Minimal Orbits of Isotropy Actions for the Classical Root Systems with Simply-Laced Dynkin Diagrams

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Abstract

Wir betrachten die Lage der eindeutigen minimalen Hauptbahn einer Isotropiewirkung eines einfach zusammenhängenden symmetrischen Raumes kompakten Typs mit Wurzelsystem A_n bzw. D_n. Nach einer Identifizierung des Bahnenraumes mit einer verallgemeinerten Weyl-Kammer, geben wir für A_n die Lage der minimalen Hauptbahn in der dominanten verallgemeinerten Weyl-Kammer explizit an. Für D_n geben wir ein Ergebnis an, welches es uns erlaubt die Lage der minimalen Hauptbahn in der dominanten verallgemeinerten Weyl-Kammer aus den Nullstellen eines Polynoms zu bestimmen, dessen Form wir explizit angeben.

Abstract

We study the position of the unique minimal principal orbit of the isotropy action of a simply-connected symmetric space of compact type with restricted root system A_n or D_n. After identifying the orbit space with a generalized Weyl chamber, we give for A_n an explicit formula for the position of the minimal principal orbit in the dominant generalized Weyl chamber. For D_n, on the other hand, we give a result that enables us to determine the position of the minimal principal orbit in the dominant generalized Weyl chamber by computing the roots of a polynomial, which we give explicitly.

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1 Introduction

An isometric action of a compact Lie group G on a Riemannian manifold M is called polar if there exists a section, i.e. a submanifold of M that intersects every orbit orthogonally. The action is called hyperpolar, if the section is flat in the induced metric. Manifolds with hyperpolar actions have various nice properties and are therefore an interesting field of study. It has been shown by A. Kollross [Koll] that in the case of hyperpolar actions on irreducible Riemannian symmetric spaces of compact type, denoted by $M = G/K$, the action has to be either of cohomogeneity one or a so called Hermann action. The latter is defined as an action of a symmetric subgroup $H \subseteq G$, i.e. a subgroup such that there exists an involution $\sigma\colon G \to G$ with the property that H is contained in the fixed point set of σ. A special case of a Hermann action is given by the action of the isotropy group K. These so called isotropy actions and their orbits are the central subject of this thesis, where we restrict our considerations to simply-connected symmetric spaces.

Isotropy actions and the geometry of their orbits have been studied extensively over the years. Particularly important for us are the following two results: In [Tas] the mean curvature of a principal orbit is calculated. Using this result, [HSTT] showed that there exists a unique principal orbit that is a minimal submanifold. The root space decomposition of symmetric spaces is a very useful tool in this context. Identifying the orbit space with the closure of a generalized Weyl chamber, enables an investigation of the orbits on the Lie algebra level.

The outlined results lead in a natural way to the problem of determining the unique point, associated to the unique minimal principal orbit of an isotropy action, in a given Weyl chamber. For some symmetric spaces of rank two the position of the minimal principal orbit has already been found, [CNV]. But for most cases it remained an open question.

In this thesis, we study the position of the minimal principal orbit for isotropy actions on symmetric spaces of compact type and rank n with root systems A_n and D_n, i.e. the classical root systems with simply-laced Dynkin diagrams. For this, we define real valued functions, denoted by ϑ_{A_n} and ϑ_{D_n}, respectively. These functions have the property that their unique extremal point coincides with the unique minimal principal orbit in the Weyl chamber.

In the case that the root system is A_n, we give an explicit formula for the point associated to the minimal principal orbit within the dominant generalized Weyl chamber.

For the root system D_n, on the other hand, the main result is a theorem that enables us to find the unique minimal principal orbit in the dominant generalized Weyl chamber by determining the roots of a polynomial of degree $\lfloor \frac{n}{2} \rfloor - 1$.

This thesis is organized as follows:

The first section gives a short introduction to the field and preliminary results. Starting with a review of some needed properties of symmetric spaces and hyperpolar actions, we then restrict our considerations to isotropy actions on simply-connected symmetric spaces of the compact type. We reprove the diagonalization of the shape operator of principal orbits given in [Ver], by adjusting the proof for the diagonalization of the shape operator of the orbits of Hermann actions given in [GT] to the special case of an isotropy action. This allows us to read off the mean curvature of the orbits from the diagonalized shape operator. For each root system Δ, we define the function ϑ_Δ on a generalized Weyl chamber, originally introduced in [CNV], which has a unique extremal point that coincides with the unique minimal principal orbit in the Weyl chamber. The investigation of the position of the critical point of ϑ_Δ within a Weyl chamber will be our approach to determine the positon of the unique minimal principal orbit. Finally, we give a formulation of the central problem in this thesis.

Section 2 deals with the solution of the posed problem for symmetric spaces with root system A_n. We state and prove the main result of this section which is an explicit formula for the position of the unique minimal principal orbit of the isotropy action within the dominant generalized Weyl chamber. As a corollary we obtain a recursive relation between the minimal principal orbit of a symmetric space of rank n and the minimal orbit of a symmetric space of rank $n+1$. Originally, the insight of the fact, that the solution of the problem for rank n can be used to construct the solution of the problem for rank $n+1$, was the foundation that led to the construction of the explicit formula in the main result.

Section 3 deals with symmetric spaces with root system D_n. The main result of this section is a theorem that enables us to determine the unique minimal principal orbit in the dominant generalized Weyl chamber from the roots of a polynomial of degree $\lfloor \frac{n}{2} \rfloor - 1$, which will be denoted by P_n. The key to the main result is an ansatz that reduces the problem from having originally n unknowns $x_1, \ldots, x_n$ to a problem with only $\lfloor \frac{n}{2} \rfloor - 1$ unknowns $\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}$. The proof starts by revealing symmetries of the components of $\nabla \vartheta_{D_n}$ with respect to certain permutations of the variables $x_1, \ldots, x_n$. This study of symmetries, together with the ansatz, lead to a non-linear system of equations (NSE) in $\lfloor \frac{n}{2} \rfloor - 1$ variables. The benifit of this derived NSE lies not only in the fact that a solution yields the unique minimal principal orbit in the considered Weyl chamber, but in particular in its relatively simple form. Finally, we show, using symmetry arguments, that the solution of the NSE is given by the roots of the polynomial P_n of degree $\lfloor \frac{n}{2} \rfloor - 1$, which we give explicitly.

We emphazise that in contrast to the first section on the preliminaries, the results in section two and three are original.

In this work we will denote the geodesic exponential map by Exp and the Lie group exponential map by exp. The parallel transport along the geodesic γ in M will be denoted by

$\gamma||_a^b \colon T_{\gamma(a)}M \to T_{\gamma(b)}M$. If the Lie group G acts on the Riemannian manifold M and $p \in M$, then we write $G \cdot p$ for the orbit through p. Further, we denote the isotropy group at p by G_p. In the following, every Riemannian manifold will be endowed with the Levi-Civita connection ∇ and we write R for the Riemannian curvature tensor. For each symmetric space of the compact type the metric is understood to be induced by a negative multiple of the Killing form on the Lie algebra of the isometry group.

2 Preliminaries

2.1 Basics

Let M be a Riemannian manifold.

Definition 2.1 *For $p \in M$ and $y \in T_pM$ The curvature operator R_y in direction y is given by the self adjoint linear map*

$$R_y \colon T_pM \to T_pM, \quad x \mapsto R(x,y)y.$$

Further, let $N \subseteq M$ be a submanifold and let $p \in N$.

Definition 2.2 *For $\xi \in \nu_p N$, the shape operator A_ξ in p in normal direction ξ is defined to be the self adjoint linear map*

$$A_\xi \colon T_pN \to T_pN, \quad x \mapsto -(\nabla_x \xi)^\top.$$

Let $\gamma\colon (-a,a) \to M$ be a geodesic with $\gamma(0) = p$ and $\gamma'(0) \in \nu_p N$. Consider the geodesic variation $\gamma_s(t)$ with $\gamma_0 = \gamma$ and $\gamma_s(0) \in N$. If in addition also $\gamma_s'(0) \in \nu_{\gamma_s(0)} N$, for all s, then the Jacobi vector field Y along γ, defined by the geodesic variation $\gamma_s(t)$, satisfies the initial conditions

$$Y(0) \in T_{\gamma(0)} N,$$

$$\frac{\nabla}{dt} Y(0) + A_{\gamma'(0)} Y(0) \in \nu_{\gamma(0)} N.$$

A Jacobi field satisfying these initial conditions is called an *N-Jacobi field.*

Definition 2.3 *Let N be a submanifold of a Riemannian manifold M. Further let $p \in N$ and $\xi \in \nu_p N$. Then the* mean curvature *of N in the point p in normal direction ξ, denoted by H_ξ, is defined to be the trace of the shape operator A_ξ.*

Remark 2.4 *Usually the mean curvature is defined slightly different in the literature: one additionally devides the trace of the shape operator through the dimension of the submanifold N, but since this factor will play no role for our considerations, we will ignore it for brevity.*

Definition 2.5 *A submanifold $N \subseteq M$ of a Riemannian manifold M is called* minimal *if for every $p \in N$ and every non-vanishing normal vector $\xi \in \nu_p N$ the trace of the shape operator in normal direction ξ vanishes, i.e. $tr(A_\xi) = 0$.*

2.2 Hyperpolar actions

Consider the isometric action of the compact Lie group G on the Riemannian manifold M. Let (G_p), for $p \in M$, be the conjugacy class of the isotropy group G_p of the G-action. We call (G_p) the *isotropy type* of the orbit $G{\cdot}p$. For closed subgroups $K, H \subseteq G$ we have the following partial ordering on the set of conjugacy classes of G: If K is conjugated to a subgroup of H we set $(K) \leq (H)$.

Definition 2.6 *For $p \in M$ the orbit $G \cdot p$ is called a* principal orbit *if there is a neighbourhood U of p such that $(G_p) \leq (G_q)$, for all $q \in U$. The point $p \in M$ is called* regular *is $G \cdot p$ is a principal orbit.*

Definition 2.7 *An isometric action of a compact Lie group G on a connected complete Riemannian manifold M is called* polar *if there exists a connected closed embedded submanifold $\Sigma \subseteq M$, called a* section*, that meets every orbit of the G-action on M and is perpendicular to every orbit it meets. The G-action is called* hyperpolar *if there exists a section that is flat in the induced metric.*

Let us state some properties of G-manifolds that admit sections:

Theorem 2.8 *Let the isometric action of the compact Lie group G on the Riemannian manifold M be polar and let Σ be a section. Then*

1. *Σ is totally geodesic.*

2. *There is a unique section through every regular point $p \in M$ which is given by $\Sigma_p = Exp_p(\nu_p G \cdot p)$, where $\nu_p G \cdot p$ denotes the normal space to the orbit $G \cdot p$ in the point p.*

3. *Every G-equivariant normal vector field on a principal orbit is parallel with respect to the normal connection.*

4. *Principal orbits have constant principal curvatures with respect to parallel normal unit vector fields.*

Proof. [PT1], page 86. □

For a principal orbit $G \cdot p$ and $v \in \nu_p G \cdot p$, a G-equivariant normal field ξ on $G \cdot p$ is defined by $\xi(g \cdot p) = (dg)_p(v)$.

Because of the third statement of the previous theorem, for a polar G-action on M and a regular point $p \in M$ the shape operator of the principal orbit $G \cdot p$ in normal direction ξ is given by

$$A_\xi X = -\nabla_X \xi.$$

2.3 Isotropy actions

In the following M is an irreducible, simply-connected, Riemannian symmetric space of compact type. The reason for restricting our considerations to simply-connected spaces lies in the fact that in such spaces all orbits of maximal volume are principal and no exceptional orbits occur [Con]. First we recall some basic properties associated to M:

Since the identity component of the isometry group of M, denoted by G, acts transitively on M, we can identify $M = G/K$, where K is the compact isotropy group of a point $p \in M$. The identification is given by $G/K \to M$, $g \cdot K \mapsto g \cdot p$. The metric on M is induced by the Killing form B of the Lie algebra of G, which we denote by $\mathfrak{g}$. We have the well-known Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{m}$ is a vector space of the same dimension as M. We can identify $\mathfrak{m}$ with the tangent space T_pM via the isomorphism $\mathfrak{m} \to T_pM$, $X \mapsto \overline{X}(p)$, where $\overline{X}$ is the induced fundamental vector field of X, given by

$$\overline{X}(q) = \frac{d}{dt}\bigg|_{t=0} \exp(tX) \cdot q,$$

for $q \in M$. We also recall the Cartan relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

The Riemannian curvature tensor in $p = eK$ is given by

$$R(X,Y)Z = -[[X,Y],Z],$$

with $X, Y, Z \in \mathfrak{m}$. We also recall that the Cartan decomposition can be characterized via Killing vector fields: If we identify $\mathfrak{g}$ with the Lie algebra of Killing vector fields on M by mapping each element of $\mathfrak{g}$ to its fundamental vector field, then we get

$$\mathfrak{k} \cong \{X \mid X \text{ is a Killing vector field on } M \text{ with } X(eK) = 0\},$$
$$\mathfrak{m} \cong \{X \mid X \text{ is a Killing vector field on } M \text{ with } (\nabla X)(eK) = 0\}.$$

Proposition 2.9 *Let M be an m-dimensional symmetric space and J a Jacobi field along the geodesic γ with $\langle J(t), \gamma'(t)\rangle = 0$. Further let $\{A_i\}$, for $i = 1, \ldots, m-1$, be an orthogonal basis of $\gamma'(0)^\perp$, such that all elements of this basis are eigenvectors of the operator $R_{\gamma'(0)} \colon \gamma'(0)^\perp \to \gamma'(0)^\perp$ with eigenvalues λ_i. Then*

$$J(t) = \sum_{i=1}^{m-1} (\delta_i \alpha_{\lambda_i} + \rho_i \beta_{\lambda_i}) \gamma||_0^t A_i,$$

where the coefficients δ_i, ρ_i are determined by the initial conditions

$$J(0) = \sum_{i=1}^{m-1} \rho_i A_i, \quad \frac{\nabla}{dt} J(0) = \sum_{i=1}^{m-1} \delta_i A_i,$$

and α_{λ_i}, β_{λ_i} are given by

$$\alpha_{\lambda_i} = \begin{cases} \dfrac{\sin(\sqrt{\lambda_i} t)}{\sqrt{\lambda_i}} & , \text{ if } \lambda_i > 0, \\ t & , \text{ if } \lambda_i = 0, \\ \dfrac{\sinh(\sqrt{-\lambda_i} t)}{\sqrt{-\lambda_i}} & , \text{ if } \lambda_i < 0, \end{cases}$$

$$\beta_{\lambda_i} = \begin{cases} \cos(\sqrt{\lambda_i} t) & , \text{ if } \lambda_i > 0, \\ 1 & , \text{ if } \lambda_i = 0, \\ \cosh(\sqrt{-\lambda_i} t) & , \text{ if } \lambda_i < 0. \end{cases}$$

An important tool for this thesis will be the real root space decomposition for symmetric spaces of compact type. For this, let $\mathfrak{a} \subseteq \mathfrak{m}$ be a maximal Abelian subalgebra, meaning that every Abelian subalgebra of $\mathfrak{m}$ containing $\mathfrak{a}$ is already equal to $\mathfrak{a}$. The rank of M is defined to be the dimension of $\mathfrak{a}$. Let for the remaining part of this thesis $\dim(\mathfrak{a}) = n$. We set

$$\mathfrak{g}_\alpha = \{Z \in \mathfrak{g}^{\mathbb{C}} \mid [H, Z] = \mathrm{i}\alpha(H)Z, \text{ for all } H \in \mathfrak{a}\}$$

and define the *restricted roots* $\Delta = \{\alpha_1, \dots, \alpha_l\}$ to be the set of one-forms $\alpha_i \colon \mathfrak{a} \to \mathbb{R}$ such that $\mathfrak{g}_{\alpha_i} \neq \{0\}$ and $\alpha_i \neq 0$. It is important to remark that, contrary to root systems of semisimple Lie algebras, for symmetric spaces also non-reduced restricted root systems occur. Non-reduced root systems have the property that there exits a root α such that an integer multiple of α, other than $-\alpha$, is also contained in the root system. In particular the non-reduced root system BC_q occurs, as for example in the quaternionic Grassmannian

$$\frac{Sp(2p + 2q)}{Sp(2p) \times Sp(2q)},$$

for $p > q$. An overview over the restricted root systems associated to symmetric spaces can be found in [Bum], page 264. Let $\Delta_+ \subseteq \Delta$ denote a choice of positive roots, i.e. a subset with the property $\Delta = \Delta_+ \cup (-\Delta_+)$ and $\Delta_+ \cap (-\Delta_+) = \emptyset$. Then the real root space decomposition is given by the following theorem, where $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$:

Theorem 2.10 *Let $\mathfrak{k}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ and $\mathfrak{m}_\alpha = \mathfrak{m} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Then the following holds:*

1. *There are the direct sum decompositions*

$$\mathfrak{k} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \sum_{\alpha \in \Delta_+} \mathfrak{k}_\alpha \quad \text{and} \quad \mathfrak{m} = \mathfrak{a} \oplus \sum_{\alpha \in \Delta_+} \mathfrak{m}_\alpha,$$

   which are orthogonal with respect to the Killing form on $\mathfrak{g}$.

2. *$\mathfrak{k}_\alpha$ and $\mathfrak{m}_\alpha$ are isomorphic: For each $X \in \mathfrak{k}_\alpha$ there exists a unique $Y \in \mathfrak{m}_\alpha$ with $[H, X] = -\alpha(H)Y$ and $[H, Y] = \alpha(H)X$, for all $H \in \mathfrak{a}$.*

3. *The root spaces are given by*

$$\mathfrak{k}_\alpha = \{X \in \mathfrak{k} \mid [H, [H, X]] = -\alpha(H)^2 X, \text{ for all } H \in \mathfrak{a}\},$$
$$\mathfrak{m}_\alpha = \{X \in \mathfrak{m} \mid [H, [H, X]] = -\alpha(H)^2 X, \text{ for all } H \in \mathfrak{a}\}.$$

We will denote by m_α the multiplicity of the root $\alpha \in \Delta_+$ which is defined to be the dimension of $\mathfrak{m}_\alpha$.

We come to the definition of the central notion of this thesis:

Definition 2.11 *For the symmetric space $M = G/K$ the action of K on G/K, given by $k \cdot gK = kgK$, is called* isotropy action.

Proposition 2.12 *Consider the isotropy action on $M = G/K$.*

1. *The action of K on G/K is hyperpolar.*

2. *A section of the action of K on G/K is given by* $\Sigma = Exp_{eK}(\mathfrak{a})$.

Now we will describe the tangent spaces of the orbits of the isotropy action for which we will make use of the root space decomposition in symmetric spaces. The following result was obtained by [Ver] but we prove it differently using the proof in [GT] for a similar result for Hermann actions and applying it to the special case of an isotropy action. In the proof we try to use a similar notation as in [GT] to make the proofs easier to compare.

For the remaining part of this section, let $w \in \mathfrak{a}$ and $p = \mathrm{Exp}_{eK}(w)$. In particular we have the geodesic $\gamma(t) = \mathrm{Exp}_{eK}(tw) \in \Sigma$.

Proposition 2.13 *The tangent space of the orbit $K \cdot p$ is given by*

$$T_pK \cdot p = \gamma||_0^1 \left(\sum_{\alpha \in \Delta_+ :\, \alpha(w) \notin \pi\mathbb{Z}} \mathfrak{m}_\alpha \right).$$

Proof. We have $T_pK \cdot p = \{X(p) \mid X \in \mathfrak{k}\}$. For $\alpha \in \Delta_+$ we denote by $\{X_i^\alpha\}_{i \in I_\alpha}$ an orthonormal basis of $\mathfrak{m}_\alpha$. Further we denote by E_i^α the parallel vector fields along γ with $E_i^\alpha(0) = X_i^\alpha(eK)$. For $X \in \mathfrak{k}$ let $Y = X|_\gamma$. As a restriction of a Killing vector field to a geodesic, Y is a Jacobi field along γ. Since $Y(t)$ is tangent to the orbit through $\gamma(t)$ for all t, it follows from the description of Jacobi fields in symmetric spaces.

$$Y(t) = \sum_{\alpha \in \Delta_+} \sum_{i \in I_\alpha} \left(c_{i,\alpha} \sin\left(\alpha(w)t\right) + d_{i,\alpha} \cos\left(\alpha(w)t\right)\right) E_i^\alpha(t). \quad (1)$$

Further, let $\{Z_i^\alpha\}_{i \in I_\alpha}$ be the orthonormal basis of $\mathfrak{k}_\alpha$ related to $\{X_i^\alpha\}$ by $[Z_i^\alpha, u] = \alpha(u) X_i^\alpha$, for all $u \in \mathfrak{a}$. We have

$$[\mathfrak{k}_\alpha, u] = \begin{cases} \mathfrak{m}_\alpha & , \text{ if } \alpha(u) \neq 0, \\ 0 & , \text{ if } \alpha(u) = 0. \end{cases}$$

For $X \in \mathfrak{k}$ we get

$$\begin{aligned}
Y(t) &= X(\mathrm{Exp}_{eK}(tw)) \\
&= \frac{d}{ds}\Big|_{s=0} \exp(sX) \cdot \mathrm{Exp}_{eK}(tw) \\
&= \frac{d}{ds}\Big|_{s=0} \exp(sX)\exp(tw)K \\
&= \frac{d}{ds}\Big|_{s=0} \exp\left(\mathrm{Ad}_{\exp(sX)}(tw)\right) K \\
&= d(\mathrm{Exp})_{tw}\left(\frac{d}{ds}\Big|_{s=0} \mathrm{Ad}_{\exp(sX)}(tw)\right) \\
&= d(\mathrm{Exp})_{tw}\left(d(\mathrm{Ad})_e\left(\frac{d}{ds}\Big|_{s=0} \exp(sX)\right)(tw)\right) \\
&= d(\mathrm{Exp})_{tw}\left(\mathrm{ad}\left(d\exp_0(X)\right)(tw)\right) \\
&= d(\mathrm{Exp})_{tw}\left(t[X, w]\right).
\end{aligned}$$

This implies for $t = 1$, together with $\mathfrak{k} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \sum_{\alpha\in\Delta_+} \mathfrak{k}_\alpha$, that

$$\begin{aligned}
T_pK \cdot p &= d(\mathrm{Exp})_w\left([\sum_{\alpha\in\Delta_+} \mathfrak{k}_\alpha, w]\right) \\
&= d(\mathrm{Exp})_w\left(\sum_{\alpha\in\Delta_+} [\mathfrak{k}_\alpha, w]\right) \\
&= d(\mathrm{Exp})_w\left(\sum_{\alpha\in\Delta_+ :\, \alpha(w)\neq 0} \mathfrak{m}_\alpha\right).
\end{aligned}$$

Now, let $\alpha(w) \neq 0$ and $\tilde{X}_\alpha \in \mathfrak{m}_\alpha$. Write $\tilde{X}_\alpha = \sum_{i\in I_\alpha} \lambda_{i,\alpha} X_i^\alpha$, with $\lambda_{i,\alpha} \in \mathbb{R}$. We use the coefficients $\lambda_{i,\alpha}$ to define $X_\alpha \in \mathfrak{k}_\alpha$ via $X_\alpha = \sum_{i\in I_\alpha} \lambda_{i,\alpha} Z_i^\alpha$. We showed that the Jacobi field $Y_\alpha = X_\alpha|_\gamma$

along γ is determined by $Y_\alpha(0) = 0$ and $\frac{\nabla}{dt} Y_\alpha(0) = [X_\alpha, w] = \alpha(w)\tilde{X}_\alpha$. From the first initial condition and the general form (1) follows $d_{i,\alpha} = 0$. Hence Y_α has to be of the form

$$Y_\alpha(t) = \sum_{i \in I_\alpha} c_{i,\alpha} \sin(\alpha(w)t) E_i^\alpha(t).$$

Differentiating yields

$$\frac{\nabla}{dt} Y_\alpha(t) = \sum_{i \in I_\alpha} c_{i,\alpha} \left\{ \cos(\alpha(w)t)\alpha(w) E_i^\alpha(t) \right. \\ \left. + \sin(\alpha(w)t) \underbrace{\frac{\nabla}{dt} E_i^\alpha(t)}_{=0} \right\}.$$

Therefore we get from the second initial condition from a comparison of coefficients that $c_{i,\alpha} = \lambda_{i,\alpha}$ and hence

$$\begin{aligned} Y_\alpha(t) &= \sum_{i \in I_\alpha} \lambda_{i,\alpha} \sin(\alpha(w)t) E_i^\alpha(t) \\ &= \sin(\alpha(w)t) \gamma||_0^1 \tilde{X}_\alpha. \end{aligned}$$

It follows that $Y_\alpha(1)$ vanishes if and only if $\alpha(w) \in \pi\mathbb{Z}$. Hence $T_pK \cdot p$ is the parallel displacement of $\sum_{\alpha \in \Delta_+ \colon \alpha(w) \notin \pi\mathbb{Z}} \mathfrak{m}_\alpha$ along γ.

□

The previous theorem motivates the following definition: The *diagram* of M is defined to be

$$D(G/K) = \{X \in \mathfrak{a} \mid \exists \alpha \in \Delta_+ \colon \alpha(X) \in \pi\mathbb{Z}\}.$$

All elements of the diagram are mapped by the exponential map to points on singular orbits, whereas all elements in $\mathfrak{a} \setminus D(G/K)$ are mapped by the exponential map to points on principal orbits. The connected components of $\mathfrak{a} \setminus D(G/K)$ are called *generalized Weyl chambers*. We call the points in $\mathfrak{a} \setminus D(G/K)$ *regular* and the points in $D(G/K)$ *singular*. Let W be a generalized Weyl chamber. Then each orbit of K intersects $\text{Exp}_{eK}(\overline{W})$ at a unique point, where $\overline{W}$ denotes the closure of W. To avoid lengthy formulations, we will from now on write Weyl chamber instead of generalized Weyl chamber. Given a choice of simple roots Δ_+, where $|\Delta_+| = \text{rank}(M) = n$, there is one distinguished Weyl chamber $\mathcal{W}_n$, with $0 < \langle \alpha, x \rangle < \pi$, for all $\alpha \in \Delta_+$ and all $x \in \mathcal{W}_n$. We call $\mathcal{W}_n$ the *dominant generalized Weyl chamber* or dominant Weyl chamber. This particular Weyl chamber has the property that its closure contains the origin of $\mathfrak{a}$.

The following theorem was given in [Ver], but we reprove it using the method in [GT].

Proposition 2.14 *Let $u \in \mathfrak{a}$ and $\alpha \in \Delta_+$ with $\alpha(w) \notin \pi\mathbb{Z}$. Then we have*

$$A_{u(p)}(v) = -\alpha(u)\cot(\alpha(w))v,$$

for all $v \in \gamma||_0^1(\mathfrak{m}_\alpha)$.

Proof. Define the geodesic $c(s) = \text{Exp}_{eK}(w + su)$ through p. We have

$$\begin{aligned} c'(0) &= \frac{d}{ds}\bigg|_{s=0} \text{Exp}_{eK}(w + su) \\ &= \frac{d}{ds}\bigg|_{s=0} \exp(w + su)K \\ &= \frac{d}{ds}\bigg|_{s=0} \exp(su)\exp(w)K \\ &= \frac{d}{ds}\bigg|_{s=0} \exp(su) \cdot p \\ &= u(p). \end{aligned}$$

For $s \in [0,1]$ let $\gamma_s(t) = \mathrm{Exp}_{eK}(t(w+su))$ be a geodesic variation with $\gamma_0 = \gamma$. For $\alpha \in \Delta_+$ with $\alpha(w) \neq \pi\mathbb{Z}$, let $\tilde{X}_\alpha \in \mathfrak{m}_\alpha$. Define $X_\alpha \in \mathfrak{k}_\alpha$ to be the vector related to $\tilde{X}_\alpha$, i.e. $[H, X] = -\alpha(H)\tilde{X}$, for all $H \in \mathfrak{a}$. Let $Y_{s,\alpha}(t) = X_\alpha(\gamma_s(t))$ be the Jacobi field along γ_s with initial values $Y_{s,\alpha}(0) = 0$ and $\frac{\nabla}{dt}Y_{s,\alpha}(0) = [X_\alpha, w + su] = \alpha(w+su)\tilde{X}_\alpha$. As in the proof of the previous lemma, we get from the general description of Jacobi fields the following:

$$Y_{s,\alpha}(t) = \sin(\alpha(w + su)t)\gamma_s||_0^1\tilde{X}_\alpha.$$

Consider $Y_\alpha(s) := Y_{s,\alpha}(1) = X_\alpha(c(s))$. Since $c(s)$ is a geodesic contained in the section $\Sigma = \mathrm{Exp}_{eK}(\mathfrak{a})$, it follows that $c(s)$ is orthogonal to the orbit $K \cdot c(s)$. Further, $Y_\alpha(s)$ is tangent to $K \cdot c(s)$. Hence, $Y_\alpha(s)$ is a $K \cdot p$ - Jacobi field and therefore $\frac{\nabla}{ds}Y_\alpha(0) + A_{u(p)}(Y_\alpha(0)) \in \nu_p K \cdot p$. The initial values of $Y_\alpha(s)$ are given by

$$Y_\alpha(0) = \sin(\alpha(w))\gamma||_0^1\tilde{X}_\alpha,$$

$$\frac{\nabla}{ds}Y_\alpha(0) = \alpha(u)\cos(\alpha(w))\gamma||_0^1\tilde{X}_\alpha.$$

In particular, both $Y_\alpha(0)$ and $\frac{\nabla}{ds}Y_\alpha(0)$ are tangent to $K \cdot p$ and from the relation $\frac{\nabla}{ds}Y_\alpha(0) + A_{u(p)}(Y_\alpha(0)) \in \nu_p K \cdot p$ together with $\alpha(w) \neq \pi\mathbb{Z}$ follows

$$A_{u(p)}(\gamma||_0^1\tilde{X}_\alpha) = -\alpha(u)\cot(\alpha(w))\gamma||_0^1\tilde{X}_\alpha.$$

□

Corollary 2.15 *The mean curvature of the principal orbit $K \cdot p$ in the normal direction $u(p) \in \mathfrak{a}$ is given by*

$$H_{u(p)}(w) = -\sum_{\alpha \in \Delta_+} m_\alpha \alpha(u)\cot(\alpha(w)).$$

Proof. The mean curvature of $K \cdot \mathrm{Exp}_{eK}(p)$ in the normal direction $u(p) \in \mathfrak{a}$ is given by the trace of $A_{u(p)}$, which can be directly obtained from the previous proposition.

□

Motivated by the proof of Lemma 3.2 in [HSTT], we define the function $\Phi\colon \mathfrak{a} \to \mathbb{R}$ by

$$\Phi(x) = -\sum_{\alpha\in\Delta_+\colon\, \alpha(w)\notin\pi\mathbb{Z}} \log\left(|\sin(\alpha(x))|^{m_\alpha}\right).$$

Lemma 2.16 *Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathfrak{a}$. Then the mean curvature of the principal orbit $K \cdot p$ in normal direction $e_i(p)$ is given by*

$$H_{e_i}(w) = \left(\nabla\Phi(w)\right)_i.$$

Proof. Partial differentiation with respect to the standard coordinates of $\mathfrak{a}$ yields

$$\frac{\partial}{\partial x_i}\Phi(x) = -\sum_{\alpha\in\Delta_+\colon\, \alpha(w)\notin\pi\mathbb{Z}} m_\alpha \cot(\alpha(x))\alpha(e_i).$$

The claim follows from the previous corollary.

□

We can also write

$$\Phi(x) = -\log\left(\prod_{\alpha\in\Delta_+\colon\, \alpha(w)\notin\pi\mathbb{Z}} |\sin(\alpha(x))|^{m_\alpha}\right),$$

which motivates the following definition:

Definition 2.17 *For $w \in \mathfrak{a}$ we set*

$$\vartheta_\Delta(w) = \prod_{\alpha\in\Delta_+} |\sin(\alpha(w))|^{\mathfrak{m}_\alpha}.$$

Remark 2.18 *The function ϑ_Δ has originally been introduced in [CNV]. Obviously $\vartheta_\Delta(w)$ vanishes if and only if $w \in D(G/K)$.*

In the setting of Lemma 2.16, we get the relation

$$H_{e_i}(w) = \left(-\nabla \log\left(\vartheta_\Delta(w)\right)\right)_i .$$

Lemma 2.19 *Let w be a point in the interior of a Weyl chamber. Then the orbit $K \cdot p$ is minimal if and only if w is a critical point of ϑ_Δ, i.e. $\nabla\vartheta_\Delta(w) = 0$.*

Proof. Since w is in the interior of a Weyl chamber, it follows $\vartheta_\Delta(w) \neq 0$. The claim follows from

$$\begin{aligned} H_{e_i}(w) &= \left(-\nabla \log\left(\vartheta_\Delta(w)\right)\right)_i \\ &= -\frac{\partial}{\partial x_i} \log\left(\vartheta_\Delta(w)\right) \\ &= -\frac{1}{\vartheta_\Delta(w)} \frac{\partial}{\partial x_i} \vartheta_\Delta(w) \\ &= -\frac{1}{\vartheta_\Delta(w)} \left(\nabla \vartheta_\Delta(w)\right)_i . \end{aligned}$$

□

Theorem 2.20 *There exists a unique principal orbit of the isotropy action on M that is a minimal submanifold of M. In particular, in each Weyl chamber there exists a unique point $w \in \mathfrak{a}$ such that $K \cdot Exp_{eK}(w)$ is the unique minimal orbit.*

Proof. [HSTT].

□

The function ϑ_Δ assigns, up to a constant, to each point $w \in \mathfrak{a}$ the volume of the orbit $K \cdot \mathrm{Exp}_{eK}(w)$, as the following theorem states:

Proposition 2.21 *For $w \in \mathfrak{a}$ we have*

$$Vol(K \cdot Exp_{eK}(w)) = c \cdot \vartheta_\Delta(w),$$

where $c \in \mathbb{R}$ is a constant.

Proof. [CNV], Theorem 3.5.

□

Remark 2.22 *In [CNV], Theorem 3.5., it is shown that*

$$c = \frac{Vol(M)}{\int_{\mathcal{W}} \vartheta_\Delta(X)dX}.$$

In particular c also depends on the scaling factor of the Killing form, which determines the Riemannian metric.

2.4 Formulation of the Problem

The stated results of the previous section lead to the following very natural problem:

Problem *Given a simply-connected symmetric space of compact type M and its restricted root system Δ. Find the unique point in each Weyl chamber corresponding to the unique minimal principal orbit of the isotropy action.*

For some symmetric spaces of low rank there are already solutions to the posed problem given in the literature. In [CNV] for example the rank two spaces

$$\frac{SU(3)}{SO(3)}, \quad \frac{Sp(2)}{U(2)}, \quad \frac{SU(4)}{S(U(2)\times U(2))}, \quad \frac{G_2}{SO(4)}$$

were investigated, having root systems A_2, C_2, C_2, G_2, respectively. As it can be seen in [CNV] the unique crtitical point of ϑ_{A_2} in the interior of the generalized Weyl chamber of A_2, which is a regular triangle, is given by the centroid. The second and third space listed above have C_2 as restricted root system but the multiplicities of the roots differ for each of these spaces. Although they have the same generalized Weyl chambers, their unique critical points in the interior of the Weyl chamber do not coincide. But still, these two critical points have in common that they lie both on the symmetry axis of the Weyl chamber, which is an isoceles triangle in this case. Concerning the relation of the unique critical point of ϑ_Δ to the symmetry of the Weyl chamber, it is important to mention that for the restricted root system BC_2, whose generalized Weyl chambers are isoceles triangles, the unique critical point of ϑ_{BC_2} doesn't even lie an the symmetry axis of the Weyl chamber.

In this thesis we will investigate the position of the unique critical point of ϑ_Δ in the interior of a Weyl chamber for the classical simply-laced restricted root systems A_n and D_n.

For A_n we will give a closed form for the unique critical point for a particular Weyl chamber that is valid for arbitrary rank n. Having found the unique critical points in one Weyl chamber, the unique critical points in all the other Weyl chambers can be obtained by the action of the generalized Weyl group.

In the case of the restricted root system D_n, we give a theorem that allows us to find the unique critical point in one certain Weyl chamber by determining the roots of a polynomial of degree $\left\lfloor \frac{n}{2} \right\rfloor - 1$. Again, the unique critical points in all the other Weyl chambers can be obtained by the action of the generalized Weyl group.

The classes of simply-connected symmetric spaces with restricted root systems A_n and D_n are given in the following table (cf. [Hel], [Bum]):

Symmetric spaces of type I:

Cartan's Class	G	K	dimension	rank	Δ
AI	SU(n)	SO(n)	$\frac{(n-1)(n+2)}{2}$	$n-1$	A_{n-1}
AII	SU$(2n)$	Sp$(2n)$	$(n-1)(2n+1)$	$n-1$	A_{n-1}
DI	SO$(2n)$	S(O$(n)\times$ O(n))	n^2	n	D_n
BII	SO$(2n+1)$	SO$(2n)$	$2n$	1	A_1
DII	SO$(2n)$	SO$(2n-1)$	$2n-1$	1	A_1
EIV	E_6	F_4	26	2	A_2

Symmetric spaces of type II:

Lie group	dimension	rank	Δ
SU(n+1)	$n(n+2)$	n	A_n
Spin(2n)	$n(2n-1)$	n	D_n

3 A_n-Problem

3.1 Result

The root system A_n can be considered as a subset of $\mathbb{R}^{n+1}$ given by

$$A_n = \{e_i - e_j \mid 1 \leq i, j \leq n+1,\ i \neq j\},$$

where $e_1, \ldots, e_{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$. In particular, A_n is contained in the hyperplane through the origin which is orthogonal to the diagonal vector $e_1 + \cdots + e_{n+1}$. A choice of positive roots is given by

$$(A_n)_+ = \{e_i - e_j \mid 1 \leq i < j \leq n+1\},$$

containing the simple roots $\alpha_1, \ldots, \alpha_n$, with $\alpha_i = e_i - e_{i+1}$.

We consider the function

$$\vartheta_{A_n}(x_1, \ldots, x_{n+1}) = \prod_{\alpha \in (A_n)_+} \sin\left(\langle \alpha, (x_1, \ldots, x_{n+1}) \rangle\right)^{m_\alpha},$$

where $x_1, \ldots, x_{n+1}$ are standard coordinates and m_α is the multiplicity of the root α. Since the Dynkin diagram of the root system A_n is simply-laced, all roots have the same multiplicity.

In the following, we will study regular solutions of the equation

$$\nabla \vartheta_{A_n}(x_1, \ldots, x_{n+1}) = 0.$$

The main result of this section is the following theorem.

Theorem 3.1 *Let $M = G/K$ be a simply-connected symmetric space of compact type with rank$(M) = n$ and restricted root system A_n.*

1. *The point*

$$L_n = \frac{\pi}{2+2n} \sum_{k=1}^{n} k(n-k+1)\alpha_k$$

is contained in the dominant generalized Weyl chamber $\mathcal{W}_n$.

2. *The unique minimal principal orbit of the isotropy action on G/K is given by*

$$K \cdot Exp_{eK}(L_n).$$

Proof. We define the function

$$\tilde{\vartheta}_{A_n}(x_1, \ldots, x_{n+1}) = \prod_{\alpha \in (A_n)_+} \sin\left(\langle \alpha, (x_1, \ldots, x_{n+1}) \rangle\right).$$

Because the multiplicities m_α are equal for all roots, it follows that

$$\begin{aligned} \nabla \vartheta_{A_n}(x_1, \ldots, x_{n+1}) =& m_\alpha \tilde{\vartheta}_{A_n}(x_1, \ldots, x_{n+1})^{m_\alpha - 1} \\ & \cdot \nabla \tilde{\vartheta}_{A_n}(x_1, \ldots, x_{n+1}). \end{aligned}$$

Therefore ϑ_{A_n} and $\tilde{\vartheta}_{A_n}$ have the same critical point in a given Weyl chamber.

Lemma 3.2 *The point L_n is regular.*

Proof. Let $1 \leq i < j \leq n+1$. Assume that L_n is singular, i.e. that there exists $m \in \mathbb{Z}$ with

$$\langle e_i - e_j, L_n \rangle = m\pi.$$

Then the relation

$$\begin{aligned} \langle e_i - e_j, L_n \rangle &= \frac{\pi}{2+2n} \sum_{k=1}^{n} k(n-k+1) \langle e_i - e_j, e_k - e_{k+1} \rangle \\ &= \frac{\pi}{2+2n} \sum_{k=1}^{n} k(n-k+1)(\delta_{i,k} - \delta_{i,k+1} - \delta_{j,k} + \delta_{j,k+1}) \\ &= \pi \frac{j-i}{1+n} \end{aligned}$$

implies $j-i=m(1+n)$. But since $1 \leq j-i \leq n$, such an integer m can not exist and we get a contradiction.

$\square$

Corollary 3.3 *The point L_n is contained in the generalized dominant Weyl chamber $\mathcal{W}_n$.*

Proof. For $1 \leq i < j \leq n+1$ the proof of the previous lemma implies

$$0 < \langle e_i - e_j, L_n \rangle < \pi.$$

$\square$

In the following, we will show that $\nabla \tilde{\vartheta}_{A_n}(L_n) = 0$, i.e. that L_n is the unique critical point of $\tilde{\vartheta}_{A_n}$ in $\mathcal{W}_n$. For this purpose we introduce some notation:

Definition 3.4 *Let $i \in \{1, \ldots, n+1\}$ and $k \in \{1, \ldots, n\}$. We define*

1) $R_{n,i}\,(x_1, \ldots, x_{n+1}) = \prod_{\substack{1 \leq k < l \leq n+1 \\ k,l \neq i}} \sin(x_k - x_l),$

2) $\zeta_n(k)\,(x_1, \ldots, x_n) = \cos(x_k) \prod_{i=1}^{k-1} \sin(x_i) \prod_{j=k+1}^{n} \sin(x_j),$

3) $V_{i,k} = \begin{cases} 1 & , \text{ for } k \geq i, \\ -1 & , \text{ for } k < i, \end{cases}$

4) $a_n(i,k)\,(x_1, \ldots, x_{n+1}) = \begin{cases} x_k - x_i & , \text{ for } k < i, \\ x_i - x_{k+1} & , \text{ for } k \geq i. \end{cases}$

To avoid lengthy notation, we will write

$$a_n(i,k)\,(x_1,\ldots,x_{n+1}) = a_n(i,k).$$

With the introduced notation, the i-th component of the gradient of $\tilde{\vartheta}_{A_n}(x_1,\ldots,x_{n+1})$ is given by

$$\begin{aligned}\frac{\partial}{\partial x_i}\tilde{\vartheta}_{A_n}(x_1,\ldots,x_{n+1}) =& R_{n,i}(x_1,\ldots,x_{n+1}) \\ & \cdot\left(\sum_{k=1}^{n} V_{i,k}\zeta_n(k)\,(a_n(i,1),\ldots,a_n(i,n))\right).\end{aligned}$$

Lemma 3.5 *We have*

$$a_n(i,k)(L_n) = \begin{cases} \dfrac{2k-2i+2}{2n+2}\pi & , \text{ for } k \geq i, \\ \\ \dfrac{2i-2k}{2n+2}\pi & , \text{ for } k < i. \end{cases}$$

Proof. The i-th component of L_n is given by

$$(L_n)_i = \frac{n-2i+2}{2n+2}\pi.$$

For $k \geq i$, we get

$$\begin{aligned} a_n(i,k)(L_n) &= (L_n)_i - (L_n)_{k+1} \\ &= \frac{2k-2i+2}{2n+2}\pi \end{aligned}$$

and for $k < i$ the relation

$$\begin{aligned} a_n(i,k)(L_n) &= (L_n)_k - (L_n)_i \\ &= \frac{2i-2k}{2n+2}\pi \end{aligned}$$

is valid.

□

The following two lemmas imply, that the first $\left\lfloor \frac{n}{2} \right\rfloor + 1$ components of $\nabla\tilde{\vartheta}_{A_n}(L_n)$ vanish.

Lemma 3.6 *Let* $i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$. *Then*

$$\sum_{k=1}^{2i-2} V_{i,k}\zeta_n(k)\left(a_n(i,1)(L_n),\dots,a_n(i,n)(L_n)\right) = 0.$$

Proof. Let $l \in \{1,\dots,i-1\}$. In particular, we have $l < i$. We will show that the l-th and the $2i-1-l$-th summand cancel each other. Since $2i-1-l \ge i$, we get

$$\begin{aligned} a_n(i,2i-1-l)(L_n) &= \frac{2(2i-1-l)-2i+2}{2+2n}\pi \\ &= \frac{2i-2l}{2+2n}\pi \\ &= a_n(i,l)(L_n). \end{aligned}$$

This implies

$$\begin{aligned} &\zeta_n(l)\left(a_n(i,1)(L_n),\dots,a_n(i,n)(L_n)\right) \\ &= \zeta_n(2i-1-l)\left(a_n(i,1)(L_n),\dots,a_n(i,n)(L_n)\right). \end{aligned}$$

Furthermore, we have $V_{i,2i-1-l} = -V_{i,l}$.

□

Lemma 3.7 *Let* $i \le \left\lfloor \frac{n}{2} \right\rfloor + 1$. *Then*

$$\sum_{k=2i-1}^{n} V_{i,k}\zeta_n(k)\left(a_n(i,1)(L_n),\dots,a_n(i,n)(L_n)\right) = 0.$$

Proof. First, let $i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and n even. We will show that for $l \in \{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - i + 1\}$ the $2i - 2 + l$-th and the $n + 1 - l$-th summand cancel each other, which implies the assertion. Because of $2i - 2 + l \geq 2i - 1 \geq i$, we get

$$a_n(i, 2i-2+l)(L_n) = \frac{2(2i-2+l) - 2i + 2}{2 + 2n}\pi = \frac{2i - 2 + 2l}{2 + 2n}\pi.$$

Since $n + 1 - l \geq n + 1 - (\left\lfloor \frac{n}{2} \right\rfloor - i + 1) = \left\lfloor \frac{n}{2} \right\rfloor + i + 1 \geq i$, we have

$$a_n(i, n-l+1)(L_n) = \frac{2(n + 1 - l) - 2i + 2}{2 + 2n}\pi = \frac{2n - 2l - 2i + 4}{2 + 2n}\pi.$$

Therefore,

$$a_n(i, 2i - 2 + l)(L_n) = \pi - a_n(i, n + 1 - l)(L_n),$$

which implies

$$\begin{aligned} \cos(a_n(i, 2i - 2 + l)(L_n)) &= \cos(\pi - a_n(i, n + 1 - l)(L_n)) \\ &= -\cos(a_n(i, n + 1 - l)(L_n)) \end{aligned}$$

and

$$\begin{aligned} \sin(a_n(i, 2i - 2 + l)(L_n)) &= \sin(\pi - a_n(i, n + 1 - l)(L_n)) \\ &= \sin(a_n(i, n + 1 - l)(L_n)). \end{aligned}$$

In particular we get

$$\begin{aligned} &\zeta_n(2i - 2 + l)\left(a_n(i, 1)(L_n), \ldots, a_n(i, n)(L_n)\right) \\ &= -\zeta_n(n + 1 - l)\left(a_n(i, 1)(L_n), \ldots, a_n(i, n)(L_n)\right). \end{aligned}$$

Because we further have $V_{i,2i-2+l} = V_{i,n+1-l}$, we showed that the $2i - 2 + l$-th and the $n + 1 - l$-th summand cancel each other.

Now, let $i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ and n odd. First we remark, that for $l \in \{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - i + 1\}$ the $2i - 2 + l$-th and the $n + 1 - l$-th summand cancel each other, which can be proved using the same argument as in 1. But for this case, it remains to consider $\zeta_n(\lfloor \frac{n}{2} \rfloor + i)\,(a_n(i,1)(L_n),\ldots,a_n(i,n)(L_n))$. We have

$$a_n\left(i, \left\lfloor \frac{n}{2} \right\rfloor + i\right)(L_n) = \frac{2(\lfloor \frac{n}{2} \rfloor + i) - 2i + 2}{2+2n}\pi = \frac{\pi}{2}$$

and therefore $\cos\left(a_n\left(i, \lfloor \frac{n}{2} \rfloor + i\right)(L_n)\right) = 0$, which implies

$$\zeta_n\left(\left\lfloor \frac{n}{2} \right\rfloor + i\right)(a_n(i,1)(L_n),\ldots,a_n(i,n)(L_n)) = 0.$$

□

Now, we show that the last $n - \lfloor \frac{n}{2} \rfloor$ components of $\nabla\tilde{\vartheta}_{A_n}(L_n)$ also vanish.

Lemma 3.8 *Let* $\lfloor \frac{n}{2} \rfloor + 1 < i \leq n+1$. *Then*

1. $a_n(i,j)(L_n) = a_n(n-i+2, n-j+1)(L_n)$,
2. $V_{i,j} = -V_{n-i+2,n-j+1}$.

Proof. To prove 1, we start by considering the case $j \geq i$. Here we have $n-j+1 < n-i+2$ and therefore

$$\begin{aligned} a_n(n-i+2, n-j+1)(L_n) &= \frac{2(n-i+2) - 2(n-j+1)}{2n+2}\pi \\ &= \frac{2j-2i+2}{2n+2}\pi \\ &= a_n(i,j)(L_n). \end{aligned}$$

In the case $j < i$ we get $n-j+1 \geq n-i+2$. Hence

$$\begin{aligned} a_n(n-i+2, n-j+1)(L_n) &= \frac{2(n-j+1) - 2(n-i+2) + 2}{2n+2}\pi \\ &= \frac{2i-2j}{2n+2}\pi \\ &= a_n(i,j)(L_n) \end{aligned}$$

and the first assertion follows.

To prove 2, we start with the case $j \geq i$. Here we have on one hand $V_{i,j} = 1$ and on the other hand $n - j + 1 < n - i + 2$, which implies $V_{n-i+2,n-j+1} = -1$. In the case $j < i$ we get $V_{i,j} = -1$ and, since $n - i + 2 \leq n - j + 1$, also $V_{n-i+2,n-j+1} = 1$.

□

Lemma 3.9 *Let* $\left\lfloor \frac{n}{2} \right\rfloor + 1 < i \leq n + 1$. *Then*

$$\sum_{k=1}^{n} V_{i,k} \zeta_n(k) \left(a_n(i,1)(L_n), \ldots, a_n(i,n)(L_n) \right) = 0.$$

Proof. First we note that $n - i + 2 < n - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + 2 = \left\lfloor \frac{n}{2} \right\rfloor + 2$ and hence $n - i + 2 \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$. For the case that n is even, we even have the stronger inequality $n - i + 2 < \left\lfloor \frac{n}{2} \right\rfloor + 1$. Using the previous lemma, we get

$$\begin{aligned}
&\sum_{k=1}^{n} V_{i,k} \zeta_n(k) \left(a_n(i,1)(L_n), \ldots, a_n(i,n)(L_n) \right)) \\
=&\sum_{k=1}^{n} \Big\{ - V_{n-i+2,n-k+1} \\
&\cdot \zeta_n(k) \left(a_n(n-i+2,n)(L_n), \ldots, a_n(n-i+2,1)(L_n) \right) \Big\} \\
=&\sum_{k=1}^{n} \Big\{ - V_{n-i+2,n-k+1} \\
&\cdot \zeta_n(n-k+1) \left(a_n(n-i+2,1)(L_n), \ldots, a_n(n-i+2,n)(L_n) \right) \Big\} \\
=&\sum_{k=1}^{n} \Big\{ - V_{n-i+2,k} \\
&\cdot \zeta_n(k) \left(a_n(n-i+2,1)(L_n), \ldots, a_n(n-i+2,n)(L_n) \right) \Big\} \\
&= 0.
\end{aligned}$$

In the last line we first changed the order of the summation, then used $n - i + 2 \leq \lfloor \frac{n}{2} \rfloor + 1$ together with Lemma 3.5 and 3.6.

$\square$

This concludes the proof of the main theorem .

$\square$

Corollary 3.10 *If L_{n-1} is the solution of*

$$\nabla \vartheta_{A_{n-1}}(x_1, \ldots, x_n) = 0$$

in the Weyl chamber $\mathcal{W}_{n-1}$, then the solution of

$$\nabla \vartheta_{A_n}(x_1, \ldots, x_{n+1}) = 0$$

in the Weyl chamber $\mathcal{W}_n$ is given by

$$L_n = Q_n + \frac{n}{n+1} L_{n-1},$$

with $Q_n = \frac{\pi}{2+2n} \sum_{k=1}^{n} k\alpha_k$.

Proof. We have

$$L_{n-1} = \frac{\pi}{2 + 2(n-1)} \sum_{k=1}^{n-1} k(n-k)\alpha_k,$$

which is an element in $\mathbb{R}^n$. The embedding of L_{n-1} into $\mathbb{R}^{n+1}$ is understood to be as follows:

$$L_{n-1} \mapsto \begin{pmatrix} L_{n-1} \\ 0 \end{pmatrix}$$

We get

$$
\begin{aligned}
Q_n + \frac{n}{n+1} L_{n-1} &= \frac{\pi}{2+2n} \sum_{k=1}^{n} k\alpha_k \\
&\quad + \frac{n}{n+1} \frac{\pi}{2+2(n-1)} \sum_{k=1}^{n-1} k(n-k)\alpha_k \\
&= \frac{\pi}{2+2n} \left(\sum_{k=1}^{n} k\alpha_k + \sum_{k=1}^{n-1} k(n-k)\alpha_k \right) \\
&= \frac{\pi}{2+2n} \sum_{k=1}^{n} k(n-k+1)\alpha_k \\
&= L_n.
\end{aligned}
$$

□

Remark 3.11 *In particular, we have a recursive formula for the solution, from one dimension to the other.*

3.2 Examples

The diagram below contains the coefficents of the simple roots $\alpha_1, \ldots, \alpha_n$ for the solution L_n:

$n = 1$:					$\frac{\pi}{4}$				
$n = 2$:				$\frac{2\pi}{6}$		$\frac{2\pi}{6}$			
$n = 3$:			$\frac{3\pi}{8}$		$\frac{4\pi}{8}$		$\frac{3\pi}{8}$		
$n = 4$:		$\frac{4\pi}{10}$		$\frac{6\pi}{10}$		$\frac{6\pi}{10}$		$\frac{4\pi}{10}$	
$n = 5$:	$\frac{5\pi}{12}$		$\frac{8\pi}{12}$		$\frac{9\pi}{12}$		$\frac{8\pi}{12}$		$\frac{5\pi}{12}$
$\ldots$:		$\ldots$		$\ldots$		$\ldots$		$\ldots$	

In the case $n = 2$, a Weyl chamber is an equilateral triangle and the minimal principal orbit is given by the centroid, as was already shown in [CNV].

4 D_n-Problem

4.1 Result

The root system D_n can be considered as a subset of $\mathfrak{a} \cong \mathbb{R}^n$ given by

$$D_n = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\},$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. A choice of positive roots is given by

$$(D_n)_+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\},$$

containing the simple roots $\alpha_1, \ldots, \alpha_n$, where

$$\alpha_i = \begin{cases} e_i - e_{i+1} & \text{for } i = 1, \ldots, n-1, \\ e_{n-1} + e_n & \text{for } i = n. \end{cases}$$

In this section, we study the function

$$\vartheta_{D_n}(x_1, \ldots, x_n) = \prod_{\alpha \in (D_n)_+} \sin\left(\langle \alpha, (x_1, \ldots, x_n) \rangle\right)^{m_\alpha},$$

where $x_1, \ldots, x_n$ are standard coordinates of $\mathbb{R}^n$ and m_α is the multiplicity of the root α. Since the Dynkin diagram for the root system D_n is simply-laced, all roots have the same multiplicity.

As in the previous section, we will study regular solutions of the equation

$$\nabla \vartheta_{D_n}(x_1, \ldots, x_n) = 0.$$

Before formulating the main result of this section, we give some definitions first:

Definition 4.1 *For $j \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$, let ξ_j be real numbers with $0 \leq \xi_i \leq 1$ and $\xi_0 = 1$. Define $L_n\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right) \in \mathfrak{a} \cong \mathbb{R}^n$ as follows:*

1. *For even n and $i \in \{1, \ldots, n\}$, we set*

$$\left(L_n\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right)\right)_i = \begin{cases} \frac{1}{2}\arccos\left(-\sqrt{\xi_{i-1}}\right) & , \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ \frac{1}{2}\arccos\left(\sqrt{\xi_{n-i}}\right) & , \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n. \end{cases}$$

2. *For odd n and $i \in \{1, \ldots, n\}$, we set*

$$\left(L_n\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right)\right)_i = \begin{cases} \frac{1}{2}\arccos\left(-\sqrt{\xi_{i-1}}\right) & , \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ \frac{\pi}{4} & , \text{ for } i = \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ \frac{1}{2}\arccos\left(\sqrt{\xi_{n-i}}\right) & , \text{ for } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n. \end{cases}$$

Remark 4.2 *For a more compact notation, we will frequently identify*

$$\xi := \left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right),$$
$$\bar{\xi} := \left(\bar{\xi}_1, \ldots, \bar{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}\right).$$

Definition 4.3 *Define the subset $\mathcal{L} \subseteq \mathfrak{a}$ by*

$$\mathcal{L} = \left\{ L_n(\xi) \;\middle|\; 1 > \xi_1 > \cdots > \xi_{\lfloor \frac{n}{2} \rfloor - 1} > 0 \right\}.$$

Definition 4.4 *Let $P_n \colon \mathbb{R} \to \mathbb{R}$ be the polynomial given by*

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} (-1)^k \mathcal{S}_k x^{\lfloor \frac{n}{2} \rfloor - 1 - k},$$

with

$$\mathcal{S}_k = \prod_{r=1}^{k} \frac{\sum_{l=r+1}^{\lfloor \frac{n}{2} \rfloor}(1 + 2n - 4l)}{\sum_{l=1}^{r}(1 + 2n - 4l)},$$

for $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.

The main result of this section is given by the following theorem.

Theorem 4.5 *Let $M = G/K$ be a simply-connected symmetric space of compact type with rank$(M) = n$ and restricted root system D_n.*

1. *The set $\mathcal{L}$ is contained in the dominant generalized Weyl chamber $\mathcal{W}_n$.*
2. *For $L_n\left(\overline{\xi}\right) \in \mathcal{L}$, the components of $\overline{\xi}$ are the roots of the polynomial P_n if and only if $K \cdot Exp_{eK}\left(L_n\left(\overline{\xi}\right)\right)$ is the unique minimal principal orbit of the isotropy action on G/K.*

Remark 4.6 1. *We remark the necessity of the assumption $L_n\left(\overline{\xi}\right) \in \mathcal{L}$ for this theorem. In particular we remark, that the proof of Conjecture 4.46 would enable us to omitt this assumption and at the same time imply that the unique minimal principal orbit is contained in $\mathcal{L}$. We explicitly emphasize, that the statement, that the unique minimal principal orbit is contained in $\mathcal{L}$, is not proven in this thesis, since Conjecture 4.46 is still open. But there is very strong evidence, partly also relying on computer simulations, that this conjecture must be true. For the precise consequences, of the validity of the mentioned conjecture, we refere the reader to the end of this section.*

2. *Since the proof of the main result of this section is rather lengthy, we want to give the reader a guideline for the most important steps in the proof:*

   After introducing an appropriate notation, oriented on the notation in the previous section, and restricting our considerations to the function $\tilde{\vartheta}_{D_n}$, which we get by setting all multiplicities in ϑ_{D_n} to 1, we decompose the i-th component of the gradient of $\tilde{\vartheta}_{D_n}$ into two factors; one denoted by $R_{n,i}$

(the R-factor) and the other one denoted by $K_{n,i}$ (the K-factor). The first factor in this decomposition will play a rather subsidiary role for our investigations. The second one, on the other hand, will be studied extensively.

We continue by introducing the premutation P_i, for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, and show in Lemma 4.16 that the action of this permutations on the variables in the argument of $K_{n,i}(x_1, \ldots, x_n)$ is equal to $-K_{n,(i+1)}(x_1, \ldots, x_n)$. Hence, this lemma gives us a method, to determine the K-factor of the $i+1$-th component of the gradient by knowing the K-factor of the i-th component. Lemma 4.16 also contains an analogue statement for going from the $n-i+1$-th component to the $n-i$-th one. At this point it is important to remark that Lemma 4.16 does not give a way to conclude from the K-factor of the $\lfloor \frac{n}{2} \rfloor$-th component of the gradient to any component in the lower part of the gradient.

In Definition 4.1 we give an ansatz (the L_n-ansatz), which reduces the problem from originally having n unknowns $x_1, \ldots, x_n$ to a problem with only $\lfloor \frac{n}{2} \rfloor - 1$ unknowns $\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}$. This ansatz can be considered as the key to the main result of this section.

One direct consequence of the L_n-ansatz is given by Lemma 4.17, which states that the first and the last component, as well as the middle component in the case of odd rank, vanish. Another consequence of the L_n-ansatz is Lemma 4.32, giving us the bridge to deduce from the K-factor of the $\lfloor \frac{n}{2} \rfloor$-th component to the K-factor of a component in the lower half of the gradient.

These developed tools enable us to do the following: After giving the explicit form of the K-factor of the second component in Proposition 4.20, which is a central result in the course of the proof, we deduce from this K-factor the explicit form of all remaining K-factors of the gradient, by *using the mentioned lemmas in the text above. This procedure leads to Proposition 4.33, and completes a major step of the proof of the main theorem.*

Proposition 4.33 translates the initial problem of determining the crtical point of the function ϑ_{D_n} to the problem of solving a non-linear system of $\lfloor \frac{n}{2} \rfloor - 1$ equations (NSE), each of which is given, up to a prefactor, by the function $F_i(n)(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1})$.

Introducing the polynomial $\hat{P}_n^{\bar{\xi}}$ in Definition 4.41, we obtain a way to reformulate the functions $F_i(n)$ in terms of this polynomial, cf. Remark 4.42.

The final crucial step in the proof of the main result is given by Lemma 4.43, which implies that the roots of the polynomial P_n, assuming that they are distinct and lie in the interval $(0, 1)$, are a solution of the (NSE) and hence give, together with the L_n-ansatz, the unique critical point of ϑ_{D_n}.

Proof of Theorem 4.5. We define the function

$$\tilde{\vartheta}_{D_n}(x_1, \ldots, x_n) = \prod_{\alpha \in (D_n)_+} \sin\left(\langle \alpha, (x_1, \ldots, x_n) \rangle\right).$$

Because the multiplicities m_α are equal for all roots, it follows that

$$\nabla \vartheta_{D_n}(x_1, \ldots, x_n) = m_\alpha \vartheta_{D_n}(x_1, \ldots, x_n)^{m_\alpha - 1} \nabla \tilde{\vartheta}_{D_n}(x_1, \ldots, x_n).$$

Therefore ϑ_{D_n} and $\tilde{\vartheta}_{D_n}$ have the same critical point in a given Weyl chamber.

Lemma 4.7 *We have*

$$\tilde{\vartheta}_{(D_n)}(x_1, \ldots, x_n) = \prod_{1 \leq p < q \leq n} \sin(x_p - x_q) \sin(x_p + x_q).$$

Proof.

$$\begin{aligned}
\tilde{\vartheta}_{(D_n)}(x_1,\dots,x_n) &= \prod_{\alpha\in(D_n)_+} \sin\left(\langle\alpha,(x_1,\dots,x_n)\rangle\right) \\
&= \prod_{\substack{\alpha=e_p\pm e_q\\ 1\le p<q\le n}} \sin\left(\langle\alpha,(x_1,\dots,x_n)\rangle\right) \\
&= \prod_{1\le p<q\le n} \sin\left(\langle e_p-e_q,(x_1,\dots,x_n)\rangle\right) \\
&\quad\cdot \sin\left(\langle e_p+e_q,(x_1,\dots,x_n)\rangle\right) \\
&= \prod_{1\le p<q\le n} \sin\left(x_p-x_q\right)\sin\left(x_p+x_q\right)
\end{aligned}$$

□

Definition 4.8 *Let $i\in\{1,\dots,n\}$ and $k\in\{1,\dots,2(n-1)\}$. We set*

1) $R_{n,i}\left(x_1,\dots,x_n\right) = \prod_{\substack{1\le p<q\le n\\ p,q\ne i}} \sin(x_p-x_q) \prod_{\substack{1\le r<s\le n\\ r,s\ne i}} \sin(x_r+x_s),$

2) $\zeta_n(k)\left(x_1,\dots,x_{2(n-1)}\right) = \cos(x_k)\prod_{i=1}^{k-1}\sin(x_i)\prod_{j=k+1}^{2(n-1)}\sin(x_j),$

3) $V_{i,k} = \begin{cases} 1 & \text{, for } k\ge i,\\ -1 & \text{, for } k<i.\end{cases}$

Remark 4.9 *We set $R_{2,i}(x_1,x_2)=1$.*

Lemma 4.10 *Let $(x_1,\dots,x_n)$ be a point in a Weyl chamber in $\mathfrak{a}$. Then*

$$R_{n,i}(x_1,\dots,x_n)\ne 0.$$

Proof. Assume that $R_{n,i}(x_1,\dots,x_n)=0$ for a regular point $(x_1,\dots,x_n)$. It follows $\vartheta_{D_n}(x_1,\dots,x_n)=0$, yielding a contradiction.

□

Definition 4.11 *For $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,n-1\}$, we define*

$$a_n^{\pm}(i,j)(x_1,\dots,x_n)=\begin{cases} x_j\pm x_i & , \text{ for } j<i,\\ x_i\pm x_{j+1} & , \text{ for } j\geq i.\end{cases}$$

Lemma 4.12 $\frac{\partial}{\partial x_i}\tilde{\vartheta}_{D_n}(x_1,\dots,x_n)$ *is equal to*

$$R_{n,i}(x_1,\dots,x_n) \cdot \sum_{k=1}^{2(n-1)} V_{i,k}\zeta_n(k)\left(a_n^-(i,1),\dots,a_n^-(i,n-1),a_n^+(i,1)\dots a_n^+(i,n-1)\right).$$

Proof. We have

$$\begin{aligned}\tilde{\vartheta}_{D_n}(x_1,\dots,x_n) &= \prod_{1\leq p<q\leq n}\sin(x_p-x_q)\sin(x_p+x_q)\\ &= \prod_{\substack{1\leq p<q\leq n\\ p,q\neq i}}\sin(x_p-x_q)\sin(x_p+x_q)\\ &\quad\cdot\prod_{\substack{1\leq p<q\leq n\\ p=i \text{ or } q=i}}\sin(x_p-x_q)\sin(x_p+x_q)\\ &= R_{n,i}(x_1,\dots,x_n)\\ &\quad\cdot\prod_{\substack{1\leq p<q\leq n\\ p=i \text{ or } q=i}}\sin(x_p-x_q)\sin(x_p+x_q).\end{aligned}$$

Since $R_{n,i}(x_1, \dots x_n)$ does not depend on x_i, we get

$$\begin{aligned}
&\frac{\partial}{\partial x_i} \tilde{\vartheta}_{D_n}(x_1, \dots, x_n) \\
&= R_{n,i}(x_1, \dots, x_n) \frac{\partial}{\partial x_i} \left(\prod_{\substack{1 \le p < q \le n \\ p=i \text{ or } q=i}} \sin(x_p - x_q) \sin(x_p + x_q) \right),
\end{aligned}$$

where

$$\prod_{\substack{1 \le p < q \le n \\ p=i \text{ or } q=i}} \sin(x_p - x_q) \sin(x_p + x_q) \tag{2}$$

is equal to

$$\begin{aligned}
&\left(\prod_{i < q \le n} \sin(x_i - x_q) \sin(x_i + x_q) \right) \\
&\cdot \left(\prod_{1 \le p < i} \sin(x_p - x_i) \sin(x_p + x_i) \right),
\end{aligned}$$

which we can write as

$$\begin{aligned}
&\left(\prod_{1 \le p < i} \sin(x_p - x_i) \right) \left(\prod_{i < q \le n} \sin(x_i - x_q) \right) \\
&\cdot \left(\prod_{1 \le p < i} \sin(x_p + x_i) \right) \left(\prod_{i < q \le n} \sin(x_i + x_q) \right).
\end{aligned}$$

Writing (2) in the form

$$\left(\prod_{1 \le p < i} \sin\left(a_n^-(i, p)\right) \right) \left(\prod_{i < q \le n} \sin\left(a_n^-(i, q-1)\right) \right)$$

$$\cdot \left(\prod_{1 \le p < i} \sin\left(a_n^+(i,p)\right) \right) \left(\prod_{i < q \le n} \sin\left(a_n^+(i,q-1)\right) \right)$$

and noting that (2) has $2(n-1)$ factors, we can deduce that differentiating (2) with respect to x_i yields

$$\frac{\partial}{\partial x_i} \left(\prod_{\substack{1 \le p < q \le n \\ p=i \text{ or } q=i}} \sin(x_p - x_q)\sin(x_p + x_q) \right)$$
$$= \sum_{k=1}^{2(n-1)} \Big\{ V_{i,k}$$
$$\cdot \zeta_n(k)\left(a_n^-(i,1),\ldots,a_n^-(i,n-1),a_n^+(i,1)\ldots a_n^+(i,n-1)\right) \Big\},$$

where the minus sign of the first $i-1$ summands arises from the fact that for $1 \le p < i$ differentiating $a_n^-(i,p) = x_p - x_i$ by x_i yields -1.

□

The previous proof motivates the following definition.

Definition 4.13 *We set*

$$K_{n,i}(x_1,\ldots,x_n)$$
$$= \sum_{k=1}^{2(n-1)} \Big\{ V_{i,k}$$
$$\cdot \zeta_n(k)\left(a_n^-(i,1),\ldots,a_n^-(i,n-1),a_n^+(i,1)\ldots a_n^+(i,n-1)\right) \Big\}.$$

Definition 4.14 *For $2 \le i \le \lfloor \frac{n}{2} \rfloor - 1$ let P_i be the permutation*

$$P_i \colon \{1,\ldots,n\} \to \{1,\ldots,n\}$$

with

$$\begin{aligned} P_i(i) &= i+1, \\ P_i(i+1) &= i, \\ P_i(n-i+1) &= n-i, \\ P_i(n-i) &= n-i+1 \end{aligned}$$

and $P_i(k) = k$, *for* $k \notin \{i, i+1, n-i+1, n-i\}$.

We will use for $2 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$ the notation

$$P_k\left(a_n^{\pm}(i,j)(x_1,\ldots,x_n)\right) := a_n^{\pm}(i,j)(x_{P_k(1)},\ldots,x_{P_k(n)}).$$

Lemma 4.15 *For* $2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ *we have*

$$\begin{aligned} &P_i\left(a_n^{\pm}(i,j)(x_1,\ldots,x_n)\right) \\ &= \begin{cases} a_n^{\pm}(i+1,j)(x_1,\ldots,x_n) & , \text{ for } j \notin \{i, n-i-1, n-i\}, \\ \pm a_n^{\pm}(i+1,j)(x_1,\ldots,x_n) & , \text{ for } j = i, \\ a_n^{\pm}(i+1,j+1)(x_1,\ldots,x_n) & , \text{ for } j = n-i-1, \\ a_n^{\pm}(i+1,j-1)(x_1,\ldots,x_n) & , \text{ for } j = n-i \end{cases} \end{aligned}$$

and

$$\begin{aligned} &P_i\left(a_n^{\pm}(n-i+1,j)(x_1,\ldots,x_n)\right) \\ &= \begin{cases} a_n^{\pm}(n-i,j)(x_1,\ldots,x_n) & , \text{ for } j \notin \{i, i+1, n-i\}, \\ a_n^{\pm}(n-i,j+1)(x_1,\ldots,x_n) & , \text{ for } j = i, \\ a_n^{\pm}(n-i,j-1)(x_1,\ldots,x_n) & , \text{ for } j = i+1, \\ \pm a_n^{\pm}(n-i,j)(x_1,\ldots,x_n) & , \text{ for } j = n-i. \end{cases} \end{aligned}$$

Proof. First we note, that the following inequality is valid:

$$\begin{aligned} n-i &\geq 2\left\lfloor \frac{n}{2} \right\rfloor - i \geq 2\left\lfloor \frac{n}{2} \right\rfloor - \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \\ &= \left\lfloor \frac{n}{2} \right\rfloor + 1 > \left\lfloor \frac{n}{2} \right\rfloor \geq i+1 \end{aligned}$$

This implies

$$i + 1 < n - i.$$

We start by proving the first equation. On the one hand we have

$$a_n^{\pm}(i,j)(x_{P_i(1)},\dots,x_{P_i(n)}) = \begin{cases} x_{P_i(j)} \pm x_{P_i(i)} & \text{, for } j < i, \\ x_{P_i(i)} \pm x_{P_i(j+1)} & \text{, for } j \geq i. \end{cases}$$

$$= \begin{cases} x_j \pm x_{i+1} & \text{, for } j < i, \\ x_{i+1} \pm x_i & \text{, for } j = i, \\ x_{i+1} \pm x_{j+1} & \text{, for } j > i \text{ and } j \notin \{n-i-1, n-i\}, \\ x_{i+1} \pm x_{n-i+1} & \text{, for } j = n-i-1, \\ x_{i+1} \pm x_{n-i} & \text{, for } j = n-i. \end{cases}$$

On the other hand we get

- For $j \notin \{i, n-i-1, n-i\}$:

$$a_n^{\pm}(i+1,j)(x_1,\dots,x_n) = \begin{cases} x_j \pm x_{i+1} & \text{, for } j < i, \\ x_{i+1} \pm x_{j+1} & \text{, for } j > i, \end{cases}$$

$$= \begin{cases} a_n^{\pm}(i,j)(x_{P_i(1)},\dots,x_{P_i(n)}) & \text{, for } j < i, \\ a_n^{\pm}(i,j)(x_{P_i(1)},\dots,x_{P_i(n)}) & \text{, for } j > i \text{ and} \\ & \quad j \notin \{n-i-1, n-i\}. \end{cases}$$

- For $j = i$:

$$a_n^{\pm}(i+1,j)(x_1,\dots,x_n)$$
$$= x_i \pm x_{i+1} = \pm a_n^{\pm}(i,j)(x_{P_i(1)},\dots,x_{P_i(n)}).$$

- For $j = n - i - 1$: Since $i + 1 < n - i$ we have

$$\begin{aligned} & a_n^{\pm}(i+1, j+1)(x_1, \ldots, x_n) \\ & = a_n^{\pm}(i+1, n-i)(x_1, \ldots, x_n) \\ & = x_{i+1} \pm x_{n-i+1} \\ & = a_n^{\pm}(i, j)(x_{P_i(1)}, \ldots, x_{P_i(n)}). \end{aligned}$$

- For $j = n - i$: The inequality $i + 1 < n - i$ implies $i + 1 \leq n - i - 1$. Therefore we have

$$\begin{aligned} & a_n^{\pm}(i+1, j-1)(x_1, \ldots, x_n) \\ & = a_n^{\pm}(i+1, n-i-1)(x_1, \ldots, x_n) \\ & = x_{i+1} \pm x_{n-i} \\ & = a_n^{\pm}(i, j)(x_{P_i(1)}, \ldots, x_{P_i(n)}). \end{aligned}$$

This proves the first equation.

To prove the second equation we note that on the one hand we have

$$\begin{aligned} & a_n^{\pm}(n-i+1, j)(x_{P_i(1)}, \ldots, x_{P_i(n)}) \\ & = \begin{cases} x_{P_i(j)} \pm x_{P_i(n-i+1)} & , \text{ for } j < n-i+1, \\ x_{P_i(n-i+1)} \pm x_{P_i(j+1)} & , \text{ for } j \geq n-i+1. \end{cases} \\ & = \begin{cases} x_j \pm x_{n-i} & , \text{ for } j < n-i+1 \text{ and } j \notin \{i, i+1, n-i\}, \\ x_{n-i} \pm x_{j+1} & , \text{ for } j \geq n-i+1, \\ x_{i+1} \pm x_{n-i} & , \text{ for } j = i, \\ x_i \pm x_{n-i} & , \text{ for } j = i+1, \\ x_{n-i+1} \pm x_{n-i} & , \text{ for } j = n-i. \end{cases} \end{aligned}$$

On the other hand we get

- For $j \notin \{i, i+1, n-i\}$:

$$
\begin{aligned}
&a_n^{\pm}(n-i, j)(x_1, \ldots, x_n) \\
&= \begin{cases} x_j \pm x_{n-i} & , \text{ for } j < n-i \text{ and } j \notin \{i, i+1\}, \\ x_{n-i} \pm x_{j+1} & , \text{ for } j > n-i, \end{cases} \\
&= \begin{cases} a_n^{\pm}(n-i+1, j)(x_{P_i(1)}, \ldots, x_{P_i(n)}) & , \text{ for } j < n-i \text{ and} \\ & \quad j \notin \{i, i+1\}, \\ a_n^{\pm}(n-i+1, j)(x_{P_i(1)}, \ldots, x_{P_i(n)}) & , \text{ for } j > n-i. \end{cases}
\end{aligned}
$$

- For $j = i$: Since $i + 1 < n - i$ we have

$$
\begin{aligned}
&a_n^{\pm}(n-i, j+1)(x_1, \ldots, x_n) \\
&= a_n^{\pm}(n-i, i+1)(x_1, \ldots, x_n) \\
&= x_{i+1} \pm x_{n-i} \\
&= a_n^{\pm}(n-i+1, j)(x_{P_i(1)}, \ldots, x_{P_i(n)}).
\end{aligned}
$$

- For $j = i + 1$: Since $i < n - i$ we have

$$
\begin{aligned}
&a_n^{\pm}(n-i, j-1)(x_1, \ldots, x_n) \\
&= a_n^{\pm}(n-i, i)(x_1, \ldots, x_n) \\
&= x_i \pm x_{n-i} \\
&= a_n^{\pm}(n-i+1, j)(x_{P_i(1)}, \ldots, x_{P_i(n)}).
\end{aligned}
$$

- For $j = n - i$:

$$
\begin{aligned}
&a_n^{\pm}(n-i, j)(x_1, \ldots, x_n) \\
&= x_{n-i} \pm x_{n-i+1} \\
&= \pm a_n^{\pm}(n-i+1, j)(x_{P_i(1)}, \ldots, x_{P_i(n)}),
\end{aligned}
$$

  which proves the second equation.

□

Lemma 4.16 *Let $2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$. Then*

1. $K_{n,i}(x_{P_i(1)}, \ldots, x_{P_i(n)}) = -K_{n,(i+1)}(x_1, \ldots, x_n)$,
2. $K_{n,(n-i+1)}(x_{P_i(1)}, \ldots, x_{P_i(n)}) = -K_{n,(n-i)}(x_1, \ldots, x_n)$.

Proof. We start by proving 1):

The k-th summand of $K_{n,i}(x_{P_i(1)}, \ldots, x_{P_i(n)})$ is equal to

$$V_{i,k}\zeta_n(k)\Big(P_i\left(a_n^-(i,1)\right), \ldots, P_i\left(a_n^-(i,n-1)\right), \\ P_i\left(a_n^+(i,1)\right), \ldots, P_i\left(a_n^+(i,n-1)\right)\Big).$$

We set

$$I_i(k) = \zeta_n(k)\Big(P_i\left(a_n^-(i,1)\right), \ldots, P_i\left(a_n^-(i,n-1)\right), \\ P_i\left(a_n^+(i,1)\right), \ldots, P_i\left(a_n^+(i,n-1)\right)\Big).$$

From Lemma 4.15 follows that for $j \in \{1, \ldots, 2(n-1)\} \setminus \{i, n-i-1, n-i, 2n-2-i, 2n-1-i\}$ the j-th argument of $I_i(k)$ is given by $a_n^-(i+1,j)$ if $j \leq n-1$ and by $a_n^+(i+1, j-n+1)$ if $j > n-1$. Further the i-th argument is given by $-a_n^-(i+1,i)$, the $n-i-1$-th by $a_n^-(i+1,n-i)$, the $n-i$-th by $a_n^-(i+1,n-i-1)$, the $n-1+i$-th by $a_n^+(i+1,i)$, the $2n-2-i$-th by $a_n^+(i+1,n-i)$ and the $2n-1-i$-th by $a_n^+(i+1,n-i-1)$.

On the other hand, the l-th summand of $K_{n,i+1}(x_1, \ldots, x_n)$ is given by

$$V_{i+1,l}\zeta_n(l)\Big(a_n^-(i+1,1), \ldots, a_n^-(i+1,n-1), \\ a_n^+(i+1,1), \ldots, a_n^+(i+1,n-1)\Big).$$

We set

$$II_{i+1}(l) = \zeta_n(l)\Big(a_n^-(i+1,1),\dots,a_n^-(i+1,n-1),$$
$$a_n^+(i+1,1),\dots,a_n^+(i+1,n-1)\Big).$$

The i-th factor of $I_i(i)$ is given by $\cos(-a_n^-(i+1,i)) = \cos(a_n^-(i+1,i))$. Hence $I_i(i) = II_{i+1}(i)$. Because we further have $V_{i,i} = 1$ and $V_{i+1,i} = -1$, the i-th summand of $K_{n,i}(x_{P_i(1)},\dots,x_{P_i(n)})$ is equal to minus one times the i-th summand of $K_{n,i+1}(x_1,\dots,x_n)$.

For $k,l \in \{1,\dots,2(n-1)\} \setminus \{i\}$, the i-th factor of $I_i(k)$ is given by $\sin(-a_n^-(i+1,i)) = -\sin(a_n^-(i+1,i))$, which is equal to minus one times the i-th factor of $II_{i+1}(l)$.

Let $k \in \{1,\dots,2(n-1)\} \setminus \{i, n-i-1, n-i, 2n-2-i, 2n-1-i\}$. Then we have $I_i(k) = -II_{i+1}(k)$. Since for $k < i$ we have $V_{i,k} = V_{i+1,k} = -1$ and for $k > i$ we get $V_{i,k} = V_{i+1,k} = 1$, we deduce that the k-th summand of $K_{n,i}(x_{P_i(1)},\dots,x_{P_i(n)})$ is equal to minus one times the k-th summand of $K_{n,i+1}(x_1,\dots,x_n)$.

The $n-i-1$-th factor of $I_i(n-i-1)$ is equal to the $n-i$-th factor of $II_{i+i}(n-i)$ and the $n-i$-th factor of $I_i(n-i-1)$ is equal to the $n-i-1$-th factor of $II_{i+1}(n-i)$. Hence $I_i(n-i-1) = -II_{i+1}(n-i)$. Since we further have $V_{i,n-i-1} = V_{i+1,n-i} = 1$ it follows that the $n-i-1$-th summand of $K_{n,i}(x_{P_i(1)},\dots,x_{P_i(n)})$ is equal to minus one times the $n-i$-th summand of $K_{n,i+1}(x_1,\dots,x_n)$.

Finally, the $2n-2-i$-th factor of $I_i(2n-2-i)$ is equal to the $2n-1-i$-th factor of $II_{i+1}(2n-1-i)$ and the $2n-1-i$-th factor of $I_i(2n-2-i)$ is equal to the $2n-2-i$-th factor of $II_{i+1}(2n-1-i)$. Hence, $I_i(2n-2-i) = -II_{i+1}(2n-1-i)$. Since further we have $V_{i,2n-2-i} = V_{i+1,2n-1-i} = 1$, it follows that the $2n-2-i$-th summand of $I_i(x_{P_i(1)},\dots,x_{P_i(n)})$is equal to minus one times the $2n-1-i$-th summand of $II_{i+1}(x_1,\dots,x_n)$.

Alltogether we proved the first statement.

Now we prove 2):

The k-th summand of $K_{n,n-i+1}(x_{P_i(1)}, \ldots, x_{P_i(n)})$ is equal to

$$\begin{aligned} &V_{n-i+1,k} \\ &\cdot \zeta_n(k)\Big(P_i\left(a_n^-(n-i+1,1)\right), \ldots, P_i\left(a_n^-(n-i+1,n-1)\right), \\ &\qquad P_i\left(a_n^+(n-i+1,1)\right), \ldots, P_i\left(a_n^+(n-i+1,n-1)\right)\Big). \end{aligned}$$

We set

$$\begin{aligned} &I_{n-i+1}(k) \\ &= \zeta_n(k)\Big(P_i\left(a_n^-(n-i+1,1)\right), \ldots, P_i\left(a_n^-(n-i+1,n-1)\right), \\ &\qquad P_i\left(a_n^+(n-i+1,1)\right), \ldots, P_i\left(a_n^+(n-i+1,n-1)\right)\Big). \end{aligned}$$

From Lemma 4.15 follows that for $j \in \{1, \ldots, 2(n-1)\} \setminus \{i, i+1, n-i, n-1+i, n+i\}$ the j-th argument of $I_{n-i+1}(j)$ is given by $a_n^-(n-i,j)$ if $j \le n-1$ and $a_n^+(n-i,j-n+1)$ if $j > n-1$. Further the i-th argument is given by $a_n^-(n-i,i+1)$, the $i+1$-th by $a_n^-(n-i,i)$, the $n-i$-th by $-a_n^-(n-i,n-i)$, the $n-1+i$-th by $a_n^+(n-i,i+1)$, the $n+i$-th by $a_n^+(n-i,i)$ and the $2n-1-i$-th by $a_n^+(n-i,n-i)$.

On the other hand, the l-th summand of $K_{n,n-i}(x_1, \ldots, x_n)$ is given by

$$\begin{aligned} V_{n-i,l}\zeta_n(l)\Big(&a_n^-(n-i,1), \ldots, a_n^-(n-i,n-1), \\ &a_n^+(n-i,1), \ldots, a_n^+(n-i,n-1)\Big). \end{aligned}$$

We set

$$II_{n-i}(l) = \zeta_n(l)\Big(a_n^-(n-i,1),\dots,a_n^-(n-i,n-1),$$
$$a_n^+(n-i,1),\dots,a_n^+(n-i,n-1)\Big).$$

The $n-i$-th factor of $I_{n-i+1}(n-i)$ is given by

$$\cos\left(-a_n^-(n-i,n-i)\right) = \cos\left(a_n^-(n-i,n-i)\right).$$

Hence $I_{n-i+1}(n-i) = II_{n-i}(n-i)$. Because we further have $V_{n-i+1,n-i} = -1$ and $V_{n-i,n-i} = 1$, the $n-i$-th summand of $K_{n,n-i+1}(x_{P_i(1)},\dots,x_{P_i(n)})$ is equal to minus one times the $n-i$-th summand of $K_{n,n-i}(x_1,\dots,x_n)$.

For $k,l \in \{1,\dots,2(n-1)\} \setminus \{n-i\}$, the $n-i$-th factor of $I_{n-i+1}(k)$ is given by $\sin\left(-a_n^-(n-i,n-i)\right) = -\sin\left(a_n^-(n-i,n-i)\right)$, which is equal to minus one times the $n-i$-th factor of $II_{n-i}(l)$.

Let $k \in \{1,\dots,2(n-1)\} \setminus \{i,i+1,n-i,n-1+i,n+i\}$. Then we have $I_{n-i+1}(k) = -II_{n-i}(k)$. Since for $k < n-i$ we have $V_{n-i+1,k} = V_{n-i,k} = -1$ and for $k > n-i$ we have $V_{n-i+1,k} = V_{n-i,k} = 1$, we deduce that the k-th summand of $K_{n,n-i+1}(x_{P_i(1)},\dots,x_{P_i(n)})$ is equal to minus one times the k-th summand of $K_{n,n-i}(x_1,\dots,x_n)$.

The i-th factor of $I_{n-i+1}(i)$ is equal to the $i+1$-th factor of $II_{n-i}(i+1)$ and the $i+1$-th factor of $I_{n-i+1}(i)$ is equal to the i-th factor of $II_{n-i}(i+1)$. Hence $I_{n-i+1}(i) = -II_{n-i}(i+1)$. Since we further have $V_{n-i+1,i} = V_{n-i,i+1} = -1$ (note that $i+1 < n-i$) it follows that the i-th summand of $K_{n,n-i+1}(x_{P_i(1)},\dots,x_{P_i(n)})$ is equal to minus one times the $i+1$-th summand of $K_{n,n-i}(x_1,\dots,x_n)$.

Finally, the $n-1+i$-th factor of $I_{n-i+1}(n-1+i)$ is equal to the $n+i$-th factor of $II_{n-i}(n+i)$ and the $n+i$-th factor of $I_{n-i+1}(n-1+i)$ is equal to the $n-1+i$-th factor of $II_{n-i}(n+i)$. Hence, $I_{n-i+1}(n-1+i) = -II_{n-i}(n+i)$. Since we further get

$V_{n-i+1,n-1+i} = V_{n-i,n+i} = 1$, it follows that the $n-1+i$-th summand of $K_{n,n-i+1}(x_{P_i(1)},\ldots,x_{P_i(n)})$ is equal to minus one times the $n+i$-th summand of $K_{n,n-i}(x_1,\ldots,x_n)$.

Hence, also the second statement is proved.

□

Lemma 4.17 *We have*

1. $K_{n,1}\left(L_n(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1})\right) = K_{n,n}\left(L_n(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1})\right) = 0$,
2. $K_{n,(\lfloor\frac{n}{2}\rfloor+1)}\left(L_n(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1})\right) = 0$, *for n odd.*

Proof. We start by proving 1).

First we will show that

$$K_{n,1}\left(L_n(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1})\right) = 0.$$

We have

$$\begin{aligned}
&K_{n,1}(x_1,\ldots,x_n)\\
&= \sum_{k=1}^{2(n-1)} V_{1,k}\zeta_n(k)\Big(a_n^-(1,1),\ldots,a_n^-(1,n-1),\\
&\qquad\qquad a_n^+(1,1),\ldots,a_n^+(1,n-1)\Big)\\
&= \sum_{k=1}^{2(n-1)} \zeta_n(k)\Big(a_n^-(1,1),\ldots,a_n^-(1,n-1),\\
&\qquad\qquad a_n^+(1,1),\ldots,a_n^+(1,n-1)\Big).
\end{aligned}$$

In the last line we used $V_{1,k}=1$, since $k\geq 1$. Further we get

$$a_n^{\pm}(1,j) = x_1 \pm x_{j+1}$$

for $j \in \{1, \ldots, n-1\}$. It follows, that for even n

$$a_n^{\pm}(1,j)\left(L_n\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right)\right)$$

is equal to

$$\frac{\pi}{2} \pm \begin{cases} \frac{1}{2}\arccos\left(-\sqrt{\xi_j}\right) & \text{, for } 1 \leq j \leq \frac{n}{2} - 1, \\ \frac{1}{2}\arccos\left(\sqrt{\xi_{n-j-1}}\right) & \text{, for } \frac{n}{2} \leq j \leq n-1 \end{cases}$$

and for odd n

$$a_n^{\pm}(1,j)\left(L_n\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right)\right)$$

is equal to

$$\frac{\pi}{2} \pm \begin{cases} \frac{1}{2}\arccos\left(-\sqrt{\xi_j}\right) & \text{, for } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1, \\ \frac{\pi}{4} & \text{, for } j = \lfloor \frac{n}{2} \rfloor, \\ \frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor \frac{n}{2} \rfloor - j}}\right) & \text{, for } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n-1. \end{cases}$$

Now we will show that the j-th and the $j+n-1$-th summand of $K_{n,1}\left(L_n\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right)\right)$ cancel each other, for $j \in \{1, \ldots, n-1\}$.

First we assume n to be even.
Then, the j-th summand of
$K_{n,1}\left(L_n\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right)\right)$ has the following properties:
The j-th factor is given by

$$\begin{cases} \cos\left(\frac{\pi}{2} - \frac{1}{2}\arccos\left(-\sqrt{\xi_j}\right)\right) & \text{, for } 1 \leq j \leq \frac{n}{2} - 1, \\ \cos\left(\frac{\pi}{2} - \frac{1}{2}\arccos\left(\sqrt{\xi_{n-j-1}}\right)\right) & \text{, for } \frac{n}{2} \leq j \leq n-1 \end{cases}$$

and the $j+n-1$-th factor is given by

$$\begin{cases} \sin\left(\frac{\pi}{2} + \frac{1}{2}\arccos\left(-\sqrt{\xi_j}\right)\right) & \text{, for } 1 \leq j \leq \frac{n}{2} - 1, \\ \sin\left(\frac{\pi}{2} + \frac{1}{2}\arccos\left(\sqrt{\xi_{n-j-1}}\right)\right) & \text{, for } \frac{n}{2} \leq j \leq n-1. \end{cases}$$

On the other hand, the $j+n-1$-th summand of $K_{n,1}\left(L_n\left(\xi_1,\dots,\xi_{\lfloor\frac{n}{2}\rfloor-1}\right)\right)$ has the following properties:
The j-th factor is given by

$$\begin{cases} \sin\left(\frac{\pi}{2}-\frac{1}{2}\arccos\left(-\sqrt{\xi_j}\right)\right) & \text{, for } 1\leq j\leq \frac{n}{2}-1, \\ \sin\left(\frac{\pi}{2}-\frac{1}{2}\arccos\left(\sqrt{\xi_{n-j-1}}\right)\right) & \text{, for } \frac{n}{2}\leq j\leq n-1 \end{cases}$$

and the $j+n-1$-th factor is given by

$$\begin{cases} \cos\left(\frac{\pi}{2}+\frac{1}{2}\arccos\left(-\sqrt{\xi_j}\right)\right) & \text{, for } 1\leq j\leq \frac{n}{2}-1, \\ \cos\left(\frac{\pi}{2}+\frac{1}{2}\arccos\left(\sqrt{\xi_{n-j-1}}\right)\right) & \text{, for } \frac{n}{2}\leq j\leq n-1. \end{cases}$$

We see that for both $1\leq j\leq\frac{n}{2}-1$ and $\frac{n}{2}\leq j\leq n-1$ the j-th factor of the j-th summand is equal to minus one times the $j+n-1$-th factor of the $j+n-1$-th summand and that the $j+n-1$-th factor of the j-th summand is equal to the j-th factor of the $j+n-1$-th summand. Since further, for $k\in\{1,\dots,2(n-1)\}\setminus\{j,j+n-1\}$, the k-th factor of the j-th summand coincides with the k-th factor of the $j+n-1$-th summand, it follows that the j-th and the $j+n-1$-th summands cancel each other.

Now we assume n to be odd.
Then, the j-th summand of $K_{n,1}\left(L_n\left(\xi_1,\dots,\xi_{\lfloor\frac{n}{2}\rfloor-1}\right)\right)$ has the following properties:
The j-th factor is given by

$$\begin{cases} \cos\left(\frac{\pi}{2}-\frac{1}{2}\arccos\left(-\sqrt{\xi_j}\right)\right) & \text{, for } 1\leq j\leq \lfloor\frac{n}{2}\rfloor-1, \\ \cos\left(\frac{\pi}{4}\right) & \text{, for } j=\lfloor\frac{n}{2}\rfloor, \\ \cos\left(\frac{\pi}{2}-\frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor\frac{n}{2}\rfloor-j}}\right)\right) & \text{, for } \lfloor\frac{n}{2}\rfloor+1\leq j\leq n-1. \end{cases}$$

and the $j+n-1$-th factor is given by

$$\begin{cases} \sin\left(\frac{\pi}{2}+\frac{1}{2}\arccos\left(-\sqrt{\xi_j}\right)\right) & \text{, for } 1 \le j \le \lfloor \frac{n}{2} \rfloor - 1, \\ \sin\left(\frac{3\pi}{4}\right) & \text{, for } j = \lfloor \frac{n}{2} \rfloor, \\ \sin\left(\frac{\pi}{2}+\frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor \frac{n}{2} \rfloor - j}}\right)\right) & \text{, for } \lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1. \end{cases}$$

On the other hand, the $j+n-1$-th summand of $K_{n,1}\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor \frac{n}{2} \rfloor -1}\right)\right)$ has the following properties:
The j-th factor is given by

$$\begin{cases} \sin\left(\frac{\pi}{2}-\frac{1}{2}\arccos\left(-\sqrt{\xi_j}\right)\right) & \text{, for } 1 \le j \le \lfloor \frac{n}{2} \rfloor - 1, \\ \sin\left(\frac{\pi}{4}\right) & \text{, for } j = \lfloor \frac{n}{2} \rfloor, \\ \sin\left(\frac{\pi}{2}-\frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor \frac{n}{2} \rfloor - j}}\right)\right) & \text{, for } \lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1 \end{cases}$$

and the $j+n-1$-th factor is given by

$$\begin{cases} \cos\left(\frac{\pi}{2}+\frac{1}{2}\arccos\left(-\sqrt{\xi_j}\right)\right) & \text{, for } 1 \le j \le \lfloor \frac{n}{2} \rfloor - 1, \\ \cos\left(\frac{3\pi}{4}\right) & \text{, for } j = \lfloor \frac{n}{2} \rfloor, \\ \cos\left(\frac{\pi}{2}+\frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor \frac{n}{2} \rfloor - j}}\right)\right) & \text{, for } \lfloor \frac{n}{2} \rfloor + 1 \le j \le n-1. \end{cases}$$

We see that also in this case for all $j \in \{1,\ldots,n-1\}$ the j-th factor of the j-th summand is equal to minus one times the $j+n-1$-th factor of the $j+n-1$-th summand and that the $j+n-1$-th factor of the j-th summand is equal to the j-th factor of the $j+n-1$-th summand. Since further, for $k \in \{1,\ldots,2(n-1)\}\setminus\{j,j+n-1\}$, the k-th factor of the j-th summand coincides with the k-th factor of the $j+n-1$-th summand, it follows that the j-th and the $j+n-1$-th summands cancel each other.

Now we will prove that

$$K_{n,n}\left(L_n(\xi_1,\ldots,\xi_{\lfloor \frac{n}{2} \rfloor -1})\right) = 0.$$

We have

$$K_{n,n}(x_1,\ldots,x_n) = \sum_{k=1}^{2(n-1)} V_{n,k}\zeta_n(k)\Big(a_n^-(n,1),\ldots,a_n^-(n,n-1),$$
$$a_n^+(n,1),\ldots,a_n^+(n,n-1)\Big),$$

with

$$V_{n,k} = \begin{cases} 1 & \text{, for } k \geq n, \\ -1 & \text{, for } k < n. \end{cases}$$

For $j \in \{1,\ldots,n-1\}$ we get

$$a_n^\pm(n,j) = x_j \pm x_n.$$

It follows, that for even n

$$a_n^\pm(n,j)\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1}\right)\right)$$

is equal to

$$\begin{cases} \frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right) & \text{, for } 1 \leq j \leq \frac{n}{2}, \\ \frac{1}{2}\arccos\left(\sqrt{\xi_{n-j}}\right) & \text{, for } \frac{n}{2}+1 \leq j \leq n-1 \end{cases}$$

and for odd n

$$a_n^\pm(n,j)\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1}\right)\right)$$

is equal to

$$\begin{cases} \frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right) & \text{, for } 1 \leq j \leq \lfloor\frac{n}{2}\rfloor, \\ \frac{\pi}{4} & \text{, for } j = \lfloor\frac{n}{2}\rfloor+1, \\ \frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor\frac{n}{2}\rfloor-j+1}}\right) & \text{, for } \lfloor\frac{n}{2}\rfloor+2 \leq j \leq n-1. \end{cases}$$

Now we will show that the j-th and the $j+n-1$-th summand of $K_{n,n}\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1}\right)\right)$ cancel each other, for $j \in \{1,\ldots,n-1\}$.

First we assume n to be even.
Then, the j-th summand of
$K_{n,n}\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1}\right)\right)$ has the following properties:
The j-th factor is given by

$$\begin{cases}\cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right)\right) & \text{, for } 1\leq j\leq\frac{n}{2},\\ \cos\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{n-j}}\right)\right) & \text{, for } \frac{n}{2}+1\leq j\leq n-1\end{cases}$$

and the $j+n-1$-th factor is given by

$$\begin{cases}\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right)\right) & \text{, for } 1\leq j\leq\frac{n}{2},\\ \sin\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{n-j}}\right)\right) & \text{, for } \frac{n}{2}+1\leq j\leq n-1.\end{cases}$$

On the other hand, the $j+n-1$-th summand of
$K_{n,n}\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1}\right)\right)$ has the following properties:
The j-th factor is given by

$$\begin{cases}\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right)\right) & \text{, for } 1\leq j\leq\frac{n}{2},\\ \sin\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{n-j}}\right)\right) & \text{, for } \frac{n}{2}+1\leq j\leq n-1\end{cases}$$

and the $j+n-1$-th factor is given by

$$\begin{cases}\cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right)\right) & \text{, for } 1\leq j\leq\frac{n}{2},\\ \cos\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{n-j}}\right)\right) & \text{, for } \frac{n}{2}+1\leq j\leq n-1.\end{cases}$$

We see that for all $j\in\{1,\ldots,n-1\}$ the j-th factor of the j-th summand is equal to the $j+n-1$-th factor of the $j+n-1$-th summand and that the $j+n-1$-th factor of the j-th summand is equal to the j-th factor of the $j+n-1$-th summand. Since further, for $k\in\{1,\ldots,2(n-1)\}\setminus\{j,j+n-1\}$, the k-th factor of the j-th summand coincides with the k-th factor of the $j+n-1$-th summand, it follows that the j-th and the $j+n-1$-th summands cancel each other because of $V_{n,j}=-1$ and $V_{n,j+n-1}=1$.

Now we assume n to be odd.
Then, the j-th summand of $K_{n,n}\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1}\right)\right)$ has the following properties:
The j-th factor is given by

$$\begin{cases} \cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right)\right) & ,\ \text{for } 1\le j\le\lfloor\frac{n}{2}\rfloor, \\ \cos\left(\frac{\pi}{4}\right) & ,\ \text{for } j=\lfloor\frac{n}{2}\rfloor+1, \\ \cos\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor\frac{n}{2}\rfloor-j+1}}\right)\right) & ,\ \text{for } \lfloor\frac{n}{2}\rfloor+2\le j\le n-1. \end{cases}$$

and the $j+n-1$-th factor is given by

$$\begin{cases} \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right)\right) & ,\ \text{for } 1\le j\le\lfloor\frac{n}{2}\rfloor, \\ \sin\left(\frac{\pi}{4}\right) & ,\ \text{for } j=\lfloor\frac{n}{2}\rfloor+1, \\ \sin\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor\frac{n}{2}\rfloor-j+1}}\right)\right) & ,\ \text{for } \lfloor\frac{n}{2}\rfloor+2\le j\le n-1. \end{cases}$$

On the other hand, the $j+n-1$-th summand of
$K_{n,n}\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor\frac{n}{2}\rfloor-1}\right)\right)$ has the following properties:
The j-th factor is given by

$$\begin{cases} \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right)\right) & ,\ \text{for } 1\le j\le\lfloor\frac{n}{2}\rfloor, \\ \sin\left(\frac{\pi}{4}\right) & ,\ \text{for } j=\lfloor\frac{n}{2}\rfloor+1, \\ \sin\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor\frac{n}{2}\rfloor-j+1}}\right)\right) & ,\ \text{for } \lfloor\frac{n}{2}\rfloor+2\le j\le n-1. \end{cases}$$

and the $j+n-1$-th factor is given by

$$\begin{cases} \cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right)\right) & ,\ \text{for } 1\le j\le\lfloor\frac{n}{2}\rfloor, \\ \cos\left(\frac{\pi}{4}\right) & ,\ \text{for } j=\lfloor\frac{n}{2}\rfloor+1, \\ \cos\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor\frac{n}{2}\rfloor-j+1}}\right)\right) & ,\ \text{for } \lfloor\frac{n}{2}\rfloor+2\le j\le n-1. \end{cases}$$

We see that also in this case for all $j\in\{1,\ldots,n-1\}$ the j-th factor of the j-th summand is equal to the $j+n-1$-th factor

of the $j+n-1$-th summand and that the $j+n-1$-th factor of the j-th summand is equal to the j-th factor of the $j+n-1$-th summand. Since further, for $k \in \{1, \dots, 2(n-1)\} \setminus \{j, j+n-1\}$, the k-th factor of the j-th summand coincides with the k-th factor of the $j+n-1$-th summand, it follows that the j-th and the $j+n-1$-th summands cancel each other because of $V_{n,j} = -1$ and $V_{n,j+n-1} = 1$.
Alltogether we proved 1).

To prove 2), we assume n to be odd. We have

$$a_n^{\pm}\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, j\right)(x_1, \dots, x_n)$$
$$= \begin{cases} x_j \pm x_{\lfloor \frac{n}{2} \rfloor + 1} & \text{, for } j < \lfloor \frac{n}{2} \rfloor + 1, \\ x_{\lfloor \frac{n}{2} \rfloor + 1} \pm x_{j+1} & \text{, for } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n-1. \end{cases}$$

It follows that

$$a_n^{\pm}\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, j\right)\left(L_n\left(\xi_1, \dots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right)\right)$$

is equal to

$$\begin{cases} \frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right) \pm \frac{\pi}{4} & \text{, for } j < \lfloor \frac{n}{2} \rfloor + 1, \\ \frac{\pi}{4} \pm \frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor \frac{n}{2} \rfloor - j}}\right) & \text{, for } \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq n-1. \end{cases}$$

To show that

$$K_{n, \lfloor \frac{n}{2} \rfloor + 1}\left(L_n\left(\xi_1, \dots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right)\right)$$

vanishes, it is enough to show that the following summands of this expression cancel each other:

a) the first with the $2(n-1)$-th

b) the $n-1$-th with the n-th

c) for $2 \leq j \leq \frac{n-1}{2}$, the j-th with the $n-j$-th

d) for $n+1 \leq j \leq \frac{3}{2}(n-1)$, the j-th with the $3n-j-2$-th

To show a) we first remark that $V_{\lfloor \frac{n}{2} \rfloor+1,1} = -1$ and $V_{\lfloor \frac{n}{2} \rfloor+1,n} = 1$. The first summand has the following properties:
The first factor is given by

$$\cos\left(\frac{\pi}{4}\right)$$

and the $2(n-1)$-th factor is given by

$$\sin\left(\frac{\pi}{4}\right).$$

On the other hand, the $2(n-1)$-th summand has the following properties:
The first factor is given by

$$\sin\left(\frac{\pi}{4}\right)$$

and the $2(n-1)$-th factor by

$$\cos\left(\frac{\pi}{4}\right).$$

Hence, the first factor of the first summand is equal to the $2(n-1)$-th factor of the $2(n-1)$-th summand and the $2(n-1)$-th factor of the first summand is equal to the first factor of the $2(n-1)$-th summand. Further, for $k \in \{1, \ldots, 2(n-1)\} \setminus \{1, 2(n-1)\}$ the k-th factor of the first summand coincides with the k-th factor of the $2(n-1)$-th summand. Statement a) follows.

To show b), we first note that $V_{\lfloor \frac{n}{2} \rfloor+1,n-1} = V_{\lfloor \frac{n}{2} \rfloor+1,n} = 1$. The $n-1$-th summand has the following properties:
The $n-1$-th factor is given by

$$\cos\left(\frac{\pi}{4} - \frac{1}{2}\arccos\left(\sqrt{\xi_0}\right)\right) = \cos\left(\frac{\pi}{4}\right)$$

and the n-th factor is given by

$$\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_0}\right)+\frac{\pi}{4}\right)=\sin\left(\frac{3\pi}{4}\right)=\sin\left(\frac{\pi}{4}\right).$$

The n-th summand has the following properties:
The $n-1$-th factor is given by

$$\sin\left(\frac{\pi}{4}\right)$$

and the n-th factor by

$$\cos\left(\frac{3\pi}{4}\right)=-\cos\left(\frac{\pi}{4}\right).$$

Hence. the $n-1$-th factor of the $n-1$-th summand is equal to minus one times the n-th factor of the n-th summand and the n-th factor of the $n-1$-th summand is equal to the $n-1$-th factor of the n-th summand. For $k\in\{1,\ldots,2(n-1)\}\setminus\{n-1,n\}$, the k-th factor of the $n-1$-th summand coincides with the k-th factor of the n-th summand. This implies b).

To show c) we assume that $2\le j\le\frac{n-1}{2}$. In this case we have $V_{\lfloor\frac{n}{2}\rfloor+1,j}=-1$ and because of the inequality

$$n-j\ge n-\frac{n-1}{2}=\frac{n}{2}+\frac{1}{2}=\left\lfloor\frac{n}{2}\right\rfloor+1,$$

it follows $V_{\lfloor\frac{n}{2}\rfloor+1,n-j}=1$.
The j-th summand has the following properties:
The j-th factor is given by

$$\cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right)-\frac{\pi}{4}\right)$$

and the $n-j$-th factor by

$$\sin\left(\frac{\pi}{4}-\frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor\frac{n}{2}\rfloor-n+j}}\right)\right).$$

Further, the $n-j$-th summand has the following properties: Using the fact that $\arccos(-x) = \pi - \arccos(x)$, we deduce that the j-th factor is given by

$$\begin{aligned}
&\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right) - \frac{\pi}{4}\right) \\
&= \sin\left(\frac{\pi}{4} - \frac{1}{2}\arccos\left(\sqrt{\xi_{j-1}}\right)\right) \\
&= \sin\left(\frac{\pi}{4} - \frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor \frac{n}{2}\rfloor - n + j}}\right)\right),
\end{aligned}$$

where we used for the last equality that $2\lfloor \frac{n}{2}\rfloor - n + j = j - 1$. Applying the same arguments, it follows that the $n-j$-th factor is given by

$$\begin{aligned}
&\cos\left(\frac{\pi}{4} - \frac{1}{2}\arccos\left(\sqrt{\xi_{2\lfloor \frac{n}{2}\rfloor - n + j}}\right)\right) \\
&= \cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right) - \frac{\pi}{4}\right).
\end{aligned}$$

As we can see, the j-th factor of the j-th summand is equal to the $n-j$-th factor of the $n-j$-th summand and the $n-j$-th factor of the j-th summand is equal to the j-th factor of the $n-j$-th summand. Since for $k \in \{1, \ldots, 2(n-1)\} \setminus \{j, n-j\}$ the k-th factor of the j-th summand coincides with the k-th factor of the $n-j$-th summand, statement c) follows.

For proving d), we assume that $n+1 \leq j \leq \frac{3}{2}(n-1)$. In this case we have $V_{\lfloor \frac{n}{2}\rfloor+1,j} = V_{\lfloor \frac{n}{2}\rfloor+1,3n-j-2} = 1$. The j-th summand has the following properties:

The j-th factor is given by

$$\begin{aligned}
&\cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-n}}\right)+\frac{\pi}{4}\right)\\
&=\cos\left(-\frac{1}{2}\arccos\left(\sqrt{\xi_{j-n}}\right)+\frac{3\pi}{4}\right)\\
&=\cos\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{j-n}}\right)-\frac{3\pi}{4}\right)\\
&=-\cos\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{j-n}}\right)+\frac{\pi}{4}\right)
\end{aligned}$$

and the $3n-j-2$-th factor by

$$\sin\left(\frac{\pi}{4}+\frac{1}{2}\arccos\left(\sqrt{\xi_{j-n}}\right)\right).$$

On the other hand, the $3n-j-2$-th summand has the following properties: The j-th factor is given by

$$\begin{aligned}
&\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-n}}\right)+\frac{\pi}{4}\right)\\
&=\sin\left(-\frac{1}{2}\arccos\left(\sqrt{\xi_{j-n}}\right)+\frac{3\pi}{4}\right)\\
&=-\sin\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{j-n}}\right)-\frac{3\pi}{4}\right)\\
&=\sin\left(\frac{\pi}{4}+\frac{1}{2}\arccos\left(\sqrt{\xi_{j-n}}\right)\right)
\end{aligned}$$

and the $3n-j-2$-th factor is given by

$$\cos\left(\frac{\pi}{4}+\frac{1}{2}\arccos\left(\sqrt{\xi_{j-n}}\right)\right).$$

Hence, the j-th factor of the j-th summand is equal to minus one times the $3n-j-2$-th factor of the $3n-j-2$-th summand and the $3n-j-2$-th factor of the j-th summand is equal to the j-th factor of the $3n-j-2$-th summand. Since further, we have that for $k \in \{1, \ldots, 2(n-1)\} \setminus \{j, 3n-j-2\}$ the k-th factor of the j-th summand coincides with the k-th factor of the $3n-j-2$-th summand, statement d) follows.

The statements a), b), c) and d) imply 2).

$\square$

Definition 4.18 *For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ let*

$$F_i(n)(\xi) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} (-1)^k(-3+2n-4k)\xi_i^{\lfloor \frac{n}{2} \rfloor - 1 - k}(a_{k-1}(\xi)+a_k(\xi)),$$

with

$$a_k(\xi) = \sum_{m=0}^{k} (-1)^{k-m} \xi_i^{k-m} S_m(\xi),$$

where $S_m(\xi)$ is the m-th elementary symmetric polynomial in the variables $\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}$. Further, we set

$$a_{-1} = 0,$$
$$a_0 = 1,$$
$$a_{\lfloor \frac{n}{2} \rfloor - 1} = 0$$

and

$$F_i(2)() = 0,$$
$$F_i(3)() = 0.$$

Remark 4.19 *For $2 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ we have the relation*

$$F_i(n)(\xi_1, \ldots, \underbrace{\xi_{i-1}}_{i-1-th}, \underbrace{\xi_i}_{i-th}, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1})$$
$$= F_{i-1}(n)(\xi_1, \ldots, \underbrace{\xi_i}_{i-1-th}, \underbrace{\xi_{i-1}}_{i-th}, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}).$$

Proposition 4.20 *We have*

$$K_{n,2}\left(L_n(\xi)\right) = \begin{cases} \dfrac{-1}{2^{n-2}}\sqrt{1-\xi_1}F_1(n)\left(\xi\right) & , \text{ for } n \text{ even,} \\ \dfrac{-1}{2^{n-2}}\sqrt{\xi_1}\sqrt{1-\xi_1}F_1(n)\left(\xi\right) & , \text{ for } n \text{ odd.} \end{cases}$$

Before we give the proof of this proposition, we state some needed intermediate results:

In the following we will denote by

$$\zeta_n(k)\left(L_n\left(\xi\right)\right)$$

the function

$$\zeta_n(k)\left(a_n^-(i,1),\ldots,a_n^-(i,n-1),a_n^+(i,1)\ldots a_n^+(i,n-1)\right)$$

evaluated at $L_n\left(\xi\right)$.

Lemma 4.21 *For $n \geq 6$ we have*

$$\zeta_n(0)\left(L_n\left(\xi\right)\right) = \zeta_{n-2}(0)\left(L_{n-2}\left(\xi\right)\right)\frac{1}{4}\left(\xi_1 - \xi_{\lfloor \frac{n}{2}\rfloor -1}\right).$$

Proof. For arbitrary $n \geq 6$ we get

$$\begin{aligned} &\zeta_n(0)\left(L_n\left(\xi\right)\right) \\ &= \zeta_{n-2}(0)\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor \frac{n-2}{2}\rfloor -1}\right)\right) \\ &\quad\cdot \sin\left((L_n(\xi))_2 - (L_n(\xi))_{\lfloor \frac{n}{2}\rfloor}\right) \\ &\quad\cdot \sin\left((L_n(\xi))_2 - (L_n(\xi))_{n-\lfloor \frac{n}{2}\rfloor+1}\right) \\ &\quad\cdot \sin\left((L_n(\xi))_2 + (L_n(\xi))_{\lfloor \frac{n}{2}\rfloor}\right) \\ &\quad\cdot \sin\left((L_n(\xi))_2 + (L_n(\xi))_{n-\lfloor \frac{n}{2}\rfloor+1}\right) \\ &= \zeta_{n-2}(0)\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor \frac{n-2}{2}\rfloor -1}\right)\right)\frac{1}{4}\left(\xi_1 - \xi_{\lfloor \frac{n}{2}\rfloor -1}\right). \end{aligned}$$

□

Lemma 4.22 *Let $n \geq 4$. Then*

$$\zeta_n(0)\left(L_n\left(\xi\right)\right) = \begin{cases} \frac{\sqrt{\xi_1}\left(1-\xi_1\right)}{4^{\lfloor \frac{n}{2} \rfloor - 1}} \prod_{i=2}^{\lfloor \frac{n}{2} \rfloor - 1} \left(\xi_1 - \xi_i\right) & \text{, for } n \text{ even,} \\ \frac{\xi_1\left(1-\xi_1\right)}{2 \cdot 4^{\lfloor \frac{n}{2} \rfloor - 1}} \prod_{i=2}^{\lfloor \frac{n}{2} \rfloor - 1} \left(\xi_1 - \xi_i\right) & \text{, for } n \text{ odd.} \end{cases}$$

Further, we have

1. $\zeta_2(0)\left(L_2()\right) = 1$,
2. $\zeta_3(0)\left(L_3()\right) = \frac{1}{4}$.

Proof. The statement for dimension $n \in \{2, 3\}$ follows from direct computation. We prove the statement for $n \geq 4$ inductively:

By direct computation we obtain

$$\zeta_4(0)\left(L_4\left(\xi_1\right)\right) = \frac{\sqrt{\xi_1}}{4}\left(1-\xi_1\right),$$
$$\zeta_5(0)\left(L_5\left(\xi_1\right)\right) = \frac{\xi_1}{2 \cdot 4}\left(1-\xi_1\right).$$

First we consider the case where n is odd. Since we want to consider only odd dimensions first, we assume the statement to be true for n and want to deduce that it is then also true for $n+2$.
Using the previous lemma, we get

$$\zeta_{n+2}(0)\left(L_{n+2}\left(\xi_1, \ldots, \xi_{\lfloor \frac{n+2}{2} \rfloor - 1}\right)\right)$$
$$= \zeta_n(0)\left(L_n\left(\xi\right)\right)\frac{1}{4}\left(\xi_1 - \xi_{\lfloor \frac{n}{2} \rfloor}\right).$$

Using the induction assumption, it follows that the term above is equal to

$$\frac{\xi_1(1-\xi_1)}{2\cdot 4^{\lfloor \frac{n}{2} \rfloor}}\left(\xi_1-\xi_{\lfloor \frac{n}{2} \rfloor}\right)\prod_{i=2}^{\lfloor \frac{n}{2} \rfloor -1}(\xi_1-\xi_i)$$

$$=\frac{\xi_1(1-\xi_1)}{2\cdot 4^{\lfloor \frac{n}{2} \rfloor}}\prod_{i=2}^{\lfloor \frac{n}{2} \rfloor}(\xi_1-\xi_i).$$

Noting that $\left\lfloor \frac{n+2}{2} \right\rfloor - 1 = \left\lfloor \frac{n}{2} \right\rfloor$, we have shown that the statement of the lemma is also true for $n+2$.

Now, we consider the case n even. As in the case before, using the previous lemma yields

$$\zeta_{n+2}(0)\left(L_{n+2}\left(\xi_1,\ldots,\xi_{\lfloor \frac{n+2}{2} \rfloor -1}\right)\right)$$

$$=\zeta_n(0)\left(L_n(\xi)\right)\frac{1}{4}\left(\xi_1-\xi_{\lfloor \frac{n}{2} \rfloor}\right).$$

Using the induction assumption, we see that this is equal to

$$\frac{\sqrt{\xi_1}(1-\xi_1)}{4^{\lfloor \frac{n}{2} \rfloor}}\left(\xi_1-\xi_{\lfloor \frac{n}{2} \rfloor}\right)\prod_{i=2}^{\lfloor \frac{n}{2} \rfloor -1}(\xi_1-\xi_i)$$

$$=\frac{\sqrt{\xi_1}(1-\xi_1)}{4^{\lfloor \frac{n}{2} \rfloor}}\prod_{i=2}^{\lfloor \frac{n}{2} \rfloor}(\xi_1-\xi_i),$$

which shows that the statement is also true for $n+2$.

□

Lemma 4.23 *Let n be odd and $\xi_1 \neq 0$. Then*

1) $\zeta_n\left(\left\lfloor\frac{n}{2}\right\rfloor\right)(L_n(\xi))$

$$= \frac{\sqrt{\xi_1}(1-\xi_1)}{2\cdot 4^{\lfloor\frac{n}{2}\rfloor-1}} \prod_{i=2}^{\lfloor\frac{n}{2}\rfloor-1}(\xi_1-\xi_i)\left(1+\sqrt{1-\xi_1}\right),$$

2) $\zeta_n\left(n+\left\lfloor\frac{n}{2}\right\rfloor-1\right)(L_n(\xi))$

$$= \frac{\sqrt{\xi_1}(1-\xi_1)}{2\cdot 4^{\lfloor\frac{n}{2}\rfloor-1}} \prod_{i=2}^{\lfloor\frac{n}{2}\rfloor-1}(\xi_1-\xi_i)\left(\sqrt{1-\xi_1}-1\right).$$

Proof. We have

$$\begin{aligned}
&\zeta_n\left(\left\lfloor\frac{n}{2}\right\rfloor\right)(L_n(\xi))\\
&= \zeta_n(0)(L_n(\xi))\cot\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right)-\frac{\pi}{4}\right)\\
&= \zeta_n(0)(L_n(\xi))\,\frac{\sqrt{1-\sqrt{\xi_1}}+\sqrt{1+\sqrt{\xi_1}}}{\sqrt{1+\sqrt{\xi_1}}-\sqrt{1-\sqrt{\xi_1}}}.
\end{aligned}$$

Using the previous lemma, we see that this is equal to

$$\begin{aligned}
&\frac{\xi_1(1-\xi_1)}{2\cdot 4^{\lfloor\frac{n}{2}\rfloor-1}}\,\frac{\sqrt{1-\sqrt{\xi_1}}+\sqrt{1+\sqrt{\xi_1}}}{\sqrt{1+\sqrt{\xi_1}}-\sqrt{1-\sqrt{\xi_1}}}\prod_{i=2}^{\lfloor\frac{n}{2}\rfloor-1}(\xi_1-\xi_i)\\
&= \frac{\sqrt{\xi_1}(1-\xi_1)}{2\cdot 4^{\lfloor\frac{n}{2}\rfloor-1}}\left(1+\sqrt{1-\xi_1}\right)\prod_{i=2}^{\lfloor\frac{n}{2}\rfloor-1}(\xi_1-\xi_i).
\end{aligned}$$

Further, we get

$$\begin{aligned}
&\zeta_n\left(n+\left\lfloor\frac{n}{2}\right\rfloor-1\right)(L_n(\xi))\\
&= \zeta_n(0)(L_n(\xi))\cot\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right)+\frac{\pi}{4}\right)\\
&= \zeta_n(0)(L_n(\xi))\,\frac{\sqrt{1-\xi_1}-1}{\sqrt{\xi_1}}.
\end{aligned}$$

Again, using the previous lemma, we obtain that this is equal to

$$\frac{\sqrt{\xi_1}\,(1-\xi_1)}{2\cdot 4^{\lfloor \frac{n}{2}\rfloor -1}}\left(\sqrt{1-\xi_1}-1\right)\prod_{i=2}^{\lfloor \frac{n}{2}\rfloor -1}(\xi_1-\xi_i)\,,$$

which concludes the proof.

□

Corollary 4.24 *Let n be odd and $\xi_1 \neq 0$. Then*

$$\begin{aligned}
1)\quad & \zeta_n\left(\left\lfloor \frac{n}{2}\right\rfloor\right)(L_n(\xi)) \\
&= \frac{1}{\sqrt{\xi_1}}\left(1+\sqrt{1-\xi_1}\right)\zeta_n(0)(L_n(\xi)) \\
2)\quad & \zeta_n\left(n+\left\lfloor \frac{n}{2}\right\rfloor -1\right)(L_n(\xi)) \\
&= \frac{1}{\sqrt{\xi_1}}\left(\sqrt{1-\xi_1}-1\right)\zeta_n(0)(L_n(\xi))\,.
\end{aligned}$$

A motivation for the following definition is given by Lemma 4.51 in the appendix.

Definition 4.25 *We set for $\xi_1 \neq 0$*

$$1)\quad \mathcal{F}_{s,s,s,s} = \frac{\xi_1-\xi_{\lfloor \frac{n}{2}\rfloor}}{2\sqrt{\xi_1}},$$

$$2)\quad \mathcal{F}_{c,s,s,s} = \frac{\left(\sqrt{1-\xi_1}+\sqrt{1-\xi_{\lfloor \frac{n}{2}\rfloor}}\right)\left(\sqrt{\xi_1}+\sqrt{\xi_{\lfloor \frac{n}{2}\rfloor}}\right)}{2\sqrt{\xi_1}},$$

$$3)\quad \mathcal{F}_{s,c,s,s} = \frac{\left(\sqrt{1-\xi_1}+\sqrt{1-\xi_{\lfloor \frac{n}{2}\rfloor}}\right)\left(\sqrt{\xi_1}-\sqrt{\xi_{\lfloor \frac{n}{2}\rfloor}}\right)}{2\sqrt{\xi_1}},$$

$$4)\quad \mathcal{F}_{s,s,c,s} = \frac{\left(\sqrt{1-\xi_1}-\sqrt{1-\xi_{\lfloor \frac{n}{2}\rfloor}}\right)\left(\sqrt{\xi_1}+\sqrt{\xi_{\lfloor \frac{n}{2}\rfloor}}\right)}{2\sqrt{\xi_1}},$$

$$5)\quad \mathcal{F}_{s,s,s,c} = \frac{\left(\sqrt{1-\xi_1} - \sqrt{1-\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\left(\sqrt{\xi_1} - \sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)}{2\sqrt{\xi_1}}.$$

Remark 4.26 *Summation yields*

$$\mathcal{F}_{c,s,s,s} + \mathcal{F}_{s,c,s,s} + \mathcal{F}_{s,s,c,s} + \mathcal{F}_{s,s,s,c} = 2\sqrt{1-\xi_1}.$$

For the proof of Proposition 4.20 we will make use of the following well-known facts for elementary symmetric polynomials.

Lemma 4.27 *For $0 \leq i \leq m$, let $S_i(x_1, \dots, x_m)$ be the i-th elementary symmetric polynomial in the variables $x_1, \dots, x_m$. Then*

$$\begin{aligned} S_i(x_1, \dots, x_m) =& S_i(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m) \\ & + x_j S_{i-1}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m). \end{aligned}$$

Further, the equality

$$\prod_{j=1}^{m} (x - x_j) = \sum_{k=0}^{m} (-1)^k S_k(x_1, \dots, x_m) x^{m-k}$$

is valid.

Using this preperation, we can now give the proof of Proposition 4.20.

Proof of Proposition 4.20. We prove this statement by induction.

A direct calcuation gives

$$\begin{aligned}
K_{2,2}\left(L_2()\right) &= 0 = -\sqrt{1-\xi_1}F_1(2)(),\\
K_{3,2}\left(L_3()\right) &= 0 = -\frac{1}{2}\sqrt{\xi_1}\sqrt{1-\xi_1}F_1(3)(),\\
K_{4,2}\left(L_4(\xi_1)\right) &= -\frac{1}{4}\sqrt{1-\xi_1}\left(5\xi_1-1\right)\\
&= -\frac{1}{4}\sqrt{1-\xi_1}F_1(4)\left(\xi_1\right),\\
K_{5,2}\left(L_5(\xi_1)\right) &= -\frac{1}{8}\sqrt{\xi_1}\sqrt{1-\xi_1}\left(7\xi_1-3\right)\\
&= -\frac{1}{8}\sqrt{\xi_1}\sqrt{1-\xi_1}F_1(5)\left(\xi_1\right).
\end{aligned}$$

First, we prove the claim for odd n, where $n \geq 5$: We get

$$\begin{aligned}
&K_{n+1,2}\left(L_{n+1}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+1}{2}\rfloor-1}\right)\right)\\
&= \sum_{k=1}^{2n} V_{i,k}\zeta_{n+1}(k)\left(L_{n+1}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+1}{2}\rfloor-1}\right)\right)\\
&= \sum_{k=1}^{\lfloor\frac{n}{2}\rfloor-1} V_{i,k}\zeta_{n+1}(k)\left(L_{n+1}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+1}{2}\rfloor-1}\right)\right)\\
&\quad + V_{i,\lfloor\frac{n}{2}\rfloor}\zeta_{n+1}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)\left(L_{n+1}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+1}{2}\rfloor-1}\right)\right)\\
&\quad + V_{i,\lfloor\frac{n}{2}\rfloor+1}\zeta_{n+1}\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)\left(L_{n+1}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+1}{2}\rfloor-1}\right)\right)\\
&\quad + \sum_{k=\lfloor\frac{n}{2}\rfloor+2}^{n+\lfloor\frac{n}{2}\rfloor-1} V_{i,k}\zeta_{n+1}(k)\left(L_{n+1}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+1}{2}\rfloor-1}\right)\right)\\
&\quad + V_{i,n+\lfloor\frac{n}{2}\rfloor}\zeta_{n+1}\left(n+\left\lfloor\frac{n}{2}\right\rfloor\right)\left(L_{n+1}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+1}{2}\rfloor-1}\right)\right)\\
&\quad + V_{i,n+\lfloor\frac{n}{2}\rfloor+1}\zeta_{n+1}\left(n+\left\lfloor\frac{n}{2}\right\rfloor+1\right)\left(L_{n+1}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+1}{2}\rfloor-1}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{k=n+\lfloor \frac{n}{2} \rfloor+2}^{2n} V_{i,k}\zeta_{n+1}(k) \left(L_{n+1}\left(\xi_1,\ldots,\xi_{\lfloor \frac{n+1}{2} \rfloor -1}\right)\right) \\
&= \mathcal{F}_{s,s,s,s} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor -1} V_{i,k}\zeta_n(k) \left(L_n\left(\xi\right)\right) \\
&+ \mathcal{F}_{c,s,s,s}\ \zeta_n\left(0\right)\left(L_n\left(\xi\right)\right) \\
&+ \mathcal{F}_{s,c,s,s}\ \zeta_n\left(0\right)\left(L_n\left(\xi\right)\right) \\
&+ \mathcal{F}_{s,s,s,s} \sum_{k=\lfloor \frac{n}{2} \rfloor+1}^{n+\lfloor \frac{n}{2} \rfloor -2} V_{i,k}\zeta_n(k) \left(L_n\left(\xi\right)\right) \\
&+ \mathcal{F}_{s,s,c,s}\ \zeta_n\left(0\right)\left(L_n\left(\xi\right)\right) \\
&+ \mathcal{F}_{s,s,s,c}\ \zeta_n\left(0\right)\left(L_n\left(\xi\right)\right) \\
&+ \mathcal{F}_{s,s,s,s} \sum_{k=n+\lfloor \frac{n}{2} \rfloor}^{2(n-1)} V_{i,k}\zeta_n(k) \left(L_n\left(\xi\right)\right) \\
&= \mathcal{F}_{s,s,s,s}\ K_{n,2}\left(L_n\left(\xi\right)\right) \\
&- \mathcal{F}_{s,s,s,s}\left(\zeta_{\lfloor \frac{n}{2} \rfloor}\left(0\right)\left(L_n\left(\xi\right)\right) + \zeta_{n+\lfloor \frac{n}{2} \rfloor -1}\left(0\right)\left(L_n\left(\xi\right)\right)\right) \\
&+ \left(\mathcal{F}_{c,s,s,s} + \mathcal{F}_{s,c,s,s} + \mathcal{F}_{s,s,c,s} + \mathcal{F}_{s,s,s,c}\right)\zeta_n\left(0\right)\left(L_n\left(\xi\right)\right) \\
&= \frac{\xi_1 - \xi_{\lfloor \frac{n}{2} \rfloor}}{2\sqrt{\xi_1}} K_{n,2}\left(L_n\left(\xi\right)\right) \\
&- \frac{\xi_1 - \xi_{\lfloor \frac{n}{2} \rfloor}}{2\sqrt{\xi_1}}\left(\zeta_{\lfloor \frac{n}{2} \rfloor}\left(0\right)\left(L_n\left(\xi\right)\right) + \zeta_{n+\lfloor \frac{n}{2} \rfloor -1}\left(0\right)\left(L_n\left(\xi\right)\right)\right) \\
&+ 2\sqrt{1-\xi_1}\ \zeta_n\left(0\right)\left(L_n\left(\xi\right)\right) \\
&= \frac{\xi_1 - \xi_{\lfloor \frac{n}{2} \rfloor}}{2\sqrt{\xi_1}} K_{n,2}\left(L_n\left(\xi\right)\right) \\
&+ \frac{\left(\xi_1 + \xi_{\lfloor \frac{n}{2} \rfloor}\right)\left(1-\xi_1\right)\sqrt{1-\xi_1}}{2\cdot 4^{\lfloor \frac{n}{2} \rfloor -1}} \prod_{i=2}^{\lfloor \frac{n}{2} \rfloor -1}\left(\xi_1 - \xi_i\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\xi_1 - \xi_{\lfloor \frac{n}{2} \rfloor}}{2\sqrt{\xi_1}} K_{n,2}\left(L_n\left(\xi\right)\right) \\
&\quad + \frac{\left(\xi_1 + \xi_{\lfloor \frac{n}{2} \rfloor}\right)\left(1-\xi_1\right)\sqrt{1-\xi_1}}{2 \cdot 4^{\lfloor \frac{n}{2} \rfloor - 1}} \\
&\quad \cdot \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 2} (-1)^k \xi_1^{\lfloor \frac{n}{2} \rfloor - 2 - k} S_k\left(\xi_2, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right) \\
&= \frac{\xi_1 - \xi_{\lfloor \frac{n}{2} \rfloor}}{2\sqrt{\xi_1}} K_{n,2}\left(L_n\left(\xi\right)\right) \\
&\quad + \frac{\xi_1\left(1-\xi_1\right)\sqrt{1-\xi_1}}{2^{n-2}} \\
&\quad \cdot \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 2} (-1)^k \xi_1^{\lfloor \frac{n}{2} \rfloor - 2 - k} S_k\left(\xi_2, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right) \\
&\quad + \frac{\xi_{\lfloor \frac{n}{2} \rfloor}\sqrt{1-\xi_1}}{2^{n-1}}\left(1-\xi_1\right) \\
&\quad \cdot \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 2} 2(-1)^k \xi_1^{\lfloor \frac{n}{2} \rfloor - 2 - k} S_k\left(\xi_2, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right) \\
&= \frac{\xi_1 - \xi_{\lfloor \frac{n}{2} \rfloor}}{2\sqrt{\xi_1}} K_{n,2}\left(L_n\left(\xi\right)\right) \\
&\quad - \frac{\sqrt{1-\xi_1}}{2^{n-2}} \\
&\quad \cdot \left(\left(1-\xi_1\right)\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 2} (-1)^{k+1} \xi_1^{\lfloor \frac{n}{2} \rfloor - 1 - k} S_k\left(\xi_2, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right)\right. \\
&\quad \left. + (-1)^{\lfloor \frac{n}{2} \rfloor - 1} \xi_1 S_{\lfloor \frac{n}{2} \rfloor - 1}\left(\xi_2, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor - 1}\right)\right) \\
&\quad - \frac{\xi_{\lfloor \frac{n}{2} \rfloor}\sqrt{1-\xi_1}}{2^{n-1}}\left(1-\xi_1\right)
\end{aligned}
$$

$$\cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor -2} 2(-1)^{k+1} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -2-k} S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor -1} \right)$$

$$= \frac{\xi_1 - \xi_{\left\lfloor \frac{n}{2} \right\rfloor}}{2\sqrt{\xi_1}} K_{n,2} \left(L_n \left(\xi \right) \right)$$

$$- \frac{\sqrt{1-\xi_1}}{2^{n-2}}$$

$$\cdot \Bigg((1-\xi_1) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor -2} (-1)^{k+1} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -1-k} S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor -1} \right)$$

$$+ (-1)^{\left\lfloor \frac{n}{2} \right\rfloor -2} \xi_1^2 S_{\left\lfloor \frac{n}{2} \right\rfloor -2} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor -1} \right)$$

$$+ (-1)^{\left\lfloor \frac{n}{2} \right\rfloor -1} \xi_1 \Big(S_{\left\lfloor \frac{n}{2} \right\rfloor -1} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor -1} \right)$$

$$+ \xi_1 S_{\left\lfloor \frac{n}{2} \right\rfloor -2} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor -1} \right) \Big) \Bigg)$$

$$- \frac{\xi_{\left\lfloor \frac{n}{2} \right\rfloor} \sqrt{1-\xi_1}}{2^{n-1}} (1-\xi_1)$$

$$\cdot \Bigg(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor -2} (-1)^k (-n+2k) \, \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -2-k} S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor -1} \right)$$

$$+ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor -2} (-1)^{k+1} (-n+2k+2) \, \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -2-k} S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor -1} \right) \Bigg)$$

$$= \frac{\xi_1 - \xi_{\left\lfloor \frac{n}{2} \right\rfloor}}{2\sqrt{\xi_1}} K_{n,2} \left(L_n \left(\xi \right) \right)$$

$$- \frac{\sqrt{1-\xi_1}}{2^{n-2}} \Bigg((1-\xi_1)$$

$$\cdot \Bigg(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor -2} (-1)^{k+1} \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 - k \right) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -1-k} S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor -1} \right)$$

$$
\begin{aligned}
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-3}(-1)^{k+2}\left(\left\lfloor \frac{n}{2}\right\rfloor-2-k\right)\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-1-k}S_k\left(\xi_2,\dots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\Bigg)\\
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-2}(-1)^{k}\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-k}S_k\left(\xi_2,\dots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-3}(-1)^{k+1}\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-k}S_k\left(\xi_2,\dots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+(-1)^{\left\lfloor \frac{n}{2}\right\rfloor-1}\xi_1\Big(S_{\left\lfloor \frac{n}{2}\right\rfloor-1}\left(\xi_2,\dots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\xi_1 S_{\left\lfloor \frac{n}{2}\right\rfloor-2}\left(\xi_2,\dots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\Big)\Bigg)\\
&-\frac{\xi_{\left\lfloor \frac{n}{2}\right\rfloor}\sqrt{1-\xi_1}}{2^{n-1}}(1-\xi_1)\Bigg(3(-1)^{\left\lfloor \frac{n}{2}\right\rfloor-2}S_{\left\lfloor \frac{n}{2}\right\rfloor-2}\left(\xi_2,\dots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-2}(-1)^{k}(-n+2k)\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-2-k}S_k\left(\xi_2,\dots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-2}(-1)^{k}(-n+2k)\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-1-k}S_{k-1}\left(\xi_2,\dots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\Bigg)\\
=&\ \frac{\xi_1-\xi_{\left\lfloor \frac{n}{2}\right\rfloor}}{2\sqrt{\xi_1}}K_{n,2}\left(L_n\left(\xi\right)\right)\\
&-\frac{\sqrt{1-\xi_1}}{2^{n-2}}\Bigg((1-\xi_1)\\
&\cdot\Bigg(\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-2}(-1)^{k+1}\left(\left\lfloor \frac{n}{2}\right\rfloor-1-k\right)\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-1-k}S_k\left(\xi_2,\dots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\sum_{k=1}^{\left\lfloor \frac{n}{2}\right\rfloor-2}(-1)^{k+1}\left(\left\lfloor \frac{n}{2}\right\rfloor-1-k\right)\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-k}S_{k-1}\left(\xi_2,\dots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-2}(-1)^{k}\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-k}S_k\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\sum_{k=1}^{\left\lfloor \frac{n}{2}\right\rfloor-2}(-1)^{k}\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-k+1}S_{k-1}\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+(-1)^{\left\lfloor \frac{n}{2}\right\rfloor-1}\xi_1\Bigg(S_{\left\lfloor \frac{n}{2}\right\rfloor-1}\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\xi_1S_{\left\lfloor \frac{n}{2}\right\rfloor-2}\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\Bigg)\Bigg)\\
&-\frac{\xi_{\left\lfloor \frac{n}{2}\right\rfloor}\sqrt{1-\xi_1}}{2^{n-1}}\Bigg((1-\xi_1)\\
&\cdot\Bigg(3(-1)^{\left\lfloor \frac{n}{2}\right\rfloor-2}S_{\left\lfloor \frac{n}{2}\right\rfloor-2}\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-2}(-1)^{k}\left(-n+2k\right)\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-2-k}\Bigg(S_k\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\xi_1S_{k-1}\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\Bigg)\Bigg)\\
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-3}(-1)^{k}3\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-2-k}S_k\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-3}(-1)^{k+1}3\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-2-k}S_k\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\Bigg)\\
=&\frac{\xi_1-\xi_{\left\lfloor \frac{n}{2}\right\rfloor}}{2\sqrt{\xi_1}}K_{n,2}\left(L_n\left(\xi\right)\right)\\
&-\frac{\sqrt{1-\xi_1}}{2^{n-2}}\Bigg((1-\xi_1)
\end{aligned}
$$

$$\cdot \left(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^{k+1} \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 - k \right) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} \left(S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \right. \right.$$

$$\left. \left. + \xi_1 S_{k-1} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \right) \right)$$

$$+ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \left(S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \right.$$

$$\left. \left. + \xi_1 S_{k-1} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \right) \right)$$

$$- \frac{\xi_{\left\lfloor \frac{n}{2} \right\rfloor} \sqrt{1 - \xi_1}}{2^{n-1}} \left((1 - \xi_1) \right.$$

$$\left(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^k (-n + 2k) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 2 - k} \left(S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \right. \right.$$

$$\left. \left. + \xi_1 S_{k-1} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \right) \right)$$

$$+ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^k 3 \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 2 - k} \left(S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \right.$$

$$\left. + \xi_1 S_{k-1} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \right)$$

$$\left. + (-1)^{\left\lfloor \frac{n}{2} \right\rfloor - 1} 3 \xi_1 S_{\left\lfloor \frac{n}{2} \right\rfloor - 2} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \right)$$

$$= \frac{\xi_1 - \xi_{\left\lfloor \frac{n}{2} \right\rfloor}}{2\sqrt{\xi_1}} K_{n,2} \left(L_n \left(\xi \right) \right)$$

$$- \frac{\sqrt{1 - \xi_1}}{2^{n-2}} \left((1 - \xi_1) \right.$$

$$
\begin{aligned}
&\cdot\left(\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-2}(-1)^{k+1}\left(\left\lfloor\frac{n}{2}\right\rfloor-1-k\right)\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-1-k}S_k\left(\xi\right)\right)\\
&+\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1}(-1)^k\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-k}S_k\left(\xi\right)\Bigg)\\
&-\frac{\xi_{\left\lfloor\frac{n}{2}\right\rfloor}\sqrt{1-\xi_1}}{2^{n-1}}\Bigg(\left(1-\xi_1\right)\\
&\cdot\Bigg(\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-2}(-1)^k\left(5-2n+4k\right)\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-2-k}S_k\left(\xi\right)\\
&+2\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-3}(-1)^k\left(\left\lfloor\frac{n}{2}\right\rfloor-2-k\right)\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-2-k}S_k\left(\xi\right)\Bigg)\\
&+\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-2}(-1)^k3\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-2-k}S_k\left(\xi\right)+(-1)^{\left\lfloor\frac{n}{2}\right\rfloor-1}3S_{\left\lfloor\frac{n}{2}\right\rfloor-1}\left(\xi\right)\Bigg)\\
=&\frac{\xi_1-\xi_{\left\lfloor\frac{n}{2}\right\rfloor}}{2\sqrt{\xi_1}}K_{n,2}\left(L_n\left(\xi\right)\right)\\
&-\frac{\sqrt{1-\xi_1}}{2^{n-2}}\Bigg(\left(1-\xi_1\right)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-2}\sum_{m=0}^{k}(-1)^{2k-m+1}\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-1-m}S_m\left(\xi\right)\\
&+\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1}(-1)^k\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-k}S_k\left(\xi\right)\Bigg)\\
&-\frac{\xi_{\left\lfloor\frac{n}{2}\right\rfloor}\sqrt{1-\xi_1}}{2^{n-1}}\\
&\cdot\Bigg(\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1}(-1)^{k+1}\left(4(k+1)-2(n+\xi_1\right)\xi^{\left\lfloor\frac{n}{2}\right\rfloor-1-k}S_{k-1}\left(\xi\right)
\end{aligned}
$$

$$
+ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \left(-3 + 2n - 4k\right) \xi^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} S_k\left(\xi\right)
$$

$$
\left. + 2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \left(1 - \xi_1\right) \sum_{m=0}^{k-2} (-1)^{k-m} \xi_1^{k-2-m} S_m\left(\xi\right) \right)
$$

$$
= \frac{\xi_1 - \xi_{\left\lfloor \frac{n}{2} \right\rfloor}}{2\sqrt{\xi_1}} K_{n,2}\left(L_n\left(\xi\right)\right)
$$

$$
- \frac{\sqrt{1 - \xi_1}}{2^{n-2}}
$$

$$
\cdot \left(\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \sum_{m=0}^{k-1} (-1)^{k-1-m} \left(1 - \xi_1\right) \xi_1^{k-1-m} S_m\left(\xi\right) \right.
$$

$$
\left. + \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} S_k\left(\xi\right) \right)
$$

$$
- \frac{\xi_{\left\lfloor \frac{n}{2} \right\rfloor} \sqrt{1 - \xi_1}}{2^{n-1}}
$$

$$
\cdot \left(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^{k+1} \left(-2n + 4(k+1)\right) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} S_{k-1}\left(\xi\right) \right.
$$

$$
+ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \left(-3 + 2n - 4k\right) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} S_k\left(\xi\right)
$$

$$
+ 2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \left(\left(1 - \xi_1\right) \right.
$$

$$
\left. \left. \cdot \sum_{m=0}^{k-2} (-1)^{k-m} \xi_1^{k-2-m} S_m\left(\xi\right) + S_{k-1}\left(\xi\right) \right) \right)
$$

$$
= \frac{\xi_1 - \xi_{\left\lfloor \frac{n}{2} \right\rfloor}}{2\sqrt{\xi_1}} K_{n,2}\left(L_n\left(\xi\right)\right)
$$

$$
\begin{aligned}
&- \frac{\sqrt{1-\xi_1}}{2^{n-2}} \left(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor -1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \left(\sum_{m=0}^{k-1} (-1)^{k-1-m} \xi_1^{k-1-m} S_m(\xi) \right.\right. \\
&\left.\left. + \sum_{m=0}^{k} (-1)^{k-m} \xi_1^{k-m} S_m(\xi) \right)\right) \\
&- \frac{\xi_{\left\lfloor \frac{n}{2} \right\rfloor} \sqrt{1-\xi_1}}{2^{n-1}} \left(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor -1} (-1)^k (-4k) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -1-k} S_{k-1}(\xi) \right. \\
&+ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor -1} (-1)^k (-3+2n-4k) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -1-k} S_k(\xi) \\
&+ 2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor -1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \left(\sum_{m=0}^{k-1} (-1)^{k-1-m} \xi_1^{k-1-m} S_{m-1}(\xi) \right. \\
&\left. + \sum_{m=0}^{k} (-1)^{k-m} \xi_1^{k-m} S_{m-1}(\xi) \right) \\
&+ (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \left(2 - 4 \left\lfloor \frac{n}{2} \right\rfloor \right) \\
&\left. \cdot \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor -1} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor -1-m} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -1-m} S_{m-1}(\xi) \right) \\
={}& \frac{\xi_1 - \xi_{\left\lfloor \frac{n}{2} \right\rfloor}}{2\sqrt{\xi_1}} K_{n,2}(L_n(\xi)) \\
&+ \frac{-\sqrt{1-\xi_1}}{2^{n-1}} \left(2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor -1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \right. \\
&\cdot \left(\sum_{m=0}^{k-1} (-1)^{k-1-m} \xi_1^{k-1-m} S_m(\xi) \right. \\
&\left. + \sum_{m=0}^{k} (-1)^{k-m} \xi_1^{k-m} S_m(\xi) \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \xi_{\left\lfloor \frac{n}{2} \right\rfloor} \\
&\cdot \Bigg(\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k (-3 + 2n - 4k)\, \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} \Big(S_{k-1}(\xi) + S_k(\xi) \Big) \\
&+ 2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \Bigg(\sum_{m=0}^{k-1} (-1)^{k-1-m} \xi_1^{k-1-m} S_{m-1}(\xi) \\
&+ \sum_{m=0}^{k} (-1)^{k-m} \xi_1^{k-m} S_{m-1}(\xi) \Bigg) \\
&+ (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \left(-3 + 2(n+1) - 4 \left\lfloor \frac{n}{2} \right\rfloor \right) \\
&\cdot \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - m} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - m} S_{m-1}(\xi) \Bigg) \Bigg) \\
= &\frac{\xi_1 - \xi_{\left\lfloor \frac{n}{2} \right\rfloor}}{2\sqrt{\xi_1}} K_{n,2}(L_n(\xi)) \\
&- \frac{\sqrt{1-\xi_1}}{2^{n-1}} \Bigg(\xi_{\left\lfloor \frac{n}{2} \right\rfloor} \\
&\cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k (-3 + 2n - 4k) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} \Bigg(S_{k-1}(\xi) + S_k(\xi) \Bigg) \\
&+ 2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \Bigg(\sum_{m=0}^{k-1} (-1)^{k-1-m} \xi_1^{k-1-m} \Bigg(S_m(\xi) \\
&+ \xi_{\left\lfloor \frac{n}{2} \right\rfloor} S_{m-1}(\xi) \Bigg) \\
&+ \sum_{m=0}^{k} (-1)^{k-m} \xi_1^{k-m} \Bigg(S_m(\xi) + \xi_{\left\lfloor \frac{n}{2} \right\rfloor} S_{m-1}(\xi) \Bigg) \Bigg) \\
&+ (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \left(-3 + 2(n+1) - 4 \left\lfloor \frac{n}{2} \right\rfloor \right)
\end{aligned}
$$

$$
\cdot \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor -1-m} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -1-m} \left(S_m(\xi) + \xi_{\left\lfloor \frac{n}{2} \right\rfloor} S_{m-1}(\xi) \right) \Bigg)
$$

$$
= \frac{\xi_1 - \xi_{\left\lfloor \frac{n}{2} \right\rfloor}}{2\sqrt{\xi_1}} K_{n,2}(L_n(\xi))
$$

$$
- \frac{\sqrt{1-\xi_1}}{2^{n-1}} \Bigg(\xi_{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k (-3 + 2n - 4k) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -1-k} \Bigg(S_{k-1}(\xi)
$$

$$
+ S_k(\xi) \Bigg)
$$

$$
+ 2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \Bigg(\sum_{m=0}^{k-1} (-1)^{k-1-m} \xi_1^{k-1-m} \Big(S_m(\xi)
$$

$$
+ \xi_{\left\lfloor \frac{n}{2} \right\rfloor} S_{m-1}(\xi) \Big)
$$

$$
+ \sum_{m=0}^{k} (-1)^{k-m} \xi_1^{k-m} \Big(S_m(\xi) + \xi_{\left\lfloor \frac{n}{2} \right\rfloor} S_{m-1}(\xi) \Big) \Bigg)
$$

$$
+ (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \left(-3 + 2(n+1) - 4 \left\lfloor \frac{n}{2} \right\rfloor \right)
$$

$$
\cdot \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor -1-m} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -1-m} \left(S_m(\xi) + \xi_{\left\lfloor \frac{n}{2} \right\rfloor} S_{m-1}(\xi) \right) \Bigg)
$$

$$
= \frac{\xi_1 - \xi_{\left\lfloor \frac{n}{2} \right\rfloor}}{2\sqrt{\xi_1}} K_{n,2}(L_n(\xi))
$$

$$
- \frac{\sqrt{1-\xi_1}}{2^{n-1}} \Bigg(\xi_{\left\lfloor \frac{n}{2} \right\rfloor}
$$

$$
\cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k (-3 + 2n - 4k) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor -1-k} \left(S_{k-1}(\xi) + S_k(\xi) \right)
$$

$$
\begin{aligned}
&+ 2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \left(\sum_{m=0}^{k-1} (-1)^{k-1-m} \xi_1^{k-1-m} S_m \left(\xi_1, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor} \right) \right. \\
&\left. + \sum_{m=0}^{k} (-1)^{k-m} \xi_1^{k-m} S_m \left(\xi_1, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor} \right) \right) \\
&+ (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \left(-3 + 2(n+1) - 4 \left\lfloor \frac{n}{2} \right\rfloor \right) \\
&\left. \cdot \sum_{m=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - m} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - m} S_m \left(\xi_1, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor} \right) \right) \\
= &\frac{\xi_1 - \xi_{\left\lfloor \frac{n}{2} \right\rfloor}}{2\sqrt{\xi_1}} K_{n,2} \left(L_n \left(\xi \right) \right) \\
&- \frac{\sqrt{1 - \xi_1}}{2^{n-1}} \left(\xi_{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k (-3 + 2n - 4k) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} \left(S_{k-1} \left(\xi \right) \right. \right. \\
&\left. + S_k \left(\xi \right) \right) \\
&+ 2 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} \left(a_{k-1} \left(\xi_1, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor} \right) \right. \\
&\left. + a_k \left(\xi_1, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor} \right) \right) \\
&\left. + (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \left(-3 + 2(n+1) - 4 \left\lfloor \frac{n}{2} \right\rfloor \right) a_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \left(\xi_1, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor} \right) \right) \\
= &- \frac{\sqrt{1 - \xi_1}}{2^{n-1}} \\
&\cdot \left(\xi_1 \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^k (-3 + 2n - 4k) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k - 1} \left(a_{k-1} \left(\xi \right) + a_k \left(\xi \right) \right. \right.
\end{aligned}
$$

$$- \frac{\xi_{\lfloor \frac{n}{2} \rfloor}}{\xi_1} \left(a_{k-1}(\xi) + a_k(\xi) - S_{k-1}(\xi) - S_k(\xi) \right) \Bigg)$$

$$+ 2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} (-1)^k \xi_1^{\lfloor \frac{n}{2} \rfloor - k} \Bigg(a_{k-1}\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor}\right)$$

$$+ a_k\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor}\right) \Bigg)$$

$$+ (-1)^{\lfloor \frac{n}{2} \rfloor} \left(-3 + 2(n+1) - 4 \left\lfloor \frac{n}{2} \right\rfloor \right) a_{\lfloor \frac{n}{2} \rfloor - 1}\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor}\right) \Bigg)$$

Now, using the easily verifiable fact that

$$a_{k-1}\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor}\right) + a_k\left(\xi_1, \ldots, \xi_{\lfloor \frac{n}{2} \rfloor}\right)$$

$$= a_{k-1}(\xi) + a_k(\xi) - \frac{\xi_{\lfloor \frac{n}{2} \rfloor}}{\xi_1} \Bigg(a_{k-1}(\xi) + a_k(\xi) - S_{k-1}(\xi) - S_k(\xi) \Bigg),$$

we see that the above expression is equal to

$$- \frac{\sqrt{1-\xi_1}}{2^{n-1}}$$

$$\cdot \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor - 1} (-1)^k (-3 + 2(n+1) - 4k) \xi_1^{\lfloor \frac{n+1}{2} \rfloor - k - 1} \Bigg($$

$$a_{k-1}\left(\xi_1, \ldots, \xi_{\lfloor \frac{n+1}{2} \rfloor - 1}\right) + a_k\left(\xi_1, \ldots, \xi_{\lfloor \frac{n+1}{2} \rfloor - 1}\right) \Bigg),$$

which is equal to

$$\frac{-1}{2^{n-1}} \sqrt{1-\xi_1} F_1(n+1)\left(\xi_1, \ldots, \xi_{\lfloor \frac{n+1}{2} \rfloor - 1}\right).$$

On the other hand, we get for even n the following, where we assume $n \geq 6$ and note that $\lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$:

$$K_{n+1,2}\left(L_{n+1}\left(\xi_1, \ldots, \xi_{\lfloor \frac{n+1}{2} \rfloor - 1}\right)\right)$$

$$
\begin{aligned}
&= K_{n+1,2}\left(L_{n+1}\left(\xi\right)\right) \\
&= \sum_{k=1}^{2n} V_{2,k}\zeta_{n+1}(k)\left(L_{n+1}\left(\xi\right)\right) \\
&= \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} V_{2,k}\zeta_{n+1}(k)\left(L_{n+1}\left(\xi\right)\right) \\
&\quad + \zeta_{n+1}(\left\lfloor \frac{n}{2} \right\rfloor)\left(L_{n+1}\left(\xi\right)\right) \\
&\quad + \sum_{k=\left\lfloor \frac{n}{2} \right\rfloor + 1}^{n+\left\lfloor \frac{n}{2} \right\rfloor - 1} V_{2,k}\zeta_{n+1}(k)\left(L_{n+1}\left(\xi\right)\right) \\
&\quad + \zeta_{n+1}(n + \left\lfloor \frac{n}{2} \right\rfloor)\left(L_{n+1}\left(\xi\right)\right) \\
&\quad + \sum_{k=n+\left\lfloor \frac{n}{2} \right\rfloor + 1}^{2n} V_{2,k}\zeta_{n+1}(k)\left(L_{n+1}\left(\xi\right)\right) \\
&= \sin\left((L_n(\xi))_2 - \frac{\pi}{4}\right)\sin\left((L_n(\xi))_2 + \frac{\pi}{4}\right) \\
&\quad \cdot \left(\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} V_{2,k}\zeta_n(k)\left(L_n\left(\xi\right)\right) + \sum_{k=\left\lfloor \frac{n}{2} \right\rfloor + 1}^{n+\left\lfloor \frac{n}{2} \right\rfloor - 1} V_{2,k}\zeta_n(k-1)\left(L_n\left(\xi\right)\right) + \right. \\
&\quad \left. \sum_{k=n+\left\lfloor \frac{n}{2} \right\rfloor + 1}^{2n} V_{2,k}\zeta_n(k-2)\left(L_n\left(\xi\right)\right) \right) \\
&\quad + \zeta_n(0)\left(L_n\left(\xi\right)\right) \\
&\quad \cdot \left(\cos\left((L_n(\xi))_2 - \frac{\pi}{4}\right)\sin\left((L_n(\xi))_2 + \frac{\pi}{4}\right) \right. \\
&\quad \left. + \sin\left((L_n(\xi))_2 - \frac{\pi}{4}\right)\cos\left((L_n(\xi))_2 + \frac{\pi}{4}\right) \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sqrt{\xi_1}}{2}\left(\sum_{k=1}^{\left\lfloor\frac{n}{2}\right\rfloor-1} V_{2,k}\zeta_n(k)\left(L_n\left(\xi\right)\right) + \sum_{k=\left\lfloor\frac{n}{2}\right\rfloor}^{n+\left\lfloor\frac{n}{2}\right\rfloor-2} V_{2,k+1}\zeta_n(k)\left(L_n\left(\xi\right)\right)\right.\\
&\quad\left. + \sum_{k=n+\left\lfloor\frac{n}{2}\right\rfloor-1}^{2n-2} V_{2,k+2}\zeta_n(k)\left(L_n\left(\xi\right)\right)\right) + \sqrt{1-\xi_1}\zeta_n(0)\left(L_n\left(\xi\right)\right)\\
&= \frac{\sqrt{\xi_1}}{2}K_{n,2}\left(L_n\left(\xi\right)\right) + \sqrt{1-\xi_1}\zeta_n(0)\left(L_n\left(\xi\right)\right)\\
&= \frac{\sqrt{\xi_1}}{2}K_{n,2}\left(L_n\left(\xi\right)\right) + \frac{\sqrt{1-\xi_1}\sqrt{\xi_1}\left(1-\xi_1\right)}{4^{\left\lfloor\frac{n}{2}\right\rfloor-1}}\prod_{i=2}^{\left\lfloor\frac{n}{2}\right\rfloor-1}\left(\xi_1-\xi_i\right)\\
&= \frac{\sqrt{\xi_1}}{2}K_{n,2}\left(L_n\left(\xi\right)\right)\\
&\quad + \frac{\sqrt{1-\xi_1}\sqrt{\xi_1}\left(1-\xi_1\right)}{2^{n-2}}\\
&\quad \cdot \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-2}\left(-1\right)^k\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-2-k}S_k\left(\xi_2,\ldots,\xi_{\left\lfloor\frac{n}{2}\right\rfloor-1}\right)\\
&= \frac{\sqrt{\xi_1}}{2}K_{n,2}\left(L_n\left(\xi\right)\right)\\
&\quad + \frac{\sqrt{1-\xi_1}\sqrt{\xi_1}}{2^{n-2}}\left(\left(1-\xi_1\right)\right.\\
&\quad \cdot \sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-2}\left(-1\right)^k\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-2-k}S_k\left(\xi_2,\ldots,\xi_{\left\lfloor\frac{n}{2}\right\rfloor-1}\right)\\
&\quad + \left(1-\xi_1\right)\left(-1\right)^{\left\lfloor\frac{n}{2}\right\rfloor-2}\left(\left\lfloor\frac{n}{2}\right\rfloor-2-\left(\left\lfloor\frac{n}{2}\right\rfloor-2\right)\right)\\
&\quad \cdot \xi_1^0 S_{\left\lfloor\frac{n}{2}\right\rfloor-2}\left(\xi_2,\ldots,\xi_{\left\lfloor\frac{n}{2}\right\rfloor-1}\right)\\
&\quad \left. + \left(-1\right)^{\left\lfloor\frac{n}{2}\right\rfloor}\xi_1^0 S_{\left\lfloor\frac{n}{2}\right\rfloor-1}\left(\xi_2,\ldots,\xi_{\left\lfloor\frac{n}{2}\right\rfloor-1}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sqrt{\xi_1}}{2} K_{n,2}\left(L_n\left(\xi\right)\right) \\
&\quad + \frac{\sqrt{1-\xi_1}\sqrt{\xi_1}}{2^{n-2}} \Bigg(\left(1-\xi_1\right) \\
&\quad \cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 2 - k} S_k\left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right) \\
&\quad + \left(1-\xi_1\right) \\
&\quad \cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^k \left(\left\lfloor \frac{n}{2} \right\rfloor - 2 - k\right) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 2 - k} S_k\left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right) \\
&\quad + \left(1-\xi_1\right) \\
&\quad \cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 3} (-1)^{k+1} \left(\left\lfloor \frac{n}{2} \right\rfloor - 2 - k\right) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 2 - k} S_k\left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right) \\
&\quad + \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^{k+1} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} S_k\left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right) \\
&\quad + \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^{k+2} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} S_k\left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right) \Bigg) \\
&= \frac{\sqrt{\xi_1}}{2} K_{n,2}\left(L_n\left(\xi\right)\right) \\
&\quad + \frac{\sqrt{1-\xi_1}\sqrt{\xi_1}}{2^{n-2}} \Bigg(\left(1-\xi_1\right) \\
&\quad \cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 2 - k} S_k\left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right) \\
&\quad + \left(1-\xi_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^k \left(\left\lfloor \frac{n}{2} \right\rfloor - 2 - k \right) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 2 - k} S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \\
&+ (1 - \xi_1) \\
&\cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^k \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 - k \right) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} S_{k-1} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \\
&+ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^{k+1} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \\
&+ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^{k+1} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - k} S_{k-1} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \Bigg) \\
= &\frac{\sqrt{\xi_1}}{2} K_{n,2} \left(L_n \left(\xi \right) \right) \\
&+ \frac{\sqrt{1 - \xi_1} \sqrt{\xi_1}}{2^{n-2}} \Bigg((1 - \xi_1) \\
&\cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^k \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 2 - k} S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \\
&+ (1 - \xi_1) \\
&\cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^k \left(\left\lfloor \frac{n}{2} \right\rfloor - 2 - k \right) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 2 - k} S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \\
&+ (1 - \xi_1) \\
&\cdot \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} (-1)^k \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 - k \right) \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} S_{k-1} \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right) \\
&+ \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (-1)^{k+1} \xi_1^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - k} S_k \left(\xi_2, \ldots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-1}(-1)^{k+1}\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-k}S_{k-1}\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\Bigg)\\
&=\frac{\sqrt{\xi_1}}{2}K_{n,2}\left(L_n\left(\xi\right)\right)\\
&+\frac{\sqrt{1-\xi_1}\sqrt{\xi_1}}{2^{n-2}}\Bigg(\left(1-\xi_1\right)\\
&\cdot\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-2}(-1)^{k}\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-2-k}\left(\left\lfloor\frac{n}{2}\right\rfloor-1-k\right)\Bigg(S_k\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\xi_1S_{k-1}\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\Bigg)\\
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-1}(-1)^{k+1}\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-1-k}\Bigg(S_k\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\\
&+\xi_1S_{k-1}\left(\xi_2,\ldots,\xi_{\left\lfloor \frac{n}{2}\right\rfloor-1}\right)\Bigg)\Bigg)\\
&=\frac{\sqrt{\xi_1}}{2}K_{n,2}\left(L_n\left(\xi\right)\right)\\
&+\frac{\sqrt{1-\xi_1}\sqrt{\xi_1}}{2^{n-2}}\Bigg(\left(1-\xi_1\right)\\
&\cdot\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-2}(-1)^{k}\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-2-k}\left(\left\lfloor\frac{n}{2}\right\rfloor-1-k\right)S_k\left(\xi\right)\\
&+\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor-1}(-1)^{k+1}\xi_1^{\left\lfloor \frac{n}{2}\right\rfloor-1-k}S_k\left(\xi\right)\Bigg)\\
&=\frac{\sqrt{\xi_1}}{2}K_{n,2}\left(L_n\left(\xi\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sqrt{1-\xi_1}\sqrt{\xi_1}}{2^{n-2}}\Bigg((1-\xi_1)\\
&\cdot\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor-1}\sum_{m=0}^{k-1}(-1)^{2k-m}\xi_1^{\lfloor\frac{n}{2}\rfloor-2-m}S_m(\xi)\\
&+\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor-1}(-1)^{k+1}\xi_1^{\lfloor\frac{n}{2}\rfloor-1-k}S_k(\xi)\Bigg)\\
=&\frac{\sqrt{\xi_1}}{2}K_{n,2}(L_n(\xi))\\
&+\frac{\sqrt{1-\xi_1}\sqrt{\xi_1}}{2^{n-2}}\\
&\cdot\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor-1}(-1)^{k+1}\xi_1^{\lfloor\frac{n}{2}\rfloor-1-k}\Bigg(\\
&\sum_{m=0}^{k-1}\left((-1)^{k-1-m}\xi_1^{k-1-m}(1-\xi_1)\right)S_m(\xi)+S_k(\xi)\Bigg)\\
=&\frac{\sqrt{\xi_1}}{2}K_{n,2}(L_n(\xi))\\
&+\frac{\sqrt{1-\xi_1}\sqrt{\xi_1}}{2^{n-2}}\\
&\cdot\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor-1}(-1)^{k+1}\xi_1^{\lfloor\frac{n}{2}\rfloor-1-k}\Bigg(\sum_{m=0}^{k-1}(-1)^{k-1-m}\xi_1^{k-1-m}S_m(\xi)\\
&+\sum_{m=0}^{k}(-1)^{k-m}\xi_1^{k-m}S_m(\xi)\Bigg)\\
=&\frac{\sqrt{\xi_1}}{2}K_{n,2}(L_n(\xi))\\
&-\frac{\sqrt{1-\xi_1}\sqrt{\xi_1}}{2^{n-2}}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor-1}(-1)^{k}\xi_1^{\lfloor\frac{n}{2}\rfloor-1-k}\Bigg(a_{k-1}(\xi)+a_k(\xi)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\sqrt{\xi_1}}{2}\frac{(-1)}{2^{n-2}}\sqrt{1-\xi_1}F_1(n)\,(\xi) \\
&\quad - \frac{\sqrt{1-\xi_1}\sqrt{\xi_1}}{2^{n-2}}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1}(-1)^k\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-1-k}\left(a_{k-1}\,(\xi)+a_k\,(\xi)\right) \\
&= \frac{-\sqrt{\xi_1}\sqrt{1-\xi_1}}{2^{n-1}}\Bigg(F_1(n)\,(\xi) \\
&\quad + 2\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1}(-1)^k\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-1-k}\left(a_{k-1}\,(\xi)+a_k\,(\xi)\right)\Bigg) \\
&= \frac{-\sqrt{\xi_1}\sqrt{1-\xi_1}}{2^{n-1}} \\
&\quad \cdot\Bigg(\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1}(-1)^k\left(-3+2n-4k\right)\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-1-k}\left(a_{k-1}\,(\xi)+a_k\,(\xi)\right) \\
&\quad + 2\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1}(-1)^k\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-1-k}\left(a_{k-1}\,(\xi)+a_k\,(\xi)\right)\Bigg) \\
&= \frac{-\sqrt{\xi_1}\sqrt{1-\xi_1}}{2^{n-1}} \\
&\quad \cdot\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor-1}(-1)^k\left(-3+2(n+1)-4k\right)\xi_1^{\left\lfloor\frac{n}{2}\right\rfloor-1-k}\left(a_{k-1}\,(\xi)+a_k\,(\xi)\right) \\
&= \frac{-\sqrt{\xi_1}\sqrt{1-\xi_1}}{2^{n-1}} \\
&\quad \cdot\sum_{k=0}^{\left\lfloor\frac{n+1}{2}\right\rfloor-1}(-1)^k\left(-3+2(n+1)-4k\right)\xi_1^{\left\lfloor\frac{n+1}{2}\right\rfloor-1-k}\Bigg(\\
&\quad a_{k-1}\left(\xi_1,\ldots,\xi_{\left\lfloor\frac{n+1}{2}\right\rfloor-1}\right)+a_k\left(\xi_1,\ldots,\xi_{\left\lfloor\frac{n+1}{2}\right\rfloor-1}\right)\Bigg) \\
&= \frac{-\sqrt{\xi_1}\sqrt{1-\xi_1}}{2^{n-1}}F_1(n+1)\left(\xi_1,\ldots,\xi_{\left\lfloor\frac{n+1}{2}\right\rfloor-1}\right)
\end{aligned}
$$

$$= K_{n+1,2}\left(L_{n+1}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+1}{2}\rfloor-1}\right)\right).$$

□

Now, the goal is to prove Proposition 4.33, which expands the result of the previous proposition to the remaining components of the considered gradient. One key of the proof of Proposition 4.33 is given by Lemma 4.32. In the following, we give some preliminary results that finally lead to the proof of Lemma 4.32.

Lemma 4.28 *Let $n \geq 4$. Then*

1) $$\zeta_{n+2,\lfloor\frac{n+2}{2}\rfloor}(0)\left(L_{n+2}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)$$
$$= \zeta_{n,\lfloor\frac{n}{2}\rfloor}(0)\left(L_n\left(\xi_2,\ldots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)\frac{1}{4}\left(\xi_1-\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right),$$

2) $$\zeta_{n+2,\lfloor\frac{n+2}{2}\rfloor+1}(0)\left(L_{n+2}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)$$
$$= \zeta_{n,\lfloor\frac{n}{2}\rfloor+1}(0)\left(L_n\left(\xi_2,\ldots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)$$
$$\cdot\frac{1}{4}\left(\xi_1-\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right),\quad \textit{for even } n,$$

3) $$\zeta_{n+2,\lfloor\frac{n+2}{2}\rfloor+2}(0)\left(L_{n+2}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)$$
$$= \zeta_{n,\lfloor\frac{n}{2}\rfloor+2}(0)\left(L_n\left(\xi_2,\ldots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)$$
$$\cdot\frac{1}{4}\left(\xi_1-\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right),\quad \textit{for odd } n.$$

Proof. For $n \geq 4$ we have

$$\zeta_{n+2,\lfloor\frac{n+2}{2}\rfloor}(0)\left(L_{n+2}\left(\xi_1,\ldots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)$$
$$= \zeta_{n,\lfloor\frac{n}{2}\rfloor}(0)\left(L_n\left(\xi_2,\ldots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)$$
$$\cdot\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right)-\frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor\frac{n+2}{2}\rfloor-1}}\right)\right)$$
$$\cdot\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor\frac{n+2}{2}\rfloor-1}}\right)-\frac{1}{2}\arccos\left(\sqrt{\xi_1}\right)\right)$$

$$\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right)+\frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor\frac{n+2}{2}\rfloor-1}}\right)\right)$$
$$\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor\frac{n+2}{2}\rfloor-1}}\right)+\frac{1}{2}\arccos\left(\sqrt{\xi_1}\right)\right)$$
$$=\zeta_{n,\lfloor\frac{n}{2}\rfloor}(0)\left(L_n\left(\xi_2,\dots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)\frac{1}{4}\left(\xi_1-\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right).$$

For even n with $n \geq 4$ we get

$$\zeta_{n+2,\lfloor\frac{n+2}{2}\rfloor+1}(0)\left(L_{n+2}\left(\xi_1,\dots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)$$
$$=\zeta_{n,\lfloor\frac{n}{2}\rfloor+1}(0)\left(L_n\left(\xi_2,\dots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)$$
$$\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right)-\frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor\frac{n+2}{2}\rfloor-1}}\right)\right)$$
$$\cdot \sin\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor\frac{n+2}{2}\rfloor-1}}\right)-\frac{1}{2}\arccos\left(\sqrt{\xi_1}\right)\right)$$
$$\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right)+\frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor\frac{n+2}{2}\rfloor-1}}\right)\right)$$
$$\cdot \sin\left(\frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor\frac{n+2}{2}\rfloor-1}}\right)+\frac{1}{2}\arccos\left(\sqrt{\xi_1}\right)\right)$$
$$=\zeta_{n,\lfloor\frac{n}{2}\rfloor+1}(0)\left(L_n\left(\xi_2,\dots,\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right)\right)\frac{1}{4}\left(\xi_1-\xi_{\lfloor\frac{n+2}{2}\rfloor-1}\right).$$

An analogous calculation for odd n yields the third statement of the lemma.

□

Lemma 4.29 *Let $n \geq 4$. Then*

1) $\zeta_{n,\lfloor \frac{n}{2} \rfloor}(0)\left(L_n(\xi)\right)$

$$= \begin{cases} \dfrac{\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor -1}}\left(1-\xi_{\lfloor \frac{n}{2} \rfloor -1}\right)}{4^{\lfloor \frac{n}{2} \rfloor -1}} \displaystyle\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor -2} \left(\xi_i - \xi_{\lfloor \frac{n}{2} \rfloor -1}\right), & n \text{ even}, \\ \dfrac{\xi_{\lfloor \frac{n}{2} \rfloor -1}\left(1-\xi_{\lfloor \frac{n}{2} \rfloor -1}\right)}{2 \cdot 4^{\lfloor \frac{n}{2} \rfloor -1}} \displaystyle\prod_{i=1}^{\lfloor \frac{n}{2} \rfloor -2} \left(\xi_i - \xi_{\lfloor \frac{n}{2} \rfloor -1}\right), & n \text{ odd}, \end{cases}$$

2) $\zeta_{n,\lfloor \frac{n}{2} \rfloor+1}(0)\left(L_n(\xi)\right)$

$$= \frac{\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor -1}}\left(1-\xi_{\lfloor \frac{n}{2} \rfloor -1}\right)}{4^{\lfloor \frac{n}{2} \rfloor -1}} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor -2} \left(\xi_i - \xi_{\lfloor \frac{n}{2} \rfloor -1}\right), \quad n \text{ even},$$

3) $\zeta_{n,\lfloor \frac{n}{2} \rfloor+2}(0)\left(L_n(\xi)\right)$

$$= \frac{\xi_{\lfloor \frac{n}{2} \rfloor -1}\left(1-\xi_{\lfloor \frac{n}{2} \rfloor -1}\right)}{2 \cdot 4^{\lfloor \frac{n}{2} \rfloor -1}} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor -2} \left(\xi_i - \xi_{\lfloor \frac{n}{2} \rfloor -1}\right), \quad n \text{ odd}.$$

Further, we have

1. $\zeta_{2,1}(0)\left(L_2()\right) = 1$,
2. $\zeta_{2,2}(0)\left(L_2()\right) = 1$,
3. $\zeta_{3,1}(0)\left(L_3()\right) = \dfrac{1}{2}$,
4. $\zeta_{3,3}(0)\left(L_3()\right) = \dfrac{1}{2}$.

Proof. The statements for the dimensions $n = 2$ and $n = 3$ follow from direct computation. The statements for $n \geq 4$ we prove inductively:

Statement 1)

Let n be even. By direct computation we obtain

$$\zeta_{4,2}(0)\left(L_4\left(\xi_1\right)\right)=\frac{\sqrt{\xi_1}}{4}\left(1-\xi_1\right).$$

We assume the first statement of the lemma to be true for n, where $n \geq 4$, and we will show that it is also true for $n+2$. From the previous lemma follows

$$\begin{aligned}
&\zeta_{n+2,\left\lfloor\frac{n+2}{2}\right\rfloor}(0)\left(L_{n+2}\left(\xi_1,\ldots,\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\right)\\
&=\zeta_{n,\left\lfloor\frac{n}{2}\right\rfloor}(0)\left(L_n\left(\xi_2,\ldots,\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\right)\frac{1}{4}\left(\xi_1-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\\
&=\frac{\sqrt{\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}}}{4^{\left\lfloor\frac{n}{2}\right\rfloor-1}}\left(1-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\\
&\quad\cdot\prod_{i=2}^{\left\lfloor\frac{n+2}{2}\right\rfloor-2}\left(\xi_i-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\frac{1}{4}\left(\xi_1-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\\
&=\frac{\sqrt{\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}}}{4^{\left\lfloor\frac{n+2}{2}\right\rfloor-1}}\left(1-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\prod_{i=1}^{\left\lfloor\frac{n+2}{2}\right\rfloor-2}\left(\xi_i-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right),
\end{aligned}$$

which is the first statement of the lemma for $n+2$ in the case that n is even.

Now, let n be odd. By direct computation we obtain

$$\zeta_{5,2}(0)\left(L_5\left(\xi_1\right)\right)=\frac{\xi_1}{2\cdot 4}\left(1-\xi_1\right).$$

We assume the first statement of the lemma to be true for n, where $n \geq 5$, and the induction step $n \to n+2$ follows from the previous lemma by an analogous argument.

Statement 2)

Let n be even. By direct computation we obtain

$$\zeta_{4,3}(0)\left(L_4\left(\xi_1\right)\right)=\frac{\sqrt{\xi_1}}{4}\left(1-\xi_1\right).$$

We assume the second statement of the lemma to be true for n, where $n \geq 4$, and we will show that it is also true for $n+2$. From the previous lemma follows

$$\begin{aligned}
&\zeta_{n+2,\left\lfloor\frac{n+2}{2}\right\rfloor+1}(0)\left(L_{n+2}\left(\xi_1,\ldots,\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\right)\\
&=\zeta_{n,\left\lfloor\frac{n}{2}\right\rfloor+1}(0)\left(L_n\left(\xi_2,\ldots,\xi_{\left\lfloor\frac{n}{2}\right\rfloor-1}\right)\right)\frac{1}{4}\left(\xi_1-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\\
&=\frac{\sqrt{\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}}}{4^{\left\lfloor\frac{n}{2}\right\rfloor-1}}\left(1-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\\
&\quad\cdot\prod_{i=2}^{\left\lfloor\frac{n+2}{2}\right\rfloor-2}\left(\xi_i-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\frac{1}{4}\left(\xi_1-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\\
&=\frac{\sqrt{\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}}}{4^{\left\lfloor\frac{n+2}{2}\right\rfloor-1}}\left(1-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\\
&\quad\cdot\prod_{i=1}^{\left\lfloor\frac{n+2}{2}\right\rfloor-2}\left(\xi_i-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right)\frac{1}{4}\left(\xi_1-\xi_{\left\lfloor\frac{n+2}{2}\right\rfloor-1}\right),
\end{aligned}$$

which is the second statement of the lemma for $n+2$.

Statement 3)

Let n be odd. Direct computation yields

$$\zeta_{5,4}(0)\left(L_5\left(\xi_1\right)\right)=\frac{\xi_1}{2\cdot 4}\left(1-\xi_1\right).$$

The induction step $n \rightarrow n+2$ follows from the previous lemma by an analogous argument as in the proof of the first two statements.

□

Corollary 4.30 *Let $n \geq 4$. Then*

$$\zeta_{n,\left\lfloor \frac{n}{2} \right\rfloor}(0)\left(L_n\left(\xi\right)\right) = \begin{cases} \zeta_{n,\left\lfloor \frac{n}{2} \right\rfloor+1}(0)\left(L_n\left(\xi\right)\right) & , \text{ for } n \text{ even}, \\ \\ \zeta_{n,\left\lfloor \frac{n}{2} \right\rfloor+2}(0)\left(L_n\left(\xi\right)\right) & , \text{ for } n \text{ odd}. \end{cases}$$

Lemma 4.31 *Let $1 \leq i \leq n$ and $1 \leq j \leq n-1$.*

1) *For even n we get* $a_n^{\pm}(i,j)\left(L_n\left(\xi_1,\ldots,\xi_{\left\lfloor \frac{n}{2} \right\rfloor-1}\right)\right)$

$$= \begin{cases} \frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right) \pm \frac{1}{2}\arccos\left(-\sqrt{\xi_{i-1}}\right), \\ \qquad\qquad \text{for } 1 \leq j < i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ \frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right) \pm \frac{1}{2}\arccos\left(\sqrt{\xi_{n-i}}\right), \\ \qquad\qquad \text{for } 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor < i \leq n, \\ \frac{1}{2}\arccos\left(\sqrt{\xi_{n-j}}\right) \pm \frac{1}{2}\arccos\left(\sqrt{\xi_{n-i}}\right), \\ \qquad\qquad \text{for } \left\lfloor \frac{n}{2} \right\rfloor < j < i \leq n, \\ \frac{1}{2}\arccos\left(-\sqrt{\xi_{i-1}}\right) \pm \frac{1}{2}\arccos\left(-\sqrt{\xi_{j}}\right), \\ \qquad\qquad \text{for } 1 \leq i \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ \frac{1}{2}\arccos\left(-\sqrt{\xi_{i-1}}\right) \pm \frac{1}{2}\arccos\left(\sqrt{\xi_{n-j-1}}\right), \\ \qquad\qquad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \leq j \leq n-1, \\ \frac{1}{2}\arccos\left(\sqrt{\xi_{n-i}}\right) \pm \frac{1}{2}\arccos\left(\sqrt{\xi_{n-j-1}}\right), \\ \qquad\qquad \text{for } \left\lfloor \frac{n}{2} \right\rfloor < i \leq j \leq n-1. \end{cases}$$

2) *For odd n we get* $a_n^{\pm}(i,j)\left(L_n\left(\xi_1,\ldots,\xi_{\left\lfloor \frac{n}{2} \right\rfloor-1}\right)\right)$

$$
= \begin{cases}
\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right) \pm \frac{1}{2}\arccos\left(-\sqrt{\xi_{i-1}}\right), \\
\qquad\qquad \textit{for } 1 \le j < i \le \left\lfloor \frac{n}{2} \right\rfloor + 1, \\
\frac{1}{2}\arccos\left(-\sqrt{\xi_{j-1}}\right) \pm \frac{1}{2}\arccos\left(\sqrt{\xi_{n-i}}\right), \\
\qquad\qquad \textit{for } 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor + 1 < i \le n, \\
\frac{1}{2}\arccos\left(\sqrt{\xi_{n-j}}\right) \pm \frac{1}{2}\arccos\left(\sqrt{\xi_{n-i}}\right), \\
\qquad\qquad \textit{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 < j < i \le n, \\
\frac{1}{2}\arccos\left(-\sqrt{\xi_{i-1}}\right) \pm \frac{1}{2}\arccos\left(-\sqrt{\xi_{j}}\right), \\
\qquad\qquad \textit{for } 1 \le i \le j \le \left\lfloor \frac{n}{2} \right\rfloor, \\
\frac{1}{2}\arccos\left(-\sqrt{\xi_{i-1}}\right) \pm \frac{1}{2}\arccos\left(\sqrt{\xi_{n-j-1}}\right), \\
\qquad\qquad \textit{for } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1 \le j \le n-1, \\
\frac{1}{2}\arccos\left(\sqrt{\xi_{n-i}}\right) \pm \frac{1}{2}\arccos\left(\sqrt{\xi_{n-j-1}}\right), \\
\qquad\qquad \textit{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 < i \le j \le n-1.
\end{cases}
$$

Proof: Follows directly from the definition of $a_n^{\pm}(i,j)$ and L_n. □

Lemma 4.32 *Let $L_n(\xi) \in \mathcal{L}$. For even n we have*

$$
K_{n,\left\lfloor \frac{n}{2} \right\rfloor}\left(L_n\left(\xi_1, \dots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right)\right)
= -K_{n,\left\lfloor \frac{n}{2} \right\rfloor + 1}\left(L_n\left(\xi_1, \dots, \xi_{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right)\right)
$$

and for odd n we have

$$K_{n,\lfloor \frac{n}{2} \rfloor}\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor \frac{n}{2} \rfloor -1}\right)\right)$$
$$= -K_{n,\lfloor \frac{n}{2} \rfloor +2}\left(L_n\left(\xi_1,\ldots,\xi_{\lfloor \frac{n}{2} \rfloor -1}\right)\right).$$

Proof: We have the following:

1. Case: $1 \leq k \leq n-1$, n even:

   (a) Case: $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$:
   It follows $n-1 \geq n-k \geq \lfloor \frac{n}{2} \rfloor$. The previous lemma yields

   $$a_n^-\left(\left\lfloor \frac{n}{2} \right\rfloor, k\right)(L_n(\xi))$$
   $$= \begin{cases} \frac{1}{2}\arccos\left(\mathcal{A}\right) & \text{, for } 1 \leq k < \left\lfloor \frac{n}{2} \right\rfloor, \\ -\frac{1}{2}\arccos\left(\hat{\mathcal{A}}\right) & \text{, for } k = \left\lfloor \frac{n}{2} \right\rfloor, \end{cases}$$

   with

   $$\mathcal{A} := \sqrt{\xi_{k-1}\xi_{\lfloor \frac{n}{2} \rfloor -1}} + \sqrt{1-\xi_{k-1}}\sqrt{1-\xi_{\lfloor \frac{n}{2} \rfloor -1}},$$

   $$\hat{\mathcal{A}} := 1 - 2\xi_{\lfloor \frac{n}{2} \rfloor -1}$$

   and

   $$a_n^-\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, n-k\right)(L_n(\xi))$$
   $$= \begin{cases} -\frac{1}{2}\arccos\left(\mathcal{A}\right) & \text{, for } n-1 \geq n-k > \left\lfloor \frac{n}{2} \right\rfloor, \\ -\frac{1}{2}\arccos\left(\hat{\mathcal{A}}\right) & \text{, for } n-k = \left\lfloor \frac{n}{2} \right\rfloor. \end{cases}$$

This implies

$$\cot\left(a_n^-\left(\left\lfloor\frac{n}{2}\right\rfloor,k\right)(L_n(\xi))\right)$$
$$=\begin{cases}\sqrt{\dfrac{1+\mathcal{A}}{1-\mathcal{A}}} & \text{, for } 1\le k<\left\lfloor\dfrac{n}{2}\right\rfloor,\\ -\sqrt{\dfrac{1+\hat{\mathcal{A}}}{1-\hat{\mathcal{A}}}} & \text{, for } k=\left\lfloor\dfrac{n}{2}\right\rfloor,\end{cases}$$

$$\cot\left(a_n^-\left(\left\lfloor\frac{n}{2}\right\rfloor+1,n-k\right)(L_n(\xi))\right)$$
$$=\begin{cases}-\sqrt{\dfrac{1+\mathcal{A}}{1-\mathcal{A}}}, & \text{for } n-1\ge n-k>\left\lfloor\dfrac{n}{2}\right\rfloor,\\ -\sqrt{\dfrac{1+\hat{\mathcal{A}}}{1-\hat{\mathcal{A}}}}, & \text{for } n-k=\left\lfloor\dfrac{n}{2}\right\rfloor.\end{cases}$$

Further, we have

$$V_{\lfloor\frac{n}{2}\rfloor,k}=\begin{cases}-1 & \text{, for } 1\le k<\left\lfloor\dfrac{n}{2}\right\rfloor,\\ 1 & \text{, for } k=\left\lfloor\dfrac{n}{2}\right\rfloor,\end{cases}$$
$$V_{\lfloor\frac{n}{2}\rfloor+1,n-k}=\begin{cases}1 & \text{, for } \left\lfloor\dfrac{n}{2}\right\rfloor<n-k\le n-1,\\ -1 & \text{, for } n-k=\left\lfloor\dfrac{n}{2}\right\rfloor.\end{cases}$$

Hence, together with Corollary 4.30 we get

$$\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}V_{\lfloor\frac{n}{2}\rfloor,k}\zeta_{n,\lfloor\frac{n}{2}\rfloor}(k)\,(L_n(\xi))$$
$$=-\sum_{k=\lfloor\frac{n}{2}\rfloor}^{n-1}V_{\lfloor\frac{n}{2}\rfloor+1,k}\zeta_{n,\lfloor\frac{n}{2}\rfloor+1}(k)\,(L_n(\xi))\,.$$

(b) Case: $\lfloor \frac{n}{2} \rfloor < k \leq n-1$:
We get $1 \leq n-k < \lfloor \frac{n}{2} \rfloor$. The previous lemma yields

$$\begin{aligned}
&a_n^-\left(\left\lfloor \frac{n}{2} \right\rfloor, k\right)(L_n(\xi)) \\
&= \frac{1}{2}\left(\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor -1}}\right) - \arccos\left(\sqrt{\xi_{n-k-1}}\right)\right) \\
&= -\frac{1}{2}\arccos(\mathcal{B})
\end{aligned}$$

with

$$\mathcal{B} := -\sqrt{\xi_{n-k-1}\xi_{\lfloor \frac{n}{2} \rfloor -1}} + \sqrt{1-\xi_{n-k-1}}\sqrt{1-\xi_{\lfloor \frac{n}{2} \rfloor -1}}$$

and

$$\begin{aligned}
&a_n^-\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, n-k\right)(L_n(\xi)) \\
&= \frac{1}{2}\left(\arccos\left(-\sqrt{\xi_{n-k-1}}\right) - \arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor -1}}\right)\right) \\
&= \frac{1}{2}\arccos(\mathcal{B}).
\end{aligned}$$

This implies

$$\cot\left(a_n^-\left(\left\lfloor \frac{n}{2} \right\rfloor, k\right)(L_n(\xi))\right) = -\sqrt{\frac{1+\mathcal{B}}{1-\mathcal{B}}},$$

$$\cot\left(a_n^-\left(\left\lfloor \frac{n}{2} \right\rfloor + 1, n-k\right)(L_n(\xi))\right) = \sqrt{\frac{1+\mathcal{B}}{1-\mathcal{B}}}.$$

Further, we have

$$\begin{aligned}
V_{\lfloor \frac{n}{2} \rfloor, k} &= 1, \\
V_{\lfloor \frac{n}{2} \rfloor + 1, n-k} &= -1.
\end{aligned}$$

Corollary 4.30 implies

$$\sum_{k=\lfloor\frac{n}{2}\rfloor+1}^{n-1} V_{\lfloor\frac{n}{2}\rfloor,k}\zeta_{n,\lfloor\frac{n}{2}\rfloor}(k)\left(L_n(\xi)\right)$$
$$= -\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor-1} V_{\lfloor\frac{n}{2}\rfloor+1,k}\zeta_{n,\lfloor\frac{n}{2}\rfloor+1}(k)\left(L_n(\xi)\right).$$

2. Case: $n \le k \le 2(n-1)$, n even:

   (c) Case $1 \le k-n+1 \le \lfloor\frac{n}{2}\rfloor$:
   We get $\lfloor\frac{n}{2}\rfloor \le 2n-k-1 \le n-1$. The previous lemma yields

$$a_n^+\left(\left\lfloor\frac{n}{2}\right\rfloor, k-n+1\right)\left(L_n(\xi)\right)$$
$$= \begin{cases} \pi - \dfrac{1}{2}\arccos\left(\mathcal{C}\right) & , \text{ for } 1 \le k-n+1 < \left\lfloor\dfrac{n}{2}\right\rfloor, \\ \dfrac{\pi}{2} & , \text{ for } k-n+1 = \left\lfloor\dfrac{n}{2}\right\rfloor. \end{cases}$$

   with

$$\mathcal{C} := \sqrt{\xi_{k-n}\xi_{\lfloor\frac{n}{2}\rfloor-1}} - \sqrt{1-\xi_{k-n}}\sqrt{1-\xi_{\lfloor\frac{n}{2}\rfloor-1}}$$

   and

$$a_n^+\left(\left\lfloor\frac{n}{2}\right\rfloor+1, 2n-k-1\right)\left(L_n(\xi)\right)$$
$$= \begin{cases} \dfrac{1}{2}\arccos\left(\mathcal{C}\right) & , \text{ for } n-1 \ge 2n-k-1 \\ & \qquad > \left\lfloor\dfrac{n}{2}\right\rfloor, \\ \dfrac{\pi}{2} & , \text{ for } 2n-k-1 = \left\lfloor\dfrac{n}{2}\right\rfloor. \end{cases}$$

This implies

$$\cot\left(a_n^+\left(\left\lfloor\frac{n}{2}\right\rfloor, k-n+1\right)(L_n(\xi))\right) = -\sqrt{\frac{1+\mathcal{C}}{1-\mathcal{C}}},$$

$$\cot\left(a_n^+\left(\left\lfloor\frac{n}{2}\right\rfloor+1, 2n-k-1\right)(L_n(\xi))\right) = \sqrt{\frac{1+\mathcal{C}}{1-\mathcal{C}}}.$$

Further, we have

$$V_{\lfloor\frac{n}{2}\rfloor, k-n+1} = -1,$$
$$V_{\lfloor\frac{n}{2}\rfloor+1, 2n-k-1} = 1.$$

We deduce

$$\sum_{k=n}^{\lfloor\frac{n}{2}\rfloor+n-1} V_{\lfloor\frac{n}{2}\rfloor,k}\zeta_{n,\lfloor\frac{n}{2}\rfloor}(k)(L_n(\xi))$$
$$= -\sum_{k=\lfloor\frac{n}{2}\rfloor+n-1}^{2(n-1)} V_{\lfloor\frac{n}{2}\rfloor+1,k}\zeta_{n,\lfloor\frac{n}{2}\rfloor+1}(k)(L_n(\xi)).$$

(d) Case $\left\lfloor\frac{n}{2}\right\rfloor < k-n+1 \le n-1$:
We get $1 \le 2n-k-1 < \left\lfloor\frac{n}{2}\right\rfloor$. The previous lemma yields

$$a_n^+\left(\left\lfloor\frac{n}{2}\right\rfloor, k-n+1\right)(L_n(\xi))$$
$$= \frac{1}{2}\left(\arccos\left(-\sqrt{\xi_{\lfloor\frac{n}{2}\rfloor-1}}\right) + \arccos\left(-\sqrt{\xi_{2n-k-2}}\right)\right)$$
$$= \frac{1}{2}\arccos(\mathcal{D})$$

with

$$\mathcal{D} := \sqrt{\xi_{2n-k-2}\xi_{\lfloor\frac{n}{2}\rfloor-1}} + \sqrt{1-\xi_{2n-k-2}}\sqrt{1-\xi_{\lfloor\frac{n}{2}\rfloor-1}}$$

and

$$a_n^+ \left(\left\lfloor \frac{n}{2} \right\rfloor + 1, 2n - k - 1 \right) (L_n(\xi))$$
$$= \frac{1}{2} \Bigg(\arccos \left(-\sqrt{\xi_{2n-k-2}} \right) + \arccos \left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor - 1}} \right) \Bigg)$$
$$= \pi - \frac{1}{2} \arccos (\mathcal{D}) .$$

This implies

$$\cot \left(a_n^+ \left(\left\lfloor \frac{n}{2} \right\rfloor , k - n + 1 \right) (L_n(\xi)) \right) = \sqrt{\frac{1 + \mathcal{D}}{1 - \mathcal{D}}},$$

$$\cot \left(a_n^+ \left(\left\lfloor \frac{n}{2} \right\rfloor + 1, 2n - k - 1 \right) (L_n(\xi)) \right) = -\sqrt{\frac{1 + \mathcal{D}}{1 - \mathcal{D}}}.$$

Further, we have

$$V_{\lfloor \frac{n}{2} \rfloor, k-n+1} = 1,$$
$$V_{\lfloor \frac{n}{2} \rfloor + 1, 2n-k-1} = -1.$$

It follows

$$\sum_{k=\lfloor \frac{n}{2} \rfloor + n}^{2(n-1)} V_{\lfloor \frac{n}{2} \rfloor, k} \zeta_{n, \lfloor \frac{n}{2} \rfloor}(k) (L_n(\xi))$$
$$= - \sum_{k=n}^{\lfloor \frac{n}{2} \rfloor + n - 2} V_{\lfloor \frac{n}{2} \rfloor + 1, k} \zeta_{n, \lfloor \frac{n}{2} \rfloor + 1}(k) (L_n(\xi)) .$$

Alltogether, we showed the first statement of the lemma.

The following cases will prove the second statement of the lemma in a similar way:

3. Case: $1 \le k \le n-1$, n odd:

   (e) Case: $1 \le k \le \lfloor \frac{n}{2} \rfloor$:
   It follows $n-1 \ge n-k \ge \lfloor \frac{n}{2} \rfloor + 1$. The previous lemma yields

$$a_n^-\left(\left\lfloor \frac{n}{2} \right\rfloor, k\right)(L_n(\xi)) = \begin{cases} \frac{1}{2}\arccos(\mathcal{A}) & , \text{ for } 1 \le k < \left\lfloor \frac{n}{2} \right\rfloor, \\ \frac{1}{2}\arccos\left(\tilde{\mathcal{A}}\right) & , \text{ for } k = \left\lfloor \frac{n}{2} \right\rfloor, \end{cases}$$

   with

$$\tilde{\mathcal{A}} := \sqrt{\xi_{\lfloor \frac{n}{2} \rfloor - 1}\xi_{\lfloor \frac{n}{2} \rfloor}} + \sqrt{1-\xi^2_{\lfloor \frac{n}{2} \rfloor - 1}}\sqrt{1-\xi^2_{\lfloor \frac{n}{2} \rfloor}}.$$

   Further, we get

$$a_n^-\left(\left\lfloor \frac{n}{2} \right\rfloor + 2, n-k\right)(L_n(\xi)) = \begin{cases} -\frac{1}{2}\arccos(\mathcal{A}) & , \text{ for } n-1 \ge n-k \\ & \qquad \ge \left\lfloor \frac{n}{2} \right\rfloor + 2, \\ -\frac{1}{2}\arccos\left(\tilde{\mathcal{A}}\right) & , \text{ for } n-k = \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

This implies

$$\cot\left(a_n^-\left(\left\lfloor\frac{n}{2}\right\rfloor, k\right)(L_n(\xi))\right)$$

$$= \begin{cases} \sqrt{\dfrac{1+\mathcal{A}}{1-\mathcal{A}}} & , \text{ for } 1 \leq k < \left\lfloor\dfrac{n}{2}\right\rfloor, \\ \sqrt{\dfrac{1+\tilde{\mathcal{A}}}{1-\tilde{\mathcal{A}}}} & , \text{ for } k = \left\lfloor\dfrac{n}{2}\right\rfloor, \end{cases}$$

$$\cot\left(a_n^-\left(\left\lfloor\frac{n}{2}\right\rfloor + 2, n-k\right)(L_n(\xi))\right)$$

$$= \begin{cases} -\sqrt{\dfrac{1+\mathcal{A}}{1-\mathcal{A}}} & , \text{ for } n-1 \geq n-k \\ & \qquad \geq \left\lfloor\dfrac{n}{2}\right\rfloor + 2, \\ -\sqrt{\dfrac{1+\tilde{\mathcal{A}}}{1-\tilde{\mathcal{A}}}} & , \text{ for } n-k = \left\lfloor\dfrac{n}{2}\right\rfloor + 1. \end{cases}$$

We have

$$V_{\lfloor \frac{n}{2} \rfloor, k} = \begin{cases} -1 & , \text{ for } 1 \leq k < \lfloor \frac{n}{2} \rfloor, \\ 1 & , \text{ for } k = \lfloor \frac{n}{2} \rfloor, \end{cases}$$

$$V_{\lfloor \frac{n}{2} \rfloor + 2, n-k} = \begin{cases} 1 & , \text{ for } n-1 \geq n-k \geq \lfloor \frac{n}{2} \rfloor + 2, \\ -1 & , \text{ for } n-k = \lfloor \frac{n}{2} \rfloor + 1. \end{cases}$$

(f) Case: $\lfloor \frac{n}{2} \rfloor + 1 \leq k \leq n-1$:
It follows $1 \leq n-k \leq \lfloor \frac{n}{2} \rfloor$. The previous lemma yields

$$a_n^-\left(\left\lfloor\frac{n}{2}\right\rfloor, k\right)(L_n(\xi)) = -\frac{1}{2}\arccos(\mathcal{B}).$$

Further, we get

$$a_n^-\left(\left\lfloor\frac{n}{2}\right\rfloor + 2, n-k\right)(L_n(\xi)) = \frac{1}{2}\arccos(\mathcal{B})$$

This implies

$$\cot\left(a_n^-\left(\left\lfloor\frac{n}{2}\right\rfloor,k\right)(L_n(\xi))\right)=-\sqrt{\frac{1+\mathcal{B}}{1-\mathcal{B}}},$$

$$\cot\left(a_n^-\left(\left\lfloor\frac{n}{2}\right\rfloor+2,n-k\right)(L_n(\xi))\right)=\sqrt{\frac{1+\mathcal{B}}{1-\mathcal{B}}}.$$

We have

$$V_{\lfloor\frac{n}{2}\rfloor,k}=1$$
$$V_{\lfloor\frac{n}{2}\rfloor+2,n-k}=-1$$

4. Case: $n\le k\le 2(n-1)$, n odd:

(g) Case: $1\le k-n+1\le\lfloor\frac{n}{2}\rfloor$:
It follows $\lfloor\frac{n}{2}\rfloor+1\le 2n-k-1\le n-1$. The previous lemma yields

$$a_n^+\left(\left\lfloor\frac{n}{2}\right\rfloor,k-n+1\right)(L_n(\xi))$$
$$=\begin{cases}\pi-\frac{1}{2}\arccos\left(\mathcal{C}\right) & \text{, for } 1\le k-n+1<\left\lfloor\frac{n}{2}\right\rfloor,\\ \pi-\frac{1}{2}\arccos\left(\tilde{\mathcal{C}}\right) & \text{, for } k-n+1=\left\lfloor\frac{n}{2}\right\rfloor,\end{cases}$$

with

$$\tilde{\mathcal{C}}:=\sqrt{\xi_{\lfloor\frac{n}{2}\rfloor-1}\xi_{\lfloor\frac{n}{2}\rfloor}}-\sqrt{1-\xi^2_{\lfloor\frac{n}{2}\rfloor-1}}\sqrt{1-\xi^2_{\lfloor\frac{n}{2}\rfloor}}.$$

Further, we get

$$a_n^+\left(\left\lfloor\frac{n}{2}\right\rfloor+2,2n-k-1\right)(L_n(\xi))$$
$$=\begin{cases}\frac{1}{2}\arccos(\mathcal{C}) & \text{, for } \left\lfloor\frac{n}{2}\right\rfloor+1 \\ & \quad < 2n-k-1 \leq n-1, \\ \frac{1}{2}\arccos\left(\tilde{\mathcal{C}}\right) & \text{, for } 2n-k-1=\left\lfloor\frac{n}{2}\right\rfloor+1.\end{cases}$$

This implies

$$\cot\left(a_n^+\left(\left\lfloor\frac{n}{2}\right\rfloor,k-n+1\right)(L_n(\xi))\right)$$
$$=\begin{cases}-\sqrt{\frac{1+\mathcal{C}}{1-\mathcal{C}}} & \text{, for } 1\leq k-n+1<\left\lfloor\frac{n}{2}\right\rfloor, \\ -\sqrt{\frac{1+\tilde{\mathcal{C}}}{1-\tilde{\mathcal{C}}}} & \text{, for } k-n+1=\left\lfloor\frac{n}{2}\right\rfloor,\end{cases}$$

$$\cot\left(a_n^+\left(\left\lfloor\frac{n}{2}\right\rfloor+2,2n-k-1\right)(L_n(\xi))\right)$$
$$=\begin{cases}\sqrt{\frac{1+\mathcal{C}}{1-\mathcal{C}}} & \text{, for } \left\lfloor\frac{n}{2}\right\rfloor+1<2n-k-1\leq n-1, \\ \sqrt{\frac{1+\tilde{\mathcal{C}}}{1-\tilde{\mathcal{C}}}} & \text{, for } 2n-k-1=\left\lfloor\frac{n}{2}\right\rfloor+1.\end{cases}$$

We have

$$V_{\left\lfloor \frac{n}{2}\right\rfloor, k-n+1} = \begin{cases} -1 & \text{, for } 1 \le k-n+1 < \left\lfloor \frac{n}{2}\right\rfloor, \\ 1 & \text{, for } k-n+1 = \left\lfloor \frac{n}{2}\right\rfloor, \end{cases}$$

$$V_{\left\lfloor \frac{n}{2}\right\rfloor +2, 2n-k-1} = \begin{cases} 1 & \text{, for } \left\lfloor \frac{n}{2}\right\rfloor + 1 < 2n-k-1 \\ & \qquad \le n-1, \\ -1 & \text{, for } 2n-k-1 = \left\lfloor \frac{n}{2}\right\rfloor + 1. \end{cases}$$

(h) Case: $\left\lfloor \frac{n}{2}\right\rfloor < k-n+1 \le n-1$:
It follows $1 \le 2n-k-1 < \left\lfloor \frac{n}{2}\right\rfloor + 1$. The previous lemma yields

$$a_n^+ \left(\left\lfloor \frac{n}{2}\right\rfloor, k-n+1 \right) (L_n(\xi)) = \frac{1}{2} \arccos(\mathcal{E}),$$

with

$$\mathcal{E} := -\sqrt{\xi_{\left\lfloor \frac{n}{2}\right\rfloor -1} \xi_{2n-k-2}} - \sqrt{1-\xi_{\left\lfloor \frac{n}{2}\right\rfloor -1}} \sqrt{1-\xi_{2n-k-2}}.$$

Further, we get

$$a_n^+ \left(\left\lfloor \frac{n}{2}\right\rfloor + 2, 2n-k-1 \right) (L_n(\xi)) = \pi - \frac{1}{2} \arccos(\mathcal{E})$$

This implies

$$\cot \left(a_n^+ \left(\left\lfloor \frac{n}{2}\right\rfloor, k-n+1 \right) (L_n(\xi)) \right) = \sqrt{\frac{1+\mathcal{E}}{1-\mathcal{E}}},$$

$$\cot \left(a_n^+ \left(\left\lfloor \frac{n}{2}\right\rfloor + 2, 2n-k-1 \right) (L_n(\xi)) \right) = -\sqrt{\frac{1+\mathcal{E}}{1-\mathcal{E}}}.$$

We have

$$V_{\left\lfloor \frac{n}{2}\right\rfloor, k-n+1} = 1$$

$$V_{\left\lfloor \frac{n}{2}\right\rfloor +2, 2n-k-1} = -1.$$

Alltogether, this implies the second statement.

□

Now, we are in the position to prove the following central proposition.

Proposition 4.33 *Let* $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. *Then*

1) $K_{n,i}\left(L_n\left(\xi\right)\right)$
$$= \begin{cases} \frac{(-1)^{i+1}}{2^{n-2}}\sqrt{1-\xi_{i-1}}F_{i-1}\left(\xi\right) & \text{, for even } n, \\ \frac{(-1)^{i+1}}{2^{n-2}}\sqrt{\xi_{i-1}}\sqrt{1-\xi_{i-1}}F_{i-1}\left(\xi\right) & \text{, for odd } n, \end{cases}$$

2) $K_{n,n-i+1}\left(L_n\left(\xi\right)\right)$
$$= \begin{cases} \frac{(-1)^{i}}{2^{n-2}}\sqrt{1-\xi_{i-1}}F_{i-1}\left(\xi\right) & \text{, for even } n, \\ \frac{(-1)^{i}}{2^{n-2}}\sqrt{\xi_{i-1}}\sqrt{1-\xi_{i-1}}F_{i-1}\left(\xi\right) & \text{, for even } n. \end{cases}$$

Proof:
1) Lemma 4.16 implies for $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$

$$\begin{aligned} &- K_{n,i+1}(x_1, \ldots, x_n) \\ &= K_{n,i}\Big(x_{P_i(1)}, \ldots, x_{P_i(i)}, x_{P_i(i+1)}, \ldots \\ &\qquad\qquad \ldots, x_{P_i(n-i)}, x_{P_i(n-i+1)}, \ldots, x_{P_i(n)}\Big) \\ &= K_{n,i}\left(x_1, \ldots, \underbrace{x_{1+i},}_{i-th} \underbrace{x_i}_{1+i-th}, \ldots, \underbrace{x_{n+1-i},}_{n-i-th} \underbrace{x_{n-i}}_{n+1-i-th}, \ldots, x_n\right). \end{aligned}$$

First, let n be even. From Lemma 4.20 follows that the statement is true for $i = 2$. Now, we assume the claim to be true for i, where $2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, and want to show that it is then also true for $i + 1$:

The consideration above implies

$$
\begin{aligned}
&K_{n,(i+1)}\left(L_n\left(\xi\right)\right)\\
&= -K_{n.i}\left(L_n\left(\xi_1,\dots,\underbrace{\xi_i}_{-1+i-th},\underbrace{\xi_{-1+i}}_{i-th},\dots,\xi_{\left\lfloor\frac{n}{2}\right\rfloor-1}\right)\right).\\
&= \frac{(-1)^{(i+1)+1}}{2^{n-2}}\sqrt{1-\xi_i}F_{i-1}\left(\xi_1,\dots,\underbrace{\xi_i}_{-1+i-th},\underbrace{\xi_{-1+i}}_{i-th},\dots,\xi_{\left\lfloor\frac{n}{2}\right\rfloor-1}\right)\\
&= \frac{(-1)^{(i+1)+1}}{2^{n-2}}\sqrt{1-\xi_i}F_i\left(\xi\right).
\end{aligned}
$$

Hence, the claim is true for all $2 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor$.

Now, let n be odd. Since in this case the proof is analogous to the even case, we omit it here.
2) Let n be even. From Lemma 4.32 follows

$$
\begin{aligned}
K_{n,n+1-\left\lfloor\frac{n}{2}\right\rfloor}\left(L_n\left(\xi\right)\right) &= K_{n,\left\lfloor\frac{n}{2}\right\rfloor+1}\left(L_n\left(\xi\right)\right)\\
&= -K_{n,\left\lfloor\frac{n}{2}\right\rfloor}\left(L_n\left(\xi\right)\right)\\
&= \frac{(-1)^{\left\lfloor\frac{n}{2}\right\rfloor}}{2^{n-2}}\sqrt{1-\xi_{\left\lfloor\frac{n}{2}\right\rfloor-1}}F_{\left\lfloor\frac{n}{2}\right\rfloor-1}\left(\xi\right),
\end{aligned}
$$

where the last equality is a consequence of 1). Hence the statement is true for $i = \left\lfloor\frac{n}{2}\right\rfloor$. Now, we assume the statement to be true for $3 \leq i \leq \left\lfloor\frac{n}{2}\right\rfloor$ and we will show that it is then also true for $i-1$:
For $2 \leq j \leq \left\lfloor\frac{n}{2}\right\rfloor - 1$, the fact that $P_j^2 = \mathrm{id}_{\{1,\dots,n\}}$ together with the second statement of Lemma 4.16 implies

$$
-K_{n,n-j}\left(x_{P_j(1)},\dots,x_{P_j(n)}\right) = K_{n,n-j+1}\left(x_1,\dots,x_n\right).
$$

Hence, because of $2 \leq i-1 \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, we have

$$\begin{aligned}
&K_{n,n-(i-1)+1}\left(L_n\left(\xi\right)\right) \\
&= -K_{n,n-i+1}\left(L_n\left(\xi_1,\ldots \underbrace{\xi_{-1+i}}_{-2+i-th}, \underbrace{\xi_{-2+i}}_{-1+i-th},\ldots,\xi_{\left\lfloor \frac{n}{2} \right\rfloor -1}\right)\right) \\
&= \frac{(-1)^{i-1}}{2^{n-2}}\sqrt{1-\xi_{i-2}}F_{i-1}\left(\xi_1,\ldots,\underbrace{\xi_{i-1}}_{-2+i},\underbrace{\xi_{1-2}}_{-1+i},\ldots,\xi_{\left\lfloor \frac{n}{2} \right\rfloor -1}\right) \\
&= \frac{(-1)^{i-1}}{2^{n-2}}\sqrt{1-\xi_{i-2}}F_{i-2}\left(\xi\right).
\end{aligned}$$

This implies statement 2) for even n.

For odd n an analogous argument, also using Lemma 4.32 and Lemma 4.16, concludes the proof.

□

An immediate consequence of Proposition 4.33 is given by the following lemma.

Lemma 4.34 *Let* $\overline{\xi}_1,\ldots,\overline{\xi}_{\left\lfloor \frac{n}{2} \right\rfloor -1}$ *with* $0 \leq \overline{\xi}_i \leq 1$ *be a solution of the system of equations given by*

$$F_i(n)\left(\xi_1,\ldots,\xi_{\left\lfloor \frac{n}{2} \right\rfloor -1}\right) = 0, \qquad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Then

$$\nabla\tilde{\vartheta}_{D_n}\left(L_n\left(\overline{\xi}_1,\ldots,\overline{\xi}_{\left\lfloor \frac{n}{2} \right\rfloor -1}\right)\right) = 0.$$

Proof: This is a direct consequence of Lemma 4.12 together with Proposition 4.33.

□

We proceed by giving necessary and sufficient conditions for $L_n\left(\xi\right)$ for being a regular point.

Lemma 4.35 *Let* $1 > \xi_1 > \xi_2 > \cdots > \xi_{\lfloor \frac{n}{2} \rfloor - 1} > 0$. *Then* $L_n(\xi)$ *is regular.*

Proof: Assume $L_n(\xi)$ is singular. Then

$$\vartheta_{D_n}(L_n(\xi)) = 0.$$

In particular there exist indices i, j with $1 \leq i < j \leq n$ such that

$$\sin(\langle e_i \pm e_j, L_n(\xi) \rangle) = 0,$$

which implies the existence of $m \in \mathbb{Z}$ with

$$\langle e_i \pm e_j, L_n(\xi) \rangle = m\pi.$$

Since

$$0 \leq (L_n(\xi))_k \leq \frac{\pi}{2}$$

for $1 \leq k \leq n$ and $\frac{1}{2}\arccos(\sqrt{x})$ is strictly monotonically decreasing, it follows that

$$0 < \langle e_i \pm e_j, L_n(\xi) \rangle < \pi.$$

Hence

$$\langle e_i \pm e_j, L_n(\xi) \rangle \neq m\pi$$

for all $m \in \mathbb{Z}$, which is a contradiction.

□

Remark 4.36 *From the fact that for* $1 > \xi_1 > \xi_2 > \cdots > \xi_{\lfloor \frac{n}{2} \rfloor - 1} > 0$, *the inequality*

$$0 < \langle e_i \pm e_j, L_n(\xi) \rangle < \pi$$

is valid follows that the set

$$\mathcal{L} := \left\{ L_n(\xi) \;\middle|\; 1 > \xi_1 > \cdots > \xi_{\lfloor \frac{n}{2} \rfloor - 1} > 0 \right\}$$

is contained in the dominant generalized Weyl chamber $\mathcal{W}_n$.

Lemma 4.37 *If $L_n(\xi)$ is regular, then $0 < \xi_i < 1$ and $\xi_i \neq \xi_j$ for $i \neq j$.*

Proof. Assume there exist indices $i \neq j$ such that $\xi_i = \xi_j$. Then

$$(L_n(\xi))_{i+1} = (L_n(\xi))_{j+1}.$$

For $e_{i+1} - e_{j+1} \in (D_n)_+$, we get

$$\langle e_{i+1} - e_{j+1}, L_n(\xi)\rangle = 0.$$

In particular,

$$\tilde{\vartheta}_{(D_n)}(L_n(\xi)) = \prod_{\alpha \in (D_n)_+} \sin(\langle \alpha, L_n(\xi)\rangle) = 0,$$

which is a contradiction to $L_n(\xi)$ being a regular point.

Now assume $\xi_i = 0$. Then we get

$$(L_n(\xi))_{i+1} = \frac{1}{2}\arccos\left(-\sqrt{0}\right) = \frac{\pi}{4}$$

and

$$(L_n(\xi))_{n-i} = \frac{1}{2}\arccos\left(\sqrt{0}\right) = \frac{\pi}{4}.$$

For $e_{i+1} - e_{n-i} \in (D_n)_+$ follows

$$\langle e_{i+1} - e_{n-i}, L_n(\xi)\rangle = 0,$$

and in the the same way as above, we get a contradiction to $L_n(\xi)$ being regular.

Finally, assume $\xi_i = 1$. Then

$$(L_n(\xi))_{i+1} = \frac{1}{2}\arccos\left(-\sqrt{1}\right) = \frac{\pi}{2}.$$

For $e_1 + e_{i+1} \in (D_n)_+$ follows

$$\langle e_1 + e_{i+1}, L_n(\xi) \rangle = \pi,$$

again, a contradiction to $L_n(\xi)$ being regular.

□

Restricted to the subset $\mathcal{L}$, we alltogether get the following statement.

Lemma 4.38 *Let $L_n\left(\overline{\xi}\right) \in \mathcal{L}$. Then*

$$F_i(n)\left(\overline{\xi}\right) = 0, \text{ for } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

if and only if

$$\nabla \tilde{\vartheta}_{D_n}\left(L_n\left(\overline{\xi}\right)\right) = 0.$$

Proof. One implication follows from Lemma 4.34.

Conversely, let

$$\nabla \tilde{\vartheta}_{D_n}\left(L_n\left(\overline{\xi}\right)\right) = 0,$$

i.e.

$$g_{n,i}\left(L_n\left(\overline{\xi}\right)\right) = 0, \text{ for } 1 \le i \le n.$$

It follows, that for all $1 \le i \le n$

$$R_{n,i}\left(L_n\left(\overline{\xi}\right)\right) K_{n,i}\left(L_n\left(\overline{\xi}\right)\right) = 0.$$

Since Lemma 4.10 implies $R_{n,i}\left(L_n\left(\overline{\xi}\right)\right) \neq 0$, we deduce

$$K_{n,i}\left(L_n\left(\overline{\xi}\right)\right) = 0, \text{ for } 1 \le i \le n.$$

Now Lemma 4.33, together with the fact that $0 < \overline{\xi}_j < 1$, implies

$$F_i(n)\left(\overline{\xi}\right) = 0, \text{ for } 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

□

Lemma 4.39 *Let* $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ *and* $n \geq 4$. *Then*

$$F_i(n)\left(\xi\right) = A(n)\xi_i^{\left\lfloor \frac{n}{2} \right\rfloor - 1} + \sum_{l=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} B(n,l)\xi_i^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - l} + C(n),$$

with

$$1)\quad A(n) := \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} (1 + 2n - 4k),$$

$$2)\quad B(n,l)(\xi) := (-1)^l \left(\left(1 + 2n - 4\left\lfloor \frac{n}{2} \right\rfloor\right) S_{l-1}\left(\xi\right) \right.$$

$$\left. + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - l} (1 + 2n - 4(k+l))\left(S_{l-1}\left(\xi\right) + S_l\left(\xi\right)\right) \right),$$

$$3)\quad C(n)(\xi) := (-1)^{\left\lfloor \frac{n}{2} \right\rfloor - 1}\left(1 + 2n - 4\left\lfloor \frac{n}{2} \right\rfloor\right) S_{\left\lfloor \frac{n}{2} \right\rfloor - 2}\left(\xi\right).$$

Proof. This can be easily seen.

□

This motivates the following definition.

Definition 4.40 *For real numbers* $\overline{\xi}_1, \ldots, \overline{\xi}_{\left\lfloor \frac{n}{2} \right\rfloor - 1}$ *we define the polynomial*

$$\hat{P}_n^{\overline{\xi}}(x) = x^{\left\lfloor \frac{n}{2} \right\rfloor - 1} + \sum_{l=1}^{\left\lfloor \frac{n}{2} \right\rfloor - 2} \frac{B(n,l)(\overline{\xi})}{A(n)} x^{\left\lfloor \frac{n}{2} \right\rfloor - 1 - l} + \frac{C(n)(\overline{\xi})}{A(n)}.$$

Remark 4.41 *For* $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ *we have the equality*

$$F_i(n)\left(\overline{\xi}_1, \ldots, \overline{\xi}_{\left\lfloor \frac{n}{2} \right\rfloor - 1}\right) = A(n)\hat{P}_n^{\overline{\xi}}(\overline{\xi}_i).$$

Lemma 4.42 *Let $L_n(\overline{\xi}) \in \mathcal{L}$. Then $\overline{\xi}_1, \ldots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}$ are the roots of $\hat{P}_n^{\overline{\xi}}$ if and only if $\overline{\xi}_1, \ldots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}$ are the roots of P_n.*

Proof. Let $\overline{\xi}_1, \ldots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}$ be the roots of $\hat{P}_n^{\overline{\xi}}$. Then

$$\hat{P}_n^{\overline{\xi}}(x) = \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor - 1} (x - \overline{\xi}_i).$$

In terms of elementary symmetric polynomials, this can also be written as

$$\hat{P}_n^{\overline{\xi}}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} (-1)^k S_k(\overline{\xi}) x^{\lfloor \frac{n}{2} \rfloor - 1 - k}.$$

A comparison of coefficients yields

$$(-1)^k S_k(\overline{\xi}) = \frac{B(n,k)(\overline{\xi})}{A(n)}, \text{ for } 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 2, \tag{3}$$

and

$$(-1)^{\lfloor \frac{n}{2} \rfloor - 1} S_{\lfloor \frac{n}{2} \rfloor - 1}(\overline{\xi}) = \frac{C(n)(\overline{\xi})}{A(n)}. \tag{4}$$

For $k = 1$ it follows

$$S_1(\overline{\xi}) = \sum_{l=1}^{\lfloor \frac{n}{2} \rfloor - 1} \frac{1 + 2n - 4(l+1)}{1 + 2n - 4}.$$

In particular, $S_1(\overline{\xi})$ is a constant. Further, for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$, we get from (3) the recursive formula

$$S_k(\overline{\xi}) = S_{k-1}(\overline{\xi}) \frac{\sum_{l=k}^{\lfloor \frac{n}{2} \rfloor - 1} (1 + 2n - 4(l+1))}{\sum_{l=1}^{k} (1 + 2n - 4l)}$$

and from (4) we deduce

$$S_{\lfloor \frac{n}{2} \rfloor -1}(\overline{\xi}) = S_{\lfloor \frac{n}{2} \rfloor -2}(\overline{\xi}) \frac{1+2n-4\left\lfloor \frac{n}{2} \right\rfloor}{\sum_{l=1}^{\lfloor \frac{n}{2} \rfloor -1}(1+2n-4l)}.$$

This implies for $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ the formula

$$S_k(\overline{\xi}) = \prod_{r=1}^{k} \frac{\sum_{l=r}^{\lfloor \frac{n}{2} \rfloor -1}(1+2n-4(l+1))}{\sum_{l=1}^{r}(1+2n-4l)}.$$

In particular, for $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, $S_k(\overline{\xi})$ are constants and hence independent of $\overline{\xi}$. From the fact that $S_k(\overline{\xi}) = \mathcal{S}_k$, follows one direction of the proof.

Conversely, let $\overline{\xi}_1, \dots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor -1}$ be the roots of P_n. Then we have

$$P_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor -1} (-1)^k S_k(\overline{\xi}) x^{\lfloor \frac{n}{2} \rfloor -1-k}.$$

In particular $\mathcal{S}_k = S_k(\overline{\xi})$ holds. By construction, the numbers $\mathcal{S}_k$ satisfy the relations

$$\mathcal{S}_{\lfloor \frac{n}{2} \rfloor -1} = \frac{1+2n-4\left\lfloor \frac{n}{2} \right\rfloor}{A(n)} \mathcal{S}_{\lfloor \frac{n}{2} \rfloor -2}$$

and for $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 2$

$$\mathcal{S}_k = \frac{1}{A(n)} \Bigg(\left(1+2n-4\left\lfloor \frac{n}{2} \right\rfloor\right) \mathcal{S}_{k-1} + \sum_{l=1}^{\lfloor \frac{n}{2} \rfloor -1-k} (1+2n-4(l+k)) \left(\mathcal{S}_{k-1} + \mathcal{S}_k\right) \Bigg).$$

This implies

$$\mathcal{S}_{\lfloor \frac{n}{2} \rfloor -1} = (-1)^{\lfloor \frac{n}{2} \rfloor -1} \frac{C(n)(\overline{\xi})}{A(n)}$$

and

$$\mathcal{S}_k = (-1)^k \frac{B(n,k)(\overline{\xi})}{A(n)},$$

for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 2$. Hence we showed $\hat{P}_n^{\overline{\xi}} = P_n$.

□

Lemma 4.43 *Let $n \geq 4$ and $L_n(\overline{\xi}) \in \mathcal{L}$. Then $\overline{\xi}_1, \ldots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}$ is a solution of the system of equations given by*

$$F_i(n)\,(\xi) = 0, \qquad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

if and only if $\overline{\xi}_1, \ldots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}$ are roots of the polynomial P_n.

Proof. Since the relation

$$F_i(n)\left(\overline{\xi}\right) = A(n)\hat{P}_n^{\overline{\xi}}(\overline{\xi}_i), \text{ for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

holds and $A(n) \neq 0$, it follows that $\overline{\xi}_1, \ldots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}$ is a solution of

$$F_i(n)\,(\xi) = 0, \qquad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

if and only if $\overline{\xi}_1, \ldots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}$ are roots of the polynomial $\hat{P}_n^{\overline{\xi}}$. The previous lemma concludes the proof.

□

Lemma 4.44 *Let $L_n\left(\overline{\xi}\right) \in \mathcal{L}$. Then $L_n\left(\overline{\xi}\right)$ is the unique critical point of $\tilde{\vartheta}_{D_n}$ in $\mathcal{W}$, if and only if $\overline{\xi}_1, \ldots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}$ are the roots of P_n.*

Proof. clear.

□

This concludes the proof of the main theorem.

□

Remark 4.45 *It is not proven yet, that the unique critical point of* $\tilde{\vartheta}_{D_n}$ *in* $\mathcal{W}_n$ *is contained in* $\mathcal{L}$*. But there is very strong evidence that the following conjecture is true, which would imply this statement. The validity of the conjecture for* $n = 4, 5, 6, 7, 8$ *can be veryfied using the examples at the end of this section.*

Conjecture 4.46 *For* $n \geq 4$*, the polynomial* P_n *has* $\lfloor \frac{n}{2} \rfloor - 1$ *distinct roots* $\overline{\xi}_1, \ldots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}$ *with* $0 < \overline{\xi_i} < 1$*.*

Remark 4.47 *The conjecture above would imply the following two theorems.*

Theorem 4.48 **(Existence) There exists a unique point* $L_n(\overline{\xi}) \in \mathcal{L}$ *such that*
$K \cdot Exp_{eK}\left(L_n\left(\overline{\xi}\right)\right)$ *is the unique minimal principal orbit of the isotropy action on* G/K*.*

Proof. Existence would follow from the validity of Conjecture 4.46. Uniqueness from [HSTT]

□*

Theorem 4.49 ** The unique minimal principal orbit of the isotropy action on* G/K *is given by* $K \cdot Exp_{eK}\left(L_n\left(\overline{\xi}\right)\right)$ *, with* $\overline{\xi}_1, \ldots, \overline{\xi}_{\lfloor \frac{n}{2} \rfloor - 1}$ *being the distinct roots of the polynomial* P_n*.*

Proof. The order of the $\overline{\xi}_i$ is not important in the previous theorem since a permutation of the $\overline{\xi}_i$ is due to the action of the Weyl group.

□*

4.2 Examples

In the following we give the polynomial P_n including the roots for $n = 4, 5, 6, 7, 8$ explicitly. We remark that the roots are all distinct and lie in the interval $(0, 1)$, respectively, substantiating Conjecture 4.46. This shows in particular, that for $n = 4, 5, 6, 7, 8$ the minimal principal orbit is an element in $\mathcal{L}$.

D_4-Problem

The root of

$$P_4(x) = x - \frac{1}{5}$$

is given by

$$\xi_1 = \frac{1}{5}.$$

D_5-Problem

The root of

$$P_5(x) = x - \frac{3}{7}$$

is given by

$$\xi_1 = \frac{3}{7}.$$

D_6-Problem

The roots of

$$P_6(x) = x^2 - \frac{14}{21}x + \frac{1}{21}$$

are given by

$$\xi_1 = \frac{7 + 2\sqrt{7}}{21},$$
$$\xi_2 = \frac{7 - 2\sqrt{7}}{21}.$$

D_7-Problem

The roots of

$$P_7(x) = x^2 - \frac{30}{33}x + \frac{5}{33}$$

are given by

$$\xi_1 = \frac{15 + 2\sqrt{15}}{33},$$
$$\xi_2 = \frac{15 - 2\sqrt{15}}{33}.$$

D_8-Problem

The roots of

$$P_8(x) = x^3 - \frac{495}{429}x^2 + \frac{135}{429}x - \frac{5}{429}$$

are given by

$$\xi_1 = \frac{8\sqrt{55}}{143} \cos\left(\frac{1}{3} \arccos\left(\frac{\sqrt{55}}{30}\right)\right) + \frac{5}{13},$$
$$\xi_2 = \frac{-8\sqrt{55}}{143} \sin\left(-\frac{1}{3} \arccos\left(\frac{\sqrt{55}}{30}\right) + \frac{\pi}{6}\right) + \frac{5}{13},$$
$$\xi_3 = \frac{-8\sqrt{55}}{143} \sin\left(\frac{1}{3} \arccos\left(\frac{\sqrt{55}}{30}\right) + \frac{\pi}{6}\right) + \frac{5}{13}.$$

Remark 4.50 *For the cubic Polynomial P_8, the* casus irreducibilis *appears.*

Appendix

Lemma 4.51 *We have*

$$\begin{aligned}
\mathcal{F}_{s,s,s,s} =& \Bigg(\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{\pi}{4}\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{\pi}{4}\right)\Bigg)^{-1} \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{c,s,s,s} =& \Bigg(\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{\pi}{4}\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{\pi}{4}\right)\Bigg)^{-1} \\
&\cdot \cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{F}_{s,c,s,s} =& \Bigg(\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{\pi}{4}\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{\pi}{4}\right)\Bigg)^{-1} \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_{s,s,c,s} =& \Bigg(\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{\pi}{4}\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{\pi}{4}\right)\Bigg)^{-1} \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_{s,s,s,c} =&\Bigg(\sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{\pi}{4}\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{\pi}{4}\right)\Bigg)^{-1} \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) - \frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \sin\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{1}{2}\arccos\left(-\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right) \\
&\cdot \cos\left(\frac{1}{2}\arccos\left(-\sqrt{\xi_1}\right) + \frac{1}{2}\arccos\left(\sqrt{\xi_{\lfloor \frac{n}{2} \rfloor}}\right)\right)
\end{aligned}
$$

Proof. One proves this lemma by applying addition theorems and using standard identities for trigonometric functions.

□

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