thus, by $\circledast$ *consistency preservation*, it follows that $A * \alpha \nvdash \bot$. Assume now that $\alpha \notin C$. Thus, by Lemma F.1, it follows that $A * \alpha \nvdash \bot$.

(b) Assume $\circledast$ satisfies *vacuity*. We will prove that $*$ satisfies *vacuity*. If $\alpha \in C$, then $A * \alpha = A \circledast \alpha$ and the rest of the proof for this case follows by $\circledast$ *vacuity*. If $\alpha \notin C$, then $A * \alpha = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$. The rest of the proof follows trivially by Lemma F.1.

Assume $\circledast$ satisfies *relevance*. We will prove that $*$ satisfies *relevance*. If $\alpha \in C$, then $A * \alpha = A \circledast \alpha$ and the rest of the proof for this case follows by $\circledast$ *relevance*. Let $\alpha \notin C$. Hence $A * \alpha = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$. The rest of the proof follows trivially by Lemma F.1.

Assume $\circledast$ satisfies *core-retainment*. We will prove that $*$ satisfies *core-retainment*. If $\alpha \in C$, then $A * \alpha = A \circledast \alpha$ and the rest of the proof for this case follows by $\circledast$ *core-retainment*. Let $\alpha \notin C$. Hence $A * \alpha = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$. The rest of the proof follows trivially by Lemma F.1.

Assume $\circledast$ satisfies *disjunctive elimination*. We will prove that $*$ satisfies *disjunctive elimination*. If $\alpha \in C$, then $A * \alpha = A \circledast \alpha$ and the rest of the proof for this case follows by $\circledast$ *disjunctive elimination*. Let $\alpha \notin C$. Hence $A * \alpha = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$. The rest of the proof follows trivially by Lemma F.1.

Assume $\circledast$ satisfies *uniformity, relative success, vacuity* and *consistency preservation*. We will prove that $*$ satisfies *uniformity*. Assume that it holds for all subsets $A' \subseteq A$, $A' \cup \{\alpha\} \vdash \bot$ if and only if $A' \cup \{\beta\} \vdash \bot$. Let $\beta \notin C$. Hence, by $\circledast$ *relative success* it follows that $A \circledast \beta = A$ and by *vacuity* that $A \vdash \neg\beta$. Hence $A \cap A \circledast \beta \vdash \neg\beta$. Therefore, by hypothesis $A \cap A \circledast \beta \vdash \neg\alpha$. Thus by $\circledast$ *uniformity* $A \cap A \circledast \alpha \vdash \neg\alpha$. From which it follows that $A \circledast \alpha \vdash \neg\alpha$. Hence, by $\circledast$ *consistency preservation*, it follows that $\alpha \notin C$. By symmetry of the case it follows that if $\alpha \notin C$, then $\beta \notin C$. Therefore $\alpha \in C$ if and only if $\beta \in C$.
Let $\alpha \in C$, then $\beta \in C$. Hence $A * \alpha = A \circledast \alpha$ and $A * \beta = A \circledast \beta$. Thus by $\circledast$ *uniformity* $A \cap A * \alpha = A \cap A * \beta$.
Let $\alpha \notin C$, then $\beta \notin C$. Then $A * \alpha = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$ and $A * \beta = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\beta)) \cup \{\beta\}$. From which it follows by Lemma F.1 that $A \cap A * \alpha = A \cap A * \beta$.

Assume $\circledast$ satisfies *weak extensionality, relative success, vacuity* and *consistency preservation*. We will prove that $*$ satisfies *weak extensionality*. Let α and β be such that $\vdash \alpha \leftrightarrow \beta$. Let $\beta \notin C$. Hence, by $\circledast$ *relative success* it follows that $A \circledast \beta = A$ and by *vacuity* that $A \vdash \neg\beta$. Hence $A \cap A \circledast \beta \vdash \neg\beta$. Therefore, by hypothesis $A \cap A \circledast \beta \vdash \neg\alpha$. Thus by $\circledast$ *weak extensionality* $A \cap A \circledast \alpha \vdash \neg\alpha$. From which it follows that $A \circledast \alpha \vdash \neg\alpha$. Hence, by $\circledast$ *consistency preservation*, it follows that $\alpha \notin C$. By symmetry of the case it follows that if $\alpha \notin C$, then $\beta \notin C$. Therefore $\alpha \in C$ if and only if $\beta \in C$.
Let $\alpha \in C$, then $\beta \in C$. Hence $A * \alpha = A \circledast \alpha$ and $A * \beta = A \circledast \beta$. Thus by $\circledast$ *weak extensionality* $A \cap A * \alpha = A \cap A * \beta$.
Let $\alpha \notin C$, then $\beta \notin C$. Then $A * \alpha = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$ and $A * \beta = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\beta)) \cup \{\beta\}$. From which it follows by Lemma F.1 that $A \cap A * \alpha = A \cap A * \beta$.

Assume $\circledast$ satisfies *relative closure*. We will prove that $*$ satisfies *relative closure*. If $\alpha \in C$, then $A * \alpha = A \circledast \alpha$ and the rest of the proof for this case follows by $\circledast$ *relative closure*. Let $\alpha \notin C$. Hence $A * \alpha = (A \setminus \bigcup(A \perp\!\!\!\perp \neg\alpha)) \cup \{\alpha\}$.

The rest of the proof follows trivially by Lemma F.1.

(c) Consider that $\circledast$ satisfies *persistence, relative success* and *vacuity*. We intend to prove that C and $*$ satisfy condition (**C** - $*$).
Let $\alpha \notin C$ and $\beta \in C$. Then, by *relative success*, $A \circledast \alpha = A$. By definition of $*$ it follows that $A * \beta = A \circledast \beta$.
If $A \nvdash \neg\alpha$, then by $\circledast$ *vacuity* it follows that $\alpha \in A \circledast \alpha$. From which it follows that $\alpha \in C$. Contradiction.
Hence $A \vdash \neg\alpha$. Therefore $A \cap A \circledast \alpha \vdash \neg\alpha$. Hence, by $\circledast$ *persistence*, $A \cap A \circledast \beta \vdash \neg\alpha$. Thus $A \cap A * \beta \vdash \neg\alpha$. ■

Proof of Theorem 8.5.1.
Let A be a consistent belief base.
((a) → (b)) Let $\circledast$ be an operator on A that satisfies *relative success, consistency preservation* and *inclusion.* Let C be the set defined by:

$$C = \{\alpha : \alpha \in A \circledast \alpha\}$$

According to Theorem 8.4.2, C satisfies *element consistency.* On the other hand, it follows from Theorem 8.4.3 (a) that there exists an operator $*$ such that $\circledast$ is the credibility-limited base revision induced by $*$ and C.
((b) → (a)) Let $\circledast$ be the operator of credibility-limited base revision induced by a revision operator $*$ and a set $C \subseteq \mathcal{L}$ that satisfies *element consistency.* Hence by Theorem 8.4.1 (a) and (b), $\circledast$ satisfies *relative success, consistency preservation* and *inclusion.* ■

Proofs of Theorems 8.5.3, 8.5.6, 8.5.9.
(Right-to-left)
Let A be a consistent belief base and $\circledast$ be an operator that satisfies *relative success, consistency preservation, inclusion, vacuity* and *uniformity.* Let C be the set defined by:

$$C = \{\alpha : \alpha \in A \circledast \alpha\}$$

It follows from Theorem 8.4.2 that C satisfies *element consistency, expansive credibility* and *uniform credibility.*
Furthermore, from Theorem 8.4.2, it follows that:
- If $\circledast$ satisfies *strict improvement*, then C satisfies *single sentence closure*;
- If $\circledast$ satisfies *disjunctive distribution*, then C satisfies *disjunctive completeness*;

From Theorem 8.4.3 (a) it follows that there exists a revision operator $*$ such that $\circledast$ is the credibility-limited base revision induced by $*$ and C. Furthermore from Theorem 8.4.3 (c) it follows that if $\circledast$ satisfies *persistence*, then C and $*$ satisfy condition (**C** - $*$).
For Theorem 8.5.3
From Observation 8.4.3 and Observation 5.2.3 it follows that $*$ is a partial meet revision.
For Theorem 8.5.6
From Observation 8.4.3 and Observation 5.2.7 it follows that $*$ is a kernel revision.

For Theorem 8.5.9
From Observation 8.4.3 and Observation 5.2.9 it follows that $*$ is a smooth kernel revision.
(Left-to-right)
Let A be a consistent belief base, $*$ be an operator on A and $C \subseteq \mathcal{L}$. Let $\circledast$ be such that:

$$A \circledast \alpha = \begin{cases} A * \alpha & \text{if } \alpha \in C \\ A & \text{otherwise} \end{cases}$$

From Theorem 8.4.1 it follows that if $*$ satisfies *success, consistency, inclusion, vacuity* and *uniformity*, then:
- $\circledast$ satisfies *inclusion* and *relative success.*
- If C satisfies *element consistency*, then $\circledast$ satisfies *consistency preservation.*
- If C satisfies *uniform credibility*, then $\circledast$ satisfies *uniformity.*
- If C satisfies *expansive credibility* and *uniform credibility*, then $\circledast$ satisfies *vacuity.*
- If C satisfies *expansive credibility* and *single sentence closure*, then $\circledast$ satisfies *strict improvement.*
- If C satisfies *expansive credibility, uniform credibility* and *disjunctive completeness*, then $\circledast$ satisfies *disjunctive distribution.*
- If C satisfies *expansive credibility* and C and $*$ satisfy condition (**C - $*$**), then $\circledast$ satisfies *vacuity.*
- If C satisfies *element consistency* and C and $*$ satisfy condition (**C - $*$**), then $\circledast$ satisfies *persistence* and *uniformity.*

For Theorem 8.5.3
If $*$ is a partial meet revision operator on A, then from Observation 5.2.3 $*$ satisfies *success, consistency, inclusion, relevance* and *uniformity.* From Observation 8.2.1 it follows that $*$ satisfies *vacuity.* Hence from Theorem 8.4.1 (a) it holds also that $\circledast$ satisfies *relevance.*

For Theorem 8.5.6
If $*$ is a kernel revision operator on A, then from Observation 5.2.7 $*$ satisfies *success, consistency, inclusion, core-retainment* and *uniformity.* From Observation 8.2.1 it follows that $*$ satisfies *vacuity.* Hence from Theorem 8.4.1 (a) it holds also that $\circledast$ satisfies *core-retainment.*

For Theorem 8.5.9
If $*$ is a smooth kernel revision operator on A, then $*$ is a kernel revision operator and $*$ satisfies *relative closure* (Observation 5.2.9). Hence from Theorem 8.4.1 (a) it holds also that $\circledast$ satisfies *core-retainment* and *relative closure.* ∎

Proof of Theorem 8.5.12.
(Right-to-left)
Let A be a consistent belief base and $\circledast$ be an operator that satisfies *relative success, consistency preservation, inclusion, vacuity, weak extensionality* and *disjunctive elimination.* Let C be the set defined by:

$$C = \{\alpha : \alpha \in A \circledast \alpha\}$$

It follows from Theorem 8.4.2 that C satisfies *element consistency, expansive credibility* and *credibility of logical equivalents.*

Furthermore, from Theorem 8.4.2, it follows that:
- If $\circledast$ satisfies *strict improvement*, then C satisfies *single sentence closure*;
- If $\circledast$ satisfies *disjunctive distribution*, then C satisfies *disjunctive completeness*;
From Observation 8.4.3 and Observation 5.2.13 it follows that there exists a basic AGM-generated base revision $*$ such that $\circledast$ is the credibility-limited base revision induced by $*$ and C. Furthermore from Theorem 8.4.3 (c) it follows that if $\circledast$ satisfies *persistence*, then C and $*$ satisfy condition (**C - $*$**).

(Left-to-right)
Let A be a consistent belief base, $C \subseteq \mathcal{L}$ and $*$ be a basic AGM-generated revision operator on A. Let $\circledast$ be such that:

$$A \circledast \alpha = \begin{cases} A * \alpha & \text{if } \alpha \in C \\ A & \text{otherwise} \end{cases}$$

From Observation 5.2.13 $*$ satisfies *success, consistency, inclusion, vacuity, weak extensionality* and *disjunctive elimination*. Hence from Theorem 8.4.1 (a) it follows that $\circledast$ satisfies *relative success, inclusion* and *disjunctive elimination*. Furthermore, it follows from Theorem 8.4.1 that:
- If C satisfies *element consistency*, then $\circledast$ satisfies *consistency preservation*.
- If C satisfies *credibility of logical equivalents*, then $\circledast$ satisfies *weak extensionality*.
- If C satisfies *expansive credibility* and *credibility of logical equivalents*, then $\circledast$ satisfies *vacuity*.
- If C satisfies *expansive credibility* and *single sentence closure*, then $\circledast$ satisfies *strict improvement*.
- If C satisfies *single sentence closure*, then $\circledast$ satisfies *weak extensionality*.
- If C satisfies *expansive credibility* and *single sentence closure*, then $\circledast$ satisfies *vacuity*.
- If C satisfies *expansive credibility, credibility of logical equivalents* and *disjunctive completeness*, then $\circledast$ satisfies *disjunctive distribution*.
- If C satisfies *expansive credibility, single sentence closure* and *disjunctive completeness*, then $\circledast$ satisfies *disjunctive distribution*.
- If C satisfies *expansive credibility* and C and $*$ satisfy condition (**C - $*$**), then $\circledast$ satisfies *vacuity*.
- If C satisfies *element consistency* and C and $*$ satisfy condition (**C - $*$**), then $\circledast$ satisfies *persistence* and *weak extensionality*. ■

Proof of Observation 8.6.1. [2]
We will prove that P-CLPMR $\subset$ SI+DD-CLPMR. That P-CLPMR $\subseteq$ SI+DD-CLPMR follows by Corollary 8.5.5 and Observations 8.2.1 and 8.2.2. Suppose by *reductio ad absurdum* that SI+DD-CLPMR $\subseteq$ P-CLPMR. Let $A = \{p, q\}$ and $-$ be a partial meet contraction on A such that $A - (p \wedge q) = \{p\}$. Let $R = \mathcal{L} \setminus Cn(q)$. Let $\sim$ be the operator of shielded base contraction induced by $-$ and R. As shown in Example 7.3.7 (c) $\sim$ is a SP+CC-SPMC but not a P-SPMC. On the other hand it follows by

[2] We note that throughout this proof we will use results of Section 8.8. However, this is not an issue because the observation that is proven here is not used in the proofs of those results that are presented further ahead.

Corollary 8.8.6 that the operator $\circledast$ defined from $\sim$ via the consistency-preserving Levi identity is a SI+DD-CLPMR. Thus $\circledast$ is a P-CLPMR. It follows by Corollary 8.8.8, Observation 5.1.2 and Theorem 8.8.9 that $\sim$ is a P-SPMC. Contradiction. The remaining items of this result can be proven reasoning in an analogous way taking into account the Corollaries 8.5.5, 8.5.8, 8.5.11 and 8.5.14 and having in mind that according to Observations 7.4.2 and 7.4.4 the shielded contraction $\sim$ presented in:

- Example 7.3.7 (a) is a SPMC, SKC, SSKC, SbAGMC, SP-SPMC, SP-SKC, SP-SSKC and SP-SbAGMC. However $\sim$ is not a CC-SPMC, CC-SKC, CC-SSKC, CC-SbAGMC, SP+CC-SPMC, SP+CC-SKC, SP+CC-SSKC and SP+CC-SbAGMC (since, as shown in that example $\sim$ does not satisfy *conjunctive constancy*);

- Example 7.3.7 (b) is a SPMC, SKC, SSKC, SbAGMC, CC-SPMC, CC-SKC, CC-SSKC and CC-SbAGMC. However $\sim$ is not a SP-SPMC, SP-SKC, SP-SSKC, SP-SbAGMC, SP+CC-SPMC, SP+CC-SKC, SP+CC-SSKC and SP+CC-SbAGMC (since, as shown in that example $\sim$ does not satisfy *success propagation*);

- Example 7.3.7 (c) is a SP+CC-SKC, SP+CC-SSKC and SP+CC-SbAGMC. However $\sim$ is not a P-SKC, P-SSKC and a P-SbAGMC (since, as shown in that example $\sim$ does not satisfy *persistence*). ■

Proof of Observation 8.6.2.
That CLPMR $\subseteq$ CLSKR, SI-CLPMR $\subseteq$ SI-CLSKR, DD-CLPMR $\subseteq$ DD-CLSKR, SI+DD-CLPMR $\subseteq$ SI+DD-CLSKR and P-CLPMR $\subseteq$ P-CLSKR follow trivially by Corollaries 8.5.5 and 8.5.11 and Observation 8.2.1 (d).
To prove that CLSKR $\nsubseteq$ CLPMR, SI-CLSKR $\nsubseteq$ SI-CLPMR, DD-CLSKR $\nsubseteq$ DD-CLPMR, SI+DD-CLSKR $\nsubseteq$ SI+DD-CLPMR and P-CLSKR $\nsubseteq$ P-CLPMR, having in mind Observation 8.6.1, it is enough to show that P-CLSKR $\nsubseteq$ CLPMR.
To do so it is enough to reason as in the proof of Observation 8.6.1, considering the shielded contraction presented in Example 7.4.1 (a) that is a P-SSKC but not a SPMC.
That CLSKR $\subseteq$ CLKR, SI-CLSKR $\subseteq$ SI-CLKR, DD-CLSKR $\subseteq$ DD-CLKR, SI+DD-CLSKR $\subseteq$ SI+DD-CLKR and P-CLSKR $\subseteq$ P-CLKR follow trivially by Corollaries 8.5.8 and 8.5.11.
To prove that CLKR $\nsubseteq$ CLSKR, SI-CLKR $\nsubseteq$ SI-CLSKR, DD-CLKR $\nsubseteq$ DD-CLSKR, SI+DD-CLKR $\nsubseteq$ SI+DD-CLSKR and P-CLKR $\nsubseteq$ P-CLSKR, having in mind Observation 8.6.1, it is enough to show that P-CLKR $\nsubseteq$ CLSKR.
To do so it is enough to reason as in the proof of Observation 8.6.1, considering the shielded contraction presented in Example 7.4.1 (b) that is a P-SKC but not a SSKC. ■

Proof of Observation 8.6.3.
That CLPMR $\subseteq$ CLbAGMR, SI-CLPMR $\subseteq$ SI-CLbAGMR, DD-CLPMR $\subseteq$ DD-CLbAGMR, SI+DD-CLPMR $\subseteq$ SI+DD-CLbAGMR and P-CLPMR $\subseteq$ P-CLbAGMR

follow trivially by Corollaries 8.5.5 and 8.5.14 and Observation 8.2.1.
To prove that CLbAGMR $\nsubseteq$ CLPMR, SI-CLbAGMR $\nsubseteq$ SI-CLPMR, DD-CLbAGMR $\nsubseteq$ DD-CLPMR, SI+DD-CLbAGMR $\nsubseteq$ SI+DD-CLPMR and P-CLbAGMR $\nsubseteq$ P-CLPMR, having in mind Observation 8.6.1, it is enough to show that P-CLbAGMR $\nsubseteq$ CLPMR.
To do so it is enough to reason as in the proof of Observation 8.6.1, considering the shielded contraction presented in Example 7.4.3 that is a P-SbAGMC but not a SPMC. ■

Proof of Observation 8.6.4.
Having in mind Observations 8.6.1 and 8.6.2, to prove all the statements of this result it is enough to show that P-CLbAGMR $\nsubseteq$ CLKR and that P-CLSKR $\nsubseteq$ CLbAGMR. To do that it is enough to reason as in the proof of Observation 8.6.1, considering the shielded contractions presented in Examples 7.4.3 and 7.4.5 (that show that P-SbAGMC $\nsubseteq$ SKC and that P-SSKC $\nsubseteq$ SbAGMC). ■

Proof of Observation 8.7.1.
Let R and C be subsets of $\mathcal{L}$ that are closed under double negation. We intend to prove that condition (**C-R**) holds if and only if condition (**R-C**) also holds.
(From left to right) $\alpha \in R$ iff $\neg\neg\alpha \in R$ iff $\neg\alpha \in C$.
(From right to left) $\alpha \in C$ iff $\neg\neg\alpha \in C$ iff $\neg\alpha \in R$. ■

Proof of Observation 8.7.2.
Let A be a belief base, R and C sets of sentences that satisfy condition (**C-R**).

(a) Assume that R satisfies closure under double negation.

Let R be a set that satisfies *non-retractability of logical equivalents* we intend to prove that C satisfies *credibility of logical equivalents*.
Let $\vdash \alpha \leftrightarrow \beta$ and assume without loss of generality that $\alpha \in C$. Hence $\neg\alpha \in R$. Therefore $\neg\beta \in R$, since R satisfies *non-retractability of logical equivalents*. Thus $\beta \in C$. By symmetry of the case it follows that if $\beta \in C$, then $\alpha \in C$. Thus $\alpha \in C$ if and only if $\beta \in C$.
Let C be a set that satisfies *credibility of logical equivalents* we intend to prove that R satisfies *non-retractability of logical equivalents*.
Let $\vdash \alpha \leftrightarrow \beta$ and assume without loss of generality that $\alpha \in R$. Hence $\neg\neg\alpha \in R$, from which it follows that $\neg\alpha \in C$. Therefore $\neg\beta \in C$, since C satisfies *credibility of logical equivalents* (and $\vdash \neg\alpha \leftrightarrow \neg\beta$). Thus $\neg\neg\beta \in R$, from which it follows that $\beta \in R$. By symmetry of the case it follows that if $\beta \in R$, then $\alpha \in R$. Thus $\alpha \in R$ if and only if $\beta \in R$.

Let R be a set that satisfies *non-retractability of tautology* we intend to prove that C satisfies *element consistency*.
Let $\alpha \vdash \perp$. Hence $\vdash \neg\alpha$. Therefore $\neg\alpha \notin R$, since R satisfies *non-retractability of tautology*. Thus $\alpha \notin C$.
Let C be a set that satisfies *element consistency* we intend to prove that R satisfies *non-retractability of tautology*.

Let $\vdash \alpha$. Hence $\{\neg\alpha\} \vdash\bot$. Thus, since C satisfies *element consistency*, $\neg\alpha \notin C$. From which it follows that $\neg\neg\alpha \notin R$. Therefore $\alpha \notin R$.

Let R be a set that satisfies *non-retractability propagation* we intend to prove that C satisfies *single sentence closure*.
Let $\alpha \in C$ and $\beta \in Cn(\alpha)$. Hence $\neg\alpha \in R$ and, by deduction, $\vdash \alpha \to \beta$. Thus $\vdash \neg\beta \to \neg\alpha$. Therefore $\neg\alpha \in Cn(\neg\beta)$. Assume by *reductio ad absurdum* that $\beta \notin C$. Hence $\neg\beta \notin R$. From which it follows that $\neg\alpha \notin R$, since R satisfies *non-retractability propagation*. Contradiction. Thus $\beta \in C$.
Let C be a set of sentences that satisfies *single sentence closure* we intend to prove that R satisfies *non-retractability propagation*.
Let $\alpha \notin R$ and suppose that $\beta \in Cn(\alpha)$. From $\alpha \notin R$ it follows that $\neg\neg\alpha \notin R$, from which it follows that $\neg\alpha \notin C$. From $\beta \in Cn(\alpha)$ it follows that $\vdash \neg\beta \to \neg\alpha$. Thus by *single sentence closure*, it follows that $\neg\beta \notin C$. Therefore $\neg\neg\beta \notin R$, from which it follows, that $\beta \notin R$.

Let R be a set that satisfies *uniform retractability with respect to* A. We intend to prove that C satisfies *uniform credibility with respect to* A.
Assume that it holds for all subsets A' of A that $A' \vdash \neg\alpha$ if and only if $A' \vdash \neg\beta$. Then, $\neg\alpha \in R$ if and only if $\neg\beta \in R$, since R satisfies *uniform retractability* with respect to A. Hence $\alpha \in C$ if and only if $\beta \in C$.
Let C be a set that satisfies *uniform credibility with respect to* A. We intend to prove that R satisfies *uniform retractability with respect to* A.
Assume that it holds for all subsets A' of A that $A' \vdash \alpha$ if and only if $A' \vdash \beta$. Hence for all subsets A' of A it holds that $A' \cup \{\neg\alpha\} \vdash\bot$ if and only if $A' \cup \{\neg\beta\} \vdash\bot$. Therefore, by *uniform credibility*, it follows that $\neg\alpha \in C$ if and only if $\neg\beta \in C$. Thus $\neg\neg\alpha \in R$ if and only if $\neg\neg\beta \in R$. Hence $\alpha \in R$ if and only if $\beta \in R$.

Let R be a set that satisfies *non-retractability upper bounding with respect to* A. We intend to prove that C satisfies *expansive credibility with respect to* A.
Assume that $\neg\alpha \notin C$. Hence $\neg\neg\alpha \notin R$, from which it follows that $\alpha \notin R$. Thus $A \vdash \alpha$, since by *non-retractability upper bounding*, $\mathcal{L} \setminus R \subseteq Cn(A)$.
Let C be a set that satisfies *expansive credibility with respect to* A. We intend to prove that R satisfies *non-retractability upper bounding with respect to* A.
Let $\alpha \in \mathcal{L} \setminus R$. Then $\neg\neg\alpha \notin R$, from which it follows that $\neg\alpha \notin C$. Hence, by *expansive credibility with respect to* A, it follows that $A \vdash \alpha$.

b) Assume that R satisfies *retractability of logical equivalents*.
Let R be a set that satisfies *conjunctive completeness* we intend to prove that C satisfies *disjunctive completeness*.
Let $\alpha \notin C$ and $\beta \notin C$. Therefore $\neg\alpha \notin R$ and $\neg\beta \notin R$. By *conjunctive completeness* it follows that $\neg\alpha \wedge \neg\beta \notin R$. Thus, by *retractability of logical equivalents*, it follows that $\neg(\alpha \vee \beta) \notin R$. Therefore $\alpha \vee \beta \notin C$.
Let C be a set that satisfies *disjunctive completeness* we intend to prove that

R satisfies *conjunctive completeness*.
Let $\alpha \notin R$ and $\beta \notin R$. Thus $\neg\neg\alpha \notin R$ and $\neg\neg\beta \notin R$, from which it follows that $\neg\alpha \notin C$ and $\neg\beta \notin C$. From which it follows, by *disjunctive completeness* that $\neg\alpha \vee \neg\beta \notin C$. Hence $\neg(\neg\alpha \vee \neg\beta) \notin R$. Thus by *retractability of logical equivalents* it follows that $\alpha \wedge \beta \notin R$. ∎

Proof of Observation 8.7.3.
Follows trivially from Lemma F.2 and Observation 8.7.2. ∎

Proof of Observation 8.7.4.
Let $\alpha \notin C$ and $\beta \in C$. We intent to prove that $A \cap A * \beta \vdash \neg\alpha$.
From $\alpha \notin C$ and $\beta \in C$ it follows by condition (**C-R**) that $\neg\alpha \notin R$ and $\neg\beta \in R$. Therefore, according to condition (**R -** $-$), $A - \neg\beta \vdash \neg\alpha$. On the other hand, by the Levi identity, $A * \beta = (A - \neg\beta) \cup \{\beta\}$. Therefore, $A \cap A * \beta = A \cap ((A - \neg\beta) \cup \{\beta\}) = (A \cap (A - \neg\beta)) \cup (A \cap \{\beta\}) = (A - \neg\beta) \cup (A \cap \{\beta\})$. Thus $A - \neg\beta \subseteq A \cap A * \beta$, from which it follows that $A \cap A * \beta \vdash \neg\alpha$. ∎

Proof of Observation 8.7.5.
Let $\alpha \notin R$ and $\beta \in R$. We intent to prove that $A - \beta \vdash \alpha$.
By condition (**R-C**), it follows that $\neg\alpha \notin C$ and $\neg\beta \in C$. Therefore, according to condition (**C -** $*$), $A \cap A * \neg\beta \vdash \neg\neg\alpha$. Thus, it follows from the Harper identity that $A - \beta \vdash \alpha$. ∎

Proof of Theorem 8.8.1.
Assume first that $\alpha \in R$. It follows that $\neg\alpha \in C$. Hence $A \sim \alpha = A - \alpha = (A * \neg\alpha) \cap A = (A \circledast \neg\alpha) \cap A$. If $\alpha \notin R$, then $\neg\alpha \notin C$. Hence $A \circledast \neg\alpha = A$ and $A \sim \alpha = A$. Thus $A \sim \alpha = A = (A \circledast \neg\alpha) \cap A$. ∎

Proof of Theorem 8.8.2.

(a) Assume first that $\alpha \notin C$. Hence $A \circledast \alpha = A$. If $\alpha \in C$, then $\neg\alpha \in R$. Thus $A \circledast \alpha = A * \alpha = (A - \neg\alpha) \cup \{\alpha\} = (A \sim \neg\alpha) \cup \{\alpha\}$.

(b) It remains to prove that: $\alpha \in C$ if and only if $A \sim \neg\alpha \nvdash \neg\alpha$.
Let $\alpha \notin C$. Then $\neg\alpha \notin R$. Hence, by $\sim$ definition, $A \sim \neg\alpha = A$. Thus, by R *non-retractability upper bounding*, $A \sim \neg\alpha \vdash \neg\alpha$.
Let $\alpha \in C$. Then $\neg\alpha \in R$. Therefore, by $\sim$ definition, $A \sim \neg\alpha = A - \neg\alpha$. On the other hand, by R *non-retractability of tautology*, $\nvdash \neg\alpha$. Thus, by $-$ *success*, $A \sim \neg\alpha \nvdash \neg\alpha$. ∎

Proof of Observation 8.8.3.
Let A be a consistent belief base and (for all $\alpha \in \mathcal{L}$)

$$A \circledast \alpha = \begin{cases} (A \sim \neg\alpha) \cup \{\alpha\} & \text{if } A \sim \neg\alpha \nvdash \neg\alpha \\ A & \text{otherwise} \end{cases}$$

Relative success and *consistency preservation* follow directly from the definition of $\circledast$.

Assume that $\sim$ satisfies *inclusion*. We intend to prove that $\circledast$ satisfies *inclusion*. It follows directly from the definition of $\circledast$ and $\sim$ *inclusion* that $A \circledast \alpha \subseteq A \cup \{\alpha\}$.

Assume that $\sim$ satisfies *inclusion* and *persistence*. We intend to prove that $\circledast$ satisfies *disjunctive distribution, persistence* and *strong regularity*.
Disjunctive distribution: Let $\alpha \notin A \circledast \alpha$ and $\beta \notin A \circledast \beta$. Hence, by definition of $\circledast$, $A \circledast \alpha = A \circledast \beta = A$. Furthermore $A \sim \neg\alpha \vdash \neg\alpha$ and $A \sim \neg\beta \vdash \neg\beta$. By $\sim$ *persistence* $A \sim \neg(\alpha \vee \beta) \vdash \neg\alpha$ and $A \sim \neg(\alpha \vee \beta) \vdash \neg\beta$. Hence $A \sim \neg(\alpha \vee \beta) \vdash \neg(\alpha \vee \beta)$. Hence $A \circledast (\alpha \vee \beta) = A$. Thus $\alpha \vee \beta \notin A \circledast (\alpha \vee \beta)$, since $A \nvdash \bot$ and by $\sim$ *inclusion* $A \vdash \neg(\alpha \vee \beta)$.
Persistence: Let $A \cap A \circledast \beta \vdash \neg\beta$. Hence $A \vdash \neg\beta$ and $A \circledast \beta \vdash \neg\beta$. If $A \sim \neg\beta \nvdash \neg\beta$, then $\beta \in A \circledast \beta$. Thus $A \circledast \beta \vdash \bot$, from which it follows, by $\circledast$ definition and deduction, that $A \sim \neg\beta \vdash \neg\beta$. Contradiction. Hence $A \sim \neg\beta \vdash \neg\beta$. From which it follows, by $\sim$ *persistence*, that $A \sim \neg\alpha \vdash \neg\beta$. By $\sim$ *inclusion* $A \sim \neg\alpha \subseteq A$. We will consider two cases:
Case 1) $A \sim \neg\alpha \vdash \neg\alpha$. Hence $A \circledast \alpha = A$. Then $A \cap A \circledast \alpha = A \vdash \neg\beta$.
Case 2) $A \sim \neg\alpha \nvdash \neg\alpha$. Hence $A \circledast \alpha = (A \sim \neg\alpha) \cup \{\alpha\}$ from which it follows that $A \cap A \circledast \alpha = (A \cap (A \sim \neg\alpha)) \cup (A \cap \{\alpha\}) = (A \sim \neg\alpha) \cup (A \cap \{\alpha\})$. Thus $A \sim \neg\alpha \subseteq A \cap A \circledast \alpha$. Therefore $A \cap A \circledast \alpha \vdash \neg\beta$.
Strong regularity: Let $A \circledast \alpha \nvdash \neg\beta$. By definition of $\circledast$ we have two cases to consider:
Case 1) $A \circledast \alpha = A$. Then $A \nvdash \neg\beta$. By $\sim$ *inclusion* $A \sim \neg\beta \nvdash \neg\beta$. Hence, by definition of $\circledast$, $\beta \in A \circledast \beta$.
Case 2) $A \circledast \alpha = (A \sim \neg\alpha) \cup \{\alpha\}$. Thus $(A \sim \neg\alpha) \cup \{\alpha\} \nvdash \neg\beta$. From which it follows that $A \sim \neg\alpha \nvdash \neg\beta$. By $\sim$ *persistence* it follows that $A \sim \neg\beta \nvdash \neg\beta$. Hence, by definition of $\circledast$, $\beta \in A \circledast \beta$.

Assume that $\sim$ satisfies *inclusion, vacuity* and *uniformity*. We intend to prove that $\circledast$ satisfies *uniformity*.
Let it be the case that for all subsets A' of A, $A' \cup \{\alpha\} \vdash \bot$ if and only if $A' \cup \{\beta\} \vdash \bot$. Hence, for all $A' \subseteq A$, $A' \vdash \neg\alpha$ if and only if $A' \vdash \neg\beta$. By $\sim$ *uniformity* $A \sim \neg\alpha = A \sim \neg\beta$. By $\sim$ *inclusion* $A \sim \neg\beta \subseteq A$.
If $A \sim \neg\alpha \vdash \neg\alpha$. Then $A \sim \neg\beta \vdash \neg\alpha$. Thus, by hypothesis, $A \sim \neg\beta \vdash \neg\beta$. Hence, by definition of $\circledast$, $A \circledast \alpha = A \circledast \beta = A$. Therefore $A \cap A \circledast \alpha = A \cap A \circledast \beta$.
If $A \sim \neg\alpha \nvdash \neg\alpha$. Then $A \sim \neg\beta \nvdash \neg\alpha$. From which it follows, by hypothesis, that $A \sim \neg\beta \nvdash \neg\beta$. Therefore, by definition of $\circledast$, $A \circledast \alpha = (A \sim \neg\alpha) \cup \{\alpha\}$ and $A \circledast \beta = (A \sim \neg\beta) \cup \{\beta\}$.
There are three cases to consider:
Case 1) $\alpha \in A$. Then $A \nvdash \neg\alpha$ (since $A \nvdash \bot$). Hence, by $\sim$ *vacuity* and *inclusion* $A \sim \neg\alpha = A$. Thus $A \sim \neg\beta = A$. Therefore $A \cap A \circledast \alpha = A \cap ((A \sim \neg\alpha) \cup \{\alpha\}) = A \cap (A \cup \{\alpha\}) = A$. Using a similar reasoning, it follows that $A \cap A \circledast \beta = A$. Hence $A \cap A \circledast \alpha = A \cap A \circledast \beta$.
Case 2) $\beta \in A$. Similar to the previous case.
Case 3) $\alpha \notin A$ and $\beta \notin A$. Then $A \cap A \circledast \alpha = A \cap ((A \sim \neg\alpha) \cup \{\alpha\}) = A \sim \neg\alpha = A \sim \neg\beta = A \cap ((A \sim \neg\beta) \cup \{\beta\}) = A \cap A \circledast \beta$.

Assume that $\sim$ satisfies *relevance*. We intend to prove that $\circledast$ satisfies *relevance*. Let $\beta \in A$ and $\beta \notin A \circledast \alpha$. Hence $A \neq A \circledast \alpha$. Thus $A \circledast \alpha = (A \sim \neg\alpha) \cup \{\alpha\}$ and $A \sim \neg\alpha \nvdash \neg\alpha$. Hence $\beta \notin A \sim \neg\alpha$. By $\sim$ *relevance* there is some A' such that $A \sim \neg\alpha \subseteq A' \subseteq A$, $A' \nvdash \neg\alpha$ but $A' \cup \{\beta\} \vdash \neg\alpha$. Let $X = A' \cup \{\alpha\}$. Thus $A \circledast \alpha \subseteq X \subseteq A \cup \{\alpha\}$. It remains to prove that:
1) $X \nvdash \bot$.
2) $X \cup \{\beta\} \vdash \bot$.
1) Assume by *reductio ad absurdum* that $X \vdash \bot$. Hence $A' \cup \{\alpha\} \vdash \bot$. Thus, by deduction, $A' \vdash \alpha \to \bot$. Hence $A' \vdash \neg\alpha$. Contradiction.
2) $X \cup \{\beta\} = A' \cup \{\alpha, \beta\}$. Since $A' \cup \{\beta\} \vdash \neg\alpha$ it follows that $A' \cup \{\alpha, \beta\} \vdash \bot$. Hence $X \cup \{\beta\} \vdash \bot$.

Assume that $\sim$ satisfies *core-retainment*. We intend to prove that $\circledast$ satisfies *core-retainment*.
Let $\beta \in A$ and $\beta \notin A \circledast \alpha$. Hence $A \neq A \circledast \alpha$. Thus $A \circledast \alpha = (A \sim \neg\alpha) \cup \{\alpha\}$ and $A \sim \neg\alpha \nvdash \neg\alpha$. Hence $\beta \notin A \sim \neg\alpha$. By $\sim$ *core-retainment* there is some A' such that $A' \subseteq A$, $A' \nvdash \neg\alpha$ but $A' \cup \{\beta\} \vdash \neg\alpha$.

Assume that $\sim$ satisfies *conjunctive constancy, relative success* and *extensionality*. We intend to prove that $\circledast$ satisfies *disjunctive distribution*.
Let $\alpha \notin A \circledast \alpha$ and $\beta \notin A \circledast \beta$. By definition of $\circledast$ it follows that $A \sim \neg\alpha \vdash \neg\alpha$ and $A \sim \neg\beta \vdash \neg\beta$. Hence by $\sim$ *relative success*, $A \sim \neg\alpha = A \sim \neg\beta = A$. Thus, by $\sim$ *conjunctive constancy*, $A \sim (\neg\alpha \wedge \neg\beta) = A$. From which it follows, by $\sim$ *extensionality*, that $A \sim \neg(\alpha \vee \beta) = A$. On the other hand $A \vdash \neg\alpha \wedge \neg\beta$. Hence $A \sim \neg(\alpha \vee \beta) \vdash \neg(\alpha \vee \beta)$. Therefore, by definition of $\circledast$, $A \circledast (\alpha \vee \beta) = A$. From $A \nvdash \bot$ it follows that $A \circledast (\alpha \vee \beta) \nvdash \alpha \vee \beta$. Hence $\alpha \vee \beta \notin A \circledast (\alpha \vee \beta)$.

Assume that $\sim$ satisfies *inclusion* and *success propagation*. We intend to prove that $\circledast$ satisfies *strict improvement*.
Let $\alpha \in A \circledast \alpha$ and $\vdash \alpha \to \beta$. Hence $\vdash \neg\beta \to \neg\alpha$. Assume by *reductio ad absurdum* that $A \sim \neg\alpha \vdash \neg\alpha$. Then, by $\sim$ *inclusion*, $A \vdash \neg\alpha$. On the other hand, it follows from $\circledast$ definition that $A \circledast \alpha = A$. Hence $\alpha \in A$. Therefore $A \vdash \bot$. Contradiction. Hence $A \sim \neg\alpha \nvdash \neg\alpha$. Thus, by $\sim$ *success propagation*, $A \sim \neg\beta \nvdash \neg\beta$. From which it follows, by definition of $\circledast$, that $\beta \in A \circledast \beta$.

Assume that $\sim$ satisfies *inclusion* and *vacuity*. We intend to prove that $\circledast$ satisfies *vacuity*.
Assume that $A \nvdash \neg\alpha$. By $\sim$ *inclusion* and *vacuity* it follows that $A \sim \neg\alpha = A \nvdash \neg\alpha$. Hence, by definition of $\circledast$, it follows that $A \circledast \alpha = (A \sim \neg\alpha) \cup \{\alpha\} = A \cup \{\alpha\}$.

Assume that $\sim$ satisfies *disjunctive elimination*. We intend to prove that $\circledast$ satisfies *disjunctive elimination*.
Let $\beta \in A$ and $\beta \notin A \circledast \alpha$. Then $A \sim \neg\alpha \nvdash \neg\alpha$ and $A \circledast \alpha = (A \sim \neg\alpha) \cup \{\alpha\}$. Thus $\beta \notin A \sim \neg\alpha$, from which it follows by *disjunctive elimination* that $A \sim \neg\alpha \nvdash \neg\alpha \vee \beta$. Hence $(A \sim \neg\alpha) \cup \{\alpha\} \nvdash \neg\alpha \vee \beta$, otherwise it would follow by deduc-

tion that $A \sim \neg\alpha \vdash \alpha \rightarrow (\neg\alpha \vee \beta)$ and consequently that $A \sim \neg\alpha \vdash \neg\alpha \vee \beta$, since $\vdash (\neg\alpha \vee \beta) \leftrightarrow (\alpha \rightarrow (\neg\alpha \vee \beta))$. Therefore $A \circledast \alpha \nvdash \neg\alpha \vee \beta$.

Assume that $\sim$ satisfies *extensionality, inclusion* and *vacuity*. We intend to prove that $\circledast$ satisfies *weak extensionality*.
Let $\vdash \alpha \leftrightarrow \beta$. Then $\vdash \neg\alpha \leftrightarrow \neg\beta$. We will prove by cases:
Case 1) $A \sim \neg\alpha \vdash \neg\alpha$. Then $A \sim \neg\alpha \vdash \neg\beta$. Thus, by $\sim$ *extensionality*, $A \sim \neg\beta \vdash \neg\beta$. Therefore, by definition of $\circledast$, $A \circledast \alpha = A \circledast \beta = A$. Hence $A \cap A \circledast \alpha = A \cap A \circledast \beta$.
Case 2) $A \sim \neg\alpha \nvdash \neg\alpha$. Then $A \sim \neg\alpha \nvdash \neg\beta$. Therefore, by $\sim$ *extensionality*, $A \sim \neg\beta \nvdash \neg\beta$. Therefore, by definition of $\circledast$, $A \circledast \alpha = (A \sim \neg\alpha) \cup \{\alpha\}$ and $A \circledast \beta = (A \sim \neg\beta) \cup \{\beta\}$.
Case 2.1) $\alpha \in A$. Therefore $A \nvdash \neg\alpha$ (since $A \nvdash \bot$) and $A \nvdash \neg\beta$. By $\sim$ *vacuity* and *inclusion* it follows that $A \sim \neg\alpha = A \sim \neg\beta = A$. Hence $A \cap A \circledast \alpha = A \cap ((A \sim \neg\alpha) \cup \{\alpha\}) = A$. By symmetry of the case it holds that $A \cap A \circledast \beta = A$. Therefore $A \cap A \circledast \alpha = A \cap A \circledast \beta$.
Case 2.2) $\beta \in A$. Follows as in the previous case.
Case 2.3) $\alpha \notin A$ and $\beta \notin A$. By $\sim$ *inclusion* it follows that $A \sim \neg\alpha \subseteq A$ and $A \sim \neg\beta \subseteq A$. Hence $A \cap A \circledast \alpha = A \cap ((A \sim \neg\alpha) \cup \{\alpha\}) = A \sim \neg\alpha$. By symmetry of the case it follows that $A \cap A \circledast \beta = A \sim \neg\beta$. Hence by $\sim$ *extensionality* it follows that $A \cap A \circledast \alpha = A \cap A \circledast \beta$.

Assume that $\sim$ satisfies *inclusion, vacuity* and *relative closure*. We intend to prove that $\circledast$ satisfies *relative closure*.
Let $\beta \in A \cap Cn(A \cap A \circledast \alpha)$. It follows trivially if $A \sim \neg\alpha \vdash \neg\alpha$. Assume now that $A \sim \neg\alpha \nvdash \neg\alpha$. Hence $A \circledast \alpha = (A \sim \neg\alpha) \cup \{\alpha\}$. Hence $\beta \in A \cap Cn(A \cap ((A \sim \neg\alpha) \cup \{\alpha\}))$. Hence $\beta \in A$ and $\beta \in Cn((A \cap (A \sim \neg\alpha)) \cup (A \cap \{\alpha\}))$. We will prove by cases:
Case 1) $\alpha \in A$. Then $A \nvdash \neg\alpha$ (since $A \nvdash \bot$). Therefore, by $\sim$ *inclusion* and *vacuity*, it follows that $A \sim \neg\alpha = A$. Thus $A \circledast \alpha = A \cup \{\alpha\}$. Hence $\beta \in A \circledast \alpha$.
Case 2) $\alpha \notin A$. Thus $\beta \in A$ and $\beta \in Cn((A \sim \neg\alpha) \cap A)$. By $\sim$ *inclusion* it follows that $\beta \in A$ and $\beta \in Cn(A \sim \neg\alpha)$. Therefore, by $\sim$ *relative closure* it follows that $\beta \in A \sim \neg\alpha$. Thus $\beta \in A \circledast \alpha$. ■

Proof of Observation 8.8.4.
Let A be a consistent belief base and $A \sim \alpha = A \cap A \circledast \neg\alpha$ (for all $\alpha \in \mathcal{L}$).
That $\sim$ satisfies *inclusion* follows directly from the definition of $\sim$.

Assume that $\circledast$ satisfies *relative success* and *consistency preservation*. We intend to prove that $\sim$ satisfies *relative success*.
If $A \sim \alpha \vdash \alpha$, then by definition of $\sim$, $A \circledast \neg\alpha \vdash \alpha$. Hence, by $\circledast$ *consistency preservation*, $\neg\alpha \notin A \circledast \neg\alpha$. Thus, by $\circledast$ *relative success*, $A \circledast \neg\alpha = A$. Therefore $A \sim \alpha = A$.

Assume that $\circledast$ satisfies *persistence*. We intend to prove that $\sim$ satisfies *persistence*.
Let $A \sim \alpha \nvdash \beta$. Then $A \cap A \circledast \neg\alpha \nvdash \beta$. Thus, by $\circledast$ *persistence*, $A \cap A \circledast \neg\beta \nvdash \beta$. Therefore, $A \sim \beta \nvdash \beta$.

Assume that $\circledast$ satisfies *relative success* and *relevance*. We intend to prove that $\sim$ satisfies *relevance*.
Let $\beta \in A$ and $\beta \notin A \sim \alpha$. Then, by definition of $\sim$, $\beta \notin A \circledast \neg\alpha$. By $\circledast$ *relative*

success it follows that $\neg\alpha \in A \circledast \neg\alpha$. By $\circledast$ *relevance*, there is some A' such that $A \circledast \neg\alpha \subseteq A' \subseteq A \cup \{\neg\alpha\}$, $A' \nvdash \bot$ but $A' \cup \{\beta\} \vdash \bot$. Let $X = A' \setminus \{\neg\alpha\}$. Hence $X \subseteq A$ and, since $\neg\alpha \in A \circledast \neg\alpha \subseteq A'$, it follows that $X \cup \{\neg\alpha\} = A'$. Therefore $A \circledast \neg\alpha \subseteq X \cup \{\neg\alpha\}$. Thus $A \sim \alpha \subseteq X \cup \{\neg\alpha\}$. To prove that $A \sim \alpha \subseteq X$ it is enough to show that $A \sim \alpha \nvdash \neg\alpha$.
Assume by *reductio ad absurdum* that $A \sim \alpha \vdash \neg\alpha$. Hence, by definition of $\sim$, $A \vdash \neg\alpha$. From $A' \subseteq A \cup \{\neg\alpha\}$ and $A' \cup \{\beta\} \vdash \bot$ it follows, by monotony, that $A \cup \{\neg\alpha, \beta\} \vdash \bot$. Contradiction, since $A \nvdash \bot$. Therefore $A \sim \alpha \nvdash \neg\alpha$. Hence $A \sim \alpha \subseteq X$. On the other hand, from $A' \nvdash \bot$, it follows that $X \cup \{\neg\alpha\} \nvdash \bot$. Thus $X \nvdash \alpha$. From $A' \cup \{\beta\} \vdash \bot$ it follows that $X \cup \{\neg\alpha, \beta\} \vdash \bot$. From which it follows, using deduction, that $X \cup \{\beta\} \vdash \alpha$.

Assume that $\circledast$ satisfies *core-retainment*. We intend to prove that $\sim$ satisfies *core-retainment*.
Let $\beta \in A$ and $\beta \notin A \sim \alpha$. Then, by definition of $\sim$, $\beta \notin A \circledast \neg\alpha$. Hence, by $\circledast$ *core-retainment*, there is some A' such that $A' \subseteq A$, $A' \nvdash \alpha$ but $A' \cup \{\beta\} \vdash \alpha$. Hence $\sim$ satisfies *core-retainment*.

Assume that $\circledast$ satisfies *uniformity*. We intend to prove that $\sim$ satisfies *uniformity*.
Let it be the case that for all subsets A' of A, $A' \vdash \alpha$ if and only if $A' \vdash \beta$. Hence, for all subsets A' of A it holds that $A' \cup \{\neg\alpha\} \vdash \bot$ if and only if $A' \cup \{\neg\beta\} \vdash \bot$. Therefore, by $\circledast$ *uniformity*, $A \cap A \circledast \neg\alpha = A \cap A \circledast \neg\beta$. Thus, by definition of $\sim$, $A \sim \alpha = A \sim \beta$.

Assume that $\circledast$ satisfies *vacuity*. We intend to prove that $\sim$ satisfies *vacuity*.
Assume that $A \nvdash \alpha$. Hence $A \nvdash \neg\neg\alpha$. Thus, by $\circledast$ *vacuity* it follows that $A \cup \{\neg\alpha\} \subseteq A \circledast \neg\alpha$. Thus $A \subseteq A \circledast \neg\alpha$. Therefore $A \sim \alpha = (A \circledast \neg\alpha) \cap A = A$.

Assume that $\circledast$ satisfies *vacuity, relative success, consistency preservation, disjunctive distribution* and *weak extensionality*. We intend to prove that $\sim$ satisfies *conjunctive constancy*. As shown above $\sim$ satisfies *inclusion* and *vacuity*.
Let $A \sim \alpha = A \sim \beta = A$. We will consider three cases:
Case 1) $A \circledast \neg\alpha \nvdash \alpha$. Then $A \sim \alpha \nvdash \alpha$. Thus $A \nvdash \alpha$. From which it follows that $A \nvdash \alpha \wedge \beta$. Hence, by $\sim$ *inclusion* and *vacuity*, then $A \sim (\alpha \wedge \beta) = A$.
Case 2) $A \circledast \neg\beta \nvdash \beta$. This case is symmetrical with the first case.
Case 3) $A \circledast \neg\alpha \vdash \alpha$ and $A \circledast \neg\beta \vdash \beta$. By $\circledast$ *consistency preservation* it follows that $\neg\alpha \notin A \circledast \neg\alpha$ and $\neg\beta \notin A \circledast \neg\beta$. Hence, by *disjunctive distribution*, it follows that $\neg\alpha \vee \neg\beta \notin A \circledast (\neg\alpha \vee \neg\beta)$. Hence, by $\circledast$ *relative success*, $A \circledast (\neg\alpha \vee \neg\beta) = A$. By definition of $\sim$, $A \sim (\alpha \wedge \beta) = A \cap A \circledast \neg(\alpha \wedge \beta)$. Thus, by $\circledast$ *weak extensionality*, it follows that $A \sim (\alpha \wedge \beta) = A \cap A \circledast (\neg\alpha \vee \neg\beta) = A$.

Assume that $\circledast$ satisfies *disjunctive elimination*. We intend to prove that $\sim$ satisfies *disjunctive elimination*.
Let $\beta \in A$ and $\beta \notin A \sim \alpha$. By definition of $\sim$ it follows that $\beta \notin A \circledast \neg\alpha$. Therefore, by $\circledast$ *disjunctive elimination*, $A \circledast \neg\alpha \nvdash (\neg\neg\alpha) \vee \beta$. Thus $A \circledast \neg\alpha \nvdash \alpha \vee \beta$. Therefore $A \sim \alpha \nvdash \alpha \vee \beta$.

Assume that $\circledast$ satisfies *weak extensionality*. We intend to prove that $\sim$ satisfies *extensionality*.
Let $\vdash \alpha \leftrightarrow \beta$. Hence $\vdash \neg\alpha \leftrightarrow \neg\beta$. By definition of $\sim$ and $\circledast$ *weak extensionality* it holds that $A \sim \alpha = A \cap A \circledast \neg\alpha = A \cap A \circledast \neg\beta = A \sim \beta$.

Assume that $\circledast$ satisfies *consistency preservation, strict improvement* and *relative success*. We intend to prove that $\sim$ satisfies *success propagation*.
Let $A \sim \alpha \vdash \alpha$ and $\vdash \alpha \rightarrow \beta$. From the latter it follows that $\vdash \neg\beta \rightarrow \neg\alpha$. By definition of $\sim$ it holds that $A \sim \alpha = A \cap A \circledast \neg\alpha$. Hence $A \circledast \neg\alpha \vdash \alpha$ and $A \vdash \alpha$. Therefore $A \vdash \beta$. On the other hand, by $\circledast$ *consistency preservation*, $\neg\alpha \notin A \circledast \neg\alpha$. From the latter and $\vdash \neg\beta \rightarrow \neg\alpha$ it follows by $\circledast$ *strict improvement* that $\neg\beta \notin A \circledast \neg\beta$. Thus by $\circledast$ *relative success* it follows that $A \circledast \neg\beta = A$. Hence $A \sim \beta = A \cap A \circledast \neg\beta = A$. Thus $A \sim \beta \vdash \beta$.

Assume that $\circledast$ satisfies *relative closure*. We intend to prove that $\sim$ satisfies *relative closure*.
Let $\beta \in A \cap Cn(A \sim \alpha)$. Hence $\beta \in A$. Furthermore, by definition of $\sim$, it follows that $(A \circledast \neg\alpha) \cap A \vdash \beta$. Thus by *relative closure* $\beta \in A \circledast \neg\alpha$. From which it follows, by definition of $\sim$, that $\beta \in A \sim \alpha$. ■

Proof of Corollary 8.8.5.
By Definition 7.3.2 and Theorem 7.3.1 $\sim$ satisfies *relative success* and *inclusion*. Thus, by Observation 8.8.3, $\circledast$ satisfies *relative success, inclusion* and *consistency preserving*. Thus, by Theorem 8.5.1, $\circledast$ is the operator of credibility-limited base revision induced by a revision operator $*$ on A and a set $C \subseteq \mathcal{L}$ that satisfies *element consistency*. Therefore, by Definition 8.5.2, $\circledast$ is a basic credibility-limited base revision operator. ■

Proof of Corollary 8.8.6.

(a) Follows trivially by Corollaries 7.3.6 and 8.5.5 and Observations 5.1.2 and 8.8.3.

(b) Follows trivially by Corollaries 7.3.11 and 8.5.8 and Observations 5.1.2 and 8.8.3.

(c) Follows trivially by Corollaries 7.3.15 and 8.5.11 and Observations 5.1.2 and 8.8.3.

(d) Follows trivially by Corollaries 7.3.19 and 8.5.14 and Observation 8.8.3. ■

Proof of Corollary 8.8.7.
By Definition 8.5.2 and Theorem 8.5.1 $\circledast$ satisfies *relative success, consistency preservation* and *inclusion*. By Observation 8.8.4 $\sim$ satisfies *inclusion* and *relative success*. Thus, by Theorem 7.3.1, $\sim$ is a shielded contraction operator on A induced by a contraction operator on A and a set R that satisfies non-retractability of tautology. Thus, by Definition 7.3.2, $\sim$ is a basic shielded base contraction. ■

Proof of Corollary 8.8.8.

(a) Follows trivially by Corollaries 7.3.6 and 8.5.5 and Observations 8.2.1 (b) and 8.8.4.

(b) Follows trivially by Corollaries 7.3.11 and 8.5.8 and Observations 8.2.1 (b) and 8.8.4.

(c) Follows trivially by Corollaries 7.3.15 and 8.5.11 and Observations 8.2.1 (b) and 8.8.4.

(d) Follows trivially by Corollaries 7.3.19 and 8.5.14 and Observation 8.8.4. ■

Proof of Theorem 8.8.9.
Let $\circledast = \mathbb{R}(\sim)$ and $\sim_2 = \mathbb{C}(\mathbb{R}(\sim))$. Then:

$$A \circledast \neg\alpha = \begin{cases} (A \sim \neg\neg\alpha) \cup \{\neg\alpha\} & \text{if } A \sim \neg\neg\alpha \nvdash \alpha \\ A & \text{otherwise} \end{cases}$$

and

$$A \sim_2 \alpha = A \cap A \circledast \neg\alpha$$

By $\sim$ *extensionality* $A \sim \neg\neg\alpha = A \sim \alpha$. There are two cases to consider:
Case 1) $A \sim \neg\neg\alpha \vdash \alpha$. Then $A \circledast \neg\alpha = A$, from which it follows that $A \sim_2 \alpha = A$. On the other hand, by $\sim$ *relative success*, $A \sim \alpha = A$. Thus $A \sim_2 \alpha = A \sim \alpha$.
Case 2) $A \sim \neg\neg\alpha \nvdash \alpha$. Then $A \circledast \neg\alpha = (A \sim \neg\neg\alpha) \cup \{\neg\alpha\} = (A \sim \alpha) \cup \{\neg\alpha\}$. Hence $A \sim_2 \alpha = A \cap ((A \sim \alpha) \cup \{\neg\alpha\})$.
Let $\beta \in A \sim \alpha$. Then, by $\sim$ *inclusion*, $\beta \in A$. Hence $\beta \in A \sim_2 \alpha$. Thus $A \sim \alpha \subseteq A \sim_2 \alpha$.
Let $\beta \in A \sim_2 \alpha$. Then $\beta \in A$ and $\beta \in (A \sim \alpha) \cup \{\neg\alpha\}$. Hence $\beta \in A \sim \alpha$ or $\beta = \neg\alpha$. If $\beta = \neg\alpha$, then $\neg\alpha \in A$. From $A \nvdash \bot$ it follows that $A \nvdash \alpha$. Then, by $\sim$ *vacuity* and *inclusion*, $A \sim \alpha = A$. Therefore $\beta \in A \sim \alpha$. Hence $A \sim_2 \alpha \subseteq A \sim \alpha$. Therefore $A \sim_2 \alpha = A \sim \alpha$. ■

Proof of Theorem 8.8.10.
Let $\sim = \mathbb{C}(\circledast)$ and $\circledast_2 = \mathbb{R}(\mathbb{C}(\circledast))$. Then:

$$A \sim \neg\alpha = A \cap A \circledast \neg\neg\alpha$$

and

$$A \circledast_2 \alpha = \begin{cases} (A \sim \neg\alpha) \cup \{\alpha\} & \text{if } A \sim \neg\alpha \nvdash \neg\alpha \\ A & \text{otherwise} \end{cases}$$

If $A \nvdash \neg\alpha$, then by $\circledast$ *vacuity* and *inclusion* $A \circledast \alpha = A \cup \{\alpha\}$ and $A \circledast \neg\neg\alpha = A \cup \{\neg\neg\alpha\}$. Thus, by $\sim$ definition, $A \sim \neg\alpha = A$. Hence $A \sim \neg\alpha \nvdash \neg\alpha$. Thus $A \circledast_2 \alpha = A \cup \{\alpha\}$. Therefore $A \circledast_2 \alpha = A \circledast \alpha$.
Assume now that $A \vdash \neg\alpha$. By $\circledast$ *relative success* $\neg\neg\alpha \in A \circledast \neg\neg\alpha$ or $A \circledast \neg\neg\alpha = A$. On the other hand, by $\circledast$ *weak extensionality* $A \cap A \circledast \neg\neg\alpha = A \cap A \circledast \alpha$. We will consider two cases:
Case 1) $A \circledast \neg\neg\alpha = A$. Hence $A \sim \neg\alpha = A$, from which it follows that $A \sim \neg\alpha \vdash \neg\alpha$. Thus $A \circledast_2 \alpha = A$.

From $A \circledast \neg\neg\alpha = A$ and $A \cap A \circledast \neg\neg\alpha = A \cap A \circledast \alpha$, it follows that $A = A \cap A \circledast \alpha$. Hence $A \subseteq A \circledast \alpha$. Since $A \vdash \neg\alpha$ and $A \nvdash \bot$ it follows, by $\circledast$ *consistency preservation*, that $\alpha \notin A \circledast \alpha$. Hence by $\circledast$ *relative success* $A \circledast \alpha = A$.
Case 2) $\neg\neg\alpha \in A \circledast \neg\neg\alpha$. Hence, by $\circledast$ *consistency preservation*, $A \circledast \neg\neg\alpha \nvdash \neg\alpha$. Thus $A \sim \neg\alpha \nvdash \neg\alpha$. Hence $A \circledast_2 \alpha = (A \sim \neg\alpha) \cup \{\alpha\} = (A \cap A \circledast \neg\neg\alpha) \cup \{\alpha\} = (A \cap A \circledast \alpha) \cup \{\alpha\}$. Let $\beta \in A \circledast_2 \alpha$. Hence $\beta \in A \cap A \circledast \alpha$ or $\beta = \alpha$. In the former case, $\beta \in A \circledast \alpha$. Assume now that $\beta = \alpha$. If $\alpha \in A \circledast \alpha$, then $\beta \in A \circledast \alpha$.
Assume by *reductio ad absurdum* that $\alpha \notin A \circledast \alpha$. Hence, by $\circledast$ *relative success* $A \circledast \alpha = A$. Thus $A \circledast_2 \alpha = A \cup \{\alpha\}$. Hence $A \circledast_2 \alpha \vdash \bot$ (since $A \vdash \neg\alpha$). Hence $(A \sim \neg\alpha) \cup \{\alpha\} \vdash \bot$. Therefore, by deduction, $A \sim \neg\alpha \vdash \neg\alpha$. Contradiction. Hence $A \circledast_2 \alpha \subseteq A \circledast \alpha$.
Let $\beta \in A \circledast \alpha$. By $\circledast$ *inclusion* $A \circledast \alpha \subseteq A \cup \{\alpha\}$. Hence $\beta \in A$ or $\beta = \alpha$. In both cases, $\beta \in A \circledast_2 \alpha$. Hence $A \circledast \alpha \subseteq A \circledast_2 \alpha$. Therefore $A \circledast_2 \alpha = A \circledast \alpha$. ■

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