Modeling the measurement uncertainty with Fuzzy approach

A.Burak Göktepe[1], Selim Altun[2]

[1]Fan River Cascade Project, Reshen, Albania
[2]Department of Civil Engineering, Ege University, İzmir, Turkey

Abstract

There are several types of uncertainty in a material characterization arisen from different sources of measurement errors, such as methodological, instrumental, and personal. As a reason of the uncertainty in material models, it is plausible to consider model parameters in an interval instead of a singleton. The probability theory is widely known method used for the consideration of uncertainties by means of a certain distribution function and confidence level concept. In this study, fuzzy logic is considered within a material characterization model to deal with the uncertainty coming from random measurement errors. Data points are treated using fuzzy numbers instead of single values to cover random measurement errors. In this context, an illustrative example, prepared with core strength-rebound hammer data obtained from a concrete structure, is solved and evaluated in detail. Results revealed that there is a potential for fuzzy logic to characterize the uncertainty in a material model arisen from measurement errors.

Introduction

Experimental analysis is the integral part of material characterization conducted for the determination of engineering parameters of materials. Basically, depending on the methodology, testing equipment, and the person performing the experiment, constant and random measurement errors are occurred throughout the experimental analyses. Methodological errors are originated from the lack of theoretical feedback and several assumptions made within the testing technique. Instrumental errors are due to weaknesses and drawbacks of testing equipments. As the name implies, personal errors are occurred as a reason of individual mistakes made by the performer. On the other hand, cumulative measurement error is considered with mathematical models that synthesize underlying elementary errors, which characterize the smallest measurement errors. Elementary error is determined by the inaccuracies in respective measurements and can be categorized into two groups, i.e. (a) constant (systematic) errors and (b) random errors. Constant errors are independent from repetition of the experiment and vary randomly within certain limits due to consistency of outcomes. They are characterized by mathematical models as well as using uniform distribution. On the other hand, random errors are caused by the inaccuracies of testing devices, and cannot be predicted easily. They are generally considered by using normal distribution [1, 2].

In the literature, there are various studies dealing with measurement uncertainties utilizing probability theory. Castrup [3] presented a methodology to calculate measurement uncertainty with statistical approach. In another study, an algorithm was developed to estimate uncertainties in situations where data samples are ambiguous [4]. Phillips et. al. [5] described an approach to better calculate measurement uncertainty using Bayesian inference. Guidelines for the statistical calculation of measurement uncertainty can be found in the written material [6]. On the other hand, several researchers focused on the consideration of measurement uncertainty by means of fuzzy set theory. In this context, Mauris, et al. [7] used a fuzzy subset for the representation of measurement uncertainty as an upper bound of a family of a probability distribution. In another notable study, α-cut concept in fuzzy set theory was utilized as the measure of measurement inexactness instead of confidence interval in statistical approach [8]. It should be noted that there are more studies on measurement uncertainty in the written material; however, a few of them is listed here due to the lack of space.

In this study, random uncertainty in measurement errors is handled by means of a fuzzy model. In this context, input data is treated using fuzzy numbers and approximate reasoning is adopted for inference process. Furthermore, an illustrative example on a material characterization problem is considered with the fuzzy model, and the results are evaluated in detail.

Problem description

Measurement uncertainty is a keynote issue influencing the confidence of a testing system or the outcomes of a modeling process. The basic way of characterizing measurement uncertainty is the probabilistic approach by treating the first two moments, i.e. mean and variance, of the probability distribution. Generally, measurement uncertainty can be grouped into two categories in terms of estimation discrepancies, namely *Type A* and *Type B*. Type A evaluation is essentially based on the statistical analysis of series of observations. Type A uncertainty (u_A) is commonly calculated by standard deviation (s_i) and the number of degrees of freedom (v_i). The most widely used techniques for the calculation of Type A uncertainties are: (a) curve fitting, (b) analysis of variance, and (c) the standard deviation of the mean of measurements as given below [6]:

$$u_A(x_i) = s(\bar{X}_i) = \sqrt{\frac{1}{n(n-1)}\sum_{k=1}^{n}\left(X_{i,k} - \bar{X}_i\right)^2} \tag{1}$$

in which, n is number of observations, X_i input quantity, $\bar{X}_i$ is the mean, and x_i denotes the input estimate. On the other hand, Type B calculation is performed by all of the relevant information, such as expert knowledge, previous observations, calibration reports, and technical specifications. In other words, Type B uncertainty (u_B), which usually coming from unsystematic sources of error, is commonly considered by experience and/or confidence interval concept in probability distributions. The following formula can be used to calculate Type B source of uncertainties [6, 9].

$$u_B = \frac{A}{\Phi^{-1}\left[\frac{1+p}{2}\right]} \tag{2}$$

where, Φ^{-1} is the inverse of the normal probability distribution function, A is the calculation interval , and p is interval probability for the range between $+\Delta A$ and - ΔA. With a simple approach, the following expression can be used on the basis of uniform probability distribution [6]:

$$u_B = \frac{A}{\sqrt{3}} \tag{3}$$

Apart from these, Eq.2 can be rewritten using the student t distribution, the estimating variance ($\pm^{2}$), and Type B degrees of freedom (v_B) as given below [6, 9]:

$$v_B = \frac{1}{2}\frac{u_B{}^2}{\pm^2(u_B)} \tag{4}$$

On the other hand, Eq.2 can be generalized for the interval probability (p) and the calculation interval (A) as given below:

$$£\left(u_B\right)=\left(\frac{\partial u_B}{\partial A}\right)£\left(A\right)+\left(\frac{\partial u_B}{\partial p}\right)£\left(p\right) \tag{5}$$

and Eq.5 finally yields the following formulation for the calculation of u_B [3, 4]:

$$\frac{\pm^{2}\left(u_B\right)}{{u_B}^2}=\frac{{u_A}^2}{A^2}+\frac{® \, e^{\Phi^2}{u_p}^2}{2\Phi^2} \tag{6}$$

Apart from these, the uncertainty in overall measurement outcomes can be evaluated using the estimated standard deviation of the result, namely *combined standard uncertainty* (u_c). This approach is generally referred to as the *law of propagation of uncertainty*. With the help of first order Taylor series approximation, the following formulation can be used for the calculation of u_c [6]:

$${u_c}^2=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)+2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j) \tag{7}$$

where, the partial derivative $\partial f / \partial x_i$ is the sensivity coefficient, $u(x_i)$ is the standard uncertainty relevant with x_i , and $u(x_i , x_j)$ is the estimated covariance with respect to x_i and x_j.

In this study, Type B (random) measurement uncertainty is handled by means of fuzzy set theory. Apart from previous fuzzy-logic-based measurement uncertainty considerations, inexactness is handled by representing the input data using triangular membership functions. Moreover, the nonlinear mapping between input and output variables of the system is established by a fuzzy inference system. Consequently, the represented methodology is illustrated by a numerical example.

Fuzzy logic

Fuzzy logic is a multi-valued logic, which reduces the system's complexity arising from the uncertainty in the form of ambiguity, by allowing intermediate values to be defined between true and false. From this point of view, fuzzy sets make possible to reason not only utilizing discrete symbols and numbers but also using ambiguous information. Fuzzy logic is a formal characterization of fuzzy set theory using logical constructs to manipulate fuzzy systems by incorporating the heuristics. Consequently, fuzzy set theory is an outstanding tool for representing the uncertainty with vagueness and data imprecision, and fuzzy logic is a way of representing the knowledge embedded in fuzzy sets as well as making human-like inferences [10, 11].

There are several inference (implication) techniques developed for use with fuzzy systems. Lukasiewicz, Mamdani, Sugeno, and Tsukamoto inference techniques are some examples to be listed. On the other hand, majority of real-life problems require a single solution instead of an inference region to draw a conclusion. In order to accomplish this, defuzzification of the solution area is a compulsory process to obtain the outcome of the problem. There are several defuzzification techniques in the written material; however, centeroid method is the most popular technique in all of them [10]. The formulation of centeroid technique is as follows:

$$x^*=\frac{\int « _{\underset{-}{A}}(x)\,x\,dx}{\int « _{\underset{-}{A}}(x)\,dx} \tag{8}$$

where, $\underline{A}$ is fuzzy set, $\mu_{\underline{A}}$ is membership function, x is input variable, and x^* denotes defuzzified output value.

Problem formulation

As mentioned before, the mapping of elements of a fuzzy set to the universe of membership values is made by the membership function. If the universe of discourse is represented by X, the fuzzy set $\underline{A}$ can be denoted (Zadeh's notation) as follows [10]:

$$\underset{\sim}{A} = \left\{ \frac{\mu_{\underset{\sim}{A}}(x_1)}{x_1} + \frac{\mu_{\underset{\sim}{A}}(x_2)}{x_2} + \ldots \right\} = \left\{ \sum_i \frac{\mu_{\underset{\sim}{A}}(x_i)}{x_i} \right\} \tag{9}$$

The most fundamental form of membership functions is the *triangular* function (Figure 1), which can be characterized as given below:

$$\mu(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ \dfrac{c-x}{c-b} & b \le x \le c \\ 0 & c \le x \end{cases} \tag{10}$$

Hence, following implementation can be written for triangular fuzzy membership functions:

$$\mu(x;a,b) = \max\left[\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b} \right), 0 \right] \tag{11}$$

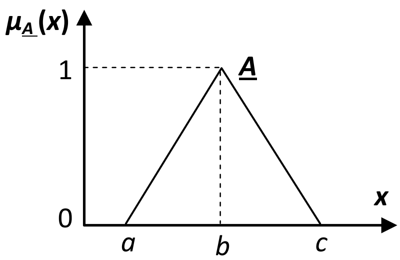

Figure 1 Triangular membership function

It should be noted for fuzzy rule-based systems that each logical proposition in the universe of discourse is characterized by a fuzzy set and the outcome of a rule is inferred using an implication technique, such as Zadeh, Mamdani, Lukasiewicz, etc. The process of implication is also referred to as the extension principle or approximate reasoning. Due to Mamdani's implication method, a fuzzy relation ($\underline{R}$) is derived from two or more fuzzy propositions ($\underline{A}$, $\underline{B}$, …) as given in Eq. 12 ($\underline{R} : \underline{A} \rightarrow \underline{B}$).

$$\mu_{\underset{\sim}{R}}(x, y, \ldots) = \min\left[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{B}}(y), \ldots \right] \tag{12}$$

Fuzzy rule-based systems consist of several rules, and the conclusion (inference) is drawn by a decomposition method. In addition, *IF* part of the rules involve conjunctive (AND) and disjunctive (OR) antecedents. A conjunctive fuzzy rule can be written as follows [10]:

(13)

IF x is $\underline{A}^1$ *AND* $\underline{A}^2$ *... AND* $\underline{A}^L$ *THEN y is* $\underline{B}^S$

then, decomposition can be made as:

$$\text{«}_{\underline{B}^S}(y) = \min\left[\text{«}_{\underline{A}^1}(x), \text{«}_{\underline{A}^2}(x), ..., \text{«}_{\underline{A}^L}(x)\right] \quad (14)$$

On the other hand, for disjunctive antecedents:

IF x is $\underline{A}^1$ *OR* $\underline{A}^2$ *... OR* $\underline{A}^L$ *THEN y is* $\underline{B}^S$ (15)

therefore, decomposition is made as follows:

$$\text{«}_{\underline{B}^S}(y) = \max\left[\text{«}_{\underline{A}^1}(x), \text{«}_{\underline{A}^2}(x), ..., \text{«}_{\underline{A}^L}(x)\right] \quad (16)$$

Consequently, graphical representation of a Mamdani fuzzy inference system is depicted in Figure 2.a. As can be seen from the figure that the outcome is a region; therefore, it is necessary to make a defuzzification in order to get single output value.

As emphasized before, the measurement uncertainty is considered by fuzzy set concept in this investigation. In detail, input parameters are characterized by triangular membership functions, and the uncertainty in the measurement process is overcome by this mean. In Figure 2.b. the fundamental difference is indicated graphically. Referring to the figure, the crossing point of the membership function and the fuzzy input variable is assumed as the input value. The following mathematical expression can be written to summarize the fundamental difference in terms of the implication philosophies:

$$\text{«}_{\underline{B}}(x_1, x_2, ..) = \min\begin{bmatrix} \min\left(junc\left(\text{«}_{\underline{A}^1}(x_1), \text{«}_{\underline{x}^1}(x_1)\right)\right), \\ \\ \min\left(junc\left(\text{«}_{\underline{A}^2}(x_2), \text{«}_{\underline{x}^2}(x_2)\right)\right), ... \end{bmatrix} \quad (16)$$

where, *junc* is the operator representing the process of the calculation of crossing points.

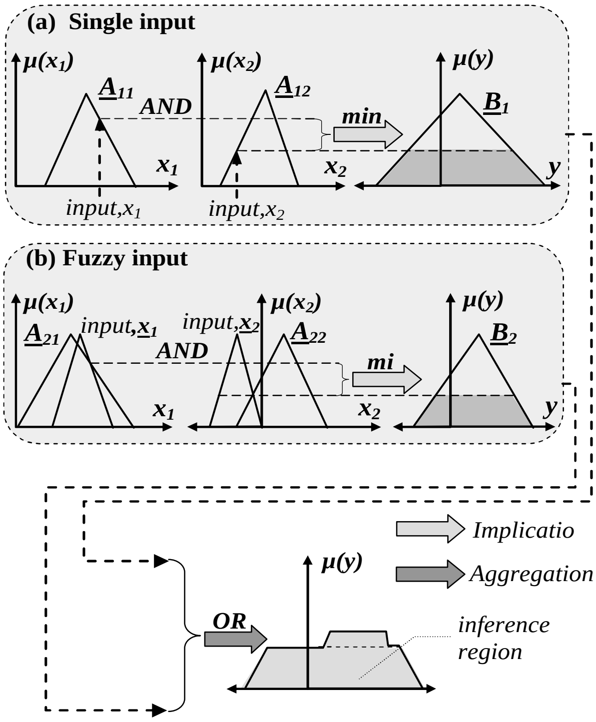

Figure 2 The fuzzy inference methodologies

Referring to Fig.3, which indicating a fuzzy partitioning ($\underline{A}ij$) and a fuzzy input ($\underline{x}_j$), the following piecewise expressions can be derived utilizing basic mathematics:

i. for $a \le x \le b$ and $0 \le y \le 1$

$$\frac{x-a}{b-a} = \frac{x-e}{f-e} \Rightarrow x = \frac{be-af}{(b+e-a-f)};\ y = \frac{x}{b-a} \tag{18}$$

$$\frac{x-a}{b-a} = \frac{x-d}{e-d} \Rightarrow x = \frac{ae-bd}{(a+e-b-d)};\ y = \frac{x}{b-a} \tag{19}$$

ii. for $b \le x \le c$ and $0 \le y \le 1$

$$\frac{x-b}{c-b} = \frac{x-d}{e-d} \Rightarrow x = \frac{be-cd}{(b+e-c-d)};\ y = \frac{x}{b-c} \tag{20}$$

$$\frac{x-b}{c-b} = \frac{x-e}{f-e} \Rightarrow x = \frac{bf-ce}{(b+f-c-e)};\ y = \frac{x}{b-c} \tag{21}$$

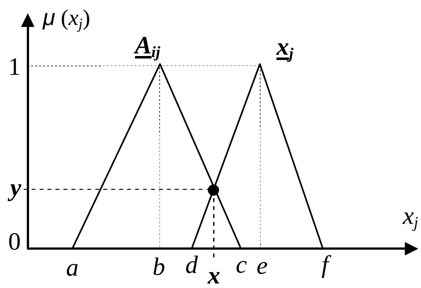

Figure 3 Illustration of fuzzy partitioning and fuzzy input

In case of one or more crossing point exists, the minimum value is considered for the determination of x and y coordinates, which are essential for the implication process. The following expression can be given to characterize the implication as well as the aggregation (for disjunctive rules) processes:

$$IR = \max\left[\mu_{\underline{B}_i}\right] = \max\left[\min\left(\mu_{\underline{A}_{ij}}, \underline{x}_j\right)\right] = \max\left[\min\left(\min(y_{ijk})\right)\right] \quad (22)$$

where, $\mu_{\underline{B}(\mathrm{i})}$ is membership function of the output fuzzy variable, $\mu_{\underline{A}(\mathrm{i,j})}$ is represents membership functions of input fuzzy variables, $\underline{x}_j$ is fuzzy input, y_{ijk} is the membership value computed at each step, and IR denotes closed inference region of the fuzzy system.

The inexactness represented by a fuzzy input is fundamentally based on the edges of the triangle, namely d, e, and f in Fig.3. Therefore, degree of the fuzziness can be adapted by changing the points. Another point must be mentioned that presented methodology handles the inexactness with fundamentally different approach from the techniques in the written material. Previous techniques depend on confidence intervals in probability distributions and α-cut concept in fuzzy set theory. Nevertheless, presented approach considers the uncertainty with using fuzzy input variables as well as changing the support (i.e. base points) of triangular fuzzy input. Therefore, a solution area comprising uncertainty effects is produced in this methodology instead of a single uncertainty value, and an additional procedure is considered for the calculation of the uncertainty.

Application

In this part of the investigation, presented methodology is considered with a numerical example. In this context, the database, which comprises rebound hammer (*RH*) and core strength (*CS*) values, is utilized for the evaluation of the methodology. The scatting of treated data is illustrated in Figure 4, and descriptive statistical parameters are given in Table 1.

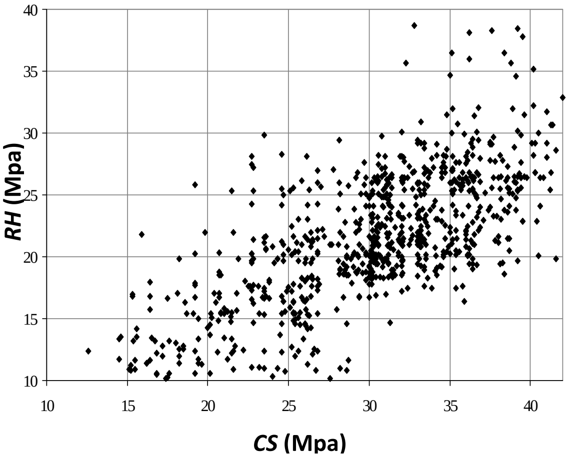

Figure 4 The evaluation database

In the first part of the verification study, Type B measurement uncertainty (u_B) is calculated due to the conventional procedure [6, 9]. The results of *RH* and *CS* free parameters are given for different confidence levels (ΔX) and error limits ($\pm\Delta X$) in Tables 2 and 3, respectively.

Table 1 Descriptive statistics of the data

Parameter	***RH***	***CS***
Average	29.72	20.98
Median	30.7	21.2
Min	12.6	5.14
Max	48.50	38.70
Std. dev.	6.74	5.91
Variance	45.42	34.87
Skewness	-0.419	-0.230
Kurtosis	-0.445	-0.127

It should be noted for this investigation that the fuzzy approach is utilized for the computation of Type B measurement uncertainties, which come from unsystematic sources of error. Therefore, only Type B uncertainties are included in the verification application.

Table 2: Results of the conventional analyses performed for *RH* variable

a*X*	u_B	a**U**/**U**						
		±10	±5	±4	±3	±2	±1	±0
90	18.05	0.22	0.16	0.15	0.14	0.14	0.13	0.13
95	15.15	-	0.18	0.17	0.15	0.14	0.13	0.13
96	14.46	-	-	0.18	0.16	0.14	0.14	0.13
97	13.68	-	-	-	0.17	0.15	0.14	0.13
98	12.76	-	-	-	-	0.16	0.14	0.13
99	11.53	-	-	-	-	-	0.15	0.13

Table 3 Results of the conventional analyses performed for *CS* variable

a*X*	u_B	a**U**/**U**						
		±10	±5	±4	±3	±2	±1	±0
90	12.75	0.24	0.18	0.18	0.17	0.17	0.16	0.16
95	10.70	-	0.21	0.19	0.18	0.17	0.11	0.16
96	10.21	-	-	0.20	0.18	0.17	0.17	0.16
97	9.66	-	-	-	0.19	0.18	0.17	0.16
98	9.01	-	-	-	-	0.19	0.17	0.16
99	8.14	-	-	-	-	-	0.18	0.16

In the second part of the application, a fuzzy model is employed to establish a correlation between *RH* and *CS* free parameters. In this context, *CS* and *RH* parameters are considered with fuzzy variables in the model. Furthermore, ΔX, and $\pm\Delta X$ parameters are taken into account by changing the support of triangular fuzzy inputs, namely *d* and *f* points in Figure 3.

As emphasized before, because presented approach handles the uncertainty with the help of a region instead of a singleton, it is not possible to make one-to-one comparison between conventional results and outcomes obtained here. However, three different ways are followed to evaluate the outcomes: (1) Comparison of outcomes of the fuzzy model obtained using singleton and fuzzy outputs, (2) Examination of the results of changing base points (*d* and *f*) of the triangular fuzzy inputs, (3) Indirect comparison with conventional results.

In order to perform the third step of the evaluation study, namely the comparison with conventional uncertainty values, the length of fuzzy confidence interval that is the half of α-cut width is computed through the following formulation:

$$«_{\sim}\left(A_{AV}\right)=\frac{\left(a_R^{AV}-a_L^{AV}\right)+\left(1-\sim\right)\left(\ _R^{AV}-\ _L^{AV}\right)}{2} \qquad (23)$$

where, $a^{AV}{}_R$, $a^{AV}{}_L$, $^{AV}{}_R$, and $^{AV}{}_L$ are points describing fuzzy trapezoidal interval A_{AV}, which are computed as given below:

$$a_{¥}^{AV}=\frac{\sum_{i=1}^{N} a_{¥}^{i}}{N} \qquad (24)$$

$$_{¥}^{AV}=\frac{\sqrt[\frac{q}{q-1}]{\sum_{i=1}^{N}\left(\ _{¥}^{i}\right)^{\frac{q}{q-1}}}}{N} \qquad (25)$$

in which, ¥=: L, R, q is the level of confidence, and N denotes data number. In Table 4, the summary of all evaluation sessions are given.

Table 4 Overall results of the evaluation study

a*X*	***RH***			***CS***		
	u_{BC}*	**u_{BS}***	**u_{BF}***	**u_{BC}***	**u_{BS}***	**u_{BF}***
90	18.05	21.36	27.67	12.75	14.56	18.81
95	15.15	18.41	22.71	10.70	11.98	11.90
96	14.46	16.75	15.23	10.21	11.63	11.74
97	13.68	15.04	19.07	9.66	11.08	11.52
98	12.76	14.48	14.32	9.01	10.92	11.36
99	11.53	13.99	11.70	8.14	10.75	11.03

** C: Conventional, S: Single input, F: Fuzzy input*

As can be derived from Table 4, fuzzy models that comprise single input and fuzzy input produced higher uncertainty values with respect to conventional analysis. Nevertheless, it is possible to conclude that the discrepancies are not excessive. On the other hand, the fuzziness in the input variables caused considerable changes in the outcomes.

Conclusion

In this investigation, fuzzy logic is used for the consideration Type B measurement uncertainty. Results revealed that there is a potential for fuzzy approach in terms of handling measurement uncertainties.

Generally, fuzzy set theory is successful for the description of systematic errors; however in this investigation, the fuzzy approach is just utilized for the calculation of random-sourced measurement uncertainties. It can also be treated for the calculation of combined measurement uncertainties with some modifications.

In this study, membership functions are selected as triangular; however, different types of membership function, such as Gaussian and Sigmoidal, can also be considered.

References

[1] Cacuci, D.G., (2003). Sensivity and Uncertainty Analysis, Theory, Vol.1, *Chapman & Hall/CRC Press.*

[2] Cacuci, D.G., (2005). Sensivity and Uncertainty Analysis: Applications to large-scale systems, Vol.2, *Chapman & Hall/CRC Pres*s.

[3] Castrup, H., (1995). Uncertainty Analysis for Risk Management, *49th ASQC Annual Quality Congress, Cincinnati*.

[4] Castrup, H., (2000). Estimating Category B Degrees of Freedom, *Measurement Science Conference*, Anaheim.

[5] Phillips, S.D., Levenson, M.S., Estler, W.T., and Eberhardt, KR, (1998). Calculation of Measurement Uncertainty Using Prior Information, *NIST Journal of Research*, Institute of Standards and Technology, **103**(6), 625-632.

[6] Taylor, B.N. and Kuyatt, C.E., (1994). *Guidelines for Evaluating and Expressing the Uncertainty of NIST Measurement Results*, Technical Note 1297.

[7] Mauris, G., Lasserre, V., and Foully, L., (2001). A Fuzzy Approach for Expression of Uncertainty in Measurement, *Measurement*, **29**, 165-177.

[8] Urbanski M.K. and Wasowski J., (2003). Fuzzy Approach to the Theory of Measurement Inexactness, *Measurement*, **34**, pp.67-74.

[9] ISO, (1995). *Guide to the Expression of Uncertainty in Measurement*,International Standards Organization, Geneva, Switzerland.

[10] Ross, T.J., (2004). *Fuzzy Logic with Engineering Applications*, 2nd Ed., McGraw Hill Co.

[11] Cox, E., (2005). *Fuzzy Modeling Tools for Data Mining and Knowledge Discovery*, Elsevier Science & Technology Books.