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# Comparative Analysis of Limit Bearing Capacity of a Continuous Beam Depending on the Character of the Load

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## ABSTRACT

Determination of the bearing capacity of a structure, as well as the assessment related to the structure failure is very valuable, not only as a simple control of beam bearing capacity, but also as a significant basis and factor in designing of structures. When the structure is exposed to the action of a proportionally increasing load, by applying the limit analysis it is possible to determine the limit failure load which is one of the bearing capacity indicators. In the case when beam systems are exposed to repeated load, the limit theorems do not yield the adequate solutions, thus the adaptation theorems which made safe limit load determination possible were developed simultaneously. Applying the limit and shakedown analysis, the analysis of bearing capacity of a continuous beam with two spans was conducted in the paper. Also displayed is the difference between the values of failure forces depending on the character of load and the beam span value in order to assess justification for application of the shakedown method in the analysis of the limit bearing capacity of the beams.

## INTRODUCTION

Engineering structures or some of their parts are exposed to various types of load, some of which acting independently. Some of these loads are constant, while other are not defined in the course of time and they belong to the group of variably repeated loads. In a large number of cases only the domain to which the variably repeated load belongs can be defined.

Limit analysis of structures is an analytic procedure which determines the maximum load parameter or load increment parameter, which can be sustained by an elasto-plastic structure. Determination of the bearing capacity of a structure, as well as the assessment of the structure failure is very valuable, not only as a simple control of beam bearing capacity, but also as a significant basis and factor in designing of structures.

The fundamental theorems of limit analysis were set by [1]. Limit theorems in designing of engineering structures were later applied by many authors, the most prominent being Greenberg and Prager [2], Horne, Neal [3], Hodge; Baker and Heyman [4], Zyczkowski.

Limit load of structures determined by application of the limit analysis is one of the indicators of bearing capacity of structure exposed to the action of proportional load. When a structure is exposed to the action of variable repeated load, the failure occurs under action of the load which is lower than the load obtained by application of the limit analysis of structures.

Application of shakedown theory in assessment of safety of elasto-plastic structures exposed to the action of variable, repeated load is important, and often indispensable. In this context the term "shakedown" introduced by Prager, means that after the onset of initial plastic deformations, the structure acts purely elastic in its further service. The opposite state, which leads to the unsafe structure, is called “non-adaptation” of the structure. The structure in this case undergoes failure due to one or both forms of failure called incremental collapse and alternating collapse. The incremental collapse occurs due to accumulation of plastic deformations during each load cycle (progressive deformation), causing reduction of structure durability, while the alternating collapse results from the repetition of plastic deformations of the opposite sign, (without accumulation of plastic deformations) causing in this manner a phenomenon of low cycle fatigue.

The first papers in this field were presented by Bleich, while Melan [5] proved the static shakedown theorem, and Koiter [6] dynamic shakedown theorem. These two theorems were successfully applied in solving of a large number of engineering problems (Maier, Corradi and Zavelani, Kaliszky, König [7], Polizzoto, Ponter and Karter).

In the recent years, the shakedown analysis of elasto-plastic structures becomes increasingly applied in the analysis of engineering problems. It is thus successfully applied in many engineering problems, such as designing of nuclear reactors, railways, civil engineering designing and safety assessment of some building structures.

The goal of this paper is to analyze the limit capacity of continuous beam in order to draw conclusions about the justification of application of shakedown method in the analysis of the beam limit capacity. Percentage discrepancy between the failure forces depending on the character of the load and the span length of the continuous beam with two spans via applying static limit theorem and shakedown analysis is presented.

## LIMIT AND SHAKEDOWN ANALYSIS

The basic theorems of limit analysis can be applied to all the types of static systems, irrespective of whether they are statistically determinate of statically indeterminate. The basic theorems of limit analysis consist of static theorem or the theorem of the lower border of limit load and kinematic theorem or the theorem of the upper border of limit load.

Static theorem is based on the static equilibrium of the observed system. A large number of distributions of bending moments meeting the equilibrium conditions as a result of the given external load can be assumed for one statically indeterminate system. Greenberg and Prager [2] named such distribution statically admissible. If on such system meets the yield condition, that is, if the bending moment has not exceeded the appropriate value it is claimed that it is also safe. A necessary requirement is that there must be at least one safe distribution of moments in the structure, which is also statically admissible. According to the static theorem, this is a sufficient condition for providing the bearing capacity.

The static theorem can be expressed in the following way: *if there exists any distribution of bending moment throughout structure which is simultaneously safe and statically admissible under the load $\lambda$P, then the value $\lambda$ must be less or equal to the factor of failure load $\lambda_C$, ($\lambda_C > \lambda$). The actual limit load ($\lambda_C P \leq P_p$) can be equal or higher than the given one.*

On the basis of this theorem, it can be concluded that under the given load $\lambda$P there is no distribution of bending moments which is simultaneously safe and statically admissible, that this $\lambda$ is higher than the factor of failure load $\lambda_C$. Also, it can be concluded that one static system can really bear the limit load without failure, considering that $\lambda_C$ is the maximum factor of load where the static equilibrium cannot be achieved without formation of plastic hinges.

Shakedown theorems have a role to set the main conditions under which the plastic yield in the structure finally ceases, regardless of how frequently and in what sequence the load was applied [10]. As well as in the limit analysis, in the shakedown analysis there are static and kinematic theorems, on whose basis it is possible to determine the safe limit load depending on the type of variable repeated load.

The bending moment of the observed cross section $j$ can be presented as:

$$\mathrm{M}_j = m_j + M_j \,, \tag{1}$$

where $\mathrm{M}_j$ is the actual bending moment of the cross section, $M_j$ is the elastic bending moment of the cross section and $m_j$ is the residual bending moment of the cross section.

Any distribution of residual bending moments, defined in this way must be statically possible in case when the structure is unloaded, because the moment $\mathrm{M}_j$ and $M_j$ must be in equilibrium with the external load[3]. Thus it can be said that the structure has adapted under the action of variable repeated load, if at some point the condition (1) has been satisfied, and all the following loads cause only elastic change of bending moments. Then it is possible to determine the value of safe limit load, which depending on the character of repeated load can be incremental limit load and alternating limit load.

On the basis of conditions (1) the static shakedown theorem can be expressed in the following form: *if there exists any distribution of residual bending moment $m_j$ throughout structure, which is statically admissible in the case with zero external loading and which also satisfies at every cross section j, it is necessary to meet one of the conditions:*

$$m_j + \lambda M_j^{max} \le \left(\mathrm{M_p}\right)_j, \; m_j + \lambda M_j^{min} \ge -\left(\mathrm{M_p}\right)_j, \; \lambda\left(M_j^{max} - M_j^{min}\right) \le 2\left(\mathrm{M_e}\right)_j \qquad (2)\ (3)\ (4)$$

*the value $\lambda$ will be less then or equal to the shakedown load factor $\lambda_S$.*

Each girder strives to adapt to the action of variable repeated load in a best possible way. Thus, if $\lambda$ exceeds the value $\lambda_S$, the unlimited plastic yield occurs, and in this case no distribution of residual moments is possible, which is a necessary condition for determination of safe limit load. Similarly, under the action of proportional load, the structure will fail when the load factor $\lambda$ reaches the value $\lambda_C$, above which the structure is not safe, and simultaneously there is a statically possible distribution of bending moments. Depending on the calculated load factor $\lambda$ it is possible to determine the safe limit load which depends on the type of variable repeated load, on the basis of meeting some of the requirements of the equations (2) and (3), as incremental conditions of plasticity and equation (4), as alternating plasticity conditions. Application of static shakedown theorem is possible only if distribution of residual bending moments is already known[3].

## ANALYSIS OF LIMIT BEARING CAPACITY OF CONTINUOUS BEAM DEPENDING ON THE CHARACTER OF THE LOAD

Applying the adequate method, and depending on the character of the load, an analysis of the limit load of continuous beam displayed on the Figure 1. was conducted. The span of the beams affects the distribution of internal forces, and therefore on the relevant condition of failure, that is, the value of the failure force. On the example of the continuous beam, a procedure of calculation of the failure force has been conducted, depending on the change of beam span, which is defined by the coefficients α and β.

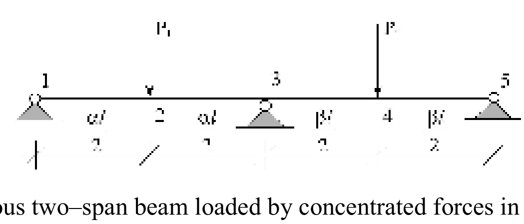


Figure 1. Continuous two–span beam loaded by concentrated forces in the middle of span

**Failure limit state**

Applying the static theorem for the observed beam, in the paper [8], was conducted a detailed analysis of limit bearing capacity. The load which leads to formation of failure mechanism, when (α≥β), that is when (α≤β) is:

$$P^{(1)}=\frac{32M_p}{3l(\alpha+\beta)}+\frac{M_p(18\beta-14\alpha)}{3\alpha l(\alpha+\beta)}, \quad P^{(2)}=\frac{32M_p}{3l(\alpha+\beta)}+\frac{M_p(18\alpha-14\beta)}{3\beta l(\alpha+\beta)} . \tag{5) (6}$$

The change of the limit failure force, in the Figure 2.(a) has been presented depending on the change of span length. Therefore, if α≥β a partial failure mechanism of the first span forms, while when β≥α a partial failure mechanism of the second span of the beam is generated, and in the case when *α*= *β* the failure mechanism forms in both spans simultaneously.

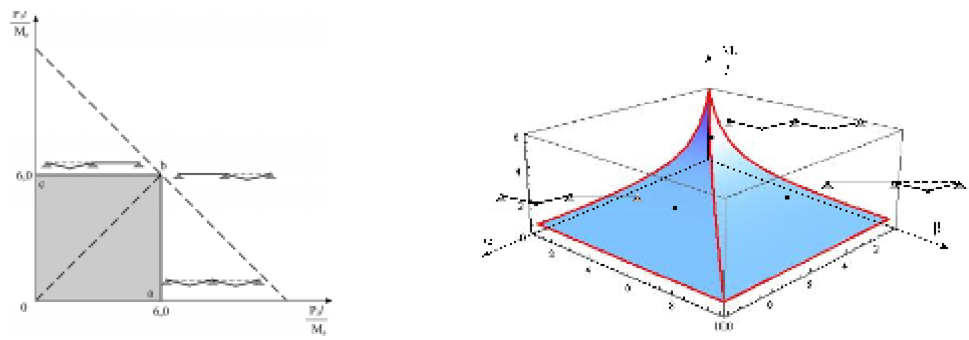


Figure 2. (a) Interaction diagram; (b) Change of the limit failure force depending on α and β

When the spans have equal lengths (α=β=1), and when the beam is simultaneously acted upon by two independent load systems $P_1$ and $P_2$, which are in arbitrary relationship, the limit bearing capacity analysis and defining of the area where the beam is safe from the onset of failure can be performed on the basis of the interaction diagram Figure 2(a). From the interaction diagram one may conclude that the failure mechanism in the second span forms when the relation of the load is $\frac{P_1}{P_2}\le 1$, while when $\frac{P_1}{P_2}\ge 1$ the failure mechanism forms in the first span of the beam. For any relation of loads lying inside the area 0abc0 there will occur no failure mechanism, and thus no beam failure. If the relation of the load is such so as to be defined by some of the segments, the failure mechanism defined by this segment is formed.

**Incremental failure load**

When the beam is exposed to the action of variable repeated load, the area in which the load acts lies within the following range: $0\le P_1\le P_1$, $0\le P_2\le P_2$. Applying the static shakedown theorem on the basis of equations (2) and (3), as well as incremental conditions of plasticity and equation (4) as alternating condition, the values of the forces causing the beam failure are obtained.

The relevant limit failure load depends on the relation of the coefficients α and β. So, if α≥β, the failure mechanism is formed in the first span, and the value of incremental failure force and residual bending moment are obtained on the basis of the expression:

$$8P_1\alpha(\alpha+\beta)+3P_2\beta^2=\frac{48M_p(\alpha+\beta)}{l}, \qquad m=\frac{P_1\alpha l(8\beta-\alpha)-6P_2\beta^2 l}{96(\alpha+\beta)}, \tag{7) (8}$$

while in the case when $\beta \geq \alpha$, the failure mechanism in the second span is formed. The incremental failure force and residual bending moment are obtained on the basis of the expression:

$$8P_2\beta(\alpha+\beta)+3P_1\alpha^2=\frac{48M_p(\alpha+\beta)}{l}, \qquad m=\frac{P_2\beta l(8\alpha-\beta)-6P_1\alpha^2 l}{96(\alpha+\beta)}. \qquad (9)\ (10)$$

On the basis of the alternating plasticity conditions (4) for the cross sections 2, 3 and 4 are obtained:

$$P_1 l\alpha(5\alpha+8\beta)+3P_2 l\beta^2=64M_e(\alpha+\beta),\ 3P_1 l\alpha^2+3P_2 l\beta^2=32M_e(\alpha+\beta),\ P_2 l\beta(8\alpha+5\beta)+3P_1 l\alpha^2=64M_e(\alpha+\beta). \qquad (11)\ (12)\ (13)$$

As $M_e=\dfrac{M_p}{\alpha_{form}}$, it is concluded that the value of alternating failure force depends on the coefficient of cross section form. Here the rectangular cross section is adopted whose form coefficient is $\alpha_{form}=1,50$.

On the basis of the expression (7) and (9), incremental failure load is obtained in one-parameter form, while on the basis of expression (8) and (10) the values of residual bending moments are obtained:

$$P_{inc}=\frac{48M_p(\alpha+\beta)}{l(8\alpha^2+8\alpha\beta+3\beta^2)}, P_{inc}=\frac{48M_p(\alpha+\beta)}{l(3\alpha^2+8\alpha\beta+8\beta^2)}, m=-\frac{M_p(\alpha^2-8\alpha\beta+6\beta^2)}{2(8\alpha^2+8\alpha\beta+3\beta^2)}, m=-\frac{M_p(\beta^2-8\alpha\beta+6\alpha^2)}{2(8\beta^2+8\alpha\beta+3\alpha^2)} \qquad (14)\ (15)\ (16)\ (17)$$

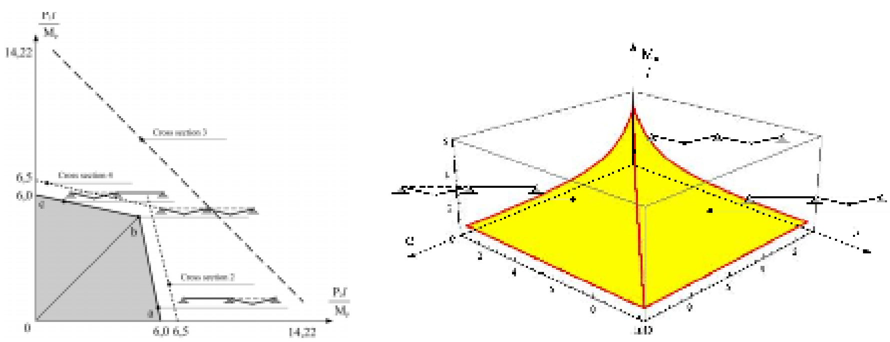


Figure 3. (a) Interaction diagram; (b) Change of incremental failure force depending on α and β

Interaction diagram (Figure 3.(a)) is constructed for the case when the beam spans are equal ($\alpha=\beta=1$), on the basis of expressions (7), (9), (11), (12) and (13). From the diagrams, it is observed that the safe area 0abc is defined on the basis of incremental failure condition.

On the basis of the expressions (14), (15), (16) and (17) the diagrams were constructed (Figure 3.(b) and Figure 4.(a)) on which the change of incremental force of failure and residual bending moment is displayed for: $1 \leq \alpha \leq 10$ and $1 \leq \beta \leq 10$. In the diagrams it is possible to observe the change of the incremental failure force and residual bending moments depending on the length of beam span, as well as of the relevant beam failure mechanism. In the diagram in Figure 4.(a) it can be observed that the value of the residual bending moment when $\alpha=\beta$ is constant, and amounts to $m=0,0263M_p$. Depending on the change of the beam span, the maximum value of the residual bending moment is $m=0,0595M_p$, and it occurs when $\alpha=1,804\beta$ whereby the failure mechanism in the first span is formed, i.e. for $\beta=1,804\alpha$ when the failure mechanism of the second span is formed.

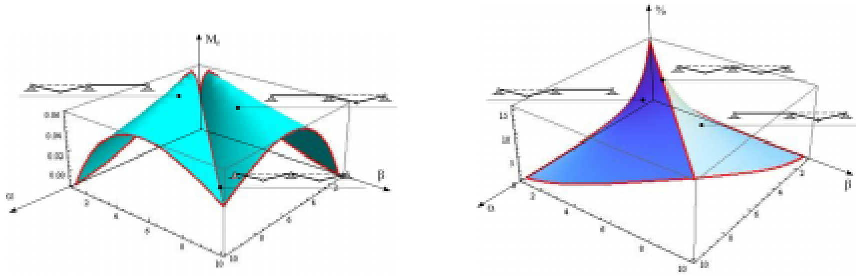

Figure 4. (a) Change of residual bending moment depending on α and β; (b) Change of limit and incremental failure force in percents depending on the change of α and β

The difference between the limit and incremental force of failure in percents, depending on the change of α and β is presented on the diagram in Figure 4. In the case when the spans of the beams are equal (α=β), this difference is the largest and amounts to 15,78%, while for α=1,804β, when the value of the residual moment is maximum, this difference is 6,90%.

**Alternating failure load**

In the further analysis of limit bearing capacity of beams it is assumed that the force in the first span of the alternating character ($-P_1 \leq P_1 \leq P_1$), while the force acting in the second span is in the range $0 \leq P_2 \leq P_2$. Applying the static shakedown theorem and failure conditions (2) and (3), when $\alpha \geq \beta$, the value of the failure force and the value of the residual bending moment are defined on the basis of the expressions:

$$8P_1 \alpha l(\alpha+\beta)+3P_2\beta^2 l=48M_p(\alpha+\beta), \qquad m=\frac{P_1\alpha l(8\beta-\alpha)-6P_2\beta^2 l}{96(\alpha+\beta)}. \qquad (18)\ (19)$$

That is, in the case when $\alpha \leq \beta$, the following is obtained:

$$3P_1\alpha^2 l+4P_2\beta l(\alpha+\beta)=24M_p(\alpha+\beta), \qquad m=\frac{P_2\beta l(8\alpha-\beta)-3P_1\alpha^2 l}{96(\alpha+\beta)}. \qquad (20)\ (21)$$

On the basis of the expression (18) and (20) the values of incremental failure force are obtained and, on the basis of the expression (19) and (21) the values of the residual bending moments:

$$P_{inc}=\frac{48M_p(\alpha+\beta)}{l(8\alpha^2+8\alpha\beta+3\beta^2)}, P_{inc}=\frac{24M_p(\alpha+\beta)}{l(3\alpha^2+4\alpha\beta+4\beta^2)}, m=-\frac{M_p(\alpha^2-8\alpha\beta+6\beta^2)}{2(8\alpha^2+8\alpha\beta+3\beta^2)}, m=-\frac{M_p(3\alpha^2-8\alpha\beta+\beta^2)}{4(3\alpha^2+4\alpha\beta+4\beta^2)}$$

(22) (23) (24) (25)

For the cross sections 2, 3 and 4 the following expressions are obtained on the basis of alternating condition of plasticity (4) of the static shakedown theorem:

$$P_1\alpha l(10\alpha+16\beta)+3P_2\beta^2 l=64M_e(\alpha+\beta),\ 3l(2\alpha^2P_1+\beta^2P_2)=32M_e(\alpha+\beta),\ 6P_1\alpha^2 l+P_2\beta l(8\alpha+5\beta)=64M_e(\alpha+\beta)$$

(26) (27) (28)

on whose basis the values of alternating forces for these characteristic cross sections are obtained, as follows:

$$P_{alt}=\frac{64M_e(\alpha+\beta)}{l(10\alpha^2+16\alpha\beta+3\beta^2)}, P_{alt}=\frac{32M_e(\alpha+\beta)}{3l(2\alpha^2+\beta^2)}, P_{alt}=\frac{64M_e(\alpha+\beta)}{l(6\alpha^2+8\alpha\beta+5\beta^2)} \qquad (29)\ (30)\ (31)$$

On the basis of the expressions (18), (19), as well as of the expressions (26), (27), (28) the interaction diagram was constructed (Figure 5.(a)) on which it can be observed that inside the area 0abc0 the beam is safe against onset of failure. This area is defined by the alternating failure condition of the cross section 2 and incremental failure condition corresponding to the formation of the second field failure mechanism.

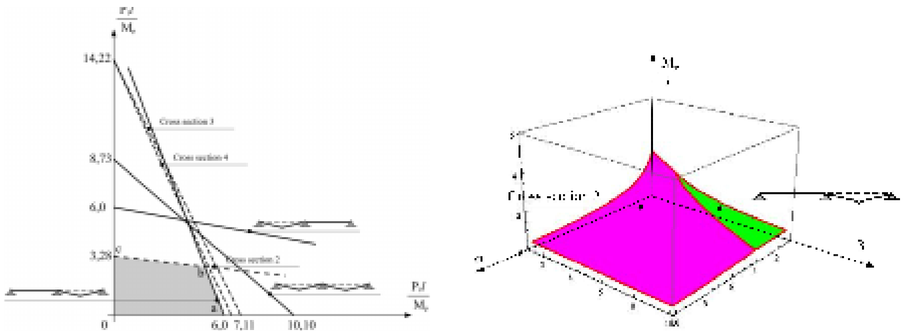


Figure 5.(a) Interaction diagram when the force is action in the middle of the first field of alternating character; (b) Change of failure force depending on α and β when the beam in the first field is loaded by the force of alternating character

On the basis of the expressions (23) and (29) a diagram was constructed (Figure 5.(b)) in the case when the failure load is defined in one-parameter form. On the diagram is presented the change of relevant condition of failure depending on the span of the beam. Thus, when $\frac{2,137}{\alpha}-\frac{5,555}{\beta}\geq 0$, the failure force is defined on the basis of alternating failure condition of the cross sections 2, and when $\frac{2,137}{\alpha}-\frac{5,555}{\beta}\leq 0$, the incremental failure condition is relevant, and the failure mechanism forms in the second span of the beam.

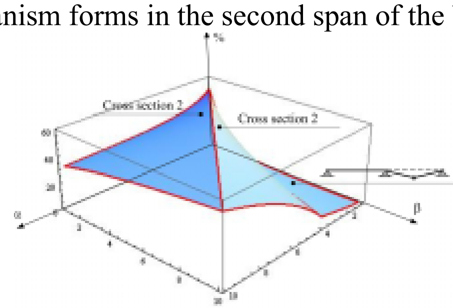


Figure 6. Change of limit and alternating failure forces depending on α and β when the beam is exposed to the action of alternating failure force in the first span

From the diagram presented in Figure 6. it can be observed that the difference between the alternating and limit forces of failure ranges between 32,74% and 50.95% when β=1, and α≥1. The largest difference between the forces of failure, when the spans are of equal lengths α=β, is 50,95%. In case when β≥α, the difference between the forces sharply decreases, so that it would be the smallest for β≥2,59α, when the failure force is defined on the basis of the incremental failure condition.

In case when both forces of alternating character are ($-P_1\leq P_1\leq P_1$, $-P_2\leq P_2\leq P_2$), the failure force is defined on the basis of the alternating condition of failure of the cross section 2, that is, of the cross section 4, for α<β, which is elaborated in detail in [8]. The largest difference between the limit and alternating force is 50,59% when the spans are of equal lengths. Depending on what span is larger, the relevant condition of failure changes, and the difference between the failure forces decreases up to 32,74%.

## CONCLUSION

The analysis of bearing capacity of linear beams exposed to proportionally increasing load and to a repeated load whose intensity lies in the previously defined range is presented on the example of the continuous beam with two spans. Applying static limit and shakedown theorem the failure load depending on the character of the load and the variations of beam span length was determined.

On the basis of Figure 4(b), where changes of limit and incremental failure force depending on the beam span value are presented it is concluded that the application of shakedown method is justified for certain relations of coefficients $\alpha$ and $\beta$, while in some cases the limit load can be determined through application of limit analysis considering that the difference between these forces is small. However, when one of the forces has alternating character, the difference between the failure forces, (Figure 6.) for some spans is up to 50,95%, so the application of shakedown analysis is obligatory.

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